ALGORITHMS FOR APPROXIMATION OF SOLUTIONS OF EQUATIONS INVOLVING NONLINEAR MONOTONE-TYPE AND MULTI-VALUED MAPPINGS

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Algorithms for Approximation of Solutions of Equations Involving Nonlinear Monotone-type and Multi-valued Mappings

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CERTIFICATE OF APPROVAL

Ph.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for award of the degree of Doctor of Philosophy degree in Mathematics.

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Dedication

To my Parents, Alhaji Abdulmalik Bello and Hajiya Bilkisu Ahmad.

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ABSTRACT

It is well know that many physically significant problems in different areas of research can be transformed into an equation of the form

$$Au = 0, \tag{0.0.1}$$

where A is a nonlinear monotone operator from a real Banach space E into its dual E^*. For instance, in optimization, if $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex, Gâteaux differentiable function and x^* is a minimizer of f, then $f'(x^*) = 0$. This gives a criterion for obtaining a minimizer of f explicitly. However, most of the operators that are involved in several significant optimization problems are not differentiable. For instance, the absolute value function $x \mapsto |x|$ has a minimizer, which, in fact, is 0. But, the absolute value function is not differentiable at 0. So, in a case where the operator under consideration is not differentiable, it becomes difficult to know a minimizer even when it exists. Thus, the above characterization only works for differentiable operators.
A generalization of differentiability called *subdifferentiability* allows us to recover the above result for non differentiable maps.
For a convex lower semi-continuous function which is not identically $+\infty$, the *subdifferential* of f at x is given by

$$\partial f(x) = \{x^* \in E^* : \langle x^*, y - x\rangle \leq f(y) - f(x) \ \ \forall\, y \in E\}. \tag{0.0.2}$$

Observe that ∂f maps E into the power set of its dual space, 2^{E^*}. Clearly, $0 \in \partial f(x)$ if and only if x minimizes f. If we set $A = \partial f$, then the inclusion problem becomes

$$0 \in Au$$

which also reduces to (0.0.1) when A is single-valued. In this case, the operator maps E into E^*. Thus, in this example, approximating zeros of A, is equivalent to the approximation of a minimizer of f.

In chapter three and four of the thesis, we give convergence results for approximating zeros of equation (0.0.1) in L_p spaces, $1 < p < \infty$, where the operator A

is *Lipschitz strongly monotone* and *generalised Φ-strongly monotone and bounded* maps respectively.

As remarked by Charles Byrne [23], most of the maps that arise in *image reconstruction* and *signal processing* are nonexpansive in nature. A more general class of nonexpansive operators is the class of k-striclty pseudo-contractive maps. In chapter five of this thesis, we prove some convergence results for a fixed point of finite family of k-striclty pseudo-contractive maps in $CAT(0)$ spaces. We also prove a convergence result for a countable family of k-striclty pseudo-contractive maps in Hilbert spaces in chapter six of the thesis.

Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable functions. An integral equation of Hammerstein has the form

$$u(x) + \int_{\Omega} k(x, y) f(y, u(y)) dy = w, \tag{0.0.3}$$

where the unknown function u and inhomogeneous function w lie in the function space E. In abstract form, the equation (0.0.3) can be written in the form

$$u + AFu = w \tag{0.0.4}$$

where $A : E \to E^*$ and $F : E^* \to E$ are monotone operators.
In general, every elliptic boundary value problem whose linear part posses a Green's function (e.g., the problem of forced oscillation of finite amplitude pendulum) can be transformed into an equation of Hammerstein type. Thus, approximating zeros of the Hammerstein-type equation in (0.0.4) (when $w = 0$) is equivalent to the approximation of solutions of some boundary value problems. Hammerstein equations also play crucial role in variational calculus and fixed point theory. In chapter seven of this thesis, we give convergence results for approximating solutions of Hammerstein-type equations in L_P spaces, $1 < p < \infty$.
In particular, we prove the following results in this thesis.

- Let $E = L_p$, $1 < p < 2$. Let $A : E \to E^*$ be a strongly monotone and Lipschitz map. For $x_0 \in E$ arbitrary, let the sequence $\{x_n\}$ be defined by:

  $$x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), n \geq 0,$$

  where $\lambda \in \Big(0, \delta\Big)$. Then, the sequence $\{x_n\}$ converges strongly to $x^* \in A^{-1}(0)$ and x^* is unique.

- Let E= L_p, $2 \leq p < \infty$. Let $A : E \to E^*$ be a Lipschitz map. Assume that there exists a constant $k \in (0, 1)$ such that A satisfies the following condition:

  $$\big\langle Ax - Ay, x - y \big\rangle \geq k\|x - y\|^{\frac{p}{p-1}},$$

and that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}$ iteratively by:

$$x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), n \geq 0,$$

where $\lambda \in (0, \delta_p)$. Then, the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Ax = 0$.

- Let $E = L_p$, $1 < p < 2$. Let $A : E \to E^*$ be a generalized Φ-strongly monotone and bounded map with $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ iteratively by:

$$x_{n+1} = J^{-1}(Jx_n - \alpha_n Ax_n), \quad n \geq 1,$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ satisfies the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. Suppose there exists $\gamma_0 > 0$ such that if $\alpha_n \leq \gamma_0$ for all $n \geq 1$. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a solution of the equation $Ax = 0$.

- Let $E = L_p$, $2 \leq p < \infty$. Let $A : E \to E^*$ be a generalized Φ-strongly monotone and bounded map with $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ iteratively by:

$$x_{n+1} = J^{-1}(Jx_n - \alpha_n Ax_n), \quad n \geq 1,$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ satisfies the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^{\frac{p}{p-1}} < \infty$. Then, there exists $\gamma_0 > 0$ such that if $\alpha_n \leq \gamma_0$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a solution of the equation $Ax = 0$.

- Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X. Let $T_i : K \to CB(K)$, $i = 1, 2, \ldots, m$, be a family of demi-contractive mappings with constants $k_i \in (0,1), i = 1, 2, \ldots, m$, such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that $T_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^{n} F(T_i)$. For arbitrary $x_1 \in K$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_m y_n^m, \qquad n \geq 1,$$

where $y_n^i \in T_i x_n$, $i = 1, 2, \ldots, m$, $\alpha_0 \in (k, 1)$, $\alpha_i \in (0,1)$, $i = 1, 2, \ldots, m$, such that $\sum_{i=0}^{m} \alpha_i = 1$, and $k := max\{k_i, i = 1, 2, \ldots, m\}$. Then, $\lim_{n\to\infty} \{d(x_n, p)\}$ exists for all $p \in \bigcap_{i=1}^{n} F(T_i)$, and $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \ldots, m$.

- Let K be a nonempty closed and convex subset of a real Hilbert space H, and $T_i : K \to CB(K)$ be a countable family of multi-valued k_i-strictly pseudo-contractive mappings; $k_i \in (0,1), i = 1, 2, ...$ such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$; and $\sup_{i\geq 1} k_i \in (0,1)$. Assume that for $p \in \bigcap_{i=1}^{\infty} F(T_i)$, $T_i(p) = \{p\}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively for arbitrary $x_0 \in K$ by

$$x_{n+1} = \lambda_0 x_n + \sum_{i=1}^{\infty} \lambda_i y_n^i,$$

where $y_n^i \in T_i x_n$, $n \geq 1$ and $\lambda_0 \in (k, 1)$; $\sum_{i=0}^{\infty} \lambda_i = 1$ and $k := \sup_{i \geq 1} k_i$. Then, $\lim_{n \to \infty} d(x_n, T_i x_n) = 0$, $i = 1, 2, \ldots$.

- Let $E = L_p$, $1 < p < 2$. Let $F : E \to E^*$ and $K : E^* \to E$ be strongly monotone and bounded maps. For $(u_0, v_0) \in E \times E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^* respectively by

$$u_{n+1} = J^{-1}(Ju_n - \alpha_n(Fu_n - v_n)),\ n \geq 0,$$

$$v_{n+1} = J_*^{-1}(J_* v_n - \alpha_n(Kv_n + u_n)),\ n \geq 0,$$

  where $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$ satisfies the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n^{\frac{q}{q-1}} < \infty$, where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that the equation $u + KFu = 0$ has a solution. Then, there exists $\gamma_0 > 0$ such that if $\alpha_n \leq \gamma_0$ for all $n \geq 1$, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

- Let $E = L_p$, $2 \leq p < \infty$. Let $F : E \to E^*$ and $K : E^* \to E$ be strongly monotone and bounded maps. For $(u_0, v_0) \in E \times E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively, by

$$u_{n+1} = J^{-1}(Ju_n - \alpha_n(Fu_n - v_n)),\ n \geq 0,$$

$$v_{n+1} = J_*^{-1}(J_* v_n - \alpha_n(Kv_n + u_n)),\ n \geq 0,$$

  where $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$ satisfies the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n^{\frac{p}{p-1}} < \infty$. Assume that the equation $u + KFu = 0$ has a solution. Then, there exists $\gamma_0 > 0$ such that if $\alpha_n \leq \gamma_0$ for all $n \geq 1$, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to u^* and v^* respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

List of publications arising from the thesis and other peer-review publications

[A] Papers Published/Accepted from the Thesis

1. C.E. Chidume, **A.U. Bello**, P. Ndambomve, *Strong and Δ-Convergence Theorems for a Finite Family of Demicontractive Mappings in CAT(0) Spaces*, **Abstr. Appl. Anal., 2014, Art. ID 805168, 6 pp.**.

2. C.E. Chidume, **A.U. Bello**, M. A. Onyido, *Convergence theorem for a countable family of multi-valued strictly pseudo-contractive mappings in Hilbert Spaces*, **International Journal of Mathematical Analysis Vol. 9, 2015, no. 27, 1331-1340**.

3. C.E. Chidume, **A.U. Bello**, B. Usman; *Iterative Algorithms For Zeros of Strongly Monotone Lipschitz Maps in Classical Banach Spaces*, **SpringerPlus, 2015, doi 10.1186/s40064-015-1044-1, 9 pp.**.

4. C.E. Chidume, C.O. Chidume, **A.U. Bello**, *An algorithm for Computing Zeros of Generalized Phi-Strongly Monotone and bounded Maps in Classical Banach Spaces*, **in press, Optimization (Taylor and Francis), doi:10.1080/02331934.2015.1074686.**

[B] Other Peer-reviewed Published/Accepted Papers

1. C.E. Chidume, P. Ndambomve, **A.U. Bello**, M.E. Okpala, *The Multiple-set Split Equality Fixed Point Problem for Finite Family of Multi-valued Demicontractive Mappings*, **International Journal of Mathematical Analysis, Vol. 9, 2015, no. 10, 453-469.**

2. C.E. Chidume, M.E. Okpala, **A.U. Bello**, P. Ndambomve, *Convergence Theorems for Finite Family of a General Class of Multi-valued Strictly Pseudo-Contractive Mappings*, **Fixed Point Theory and Appl. (Springer-Verlag), 2015, DOI 10.1186/s13663-015-0365-7.**

3. C.E. Chidume, P. Ndambomve, **A.U. Bello**, *The Multiple-set Split Equality Fixed Point Problem for Multi-valued Demicontractive Mappings in Hilbert Spaces*, (Accepted for publication, Dec. 2014), to appear in: **Journal of Nonlinear Analysis and Optimization**.

[C] Other Papers Still in the Refereeing Process

1. C.E. Chidume, **A.U. Bello**, *Approximation of solutions of Hammerstein equations with strongly monotone and bounded operators in classical Banach spaces*, Submitted, **Proceedings of the American Mathematical Society**.

2. C.E. Chidume, **A.U. Bello**, P. Ndambomve, M.E. Okpala, *Strong Convergence Theorem for Fixed Poins of Nearly Uniformly L-Lipschitzian Asymptotically generalized Hemicontractive Mappings*, Submitted, **International Journal of Mathematical Analysis**.

Contents

CHAPTER 1

General introduction

1.1 Some Motivation

The contents of this thesis fall within the general area of nonlinear functional analysis, an area which has attracted the attention of prominent mathematicians due to its diverse applications in numerous fields of sciences. The contributions of this thesis concentrate mainly on the following three important topics. Namely;

- Approximation of zeros of nonlinear monotone mappings in classical Banach spaces.
- Approximation of fixed points of a finite family of k-strictly pseudo-contractive mappings in $CAT(0)$ spaces, and a countable family of k-strictly pseudo-contractive maps in Hilbert spaces.
- Approximating solutions of Integral equations of Hammerstein-type with monotone operators in Banach spaces.

1.1.1 Approximation of zeros of nonlinear mappings of monotone-type in classical Banach spaces

It is well known that many physically significant problems in different areas of research can be transformed into an equation of the form

$$Au = 0, \tag{1.1.1}$$

where A is a nonlinear monotone operator defined on a real Banach space E. Let H be a real inner product space. A mapping $A : D(A) \subset H \to H$ is called *monotone* if for each $x, y \in D(A)$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq 0,$$

and is called *strongly monotone* if there exists $k \in (0,1)$ such that for all $x, y \in D(A)$, the following inequality holds:

$$\langle Ax - Ay, x - y\rangle \geq k\|x - y\|^2.$$

Monotone mappings were studied in Hilbert spaces by Zarantonello [118], Minty [83], Kačurovskii [69] and a host of other authors. Interest in such mappings stems mainly from their usefulness in numerous applications. Consider, for example, the following: Let $f : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *sub-differential* of f at $x \in H$ is defined by

$$\partial f(x) = \big\{x^* \in H : f(y) - f(x) \geq \big\langle y - x, x^*\big\rangle \ \ \forall\ y \in H\big\}.$$

It is easy to check that $\partial f : H \to 2^H$ is a *monotone operator* on H, and that $0 \in \partial f(x)$ *if and only if* x *is a minimizer of* f. Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of f. In a case where the operator A is single valued, the inclusion $0 \in Au$ reduces to equation (1.1.1).

The extention of the monotonicity definition to operators from real Banach space into its dual has been the beginning of nonlinear functional analysis as remarked by Pascali and Sburian [91] as follows:

> The extension of the monotonicity definition to operators from a Banach space into its dual has been the starting point for the development of nonlinear functional analysis .... The monotone maps constitute the most manageable class, because of the very simple structure of the monotonicity condition. The monotone mappings appear in a rather wide variety of contexts, since they can be found in many functional equations. Many of them appear also in calculus of variations, as subdifferential of convex funtions (Pascali and Sburian [91], p.101).

Unlike as in the case of Hilbert spaces, where the operator A maps H to H (in this case $H = H^*$ by the virtue of Reiz representation theorem), in arbitrary real Banach space E, the extension of monotonicity is split into two cases; a case where A maps E to E in which A shall be called *accretive*, and the other case where A maps E to E^* (the dual of E) in which it retains its name as *monotone*.

Let E be a real normed space with dual E^*. An operator $A : E \longrightarrow E$ is said to be *accretive* if and only if $\forall\ x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geqslant 0,$$

where J is the normalized duality mapping on E defined by

$$J(x) = \{j(x) \in E^* : \langle j(x), x\rangle = \|j(x)\|\|x\|,\ \|j(x)\| = \|x\|\}.$$

and is called *strongly accretive* if and only if there exists $k \in (0,1)$, and for each $x, y \in D(A)$ there exists $j(x - y) \in J(x - y)$ such that the following inequality holds:

$$\langle Ax - Ay, j(x - y)\rangle \geq k\|x - y\|^2.$$

An operator $A : E \longrightarrow E^*$ is said to be *monotone* if and only if

$$\langle Ax - Ay, x - y\rangle \geqslant 0 \ \forall \ x, y \in E.$$

and is called *strongly monotone* if and only if there exists $k \in (0, 1)$ such that for each $x, y \in E$, the following inequality holds:

$$\langle Ax - Ay, x - y\rangle \geq k\|x - y\|^2.$$

In equation (1.1.1), setting $T = I - A$ we obtain that zeros of A are precisely the fixed points of the operator T (i.e., $Au = 0$ if and only if $Tu = u$). In the case that A maps E to E the operator T is called *pseudo-contractive* whenever the operator A is accretive.

Accretive operators were introduced independently in 1967 by Browder [14] and Kato [73]. Interest in such mappings stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in Banach spaces. For accretive-type operator A, solutions of the equation $Au = 0$, in many cases, represent *equilibrium state* of some dynamical system. The examples below show how some problems in applications can be transformed into an equation of the form (1.1.1).

Evolution Equations: Consider the following diffussion equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) & = \triangle u(t, x) + g(u(t, x)), \ t \geq 0, \ x \in \Omega, \\ u(t, x) & = 0, \quad t \geq 0, \ x \in \partial\Omega, \\ u(0, x) & = u_0(x), \ u_0 \in L^2(\Omega), \end{cases} \tag{1.1.2}$$

where Ω is an open smooth subset of $\mathbb{R}^n$.

By simple transformation i.e., by setting $v(t) = u(t, .)$, where

$$v : [0, +\infty) \longrightarrow L^2(\Omega)$$

is defined by $v(t)(x) = u(t, x)$ and $f(\varphi)(x) = g(\varphi(x))$, where

$$f : L^2(\Omega) \longrightarrow L^2(\Omega),$$

we see that equation (1.1.2) is equivalent to

$$\begin{cases} v'(t) & = Av(t) + f(v(t)), \quad t \geq 0, \\ v(0) & = u_0. \end{cases} \tag{1.1.3}$$

Setting f to be identically zero, at an equilibrium state (i.e., when the system becomes independent of time) we see that equation (1.1.3) reduces to

$$Av = 0.$$

Thus, approximatig zeros of equation (1.1.1) or equivalently fixed points of T, where $T = I - A$, is equivalent to the approximation of solutions of the diffusion equation (1.1.2) at an equilibrium state.

Optimization: Consider the following optimization problem:

$$\text{find } x^* \in E \text{ such that } f(x^*) \leq f(x) \ \ \forall\, x \in E, \tag{1.1.4}$$

where $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a map and E is a real normed linear space. It is well known that if the function f is differentiable and x^* exists, then $f'(x^*) = 0$. This gives a criterion for obtaining a minimizer explicitly. However, most of the operators that are involved in several significant optimization problems are not differentiable in the usual sense. For instance, the absolute value function $x \mapsto |x|$ has a minimizer, which, in fact, is 0. But, the absolute value function is not differentiable at 0. So, in a case where the operator under consideration is not differentiable, it becomes difficult to compute a minimizer even when it exists. Thus, the above result only works for differentiable operators.
A generalization of differentiability called *subdifferentiability* allows us to recover in a sense, the above result for non differentiable maps.
Let E be a real normed linear space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex and proper function (i.e., f is not identically ∞). Then, the *sub-differential* of f at x denoted by $\partial f(x)$ is defined by

$$\partial f(x) = \{x^* \in E^* : \langle x^*, y - x\rangle \leq f(y) - f(x) \ \ \forall\, y \in E\}. \tag{1.1.5}$$

It is easy to see that $0 \in \partial f(x)$ if and only if x minimizes f. If we set $A = \partial f$, then the optimization problem (1.1.4) reduces to the inclusion problem

$$0 \in Au$$

which also reduces to (1.1.1) when A is single-valued. In this case, the operator A maps E into E^*. Thus, approximating zeros of A, is equivalent to the approximation of a minimizer of f.

1.2 Approximation Methods for the Zeros of Nonlinear Mappings of Accretive-type

We recall that in Hilbert spaces accretive and monotone operators coincide. A monotone operator from a real Hilbert space H into itself is said to be *maximal monotone* if $R(I + \lambda A) = H \ \forall\ \lambda > 0$. For the approximation of zeros of maximal monotone operators in Hilbert space, assuming existence, Martinet [82], introduced the so-called *proximal point algorithm* which was further studied by Rockafellar [99] and a host of other authors (see e.g., Reich [24, 94, 98], Ishikawa [80], Takahashi and Ueda [108]). Specifically, given $x_k \in H$, an approximation of a solution of

(1.1.1), the proximal point algorithm generates the next iterate x_{k+1} by solving the following equation:

$$x_{k+1} = \left(1 + \frac{1}{\lambda_k} A\right)^{-1} (x_k) + e_k, \tag{1.2.1}$$

where $\lambda_k > 0$ is a regularizing parameter. If the sequence $\{\lambda_k\}_{k=1}^{\infty}$ is bounded from above, then the resulting sequence $\{x_k\}_{k=1}^{\infty}$ of proximal point iterates converges *weakly* to a solution of (1.1.1), provided that a solution exists (Rockafellar [99]). Rockafellar then posed the following question.

- Does the proximal point algorithm always converge strongly?

This question was resolved in the negative by Güler [66], who produced a proper closed convex function g in the infinite-dimensional Hilbert space l_2 for which the proximal point algorithm converges weakly but not strongly. This naturally raises the following question.

- Can the proximal point algorithm be modified to guarantee strong convergence?

