Electro-Magnetic Duality, Magnetic monopoles and Topological Insulators.

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By

NOUNAHON Mahougnon Philippe

Supervisor: Popov Todor

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African University of Science and Technology (AUST)
www.aust.edu.ng
P.M.B 681, Garki, Abuja F.C.T
Nigeria.

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Abstract

The Maxwell equations of electrodynamics acquire an additional symmetry if one assumes the existence of hypothetical particles-magnetic monopoles, carrying a magnetic charge. The additional internal symmetry is the electromagnetic duality generated by the rotations in the space of electric and magnetic charges.

In this project we revise the electromagnetic duality in his global aspect starting with the celebrated Dirac monopole, a singular solution in a slightly modified Maxwell theory. We then take account of the new insight on the duality in the broken $SO(3)$ gauge theory where the magnetic monopoles arose as finite-energy smooth solution (found by 't Hooft and Polyakov). The stability of these monopoles is guaranteed by the conservation of topological invariants, i.e., these are topologically protected states. The spectrum of the gauge theory states enjoys a symmetry between the electrically charged gauge boson and the magnetic monopole, manifesting a quantum electro-magnetic duality which turns out to be a part of larger $SL(2,\mathbb{Z})$-group symmetry acting on the 2-dimensional charge lattice.

Recently the idea of magnetic monopoles and dyons was revived by the discovery of new kind of materials known as topological insulators. The theoretical considerations in the modified axion electrodynamics show that the electric charges on the boundary of a topological insulator induce mirror images carrying magnetic charges. We consider carefully the mirror images in the case of topological insulator with planar and spherical boundary. We then provide a description of the induced mirror images in a manifestly $SL(2,\mathbb{Z})$-covariant form.

Contents

Chapter 1

Introduction

This thesis will consider abelian $U(1)$ and non-abelian $SO(3)$ gauge theories allowing for states carrying units of magnetic charge, the so called magnetic monopoles. The existence of such magnetic monopoles was first suggested by Dirac as a speculation and an outcome of a thought experiment [3]. It attracts so much attention because if a monopole with magnetic charge g exists in nature, would automatically imply the quantization of the electric charge. In fact, the requirement that the wave-function solving the Schrodinger in the presence of monopole is single-valued function implies the *Dirac quantization condition*

$$eg = nh \qquad n \in \mathbb{Z}$$

where h stays for the Plank constant $h = 2\pi\hbar$ and then all electric charges are multiples of a minimal electric charge $e = \frac{h}{g}$.

The presence of magnetic charges would restore the broken symmetry between electric and magnetic charges, and the extended Maxwell equations enjoys *electromagnetic duality*, an exchange symmetry of the electric and magnetic components of the electromagnetic field. The electrodynamics with magnetic monopoles becomes highly symmetric, reducing the difference between "electric" and "magnetic" to a matter of convention.

The Dirac monopole suffers from one defect, it is described by a singular potential. It is only in the '70 after the advent of the non-abelian gauge theories when 't Hooft [9] and Polyakov [7] independently discovered that in a model with non-abelian gauge group G spontaneously broken to a $U(1)$ through the Higgs mechanism there exists a non-singular non-perturbative solution with a finite energy which is looking from outside like a Dirac monopole. This finite energy solution is called 't Hooft-Polyakov monopole, it is a static field configuration with potential non-vanishing at spatial infinity and Higgs field asymptotically approaching one of the Higgs vacuums. The stability of the finite energy solution is guaranteed by the conservation of *topological charges*, these are topological invariants of the field configuration. The magnetic charge of the 't Hooft-Polyakov monopole is a topological charge. The mass of the magnetic monopole has a lower bound, found by Bogomol'nyi [1]; the states that saturate the bound are called BPS-states (after Bogomol'nyi-Prasad-Sommerfeld). Montonen and Olive put forward a conjecture [5] that there should exist a dual "magnetic" gauge theory in which the roles of the massive gauge bosons and the magnetic BPS-monopoles are exchanged. This is an atractive possibility, since due to the Dirac quantization condition if the coupling constant e of the original theory is small, the coupling constant $g = \frac{h}{e}$ of the dual "magnetic" theory must be large, and *vice versa*. Therefore the strong coupling regime of a gauge theory will be controlled by the weak coupling regime of its dual gauge theory. An additional term in the Lagrangian of the gauge theory describes the Witten effect, forcing the magnetic monopoles to pick up non-trivial electric charges. Thus the excitation of the theory become *dyons*, these are particles with both electric and magnetic charge. The Montonen-Olive conjecture exchanging electric charged states with magnetic ones provides a quantum electromagnetic duality which can be enhanced to a larger group $SL(2,\mathbb{Z})$ operating on the lattice of dyons charges, that is, the two-dimensional lattice spanned by the quantized electric and magnetic charges.

Magnetic monopoles came back to the scene in a new guise after the discovery of the topological insulators. Topological insulators are new electronic materials insulating in bulk but having gapless edge or surface states which are protected against the opening up of a gap as long as the time-reversal symmetry is respected. Recently, Xiao-Liang Qi, Taylor Hughes and Shou-Cheng Zhang [10] proposed that the topological band insulator can provide a condensed-matter example of axionic electromagnetism. The axion field θ is now disguised as a parameter of the medium, together with the permittivity ϵ and the permeability μ. Only two disconected value of θ are compatible with the time-reversal symmetry of the problem; these are $\theta = 0$ corresponding to an ordinary (trivial) insulator and $\theta = \pi$ corresponding to a topological insulator. In the work [11] it was argued that an electric charge near the interface of topological insulators (with an ordinary insulator) induces as a mirror image a magnetic monopole in the bulk.

We do a carefull analysis of the induces charges of a topological insulator with a planar and spherical boundary. Finally we apply the ideas of the Montonen-Olive duality in the context of topological insulators providing a description of the induced mirror images into a $SL(2,\mathbb{Z})$-covariant form clarifing the meaning of the image charges.

Chapter 2

Electro-Magnetic Duality in Maxwell Theory

2.1 Maxwell Theory with Magnetic Charges

Maxwell's equations in the vacuum are given by :

$$\begin{array}{rclrcl} \overrightarrow{\nabla} \cdot \overrightarrow{E} & = & \vec{0}, & \overrightarrow{\nabla} \cdot \overrightarrow{B} & = & \vec{0} \\ \overrightarrow{\nabla} \wedge \overrightarrow{E} & = & -\frac{\partial \overrightarrow{B}}{\partial t}, & \overrightarrow{\nabla} \wedge \overrightarrow{B} & = & \frac{\partial \overrightarrow{E}}{\partial t} \end{array}$$

These equations are invariant under the electromagnetic duality transform

$$(\vec{E}, \vec{B}) \to (\vec{B}, -\vec{E}) \; .$$

Indeed, one can enlarge this duality to a rotation group. In order to see this, it is convenient to write Maxwell equations in a manifestly Lorentz covariant form, by introducing the field-strength $F^{\mu\nu}$ given by :

$$F^{i0} = E^i \quad F^{ij} = -\epsilon^{ijk} B^k \tag{2.1.1}$$

using $g^{\mu\nu} = (+ - --)$ and $\epsilon^{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta} = 1$. Defining the dual $*F^{\mu\nu}$ of the tensor $F^{\mu\nu}$ by $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ we get

$$*F^{i0} = B^i \tag{2.1.2}$$

then the free Maxwell's equations become

$$\partial_\mu F^{\mu\nu} = 0 \qquad \partial_\mu * F^{\mu\nu} = 0 \tag{2.1.3}$$

These two real equations can be combined in a single complex equation

$$\partial_\mu(F^{\mu\nu} + i * F^{\mu\nu}) = 0 \tag{2.1.4}$$

It easy to see this equation is invariant under complex multiplication

$$F^{\mu\nu} + i * F^{\mu\nu} \to \exp(i\varphi)(F^{\mu\nu} + i * F^{\mu\nu}) \tag{2.1.5}$$

where φ is a constant phase. In terms of the electric and magnetic fields,

$$E^i + iB^i \to \exp(i\varphi)(E^i + iB^i) \tag{2.1.6}$$

$$\Longrightarrow \{E^i \to E^i \cos\varphi - B^i \sin\varphi; B^i \to E^i \sin\varphi + B^i \cos\varphi\}$$

Taking $\varphi = -\frac{\pi}{2}$, it gives the previous particular transformation

$$(\vec{E}, \vec{B}) \to (\vec{B}, -\vec{E}) \,. \tag{2.1.7}$$

This beautiful duality transformation is lost when we consider Maxwell's equation in the presence of matter,

$$\partial_\mu F^{\mu\nu} = j_e^\nu \qquad \partial_\mu * F^{\mu\nu} = 0 \tag{2.1.8}$$

These equation are clearly not invariant under (2.1.4). In order to restore the symmetry in the presence of matter, in 1931 Dirac [3] ”postulated” the existence of a particles with magnetic charges and called them magnetic monopoles. Schwinger and Zwanziger followed the Dirac idea, considering the possibility of particles having both electric and magnetic charges, and called them dyons. In either case, Maxwell's equations read

$$\partial_\mu F^{\mu\nu} = j_e^\nu \qquad \partial_\mu * F^{\mu\nu} = j_m^\nu \tag{2.1.9}$$

where j_m^ν is the magnetic current. As before, these equations can be concombined as

$$\partial_\mu(F^{\mu\nu} + i * F^{\mu\nu}) = (j_e^\nu + ij_m^\nu) \tag{2.1.10}$$

where $\mu_0 = c = 1$. And it's invariant under (2.1.4) if the currents transform as

$$j_e^\nu + ij_m^\nu \to \exp(i\varphi)(j_e^\nu + ij_m^\nu) \,. \tag{2.1.11}$$

If the currents result from point particles, each one with electric and magnetic charge (q_a, g_a), we must have that

$$(q_a + ig_a) \to \exp(i\varphi)(q_a + ig_a) \tag{2.1.12}$$

2.2 Dirac Monopole

In electromagnetism without magnetic monopoles ,

$$\partial_\mu * F^{\mu\nu} = 0 \tag{2.2.1}$$

which implies that the field strength $F^{\mu\nu}$ can be written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.2.2}$$

where A_μ is well defined vector function in all space time. If A_μ solve the Maxwell equation (2.2.1) then

$$A'_\mu = A_\mu + \partial_\mu \alpha \tag{2.2.3}$$

is also solution (giving the same electric $\overrightarrow{B}$ and magnetic $\overrightarrow{B}$ fields).

The vector potential A_μ plays a central role in the quantum theory : a particle with mass m and electric charge q satisfies the Shrodinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}(i\hbar\overrightarrow{\nabla} - q\overrightarrow{A})^2\psi + qA_0\psi\ . \tag{2.2.4}$$

In order that this equation to be invariant under the gauge transformation(2.2.3), the wave function must be transform as

$$\psi' = \exp(-i\frac{\alpha q}{\hbar})\psi \tag{2.2.5}$$

Consider now a magnetic monopole at the origin . It will produce a magnetic field

$$\overrightarrow{B}_m = \frac{g}{4\pi r^3}\vec{r} \Longrightarrow \overrightarrow{\nabla}\cdot\overrightarrow{B}_m = g\delta^3(\vec{r}) \tag{2.2.6}$$

Since $\overrightarrow{\nabla}\cdot\overrightarrow{B}_m \neq 0$ we can not have an $\overrightarrow{A}$ regular for all the space and satisfying the equations (2.2.2) and (2.2.6). However, we can use the ambiguity (2.2.3) and use the vector potential in one part of space and another vector potential in the other part of space.
let's work in spherical coordinates and take

$$\overrightarrow{A}_N = \frac{g}{4\pi r}\frac{1-\cos(\theta)}{\sin(\theta)}\hat{e_\theta} \tag{2.2.7}$$

which satisfies $\overrightarrow{\nabla} \wedge \overrightarrow{A}_N = \overrightarrow{B}_m$. It is obvious that A_N is well defined in the whole space, except for $\theta = \pi$. Now let us take

$$\overrightarrow{A}_S = -\frac{g}{4\pi r}\frac{1-\cos(\theta)}{\sin(\theta)}\hat{e_\theta} \tag{2.2.8}$$

which is well defined on all space except $\theta = 0$. Since

$$\overrightarrow{A}_S = \overrightarrow{A}_N + \overrightarrow{\nabla}\alpha \Longrightarrow \alpha(\phi) = -\frac{g}{2\pi}\phi \tag{2.2.9}$$

in the region where both are well defined, we can conclude that $\overrightarrow{\nabla} \wedge \overrightarrow{A}_S = \overrightarrow{B}_m$. Therefore $\overrightarrow{B}_m$ can be written as a curl of the potential $\overrightarrow{A}_N$ in the north hemisphere and $\overrightarrow{A}_S$ in the south hemisphere. In each hemisphere we shall have a wave-funtion ψ_N and ψ_S, which will differ by phase (2.2.5). We know from quantum mechanics that wave-functions must single-valued. But from eq. (2.2.5) we can conclude that ψ_N and ψ_S can only be simultaneously single-valued if

$$\exp(-i\frac{q\alpha(\phi)}{\hbar}) = \exp(-i\frac{q\alpha(\phi+2\pi)}{\hbar}) \tag{2.2.10}$$

From (2.2.8) we see that the periodicity of the $\exp$-function implies that

$$qg = 2\pi n\hbar \quad n \in \mathbb{Z} \tag{2.2.11}$$

This can be extend to the situation with various electric charges and magnetic monopoles, which result in the condition

$$q_i g_j = 2\pi n_{ij}\hbar \quad n_{ij} \in \mathbb{Z} \tag{2.2.12}$$

This quantization condition has every important consequence; suppose that at least one magnetic monopole exist in the whole Universe, with a magnetic charge $g = g_0$. Then, condition (2.2.12), would imply that all particles would have electric charges of the form

$$q_i = n_i q_0 \quad \text{where} \quad q_0 = \frac{2\pi\hbar}{g_0} \quad n_i \in \mathbb{Z} \tag{2.2.13}$$

So the electric charge would be integer multiples of a fundamental charge q_0. If one considers the more genra case with not just magnetic monopoles but also dyons, it results the Dirac- Schwinger-Zwanziger quantization condition

$$q_i g_j - q_j g_i = 2\pi n_{ij}\hbar \tag{2.2.14}$$

A good property of this more general condition is that it is invariant under the duality transformation (2.1.6). This can be easily seen by noting that this condition is the imaginary part of $(q_i + ig_i)(q_j + ig_j)^*$ which is manifestly invariant under duality (2.1.12).