Solodov and Svaiter [105] were the first to propose a modification of the proximal point algorithm which guarantees strong convergence in a real Hilbert space. Their algorithm is as follows:
Choose arbitrary $x^0 \in H$ and $\sigma \in [0, 1)$. At iteration k, having x_k choose $\mu_k > 0$ and find (y_k, v_k) an inexact solution of $0 \in Tx + \mu_k(x - x_k)$, with tolerance σ. Define

$$C_k := \{z \in H : \langle z - y^k, v^k \rangle \leq 0\},$$

$$Q_k := \{z \in H : \langle z - x^k, x^0 - x^k \rangle \leq 0\}.$$

Take

$$x_{k+1} = P_{C_k \cap Q_k}(x^0).$$

The authors themselves noted ([105], p.195) that "...at each iteration, there are two subproblems to be solved...": Firstly, to find an inexact solution of the proximal point algorithm. Secondly, to find the projection of x^0 onto $C_k \cap Q_k$, the intersection of the two half spaces. They also acknowledged that these two subproblems constitute a serious drawback in their algorithm. This method of Solodov and Svaiter is part of the so-called CQ-method which has been studied by various authors.

Several authors have successfully extended the results of Martinet [82], and Rockafellar [99], to a more general space than Hilbert space in a case where the operator A is accretive. (see e.g., Reich [24, 94, 98], Bruck [19], Browder [14, 18], Takahashi [71], and a host of other authors).

Remark 1.2.1 *We remark here that while many convergence results have appeared on the extention of the results of Martinet [82], and Rockafellar [99] to a more*

general space than Hilbert space in a case where the operator A, is accretive, most of the convergence results obtained are weak convergence, in a case where strong convergence is obtained, virtually all the algorithms use CQ-method introduced by Solodov and Svaiter [105], which is not suitable for implementation in applications as Solodov and Svaiter acknowledged themselves.

We recall that a point $x \in E$ is said to be *fixed point* of a map $T : E \to E$ if $Tx = x$. The set of of fixed points of T is denoted by $F(T)$. A map T is said to be *Lipschitz* if there exists $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in E$. If $L = 1$, then T is called *nonexpansive*. Also, a map $T : E \to E$ is said to be *strongly pseudo-contractive* if $(I - T)$ is strongly accretive.

In 1986, Chidume [30], proved the following strong convergence theorem for Lipschitz strongly pseudo-contractive mappings in L_P spaces, $2 \leq p < \infty$.

Theorem 1.2.2 *Let $E = L_p,\ 2 \leq p < \infty$, and $K \subset E$ be nonempty closed convex and bounded. Let $T : K \to K$ be a strongly pseudo-contractive and Lipschitz map. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,\ n \geq 0, \tag{1.2.2}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $(ii) \sum_{n=1}^{\infty} \alpha_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique fixed point of T

The iteration formula (1.2.2) is the so-called *Mann iteration formula* in the light of Mann in [81], to approximate fixed points of nonexpansive maps. Replacing T by $I - A$ in Theorem 1.2.2, the following theorem for approximating the unique solution of $Au = 0$ when $A : E \to E$ is a strongly accretive and Lipschitz map is easily proved.

Theorem 1.2.3 *Let $E = L_p,\ 2 \leq p < \infty$. Let $A : E \to E$ be a strongly accretive and Lipschitz map. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by*

$$x_{n+1} = x_n - \alpha_n A x_n,\ n \geq 0, \tag{1.2.3}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $(ii) \sum_{n=1}^{\infty} \alpha_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique solution of $Au = 0$.

The main tool used in the proof of Theorem 1.2.2 is an inequality of Bynum [22]. This theorem signalled the return to extensive research efforts on inequalities in Banach spaces and their applications to iterative methods for solutions of nonlinear equations. Consequently, Theorem 1.2.2 has been generalized and extended in various directions, leading to flourishing areas of research, for the past thirty years or so, for numerous authors (see e.g., Censor and Riech [24], Chidume [26, 27], Chidume and Ali [31], Chidume and Chidume [36, 37], Chidume and Osilike [48], Deng [56], Moudafi [84, 85, 86, 87], Zhou and Jia [120], Liu [80], Qihou [92], Berinde *et al.* [7], Reich [94, 95, 96], Reich and Sabach [97, 98], Weng [109], Xiao [111], Xu [113, 116, 117], Xu and Roach [114], Xu [115], Zhu [121] and a host of other authors).

Recent monographs emanating from these researches include those by Berinde [6], Chidume [29], Goebel and Reich [65], and William and Shahzad [110].

Taking into account the references mentioned above (and the references contained therein), it is readily clear that much has been done on the approximation of zeros of mappings of accretive-type. However, little has been done in the case where the operator A is *monotone* (i.e., A maps E into E^*). This is, perhaps, because of the following two major difficulties.

- Well defineness of the scheme: If we consider, for instance, the Mann recursion formula for approximatig zeros of accretive operators which is given by, $x_0 \in E$ and
$$x_{n+1} = x_n - \alpha_n A x_n, \ n \geq 0,$$
we see that in the case of monotone operators this formula is not applicable, simply because of the fact that it is not well defined (i.e., we are adding two elements from two different vector spaces. i.e., $x_n \in E$ and $Ax_n \in E^*$). So there is a need to develope a scheme that is well defined and simple to implement in any possible application.

- Inequalities: Most of the inequalities developed for proving convergence results for iterative schemes for zeros of accretive operators are not applicable in the case of monotone operators as they involved the generalized duality mappings, where as the definition of monotone operators does not involve the generalized duality mappings.

1.3 Iterative methods for zeros of monotone-type mappings

In trying to overcome these two major difficulties, recently, many authors have successfully employed the notion of *suppressive operators* introduced by Alber [2] and Bregman [8] respectively, to approximate zeros of monotone operators (see e.g., Aoyama *et al.* [5], Kamimura *et al.* [72], Takahashi [71], Zegeye and Shahzad [119] and the references contained therein). A typical example of the algorithms used by most of these authors is contained in the following result of Zegeye and Shahzad [119]. We first remark that a map $A : E \to E^*$ is said to be *γ-inverse strongly monotone* if there exists $\gamma \in (0, 1)$ such that for all $x, y \in E$ the following inequality holds
$$\langle Ax - Ay, (x - y)\rangle \geq \gamma \|Ax - Ay\|^2.$$

Theorem 1.3.1 *(Zegeye and Shahzad [119]) Let E be uniformly smooth and 2-uniformly convex real Banach space with dual E^*. Let $A : E \longrightarrow E^*$ be a γ-inverse strongly monotone mapping and $T : E \longrightarrow E$ be relatively weak nonexpansive mapping with $A^{-1}(0) \cap F(T) \neq \emptyset$. Assume that $0 < \alpha_n \leq b_0 := \frac{\gamma c^2}{2}$, where c is the*

constants from the Lipschitz property of J^{-1}, then the sequence generated by

$$\begin{cases} x_0 & \in & K, choosen \quad arbitrary; \\ y_n & = & J^{-1}(Jx_n - \alpha_n Ax_n); \\ z_n & = & Ty_n, \\ H_0 & = & \{v \in K : \phi(v, z_0) \leq \phi(v, y_0) \leq \phi(v, x_0)\}; \\ H_n & = & \{v \in H_{n-1} \cap W_{n-1} : \phi(v, z_n) \leq \phi(v, y_n) \leq \phi(v, x_n)\}; \\ W_0 & = & E; \\ W_n & = & \{v \in W_{n-1} \cap H_{n-1} : \langle x_n - v, jx_0 - jx_n \rangle \geq 0\}; \\ x_{n+1} & = & \Pi_{H_n \cap W_n}(x_0), n \geq 0, \end{cases} \tag{1.3.1}$$

converges strongly to $\Pi_{F(T)\cap A^{-1}(0)}x_0$, where $\Pi_{F(T)\cap A^{-1}(0)}$ is the generalised projection from E onto $F(T) \cap A^{-1}(0)$.

In the above theorem J is the duality mapping on E and $\phi : E \times E^* \to \mathbb{R}$ is the suppressive operator introduced by Alber in [2], which is given by

$$\phi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2.$$

Remark 1.3.2 *We point out the major weaknesses in scheme (1.3.1).*

- *The duality mapping J (resp. J^{-1}) is not known precisely in any space more general than L_p spaces, $1 < p < \infty$. Therefore, the value of J (resp. J^{-1}) cannot be computed in spaces more general than L_P spaces.*
- *At each step of scheme (1.3.1), one has to compute the inverse of the duality mapping which like the duality mapping itself, is not known in spaces more general than L_P spaces. One has to compute some sets (e.g., H_n and W_n) which are quite difficult to obtain as they involve generalized projections.*

Even though the approximation method used in Thoerem 1.3.1 yields strong convergence to a solution of the problem under consideration, it is clear that it is not easy to be used in application.
In chapter three and four of this thesis we shall give one-step iterative algorithm that does not involve projections for approximating zeros of Lipschitz strongly monotone operators and bounded generalised Φ-monotone operators, respectively, in L_p spaces, $1 < p < \infty$.

1.4 Approximation of fixed points of a finite family of k-strictly pseudo-contractive mappings in $CAT(0)$ spaces

An important class of nonlinear operators is the class of *nonexpansive mappings*. We recall that an operator $T : D(T) \subset E \longrightarrow E$ is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\| \text{ for all } x, y \in D(T),$$

where $D(T)$ is the domain of T. Nonexpansive operators surface in many important real world applications such as image reconstruction, signal processing, e.t.c. The following quotation further shows the importance of iterative methods for approximating fixed points of nonexpansive mappings.

"*Many well known algorithms in signal processing and image reconstruction are iterative in nature. A wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the KM iteration procedure, for particular choice of ne operator...*" (Charles Byrne, [23]).

Note that *KM* in the above quotation stands for *Krasnoselskii method* and *ne* stands for nonexpansive.

For $x_0 \in E$, the recursion formula defined by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \ n \geqslant 0, \tag{1.4.1}$$

is called the *Krasnoselskii* formula, while the formula defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ n \geqslant 0, \tag{1.4.2}$$

is called the *Mann* iteration formula. The Mann iterative method has been successfully employed in approximating fixed points (when they exist) of nonexpansive mappings. This success does not carry over to the more general class of Lipschitz pseudo-contractions (see Chidume and Mutangadura [45]). An important superclass of the class of nonexpansive mappings and a subclass of the class of Lipschitz pseudo-contractive mappings is the class of *k-strictly pseudo-contractive* mappings introduced by Browder and Petryshyn in Hilbert spaces in [18]. They defined the map in *Hilbert and Banach spaces*, respectively, as follows.

Let K be a nonempty subset of a real Hilbert space H. A map $T : K \to H$ is called *k-strictly pseudo-contractive* if there exists $k \in (0,1)$ such that

$$\|Tx - Ty\|^2 \quad \leq \quad \|x-y\|^2 + k\|x - y - (Tx - Ty)\|^2 \qquad \forall\, x, y \in K. \tag{1.4.3}$$

It is easy to see that every nonexpansive map is also pseudo-contractive.

Let K be a nonempty subset of a real normed space E. A map $T : K \to E$ is called *k-strictly pseudo-contractive* (see, *e.g.*, [29], p.145; [17]) if there exists $k \in (0,1)$ such that for all $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \quad \leq \quad \|x-y\|^2 - k\|x - y - (Tx - Ty)\|^2. \tag{1.4.4}$$

It can be trivially shown that in Hilbert spaces (1.4.3) and (1.4.4) are equivalent.

The class of k-strictly pseudo-contractive operators is important for the following two reasons; firstly, it is an important generalization of nonexpansive maps, and secondly, it helps to have better understanding of the class of Lipschitz pseudo-contractive mappings.

1.5 Fixed point of multivalued maps

Interest in fixed point theory for *multi-valued nonlinear mappings* stems, perhaps, mainly from their usefulness in real-world applications, such as in *Game Theory* and *Market Economy* and in other areas of mathematics, such as in *Non-Smooth Differential Equations*. We give below some examples that show the connection between fixed point theory and some of the areas of applications in sciences.

1.5.1 Game Theory and Market Economy

In game theory and market economy, the existence of equilibrium was uniformly obtained by the application of a fixed point theorem. In fact, Nash [88, 89] showed the existence of equilibria for non-cooperative static games as a direct consequence of Brouwer [13] or Kakutani [70] fixed point theorem. More precisely, under some regularity conditions, given a game, there always exists a *multi-valued map* whose fixed points coincide with the equilibrium points of the game. A model example of such an application is the *Nash equilibrium theorem* (see, *e.g.,* [88]).

Consider a game $G = (u_n, K_n)$ with N players denoted by n, $n = 1, \cdots, N$, where $K_n \subset \mathbb{R}^{m_n}$ is the set of possible strategies of the n'th player and is assumed to be nonempty, compact and convex and $u_n : K := K_1 \times K_2 \cdots \times K_N \to \mathbb{R}$ is the payoff (or gain function) of the player n and is assume to be continuous. The player n can take *individual actions*, represented by a vector $\sigma_n \in K_n$. All players together can take a *collective action*, which is a combined vector $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$. For each n, $\sigma \in K$ and $z_n \in K_n$, we will use the following standard notations:

$$K_{-n} := K_1 \times \cdots \times K_{n-1} \times K_{n+1} \times \cdots \times K_N,$$

$$\sigma_{-n} := (\sigma_1, \cdots, \sigma_{n-1}, \sigma_{n+1}, \cdots, \sigma_N),$$

$$(z_n, \sigma_{-n}) := (\sigma_1, \cdots, \sigma_{n-1}, z_n, \sigma_{n+1}, \cdots, \sigma_N).$$

A strategy $\bar{\sigma}_n \in K_n$ permits the n'th player to maximize his gain *under the condition* that the *remaining players* have chosen their strategies σ_{-n} if and only if

$$u_n(\bar{\sigma}_n, \sigma_{-n}) = \max_{z_n \in K_n} u_n(z_n, \sigma_{-n}).$$

Now, let $T_n : K_{-n} \to 2^{K_n}$ be the multi-valued map defined by

$$T_n(\sigma_{-n}) := \operatorname*{Arg\,max}_{z_n \in K_n} u_n(z_n, \sigma_{-n}) \ \forall\, \sigma_{-n} \in K_{-n}.$$

Definition. A collective action $\bar{\sigma} = (\bar{\sigma}_1, \cdots, \bar{\sigma}_N) \in K$ is called a *Nash equilibrium point* if, for each n, $\bar{\sigma}_n$ is the best response of the n'th player to the action $\bar{\sigma}_{-n}$ made by the remaining players. That is, for each n,

$$u_n(\bar{\sigma}) = \max_{z_n \in K_n} u_n(z_n, \bar{\sigma}_{-n}) \tag{1.5.1}$$

or equivalently,

$$\bar{\sigma}_n \in T_n(\bar{\sigma}_{-n}). \tag{1.5.2}$$

This is equivalent to $\bar{\sigma}$ is a fixed point of the multi-valued map $T : K \to 2^K$ defined by

$$T(\sigma) := [T_1(\sigma_{-1}), T_2(\sigma_{-2}), \cdots, T_N(\sigma_{-N})].$$

From the point of view of social recognition, game theory is perhaps the most successful area of application of fixed point theory of multi-valued mappings. However, it has been remarked that the applications of this theory to equilibrium are mostly static: they enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a non-equilibrium point and convergent to equilibrium solution. This is part of the problem that is being addressed by iterative methods for fixed point of multi-valued mappings.

1.5.2 Non-smooth Differential Equations

The mainstream of applications of fixed point theory for multi-valued maps has been initially motivated by the problem of differential equations (DEs) with discontinuous right-hand sides which gave birth to the existence theory of differential inclusion (DIs). Here is a simple model for this type of application.

Consider the initial value problem

$$\frac{du}{dt} = f(t, u), \ a.e.\, t \in I := [-a, a], \ u(0) = u_0. \tag{1.5.3}$$

If $f : I \times \mathbb{R} \to \mathbb{R}$ is discontinuous with bounded jumps, measurable in t, one looks for *solutions* in the sense of Filippov [63] which are solutions of the differential inclusion

$$\frac{du}{dt} \in F(t, u), \ a.e.\, t \in I, \ u(0) = u_0, \tag{1.5.4}$$

where

$$F(t, x) = [\liminf_{y \to x} f(t, y), \ \limsup_{y \to x} f(t, y)]. \tag{1.5.5}$$

Now, set $H := L^2(I)$ and let $N_F : H \to 2^H$ be the *multi-valued Nemystkii operator* defined by

$$N_F(u) := \{v \in H \ : \ v(t) \in F(t, u(t)) \ a.e. \ on \ I\}.$$

Finally, let $T : H \to 2^H$ be the multi-valued map defined by $T := N_F \circ L^{-1}$, where L^{-1} is the inverse of the derivative operator $Lu = u'$ given by

$$L^{-1}v(t) := u_0 + \int_0^t v(s)ds.$$

One can see that problem (1.5.4) reduces to the fixed point problem: $u \in Tu$.

Finally, a variety of fixed point theorems for multi-valued maps, with non empty and convex values is available to conclude the existence of solution. We used a first order differential equation as a model for simplicity of presentation but this approach is most commonly used with respect to second order boundary value problems for ordinary differential equations or partial differential equations. For more about these topics, one can consult [25, 55, 61, 64] and references therein as examples.

1.6 Iterative methods for fixed points of some nonlinear multi-valued mappings

Let E be a real normed linear space and K be a nonempty subset of E. The set K is called *proximinal* (see *e.g.*, [90, 101, 106]) if for each $x \in E$, there exists $u \in K$ such that

$$d(x, u) = \inf\{\|x - y\| : y \in K\} = d(x, K),$$

where $d(x, y) = \|x-y\|$ for all $x, y \in E$. Let $CB(K)$ and $P(K)$ denote the families of nonempty closed bounded subsets of K and nonempty proximinal bounded subsets of K, respectively. The *Hausdorff metric* on $CB(K)$ is defined by:

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

for all $A, B \in CB(K)$. Let $T : D(T) \subseteq E \to CB(E)$ be a *multi-valued mapping* on E. A point $x \in D(T)$ is called a *fixed point of* T if and only if $x \in Tx$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called L- *Lipschitzian* if there exists $L > 0$ such that

$$H(Tx, Ty) \leq L\|x - y\| \ \ \forall\ x, y \in D(T). \tag{1.6.1}$$

When $L \in (0, 1)$ in (1.6.1), we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.

Several results on the approximation of fixed points of *multi-valued nonexpansive* mappings in real Hilbert spaces have appeared in the literature (see e.g., Abbas *et al.* [1], Khan *et al.* [74], Panyanak [90], Sastry and Babu [101], Song and wong [106] and the references contained therein). For their generalizations (see e.g., Chidume *et al.* [39], Chidume and Ezeora [41] and the references contained therein). In [101], Sastry and Babu proved the following result for multi-valued nonexpansive mappings:

Theorem 1.6.1 *(Sastri and Babu [101]) Let H be real Hilbert space, K be a nonempty, compact and convex subset of H, and $T : K \longrightarrow CB(K)$ be a multi-valued nonexpansive map with a fixed point p. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and*

(iii) $\sum \alpha_n\beta_n = \infty$. *where* α_n *and* β_n *are sequences of real numbers. Let* $x^* \in F(T)$, *then the sequence defined by*

$$\begin{cases} y_n & = (1-\beta_n)x_n + \beta_n z_n,\ z_n \in Tx_n,\ \|z_n - x^*\| = (x^*, Tx_n), \\ x_{n+1} & = (1-\alpha_n)x_n + \alpha_n u_n,\ u_n \in Ty_n,\ \|u_n - x^*\| = d(y_n, x^*), \end{cases} \tag{1.6.2}$$

converges strongly to a fixed point of T.

In [90], Panyanak extended the result of Sastry and Babu to a uniformly convex real Banach spaces. He proved the following result.

Theorem 1.6.2 *(Panyanak, [90]) Let* E *be a uniformly convex real Banach space,* K *be a nonempty, closed, bounded and convex subset of* E, *and* $T : D(T) \subseteq E \to CB(K)$ *a multi-valued nonexpansive map with a fixed point* p. *Assume that* (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ *and* (iii) $\sum \alpha_n\beta_n = \infty$. *where* α_n *and* β_n *are sequences of real numbers. Then, the sequence defined by (1.6.2) converges strongly to a fixed point of* T.

Remark 1.6.3 *In the recursion formular (1.6.2) the authors imposed condition that,* $z_n \in Tx_n$ *such that* $\|z_n - x^*\| = (x^*, Tx_n)$. *The existence of such* z_n *in each step of the iteration process is guaranteed when* Tx_n *is proximinal. In general to pick* z_n *is very difficult and hence this makes the iterative process to be inconvenient in any possible application.*

Chidume *et al.*, [39], introduced multi-valued k-strictly pseudo-contractive mappings. They gave the following definition.

Definition 1.6.4 *A multi-valued map* $T : D(T) \subset H \to CB(H)$ *is called* k-strictly pseudo-contractive *if there exists* $k \in (0,1)$ *such that for all* $x, y \in D(T)$,

$$\big(H(Tx, Ty)\big)^2 \quad \leq \quad \|x-y\|^2 + k\|x - y - (u-v)\|^2 \ \forall u \in Tx, v \in Ty.$$

They constructed a Krasnoselskii-type algorithm and showed that it converges strongly to a fixed point of T under some additional mild condition. More precisely, they proved the following result.

Theorem 1.6.5 *(Chidume et al. [39]) Let* K *be a nonempty, closed and convex subset of a real Hilbert space* H. *Suppose that* $T : K \to CB(K)$ *is a multi-valued* k*-strictly pseudo-contractive mapping such that* $F(T) \neq \emptyset$. *Assume that* $Tp = \{p\}$ *for all* $p \in F(T)$. *Suppose that* T *is semi-compact and continuous. Let* $\{x_n\}$ *be a sequence defined iteratively from* $x_0 \in K$ *by*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n,\ n \geq 0, \tag{1.6.3}$$

where $y_n \in Tx_n$ *and* $\lambda \in (0, 1-k)$. *Then, the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T .

Remark 1.6.6 *This result of Chidume et al. is an important improvement of several results in the literature. It deals with the class of multi-valued k-strictly pseudo-contractive mappings which is an important generalization of the class of multi-valued nonexpansive mappings. Moreover, the condition $z_n \in Tx_n$ such that $\|z_n - x^*\| = (x^*, Tx_n)$ imposed by Sastry and Babu in the recusion formular (1.6.2) is dispensed with in the theorem of Chidume et al. [39].*

Later on, Chidume et al. [40] extended their result to q-uniformly smooth real Banach space. The following is their main result.

Theorem 1.6.7 *(Chidume et al. [40]) Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Let $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ and such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is continuous and semi-compact. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{1.6.4}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

This leads us to the following important question.

Question: *Can an iterative algorithm be obtained to approximate fixed points of multi-valued k-strictly pseudo-contractive mappings in a more general metric space? That is, can we obtain the analogue of the results of [39] in important space that do necessarily have a norm?*

In chapter five of this thesis, we answer the above question in the affirmative by constructing a Krasnoselskii-type algorithm that converges strongly to a fixed point of T in a complete $CAT(k)$ space, $k \leq 0$, which has been studied by various world-class mathematicians (see e.g., Bridson and Haefliger [12], Bruhat [20], Burago *et al.* [21], Kirk [75, 76, 77]).

In chapter six of this thesis, we also prove a convergence result for a countable family of k-strictly pseudo-contractive mappings in Hilbert spaces.

1.7 Hammerstein Integral Equations

Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of Hammerstein-type has the form

$$u(x) + \int_\Omega k(x, y) f(y, u(y)) dy = w(x), \tag{1.7.1}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions. If we define $F : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ and

$K : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ by

$$Fu(y) = f(y, u(y)),\ y \in \Omega,$$

and

$$Kv(x) = \int_\Omega k(x, y)v(y)dy,\ x \in \Omega,$$

respectively, where $\mathcal{F}(\Omega, \mathbb{R})$ is a space of measurable real-valued functions defined from Ω to $\mathbb{R}$, then equation (1.7.1) can be put in an abstract form

$$u + KFu = w. \tag{1.7.2}$$

Without loss of generality we can assume that $w \equiv 0$ so that (1.7.2) becomes

$$u + KFu = 0. \tag{1.7.3}$$

Indeed, if $w \neq 0$, then $u - w + KFu = 0$. setting $h = u - w$ we obtain that

$$h + K\overline{F}h = 0,$$

where $\overline{F}(h) = F(h + w)$.

Interest in (1.7.1) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear part posses Green's function can, as a rule, be transformed into the form (1.7.1) (see e.g., Pascali and Sburian [91], chapter 4, p. 164). Among these, we mention the problem of the forced ocsillation of finite amplitude of a pendulum.

Example. We consider the problem of the pendulum

$$\begin{cases} \frac{d^2v(t)}{dt^2} + a^2 \sin v(t) \quad = z(t), \quad t \in [0, 1], \\ \\ v(0) = v(1) = 0. \end{cases} \tag{1.7.4}$$

where the driving force z is odd. The constant a ($a \neq 0$) depends on the length of the pendulum and gravity. Since the Green's function of the problem

$$v''(t) = 0;\ v(0) = v(1) = 0$$

is the function defined by

$$k(t, s) = \begin{cases} t(1 - s),\ 0 \leq t \leq s \leq 1, \\ s(1 - t),\ 0 \leq s \leq t \leq 1, \end{cases} \tag{1.7.5}$$

it follows that problem (1.7.4) is equivalent to the nonlinear integral equation

$$v(t) = \int_0^1 k(t, s)[z(s) - a^2 \sin v(s)]ds,\ t \in [0, 1]. \tag{1.7.6}$$

Setting $g(t) = \int_0^1 k(t,s)z(s)ds$ and $u(t) = v(t) - g(t)$, then we have

$$u(t) + \int_0^1 k(t,s)a^2 \sin(u(s) + g(s))ds = 0$$

which is in Hammerstein equation form

$$u(t) + \int_0^1 k(t,s)f(s,u(s))ds = 0,$$

where $f(s,t) = a^2 \sin(t + g(s))$.

Equations of Hammerstein-type play a crucial role in the theory of optimal control system and in automation and network theory (see e.g., Dolezale [60]). Several existence results have been proved for equations of Hammerstein-type (see e.g., Brézis and Browder [9, 10, 11], Browder [15], Browder, De Figueiredo and Gupta [16]).

1.8 Approximating solutions of equations of Hammerstein-type

In general, equations of Hammerstein-type are nonlinear and there is no known method to find a close form solutions for them. Consequently, methods of approximating solutions of such equations are of interest.

Let H be a real Hilbert space. A *nonlinear* operator $A : H \to H$ is said to be *angle-bounded* with angle $\beta > 0$ if and only if

$$\langle Ax - Ay, z - y\rangle \leq \beta\langle Ax - Ay, x - y\rangle \tag{1.8.1}$$

for any triple elements $x, y, z \in H$. For $y = z$ inequality (1.8.1) implies the monotonicity of A.

A monotone *linear* operator $A : H \to H$ is said to be *angle bounded* with angle $\alpha > 0$ if and only if

$$|\langle Ax, y\rangle - \langle Ay, x\rangle| \leq 2\alpha\langle Ax, x\rangle^{\frac{1}{2}}\langle Ay, y\rangle^{\frac{1}{2}} \tag{1.8.2}$$

for all $x, y \in H$. In the special case where the operator is angle bounded Brézis and Browder [9, 11] proved the strong convergence of a suitably defined Galerkin approximation to a solution of (1.7.2). In fact, they prove the following theorem.

Theorem 1.8.1 (Brézis and Browder [11]) *Let H be a separable Hilbert space and C be a closed subspace of H. Let $K : H \to C$ be a bounded continuous monotone operator and $F : C \to H$ be angle-bounded and weakly compact mapping. For a given $f \in C$, consider the Hammerstein equation*

$$(I + KF)u = f \tag{1.8.3}$$

and its nth Galerkin approximation given by

$$(I + K_n F_n)u_n = P^* f, \tag{1.8.4}$$

where $K_n = P_n^* K P_n : H \to C$ *and* $F_n = P_n F P_n^* : C_n \to H$.

Then, for each $n \in \mathbb{N}$, *the Galerkin approximation (1.8.4) admits a unique solution* u_n *in* C_n *and* $\{u_n\}$ *converges strongly in* H *to the unique solution* $u \in C$ *of the equation (1.8.3).*

In the theorem above all the symbols used have their usual meanings (see e.g., [91]).

It is obvious that if an iterative algorithm can be developed for the approximation of solutions of equation of Hammerstein-type (1.7.3), this will certainly be preferred.
Attempts have been made to approximate solutions of equations of Hammerstein-type using Mann-type iteration scheme. However, the results obtained were not satisfactory (see e.g., [49]). The recurrance formulas used in early attempts involved K^{-1} which is also required to be strongly monotone, and this, apart from limiting the class of mappings to which such iterative schemes are applicable, it is also not convenient in applications. Part of the difficulty is the fact that the composition of two monotone operators need not to be monotone. It suffices to take $K : \mathbb{R}^2 \to \mathbb{R}^2$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$K = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

The first satisfactory results on *iterative methods* for approximating solutions of Hammerstein equations, as far as we know, were obtained by Chidume and Zegeye [51, 52, 53]. Under the setting of a real Hilbert space H, for $F, K : H \to H$, they defined an auxillary map on the Cartesian product $E := H \times H$, $T : E \to E$ by

$$T[u, v] = [Fu - v, Kv + u].$$

We note that

$$T[u, v] = 0 \Longleftrightarrow \text{u solves (1.7.3) and } v = Fu.$$

With this, they were able to obtain strong convergence of an iterative scheme defined in the Cartesian product space E to a solution of Hammerstein equation (1.7.3). Extensions to a real Banach space setting were also obtained.

Let X be a real Banach space and $F, K : X \to X$ be *accretive-type* mappings. Let $E := X \times X$. The same authors (see [51, 52]) defined $T : E \to E$ by

$$T[u, v] = [Fu - v, Kv + u]$$

and obtained strong convergence theorems for solutions of Hammerstein equations under various continuity conditions in the Cartesian product space E.
The method of proof used by Chidume and Zegeye provided the clue to the establishement of the following couple explicit algorithm for computing a solution of the

equation $u + KFu = 0$ in the original space X. With initial vectors $u_0, v_0 \in X$, sequences $\{u_n\}$ and $\{v_n\}$ in X were defined iteratively as follows:

$$u_{n+1} = u_n - \alpha_n(Fu_n - v_n),\ n \geq 0, \tag{1.8.5}$$

$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n),\ n \geq 0, \tag{1.8.6}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying appropriate conditions. The recursion formulas (1.8.5) and (1.8.6) had been used successfully to approximate solutions of Hammerstein equations involving nonlinear accretive-type mappings. Following this, Chidume and Djitte [43, 44] studied this explicit couple iterative algorithm and proved several strong convergence theorems.

We remark here that even though monotone-type operators have more applications than accretive-type operators in Banach spaces, virtually all the results on the approximation of solutions of Hammerstein equations are either proved in Hilbert spaces or in a Banach space in the case where the operators K and F are accretive-type mappings (see [42], [46], [48] and [50]). To the best of our knowledge, there is no single result on the approximation of solutions of Hammerstein-type equations in Banach spaces (in the case where the operators K and F are monotone-type operators) that has appeared in the literature. Perhaps, part of the problem is that since the operator F maps E to E^* and K maps E^* to E the recursion formulas used for accretive-type mappings may no longer make sense.

In chapter seven, we proved convergence results for solutions of equations of Hammerstein-type in L_p spaces, $1 < p < \infty$, in the case where the operators K and F are of monotone-type using Mann-type algorithms.

CHAPTER 2

Preliminaries

In this chapter, we give some fundamental definitions and results that shall be used subsequently in the thesis. While we give proof of some of the results presented in this chapter, the proof of the rest can be found in the references mentioned in the result.

2.1 Duality Mappings and Geometry of Banach Spaces

Definition 2.1.1 *A real normed linear space E is said to be uniformly convex if for any $\varepsilon \in (0, 2]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for each $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$, we have that $\|\frac{1}{2}(x + y)\| \leq 1 - \delta$.*

Definition 2.1.2 *A normed linear space E is said to be* strictly convex *if for all $x, y \in E$ $x \neq y$, $\|x\| = \|y\| = 1$, the following inequality holds*

$$\|\alpha x + (1 - \alpha)y\| < 1 \text{ for all } \alpha \in (0, 1).$$

Remark 2.1.3 *Every uniformly convex space is stricly convex. However the converse may not hold (see e.g., [29]). Moreover, it is well known that every uniformly convex space is reflexive.*

L_p spaces, $1 < p < \infty$, and l_p spaces, $1 < p < \infty$, are both uniformly convex spaces. Thus, stricly convex spaces.

Let E be a real normed linear space with dual E^* and let $S := \{x \in E : \|x\| = 1\}$. The space E is said to have *Gâteaux differentiable norm,* in this case, E is called *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1.1}$$

exists for all $x, y \in S$. The space E is said to have *uniformly Gâteaux differentiable norm* if for each $y \in S$, the limit in (2.1.1) is attained uniformly for $x \in S$. If the limit exists uniformly for all $x, y \in S$, E is said to be *uniformly smooth.*

Let E be a real normed linear space of $dim(E) \geq 2$, where $dim(E)$ denotes the dimension of the space E. The *modulus of smoothness* of E , ρ_E, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

A normed linear space E is called *uniformly smooth* if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$. It is well known (see, *e.g.* [29], [79]) that ρ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth.* Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where

$$L_p \text{ (or } l_p) \text{ or } W_p^m \text{ is } \begin{cases} 2-\text{uniformly smooth} & \text{if} \quad 2 \leq p < \infty; \\ p-\text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{cases}$$

Lemma 2.1.4 (Lindenstrauss and Tzafriri, [79]) *In L_p(or ℓ_p) spaces, $1 < p < \infty$,*

$$\rho_{L_p}(\tau) = \begin{cases} (1+\tau^p)^{\frac{1}{p}} - 1 < \frac{1}{p}\tau^p; & 1 < p < 2 \\ \frac{p-1}{2}\tau^2 + o(\tau^2) < \frac{p-1}{2}\tau^2; & p \geq 2. \end{cases}$$

Definition 2.1.5 *Let E be a real normed linear space and $p > 1$, Then, the generalized duality map $J_p : E \longrightarrow 2^{E^*}$ is defined by*

$$J_p(x) \quad = \quad \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \ ||x^*|| = ||x||^{p-1}\}, \tag{2.1.2}$$

where $\langle ., . \rangle$ denotes the duality pairing between elements of E and E^* (see e.g., [54]).

For $p = 2$, we have from (2.1.2) that,

$$J_2(x) \quad = \quad \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \ ||x^*|| = ||x||\}.$$

J_2 is called the *normalized duality mapping* on E and is simply denoted by J.

We make the following remarks.

- The normalized duality mapping exists in any Banach space and its domain is whole E.
- In Hilbert spaces, normalized duality mappings are precisely the identity maps, while in L_P spaces, $1 < P < \infty$, the duality map is given by

$$J(f) = |f|^{p-1} \cdot sign \frac{f}{\|f\|^{p-1}}.$$

- The value of the duality mappings in spaces higher than L_P spaces is not known hitherto.

Lemma 2.1.6 (see e.g., [4], p.34) *If E is a strictly convex space, then J is a strictly monotone operator. If E^* is strictly convex, then J is single-valued.*

Remark 2.1.7 *From Lemma 2.1.6, we can infer that, if E is uniformly convex (hence strictly convex and reflexive) and E^* is strictly convex, then the inverse of the normalized duality map $J^{-1} : E^* \to E$ is well defined.*

We give the following result which gives the relation between the inverse of the normalized duality mapping J^{-1}, and the duality mapping J_* on E^*

Lemma 2.1.8 (see e.g., [4], p.36) *Let E be a reflexive strictly convex Banach space with strictly convex dual space E^*. If $J_p : E \to E^*$ and $J_q^* : E^* \to E$ are the duality mappings on E and E^*, respectively, such that $\frac{1}{p} + \frac{1}{q} = 1$, then $J_p^{-1} = J_q^*$.*

In Lemma 2.1.8 above, if $p = 2$, then it is readily clear that $J^{-1} = J_*$, where J_* is the normalized duality mapping on E^*.

Lemma 2.1.9 (see e.g., [29], p. 55) *Let $E = L_p$. Then, the following inequalities hold:*

If $1 < p < 2$, then we have for all x, y in L_p, and some constant $c_p > 0$,

$$\|x + y\|^2 \quad \geq \quad \|x\|^2 + 2\langle y, j(x)\rangle + c_p\|y\|^2, \tag{2.1.3}$$

$$\langle x - y, J(x) - J(y)\rangle \geq (p - 1)\|x - y\|^2. \tag{2.1.4}$$

Observe that inequality (2.1.4) yields

$$||J^{-1}(x) - J^{-1}(y)|| \leq L_1||x - y||,$$

where $L_1 := \frac{1}{p-1}$.

Lemma 2.1.10 (Alber and Ryazantseva [4], p.48) *Let $E = L_p$, $2 \leq p < \infty$. Then, J^{-1} is Hölder continuous on each bounded set, i.e., $\forall u, v \in E^*$ such that $\|u\| \leq R$ and $\|v\| \leq R$, the following inequality holds:*

$$\|J^{-1}(u) - J^{-1}(v)\| \leq m_p\|u - v\|^{\frac{1}{p-1}},$$

where $m_p := (2^{p+1}Lpc_2^p)^{\frac{1}{p-1}}$, for some constants $c_2 > 0,\ L \in (1, 1.7)$.

Proof This follows from the following inequality which holds on bounded sets:

$$\langle Jx - Jy, x - y\rangle \geq \frac{\|x - y\|^p}{2^{p+1}Lpc_2^p}, \quad c_2 = 2\max\{1, R\},\ L \in (1, 1.7). \tag{2.1.5}$$

(see e.g., Alber and Ryazantseva [4], p.48). ■

Definition 2.1.11 *Let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex function (i.e., f is not identically $+\infty$). Then, the sub-differential operator $\partial f : D(f) \subset E \to 2^{E^*}$, is defined by*

$$\partial f(x) = \{x^* \in E^* : \langle y - x, x^* \rangle \leq f(y) - f(x) \ \forall y \in E\}.$$

We remarked that if x is not in $D(f)$ then $\partial f(x) = \emptyset$.

Lemma 2.1.12 *Let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function defined by*

$$f(x) = \frac{1}{2}\|x\|^2 \quad \forall\ x \in E.$$

Then, for each $x \in E$, $\partial f(x) = J(x)$, where J is the duality map on E.

Indeed, let $x^* \in J(x)$. Then, for any $y \in E$ we have

$$\begin{aligned}\langle y - x, x^* \rangle = \langle y, x \rangle - \|x\|^2 &\leq \|y\|\|x\| - \|x\|^2 \\ &\leq \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \\ &= f(y) - f(x).\end{aligned}$$

Thus we have $x^* \in \partial f(x)$. Conversely, for $x^* \in \partial f(x)$ we have

$$\langle y - x, x^* \rangle \leq f(y) - f(x) \ \forall y \in E.$$

For $t \in (0, 1)$, set $y = x + ty$, then we have

$$\langle x^*, y \rangle \leq \frac{1}{2t}(\|x + ty\|^2 - \|x\|^2) \leq \|x\|\|y\| + \frac{t}{2}\|y\|^2.$$

As $t \to 0^+$ we have $\langle x^*, y \rangle \leq \|x\|\|y\|$, which implies $\|x^*\| \leq \|x\|$. Also using the fact that $x^* \in \partial f(x)$ and setting $y = x - tx, t \in (0, 1)$, we have

$$2t\langle -x, x^* \rangle \leq \|x - tx\|^2 - \|x\|^2 = (t^2 - 2t)\|x\|^2.$$

So we have $(2 - t)\|x\|^2 \leq 2\langle x, x^* \rangle$. Now as $t \to 0^+$ we obtained

$$\|x\|^2 \leq \langle x, x^* \rangle \leq \|x\|\|x^*\| \text{ which implies } \|x\| \leq \|x^*\|.$$

Therefore, we have $\|x\| = \|x^*\|$ and $\langle x, x^* \rangle = \|x\|^2$. Thus, $x^* \in J(x)$.

The following theorem gives the general form of Lemma 2.1.12.