2.3 Angular Momentum of EM field

We begin by examining the simplest possible solution of a particle of mass m and electric charge q moving in the field of a magnetic monopole of strength g located at the origin

$$\overrightarrow{B} = \frac{g}{4\pi r^3}\vec{r}\,.$$

The equation of motion of the particle reads

$$m\ddot{\vec{r}} = q\dot{\vec{r}} \wedge \overrightarrow{B}\,. \tag{2.3.1}$$

Remark that the magnetic field is spherically symmetric and one therefore expects something like the conservation of angular momentum. However the orbital momentum of the charge q alone is not conserved because the force (2.3.1) is not central. The rate of change of orbital angular momentum :

$$\frac{d}{dt}(\vec{r} \wedge m\dot{\vec{r}}) = \vec{r} \wedge m\ddot{\vec{r}} \tag{2.3.2}$$

$$= \frac{qg}{4\pi r^3}\vec{r} \wedge (\dot{\vec{r}} \wedge \vec{r}) \tag{2.3.3}$$

$$= \frac{d}{dt}(\frac{qg}{4\pi}\hat{r}) \tag{2.3.4}$$

where $\hat{r}$ is the unit vector $\hat{r} := \frac{\vec{r}}{r}$.

This result, due to Poincaré, suggests that one can define the total angular momentum to be

$$\overrightarrow{J} = \vec{r} \wedge m\dot{\vec{r}} - \frac{qg}{4\pi}\frac{\vec{r}}{r} \tag{2.3.5}$$

and this expression will be intergral of motion, i.e., conserving quantity.

To give a physical interpretation to the second term in the equation (2.3.5) we must consider the other possible source of angular momentum, which is the electromagnetic field. Classicaly the angular momentum of the electromagnetic field is obtained by integrating the moment of the Poynting vector, $\overrightarrow{E} \wedge \overrightarrow{B}$, over all space

$$\overrightarrow{J}_{em} = \int_{\mathbb{R}^{\not{3}}} d^3r \; \vec{r} \wedge (\overrightarrow{E} \wedge \overrightarrow{B}) \; . \tag{2.3.6}$$

Here $\overrightarrow{B}$ is the radial field given and $\overrightarrow{E}$ is the field due to the electric charge q at $\vec{r'}$, thus :

$$\begin{aligned}
J^i_{em} &= \int_{\mathbb{R}^3} d^3r \frac{g}{4\pi r^3} \vec{r} \wedge (\overrightarrow{E} \wedge \vec{r}) \\
&= \int_{\mathbb{R}^3} d^3r \frac{g}{4\pi r^3} \epsilon_{ijk} r^j (\overrightarrow{E} \wedge \vec{r})^k \\
&= \int_{\mathbb{R}^3} d^3r \frac{g}{4\pi r^3} \epsilon_{ijk} \epsilon_{klm} r^j E^l r^m \\
&= \int_{\mathbb{R}^3} d^3r \frac{g}{4\pi r^3} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) r^j E^l r^m \\
&= \int_{\mathbb{R}^3} d^3r \frac{g}{4\pi r^3} (r^j E^i r^j - r^j E^j r^i) \\
&= \int_{\mathbb{R}^3} d^3r E^j \frac{g}{4\pi r} (\delta_{ij} - \hat{r^i}\hat{r^j}) \\
&= \int_{\mathbb{R}^3} d^3r E^j \frac{\partial}{\partial r^j} \left(\frac{g}{4\pi} \hat{r^i} \right) \\
&= \left[\frac{E^j g}{4\pi} \hat{r^i} \right]_{S_\infty} - \int_{\mathbb{R}^3} d^3r \left(\frac{g}{4\pi} \hat{r^i} \right) \nabla \cdot E^j \\
&= -\int_{\mathbb{R}^3} d^3r \left(\frac{g}{4\pi} \hat{r^i} \right) q\delta^3(r - r') \\
&= -\frac{qg}{4\pi} \hat{r^i}
\end{aligned} \tag{2.3.7}$$

Here we use the unit vector $\hat{r^i} = \frac{r^i}{r}$ and the fact that $\nabla \cdot E = q\delta^3(r - r')$ then

$$\overrightarrow{J}_{em} = -\frac{qg}{4\pi r} \vec{r} \tag{2.3.8}$$

Thus we conclude that the total angular momentum which is conserved is indeed the sum of the orbital angular momentum of the particle and the angular momentum of the electromagnetic field. Moreover, requiring that the angular momentum be quantized in unit $\frac{\hbar}{2}$ yields the Dirac quantization condition

$$|\overrightarrow{J}_{em}| = \frac{n\hbar}{2} \tag{2.3.9}$$

$$|-\frac{qg}{4\pi r}\vec{r}| = \frac{n\hbar}{2} \tag{2.3.10}$$

$$\frac{qg}{4\pi} = \frac{n\hbar}{2} \tag{2.3.11}$$

Hence we got the Dirac quantization condition

$$qg = 2\pi n\hbar, \qquad n \in \mathbb{Z}. \tag{2.3.12}$$

2.4 CP-symmetry and Dyon Quantization

Dyons are particles that carry both electric and magnetic charges. Consider dyon with charges (q_1, g_1) fixed at the origin with another with charges (q_2, g_2) orbiting about it . Angular momentum analysis of (2.3.6) can be repeated. The contribution of the electromagnetic field to the angular momentum is now

$$\frac{q_1 g_2 - q_2 g_1}{4\pi r}\vec{r} \tag{2.4.1}$$

Imposing the quantization condition (2.3.9) yields

$$\frac{q_1 g_2 - q_2 g_1}{4\pi} = \frac{1}{2}n\hbar \tag{2.4.2}$$

so if deal with two dyons of charges $(q = e\hbar, g)$ and $(q' = e'\hbar, g')$ we got

Hence we got the Zwanziger-Schwinger quantization condition about the dyon charges

$$eg' - e'g = 2\pi n\ . \tag{2.4.3}$$

In Dirac's symmetrized form of electromagnetodynamics, the Maxwell equations are replaced by

$$\overrightarrow{\nabla} \cdot \overrightarrow{E}(t, \vec{r}) = \frac{\rho_e}{\epsilon_0} \tag{2.4.4}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{B}(t, \vec{r}) = \mu_0 \rho_m \tag{2.4.5}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{E}(t, \vec{r}) = -\mu_0 \vec{j}_m - \frac{\partial \overrightarrow{B}(t, \vec{r})}{\partial t} \tag{2.4.6}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{B}(t, \vec{r}) = \mu_0 \vec{j}_e + \frac{1}{c^2}\frac{\partial \overrightarrow{E}(t, \vec{r})}{\partial t} \tag{2.4.7}$$

We now study the charge conjugation C and the parity conjugation P of these equations. The constants t, c, μ_0, ϵ_0 are ordinary scalar and are therefore unaffected by coordinates changes. The charge density transforms sililarly as the charge $\rho_e \approx q$ and the electric currents read

$$\vec{j_e} = \rho_e \frac{d\vec{r}}{dt} \; . \tag{2.4.8}$$

Under charge conjugation the electric charge changes its sign

$$C : q \to q' = -q \tag{2.4.9}$$

so do the electric charge densities and currents

$$C : \rho_e \to \rho_e' \;\; = \;\; -\rho_e \tag{2.4.10}$$
$$C : \vec{j_e} \to \vec{j_e'} \;\; = \;\; -\vec{j_e} \tag{2.4.11}$$

The coordinates are not affected by the charge conjugation

$$C : \vec{\nabla} \to \vec{\nabla}' \;\; = \;\; \vec{\nabla} \tag{2.4.12}$$
$$C : \frac{\partial}{\partial t} \to \frac{\partial}{\partial t'} \;\; = \;\; \frac{\partial}{\partial t} \tag{2.4.13}$$

The conjugation C applied to Maxwell equation (2.4.4) yields

$$C : \vec{E} \to \vec{E}' = -\vec{E} \; .$$

Next from eq. (2.4.7) we got

$$C : \vec{B} \to \vec{B}' = -\vec{B}$$

and then from the Maxwell equations (2.4.5) and (2.4.6) we haven't used yet one sees that the magnetic charges and currents also change sign under the charge conjugation

$$C : \rho_m \to \rho_m' = -\rho_m \qquad j_m \to j_m' = -j_m \; . \tag{2.4.14}$$

Under parity symmetry $P : \vec{r} \to \vec{r'} = -\vec{r}$ we get

$$\rho_e \to \rho_e' \;\; = \;\; \rho_e \tag{2.4.15}$$
$$\vec{j_e} \to \vec{j_e'} \;\; = \;\; -\vec{j_e} \tag{2.4.16}$$
$$\vec{\nabla} \to \vec{\nabla}' \;\; = \;\; -\vec{\nabla} \tag{2.4.17}$$
$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t'} \;\; = \;\; \frac{\partial}{\partial t} \tag{2.4.18}$$

Then the Maxwell equation (2.4.4) implies the electric field $\overrightarrow{E}$ transforms as a vector

$$P : \overrightarrow{E} \to \overrightarrow{E}' = -\overrightarrow{E}$$

whereas the eq. (2.4.7) implies that the magnetic field transforms as a pseudovector (axial vector)

$$P : \overrightarrow{B} \to \overrightarrow{B}' = \overrightarrow{B} \ .$$

Finally from Maxwell equations (2.4.5) and (2.4.6) we conclude that under the parity conjugation P the magnetic charges and currents transform as

$$\rho_m \to \rho_m' = -\rho_m \qquad (2.4.19)$$
$$\vec{j}_m \to \vec{j}_m' = \vec{j}_m \qquad (2.4.20)$$
$$(2.4.21)$$

To resume: the dyonic charge under the charge conjugation C and the parity conjugation P will transform as, respectively

$$C : (q, g) \longrightarrow (-q, -g) \qquad (2.4.22)$$
$$P : (q, g) \longrightarrow (q, -g) \qquad (2.4.23)$$

Consequantly under conjugation CP the dyonic charge is transformed as

$$CP : (q, g) \longrightarrow (-q, g) \qquad (2.4.24)$$

The quantization condition alone

$$qg' - q'g = 2\pi n$$

is not enough to fix the values of the charge of the monopole but it implies a condition for the difference

$$q - q' = ne = n(2\pi/g) \ .$$

The difference $q - q'$ to be integer means that the charges q lie on a "charge lattice", the minimal electric charge e being the period of the lattice

$$Q_e := \{q = ne + re | n \in \mathbb{Z}, 0 \leq r < 1\} = \mathbb{Z}e + re$$

where $0 \leq r < 1$ since $0 \leq \theta < 2\pi$ and $r = \frac{\theta}{2\pi}$.

Under the conjugation CP the lattice Q_e gets reflected

$$CP : Q_e \to -Q_e$$

and this conjugation is a symmetry (that is the lattice Q_e is the same as the lattice $-Q_e$) if and only if $r = 0$ or $r = 1/2$. Meaning that the points of $Q_e \subset \mathbb{R}$ are with integer or half-integer coordinates.

We conclude that the electric charge spectrum of the dyon in CP non-violated theory can be written as :

$$q = ne \quad \text{or} \quad q = (n + \frac{1}{2})e \tag{2.4.25}$$

Remark : If we assume that $q = 0$ is in the spectrum, the half-integers will be eliminated.

Chapter 3

’t Hooft-Polyakov monopole

’t Hooft-Polyakov found a classical static solution of Georgi-Glashow $SO(3)$ gauge model with a triplet of Higgs field $\overrightarrow{\phi}$, which was finite energy and represents a non singular model of a magnetic monopole with an internal structure.

3.1 Georgi-Glashow model

We will start by considering Yang-Mills theory with a gauge group $SO(3)$ and the scalar Higgs field in the adjoint representation which is known under the name Georgi-Glashow model. This theory is an exemple of Yang-Mills-Higgs theory. The gauge group $SO(3)$ is broken to a $U(1)$ factor. This factor can be identified as the electromagnetic theory. The $U(1)$ factor is compact (i.e. isomorphic to the circle) and then it can be shown that these theories always possess magnetic monopole solutions. The Lagrangian of Georgi-Glashow model reads

$$L = -\frac{1}{4}\overrightarrow{G}^{\mu\nu} \cdot \overrightarrow{G}_{\mu\nu} + \frac{1}{2}D^{\mu}\overrightarrow{\phi} \cdot D_{\mu}\overrightarrow{\phi} - V(\phi) \tag{3.1.1}$$

Here the gauge field-strength $G^a_{\mu\nu}$ is defined through the vector gauge potential W^a_ν taking values in the $SO(3)$ Lie algebra with structure constants ϵ_{abc}

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - e\epsilon_{abc}W^b_\mu W^c_\nu \, . \tag{3.1.2}$$

The three components of the Higgs field $\overrightarrow{\phi} = (\phi_1, \phi_2, \phi_3)$ transform in the adjoint $SO(3)$-representaion, but ϕ_a is a scalar with respect to the Lorentz transformations. The components ϕ_a are minimally coupled to the gauge field through the gauge-covariant derivative

$$D_\mu \phi_a = \partial_\mu \phi_a - e\epsilon_{abc} W_\mu^b \phi_c \ . \tag{3.1.3}$$

The field $\overrightarrow{\phi}$ is subject to the Higgs potential

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - a^2)^2 \quad \text{with} \quad \phi^2 = \overrightarrow{\phi} \cdot \overrightarrow{\phi} \tag{3.1.4}$$

where λ is assumed to be non-negative constant and a is a reel number.

The equations of motion are obtained by variation of the Lagrangian (3.1.1) with respect to W_μ^a and ϕ_a

$$D_\nu G_a^{\mu\nu} = -e\epsilon_{abc}\phi_b D^\mu \phi_c \tag{3.1.5}$$

$$D^\mu D_\mu \phi_a = -\lambda \phi_a(\phi^2 - a^2) \tag{3.1.6}$$

Further we have the Bianchi identity

$$D_\mu * G^{\mu\nu} = 0 \tag{3.1.7}$$

with $\overrightarrow{G}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\overrightarrow{G}_{\alpha\beta}$. The associated symmetric tensor

$$\Theta^{\mu\nu} = -G^{\mu\lambda} \cdot \ G_\lambda^\nu + D^\mu \phi \cdot D^\nu \phi - \eta^{\mu\nu} L \tag{3.1.8}$$

Analogously to the Maxwell case, the non-abelian electric and magnetic field are defined as

$$G^{i0} = E^i \qquad * G^{i0} = B^i \ . \tag{3.1.9}$$

Then the total energy can be written as

$$E = \int d^3x \Theta^{00} = \int d^3x \{ \frac{1}{2}[(E_i)^2 + (B_i)^2 + (D_0\phi)^2 + (D_i\phi)^2] + V(\phi) \} \ . \tag{3.1.10}$$

Note that $\Theta^{00} \geq 0$ and vanishes if and only if

$$E_i = B_i = 0 \qquad D^\mu \phi_a = 0 \quad V(\phi) = 0 \ . \tag{3.1.11}$$

A field configuration which satisfies (3.1.11) everywhere has total energy zero $E = 0$ and it is the vacuum configuration whereas a configuration satisfying only the conditions

$$D^\mu \phi_a = 0 \qquad\qquad V(\phi) = 0 \tag{3.1.12}$$

is called a Higgs vacuum. The condition $V(\phi) = 0$ implies that $\phi^2 = a^2$ for the Higgs vacuum, i.e., geometrically the Higgs vacua are located on a two-dimensional sphere S^2.

Consider a small pertubation φ along the ϕ_3 direction around the constant vector $\vec{a} := (0, 0, a)$ such as

$$\overrightarrow{\phi} = (0, 0, a + \varphi) \tag{3.1.13}$$

The expansion of the Higgs potential in φ reads

$$V(a + \varphi) = \lambda(\varphi^2 + a^2 + a\varphi^3 + \varphi^4) \, .$$

The covariant derivative $D^\mu \overrightarrow{\phi}$ of the Higgs field is expressed in terms of φ as

$$\begin{aligned} D_\mu \phi_1 &= -eW_2^\mu (a + \varphi) && (3.1.14) \\ D_\mu \phi_2 &= eW_1^\mu (a + \varphi) && (3.1.15) \\ D_\mu \phi_3 &= \partial_\mu \varphi && (3.1.16) \end{aligned}$$

thus its square gives rise to the terms

$$\frac{1}{2} D^\mu \overrightarrow{\phi} D_\mu \overrightarrow{\phi} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2} e^2 (\varphi^2 + a^2 + 2a\varphi)[(W_\mu^1)^2 + (W_\mu^2)^2]$$

Let us introduce new fields as linear combinations of the old ones $W_\mu^\pm = \frac{W_\mu^1 \pm i W_\mu^2}{\sqrt{2}}$. Then the Lagrangian becomes

$$\begin{aligned} L &= -\frac{1}{4} \overrightarrow{G}^{\mu\nu} \overrightarrow{G}_{\mu\nu} + \frac{1}{2}(\partial_\mu \vec{\varphi})^2 + \frac{1}{2}(\frac{a\hbar\sqrt{2\lambda}}{\hbar})^2 \varphi^2 + (\frac{ae\hbar}{\hbar})^2 W^+ W^- \\ &+ (2a\varphi + \varphi^2) W^+ W^- + \frac{\lambda}{4}(\varphi^4 + 4a\varphi^3) && (3.1.17) \end{aligned}$$

from where the masses $m_H = a\hbar\sqrt{2\lambda}$ and $m_W = ae\hbar$ can be read off as the properly normalized coefficients in front of the quadratic terms.

We see that the symmetry $SO(3)$ is spontaneously broken down to $U(1)$. Consequently, after symmetry breaking we are left with a $U(1)$ gauge theory which has all the characteristics of Maxwell's electromagnetic theory.

The Lagrangian (3.1.17) after the symmetry breakdown contains: a massless vector boson $A_\mu = \frac{1}{a}\vec{a} \cdot \overrightarrow{W}_\mu$ which we will identify with the photon; a massive scalar field $\varphi = \frac{1}{a}\vec{a} \cdot \overrightarrow{\phi}$ of mass m_H; two massive vector bosons

$W_\mu^\pm$ of mass m_W. The quantum numbers of the particles are given in the following table

particles	mass	spin	electric charge
Higgs particle φ	$a\hbar\sqrt{2\lambda}$	0	0
Photon A_μ	0	$\hbar$	0
massive bosons $W_\mu^\pm$	$ae\hbar$	$\hbar$	$\pm e\hbar$

The electric charge is obtained by comparing the $SO(3)$ covariant derivative with electromagnetic covariant derivative $D_\mu = \partial_\mu + i\frac{Q}{\hbar}A_\mu$, we find

$$Q = \frac{\hbar\overrightarrow{\phi}\cdot\overrightarrow{T}}{a} \tag{3.1.18}$$

where $\overrightarrow{T}$ are the $SO(3)$ generators and $|\overrightarrow{\phi}| = a$ in the vacuum.

3.2 Finite Energy solutions

For a given field configuration the energy is defined as $E = \int d^3x\Theta^{00}$ where Θ^{00} is the energy density. The finite-energy mean that the integral is convergent. The finite-energy requirement forces the fields to be in the Higgs vacuum asymptotically at large distances. Physically one would expect the solution with a lowest non-zero energy to be time independent and to have a high degree of symmetry. These conditions are satisfied by taking :

$$\phi^2 = a^2 \qquad W_\mu^a = 0 \qquad E_i^a = \pi_i^a = 0 \tag{3.2.1}$$

3.3 t' Hooft-Polyakov ansatz

Using some symmetry considerations 't Hooft-Polyakov constructed the monopole solution, starting from a spherical radially symmetry ansatz for the Higgs field $\overrightarrow{\phi}$ and the gauge field $\overrightarrow{W}_\mu$

$$\phi_a = \frac{r^a}{r^2}H(\xi)\,, \tag{3.3.1}$$

$$W_a^i = -\epsilon_{aij}\frac{r^j}{er^2}(1-K(\xi))\,, \tag{3.3.2}$$

$$W_a^0 = 0\,. \tag{3.3.3}$$

Where H and K are some arbitrary functions and the variable $\xi := aer$.

Plugging this ansatz in the expression for the energy (3.1.10) and using the non-zero radial component $\phi(\xi) = \frac{aH(\xi)}{\xi}$ together with $V(\phi) = \frac{\lambda a^4}{4\xi^4}(H^2 - \xi^2)^2$

$$\begin{aligned} \frac{1}{2} D_i \overrightarrow{\phi} D_i \overrightarrow{\phi} &= \frac{a^4 e^2}{\xi^4}[\frac{1}{2}(\xi \frac{dH}{d\xi} - H)^2 + H^2(1-K)^2] \\ \frac{1}{2} \overrightarrow{B}_i \overrightarrow{B}_i &= \frac{a^4 e^2}{\xi^4}[(\xi \frac{dK}{d\xi} - (1-K))^2 + \frac{1}{2}(1-K)^4] \end{aligned}$$

we get

$$E = \frac{4\pi a}{e} \int_0^\infty \frac{d\xi}{\xi^2}[\xi^2 (\frac{dK}{d\xi})^2 + \frac{1}{2}(\xi \frac{dH}{d\xi} - H)^2 + \frac{1}{2}(K^2 - 1)^2 + K^2 H^2 + \frac{\lambda}{4e^2}(H^2 - \xi^2)^2] \,. \tag{3.3.4}$$

Since each term in the integrand has a finite contribution: as ξ goes to zero $(\xi \to 0)$ the second and third term in the integrand implies

$$H \to 0 \quad K = \pm 1 \quad (\text{ we shall use 1 }) \,.$$

At large radius the solution must satisfy the point monopole condition $D_\mu \phi = 0$ and $\frac{\partial V}{\partial \phi} = 0$. The scalar field must be such that $V(\phi)$ is minimal. So

$$\phi^2 = a^2 \Longrightarrow \xi \to \infty \,.$$

Thus $H \to \pm\infty$ (we shall use $+\infty$) and the third term in the integrand implies that $K \to 0$.

To avoid singularity at the origin and achieve non-trivial spatial condition $\phi = a$, the functions H and K must satisfy the following boundary condition:

$$\begin{array}{lll} K(\xi) \to 1 & H(\xi) \to 0 & \text{when} \quad \xi \to 0 \\ K(\xi) \to 0 & H(\xi) \to \xi & \text{when} \quad \xi \to \infty \,. \end{array} \tag{3.3.5}$$

Extremizing E by use of the Euler-Lagrange equation one obtain the following ordinary differential equations in ξ. The same result is obtained if we plug the ansatz in the equations of motion.

$$\frac{\partial E}{\partial K} - \frac{d}{d\xi}\left(\frac{\partial E}{\partial K'}\right) = 0 \qquad \frac{\partial E}{\partial H} - \frac{d}{d\xi}\left(\frac{\partial E}{\partial H'}\right) = 0$$

where $K' = \frac{dK}{d\xi}$ and $H' = \frac{dH}{d\xi}$

$$\begin{array}{rcl} \xi^2 \frac{d^2K}{d\xi^2} & = & KH^2 + K(K^2-1) \\ \xi^2 \frac{d^2H}{d\xi^2} & = & 2K^2H + \frac{\lambda}{e^2}H(H^2-\xi^2) \end{array} \tag{3.3.6}$$

It is worth noting that the existence of solution for the system (3.3.6) of the equations was first indicated by numerical simulations.

We remark that in the limit $\xi \to \infty$ using the boundary conditions (3.3.5)

$$\frac{d^2K}{d\xi^2} = K \tag{3.3.7}$$

$$\frac{d^2h}{d\xi^2} = 2\frac{\lambda}{e^2}h \tag{3.3.8}$$

where $h = H - \xi$ and we have used the fact that $\frac{H}{\xi} \sim 1$ and $K \sim 0$. One can readily solve the equations (3.3.7) and (3.3.8). The solutions compatibles with boundary conditions read

$$K = \exp(-\xi) = \exp(-\frac{m_W}{\hbar}r)$$

$$H = \exp(-\frac{m_H}{\hbar}r)$$

Here m_H and m_W stand for the masses of the Higgs and the vector boson W particles, respectively, these were read from the Lagrangian (3.1.17). The pace of the asymptotic form is thus given by the Compton wavelengths $\frac{\hbar}{m_W}$ or $\frac{\hbar}{m_H}$ of the massive particle associated to the field in question. This mean that the solution describes an object of finite size given by the largest of the Compton wavelengths.

However, in order to obtain the value of the magnetic charge of 't Hooft-Polyakov monopole, one need only use the boundary condition at $\xi \to \infty$ which implies that

$$W_a^i \to -\epsilon_{aij}\frac{r^j}{er^2} \qquad \phi_a \to a\frac{r^a}{er} \qquad \text{for} \qquad \xi \to \infty \tag{3.3.9}$$

and therefore

$$G_a^{ij} \approx \frac{1}{er^4}\epsilon_{aij}r^ar^k = \frac{1}{ear^3}\epsilon_{ijk}r^k\phi_a$$

$$G_a^{ij} = \frac{1}{a}\phi_a \epsilon_{ijk} \frac{1}{er^3} r^k$$

Using the fact that $F_{oi} = 0$ (static solution) and $\overrightarrow{G}^{\mu\nu} = \frac{1}{a}\overrightarrow{\phi} \cdot \overrightarrow{F}^{\mu\nu}$. So

$$G_a^{ij} = \frac{1}{a}\phi_a F^{ij} = \frac{1}{a}\phi_a \epsilon^{ijk} \frac{1}{er^3} r^k$$

hence

$$F_{ij} = \epsilon_{ijk} \frac{r^k}{er^3} \tag{3.3.10}$$

Therefore the asymptotic magnetic field is

$$F^{\mu\nu} = \frac{\overrightarrow{\phi} \cdot \overrightarrow{G}^{\mu\nu}}{a} \Longrightarrow *F^{\mu\nu} = \frac{\overrightarrow{\phi} \cdot *\overrightarrow{G}^{\mu\nu}}{a}$$

then

$$*F^{0i} = \frac{*G^{0i}\phi}{a} \Longrightarrow B^i = -\frac{1}{e}\frac{r^i}{r^3}$$

Hence

$$\overrightarrow{B} = -\frac{1}{er^3}\vec{r} \tag{3.3.11}$$

Comparing with equation (2.2.6) we see that the magnetic charge of the 't Hooft-Polyakov monopole is

$$g = -\frac{4\pi}{e} \tag{3.3.12}$$

Since the electromagnetic $U(1)$ is embedded in $SO(3)$, it is generated by T_3 element which has eigenvalues $\pm 1, 0$ in the 3 dimensional irreducible representation and $\pm\frac{1}{2}$ in 2 dimensional irreducible representation. Also by Noether procedure one obtains $q = e\hbar T_3$. Therefore the smallest charge which enter in the theory is $q_0 = e\frac{\hbar}{2}$. Thus we conclude that 't Hooft-Polyakov monopole satisfies Dirac quantization condition (2.2.11)

$$q_0 g = -2\pi\hbar \tag{3.3.13}$$

and g assume the lowest value compatible with Dirac quantization.

Note that q_0 is the charge of another particle since the 't Hooft-Polyakov monopole has no electric charge. This electrical neutrally monopole is obtained due to the fact that we imposed the condition $W_a^0 = 0$.

3.4 Topological Charges

We will now prove that the Dirac quantization condition holds not just for 't Hooft-Polyakov but for any monopole solution. To do it, we take such field ϕ that satisfy $\phi_a^2 = a^2$ and $D^\mu \phi = 0$ asymptotically. From

$$D_\mu \overrightarrow{\phi} = \partial \overrightarrow{\phi} - e\overrightarrow{W}_\mu \wedge \overrightarrow{\phi} = 0 \Longrightarrow \overrightarrow{\phi} \wedge \overrightarrow{W}_\mu = -\frac{1}{e}\partial_\mu \overrightarrow{\phi} \ . \tag{3.4.1}$$

The multiplication by $\overrightarrow{\phi}\wedge$

$$\overrightarrow{\phi} \wedge (\overrightarrow{\phi} \wedge \overrightarrow{W}_\mu) = -\frac{1}{e}(\overrightarrow{\phi} \wedge \partial_\mu \overrightarrow{\phi})$$

produces the expression

$$(\overrightarrow{\phi} \cdot \overrightarrow{W}_\mu)\overrightarrow{\phi} - \phi^2 \overrightarrow{W}_\mu = -\frac{1}{e}(\overrightarrow{\phi} \wedge \partial_\mu \overrightarrow{\phi}) \ .$$

Hence the gauge field asymptotically has the form

$$\overrightarrow{W}_\mu = \frac{1}{ea^2}(\overrightarrow{\phi} \wedge \partial_\mu \overrightarrow{\phi}) - \frac{1}{a} A_\mu \overrightarrow{\phi} \tag{3.4.2}$$

where we denote by A_μ the projection of W_μ along $\overrightarrow{\phi}$, $A_\mu := \frac{\overrightarrow{\phi} \cdot \overrightarrow{W}_\mu}{a}$.