Theorem 2.1.13 (see e.g., [29], p.32) *For $p \geq 1$, J_p is the sub-differential of the functional $\frac{1}{p}\|x\|^p$.*

2.2 Some Nonlinear Functionals and Operators

Let E be a smooth real Banach space with dual E^*. The function $\phi : E \times E \to \mathbb{R}$, defined by,

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for } x, y \in E, \tag{2.2.1}$$

where J is the normalized duality mapping from E into 2^{E^*} was introduced by Alber in [2], and it has been studied by Alber and Guerre-Delabriere [3], Kamimura and Takahashi [71], Reich [93] and a host of other authors. If $E = H$, a real Hilbert space, then equation (2.2.1) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \quad \text{for } x, y \in E. \tag{2.2.2}$$

Lemma 2.2.1 (Kamimura and Takahashi, [71]) *Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 2.2.2 *Let $E = L_p,\ 2 \leq p < \infty$. Then, the following inequality holds:*

$$||x - y||^2 \geq \phi(x, y) - p||x||^2 \ \forall\ x, y \in E.$$

Proof The following inequality holds for all $x, y \in L_p,\ p \geq 2$, (see e.g., Chidume [29], p. 54):

$$||x + y||^2 \leq ||x||^2 + 2\langle y, J(x)\rangle + (p - 1)||y||^2.$$

Interchanging x and y, we obtain:

$$||x + y||^2 \leq ||y||^2 + 2\langle x, J(y)\rangle + (p - 1)||x||^2.$$

Replacing y by $(x + y)$ and x by $(-x)$ we get:

$$||y||^2 \leq ||x + y||^2 - 2\langle x, J(x + y)\rangle + (p - 1)||x||^2,$$

which implies,

$$\begin{aligned} ||x + y||^2 &\geq ||y||^2 + 2\langle x, J(x + y)\rangle + ||x||^2 - p||x||^2 \\ &= (||x||^2 + 2\langle x, J(x + y)\rangle + ||y||^2) + 2\langle x, J(y)\rangle - 2\langle x, J(y)\rangle - p||x||^2. \end{aligned}$$

Replacing y by $(-y)$ and using the fact that the normalized duality map is monotone, we obtain:

$$\begin{aligned} ||x - y||^2 &\geq (||x||^2 - 2\langle x, J(y)\rangle + ||y||^2) + 2[\langle x, J(y) - J(y - x)\rangle] - p||x||^2 \\ &= \phi(x, y) + 2[\langle x, J(y) - J(y - x)\rangle] - p||x||^2 \\ &\geq \phi(x, y) - p||x||^2, \end{aligned}$$

establishing the lemma. ■

Let $V : E \times E^* \to \mathbb{R}$ be a map defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2. \tag{2.2.3}$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}(x^*)) \ \forall x \in X, \ x^* \in X^*. \tag{2.2.4}$$

Lemma 2.2.3 (Alber, [2]) *Let E be a reflexive striclty convex Banach space with with striclty convex dual E^*. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*) \tag{2.2.5}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Proof For arbitrary $x \in E, \ x^*, y^* \in E^*$, we have

$$\begin{aligned} V(x, x^*) &= \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \\ &= \|x\|^2 - 2\langle x, x^* + y^*\rangle + \|x^* + y^*\|^2 + \|x^*\|^2 - \|x^* + y^*\|^2 + 2\langle x, y^*\rangle \\ &= V(x, x^* + y^*) + \|x^*\|^2 - \|x^* + y^*\|^2 + 2\langle x, y^*\rangle \end{aligned}$$

Using the sub-differential inequality, and the fact that $\partial f(\frac{1}{2}\|\cdot\|^2) = J_* = J^{-1}$ (see Lemmas 2.1.12 and 2.1.8), where $\|\cdot\|_*$ and J_* are the norm and the normalized duality map of E^* respectively, we have

$$\begin{aligned} V(x, x^*) &\leq V(x, x^* + y^*) - 2\langle J^{-1}x^*, y^*\rangle - 2\langle -x, y^*\rangle \\ &\leq V(x, x^* + y^*) - 2\langle J^{-1}x^* - x, y^*\rangle. \end{aligned}$$

The proof is complete. ■

A map $A : E \to E^*$ is called *monotone* if for all $x, y \in E$, the following inequality holds:

$$\big\langle Ax - Ay, x - y\big\rangle \geq 0. \tag{2.2.6}$$

The mapping A is called *maximal monotone* if the graph of A is not properly contained in any other graph of monotone operator defined on E. i.e., A is maximal monotone if for any $(u, v) \in E \times E^*$ such that

$$\big\langle Ax - v, x - u\big\rangle \geq 0 \ \forall x \in D(A)$$

we have $u \in D(A)$ and $Au = v$.

The mapping A is called *strongly monotone* if there exists $k \in (0, 1)$ such that for all $x, y \in E$, the following inequality holds:

$$\big\langle Ax - Ay, x - y\big\rangle \geq k\|x - y\|^2. \tag{2.2.7}$$

A mapping $A : E \to E$ is called *generalized $\Phi -$ strongly monotone* if there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \Phi(\|x - y\|) \ \forall \ x, y \in D(A). \tag{2.2.8}$$

If $\Phi(t) = \phi(t)t$, where $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$, then A is called $\phi -$ *strongly monotone*.

Every strongly monotone operator is generalized Φ-strongly monotone (by setting $\Phi(t) = kt^2$, $k \in (0, 1)$), and every generalized Φ-strongly monotone operator is obviously monotone.

Simillarly, a map $A : E \to E$ is called *accretive* if for all $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\big\langle Ax - Ay, j(x - y)\big\rangle \geq 0. \tag{2.2.9}$$

A is called *strongly accretive* if there exists $k \in (0, 1)$ such that for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\big\langle Ax - Ay, j(x - y)\big\rangle \geq k\|x - y\|^2. \tag{2.2.10}$$

A mapping $A : E \to E$ is called *generalized* $\Phi -$ *strongly accretive* if there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \Phi(\|x - y\|) \ \forall \ x, y \in D(A).$$

If $\Phi(t) = \phi(t)t$, where $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$, then the mapping A is called $\phi -$ *strongly accretive*.

Remark 2.2.4 *If u is a solution of the equation $Au = 0$, where A is a generalized Φ-strongly monotone or generalized Φ-strongly accretive, then u is unique. Indeed, if $u_1, u_2 \in E$ are two different solutions of $Au = 0$ (i.e., $u_1 \neq u_2$ such that $Au_1 = Au_2 = 0$), then $0 = \langle Au_1 - Au_2, (u_1 - u_2)\rangle \geq \Phi(\|u_1 - u_2\|) > 0$ since $u_1 \neq u_2$ and $\Phi(t) > 0$ for $t > 0$. Hence a contradiction. Thus, $u_1 = u_2$. However, if A is just monotone, it is easy to see that the solution of the equation $Au = 0$ is not necessarily unique.*

Lemma 2.2.5 (Kato, [29]) *Let E be real Banach space and let J be the normalized duality map. Then for any $x, y \in E$, the following are equivalent:*

(i) $\|x\| \leq \|x + y\| \ \forall \ \lambda > 0$
(ii) there exists $u^* \in Jx$ such that $\langle y, u^*\rangle \geq 0$.

By virtue of Lemma 2.2.5, it can be shown that A is accretive if and only if for all $\lambda > 0$ and for all $x, y \in E$

$$\|x - y\| \leq \|x - y + \lambda(Ax - Ay)\|. \tag{2.2.11}$$

A map $T : E \to E$ is said to be *peudo-contractive if $I - T$ is accretive*. Setting $A = I - T$, it is easy to see that fixed points of T are precisely the zeros of A.

Let E be a real normed linear space. A map $T : E \to E$ is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\| \text{ for all } x, y \in E.$$

It is easy to see that every nonexpansive map is also pseudo-contractive.

Let K be a nonempty subset of a real Hilbert space H. A map $T : K \to H$ is called *k-strictly pseudo-contractive* if and only if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \quad \leq \quad \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2 \qquad \forall x, y \in K. \quad (2.2.12)$$

Let K be a nonempty subset of a real normed space E. A map $T : K \to E$ is called *k-strictly pseudo-contractive* (see, *e.g.*, [29], p.145; [17]) if there exists $k \in (0, 1)$ such that for all $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \quad \leq \quad \|x - y\|^2 - k\|x - y - (Tx - Ty)\|^2. \qquad (2.2.13)$$

In Hilbert spaces, (2.2.12) and (2.2.13) are equivalent.

Let E be a real normed linear space and $A : E \to 2^E$ be a multi-valued map. Then, the *domain* and *range* of T are defined by

$$D(T) = \{x \in E : Tx \neq \emptyset\},$$

and

$$R(T) = \bigcup_{x \in D(T)} Tx,$$

respectively. A point $x \in E$ is called a fixed point of T if $x \in Tx$. The set $F(T) = \{x \in E : x \in Tx\}$ is called the fixed point set of T.

Let (E, d) be a metric space. We denote by $CB(E)$, the collection of all nonempty closed and bounded subsets of E. Let H be the Hausdorff metric with respect to the metric distance d, i.e.,

$$H(A, B) = max\left\{\sup_{a \in A} d(a, B), \quad \sup_{b \in B} d(A, b)\right\},$$

for all $A, B \in CB(E)$, where $d(a, B) = \inf_{b \in B} d(a, b)$ is the distance from the point a to the subset B. A multi-valued mapping $T : E \to 2^E$ is said to be

(i) Nonexpansive if and only if

$$H(Tx, Ty) \leqslant d(x, y) \ \forall \ x, y \in E;$$

(ii) Quasi-nonexpansive if and only if $F(T) \neq \emptyset$ and

$$H(Tx, p) \leqslant d(x, p) \ \forall \ x \in E, \ p \in F(T);$$

(iii) Demi-contractive if and only if $F(T) \neq \emptyset$ and there exists $k \in (0,1)$ such that

$$H(Tx,Tp)^2 \leqslant d(x,p)^2 + kd(x,Tx)^2 \ \forall\ x \in E,\ p \in F(T),$$

where $H(Tx,Tp)^2 = [H(Tx,Tp)]^2$ and $d(x,p)^2 = [d(x,p)]^2$.

(iv) Hemi-contractive if $k = 1$ in (iii) above, i.e.,

$$H(Tx,Tp)^2 \leqslant d(x,p)^2 + d(x,Tx)^2 \ \forall\ x \in E,\ p \in F(T).$$

It is clear that, every multi-valued nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive, and every quasi-nonexpansive mapping is demi-contractive mapping.

The following example shows that the class of demi-contractive mappings strictly contains the class of quasi-nonexpansive mappings.

Example 2.2.6 *Let $X = \mathbb{R}$ (the set of real numbers with the usual metric). Define $T : E \to 2^E$ by*

$$Tx = \begin{cases} [-3x, -\frac{5x}{2}],\ x \in [0,\infty), \\ [-\frac{5x}{2}, -3x],\ x \in (-\infty, 0]. \end{cases} \tag{2.2.14}$$

Then, $F(T) = \{0\}$, and T is demi-contractive mapping which is not quasi-nonexpansive.

Indeed, for each $x \in (-\infty,0) \cup (0,\infty)$, we have

$$H(Tx,T0)^2 = |-3x-0|^2 = 9|x-0|^2,$$

which implies that T is not quasi-nonexpansive.

We also have that;

$$d(x,Tx)^2 = \left|x - (-\frac{5}{2}x)\right|^2 = \frac{49}{4}|x|^2.$$

Thus,

$$H(Tx,T0)^2 = |x-0|^2 + 8|x-0|^2 = |x-0|^2 + \frac{32}{49}d(x,Tx)^2.$$

Hence, T is a demi-contractive mapping with constant $k = \frac{32}{49} \in (0,1)$.

2.3 Some Important Results about Geodesic Spaces

Let (E,d) be a metric space. A geodesic path joining $x \in E$ and $y \in E$ is a continuous map c from a closed interval $[0,l] \subset \mathbb{R}$ to E such that $c(0) = x,\ c(l) = y$ and $d(c(t),c(t^{'})) = |t - t^{'}|$ for all $t, t^{'} \in [0,l]$. In particular, the mapping c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic segment joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. The space (E,d) is called a geodesic space if any two points of E are joined by a geodesic,

and E is said to be uniquely geodesic if there is exactly one geodesic joining x and $y \in E$. A subset K of E is said to be convex if for all $x,\ y \in K$, the segment $[x, y]$ remains in K.
A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (E, d) consists of three points in E (the vertices of $\triangle$), and a geodesic segment between each pair of points (the edges $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (E, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) = \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(x_i, x_j) = d(\overline{x_i}, \overline{x_j})$, for $i, j \in \{1, 2, 3\}$. A geodesic metric space E is called a $CAT(0)$ space if all geodesic triangles satisfy the following comparison axiom:
Let $\triangle$ be a geodesic triangle in E, and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then, $\triangle$ is said to satisfy $CAT(0)$ inequality, if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leqslant d(\overline{x}, \overline{y}).$$

If x, y_1, y_2 are points in $CAT(0)$ space, and if y_0 is the midpoint of the segment $[y_1, y_2]$, then, the $CAT(0)$ inequality implies

$$d(x, y_0)^2 \leqslant \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \qquad (CN)$$

This is the (CN) inequality of Bruhat and Tits [20]. In fact, (cf.[12], p.163), a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality.
We now collect some elementary facts about $CAT(0)$ spaces.

Lemma 2.3.1 (See e.g., [57]) *Let (E, d) be a $CAT(0)$ space. Then,*

(i) (E, d) is uniquely geodesic.

(ii) For all $x, y \in E$, and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1 - t)d(x, y). \qquad (2.3.1)$$

For convenience, from now on, we shall use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (2.3.1).
Also, for $\alpha_1, \alpha_2, \alpha_3 \in (0, 1)$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$, and $x_1, x_2, x_3 \in E$, we will use the notation $\alpha_1 x_1 \oplus \alpha_2 x_2 \oplus \alpha_3 x_3$ to denote the unique point satisfying

$$\begin{aligned} d(x_1, z) &= (\alpha_2 + \alpha_3)d(x_1, \alpha_2^{'}x_2 \oplus \alpha_3^{'}x_3), \text{ and} \\ d(\alpha_2^{'}x_2 \oplus \alpha_3^{'}x_3, z) &= \alpha_1 d(x_1, \alpha_2^{'}x_2 \oplus \alpha_3^{'}x_3),\ \alpha_i^{'} := \frac{\alpha_i}{(\alpha_2 + \alpha_3)},\ i = 2, 3. \end{aligned} \qquad (2.3.2)$$

In particular, taking $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$, we compute the point $\frac{1}{3}x_1 \oplus \frac{1}{3}x_2 \oplus \frac{1}{3}x_3$ as follows:

From the illustration above, $\frac{1}{3}x_1 \oplus \frac{1}{3}x_2 \oplus \frac{1}{3}x_3 := \frac{1}{3}x_1 \oplus \frac{2}{3}\left(\frac{1}{2}x_2 \oplus \frac{1}{2}x_3\right)$, where $\frac{1}{2}x_2 \oplus \frac{1}{2}x_3$ denotes the unique point $z_1 \in [x_2, x_3]$ such that $d(x_2, z_1) = \frac{1}{2}d(x_2, x_3)$, and $d(z_1, x_3) = \frac{1}{2}d(x_2, x_3)$.

Thus, we have $\frac{1}{3}x_1 \oplus \frac{1}{3}x_2 \oplus \frac{1}{3}x_3 := \frac{1}{3}x_1 \oplus \frac{2}{3}z_1$, where $\frac{1}{3}x_1 \oplus \frac{2}{3}z_1$ denotes the unique point $z_2 \in [x_1, z_1]$ satisfying $d(x_1, z_2) = \frac{2}{3}d(x_1, z_1)$, and $d(z_2, z_1) = \frac{1}{3}d(x_1, z_1)$. Hence we have $z_2 := \frac{1}{3}x_1 \oplus \frac{1}{3}x_2 \oplus \frac{1}{3}x_3$.

Extending this notation up to some $n \geqslant 3$, we use $\sum_{i=1}^{n} \oplus \alpha_i x_i$ to denote the unique point $z \in \left[x_1, \sum_{i=2}^{n} \oplus \frac{\alpha_i}{\sigma} x_i\right]$ satisfying

$$\begin{aligned} d(x_1, z) &= \sigma d\left(x_1, \sum_{i=2}^{n} \oplus \frac{\alpha_i}{\sigma} x_i\right), \\ d\left(\sum_{i=2}^{n} \oplus \frac{\alpha_i}{\sigma} x_i, z\right) &= \alpha_1 d\left(x_1, \sum_{i=2}^{n} \oplus \frac{\alpha_i}{\sigma} x_i\right), \end{aligned} \tag{2.3.3}$$

where $\alpha_i \in (0,1),\ i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} \alpha_i = 1,\ x_i \in E,\ i = 1, 2, \ldots, n$, $\sigma = \sum_{i=2}^{n} \alpha_i = (1 - \alpha_1)$.

Lemma 2.3.2 (See e.g., [57], lemmas 2.4 **and** 2.5**)** *Let (E, d) be a $CAT(0)$ space. For $x, y \in E$, and $t \in [0,1]$, the following inequalities hold:*

(i) $d((1-t)x \oplus ty, z) \leqslant (1-t)d(x,z) + td(y,z)$;

(ii) $d((1-t)x \oplus ty, z)^2 \leqslant (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2$;

where $d(x,z)^2 = (d(x,z))^2$.

Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space E. For $x \in E$, we set $r(x, \{x_n\}) = \limsup d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\})\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is well known that in a $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point.

Definition 2.3.3 *A sequence $\{x_n\}$ in a $CAT(0)$ space E is said to Δ-converge to $x \in E$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim x_n = x$ and x is called the Δ-limit of $\{x_n\}$.*

Lemma 2.3.4 *(i) (See e.g., [77]) Every bounded sequence in a complete $CAT(0)$ space has a Δ-convergent subsequence.*

(ii) (See e.g., [59]) If C is a nonempty closed and convex subset of a complete $CAT(0)$ space, and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.

(iii) (See e.g., [57]) If $\{x_n\}$ is a bounded sequence in a complete $CAT(0)$ E, with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.

Lemma 2.3.5 (Tan and Xu, [107]) *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq a_n + \sigma_n, \quad n \geq 0, \tag{2.3.4}$$

such that $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then, $lim_{n\to\infty} a_n$ exists. If, in addition, the sequence $\{a_n\}$ has a subsequence that converges to 0, then the sequence $\{a_n\}$ converges to 0.

CHAPTER 3

Krasnoselskii-Type Algorithm For Zeros of Strongly Monotone Lipschitz Maps in Classical Banach Spaces

3.1 Introduction

In this chapter, we construct and prove strong convergence of a Krasnoselskii-type sequence to the unique zero of Lipschitz strongly monotone operator in L_P spaces, $1 < p < \infty$. Furthermore, our technique of proof is of independent interest. We first recall the follwing useful definitions and Lemmas.

Definition 3.1.1 *An operator $T : E \to E^*$ is called ψ-strongly monotone if there exists a continuous, strictly increasing function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(0) = 0$ such that*

$$\big\langle Tx - Ty, x - y \big\rangle \geq \psi(\|x - y\|)\|x - y\| \quad \forall\ x, y \in D(T). \tag{3.1.1}$$

Definition 3.1.2 *Let E be a real normed linear space and $T : E \to E^*$ be a map. Then, T is said to be hemicontinuous if for all $x_1, x_2, y \in E$ and $\lambda \in \mathbb{R}$ the function $\mathbb{R} \to \mathbb{R}$ defined by $\lambda \to \langle y, T(x_1 + \lambda x_2)\rangle$ is continuous.*

Lemma 3.1.3 *Let $T : E \to E^*$ be a hemicontinuous ψ-strongly monotone operator. Then, $R(T) = E^*$.*

Proof See chapter III, page 48 of [91]. ■

Lemma 3.1.4 (Alber, [2]) *Let E be a reflexive striclty convex and smooth Banach space with E^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*) \tag{3.1.2}$$

for all $x \in E$ and $x^*, y^* \in E^*$, where $V(x, x^*) = \|x\|^2 - 2\langle x, J^{-1}x^*\rangle + \|x^*\|^2$ for all $(x, x^*) \in E \times E^*$.

Lemma 3.1.5 *Let $E = L_p$, $1 < p \leq 2$. Then J^{-1} is Lipschitz, i.e., there exists $L_1 > 0$ such that for all $u, v \in E^*$, the following inequality holds:*

$$\|J^{-1}(u) - J^{-1}(v)\| \leq L_1\|u - v\|. \tag{3.1.3}$$

Lemma 3.1.6 (Alber and Ryanzantseva [4], p.48) *Let $E = L_p$, $2 \leq p < \infty$. Then, the inverse of the normalized duality map $J^{-1} : E^* \to E$ is Hölder continuous on balls. i.e., $\forall\ u, v \in X^*$ such that $\|u\| \leq R$ and $\|v\| \leq R$,2*

$$\|J^{-1}(u) - J^{-1}(v)\| \leq m_p\|u - v\|^{\frac{1}{p-1}}, \tag{3.1.4}$$

where $m_p := (2^{p+1}Lpc_2^p)^{\frac{1}{p-1}} > 0$, for some constant $c_2 > 0$.

Proof This follows from the following inequality which holds on bounded sets:

$$\langle Jx - Jy, x - y\rangle \geq \frac{\|x - y\|^p}{2^{p+1}Lpc_2^p}, \quad c_2 = 2\max\{1, R\},\ L \in (1, 1.7). \tag{3.1.5}$$

(see e.g., Alber and Ryazantseva [4], p.48). ■

3.2 Convergence in L_P spaces, $1 < p < 2$

In the sequel, k is the strong monotonicity constant of A, $L > 0$ is its Lipschitz constant, and $\delta := \frac{k}{2(L_1+1)(L+1)^2}$, where L_1 is the Lipschitz constant of J^{-1}.

Theorem 3.2.1 *Let $E = L_p$, $1 < p < 2$. Let $A : E \to E^*$ be a strongly monotone and Lipschitz map. For $x_0 \in E$ arbitrary, let the sequence $\{x_n\}$ be defined by:*

$$x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), n \geq 0, \tag{3.2.1}$$

where $\lambda \in \Big(0, \delta\Big)$. Then, the sequence $\{x_n\}$ converges strongly to the unique zero of A.