With the help of eq. (3.4.2) we can obtain the gauge field-strength

$$\overrightarrow{G}_{\mu\nu} = \partial_\mu \overrightarrow{W}_\nu - \partial_\nu \overrightarrow{W}_\mu - e\overrightarrow{W}_\mu \wedge \overrightarrow{W}_\nu$$

as a function of the new broken potential A_μ and the Higgs fileds.

To this end we calculate step by step the needed pieces, starting by

$$\partial_\mu \overrightarrow{W}_\nu = \frac{1}{a^2 e}(\partial_\mu \overrightarrow{\phi} \wedge \partial_\nu \overrightarrow{\phi} + \overrightarrow{\phi} \wedge \partial_\mu \partial_\nu \overrightarrow{\phi}) + \frac{1}{a}(A_\nu \partial_\mu \overrightarrow{\phi} + \overrightarrow{\phi} \partial_\mu A_\nu)$$

then its antisymmetrization reads

$$\partial_\mu \overrightarrow{W}_\nu - \partial_\nu \overrightarrow{W}_\mu = \frac{2}{ea^2}(\partial_\mu \overrightarrow{\phi} \wedge \partial_\nu \overrightarrow{\phi}) + \frac{1}{a}\overrightarrow{\phi}(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

and the non-linear term yields

$$\overrightarrow{W}_\mu \wedge \overrightarrow{W}_\nu = (\frac{1}{a^2 e}\overrightarrow{\phi} \wedge \partial_\mu \overrightarrow{\phi} + \frac{1}{a}\overrightarrow{\phi} A_\mu) \wedge (\frac{1}{a^2 e}\overrightarrow{\phi} \wedge \partial_\nu \overrightarrow{\phi} + \frac{1}{a}\overrightarrow{\phi} A_\nu)$$

And then for the gauge field-strength we got the expression

$$\overrightarrow{G}_{\mu\nu} = \frac{1}{a^2 e}(\partial_\mu \overrightarrow{\phi} \wedge \partial_\nu \overrightarrow{\phi}) + \frac{1}{a}\overrightarrow{\phi}(\partial_\mu A_\nu - \partial_\nu A_\mu) \; .$$

Remarkably enough the field-strength $\overrightarrow{G}_{\mu\nu}$ points in the $\overrightarrow{\phi}$-direction which follows from the alternative form

$$\overrightarrow{G}_{\mu\nu} = \frac{1}{a}\overrightarrow{\phi}[\frac{\overrightarrow{\phi}}{a^3 e}(\partial_\mu \overrightarrow{\phi} \wedge \partial_\nu \overrightarrow{\phi}) + \partial_\mu A_\nu - \partial_\nu A_\mu]$$

Then we can define a new field-strength $F_{\mu\nu}$ for the gauge $U(1)$ theory after the symmetry breaking

$$\overrightarrow{G}_{\mu\nu} = \frac{1}{a}F_{\mu\nu}\overrightarrow{\phi} \tag{3.4.3}$$

where

$$F_{\mu\nu} = \frac{1}{a^3 e}\overrightarrow{\phi} \cdot (\partial_\mu \overrightarrow{\phi} \wedge \partial_\nu \overrightarrow{\phi}) + \partial_\mu A_\nu - \partial_\nu A_\mu \; . \tag{3.4.4}$$

Let's check whether $F_{\mu\nu}$ satisfies Maxwell's equation. Since

$$D_\nu \overrightarrow{G}_{\mu\nu} = \frac{1}{a}(D_\nu F_{\mu\nu} \cdot \overrightarrow{\phi} + F_{\mu\nu} D_\nu \overrightarrow{\phi})$$

or $\quad D_\nu \overrightarrow{\phi} = 0$ and $D_\nu \overrightarrow{G}_{\mu\nu} = -e\overrightarrow{\phi} \wedge D_\mu \overrightarrow{\phi} = \vec{0}$. Thus

$$\partial_\nu F^{\mu\nu} = 0 \tag{3.4.5}$$

Using Bianchi identity one obtains

$$D_\mu * \overrightarrow{G}^{\mu\nu} = 0 \quad \text{where} \quad * \overrightarrow{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\overrightarrow{G}_{\lambda\rho}.$$

Thus

$$D_\mu * G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} D_\mu \overrightarrow{G}_{\lambda\rho} = 0$$

$$D_\mu * G^{\mu\nu} = \frac{1}{a}\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}[D_\mu \overrightarrow{\phi} F_{\lambda\rho} + \overrightarrow{\phi}\partial_\mu F_{\lambda\rho}] = 0$$

$$\partial_\mu(\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}) = 0$$

Hence

$$\partial_\mu * F^{\mu\nu} = 0 \tag{3.4.6}$$

We note that asymptotically the only non-zero component of the $\overrightarrow{G}_{\mu\nu}$ is the component in the $\overrightarrow{\phi}$ direction which is the generator of the electromagnetic $U(1)$ and satisfies Maxwell equation (3.4.5) and (3.4.6). Now let's consider a global characteristic of the field vacuum and study the magnetic flux, g_Σ through the closed surface Σ. The magnetic flux through Σ measures the magnetic charge. Also by Maxwell's equations g_Σ will be non-zero only if the surface Σ encloses a region in which $\overrightarrow{B} = \overrightarrow{\nabla} \wedge \overrightarrow{A}$ fails we deal with a potential monopole

$$g_\Sigma = \int_\Sigma \overrightarrow{B} \cdot d\overrightarrow{S} \ .$$

Using eq. (3.4.4) and the fact that the contribution of A_μ vanishes by Stokes' theorem one obtain

$$g_\Sigma = \int_\Sigma \overrightarrow{B} \cdot d\overrightarrow{S} = -\frac{1}{2ea^2}\int_\Sigma \epsilon_{ijk}\overrightarrow{\phi} \cdot (\partial_j \overrightarrow{\phi} \wedge \partial_k \overrightarrow{\phi})dS_i \qquad (3.4.7)$$

Where

$$B_i = -\frac{1}{ea^2}\epsilon_{ijk}\overrightarrow{\phi}(\partial_j \overrightarrow{\phi} \wedge \partial_k \overrightarrow{\phi}) + \frac{1}{a}\overrightarrow{\phi}(\partial_j A_k - \partial_k A_j)$$

Notice that the derivative $\partial_j \overrightarrow{\phi}$ occuring in (3.3.7) are those tangential to Σ and only contribute to the integral so that the magnetic within Σ depends only on the behaviour of Σ and $\overrightarrow{\phi}$. In fact, if we consider a slightly different Higgs field satisfying (3.1.11)

$$\overrightarrow{\phi}' = \overrightarrow{\phi} + \delta\overrightarrow{\phi} \qquad \overrightarrow{\phi} \cdot \delta\overrightarrow{\phi} = 0 \qquad (3.4.8)$$

Then

$$\delta[\overrightarrow{\phi} \cdot (\partial_j \overrightarrow{\phi} \wedge \partial_k \overrightarrow{\phi})] = 3\delta\overrightarrow{\phi} \cdot (\partial_j \overrightarrow{\phi} \wedge \partial_k \overrightarrow{\phi}) + \partial_j[\overrightarrow{\phi} \cdot (\delta\overrightarrow{\phi} \wedge \partial_k \overrightarrow{\phi})] - \partial_k[\overrightarrow{\phi} \cdot (\delta\overrightarrow{\phi} \wedge \partial_j \overrightarrow{\phi})]$$

The integral of the last two terms in this expression vanishes by Stokes theorem. Moreover, from (3.3.8) we have $\overrightarrow{\phi} \cdot (\partial_j \overrightarrow{\phi} \wedge \partial_k \overrightarrow{\phi}) = 0$ thus $\partial_j \overrightarrow{\phi} \wedge \partial_k \overrightarrow{\phi}$ is parallel to $\overrightarrow{\phi}$. Hence $\delta\overrightarrow{\phi} \cdot (\partial_j \overrightarrow{\phi} \wedge \partial_k \overrightarrow{\phi}) = 0$. Consequently,$\delta g_{\sum} = 0$. This is a funda mental result. It extends to any change in $\overrightarrow{\phi}$ which can be built up by a small continuous deformation. Such a deformation is called a homotopy. Consequently g_Σ is time independent, gauge invariant, and unchanged under any continuous deformation of the surface Σ containing the monopole or monopoles and hence the magnetic charge is additive in domain, in which unshaded region are close to the Higgs vacuum . Thus

$$g_\Sigma = \sum_{i=1}^{n} g_i \tag{3.4.9}$$

Then, the magnetic charge will be

$$g = -\frac{4\pi}{e} N_\Sigma \tag{3.4.10}$$

where

$$N_\Sigma = \frac{1}{8\pi a^3} \int_\Sigma \epsilon_{ijk} \overrightarrow{\phi} (\partial_j \overrightarrow{\phi} \wedge \partial_k \overrightarrow{\phi}) dS_i$$

N_Σ has the geometric interpretation of being the number of times Σ is wrapped about the sphere M_0 by the map $\overrightarrow{\phi} : \Sigma \to M_0$, which is the classical definition of the degree of a map. It easy to see that N_Σ can only take an integer value, since the integrand is Jacobian of $\overrightarrow{\phi}$. To prove that every integer N may be realised for a suitable $\overrightarrow{\phi}$ we consider

$$\overrightarrow{\phi}_N(\vec{r}) = [\cos(N\varphi)\sin(\theta), \sin(N\varphi)\sin(\theta), \cos(\theta)] \tag{3.4.11}$$

where (r, θ, φ) are spherical polar coordinates. Thus

$$dS_i = \frac{1}{2} \epsilon_{ijk} \frac{\partial x^j}{\partial \xi^\alpha} \frac{\partial x^k}{\partial \xi^\beta} \epsilon^{\alpha\beta} d^2\xi$$

$$\int dS_i \epsilon_{ijk} \overrightarrow{\phi} (\partial_j \overrightarrow{\phi} \wedge \partial_k \overrightarrow{\phi}) = \frac{1}{2} \int d^2\xi \epsilon^{\alpha\beta} \epsilon_{abc} \phi^a \frac{\partial \phi^b}{\partial \xi^\alpha} \frac{\partial \phi^c}{\partial \xi^\beta}$$

where

$$J = \epsilon_{abc} \phi^a \frac{\partial \phi^b}{\partial \xi^\alpha} \frac{\partial \phi^c}{\partial \xi^\beta}$$

is the Jacobian and

$$\phi_1 = \sin\theta \cos N\varphi \ , \quad \phi_2 = \sin\theta \sin N\varphi \ , \quad \phi_3 = \cos\theta$$

computing J by

$$J = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 \\ \partial_\theta \phi_1 & \partial_\theta \phi_2 & \partial_\theta \phi_3 \\ \partial_\varphi \phi_1 & \partial_\varphi \phi_2 & \partial_\varphi \phi_3 \end{vmatrix}$$

one obtain $J = N \sin\theta$ hence doing the intergration in spherical coordinates

$$\frac{1}{2}\int d^2\xi N \sin\theta = 2\pi N \ .$$

Therefore

$$N_{\sum} = N \tag{3.4.12}$$

for any integer and yields N in equation (3.3.10). Hence the magnetic charge is topologically conserved and quantized in units of $\frac{4\pi}{e}$, i.e.,

$$g = \frac{4\pi}{e} n_m \tag{3.4.13}$$

Moreover the smallest electric charge is $q_0 = \frac{e\hbar}{2}$, thus

$$q_0 g = -2\pi\hbar n_m \tag{3.4.14}$$

and therefore we see that Dirac's quantization holds for any magnetic monopole of this theory.

3.5 Bogomol'nyi-Prasad-Sommerfeld(BPS) state

We would like calculate the mass of an arbitrary finite energy solution. To do so, we will first use the fact that in the rest frame the mass coincides with the energy, $M = E$. Notice that in the Higgs vacuum the electromagnetic tensor $F_{\mu\nu} = \frac{1}{a}\overrightarrow{\phi} \cdot \overrightarrow{G}_{\mu\nu}$. For any solution the magnetic charge is naturally

$$g = \int_{\Sigma} \overrightarrow{B} \cdot d\vec{S} = \frac{1}{a}\int B_k^a \phi^a dS^k = \frac{1}{a}\int_{\mathbb{R}^3} B_k^a (D_k\phi)^a d^3r \ . \tag{3.5.1}$$

here the surface integral is to be understood as taken in limiting sense over the sphere at spacial infinity and we have used the Bianchi identity and Stokes theorem. Similary using equation of motion and Bianchi identity, one finds the electric charge of a solution

$$q = \int_{\Sigma} \overrightarrow{E} \cdot d\vec{S} = \frac{1}{a}\int_{\mathbb{R}^3} E_k^a (D_k\phi)^a d^3r \ . \tag{3.5.2}$$

Therefore for a given static field configuration the mass (3.1.10) is

$$M = \int_{\mathbb{R}^3} [\frac{1}{2}(\overrightarrow{E}_k \cdot \overrightarrow{E}_k + \overrightarrow{B}_k \cdot \overrightarrow{B}_k + D_k\overrightarrow{\phi} \cdot D_k\overrightarrow{\phi}) + V(\phi)] d^3r \ .$$

Neglecting positively definite ters we get the lower bound

$$M \geq \frac{1}{2}\int_{\mathbb{R}^3}(\overrightarrow{E}_k \cdot \overrightarrow{E}_k + \overrightarrow{B}_k \cdot \overrightarrow{B}_k + D_k\overrightarrow{\phi} \cdot D_k\overrightarrow{\phi})d^3r$$

Next step in the estimation is to add and subtract $E_k^a D_k\phi \sin\theta$ and $B_k^a D_k\phi\cos\theta$ with an arbitraty angle θ to the integrand

$$\begin{aligned} M &\geq \frac{1}{2}\int_{\mathbb{R}^3}\left\{[E_k^a - (D_k\phi)^a\sin(\theta)]^2 + [B_k^a - (D_k\phi)^a\sin(\theta)]^2\right\}d^3r \\ &+ \int_{\mathbb{R}^3}[E_k^a(D_k\phi)^a\sin(\theta) + B_k^a(D_k\phi)^a\cos(\theta)]d^3r \end{aligned} \quad (3.5.3)$$

Once again we can drop some non-negative terms

$$M \geq \sin(\theta)\int_{\mathbb{R}^3}E_k^a(D_k\phi)^a d^3r + \cos(\theta)\int_{\mathbb{R}^3}B_k^a(D_k\phi)^a d^3r$$

Using (3.5.1) and (3.5.2) one finally obtains the mass bound

$$M \geq a(q\sin\theta + g\cos\theta)\ . \quad (3.5.4)$$

The sharpest bound occurs when the right side is a maximum, which happens for $\tan(\theta) = \frac{q}{g}$. Plugging this back into (3.5.4) we find the *Bogomol'nyi bound*

$$M \geq a\sqrt{q^2 + g^2} = |a(q + ig)| \quad (3.5.5)$$

This is an important result. It holds for any finite energy solutions of the equation of motion.

A natural question is "What are the solutions that saturates the Bogomol'nyi bound?" From an inspection at the way we derived the bound we conclude that such kind of solution, with electric and magnetic charges (q, g) must satisfy the following equations throughout the space

$$D_0\phi = 0 \quad (3.5.6)$$
$$E_k^a = (D_k\phi)^a\sin\theta \quad (3.5.7)$$
$$B_k^a = (D_k\phi)^a\cos\theta \quad (3.5.8)$$

where $\tan\theta = \frac{q}{g}$. We will now consider a static solutions which saturates the bound. Static solutions satisfy

$$E_k^a = 0 \qquad (D_0\phi)^a = 0\ . \quad (3.5.9)$$

In particular they have no electric charge hence $\sin\theta = 0$. This mean that $\cos\theta = \pm 1$ correlated to the sign of the magnetic charge. Then the saturation of the bound should require that the so called *Bogomol'nyi equation* holds

$$B_k^a \;=\; \pm (D_k\phi)^a \; . \tag{3.5.10}$$

at the top of the vanishing of the Higgs potential $V(\phi) = 0$.

The solutions saturating the Bogomol'nyi bound are called BPS-states, after the names of Bogomol'nyi, Prasad and Sommerfeld.

The Bogomol'nyi equation (3.5.10) is compatible with the vanishing of the potential $V(\phi) = 0$ only if the parameter λ vanishes. However this condition must be understood as limit $\lambda \to 0$, in order to retain the boundary condition

$$\phi^2 \to a^2 \quad \text{as} \qquad r \to \infty \tag{3.5.11}$$

responsible for the spontaneous breaking. Note that $\lambda \to 0$ implies that the scalar field is massless.

The equation of motion for the Yang-Mills-Higgs system with $\lambda = 0$ for static solution follows from Bianchi identity and the Bogomol'nyi equation. Indeed one has $\overrightarrow{G}^{0i} = -\overrightarrow{E}^i$ hence

$$0 = D_i * \overrightarrow{G}^{0i} = D_i \overrightarrow{B}^i = D_i D^i \overrightarrow{\phi}$$

which together with $D_0 \overrightarrow{\phi} = 0$ implies the equation of motion (3.1.6) with $\lambda = 0$

$$D_\mu D^\mu \overrightarrow{\phi} = 0 \; .$$

The advantage of the Bogomol'nyi equation lies in its simplicity. In fact, it is not hard to find an explicit solution to the Bogomol'nyi in the 't Hooft-Polyakov ansatz. Recall

$$B_a^i = -\frac{\delta^{ai}}{er^2}\xi K' + \frac{r^i r^a}{er^4}(\xi K' + 1 - K^2)$$

$$(D_i\phi)_a = \frac{\delta^{ai}}{er^2} HK + \frac{r^i r^a}{er^4}(\xi H' - H(1+K))$$

Inserting in the $B_a^i = \pm (D^i\phi)^a$ one get

$$\xi \frac{dK}{d\xi} \;=\; -KH \tag{3.5.12}$$

$$\xi \frac{dH}{d\xi} \;=\; H + 1 - K^2 \tag{3.5.13}$$

We introduce the new functions

$$H = -1 - \xi h(\xi) \qquad K = \xi k(\xi) \tag{3.5.14}$$

in which the system got simplified

$$\left\{ \begin{array}{ccc} \frac{dh}{d\xi} & = & k^2 \\ \frac{dk}{d\xi} & = & hk \end{array} \right. \qquad \Longrightarrow \qquad \frac{d}{d\xi}(k^2 - h^2) = 0$$

Therefore $k^2 - h^2 = C$ with a constant C which is determined by imposing the boundary condition

$$\lim_{\xi \to 0} k(\xi) = 0 \quad \lim_{\xi \to 0} h(\xi) = -1 \ .$$

The latter boundary conditions require $C = -1$ hence $h^2 - k^2 = 1$ and finally

$$\frac{dh}{d\xi} = h^2 - 1$$

with a solution given by $h(\xi) = -\coth(\xi + \beta)$ where β is a constant to be determined. Then for the other function $k(\xi)$ we get

$$\frac{dk}{d\xi} = -k \coth(\xi + \beta)$$

whose solution is $k(\xi) = \frac{B}{\sinh(\xi+\beta)}$. where B is another constant.

The finiteness of the energy require that $\lim_{\xi \to 0} k^2 = 1$. This limit is possible only if $\beta = 0$. Then $B^2 = 1$. Choosing $B = 1$ we conclude that

$$h(\xi) = -\coth \xi \qquad k(\xi) = \frac{1}{\sinh \xi}$$

and then got a solution called the BPS-monopole

$$K = \frac{\xi}{\sinh \xi} \qquad H = \frac{\xi}{\tanh \xi} - 1 \ . \tag{3.5.15}$$

Notice that for the BPS-monopole the mass density is

$$(D_i \overrightarrow{\phi})^2 = D_i(\overrightarrow{\phi} D_i \overrightarrow{\phi})$$

$$\nabla^2(\phi^2) = \frac{1}{r}\frac{d}{dr^2}\left(\frac{H^2}{e^2 r}\right) \ .$$

But $H(\xi) = \frac{1}{6}\xi^2 + O(\xi^2)$ for small ξ ($\xi \sim 0$). Hence

$$(D_i \overrightarrow{\phi})^2 = \frac{a^4 e^2}{6\xi} \frac{d^2}{d\xi^2}(\xi^3) = a^4 e^2 \tag{3.5.16}$$

Therefore the mass density at the origin is finite, not merely integrable.

Chapter 4

Duality Conjectures

4.1 Montonen-Olive Conjecture

We have seen that the mass for the BPS-monopole satisfying the BPS is $m_M = \frac{4\pi a}{e}$. Due to the spontaneous symmetry breaking, the gauge particles $W_\mu^\pm$, which have electric charge $q_{W_\mu^\pm} = \pm e\hbar$ will acquire masses $m_{W^\pm} = ae\hbar = aq$ and the W_3 remains massless like the Higgs.

particle	electric charge	magnetic charge	mass	spin
photon	0	0	0	± 1
Higgs	0	0	0	0
$W^\pm$	$\pm e\hbar$	0	aq	1
$M_\pm$	0	$\pm\frac{4\pi}{e}$	ag	0

From this table we observe the following features
• all particles saturate the Bogomol'nyi bound $m = |a(q + ig)|$
• the mass of the BPS-monopole $(W^\pm)$ of one theory with coupling constant e is equal to the mass of the $W^\pm$(BPS-monopole) of a dual theory with coupling $e' = \frac{4\pi}{e\hbar}$.

Based on these observations, Montonen and Olive conjecture that at the quantum level, the monoples of one theory would be described by the $W^\pm$ particles of the dual theory. Similary the $W^\pm$'s of the original theory would be the monopoles of the dual theory.

So far there is no rigorous proof for this conjecture. Note also that this duality conjecture is not a symmetry, because it relates a theory with one value of the coupling constant with the same theory with a different coupling constant.

4.2 The Witten Effect

The Witten effect refers to the shift of the allowed electric charges carried by magnetic monopoles. In other words, the dyonic spectrum of the theory depends on the CP violating θ vacuum parameter. The θ parameter enters throught the addition of a topological term to the Lagrangian

$$L_\theta = \frac{\theta e^2}{32\pi^2} * \overrightarrow{G}^{\mu\nu} \cdot \overrightarrow{G}_{\mu\nu} \tag{4.2.1}$$

often called the θ-term.

The θ-term is a total derivative and therefore does not affect the field equation. Let's introduce this θ-term in the Georgi-Glashow model and as consequence, we will show that the electric charge of a dyon gets an extra contribution. To do this we consider a gauge rotation with a small angle φ around the direction of the gauge field ϕ^a with the gauge parameter $\Xi^a = \frac{\phi^a}{a}$, where ϕ^a is the Higgs field in the monopole background. At spatial infinity this is a gauge transformation corresponding to the unbroken $U(1)$. Its generator corresponds to $U(1)$ electric charge defined in (3.1.18) $Q_e = \frac{e}{a} T_a \phi_a \hbar = e T_3 \hbar$. Under this transformation the Higgs field is left invariant while the vector potential gets transformed as follows

$$\delta W^a_\mu = -\frac{1}{ea} D_\mu \phi^a \ . \tag{4.2.2}$$

The generator of this transformation is obtained from the Lagrangian (3.1.1) of the Georgi-Glashow with the addition of the θ-term

$$L_{GG} \longrightarrow L_{GG} + L_\theta$$

and is given by

$$N = \int_{\mathbb{R}^3} d^3x \left(\frac{\partial L}{\partial(\partial_0 W^a_\mu)} \delta W^a_\mu \right) \tag{4.2.3}$$

The θ-extended Georgi-Glashow Lagrangian reads

$$L_{GG} = -\frac{1}{4} \overrightarrow{G}^{\mu\nu} \cdot \overrightarrow{G}_{\mu\nu} + \frac{\theta e^2}{32\pi^2} * \overrightarrow{G}^{\mu\nu} \cdot \overrightarrow{G}_{\mu\nu} + \frac{1}{2} D_\mu \overrightarrow{\phi} \cdot D^\mu \overrightarrow{\phi} - V(\phi) \tag{4.2.4}$$

thus on the top of the contribution

$$\frac{\partial(G^{\mu\nu}_a G^a_{\mu\nu})}{\partial(\partial_0 W^a_i)} = -4 G^{oi}_a = -4 E^i_a$$

one has also the contribution

$$\frac{\partial(*G_a^{\mu\nu}G_{\mu\nu}^a)}{\partial(\partial_0 W_i^a)} = -4 * G_a^{i0} = -4B_a^i \ .$$

Then

$$N = \int d^3x \frac{\partial L}{\partial(\partial_0 W_i^a)} \delta W_i^a = -\frac{1}{ea}\int_{\mathbb{R}^3} [E_a^i D_i \phi^a - \frac{\theta e^2}{8\pi^2} B_a^i D_i \phi^a] d^3x$$

From the definitions (3.5.1) and (3.5.2) of the magnetic and electric charges we get the expression

$$N = -\frac{1}{e}(q - \frac{\theta e^2}{8\pi^2} g) \ .$$

Since $\frac{Q_e}{e}$ is an integer, the finite $U(1)$ transformation generated by rotation on 2π

$$\exp\left(i2\pi\frac{Q_e}{e}\right) = 1$$

must be equal to the identity. This implies that the same finite transformation generated by N must also be equal to 1 and therefore also N must be an integer

$$\exp(2i\pi N) = 1 \Longrightarrow N = -\frac{1}{e}(q - \frac{\theta e^2}{8\pi^2} g) = -n \ .$$

Taking $g = \frac{4\pi}{e} m$ we get

$$q = ne - m\frac{\theta e}{2\pi} \ . \qquad (4.2.5)$$

Here $m, n \in \mathbb{Z}$ and we have not restricted ourselves to a monopole with topological charge $k = 1$, but we have allowed for any value of $k = m$.

In absence of θ-term the electric charge is quantized in agreement with (3.4.4), while in presence of a θ-term, one gets an extra term proportional to θ and to the magnetic charge of the dyon. The translation $\theta \to \theta + 2\pi$ (4.2.5) implies that if dyon with a certain value of the electric charge n exist , then dyons with any integer value must exist. In conclusion the electric charge of the dyon is not only quantized, but dyons with any integer value n of the electric charge must exist in the spectrum. This is a consequence of the θ periodicity which will be important in the following.

4.3 $SL(2,\mathbb{Z})$- Duality

The $SL(2,\mathbb{Z})$-duality conjecture means that the dyonic spectrum of our theory should be invariant under the $SL(2,\mathbb{Z})$ transformations.

The Georgi-Glashow model with the addition of the θ-term can be rewritten in more compact way in terms of gauge fields normalized in a way to include the gauge coupling constant in the rescaled field

$$\overrightarrow{W}_\mu \longrightarrow e\overrightarrow{W}_\mu \, . \tag{4.3.1}$$

With the help of the complex coupling constant

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2} = \frac{\theta}{2\pi} - i\frac{g}{e} \tag{4.3.2}$$

and the complex quantity $\mathcal{G}_a^{\mu\nu} = G_a^{\mu\nu} + i * G_a^{\mu\nu}$ the Lagrangian (4.2.4) can be written as

$$L = -\frac{1}{32\pi}\mathrm{Im}[\tau\mathcal{G}_a^{\mu\nu}\mathcal{G}^a_{\mu\nu}] + \frac{1}{2}(D_\mu\phi)^2 - V(\phi) \tag{4.3.3}$$

After the rescaling (4.3.1) we have to reformulate the mass formula in eq. (3.5.5) as follows

$$M = \frac{a}{e}|q + ig| = a|n - \tau m| \, . \tag{4.3.4}$$

Let g be an element of $SL(2,\mathbb{Z})$ which is defined as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - cb = 1$.

The parameter τ transforms as

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d} \, . \tag{4.3.5}$$

The group $SL(2,\mathbb{Z})$ is generated by two transformations, the S-transformation

$$T : \tau \longrightarrow \tau + 1 \qquad \text{with matrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{4.3.6}$$

and the T-transformation

$$S : \tau \longrightarrow \frac{-1}{\tau} \qquad \text{with matrix} \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.3.7}$$

The generator T generates a shift of θ over 2π, while for $\theta = 0$ the S generator corresponds to the original Montonen-Olive duality transformation $(e \rightarrow g = -\frac{4\pi}{e})$. Indeed, when the θ-term is not present, i.e., $\theta = 0$ so the complex parameter is completely imaginary

$$\tau = -i\frac{g}{e} \ .$$

We have a basis (f_1, f_2) in the complex plane $\mathbb{C}$, such that $f_1 = e$, $f_2 = -ig$ and the ratio $\tau = \frac{f_2}{f_1}$. The Montonen-Olive duality $e \leftrightarrow g$ then yields a new basis $f_1' = g$, $f_2' = -ie$ having ratio

$$\tau' = \frac{f_2'}{f_1'} = -i\frac{e}{g} = -\frac{1}{\tau}$$

thus coinciding with the S-transformation (4.3.7).