Proof Let $\psi(t) = kt$ in inequality (3.1.1). By Lemma 3.1.3, $A^{-1}(0) \neq \emptyset$, which implies Remark 2.2.4 that $A^{-1}(0) = \{x^*\}$ for some $x^* \in E$ (since every strongly monotone operators is also generalized monotone). Using the definition of x_{n+1}, equation (2.2.4) and inequality (3.1.2) with $y^* = \lambda Ax_n$, we compute as follows:

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= V(x^*, Jx_n - \lambda Ax_n) \\
&\leq V(x^*, Jx_n) - 2\lambda\langle J^{-1}(Jx_n - \lambda Ax_n) - x^*, Ax_n - Ax^*\rangle \\
&= \phi(x^*, x_n) - 2\lambda\langle x_n - x^*, Ax_n - Ax^*\rangle \\
&+ 2\lambda\langle x_n - x^*, Ax_n - Ax^*\rangle \\
&- 2\lambda\langle J^{-1}(Jx_n - \lambda Ax_n) - x^*, Ax_n - Ax^*\rangle \\
&= \phi(x^*, x_n) - 2\lambda\langle x_n - x^*, Ax_n - Ax^*\rangle \\
&- 2\lambda\langle J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n), Ax_n - Ax^*\rangle.
\end{aligned}$$

Using the strong monotonocity of A, Lipschitz property of J^{-1} (see Lemma 3.1.5) and the Lipschitz property of A with Lipschitz constants L_1 and L, respectively, we have :

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\leq \phi(x^*, x_n) - 2\lambda k\|x_n - x^*\|^2 \\
&+ 2\lambda\|J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n)\|\|Ax_n - Ax^*\| \\
&\leq \phi(x^*, x_n) - 2\lambda k\left\|x_n - x^*\right\|^2 + 2\lambda^2 L_1 L^2\left\|x_n - x^*\right\|^2 \\
&\leq \phi(x^*, x_n) - \lambda k\|x_n - x^*\|^2.
\end{aligned}$$

Thus, $\phi(x^*, x_n)$ converges, since it is monotone decreasing and bounded below by zero. Consequently,

$$\lambda k\|x_n - x^*\|^2 \leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}) \to 0, \ as \ n \to \infty.$$

This yields $x_n \to x^* \ \ as \ \ n \to \infty$. ■

3.3 Convergence in L_p spaces, $2 \leq p < \infty$.

Remark 3.3.1 *We remark that for $E = L_p$, $2 \leq p < \infty$, if $A : E \to E^*$ satisfies the following conditions: there exists $k \in (0, 1)$ such that*

$$\left\langle Ax - Ay, x - y\right\rangle \geq k\|x - y\|^{\frac{p}{p-1}} \ \forall \ x, y \in E, \tag{3.3.1}$$

and $A^{-1}(0) \neq \emptyset$, then the Krasnoselskii-type sequence (3.2.1) converges strongly to the unique solution of $Au = 0$. In fact, we prove the following theorem.

In the next theorem, we set $\delta_p := \left(\frac{k}{2m_p L^{\frac{p}{p-1}}}\right)^{p-1}$.

Theorem 3.3.2 *Let $E = L_p$, $2 \leq p < \infty$. Let $A : E \to E^*$ be a Lipschitz map. Assume that there exists a constant $k \in (0, 1)$ such that A satisfies the following condition:*

$$\left\langle Ax - Ay, x - y\right\rangle \geq k\|x - y\|^{\frac{p}{p-1}}, \tag{3.3.2}$$

and that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_0 \in X$, define the sequence $\{x_n\}$ iteratively by:

$$x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), n \geq 0, \tag{3.3.3}$$

where $\lambda \in (0, \delta_p)$. Then, the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Ax = 0$.

Proof We first prove that $\{x_n\}$ is bounded. This proof is by induction.
Then, there exists $r > 0$ such that $\phi(x^*, x_1) \leq r$, where x^* is unique solution of $Ax = 0$. Suppose that $\phi(x^*, x_n) \leq r$, for some $n \geq 1$. We prove that $\phi(x^*, x_{n+1}) \leq r$.

Using equation (2.2.4) and inequality (3.1.2) with $y^* = \lambda Ax_n$, we have:

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(Jx_n - \lambda Ax_n)) = V(x^*, Jx_n - \lambda Ax_n) \\
&\leq V(x^*, Jx_n) - 2\langle J^{-1}(Jx_n - \lambda Ax_n) - x^*, \lambda Ax_n\rangle \\
&= V(x^*, Jx_n) - 2\lambda\langle x_n - x^*, Ax_n - Ax^*\rangle \\
&\quad -2\lambda\langle J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n), Ax_n - Ax^*\rangle. \\
&\leq \phi(x^*, x_n) - 2\lambda\langle x_n - x^*, Ax_n - Ax^*\rangle \\
&\quad +2\lambda\|J^{-1}(Jx_n - \lambda Ax_n) - J^{-1}(Jx_n)\|\|Ax_n - Ax^*\|.
\end{aligned}$$

Using condition (3.3.2) on A and inequality (3.1.4), we obtain:

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\leq \phi(x^*, x_n) - 2k\lambda\|x_n - x^*\|^{\frac{p}{p-1}} + 2\lambda\lambda^{\frac{1}{p-1}} m_p\|Ax_n\|^{\frac{1}{p-1}}\|Ax_n - Ax^*\| \\
&\leq \phi(x^*, x_n) - 2k\lambda\|x_n - x^*\|^{\frac{p}{p-1}} + 2\lambda\lambda^{\frac{1}{p-1}} m_p\|Ax_n - Ax^*\|^{\frac{p}{p-1}}. \\
&\leq \phi(x^*, x_n) - 2k\lambda\|x_n - x^*\|^{\frac{p}{p-1}} + 2\lambda\lambda^{\frac{1}{p-1}} m_p L^{\frac{p}{p-1}}\|x_n - x^*\|^{\frac{p}{p-1}} \\
&\leq \phi(x^*, x_n) - k\lambda\|x_n - x^*\|^{\frac{p}{p-1}} \\
&\leq r.
\end{aligned}$$

Hence, by induction, $\{x_n\}$ is bounded. We now prove that $\{x_n\}$ converges strongly to $x^* = A^{-1}(0)$. From the same computation as above, we have that:

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) - \lambda k\|x_n - x^*\|^{\frac{p}{p-1}},$$

which implies $\phi(x^*, x_n)$ is decreasing and bounded below by zero, so the limit of $\phi(x^*, x_n)$ exists. Therefore,

$$0 \leq \lim\left(\lambda k\|x_n - x^*\|^{\frac{p}{p-1}}\right) \leq \lim\left(\phi(x^*, x_n) - \phi(x^*, x_{n+1})\right) = 0.$$

Hence, $x_n \to x^*$ as $n \to \infty$. ■

All the results of this chapter are the results obtained in [33], which was **published** in **SpringerPlus**, June, 2015.

CHAPTER 4

An Algorithm for Computing Zeros of Generalized Phi-Strongly Monotone and bounded Maps in Classical Banach Spaces

4.1 Introduction

In this chapter, we construct and prove strong convergence of a Mann-type sequence to the unique zero of Generalized Phi-Strongly Monotone and bounded Maps in L_P spaces, $1 < p < \infty$. We first recall the following lemma.

Lemma 4.1.1 (see e.g., [29], p. 55) *Let $E = L_p$, $1 < p < 2$, then the following inequalities hold for all x, y in L_p, and some constant $c_p > 0$.*

$$\|x+y\|^2 \quad \geq \quad \|x\|^2 + 2\langle y, J(x)\rangle + c_p\|y\|^2, \tag{4.1.1}$$

$$\langle x-y, J(x)-J(y)\rangle \geq (p-1)\|x-y\|^2. \tag{4.1.2}$$

Let $E = L_p$, $1 < p < 2$. Define $\phi_p : E \times E \to \mathbb{R}$ by

$$\phi_p(x,y) = \|y\|^2 - 2\langle x, J(y)\rangle + c_p\|x\|^2, \tag{4.1.3}$$

where c_p is the constant appearing in inequality (4.1.1). Then, from (4.1.1) we have that

$$\|x-y\|^2 \geq \phi_p(x,y). \tag{4.1.4}$$

Also, following the pattern of proof of Lemma 3.1.4 (which was proved in Chapter 2), the following inequality can be established:

$$V_p(x,x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V_p(x, x^* + y^*) \tag{4.1.5}$$

for all $x \in E$ and $x^*, y^* \in E^*$, where $V_p(x,x^*) = \phi_p(x, J^{-1}x^*)$. Moreover, it can be easily seen that

$$\phi(x,y) = \phi_p(x,y) + (1-c_p)\|x\|^2. \tag{4.1.6}$$

Thus, the following inequality follows from inequality (2.2.2).

$$\|y\| \leq \sqrt{\phi_p(x,y) + (1-c_p)\|x\|} + \|x\| \ \forall\ x, y \in E. \tag{4.1.7}$$

It is also easy to see from inequality (2.2.2) and equation (4.1.6) that

$$(c_p - 1)\|x^*\| \leq \phi_p(x,y). \tag{4.1.8}$$

4.2 Convergence Theorems in L_p spaces, $1 < p < 2$

Theorem 4.2.1 *Let $E = L_p$, $1 < p < 2$. Let $A : E \to E^*$ be a generalized Φ-strongly monotone and bounded map with $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ iteratively by:*

$$x_{n+1} = J^{-1}(Jx_n - \alpha_n Ax_n), \quad n \geq 1, \tag{4.2.1}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ satisfies the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. Then, there exists $\gamma_0 > 0$ such that if $\alpha_n \leq \gamma_0\ \forall\ n \geq 1$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a solution of the equation $Ax = 0$.

Proof We first prove that $\{x_n\}_{n=1}^{\infty}$ is bounded. This proof is by induction. Let $\psi(t) = kt$ in inequality (3.1.1). By Lemma 3.1.3, $A^{-1}(0) \neq \emptyset$, which implies by Remark 2.2.4 that $A^{-1}(0) = \{x^*\}$ for some $x^* \in E$ (since every strongly monotone operators is also generalized monotone). Let $r > 0$ be such that $\phi_p(x^*, x_1) \leq r$. Since A is bounded, define:

$$M_0 := 2L_1 \sup\{\|Ax\|^2 : \|x\| \leq \sqrt{r + (1-c_p)\|x^*\|} + \|x^*\|\} + 1, \tag{4.2.2}$$

where $L_1 > 0$ is the Lipschitz constant of J^{-1}. Futhermore, define the following constants.

$$h_p := \Big[\Big(\frac{1}{L_2}\Big)\Big(\frac{1}{2}(p-1)\sqrt{r}\Big)\Big]^{p-1}; \quad \gamma_0 := \frac{1}{2} min\Big\{1, \frac{\sqrt{r}(p-1)}{M_0}, \frac{\Phi(h_p)}{2M_0}\Big\}. \tag{4.2.3}$$

where $L_2 := m_p$ is the constant of Hölder continuity of J (obtained from inequality (3.1.4)). We show that $\phi_p(x^*, x_n) \leq r\ \forall\ n \geq 1$. By construction, $\phi_p(x^*, x_1) \leq r$. Suppose that $\phi_p(x^*, x_n) \leq r$ for some $n \geq 1$. This implies, from inequality (4.1.7), that $\|x_n\| \leq \sqrt{r + (1-c_p)\|x^*\|} + \|x^*\|$. We prove that $\phi_p(x^*, x_{n+1}) \leq r$. Assume this is not the case. Then, $\phi_p(x^*, x_{n+1}) > r$. Using the definition of x_{n+1} and inequality (4.1.5) with $y^* = \alpha_n Ax_n$ we obtain:

$$\begin{aligned}
\phi_p(x^*, x_{n+1}) &= \phi_p(x^*, J^{-1}(Jx_n - \alpha_n Ax_n)) = V_p(x^*, Jx_n - \alpha_n Ax_n) \\
&\leq V_p(x^*, Jx_n) - 2\langle J^{-1}(Jx_n - \alpha_n Ax_n) - x^*, \alpha_n Ax_n\rangle \\
&= V_p(x^*, Jx_n) - 2\alpha_n\langle x_n - x^*, Ax_n - Ax^*\rangle \\
&\quad -2\alpha_n\langle J^{-1}(Jx_n - \alpha_n Ax_n) - J^{-1}(Jx_n), Ax_n - Ax^*\rangle. \\
&\leq \phi_p(x^*, x_n) - 2\alpha_n\langle x_n - x^*, Ax_n - Ax^*\rangle \\
&\quad +2\alpha_n\|J^{-1}(Jx_n - \alpha_n Ax_n) - J^{-1}(Jx_n)\|\|Ax_n - Ax^*\|.
\end{aligned}$$

Using the fact that A is generalized Φ-strongly monotone and that J^{-1} is Lipschitz (see Lemma 3.1.5, Chapter 3) we obtain:

$$\begin{aligned}\phi_p(x^*, x_{n+1}) &\leq \phi_p(x^*, x_n) - 2\alpha_n\Phi(\|x_n - x^*\|) + 2\alpha_n\alpha_n L_1\|Ax_n\|^2 \quad (4.2.4)\\ &\leq \phi_p(x^*, x_n) - 2\alpha_n\Phi(\|x_n - x^*\|) + \alpha_n\gamma_0 M_0.\end{aligned}$$

From recursion formular (4.2.1), inequalities (4.1.2) and (7.1.7) which are valid for all $x, y \in L_p, 1 < p < 2$, we obtain that,

$$\begin{aligned}||J(x_n) - Jx^*|| &= ||Jx_{n+1} - Jx^* + \alpha_n Ax_n||\\ &\geq (p-1)||x_{n+1} - x^*|| - \gamma_0 M_0\\ &> (p-1)\sqrt{r} - \gamma_0 M_0 \geq \frac{1}{2}(p-1)\sqrt{r}.\end{aligned}$$

Using the fact that J is Hölder continuous (obtained from inequality (3.1.4)), since J^{-1} on $L_p, 2 \leq p < \infty$ is J on $L_p, 1 < p < 2$, we have:

$$||x_n - x^*|| \geq \Big[\Big(\frac{1}{L_2}\Big)\Big(\frac{1}{2}(p-1)\sqrt{r}\Big)\Big]^{p-1} = h_p.$$

Hence,

$$\Phi(||x_n - x^*||) \geq \Phi(h_p).$$

Substituting this inequality in inequality (4.2.4), we obtain that

$$\phi_p(x^*, x_{n+1}) \leq \phi_p(x^*, x_n) - \alpha_n\Phi(h_p) + \frac{1}{2}\alpha_n\Phi(h_p). \quad (4.2.5)$$

Hence, we have that:

$$r \leq r - \frac{1}{2}\alpha_n\Phi(h_p) < r,$$

a contradiction. Hence, $\phi_p(x^*, x_{n+1}) \leq r$. By induction, $\phi_p(x^*, x_n) \leq r \ \forall \ n \geq 1$. Consequently, from inequality (4.1.7) and (4.1.8), we have that $\{x_n\}_{n=1}^{\infty}$ is bounded. We now prove that $\{x_n\}_{n=1}^{\infty}$ converges strongly to x^*. Repeating the same method of computation as above with ϕ instead of ϕ_p, the boundedness of the sequence $\{x_n\}_{n=1}^{\infty}$, and Lemma (3.1.4), there exists $M > 0$ such that:

$$\phi(x^*, x_{n+1}) \leqslant \phi(x^*, x_n) - 2\alpha_n\Phi(\|x_n - x^*\|) + \alpha_n^2 M.$$

By Lemma 2.3.5, we obtain that $\lim \phi(x^*, x_n)$ exists. Futhermore, using the condition condition $\sum \alpha_n = \infty$, we obtain that $\liminf \Phi(\|x_n - x^*\|) = 0$. From the properties of Φ, it is easy to see that $\liminf \|x_n - x^*\| = 0$. In fact, if this is not the case, then we have $\liminf \|x_n - x^*\| > 0$. Thus, there exist $\epsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - x^*\| \geqslant \epsilon_0 \ \forall \ n \geqslant 1$. Since Φ is strictly increasing, we have $\Phi(\|x_{n_k} - x^*\|) \geqslant \Phi(\epsilon_0) > 0$. Taking $\liminf$ on both sides, we have a contradiction since $\Phi(\epsilon_0) > 0$. Hence, $\liminf \|x_n - x^*\| = 0$. So, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^*$. Using the definition of ϕ and the continuity of J, we have $\phi(x^*, x_{n_k}) = \|x^*\| - 2\langle x^*, J(x_{n_k})\rangle + \|x_{n_k}\| \to 0$ as $k \to \infty$. Thus, again by Lemma 2.3.5, we obtain that $\lim \phi(x^*, x_n) = 0$, and by Lemma 2.2.1, we conclude that, $\lim \|x_n - x^*\| = 0$. This completes the proof. ■

4.3 Convergence Theorems in L_p spaces, $2 \leq p < \infty$

We prove the following theorem.

Theorem 4.3.1 *Let $E = L_p$, $2 \leq p < \infty$. Let $A : E \to E^*$ be a generalized Φ-strongly monotone and bounded map with $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ iteratively by:*

$$x_{n+1} = J^{-1}(Jx_n - \alpha_n Ax_n), \quad n \geq 1, \tag{4.3.1}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ satisfies the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^{\frac{p}{p-1}} < \infty$. Then, there exists $\gamma_0 > 0$ such that if $\alpha_n \leq \gamma_0 \ \forall \ n \geq 1$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique solution of the equation $Ax = 0$.

Proof We first prove that $\{x_n\}_{n=1}^{\infty}$ is bounded. This proof is by induction.
Let x^* be the unique solution of $Ax = 0$. Clearly, there exists $r > 0$ such that

$$r \geq \max\{4p\|x^*\|^2, \phi(x^*, x_1)\}. \text{ Thus, } \phi(x^*, x_1) \leq r. \tag{4.3.2}$$

Since A is bounded, define:

$$M_0 := 2m_p \sup\{\|Ax\|^{\frac{p}{p-1}} : \|x\| \leq \|x^*\| + \sqrt{r}\} + 1 < \infty, \tag{4.3.3}$$

where

$$m_p := \Big(\frac{1}{2^{p+1}Lpc_2^p}\Big)^{\frac{1}{p-1}}, \quad c_2 = 2\max\{1, R\}, L \in (1, 1.7). \tag{4.3.4}$$

Also, define the following constants:

$$\delta_p := \Big(\frac{\frac{1}{2}(p-1)r}{L_3}\Big)^{\frac{1}{p-1}}; \quad h_p := \Big(\frac{1}{L_3}\Big)\Big(\frac{1}{2}m_p(\sqrt{r - 4p\|x^*\|^2})^{p-1}\Big), \tag{4.3.5}$$

where L_3 is the Lipschitz constant of J on bounded sets. Define

$$\gamma_0 := min\frac{1}{2}\Big\{1, \frac{m_p(\sqrt{r - 4p\|x^*\|^2})^{p-1}}{2M_0}, \frac{\Phi(h_p)}{M_0}\Big\}.$$

We show that $\phi(x^*, x_n) \leq r \ \forall \ n \geq 1$. By construction, $\phi(x^*, x_1) \leq r$. Suppose that $\phi(x^*, x_n) \leq r$ for some $n \geq 1$. We prove that $\phi(x^*, x_{n+1}) \leq r$. Suppose this is not the case. Then, $\phi(x^*, x_{n+1}) > r$. Following the method of proof of Theorem 4.2.1, we obtain that:

$$\begin{aligned} \phi(x^*, x_{n+1}) \quad \leq \quad & \phi(x^*, x_n) - 2\alpha_n\langle x_n - x^*, Ax_n - Ax^*\rangle \\ & +2\alpha_n\|J^{-1}(Jx_n - \alpha_n Ax_n) - J^{-1}(Jx_n)\|\|Ax_n - Ax^*\|. \end{aligned}$$

Using the fact that A is generalized Φ-strongly monotone and that J^{-1} is Hölder continuous on balls, we obtain, since $\{x_n\}$ is bounded by our induction hypothesis:

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) - 2\Phi(\|x_n - x^*\|) + 2\alpha_n\alpha_n^{\frac{1}{p-1}}m_p\|Ax_n\|^{\frac{p}{p-1}}. \tag{4.3.6}$$

Observe that, from the recurrance relation (4.3.1), inequality (3.1.4) and the fact that J is Lipschitz (see inequality (4.1.2)), we have:

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|J^{-1}(Jx_n-\alpha_n Ax_n)-J^{-1}Jx^*\| \\ &\leq m_p(\|Jx_n-Jx^*\|+\gamma_0 M_0)^{\frac{1}{p-1}} \\ &\leq m_p(L_3(\|x^*\|+\sqrt{r}+1)+\gamma_0 M_0)^{\frac{1}{p-1}},\end{aligned}$$

which implies that $\|x_{n+1}\| \leq R$ for some constant $R > 0$. Now, from recursion formula (4.3.1), Lemma 2.2.2, and the fact that $\phi(x^*, x_{n+1}) > r$ we obtain that

$$\begin{aligned}||J(x_n)-Jx^*|| &= ||Jx_{n+1}-Jx^*+\alpha_n Ax_n|| \\ &\geq m_p||x_{n+1}-x^*||^{p-1}-\gamma_0 M_0 \\ &> m_p(\sqrt{r-4p\|x^*\|^2})^{p-1}-\gamma_0 M_0 \geq \frac{1}{2}m_p(\sqrt{r-4p\|x^*\|^2})^{p-1}.\end{aligned}$$

Using the fact that J is Lipschitz with Lipschitz constant L_3, we obtain that

$$||x_n-x^*|| \geq \left(\frac{1}{L_3}\right)\left(\frac{1}{2}m_p(\sqrt{r-4p\|x^*\|^2})^{p-1}\right) = h_p.$$

Hence,

$$\Phi(||x_n-x^*||) \geq \Phi(h_p).$$

Substituting in inequality (4.3.6), we obtain that

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) - \alpha_n\Phi(h_p) + \frac{1}{2}\alpha_n\Phi(h_p). \tag{4.3.7}$$

Hence, we have that:

$$r \leq r - \frac{1}{2}\alpha_n\Phi(h_p) < r,$$

a contradiction. Hence, $\phi(x^*, x_{n+1}) \leq r$. By induction, $\phi(x^*, x_n) \leq r \; \forall \, n \geq 1$. From inequality (2.2.2), $\{x_n\}$ is bounded. We now prove that $\{x_n\}_{n=1}^{\infty}$ converges strongly to x^*. Using inequality (4.3.6) and the boundedness of the sequence $\{x_n\}_{n=1}^{\infty}$, there exists $M > 0$ such that:

$$\phi(x^*, x_{n+1}) \leqslant \phi(x^*, x_n) - 2\alpha_n\Phi(\|x_n-x^*\|) + \alpha_n^{\frac{p}{(p-1)}}M.$$

The conditon $\sum_{n=1}^{\infty}\alpha_n^{\frac{p}{p-1}} < \infty$ implies, by Lemma 2.3.5, that $\lim\limits_{n\to\infty}\phi(x^*, x_n)$ exists. The condition $\sum_{n=1}^{\infty}\alpha_n = \infty$ now implies that $\liminf \Phi(\|x_n-x^*\|) = 0$, which further implies (using the properties of Φ) that $\liminf \|x_n-x^*\| = 0$. Hence, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that, (as in the proof of Theorem 7.2.1), $x_{n_k} \to x^*$ as $k \to \infty$. Futhermore, from

$$\phi(x^*, x_{n_k}) = \|x^*\|^2 - 2\langle x^*, J(x_{n_k})\rangle + \|x_{n_k}\|^2$$

and the fact that J is continuous, we obtain that, $\{\phi(x^*, x_n)\}_{n=1}^{\infty}$ has a subsequence which converges to 0. Thus, by Lemma 2.3.5, $\{\phi(x^*, x_n)\}_{n=1}^{\infty}$ converges strongly to 0. Applying Lemma 2.2.1, we obtain that $||x_n-x^*|| \to 0$ as $n \to \infty$, completing the proof. ■

Remark 4.3.2 *It is easy to see that our theorems hold for ϕ- strongly monotone and bounded operators and for k-strongly monotone and bounded operators in L_p spaces, $1 < p < \infty$, by simply setting $\Phi(s) = s\phi(s)$ and $\Phi(s) = ks^2$, respectively, in Theorems 4.2.1 and 4.3.1.*

A prototype of the parameter in our theorems is the canonical choice
$\alpha_n = \frac{1}{n},\ n \geq 1$.

All the results of this chapter are the results obtained in [38], which was accepted for publication in **Optimization (Taylor and Francis)**.

CHAPTER 5

Strong and Δ-Convergence Theorems for Common Fixed Point of a Finite Family of Multivalued Demi-Contractive Mappings in CAT(0) Spaces

5.1 Introduction

In this chapter, we prove strong and Δ-convergence theorems for common fixed point of a finite family of multivalued demi-contractive mappings in a complete $CAT(0)$ space. The results in this chapter extend and improve the results of Chidume and Ezeora [41], Chidume *et al.* [39], Isiogugu and Osilike [68], and complement the results of Dhompongsa and Panyanak [57], Dhompongsa *et al.* [58], Leustean [78], Shahzad and Markin [102], Sokhuma [104], and results of a host of other authors on iterative approximation of fixed points in $CAT(0)$ spaces.

We start by recalling the following definition and lemmas.

Definition 5.1.1 *A mapping $T : K \to CB(K)$ is called semi-compact if, for any sequence $\{x_n\}$ in K such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.*

Remark 5.1.2 *If K is compact, then every multi-valued mapping $T : K \to CB(K)$ is semi-compact.*

Lemma 5.1.3 (See e.g., [57]) *Let (E, d) be a $CAT(0)$ space. Then,*

(i) *(E, d) is uniquely geodesic.*

(ii) *For each $x, y \in E$, and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1 - t)d(x, y). \tag{5.1.1}$$

Lemma 5.1.4 (See e.g., [57], Lemmas 2.4 **and** 2.5**)** *Let (E, d) be a $CAT(0)$ space. For $x, y, z \in E$, and $t \in [0, 1]$, the following inequalities hold:*

(*i*) $d((1-t)x \oplus ty, z) \leqslant (1-t)d(x, z) + td(y, z)$;

(*ii*) $d((1-t)x \oplus ty, z)^2 \leqslant (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2$;

where $d(x, z)^2 = (d(x, z))^2$.

Lemma 5.1.5 (*i*) *(See e.g., [77]) Every bounded sequence in a complete $CAT(0)$ space has a Δ-convergent subsequence.*

(*ii*) *(See e.g., [59]) If C is a nonempty closed and convex subset of a complete $CAT(0)$ space, and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

(*iii*) *(See e.g., [57]) If $\{x_n\}$ is a bounded sequence in a complete $CAT(0)$, with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

5.2 Main Results

Lemma 5.2.1 *Let E be a CAT(0) space. Let $\{x_i,\ i = 1, 2, \ldots, n\} \subset E$, and $\alpha_i \in (0, 1),\ i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} \alpha_i = 1$. Then, the following inequality holds:*

$$d\left(\sum_{i=1}^{n} \oplus \alpha_i x_i, z\right)^2 \leqslant \sum_{i=1}^{n} \alpha_i d(x_i, z)^2 - \sum_{i,j=1, i\neq j}^{n} \alpha_i \alpha_j d(x_i, x_j)^2,\ \forall z \in E. \tag{5.2.1}$$

Proof The proof is by induction. For $n = 2$, the result follows from Lemma 5.1.4(*ii*). For simplicity, we shall give the proof for $n = 3$. From Lemma 5.1.4(*ii*), we have that

$$\begin{aligned}
d\left(\sum_{i=1}^{3} \oplus \alpha_i x_i, z\right)^2 &= d(\alpha_1 x_1 \oplus (\alpha_2 + \alpha_3)(\alpha_2^{'} x_2 \oplus \alpha_3^{'} x_3), z)^2,\ \alpha_i^{'} := \frac{\alpha_i}{(\alpha_2 + \alpha_3)},\ i \geqslant 2 \\
&\leqslant \alpha_1 d(x_1, z)^2 + (\alpha_2 + \alpha_3) d(\alpha_2^{'} x_2 \oplus \alpha_3^{'} x_3, z)^2 \\
&- \alpha_1(\alpha_2 + \alpha_3) d(x_1, \alpha_2^{'} x_2 \oplus \alpha_3^{'} x_3)^2 \\
&\leqslant \alpha_1 d(x_1, z)^2 + (\alpha_2 + \alpha_3)[\alpha_2^{'} d(x_2, z)^2 \\
&+ \alpha_3^{'} d(x_3, z)^2 - \alpha_2^{'} \alpha_3^{'} d(x_2, x_3)^2] \\
&- \alpha_1(\alpha_2 + \alpha_3)\left[\alpha_2^{'} d(x_1, x_2)^2 + \alpha_3^{'} d(x_1, x_3)^2 - \alpha_2^{'} \alpha_3^{'} d(x_2, x_3)^2\right] \\
&= \sum_{i=1}^{3} \alpha_i d(x_i, z)^2 - \alpha_2 \alpha_3^{'} d(x_2, x_3)^2 - \alpha_1 \alpha_2 d(x_1, x_2)^2 \\
&- \alpha_1 \alpha_3 d(x_1, x_3)^2 - \alpha_1 \alpha_2 \alpha_3^{'} d(x_2, x_3)^2
\end{aligned}$$

$$= \sum_{i=1}^{3} \alpha_i d(x_i, z)^2 - \sum_{i,j=1, i\neq j}^{3} \alpha_i \alpha_j d(x_i, x_j)^2.$$

Now, suppose (5.2.1) holds up to some $k \geqslant 3$, i.e.,

$$d\left(\sum_{i=1}^{k} \oplus \alpha_i x_i, z\right)^2 \leqslant \sum_{i=1}^{k} \alpha_i d(x_i, z)^2 - \sum_{i,j=1, i\neq j}^{k} \alpha_i \alpha_j d(x_i, x_j)^2.$$

Then, again from Lemma 5.1.4 we have

$$\begin{aligned} d\left(\sum_{i=1}^{k+1} \oplus \alpha_i x_i, z\right)^2 =& d\left(\alpha_1 x_1 \oplus \sigma\left(\sum_{i=2}^{k+1} \oplus \frac{\alpha_i}{\sigma} x_i\right), z\right)^2, \ \sigma = \sum_{i=2}^{k+1} \alpha_i, \\ &\leqslant \alpha_1 d(x_1, z)^2 + \sigma d\left(\sum_{i=2}^{k+1} \oplus \frac{\alpha_i}{\sigma} x_i, z\right)^2 \\ &- \alpha_1 \sigma d\left(x_1, \sum_{i=2}^{k+1} \oplus \frac{\alpha_i}{\sigma} x_i\right)^2 \\ &= \alpha_1 d(x_1, z)^2 + \sigma d\left(\sum_{i=1}^{k} \oplus \frac{\alpha_{i+1}}{\sigma} x_{i+1}, z\right)^2 \\ &- \alpha_1 \sigma d\left(x_1, \sum_{i=1}^{k} \oplus \frac{\alpha_{i+1}}{\sigma} x_{i+1}\right)^2. \end{aligned}$$

Using the induction hypothesis, we have

$$\begin{aligned} d\left(\sum_{i=1}^{k+1} \oplus \alpha_i x_i, z\right)^2 \quad \leqslant \quad & \alpha_1 d(x_1, z)^2 + \sum_{i=1}^{k} \alpha_{i+1} d(x_{i+1}, z)^2 \\ - \quad & \sum_{i=1, j, i\neq j}^{k} \frac{\alpha_{i+1}\alpha_{j+1}}{\sigma} d(x_{i+1}, x_{j+1})^2 \\ - \quad & \sum_{i=1}^{k} \alpha_1 \alpha_{i+1} d(x_1, x_{i+1})^2 \\ + \quad & \sum_{i=1, j, i\neq j}^{k} \frac{\alpha_1 \alpha_{i+1}\alpha_{j+1}}{\sigma} d(x_{i+1}, x_{j+1})^2 \\ = \quad & \sum_{i=1}^{k+1} \alpha_i d(x_i, z)^2 - \sum_{i,j=1, i\neq j}^{k+1} \alpha_i \alpha_j d(x_i, x_j)^2. \end{aligned}$$

Hence, by induction we have that inequality (5.2.1) holds for all $n \geq 1$. The proof is complete. ■

Lemma 5.2.2 *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space E. Let $T_i : K \to CB(K)$, $i = 1, 2, \ldots, m$ be a family of multi-valued demi-contractive mappings with constants $k_i \in (0, 1), i = 1, 2, \ldots, m$ such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that $T_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^{n} F(T_i)$. For arbitrary $x_1 \in K$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_m y_n^m, \qquad n \geq 1, \tag{5.2.2}$$

where $y_n^i \in T_i x_n$, $i = 1, 2, \ldots, m$, $\alpha_0 \in (k, 1)$, $\alpha_i \in (0, 1)$, $i = 1, 2, \ldots, m$, such that $\sum_{i=0}^{m} \alpha_i = 1$, and $k := max\{k_i, i = 1, 2, \ldots, m\}$. Then, $\lim_{n\to\infty} \{d(x_n, p)\}$ exists for all $p \in \bigcap_{i=1}^{n} F(T_i)$, and $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \ldots, m$.

Proof Let $p \in \bigcap_{i=1}^{n} F(T_i)$. By Lemma 5.2.1 and the fact that T_i is k-strictly pseudo-contractive for $i = 1, 2, 3 \ldots$, we have

$$\begin{aligned}
d(x_{n+1}, p)^2 &= d(\alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_m y_n^m, p)^2 \\
&\leq \alpha_0 d(x_n, p)^2 + \sum_{i=1}^{m} \alpha_i d(y_n^i, p)^2 \\
&- \sum_{i=1}^{m} \alpha_0 \alpha_i d(x_{n_0}, y_n^i) - \sum_{i,j=2, i\neq j}^{m} \alpha_i \alpha_j d(y_n^i, y_n^j) \\
&\leq \alpha_0 d(x_n, p)^2 + \sum_{i=1}^{m} \alpha_i (H(T_i x_n, Tp))^2 - \sum_{i=1}^{m} \alpha_0 \alpha_i d(x_n, y_n^i)^2 \\
&\leq \alpha_0 d(x_n, p)^2 + \sum_{i=1}^{m} \alpha_i d(x_n, p)^2 + \sum_{i=1}^{m} k_i \alpha_i d(x_n, y_n^i)^2 - \sum_{i=1}^{m} \alpha_i \alpha_0 d(x_n, y_n^i)^2 \\
&\leq d(x_n, p)^2 - (\alpha_0 - k) \sum_{i=1}^{m} \alpha_i d(x_n, y_n^i)^2 \\
&\leq d(x_n, p)^2 - (\alpha_0 - k) \sum_{i=1}^{m} \alpha d(x_n, y_n^i)^2 \leq d(x_n, p)^2, \ \alpha = \min_{0\leq i\leq m} \alpha_i,
\end{aligned}$$

which shows that $\{d(x_n, p)\}$ is non-increasing and bounded. Hence, its limit exists.

Moreover, we have that

$$\alpha(\alpha_0 - k) \sum_{i=1}^{m} d(x_n, y_n^i)^2 \leq d(x_1, p)^2 < \infty.$$

Therefore, $\lim_{n\to\infty} d(x_n, y_n^i) = 0 \ \forall \ i = 1, 2, ..., m$. Consequently,

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0 \ \forall \ i = 0, 1, \ldots, m.$$

■

Theorem 5.2.3 *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space. Let $T_i : K \to CB(K), i = 1, 2, \ldots, m$ be a family of demi-contractive mappings with constants $k_i \in (0, 1)$, $i = 1, 2, \ldots, m$ such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that T_i is $\Delta-$demi-closed at 0 for all $i = 1, 2, \ldots, m$ and $T_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^{n} F(T_i)$. For arbitrary $x_1 \in K$, define a sequence x_n by*

$$x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_m y_n^m, \quad n \geq 1,$$

where $y_n^i \in T_i x_n$, $i = 1, 2, \ldots, m$, $\alpha_0 \in (k, 1)$, $\alpha_i \in (0, 1)$, $i = 1, 2, \ldots, m$ such that $\sum_{i=0}^{m} \alpha_i = 1$ and $k := max\{k_i, i = 1, 2, ..., m\}$. Then, $\{x_n\}$ $\Delta-$ converges to a point $p \in \bigcap_{i=1}^{m} F(T_i)$.

Proof Define $W_\Delta(x_n) := \cup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We shall show that, $W_\Delta(x_n) \subseteq \bigcap_{i=1}^{m} F(T_i)$, and that $W_\Delta(x_n)$ consists of exactly one point.
Let $u \in W_\Delta(x_n)$, this implies that there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Since by Lemma 5.2.2 $\{u_n\}$ is bounded, this implies from Lemma 5.1.5((i) and (ii)) that, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in K$.

Using Lemma 5.2.2 and the fact that $T_i, i = 1, 2, ..., m$ is $\Delta-$demi-closed at 0 for all $i = 1, 2, ..., m$ we have that $v \in \bigcap_{i=1}^{m} F(T_i)$, and hence, $d(u_n, v)$ converges. Lemma 5.1.5(iii) implies that u = v. Thus, we have $W_\Delta(x_n) \subseteq \bigcap_{i=1}^{m} F(T_i)$.
We now show that $W_\Delta(x_n)$ consists of exactly one point. Let $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ be arbitrary subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$ Since $u \in W_\Delta(x_n) \subseteq \bigcap_{i=1}^{m} F(T_i)$, we have by Lemma 5.2.2 that $d(x_n, u)$ converges. Lemma 5.1.5(iii) implies that u = v. The proof is complete. ■

Corollary 5.2.4 *Let K, E, $T_i, i = 1, 2, \ldots, m$ and $\{x_n\}$ be as in Theorem 5.2.3. Suppose there exists $i_0 \in \{1, 2, ..., m\}$ such that T_{i_0} is semi-compact, then $\{x_n\}$ converges strongly to a common fixed point of $T_i, i = 1, 2, ..., m$.*

Proof Since by Lemma 5.2.2 $d(x_n, T_{i_0} x_n) \to 0$, and T_{i_0} is semi-compact, then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $\{u_n\} \to u \in K$, which implies $\Delta - \lim_{n\to\infty} u_n = u \in K$. By Theorem 5.2.3 we have that $u \in \bigcap_{i=1}^{n} F(T_i)$, which implies by Lemma 5.2.2 that, $x_n \to u$. ■

Corollary 5.2.5 *Let K be a nonempty compact convex subset of a complete $CAT(0)$ space. Let $T_i : K \to CB(K), i = 1, 2, ..., m$ be a family of demi-contractive mappings with constants $k_i \in (0, 1)$, $i = 1, 2, ..., m$ such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that T_i is $\Delta-$demi-closed at 0 for all $i = 1, 2, ..., m$ and $T_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^{n} F(T_i)$. For arbitrary $x_1 \in K$, define a sequence x_n by*

$$x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus ... \oplus \alpha_m y_n^m, \quad n \geq 1,$$

where $y_n^i \in T_i x_n,\ i = 1, 2, \ldots, m,\ \alpha_0 \in (k, 1), \alpha_i \in (0, 1),\ i = 1, 2, \ldots, m$ *such that* $\sum_{i=0}^{m} \alpha_i = 1$ *and* $k := max\{k_i, i = 1, 2, ..., m\}$. *Then,* $\{x_n\}$ *converges strongly to some point* $p \in \bigcap_{i=1}^{m} F(T_i)$.

Proof The proof follows from the fact that, if K is compact, then every multi-valued mapping $T : K \to CB(K)$ is semi-compact (see remark 5.1.2). Thus, the conclusion follows from Corollary 5.2.4. ■

Corollary 5.2.6 *Let* K *be a nonempty closed convex subset of a complete* $CAT(0)$ *space. Let* $T_i : K \to CB(K), i = 1, 2, ..., m$ *be a family of quasi-nonexpansive mappings such that* $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. *Suppose that* T_i *is* $\Delta-$*demi-closed at* 0 *for all* $i = 1, 2, ..., m$ $T_i(p) = \{p\}$ *for all* $p \in \bigcap_{i=1}^{n} F(T_i)$, *and there exists* $i_0 \in \{1, 2, ..., m\}$ *such that* T_{i_0} *is semi-compact. For arbitrary* $x_1 \in K$, *define a sequence* x_n *by*

$$x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus ... \oplus \alpha_m y_n^m, \quad n \geq 1,$$

where $y_n^i \in T_i x_n,\ i = 1, 2, \ldots, m,\ \alpha_0 \in (k, 1), \alpha_i \in (0, 1),\ i = 1, 2, \ldots, m$ *such that* $\sum_{i=0}^{m} \alpha_i = 1$. *Then,* $\{x_n\}$ *converges strongly to some point* $p \in \bigcap_{i=1}^{m} F(T_i)$.

Remark 5.2.7 *It is worth mentioning that our result is true for all* $CAT(k)$ *spaces,* $k \leqslant 0$. *Since for* $k \leqslant k'$, $CAT(k) \subseteq CAT(k')$ *see (Bridson and Haefliger [12]).*

Remark 5.2.8 *Our results extend the results of Chidume and Ezeora [41] to a more general space than Hilbert space (*$CAT(0)$ *spaces). Futhermore, the condition imposed on* $\lambda_i,\ i = 0, 1, 2, \ldots, m$ *in Theorem 2.2 of [41] (*$\lambda_i \in (k, 1),\ i = 0, 1, 2, \ldots, m$ *such that* $\sum_{i=0}^{m} \lambda_i = 1$*) restricts the class of operators for which the theorem is applicable. In our result, the condition is reduced to* $\lambda_0 \in (k, 1),\ \lambda_i \in (0, 1),\ i = 1, 2, \ldots, m$ *such that* $\sum_{i=0}^{m} \lambda_i = 1$, *thereby making our results to be applicable to all classes of demi-contractive mappings.*

Remark 5.2.9 *It is worth mentioning that the result proved in Lemma 5.2.1 is of special interest.*

Remark 5.2.10 *The result of Chidume et al. (Theorem 3.1 of [39]), Isiogugu and Osilike (Theorem 3.1 of [68]) are special cases of our results.*

Remark 5.2.11 *All the results of this chapter are the results obtained in [34], which was published in:* **Abstract and Applied Analysis**.