If we consider the states saturating the BPS bound $M^2 = a^2|n - \tau m|^2$ the the allowed values for the dyons charges (g, q) belong to a two dimensional charge lattice $Q \cong \mathbb{Z}^2$ which is naturally embedded in the complex plane $\mathbb{C}$

$$Q = q + ig \quad \text{with} \quad \begin{array}{l} g = m\frac{4\pi}{e} \\ q = ne - m\frac{e\theta}{2\pi} \end{array} \quad \text{for} \quad m, n \in \mathbb{Z} \ . \tag{4.3.8}$$

The BPS mass formula (4.3.4) is invariant under $SL(2,\mathbb{Z})$-duality transformation which transform as

$$(m, n) \longrightarrow (m', n') = (m, n)g^{-1} \tag{4.3.9}$$

where

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \ .$$

The $SL(2,\mathbb{Z})$ transformations keep the lattice Q (4.3.8) invariant. Indeed the BPS mass formula can be written as

$$M^2 = 4\pi a^2 (m, n) \begin{pmatrix} \tau\bar{\tau} & -\mathrm{Re}\tau \\ -\mathrm{Re}\tau & 1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \ . \tag{4.3.10}$$

The only effect of the $SL(2,\mathbb{Z})$-transformation (4.3.9) will be the change (4.3.5) of the modular parameter $\tau \to \tau' = \frac{a\tau+b}{c\tau+d}$ of the lattice Q. Note that since $e^2 > 0$, the parameter τ belong to the upper half-plane, $\mathrm{Im}\tau > 0$. Taking into account the θ-periodicity, we can restrict $-\frac{1}{2} \leq \mathrm{Re}\tau \leq \frac{1}{2}$. Furthermore by $SL(2,\mathbb{Z})$-transformation, or equivalently a sequence of T (4.3.6) and S (4.3.7) transformations any modular parameter $\tau \in \mathbb{C}$ can be brought into the fundamental domain

$$D = \{\tau | \mathrm{Im}\tau > 0, |\mathrm{Re}| \leq \frac{1}{2}, |\tau| > 1\} . \qquad (4.3.11)$$

Chapter 5

Topological Insulators

5.1 Axion Electrodynamics

The Maxwell Lagrangian can be modified by an axion term describing a magneto-electric effect

$$\Delta L_{axion} = \frac{\theta}{2\pi}\frac{\alpha c\epsilon}{2\pi}\overrightarrow{E}\cdot\overrightarrow{B} \tag{5.1.1}$$

In the resulting extended electrodynamics the electric field induces a magnetic polarization, and the magnetic field induces electric polarization. The field $\theta(x)$ is called axion and the resulting "axion electrodynamics" is defined by the action

$$S = \int d^3x dt[\frac{1}{8\pi}(\epsilon\overrightarrow{E}^2 - \frac{1}{\mu}\overrightarrow{B}^2) + \frac{\theta}{2\pi}\frac{\alpha c\epsilon}{2\pi}\overrightarrow{E}\cdot\overrightarrow{B}]$$

where ϵ and μ are permittivity and permeability of the medium. In matter the speed of light is $c = \frac{1}{(\mu\epsilon)^{1/2}}$. The electric and magnetic field are given by $\overrightarrow{E} = -\overrightarrow{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$ and $\vec{B} = \overrightarrow{\nabla}\wedge\vec{A}$ thus the extended Lagrangian in terms of field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is

$$S = \epsilon c^2 \int d^4x \left(\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{\theta}{2\pi}\frac{\alpha}{8\pi}F_{\mu\nu} * F^{\mu\nu}\right). \tag{5.1.2}$$

The parameter $\alpha = \frac{e^2}{c\hbar}$ is the fine structure constant, and θ is the phenomenological axionic parameter. The permittivity ϵ varies for different dielectric materials. The axionic Maxwell equations in the vacuum when θ is a function on space and time are obtained by variation of the action S

$$\partial_\mu F^{\mu\nu} = -\frac{\alpha}{\pi}\partial_\mu \theta (*F^{\mu\nu})$$

The additional term revises both the Gauss' law and Ampere's law in the Maxwell's equations by adding extra terms

$$\begin{aligned} \overrightarrow{\nabla} \cdot \overrightarrow{E} &= \frac{c\alpha}{\pi}(\overrightarrow{\nabla}\theta \cdot \overrightarrow{B}) && (5.1.3) \\ \overrightarrow{\nabla} \wedge \overrightarrow{B} &= \frac{1}{c^2}\partial_t \overrightarrow{E} - \frac{\alpha}{\pi c}(\overrightarrow{B}\partial_t\theta + \overrightarrow{\nabla}\theta \wedge \overrightarrow{E}) \,. && (5.1.4) \end{aligned}$$

Importantly, when expressed in terms of the vector potential $\overrightarrow{E} \cdot \overrightarrow{B}$ is a total derivative, so a constant θ has no effect on the electrodynamics (seen also directly from eqs. (5.1.3) and (5.1.4)). However, to an interface across which the θ changes by $\Delta\theta$ one associates a surface Hall conductivity $\sigma_{xy} = \Delta\theta\frac{e^2}{2\pi h}$.

One can continue to work with the conventional Maxwell equations

$$\begin{array}{rclcrcl} \overrightarrow{\nabla} \cdot \overrightarrow{D} &=& \rho_e & \quad & \overrightarrow{\nabla} \wedge \overrightarrow{H} &=& \partial_t \overrightarrow{D} + \vec{j_e} \\ \overrightarrow{\nabla} \cdot \overrightarrow{B} &=& \rho_m & \quad & \overrightarrow{\nabla} \wedge \overrightarrow{E} &=& -\partial_t \overrightarrow{B} - \vec{j_m} \end{array} \qquad (5.1.5)$$

at the expense of redefinition of the *constitutive relations*

$$\begin{aligned} \overrightarrow{D} &= \epsilon \overrightarrow{E} - \frac{\epsilon\alpha\theta}{\pi}(c\overrightarrow{B}) && (5.1.6) \\ c\overrightarrow{H} &= \frac{c\overrightarrow{B}}{\mu} + \frac{\alpha\theta\overrightarrow{E}}{\pi\mu} && (5.1.7) \end{aligned}$$

The normalized field $P_3(x) = \theta(x)/2\pi$ is called the magneto-electric polarization.

5.2 Topological Insulators with Planar Boundary

Topological insulators are electronic materials that behave like insulators or semiconductors in the bulk, but are surrounded by a topologically protected conduting layer near the surface of the materials. One of the predicted features in these materials is the response to external electromagnetic field, the magneto-electric effect. Recently, it has been argued that the low energy effective action for so called *topological insulator* is described by the "axion electrodynamics"[10]. A key manifestation of the axion term (5.1.1) is the Witten effect: a unit magnetic monopole placed inside a medium with $\theta \neq 0$ is predicted to bind a (generally fractional) electric charge $-e(\frac{\theta}{2\pi} + n)$ with n integer.

The dielectric material-insulator in the axion electrodynamic is characterized by 3 parameters: permittivity ϵ, permeability μ and the field θ. For a periodic system, there are only two values of the field θ for which the system is time-reversal invariant

$$\theta = 0 \Longrightarrow \text{trivial insulator} \tag{5.2.1}$$

$$\theta = \pi \Longrightarrow \text{non-trivial insulator (topological insulator)} \tag{5.2.2}$$

There is no time-reversal invariant perturbation that can switch from trivial to non-trivial topological insulator, that's why one says that the states are topologically protected. We deal with topological band theory based on Z_2 topological band invariant of single particles states.

Following [11] we consider a planar interface between a topological insulator (with non trivial μ_2 and ϵ_2 as well as $\theta = \pi$, that is $P_3 = \frac{1}{2}$) and trivial insulator (with ϵ_1, μ_1 and $P_3 = 0$). Consider a single static point charge q inside the trivial insulator at distance d away from the surface of the topological insulator. Let's find the electromagnetic field made by the point particle of electric charge q for general μ and ϵ at position $z = d$ by the image method. Notice that the Topological Insulator occupy the $z < 0$ region of the space. The curl of the electric and magnetic field are vanishing outside the interface, $rot\overrightarrow{E} = 0$ and $rot\overrightarrow{B} = 0$. So one can introduce the electric and magnetic potentials, ϕ_e and ϕ_m

$$\overrightarrow{E} = -\overrightarrow{\nabla}\phi_e \qquad \overrightarrow{B} = -\overrightarrow{\nabla}\phi_m \,.$$

Above the interface they are given by (let's weight all electric charge by ϵ_0 for convenience , and also use ϵ_1 for q on both sides of the interface, these are just definitions of our mirror charges)

$$\phi_e^I = \frac{q}{4\pi\epsilon_1 R_1} + \frac{q_2}{4\pi\epsilon_0 R_2} \quad \phi_m^I = \frac{g_2}{4\pi R_2} \tag{5.2.3}$$

$$\phi_e^{II} = \frac{q}{4\pi\epsilon_1 R_1} + \frac{q_1}{4\pi\epsilon_0 R_1} \quad \phi_m^{II} = \frac{g_1}{4\pi R_1} \tag{5.2.4}$$

Here $q_{1,2}$ and $g_{1,2}$ are the mirror charges located a distance d above (for q_1, g_1) and below (for q_2, g_2) the interface. $R_1^2 = x^2 + y^2 + (d - z)^2$ and $R_2^2 = x^2 + y^2 + (d + z)^2$. Maxwell's equations in the absence of surface currents or charges as usual demand continuty of $D_\perp$, $B_\perp$, $H_\parallel$ and $E_\parallel$. Since at $z = 0$ $R_1 = R_2$ and $\partial_z R_1 = -\partial_z R_2$ we get th system

$$\begin{aligned} (q - \frac{\epsilon_1}{\epsilon_0} q_2) &= (\frac{\epsilon_2}{\epsilon_1} q + \frac{\epsilon_2}{\epsilon_0} q_1) - \frac{\epsilon_0 \alpha \theta}{\pi}(c g_1) && (5.2.5) \\ g_1 &= -g_2 && (5.2.6) \\ \frac{g_2}{\mu_1} &= \frac{g_1}{\mu_2} + \frac{\alpha\theta}{\pi} \frac{\frac{q}{\epsilon_1} + \frac{q_1}{\epsilon_0}}{\mu_0 c} && (5.2.7) \\ q_1 &= q_2 && (5.2.8) \end{aligned}$$

Solving these equations we get the mirrors charges

$$q_2 = q_1 = \frac{\epsilon_0}{\epsilon_1} \frac{(\epsilon_1 - \epsilon_2)(\frac{\mu_0}{\mu_1} + \frac{\mu_0}{\mu_2}) - \epsilon_0 \alpha^2 \frac{\theta^2}{\pi^2}}{(\epsilon_1 + \epsilon_2)(\frac{\mu_0}{\mu_1} + \frac{\mu_0}{\mu_2}) + \epsilon_0 \alpha^2 \frac{\theta^2}{\pi^2}} q \qquad (5.2.9)$$

$$g_2 = -g_1 = \frac{1}{\epsilon_0} \frac{2\alpha \frac{\theta}{\pi} q}{(\epsilon_1 + \epsilon_2)(\frac{\mu_0}{\mu_1} + \frac{\mu_0}{\mu_2}) + \epsilon_0 \alpha^2 \frac{\theta^2}{\pi^2}} \qquad (5.2.10)$$

The system consisting a charge q and a magnetic charge g gives rise to angular momentum

$$\overrightarrow{J} = \frac{qg}{4\pi r} \vec{r}$$

For the interface , we calculate the contribution to the angular momentum in the two regions independently. Inside the topological insulator both electric and magnetic charge are pointing radially outward from the point at $z = d$, so the angular momentum vanishes ($\overrightarrow{E}$ and $\overrightarrow{B}$ are parallel and so pointing vector vanishes). For the region above the interface, we get a non-zero contribution to the angular momentum due to the charge (monopole) system formed by the original charge q at $z = d$ and the mirror magnetic charge g_2 at $z = -d$ (the electric mirror charge q_2 at $z = -d$ does not contribute, as the charge at $z = d$ is purely electric). Then

$$J_z = -\frac{q g_2}{8\pi} = -\frac{1}{4\pi c} \frac{\frac{\alpha\theta}{\pi}}{(\epsilon_1 + \epsilon_2)(\frac{\mu_0}{\mu_1} + \frac{\mu_0}{\mu_2}) + \epsilon_0 \alpha^2} q_2 = -\frac{\alpha^2 \hbar}{(\frac{\epsilon_1}{\epsilon_0} + \frac{\epsilon_2}{\epsilon_0})(\frac{\mu_0}{\mu_1} + \frac{\mu_0}{\mu_2}) + \alpha^2} \qquad (5.2.11)$$

where $\alpha = \frac{q_2}{4\pi\epsilon_0\hbar c}$ in SI, we use the fact that we get equal contributions to the angular momentum from lower and the upper half plane, as in our case we only get a contribution from the upper half plane, the angular momentum to the charge and mirror charge system is exactly half of what it would be for charge and monopole pair. The angular momentum reads

$$\vec{r} \wedge (\overrightarrow{E} \wedge \overrightarrow{B}) \sim d \frac{\vec{r} \wedge (\vec{r} \wedge \hat{e_z})}{|\vec{r} - d\hat{e_z}|^2 |\vec{r} + d\hat{e_z}|^2} \tag{5.2.12}$$

which expression is symmetric under $\vec{r} \rightarrow -\vec{r}$. Therefore the statistical angle is

$$|\theta_S| = 2\pi \frac{J_z}{\hbar} = 2\pi \frac{\alpha^2}{(\frac{\epsilon_1}{\epsilon_0} + \frac{\epsilon_2}{\epsilon_0})(\frac{\mu_0}{\mu_1} + \frac{\mu_0}{\mu_2}) + \alpha^2} \,. \tag{5.2.13}$$