CHAPTER 6

Convergence Theorem for a Countable Family of Multi-Valued Strictly Pseudo-Contractive Mappings in Hilbert Spaces

6.1 Introduction

In this chapter, a Krasnoselskii-type algorithm is constructed and proved to be an approximate fixed point sequence for a countable family of multi-valued strictly pseudo-contractive mappings in a real Hilbert space. Under some additional mild conditions, the sequence is proved to converge strongly to a common fixed point of the family.
Our theorems complement and improve the results of Chidume and Ezeora [41], Abbas *et al.* [1], Chidume *et al.* [39] and a host of other important results. Before proving the the main results of this chapter we start with the following definition and lemmas which shall be used subsequently in the chapter.

Definition 6.1.1 *A mapping $T : K \to CB(K)$ is called semi-compact if, for any sequence $\{x_n\}$ in K such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.*

Remark 6.1.2 *If K is compact, then every multi-valued mapping $T : K \to CB(K)$ is semi-compact.*

Lemma 6.1.3 *Let H be a real Hilbert space. Let $\{x_i, i = 1, ..., m\} \subset H$. For $\alpha_i \in (0, 1), i = 1, ..., m$ such that $\sum_{i=1}^{m} \alpha_i = 1$, the following identity holds:*

$$\left\| \sum_{i=1}^{m} \alpha_i x_i \right\|^2 = \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \sum_{i,j=1, i\neq j}^{m} \alpha_i \alpha_j \|x_i - x_j\|^2 . \tag{6.1.1}$$

The proof of Lemma 6.1.3 can be found in [62]. We now state and prove its generalization.

Lemma 6.1.4 *Let H be a real Hilbert space. Let $\{x_i\}_{i=1}^{\infty} \subset H$ and $\alpha_i \in (0,1), i = 1, 2, ...,$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. If $\{x_i\}_{i=1}^{\infty}$ is bounded, then*

$$\left\| \sum_{i=1}^{\infty} \alpha_i x_i \right\|^2 = \sum_{i=1}^{\infty} \alpha_i \|x_i\|^2 - \sum_{i,j=1, i\neq j}^{\infty} \alpha_i \alpha_j \|x_i - x_j\|^2. \tag{6.1.2}$$

Proof We observe that setting $M := \sup_{i\geq 1} \|x_i\|$, we have $\sum_{i=1}^{\infty} \alpha_i \|x_i\|^2 < \infty$. Moreover, since $\sum_{i=1}^{\infty} \alpha_i = 1$, we have

$$\sum_{i=1}^{n} \alpha_i = 1 - \sum_{i=n+1}^{\infty} \alpha_i$$

Thus setting

$$\alpha_i^n = \frac{\alpha_i}{1 - \sum_{i=n+1}^{\infty} \alpha_i}, \text{ we see that } \sum_{i=1}^{n} \alpha_i^n = 1. \tag{6.1.3}$$

We also have from Lemma 6.1.3 that, for each $n \geq 1$,

$$\left\| \sum_{i=1}^{n} \alpha_i^n x_i \right\|^2 = \sum_{i=1}^{n} \alpha_i^n \|x_i\|^2 - \sum_{i,j=1, i\neq j}^{n} \alpha_i^n \alpha_j^n \|x_i - x_j\|^2. \tag{6.1.4}$$

Since,

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{n} \alpha_i^n x_i \right\|^2 = \left\| \sum_{i=1}^{\infty} \alpha_i x_i \right\|^2$$

and

$$\lim_{n\to\infty} \Big(\sum_{i=1}^{n} \alpha_i^n \|x_i\|^2 - \sum_{i,j=1, i\neq j}^{n} \alpha_i^n \alpha_j^n \|x_i - x_j\|^2 \Big) = \Big(\sum_{i=1}^{\infty} \alpha_i \|x_i\|^2 - \sum_{i,j=1, i\neq j}^{\infty} \alpha_i \alpha_j \|x_i - x_j\|^2 \Big).$$

We have,

$$\left\| \sum_{i=1}^{\infty} \alpha_i x_i \right\|^2 = \sum_{i=1}^{\infty} \alpha_i \|x_i\|^2 - \sum_{i,j=1, i\neq j}^{\infty} \alpha_i \alpha_j \|x_i - x_j\|^2 \tag{6.1.5}$$

This proves the result. ■

The following result is proved in [39] .

Lemma 6.1.5 *Let $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping, then T is Lipschitz with Lipschitz constant $\frac{1+\sqrt{k}}{1-\sqrt{k}}$.*

Remark 6.1.6 *In Lemma 1.1 of [39], the authors required that for each $x \in K$, Tx be weakly closed. However, it was later on proved in Lemma 3.7 of [47] that, this condition is not necessary. Thus, it is dispensed with in Lemma 6.1.5.*

As a consequence of Lemma 6.1.5, we obtain the following lemma.

Lemma 6.1.7 *Let $T_i : K \to CB(K)$ be a countable family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0,1), i = 1,2,...$. If $\sup_{i\geq 1} k_i \in (0,1)$, then T_i is uniformly Lipschitz; that is, there exists $L > 0$, such that*

$$H(T_i x, T_i y) \leq L\|x - y\| \; \forall \, x, y \in K$$

Proof From Lemma 6.1.5, we have that T_i is Lipschitz for each i, with Lipschitz constant $L_i = \frac{1+\sqrt{k_i}}{1-\sqrt{k_i}}$. Now, since $k_i \in (0,1), i = 1,2,...$, setting $k := \sup_{i\geq 1} k_i$, we have,

$$H(T_i x, T_i y) \leq \Big(\frac{1+\sqrt{k_i}}{1-\sqrt{k_i}}\Big)\Big\|x - y\Big\| \leq \Big(\frac{1+\sqrt{k}}{1-\sqrt{k}}\Big)\Big\|x - y\Big\|.$$

Hence, T_i is uniformly Lipschitz. ■

6.2 Main Results

We now prove the following theorem.

Theorem 6.2.1 *Let K be a nonempty closed and convex subset of a real Hilbert space H, and $T_i : K \to CB(K)$ be a countable family of multi-valued k_i-strictly pseudo-contractive mappings; $k_i \in (0,1), i = 1,2,...$ such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$; and $\sup_{i\geq 1} k_i \in (0,1)$. Assume that for $p \in \bigcap_{i=1}^{\infty} F(T_i)$, $T_i(p) = \{p\}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively for arbitrary $x_0 \in K$ by*

$$x_{n+1} = \lambda_0 x_n + \sum_{i=1}^{\infty} \lambda_i y_n^i, \tag{6.2.1}$$

where $y_n^i \in T_i x_n$, $n \geq 1$ and $\lambda_0 \in (k,1)$; $\sum_{i=0}^{\infty} \lambda_i = 1$ and $k := \sup_{i\geq 1} k_i$. Then, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, $i = 1,2,...$.

Proof We first show that $\{x_n\}$ is well defined for each $n \geq 1$. It suffices to show that $\sum_{i=1}^{\infty} \lambda_i \|y_n^i\| < \infty$ for each $n \in \mathbb{N}$. Indeed, for $p \in \bigcap_{i=1}^{\infty} F(T_i)$, we have,

$$\|y_n^i - p\| \leq \sup_{i\geq 1} \|y_n^i - p\| = H(T_i x_n, \{p\}) \leq L\|x_n - p\| := M_n.$$

Thus $\|y_n^i\| - \|p\| \leq \|y_n^i - p\| \leq M_n$. That is $\|y_n^i\| \leq M_n + \|p\|$ Therefore, for each n, $\{y_n^i\}$ is bounded. Hence,

$$\sum_{i=1}^{\infty} \lambda_i \|y_n^i\| \leq (M_n + \|P\|) \sum_{i=1}^{\infty} \lambda_i = (M_n + \|P\|)(1 - \lambda_0) < \infty.$$

Using Lemma 6.1.4, the fact that T_i is strictly pseudo-contractive for each $i = 1, 2, \ldots$ and the fact that $\lambda_0 \in (k, 1)$ we have:

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|\lambda_0(x_n - p) + \sum_{i=1}^{\infty} \lambda_i (y_n^i - p)\right\|^2 \\
&= \lambda_0 \|x_n - p\|^2 + \sum_{i=1}^{\infty} \lambda_i \|y_n^i - p\|^2 - \sum_{i=1}^{\infty} \lambda_i \lambda_0 \|x_n - y_n^i\|^2 \\
&- \sum_{i,j=1, i\neq j}^{\infty} \lambda_i \lambda_j \|y_n^i - y_n^j\|^2 \\
&\leq \lambda_0 \|x_n - p\|^2 + \sum_{i=1}^{\infty} \lambda_i \|y_n^i - p\|^2 - \sum_{i=1}^{\infty} \lambda_i \lambda_o \|x_n - y_n^i\|^2 \\
&\leq \lambda_0 \|x_n - p\|^2 + \sum_{i=1}^{\infty} \lambda_i (H(T_i x_n, T_i p))^2 - \sum_{i=1}^{\infty} \lambda_i \lambda_o \|x_n - y_n^i\|^2 \\
&\leq \lambda_0 \|x_n - p\|^2 + \sum_{i=1}^{\infty} \lambda_i \Big(\|x_n - p\|^2 + k\|x_n - y_n^i\|^2\Big) \\
&- \sum_{i=1}^{\infty} \lambda_i \lambda_0 \|x_n - y_n^i\|^2 \\
&= \sum_{i=0}^{\infty} \lambda_i \|x_n - p\|^2 - \sum_{i=1}^{\infty} \lambda_i (\lambda_0 - k) \|x_n - y_n^i\|^2 \qquad (6.2.2) \\
&= \|x_n - p\|^2 - \sum_{i=1}^{\infty} \lambda_i (\lambda_0 - k) \|x_n - y_n^i\|^2. \qquad (6.2.3)
\end{aligned}$$

Thus, $\lim_{n\to\infty} \|x_n - p\|$ exists. Moreover,

$$\sum_{i=1}^{\infty} \lambda_i \|x_n - y_n^i\|^2 \leq \frac{1}{\lambda_0 - k}\Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big),$$

which implies that

$$\lambda_i \|x_n - y_n^i\|^2 \leq \frac{1}{\lambda_0 - k}\Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big).$$

Therefore,

$$\lambda_i \sum_{n=1}^{\infty} \|x_n - y_n^i\|^2 \leq \frac{1}{(\lambda_0 - k)} \sum_{n=1}^{\infty} \Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big) < \infty.$$

Hence, $\lim_{n\to\infty} \|x_n - y_n^i\| = 0 \ \forall i = 1, 2, \ldots$. Since $y_n^i \in T_i x_n, i = 1, 2, \ldots$, it follows that

$$0 \leq d(x_n, T_i x_n) \leq \|x_n - y_n^i\|.$$

Thus, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0 \ \ \forall\, i = 1, 2, \ldots$. ■

The proof is complete.

Theorem 6.2.2 *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T : K \to CB(K)$ be a countable family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0,1), i = 1,2,...,$ such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that for $p \in \bigcap_{i=1}^{\infty} F(T_i)$, $T_i(p) = \{p\}$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = \lambda_0 x_n + \sum_{i=1}^{\infty} \lambda_i y_n^i \tag{6.2.4}$$

where $y_n^i \in T_i x_n$, $n \geq 1$ and $\lambda_i \in (k,1), i = 1,2,...$ such that $\sum_{i=0}^{\infty} \lambda_i = 1$ and $k := \sup_{i \geq 1} k_i$. Suppose $\exists\, i_0 \in \mathbb{N}$ such that T_{i_0} is semi-compact. Then, the sequence $\{x_n\}$ converges strongly to an element of $\bigcap_{i=1}^{\infty} F(T_i)$.

Proof From Theorem 6.2.1, we have $\lim_{n\to\infty} d(x_n, T_i x_n) = 0, i = 1,2,...$. Since T_{i_0}, is semi-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q \in K$ as $k \to \infty$. Also, by continuity of $T_i, i = 1,2,...$, we have $d(x_{n_k}, T_i x_{n_k}) \to d(q, T_i q), i = 1,2,...$ as $k \to \infty$. Therefore, $d(q, T_i q) = 0, i = 1,2,...$, and so $q \in F(T_i)$, $i = 1,2,...$ Thus, from (6.2.2), we have that $\lim_{n\to\infty} \|x_n - q\|$ exists; and since $x_{n_k} \to q$, it follows that $\lim_{n\to\infty} \|x_n - q\| = 0$. So, $\{x_n\}$ converges strongly to $q \in \bigcap_{i=1}^{\infty} F(T_i)$. ■

The following corollary follows from Theorem 6.2.2 and Remark 6.1.2

Corollary 6.2.3 *Let K be a nonempty, compact and convex subset of a real Hilbert space H and $T : K \to CB(K)$ be a countable family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0,1), i = 1,2,...,$ such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and $\sup_{i \geq 1} k_i \in (0,1)$. Assume that for $p \in \bigcap_{i=1}^{\infty} F(T_i)$, $T_i(p) = \{p\}$. Let $\{x_n\}$ be a sequence defined iteratively for arbitrary $x_0 \in K$ by*

$$x_{n+1} = \lambda_0 x_n + \sum_{i=1}^{\infty} \lambda_i y_n^i \tag{6.2.5}$$

where $y_n^i \in T_i x_n$, $n \geq 1$ and $\lambda_0 \in (k,1), i = 1,2,...$; $\sum_{i=0}^{\infty} \lambda_i = 1$ and $k := \sup_{i \geq 1} k_i$, then the sequence $\{x_n\}$ converges strongly to an element of $\bigcap_{i=1}^{\infty} F(T_i)$.

Remark 6.2.4 *It is worth mentioning that the recursion formulas studied in this chapter are of the Krasnoselskii-type which is well known to be superior to the recursion formula of either the Mann algorithm or the so-called Ishikawa-type algorithm (see [67]).*

Remark 6.2.5 *Our theorems extend the results of Chidume et al. [39] and Chidume and Ezeora [41] from a single multi-valued strictly pseudo- contractive mappings and a finite family of multi-valued strictly pseudo-contractive mappings, respectively, to a countable family of multi-valued strictly pseudo-contractive mappings. Furthermore, under the setting of Hilbert spaces, our theorems and corollaries improve the convergence theorems for multi-valued nonexpansive mappings studied in Sastry and Babu [101], Panyanak [90] to a more general class of multi-valued*

strictly pseudo-contractive mappings. Also, in our algorithms, $y_n \in Tx_n$ is arbitrary and is not required to satisfy the very restrictive condition, "$y_n \in Tx_n$ such that $\|y_n - x^*\| = d(x^*, Tx_n)$" imposed in [74] and [103].

All the results of this chapter are the results obtained in [35] which was published in: International Journal of Mathematical Analysis.

CHAPTER 7

Approximation of Solutions of Hammerstein Equations with Strongly Monotone and Bounded Operators in Classical Banach Spaces

7.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of Hammerstein-type has the form

$$u(x) + \int_\Omega k(x,y) f(y, u(y)) dy = w(x), \tag{7.1.1}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions. If we define $F : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ and $K : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ by

$$Fu(y) = f(y, u(y)),\ x \in \Omega,$$

and

$$Kv(x) = \int_\Omega k(x,y) v(y) dy,\ x \in \Omega,$$

respectively, where $\mathcal{F}(\Omega, \mathbb{R})$ is a space of measurable real-valued functions defined from Ω to $\mathbb{R}$, then equation (7.1.1) can be put in an abstract form

$$u + KFu = w. \tag{7.1.2}$$

Without loss of generality we may assume that $w \equiv 0$ so that (7.1.2) becomes

$$u + KFu = 0. \tag{7.1.3}$$

In this chapter, we shall construct a coupled iterative process and prove its strong convergence to a solution of the Hammerstein equation (7.1.3) in L_p spaces, $1 < p <$

∞, where the operators $F : L_p \to {L_p}^*$ and $K : {L_p}^* \to L_p$ are strongly monotone and bounded operators. Futhermore, our technique of proof is of independent interest. Before stating the main results of this chapter we start with the following definitions and lemmas that will be useful subsequently.

Lemma 7.1.1 (see e.g., [29], p. 55) *Let $E = L_p$, $1 < p < 2$, then the following inequalities hold for all x, y in L_p, and some constant $c_p > 0$.*

$$\|x+y\|^2 \quad \geq \quad \|x\|^2 + 2\langle y, j(x)\rangle + c_p\|y\|^2, \tag{7.1.4}$$

$$\langle x-y, J(x)-J(y)\rangle \geq (p-1)\|x-y\|^2. \tag{7.1.5}$$

Observe that this inequality yields

$$||J^{-1}(x) - J^{-1}(y)|| \leq L_1||x-y||,$$

where $L_1 := \frac{1}{p-1}$.

Let $E = L_p$, $1 < p < 2$. Define $\phi_p : E \times E \to \mathbb{R}$ by

$$\phi_p(x,y) = \|y\|^2 - 2\langle x, j(y)\rangle + c_p\|x\|^2, \tag{7.1.6}$$

where c_p is the constant appearing in inequality (7.1.4). Then, from (7.1.4) we have that

$$\|x-y\|^2 \geq \phi_p(x,y). \tag{7.1.7}$$

Also, following the pattern of proof of Lemma 2.2.3, the following inequality can be established.

$$V_p(x,x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V_p(x, x^*+y^*) \tag{7.1.8}$$

for all $x \in E$ and $x^*, y^* \in E^*$, where $V_p(x,x^*) = \phi_p(x, J^{-1}x^*)$. Moreover, it can be easily seen that

$$\phi(x,y) = \phi_p(x,y) + (1-c_p)\|x\|^2. \tag{7.1.9}$$

Thus, the following inequality follows from inequality (2.2.2).

$$\|y\| \leq \sqrt{\phi_p(x,y) + (1-c_p)\|x\|} + \|x\| \ \forall\ x, y \in E. \tag{7.1.10}$$

7.2 Convergence Theorems in L_p spaces, $1 < p < 2$

In the following we assume that $R(F) = D(K) = E^*$.

Theorem 7.2.1 *Let $E = L_p$, $1 < p < 2$. Let $F : E \to E^*$ and $K : E^* \to E$ be strongly monotone and bounded maps. For $(u_0, v_0) \in E \times E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^* respectively by*

$$u_{n+1} = J^{-1}(Ju_n - \alpha_n(Fu_n - v_n)),\ n \geq 0, \tag{7.2.1}$$

$$v_{n+1} = J_*^{-1}(J_* v_n - \alpha_n(Kv_n + u_n)),\ n \geq 0, \tag{7.2.2}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ *satisfies the following conditions:* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n^{\frac{q}{q-1}} < \infty$, *where* q *is such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that the equation* $u + KFu = 0$ *has a solution. Then, there exists* $\gamma_0 > 0$ *such that if* $\alpha_n \leq \gamma_0$ *for all* $n \geq 1$, *the sequences* $\{u_n\}_{n=1}^{\infty}$ *and* $\{v_n\}_{n=1}^{\infty}$ *converge strongly to* u^* *and* v^*, *respectively, where* u^* *is the solution of* $u + KFu = 0$ *with* $v^* = Fu^*$.

Proof We first prove that the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are bounded. The proof is by induction. For (u_n, v_n), $(u^*, v^*) \in L_p \times L_q$ where u^* is the solution of (7.1.3) with $v^* = Fu^*$, set $w_n = (u_n, v_n)$ and $w^* = (u^*, v^*)$. Define $\Phi_p : (E \times E^*) \times (E \times E^*) \to \mathbb{R}$ and $\Phi : (E \times E^*) \times (E \times E^*) \to \mathbb{R}$ by

$$\Phi_p(w_1, w_2) = \phi_p(u_1, u_2) + \phi(v_1, v_2) \tag{7.2.3}$$

and

$$\Phi(w_1, w_2) = \phi(u_1, u_2) + \phi(v_1, v_2), \tag{7.2.4}$$

respectively, where $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$. Let $E \times E^*$ be endowed with the norm $\|(u, v)\| = (\|u\|_E^2 + \|v\|_{E^*}^2)^{\frac{1}{2}}$. Let $r > 0$ be such that

$$r \geq \max\ \{\Phi_p(w^*, w_0),\ 6q\|v^*\|^2\}. \tag{7.2.5}$$

Since F and K are bounded, define

$$M_1 := L_1 \sup\{\|Fu - v\|^2 : \|u\| \leq \sqrt{r + (1 - c_p)\|u^*\|} + \|u^*\|;\ \|v\| \leq \|v^*\| + \sqrt{r}\} + 1;$$

and

$$M_2 := m_q \sup\{\|Kv + u\|^{\frac{q}{q-1}} : \|u\| \leq \sqrt{r + (1 - c_p)\|u^*\|} + \|u^*\|;\ \|v\| \leq \|v^*\| + \sqrt{r}\} + 1,$$

where c_p is the constant appearing in inequality (7.1.4), m_q $(2 \leq q < \infty)$ is the Hölder continuity constant appearing in Lemma 2.2.3 (Chapter 3) and L_1 is the Lipschitz constant of J^{-1}. Define

$$\gamma_0 := \left[\frac{1}{2} min\left\{\frac{kr}{6M_1}, \frac{kr}{6M_2}\right\}\right]^{q-1},$$

where $k = \min\{k_1, k_2\}$, k_1 and k_2 are the constants of strong monotonicity of F and K, respectively. We claim that $\Phi_p(w^*, w_n) \leq r\ \forall\ n \geq 1$. Indeed, by construction, we have $\Phi_p(w^*, w_0) \leq r$. Suppose that $\Phi_p(w^*, w_n) \leq r$ for some $n \geq 1$. This implies that

$$\phi_p(u_*, u_n) + \phi(v^*, v_n) \leq r,\ \forall\ n \geq 1.$$

So, from inequalities (7.1.10) and (2.2.2), we have:

$$\|u_n\| \leq \sqrt{r + (1 - c_p)\|u^*\|} + \|u^*\| \ \text{ and } \|v_n\| \leq \|v^*\| + \sqrt{r},\ \forall\ n \geq 1,$$

respectively. We now prove that $\Phi_p(w^*, w_{n+1}) \leq r$. Using the definition of u_{n+1} and inequality (7.1.8) with $y^* = \alpha_n(Fu_n - v_n)$ we obtain:

$$\begin{aligned}
\phi_p(u^*, u_{n+1}) &= \phi_p(u^*, J^{-1}(Ju_n - \alpha_n(Fu_n - v_n))) = V_p(x^*, Ju_n - \alpha_n(Fu_n - v_n)) \\
&\leq V_p(u^*, Ju_n) - 2\langle J^{-1}(Ju_n - \alpha_n(Fu_n - v_n)) - u^*, \alpha_n(Fu_n - v_n)\rangle \\
&= V_p(u^*, Ju_n) - 2\alpha_n\langle u_n - u^*, (Fu_n - v_n)\rangle \\
&\quad -2\alpha_n\langle J^{-1}(Ju_n - \alpha_n(Fu_n - v_n)) - J^{-1}(Ju_n), (Fu_n - v_n)\rangle. \\
&\leq \phi_p(u^*, u_n) - 2\alpha_n\langle u_n - u^*, (Fu_n - v_n)\rangle \\
&\quad +2\alpha_n\|J^{-1}(Ju_n - \alpha_n(Fu_n - v_n)) - J^{-1}(Ju_n)\|\|(Fu_n - v_n)\|.
\end{aligned}$$

Observe that

$$\langle u_n - u^*, (Fu_n - v_n)\rangle = \langle u_n - u^*, (Fu_n - Fu^*)\rangle + \langle u_n - u^*, (Fu^* - v_n)\rangle.$$

Now, using the fact that A is strongly monotone and that J^{-1} is Lipschitz we obtain:

$$\begin{aligned}
\phi_p(u^*, u_{n+1}) &\leq \phi_p(u^*, u_n) - 2\alpha_n k_1\|u_n - u^*\|^2 + 2\alpha_n\alpha_n L_1\|Fu_n - v_n\|^2 \\
&+ 2\alpha_n\langle u_n - u^*, (v_n - Fu^*)\rangle
\end{aligned}$$

Using inequality (7.1.7), definition of M_1 and the fact that $k = \min\{k_1, k_2\}$ we have

$$\begin{aligned}
\phi_p(u^*, u_{n+1}) &\leq (1 - 2\alpha_n k)\phi_p(u^*, u_n) + 2\alpha_n\alpha_n M_1 \\
&+ 2\alpha_n\langle u_n - u^*, (v_n - Fu^*)\rangle.
\end{aligned} \tag{7.2.6}$$

Similarly, using Lemma 3.1.4 with $y_n = \alpha_n(Kv_n + u_n)$ we obtain:

$$\begin{aligned}
\phi(v^*, v_{n+1}) &\leq \phi(v^*, v_n) - 2\alpha_n k\|v_n - v^*\|^2 \\
&+ 2\alpha_n\|J_*^{-1}(J_*v_n - \alpha_n(Kv_n + u_n)) - J_*^{-1}(J_*u_n)\|\|(Kv_n + u_n)\| \\
&+ 2\alpha_n\langle v_n - v^*, -(Kv^* + u_n)\rangle.
\end{aligned}$$

Using the fact that J_*^{-1} is Hölder continuous on bounded sets, the definition of M_2 and Lemma 2.2.2 we have

$$\begin{aligned}
\phi(v^*, v_{n+1}) &\leq (1 - 2\alpha_n k)\phi(v^*, v_n) + 2\alpha_n\alpha_n^{\frac{1}{q-1}} M_2 + 2\alpha_n qk\|v^*\|^2 \\
&+ 2\alpha_n\langle v_n - v^*, -(Kv_n + u_n)\rangle.
\end{aligned} \tag{7.2.7}$$

Adding (7.2.6) and (7.2.7), we have

$$\begin{aligned}
\Phi_p(w^*, w_n) &\leq (1 - 2\alpha_n k)\Phi_p(w^*, w_n) + 2\alpha_n\alpha_n M_1 \\
&+ 2\alpha_n\alpha_n^{\frac{1}{q-1}} M_2 + 2\alpha_n kq\|v^*\|^2 \\
&\leq (1 - 2\alpha_n k)\Phi_p(w^*, w_n) + 2\alpha_n\gamma_0 M_1 \\
&+ 2\alpha_n\gamma_0 M_2 + 2\alpha_n kq\|v^*\|^2.
\end{aligned}$$

Using definition of γ_0, induction hypothesis and inequality (7.2.5) we have:

$$\begin{aligned}\Phi_p(w^*, w_n) &\leq (1-2\alpha_n k)r + \frac{\alpha_n k}{3} + \frac{\alpha_n k}{3} + \frac{\alpha_n k}{3} \\ &\leq (1-\alpha_n k)r \leq r.\end{aligned}$$

Hence, $\Phi_p(w^*, w_{n+1}) \leq r$. By induction, $\Phi(w^*, w_n) \leq r \ \forall \ n \geq 1$. Consequently, we have $\phi_p(u^*, u_{n+1}) \leq r$ and $\phi(v^*, v_{n+1}) \leq r$. Thus from inequalities (7.1.10) and (2.2.2) we have that $\{u_n\}$ and $\{v_n\}$ are bounded, respectively.

We now prove that $\Phi(w^*, w_{n+1}) \to 0$ as $n \to \infty$. If we set $M_0 = L_1 \sup\{\|Fu_n - v_n\|^2\}$, following the same method of computation above with $\phi(u^*, u_{n+1})$ instead of $\phi_p(u^*, u_{n+1})$ and using Lemma 2.2.3 obtain that

$$\begin{aligned}\phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) - 2\alpha_n k\|u_n - u^*\|^2 + 2\alpha_n^2 M_0 \\ &+ 2\alpha_n\langle u_n - u^*, (v_n - Fu^*)\rangle\end{aligned} \tag{7.2.8}$$

Simillarly, we have

$$\begin{aligned}\phi(v^*, v_{n+1}) &\leq \phi(v^*, v_n) - 2\alpha_n k\|v_n - v^*\|^2 + 2\alpha_n \alpha_n^{\frac{1}{q-1}} M_2 \\ &+ 2\alpha_n\langle v_n - v^*, -(Kv_n^* + u_n)\rangle\end{aligned} \tag{7.2.9}$$

Adding (7.2.8) and (7.2.9) we obtain:

$$\begin{aligned}\Phi_p(w^*, w_{n+1}) &\leq \Phi_p(w^*, w_n) - 2\alpha_n k(\|u_n - u^*\|^2 + \|v_n - v^*\|^2) \\ &+ 2\alpha_n^2 M_0 + 2\alpha_n^{\frac{q}{q-1}} M_2.\end{aligned} \tag{7.2.10}$$

By Lemma 2.3.5, we obtain that $\lim \Phi(w^*, w_n)$ exists. Futhermore, using the condition $\sum \alpha_n = \infty$, we obtain that $\liminf(\|u_n - u^*\|^2 + \|v_n - v^*\|^2) = 0$. So, there exist a subsequences $\{u_{n_k}\}$ of $\{u_n\}$, $\{v_{n_k}\}$ of $\{v_n\}$ such that $u_{n_k} \to u^*$ and $v_{n_k} \to v^*$. Using the definition of ϕ and the continuity of J, we have $\phi(u^*, u_{n_k}) = \|x^*\| - 2\langle x^*, J(u_{n_k})\rangle + \|u_{n_k}\| \to 0$ as $k \to \infty$. Simillarly, using the continuity of J_* we have $\phi(v^*, v_{n_k}) = \|x^*\| - 2\langle x^*, J_*(v_{n_k})\rangle + \|v_{n_k}\| \to 0$ as $k \to \infty$. Thus, again by Lemma 2.3.5, we obtain that $\lim \Phi(w^*, w_n) = 0$. Hence we have $\lim \phi(u^*, u_n) = \lim \phi(v^*, v_n) = 0$. Therefore, by Lemma 2.2.1, we conclude that, $\lim \|u_n - u^*\| = \lim \|v_n - v^*\| = 0$. This completes the proof. ■

7.3 Convergence Theorems in L_p spaces, $p \geq 2$

In the theorem below, we assume $R(F) = D(K) = E^*$. We now prove the following theorem.

Theorem 7.3.1 *Let $E = L_p$, $2 \leq p < \infty$. Let $F : E \to E^*$ and $K : E^* \to E$ be strongly monotone and bounded maps. For arbitrary $(u_0, v_0) \in E \times E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively, by*

$$u_{n+1} = J^{-1}(Ju_n - \alpha_n(Fu_n - v_n)), \ n \geq 0, \tag{7.3.1}$$

$$v_{n+1} = J_*^{-1}(J_* v_n - \alpha_n(Kv_n + u_n)),\ n \geq 0, \tag{7.3.2}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ *satisfies the following conditions:* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n^{\frac{p}{p-1}} < \infty$. *Assume that the equation* $u + KFu = 0$ *has a solution. Then, there exists* $\gamma_0 > 0$ *such that if* $\alpha_n \leq \gamma_0$ *for all* $n \geq 1$, *the sequences* $\{u_n\}_{n=1}^{\infty}$ *and* $\{v_n\}_{n=1}^{\infty}$ *converge strongly to* u^* *and* v^* *respectively, where* u^* *is the solution of* $u + KFu = 0$ *with* $v^* = Fu^*$.

Proof We first prove that the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are bounded. The proof is by induction. For (u_n, v_n), $(u^*, v^*) \in L_p \times L_q$ where u^* is the solution of (7.1.3) with $v^* = Fu^*$, set $w_n = (u_n, v_n)$ and $w^* = (u^*, v^*)$. Define $\Phi_q : (E \times E^*) \times (E \times E^*) \to \mathbb{R}$ and $\Phi : (E \times E^*) \times (E \times E^*) \to \mathbb{R}$ by

$$\Phi_q(w_1, w_2) = \phi(u_1, u_2) + \phi_q(v_1, v_2) \tag{7.3.3}$$

and

$$\Phi(w_1, w_2) = \phi(u_1, u_2) + \phi(v_1, v_2), \tag{7.3.4}$$

respectively, where $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2)$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $E \times E^*$ be endowed with the norm $\|(u,v)\| = (\|u\|_E^2 + \|v\|_{E^*}^2)^{\frac{1}{2}}$. Let $r > 0$ be such that

$$r \geq \max\ \{\Phi_q(w^*, w_0),\ 6p\|v^*\|^2\}. \tag{7.3.5}$$

Since F and K are bounded, define

$$M_1 := m_p \sup\{\|Fu - v\|^2 : \|u\| \leq \|u^*\| + \sqrt{r};\ \|v\| \leq \sqrt{r + (1 - c_q)\|v^*\|} + \|v^*\|\} + 1;$$

and

$$M_2 := L_1 \sup\{\|Kv + u\|^{\frac{q}{q-1}} : \|u\| \leq \|u^*\| + \sqrt{r};\ \|v\| \leq \sqrt{r + (1 - c_q)\|v^*\|} + \|v^*\|\} + 1,$$

where c_q is the constant appearing in inequality (7.1.4), m_p is the Hölder continuity constant appearing in Lemma 2.2.3 and L_1 is the Lipschitz constant of J^{-1}. Define

$$\gamma_0 := \left[\frac{1}{2} min\left\{\frac{kr}{6M_1}, \frac{kr}{6M_2}\right\}\right]^{p-1},$$

where $k = \min\{k_1, k_2\}$, k_1 and k_2 are the constants of strong monotonicity of F and K, respectively. We claim that $\Phi_q(w^*, w_n) \leq r\ \forall\ n \geq 1$. Indeed, by construction, we have $\Phi_q(w^*, w_0) \leq r$. Suppose that $\Phi_q(w^*, w_n) \leq r$ for some $n \geq 1$. This implies that

$$\phi(u_*, u_n) + \phi_q(v^*, v_n) \leq r,\ \forall\ n \geq 1.$$

So, from inequalities (7.1.10) and (2.2.2), we have:

$$\|u_n\| \leq \|u^*\| + \sqrt{r} \ \text{ and } \|v_n\| \leq \sqrt{r + (1 - c_q)\|v^*\|} + \|v^*\|,\ \forall\ n \geq 1,$$

respectively. We prove that $\Phi_q(w^*, w_{n+1}) \leq r$. Using the definition of u_{n+1} and inequality (7.1.8) with $y^* = \alpha_n(Fu_n - v_n)$, we obtain that:

$$\begin{aligned}
\phi(u^*, u_{n+1}) &= \phi(u^*, J^{-1}(Ju_n - \alpha_n(Fu_n - v_n))) = V(x^*, Ju_n - \alpha_n(Fu_n - v_n)) \\
&\leq V(u^*, Ju_n) - 2\langle J^{-1}(Ju_n - \alpha_n(Fu_n - v_n)) - u^*, \alpha_n(Fu_n - v_n)\rangle \\
&= V(u^*, Ju_n) - 2\alpha_n\langle u_n - u^*, (Fu_n - v_n)\rangle \\
&\quad -2\alpha_n\langle J^{-1}(Ju_n - \alpha_n(Fu_n - v_n)) - J^{-1}(Ju_n), (Fu_n - v_n)\rangle. \\
&\leq \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, (Fu_n - v_n)\rangle \\
&\quad +2\alpha_n\|J^{-1}(Ju_n - \alpha_n(Fu_n - v_n)) - J^{-1}(Ju_n)\|\|(Fu_n - v_n)\|.
\end{aligned}$$

Observe that

$$\langle u_n - u^*, (Fu_n - v_n)\rangle = \langle u_n - u^*, (Fu_n - Fu^*)\rangle + \langle u_n - u^*, (Fu^* - v_n)\rangle.$$

Now, using the fact that A is strongly monotone and that J^{-1} is Hölder continuous we obtain:

$$\begin{aligned}
\phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) - 2\alpha_n k_1\|u_n - u^*\|^2 + 2\alpha_n\alpha_n^{\frac{1}{p-1}} m_p\|Fu_n - v_n\|^{\frac{p}{p-1}} \\
&+ 2\alpha_n\langle u_n - u^*, (v_n - Fu^*)\rangle
\end{aligned}$$

Using Lemma 2.2.2, definition of M_1 and the fact that $k = \min\{k_1, k_2\}$ we have

$$\begin{aligned}
\phi(u^*, u_{n+1}) &\leq (1 - 2\alpha_n k)\phi(u^*, u_n) + 2\alpha_n\alpha_n^{\frac{1}{p-1}} M_1 + 2\alpha_n kp\|v^*\|^2 \\
&+ 2\alpha_n\langle u_n - u^*, (v_n - Fu^*)\rangle.
\end{aligned} \tag{7.3.6}$$

Similarly, using inequality (7.1.8) with $y^* = \alpha_n(Kv_n + u_n)$ we obtain:

$$\begin{aligned}
\phi_p(v^*, v_{n+1}) &\leq \phi_p(v^*, v_n) - 2\alpha_n k\|v_n - v^*\|^2 \\
&+ 2\alpha_n\|J_*^{-1}(J_*v_n - \alpha_n(Kv_n + u_n)) - J_*^{-1}(J_*u_n)\|\|(Kv_n + u_n)\| \\
&+ 2\alpha_n\langle v_n - v^*, -(Kv^* + u_n)\rangle.
\end{aligned}$$

Using the fact that J_*^{-1} is Lipschitz, the definition of M_2 and inequality (7.1.7) we obtain:

$$\begin{aligned}
\phi_p(v^*, v_{n+1}) &\leq (1 - 2\alpha_n k)\phi_p(v^*, v_n) + 2\alpha_n^2 M_2 \\
&+ 2\alpha_n\langle v_n - v^*, -(Kv_n + u_n)\rangle.
\end{aligned} \tag{7.3.7}$$

Adding (7.3.6) and (7.3.7), we have

$$\begin{aligned}
\Phi_p(w^*, w_{n+1}) &\leq (1 - 2\alpha_n k)\Phi_p(w^*, w_n) + 2\alpha_n\alpha_n M_2 \\
&+ 2\alpha_n\alpha_n^{\frac{1}{q-1}} M_1 + 2\alpha_n kp\|v^*\|^2 \\
&\leq (1 - 2\alpha_n k)\Phi_p(w^*, w_n) + 2\alpha_n\gamma_0 M_2 \\
&+ 2\alpha_n\gamma_0 M_1 + 2\alpha_n kp\|v^*\|^2.
\end{aligned} \tag{7.3.8}$$

Using definition of γ_0, induction hypothesis and inequality (7.3.5) we have:

$$\Phi_p(w^*, w_{n+1}) \leq (1 - 2\alpha_n k)r + \frac{\alpha_n k}{3} + \frac{\alpha_n k}{3} + \frac{\alpha_n k}{3} \leq (1 - \alpha_n k)r \leq r \quad (7.3.9)$$

Hence, $\Phi_p(w^*, w_{n+1}) \leq r$. By induction, $\Phi(w^*, w_n) \leq r \ \forall \ n \geq 1$. Consequently, we have $\phi_p(v^*, v_{n+1}) \leq r$ and $\phi(u^*, u_{n+1}) \leq r$. Thus from inequalities (7.1.9) and (2.2.2) we have that $\{u_n\}$ and $\{v_n\}$ are bounded, respectively.

We now prove that $\Phi(w^*, w_{n+1}) \to 0$ as $n \to \infty$. If we set $M_0 = m_p \sup\{\|Fu_n - v_n\|^{\frac{p}{p-1}}\}$, following the same method of computation above we obtain:

$$\begin{aligned} \phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) - 2\alpha_n k\|u_n - u^*\|^2 + 2\alpha_n^{\frac{p}{p-1}} M_0 \\ &+ 2\alpha_n \langle u_n - u^*, (v_n - Fu^*)\rangle \end{aligned} \quad (7.3.10)$$

Simillarly, we have

$$\begin{aligned} \phi(v^*, v_{n+1}) &\leq \phi(v^*, v_n) - 2\alpha_n k\|v_n - v^*\|^2 + 2\alpha_n^2 M_2 \\ &+ 2\alpha_n \langle v_n - v^*, -(Kv_n^* + u_n)\rangle \end{aligned} \quad (7.3.11)$$

Adding (7.3.10) and (7.3.11) we obtain:

$$\begin{aligned} \Phi_p(w^*, w_{n+1}) &\leq \Phi_p(w^*, w_n) - 2\alpha_n k(\|u_n - u^*\|^2 + \|v_n - v^*\|^2) \\ &+ 2\alpha_n^{\frac{p}{p-1}} M_0 + 2\alpha_n^2 M_2. \end{aligned} \quad (7.3.12)$$

By Lemma 2.3.5, we obtain that $\lim \Phi(w^*, w_n)$ exists. Futhermore, using the condition $\sum \alpha_n = \infty$, we obtain that $\liminf(\|u_n - u^*\|^2 + \|v_n - v^*\|^2) = 0$. So, there exist a subsequences $\{u_{n_k}\}$ of $\{u_n\}$, $\{v_{n_k}\}$ of $\{v_n\}$ such that $u_{n_k} \to u^*$ and $v_{n_k} \to v^*$. Using the definition of ϕ and the continuity of J, we have $\phi(u^*, u_{n_k}) = \|x^*\| - 2\langle x^*, J(u_{n_k})\rangle + \|u_{n_k}\| \to 0$ as $k \to \infty$. Simillarly, using the continuity of J_* we have $\phi(v^*, v_{n_k}) = \|x^*\| - 2\langle x^*, J_*(v_{n_k})\rangle + \|v_{n_k}\| \to 0$ as $k \to \infty$. Thus, again by Lemma 2.3.5, we obtain that $\lim \Phi(w^*, w_n) = 0$. Hence we have $\lim \phi(u^*, u_n) = \lim \phi(v^*, v_n) = 0$. Therefore, by Lemma 2.2.1, we conclude that, $\lim \|u_n - u^*\| = \lim \|v_n - v^*\| = 0$. This completes the proof. ■

Remark 7.3.2 *Since every Lipschitz map is bounded, it is easy to see that our theorems hold for Lipschitz ϕ- strongly monotone operators in L_p spaces, $1 < p < \infty$.*

A prototype of the parameter in our theorems is the canonical choice $\alpha_n = \frac{1}{n}$, $n \geq 1$.

All the results of this chapter are the results obtained in [32], which was submitted to the Proceedings of the American Mathematical Society for consideration for publication.

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