5.3 Spherical Topological Insulators

We consider a spherical topological insulator of radius a and magneto-electric polarization P_3 centered at the origin, and a point electric charge q is located at $(0, 0, d)$, with $d > a$. The constants ϵ_1 and μ_1 are the dielectric constant and the magnetic permeability outside the sphere, respectively ϵ_2 and μ_2 the corresponding quantity inside the sphere.It's shown that the electric and magnetic fields inside and outside the sphere can be viewed as induced by a point electric charge or magnetic monople plus a line of image electric or magnetic charge density. Inside the sphere , the total electric charge and total magnetic charge vanish. Both inside and outside the sphere, the curl of electric and magnetic fields is zero, thus we can find a scalar potentials in both region:

$$\overrightarrow{E}^i = -\overrightarrow{\nabla}\phi_e^i \tag{5.3.1}$$

$$\overrightarrow{B}^i = -\overrightarrow{\nabla}\phi_m^i \tag{5.3.2}$$

where $i = 1, 2$ stand for outside and inside region. The most general solution for the potential in equation (5.3.1) and (5.3.2) can be written in terms of Legendre polynomials since the problem has azimuthal symmetry(there will be no dependence on the azumuthal angle φ). Moreover both inside and outside the sphere, the electric and magnetic scalar potential satisfy the Laplace's equation and are expressed as:

$$\phi_e^1 = \frac{q}{\epsilon_1}\sum_{l=0}^{\infty}\frac{r^l}{d^{l+1}}P_l(\cos\theta) + \sum_{l=0}^{\infty}A_l(\frac{a}{r})^{l+1}P_l(\cos\theta) \quad (5.3.3)$$

$$\phi_e^2 = \sum_{l=0}^{\infty}B_l(\frac{r}{a})^l P_l(\cos\theta) \quad (5.3.4)$$

$$\phi_m^1 = \sum_{l=0}^{\infty}C_l(\frac{a}{r})^{l+1}P_l(\cos\theta) \quad (5.3.5)$$

$$\phi_m^2 = \sum_{l=0}^{\infty}D_l(\frac{r}{a})^l P_l(\cos\theta) \quad (5.3.6)$$

Matching the boundary condition for the interface between trivial (vacuum) and topological insulator means to continuity of the transversal (to the boundary) components $\overrightarrow{D}_\perp$ and $\overrightarrow{B}_\perp$ as well as parallel components $\overrightarrow{H}_\parallel$ and $\overrightarrow{E}_\parallel$):

$$\overrightarrow{D}_\perp^1 = \overrightarrow{D}_\perp^2 \quad (5.3.7)$$

$$\overrightarrow{B}_\perp^1 = \overrightarrow{B}_\perp^2 \quad (5.3.8)$$

$$\overrightarrow{E}_\parallel^1 = \overrightarrow{E}_\parallel^2 \quad (5.3.9)$$

$$\overrightarrow{H}_\parallel^1 = \overrightarrow{H}_\parallel^2 \quad (5.3.10)$$

or in more details

$$\epsilon_1\frac{\partial\phi_e^1(a)}{\partial r} = \epsilon_2\frac{\partial\phi_e^2(a)}{\partial r} - 2\alpha P_3\frac{\partial\phi_m^2(a)}{\partial r} \quad (5.3.11)$$

$$\frac{\partial\phi_m^1(a)}{\partial r} = \frac{\partial\phi_m^2(a)}{\partial r} \quad (5.3.12)$$

$$\frac{\partial\phi_e^1(a)}{\partial\theta} = \frac{\partial\phi_e^2(a)}{\partial\theta} \quad (5.3.13)$$

$$\frac{1}{\mu_1}\frac{\partial\phi_m^1(a)}{\partial\theta} = \frac{1}{\mu_2}\frac{\partial\phi_m^2(a)}{\partial\theta} + 2\alpha P_3\frac{\partial\phi_e^2(a)}{\partial\theta} \quad (5.3.14)$$

Plugging the expressions of the potentials we obtain:

$$
\begin{aligned}
\frac{\partial\phi_e^1(a)}{\partial r} &= \frac{q}{\epsilon_1}\sum_l \frac{la^l a^{-1}}{d^{l+1}}P_l(\cos\theta) - \sum_l (l+1)A_l a^{-1} \\
\frac{\partial\phi_a^2(a)}{\partial r} &= \sum_l lB_l a^{-1}P_l(\cos\theta) \\
\frac{\partial\phi_m^1(a)}{\partial r} &= -\sum_l (l+1)C_l a^{-1}P_l(\cos\theta) \\
\frac{\partial\phi_m^2(a)}{\partial r} &= \sum_l lD_l a^{-1}P_l(\cos\theta) \\
\frac{\partial\phi_e^1(a)}{\partial\theta} &= \frac{q}{\epsilon_1}\sum_l \frac{a^l}{d^{l+1}}\frac{d}{d\theta}(P_l(\cos\theta)) + \sum_l A_l\frac{d}{d\theta}(P_l(\cos\theta)) \\
\frac{\partial\phi_e^2(a)}{\partial\theta} &= \sum_l B_l\frac{d}{d\theta}(P_l(\cos\theta)) \\
\frac{\partial\phi_m^1(a)}{\partial\theta} &= \sum_l C_l\frac{d}{d\theta}(P_l(\cos\theta)) \\
\frac{\partial\phi_m^2(a)}{\partial\theta} &= \sum_l D_l\frac{d}{d\theta}(P_l(\cos\theta))
\end{aligned}
$$

And we get the following equations:

$$
\begin{aligned}
ql\frac{a^l}{d^{l+1}} - \epsilon_1(l+1)A_l &= l\epsilon_2 B_l - 2\alpha P_3 lD_l \\
C_l &= -\frac{l}{(l+1)}D_l \\
\frac{q}{\epsilon_1}\frac{a^l}{d^{l+1}} + A_l &= B_l \\
\frac{1}{\mu_1}C_l &= \frac{1}{\mu_2}D_l + 2\alpha P_3\epsilon_1 B_l \\
D_l &= -\frac{\mu_1\mu_2(l+1)}{l\mu_2+\mu_1(l+1)}2\alpha P_3 B_l
\end{aligned}
$$

After some straightforward calculations we get:

$$
\begin{aligned}
A_l &= \frac{q}{\epsilon_1}\frac{a^l}{d^{l+1}}\left[\frac{(l\epsilon_1 - l\epsilon_2)[\frac{l}{\mu_1}+\frac{l+1}{\mu_2}] - (2\alpha P_3)^2 l(l+1)}{(2\alpha P_3)^2 l(l+1) + [\epsilon_1(l+1)+l\epsilon_2][\frac{l}{\mu_1}+\frac{(l+1)}{\mu_2}]}\right] \\
B_l &= \frac{q}{\epsilon_1}\frac{a^l}{d^{l+1}}\left[\frac{(l\epsilon_1 - l\epsilon_2)[\frac{l}{\mu_1}+\frac{l+1}{\mu_2}] - (2\alpha P_3)^2 l(l+1)}{(2\alpha P_3)^2 l(l+1) + [\epsilon_1(l+1)+l\epsilon_2][\frac{l}{\mu_1}+\frac{(l+1)}{\mu_2}]} + 1\right] \\
C_l &= q\frac{a^l}{d^{l+1}}\frac{2\alpha P_3 l(2l+1)}{(2\alpha P_3)^2 l(l+1) + [\epsilon_1(l+1)+l\epsilon_2][\frac{l}{\mu_1}+\frac{(l+1)}{\mu_2}]} \\
D_l &= q\frac{a^l}{d^{l+1}}\frac{-2\alpha P_3(2l+1)(l+1)}{(2\alpha P_3)^2 l(l+1) + [\epsilon_1(l+1)+l\epsilon_2][\frac{l}{\mu_1}+\frac{(l+1)}{\mu_2}]}
\end{aligned}
$$

The fields here could be considered to be generated by a point image electric charge, magnetic monople, and a line of image electric or magnetic charges. Now for simplicity we take $\epsilon_1 = \epsilon_2 = \mu_1 = \mu_2 = 1$ and consider a potentials in the region $a < z < d$ assuming that the magnetic field is generated by an image magnetic monople of the magnitude $g_2 = \frac{a}{d}g'$ at the reverse point (0,0,$\frac{a^2}{d}$) of the applied charge, and a line of image magnetic charge density stretching from the center of the sphere to the reverse point, with magnetic charge density

$$\eta_2(z) = \frac{g'}{a}[c_1(\frac{zd}{a^2})^{-t_1} + c_2(\frac{zd}{a^2})^{-t_2}] \qquad (0 \le z \le \frac{a^2}{d}) \tag{5.3.15}$$

The potential induced by the image magnetic monopole and magnetic charge density is

$$\sum_l g'\frac{a^{2l+1}}{(zd)^{l+1}}[\frac{c_1}{-t_1+l+1} + \frac{c_2}{-t_2+l+1} + 1]P_l(\cos\theta) \tag{5.3.16}$$

where $g_2 = \frac{a}{d}\frac{\alpha P_3}{1+(\alpha P_3)^2}q$; $t_{1,2} = \frac{1}{2}[1 \pm \frac{\alpha P_3}{\sqrt{1+(\alpha P_3)^2}}]$; $c_{1,2} = -\frac{1}{4}[1 \pm \frac{\alpha P_3}{\sqrt{1+(\alpha P_3)^2}}]$
Further we assume the electric field induced by the topological insulator can be regarded as being generated by an image electric charge $q = \frac{a}{d}q'$ at the reverse point and a line of image electric charge density stretching from the center of the sphere to the reverse point, with an electric charge density:

$$\rho_2(z) = \frac{q'}{a}[c_1(\frac{zd}{a^2})^{-t_1} + c_2(\frac{zd}{a^2})^{-t_2}] \qquad (0 \le z \le \frac{a^2}{d}) \tag{5.3.17}$$

V

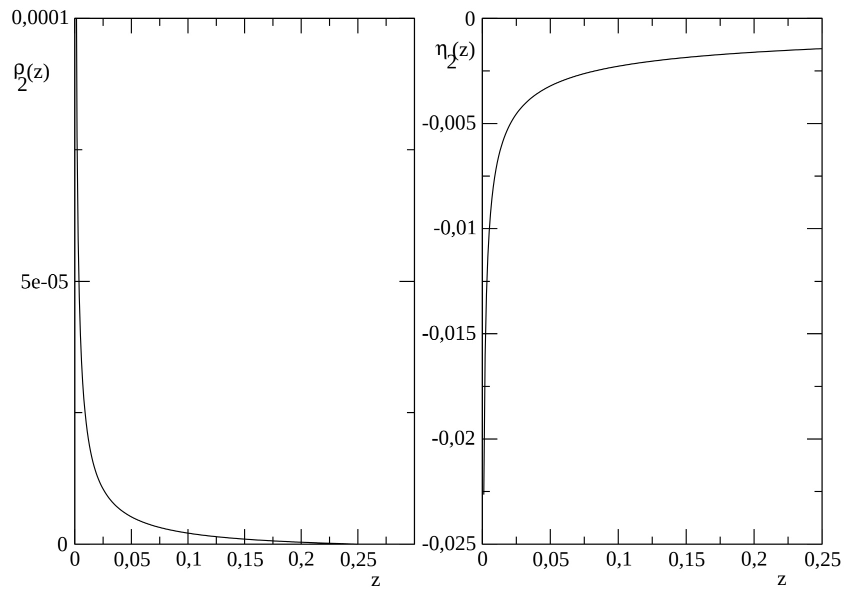

Figure 5.1: The image magnetic(right) and electric(left) charge densities as seen from outside the sphere. The parameters are taken as $a = 1, d = 4, P_3 = \frac{1}{2}, q = 1$.

We easily verify that the integrals of the image electric and magnetic charge densities cancel the point image electric and magnetic monopole, as expected.

Now consider the fields inside the sphere and assuming that the magnetic field is generated by an image magnetic monopole of the magnitude g_1 at (0,0,d), and a line of image charge density stretching from (0,0,d) to infinity along z axis, with a magnetic charge density :

$$\eta_1(z) = \frac{g_1}{d}[c_1(\frac{z}{d})^{-t_1} + c_2(\frac{z}{d})^{-t_2}]; (d \leq z < \infty) \tag{5.3.18}$$

The potential induced by the image magnetic monopole and magnetic charge density is

$$\sum_l g_1 \frac{r^l}{d^{l+1}}[\frac{c_1}{t_1 + l} + \frac{c_2}{t_2 + l} + 1]P_l(\cos\theta) \tag{5.3.19}$$

By matching with the existing solution we easily obtain

$$g_1 = -\frac{\alpha P_3}{1 + (\alpha P_3)^2} q \tag{5.3.20}$$

$$t_{1,2} = \frac{1}{2}[1 \pm \frac{\alpha P_3}{\sqrt{1 + (\alpha P_3)^2}}] \tag{5.3.21}$$

$$c_{1,2} = \frac{1}{4}[1 \pm \frac{\alpha P_3}{\sqrt{1 + (\alpha P_3)^2}}]. \tag{5.3.22}$$

Then we assume the electric field can be regarded as being generated by the effective charge $\frac{q}{\epsilon_1}$ plus an image electic charge q_1 at (0,0,d) and a line of image electric charge density stretching from (0,0,d) to infinity along z axis,with an electric charge density

$$\rho_1 = \frac{q_1}{d}[c_1(\frac{z}{d})^{-t_1} + c_2(\frac{z}{d})^{-t_2}] \qquad (0 \leq z < \infty) \tag{5.3.23}$$

And then we obtain

$$q_1 = -\frac{(\alpha P_3)^2}{1+(\alpha P_3)^2} q \tag{5.3.24}$$

$$t_{1,2} = \frac{1}{2}[1 \pm \frac{\alpha P_3}{\sqrt{1+(\alpha P_3)^2}}] \tag{5.3.25}$$

$$c_{1,2} = \pm\frac{1}{4\alpha P_3\sqrt{1+(\alpha P_3)^2}}. \tag{5.3.26}$$

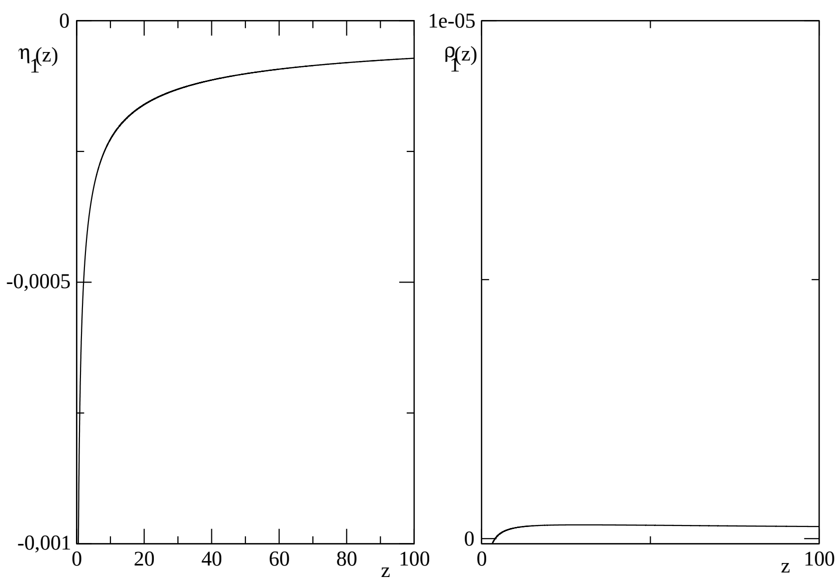

Figure 5.2: The image magnetic(left) and electric(right) charge densities as seen from inside the sphere. The parameters are taken as $a = 1, d = 4, P_3 = \frac{1}{2}, q = 1$.

5.4 Topological insulator and $SL(2,\mathbb{Z})$

The constitutive relations (5.1.6) can be brought to a beautiful and concise matrix form. We first rewrite (5.1.6) as

$$\overrightarrow{D} = \epsilon\overrightarrow{E} - \frac{\theta}{2\pi}(2\alpha\epsilon c\overrightarrow{B}) \qquad (5.4.1)$$

$$\overrightarrow{H} = \frac{\overrightarrow{B}}{\mu} + \frac{\theta}{2\pi}(\alpha\epsilon c\overrightarrow{E}) \qquad (5.4.2)$$

Then defining the normalizing constant $\kappa = 2\alpha\epsilon c$

$$\begin{array}{rcl} \overrightarrow{D} &=& \frac{\kappa}{\epsilon c^2}[\ (\frac{\theta^2}{4\pi^2} + (\frac{1}{2\alpha})^2)(\kappa\overrightarrow{E}) - \frac{\theta}{2\pi}\overrightarrow{H}] \\ \kappa\overrightarrow{B} &=& \frac{\kappa}{c^2\epsilon}[\ -\frac{\theta}{2\pi}(\kappa\overrightarrow{E}) + \overrightarrow{H}] \end{array} \qquad (5.4.3)$$

and replacing $c^2\epsilon = 1/\mu$ we obtain

$$\begin{pmatrix} \overrightarrow{D} \\ \kappa\overrightarrow{B} \end{pmatrix} = \mathcal{M}\begin{pmatrix} \kappa\overrightarrow{E} \\ \overrightarrow{H} \end{pmatrix} \qquad \mathcal{M} = \kappa\mu\begin{pmatrix} \frac{\theta^2}{4\pi^2} + (\frac{1}{2\alpha})^2 & -\frac{\theta}{2\pi} \\ -\frac{\theta}{2\pi} & 1 \end{pmatrix}. \qquad (5.4.4)$$

It is worth noting the parallel of the matrix $\mathcal{M}$ with the BPS-mass formula (4.3.10) which explains the underlying $SL(2,\mathbb{Z})$-symmetry.

Classically the constitutive relations are invariant under shifts of θ by any constant, $\theta = \theta' + C$ together with

$$\begin{pmatrix} \overrightarrow{D} \\ \kappa\overrightarrow{B} \end{pmatrix} = \Lambda\begin{pmatrix} \overrightarrow{D}' \\ \kappa\overrightarrow{B}' \end{pmatrix}, \qquad \begin{pmatrix} \kappa\overrightarrow{E} \\ \overrightarrow{H} \end{pmatrix} = (\Lambda^T)^{-1}\begin{pmatrix} \kappa\overrightarrow{E}' \\ \overrightarrow{H}' \end{pmatrix} \qquad (5.4.5)$$

$$\begin{pmatrix} \rho_e \\ \kappa\rho_m \end{pmatrix} = \Lambda\begin{pmatrix} \rho'_e \\ \kappa\rho'_m \end{pmatrix}, \qquad \begin{pmatrix} \vec{j}_e \\ \kappa\vec{j}_m \end{pmatrix} = \Lambda\begin{pmatrix} \vec{j}'_e \\ \kappa\vec{j}'_m \end{pmatrix}$$

where the matrix Λ stands for

$$\Lambda = \begin{pmatrix} 1 & -\frac{C}{2\pi} \\ 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad (\Lambda^T)^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{C}{2\pi} & 1 \end{pmatrix}$$

Classically we dispose with this symmetry to set θ to zero however quantum mechanically shifts in θ are only a symmetry if C is an integer multiple of

2π. As long as the electric and magnetic fluxes are properly quantized the normalized action $\frac{S_\theta}{\hbar}$ apearing in the path integral is an integer multiple of θ, thus a shift of θ by $2\pi n$ do not alter the path integral. Thus only the values of θ between 0 and 2π are physically distinct. Among this values only $\theta = 0$ and $\theta = \pi$ give a time-reversal symmetric theory. The shift $\theta' = \theta - 2\pi$ gives the T-transformation (4.3.6) of $SL(2,\mathbb{Z})$.

Suppose now that we are in trivial insulators with $\theta = 0$. Then the Maxwell equations (5.1.5) are invariant under the rotation

$$O_\beta = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$$

if the quatities transforms simultanously

$$\begin{pmatrix} \overrightarrow{D} \\ \kappa\overrightarrow{B} \end{pmatrix} = O_\beta \begin{pmatrix} \overrightarrow{D}' \\ \kappa\overrightarrow{B}' \end{pmatrix} , \qquad \begin{pmatrix} \kappa\overrightarrow{E} \\ \overrightarrow{H} \end{pmatrix} = O_\beta \begin{pmatrix} \kappa\overrightarrow{E}' \\ \overrightarrow{H}' \end{pmatrix}$$

$$\begin{pmatrix} \rho_e \\ \kappa\rho_m \end{pmatrix} = O_\beta \begin{pmatrix} \rho'_e \\ \kappa\rho'_m \end{pmatrix} , \qquad \begin{pmatrix} \vec{j}_e \\ \kappa\vec{j}_m \end{pmatrix} = O_\beta \begin{pmatrix} \vec{j}'_e \\ \kappa\vec{j}'_m \end{pmatrix}$$

Classically rotation on arbitrary angle $\beta \in [0, 2\pi)$ are symmetry. But the quantization of charges forces the total electric charge q_e to be multiple of the electron electric charge e

$$q_e = \int d\vec{S} \cdot \vec{D} = n_e e \qquad n_e \in \mathbb{Z}$$

and the total magnetic charge q_m to be a multiple of the minimal magnetic charge $g = \frac{e}{2\alpha}$

$$q_m = \int d\vec{S} \cdot \vec{B} = n_m g \qquad n_m \in \mathbb{Z} \, .$$

The integrations are done on a closed surface containing all charges.

Only the special rotation with angle $\beta = \pi/2$,

$$O_{\pi/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = S$$

which is the duality transformation (2.1.7), is compatible with the quantization. This transformation is reffered to as S-transformation (4.3.7). Note that S leaves the constitutive relations (5.1.6) invariant if we exchange the values of the parameters as follows

$$\epsilon/\kappa \quad \leftrightarrow \quad \kappa\mu \ .$$

The speed of light $c = \frac{1}{\epsilon\mu}$ is invariant under S-transformation.

The full quantum mechanical duality group $SL(2,\mathbb{Z})$ is written as a sequence of S and T-transformation. Let us now denote by Λ the generic element in $SL(2,\mathbb{Z})$

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{such that} \qquad ad - bc = 1 \ .$$

The action of general $\Lambda \in SL(2,\mathbb{Z})$ transformation on the three parameters ϵ, μ, and θ

From the constitutive relations (5.4.4) follows that the transformation (5.4.5) with generic $\Lambda \in SL(2,\mathbb{Z})$ is a symmetry if and only if

$$\mathcal{M} = \Lambda\mathcal{M}'\Lambda^T \qquad \mathcal{M} = \mu\kappa \begin{pmatrix} \frac{\theta^2}{4\pi^2} + (\frac{1}{2\alpha})^2 & -\frac{\theta}{2\pi} \\ -\frac{\theta}{2\pi} & 1 \end{pmatrix} \tag{5.4.6}$$

Note that the speed of light is $SL(2,\mathbb{Z})$-invariant, due to $\det\mathcal{M} = \frac{1}{c^2}$.

The transformation (5.4.6) of the matrix $\mathcal{M}$ is equivalent to the modular transformation

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

of the complexified parameter

$$\tau = \frac{\theta}{2\pi} + i\frac{1}{2\alpha}$$

With the new formalism using the matrix $\mathcal{M}$ we can repeat and generalize the calculations for the planar insulator [11] (given in section 5.2) and determine the mirror charges in manifestly $SL(2,\mathbb{Z})$-covariant form.

Suppose we have a planar interface $z = 0$ between the two insulators with parameters ϵ_1, μ_1 and θ_1 and ϵ_2, μ_2 and θ_2, respectively. Equivalently the insulators are specified by the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$. Suppose now that the test charge $\vec{q}$ at distance d from the interface has two components

$$\vec{q} = \begin{pmatrix} q_e \\ \kappa q_m \end{pmatrix} \ .$$

We introduce the electric and magnetic potentials $\Phi_{e,m}$

$$\vec{D} = -\vec{\nabla}\Phi_e \qquad \vec{B} = -\vec{\nabla}\Phi_m \ .$$

Above the interface $z > 0$ these potentials are given by

$$\Phi_e^I = \frac{q_e}{R_1} + \frac{q_e^{(2)}}{R_2} \qquad \Phi_m^I = \frac{q_m}{R_1} + \frac{q_m^{(2)}}{R_2}$$

and below the interface $z < 0$ are given by

$$\Phi_e^{II} = \frac{q_e}{R_1} + \frac{q_e^{(1)}}{R_1} \qquad \Phi_m^{II} = \frac{q_m}{R_1} + \frac{q_m^{(1)}}{R_1}$$

where $q^(1)_{e,m}$ ($q^(2)_{e,m}$) are the mirror charges located at a distance d above (below) the interface and $R_{1,2}^2 = x^2 + y^2 + (d \mp z)^2$.

The continuity on the boundary of the transvesal components of $\vec{B}$ and $\vec{D}$ implies relations about the mirror charges

$$\begin{pmatrix} D_\perp^{(1)} \\ \kappa B_\perp^{(1)} \end{pmatrix} = \begin{pmatrix} D_\perp^{(2)} \\ \kappa B_\perp^{(2)} \end{pmatrix} \qquad \Rightarrow \qquad \vec{q}^{(1)} = -\vec{q}^{(2)}$$

Similarly, the continuity of the parallel components of $\vec{E}$ and $\vec{H}$

$$\begin{pmatrix} \kappa E_\parallel^{(1)} \\ H_\parallel^{(1)} \end{pmatrix} = \begin{pmatrix} \kappa E_\parallel^{(2)} \\ H_\parallel^{(2)} \end{pmatrix} \qquad \Rightarrow \qquad \mathcal{M}_1^{-1} \begin{pmatrix} D_\parallel^{(1)} \\ \kappa B_\parallel^{(1)} \end{pmatrix} = \mathcal{M}_2^{-1} \begin{pmatrix} D_\parallel^{(2)} \\ \kappa B_\parallel^{(2)} \end{pmatrix}$$

and therefore if we denote by $\mathcal{T} = \mathcal{M}_1 \mathcal{M}_2^{-1}$

$$\begin{pmatrix} D_\parallel^{(1)} \\ \kappa B_\parallel^{(1)} \end{pmatrix} = \mathcal{T} \begin{pmatrix} D_\parallel^{(2)} \\ \kappa B_\parallel^{(2)} \end{pmatrix} \qquad \Rightarrow \qquad \vec{q} + \vec{q}^{(2)} = \mathcal{T}(\vec{q} + \vec{q}^{(1)})$$

The $SL(2,\mathbb{Z})$-transformation law of the matrix $\mathcal{M}$ implies the transformation properties of the matrix $\mathcal{T}$

$$\mathcal{T} = \Lambda \mathcal{T}' \Lambda^{-1} \tag{5.4.7}$$

and these expressions are manifestly covariant.

The covariant system of equations

$$\vec{q}^{(1)} = -\vec{q}^{(2)} \qquad (\vec{q} + \vec{q}^{(2)}) = \mathcal{T}(\vec{q} + \vec{q}^{(1)}) \tag{5.4.8}$$

has a unique solution

$$\vec{q}^{(2)} = -\vec{q}^{(1)} = (\mathcal{T}+1)^{-1}(\mathcal{T}-1)\vec{q} \tag{5.4.9}$$

Let us fix $\epsilon_0 = \mu_0 = c_0 = 1$ and compare the latter results with the mirror charges (5.2.10) and (5.2.9) (see also [11]). We observe that for the magnetic charges $q_m^{(1)} = -q_m^{(2)}$ the latter expression is in perfect agreement with $g_1 = -g_2$ (5.2.10) whereas for the electric charges there seems to be a disagreement with the values obtain in (5.2.9). In particular, the mirror electric charges $q_e^{(1)} = -q_e^{(2)}$ are equal opposite while $q_1 = q_2$ in (5.2.9).

The apparent paradox can be resolved by noting that q_1 and q_2 are not the induced electric mirror charges. Proper electric charges are sources of the flux of the displacement $\vec{D}$. The electric mirror charges q_1 and q_2 were defined as sources of the flux of the electric field $\vec{E}$ instead.

The discrepancy is due to the modified constitutive relations in the presence of the θ term and the two definitions differ by a multiple of the magnetic charge.

So the proper mirror electric charges are related to the quantities calculated in (5.2.9) by

$$\frac{1}{\epsilon_2}(q + q_e^{(1)} + 2\alpha q_m^{(1)}\frac{\theta_2}{2\pi}) = (\frac{q}{\epsilon_1} + q_1)$$

and

$$\frac{1}{\epsilon_1}(q_e^{(2)} + 2\alpha q_m^{(2)}\frac{\theta_2}{2\pi}) = q_2$$

Thus we obtain after substitution the compatibility of the two expressions.

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