CONTRIBUTIONS TO THE CONTROL THEORY OF SOME PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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Ph.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirements for the award of the degree of Doctor of Philosophy Degree in Mathematics.

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Dedication

To My **Lord and Savior Jesus Christ** for His Unmerited Favour without which I would not have been here and for giving me the wisdom, inspiration and strength to start and complete this thesis.
To my Family and to my fiancée, **Tiako Fani Michele Doriane**, whose love, care, vulnerability, wisdom, strength, unceasing encouragements, motivation, and support have inspired me to be the best I can be.

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Doing Mathematics may be fun. But, *it takes more than just a piece of paper, a pen, and the ability to recognize patterns, and put together implications and equivalences to write a logical, sound and correct mathematical proof. Unfortunately, most often, a substantial part of what one may need has neither a material nor a Mathematical equivalent.*
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Patrice Ndambomve

Abstract

This thesis is a contribution to Control Theory of some Partial Functional Integrodifferential Equations in Banach spaces. It is made up of two parts: controllability and existence of optimal controls. In the first part, we consider the dynamical control systems given by the following models that arise in the analysis of heat conduction in materials with memory, and viscoelasticity, and take the form of a:

- Partial functional integrodifferential equation subject to a nonlocal initial condition in a Banach space $(X, \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds + f(t, x(t)) + Cu(t) \\ \qquad\qquad \text{for t} \in \text{I} = [0, \text{b}], \\ \\ x(0) = x_0 + g(x), \end{cases} \tag{0.0.1}$$

  where $x_0 \in X$, $g : \mathcal{C}(I, X) \to X$ and $f : I \times X \to X$ are functions satisfying some conditions; $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on X; for $t \geq 0$, $B(t)$ is a closed linear operator with domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$. The control u belongs to $L^2(I, U)$ which is a Banach space of admissible controls, where U is a Banach space.

- Partial functional integrodifferential equation with finite delay in a Ba-

nach space $(X, \ \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) \ + \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t,x_t) + Cu(t)\,, \\ \qquad\qquad \text{for t} \in \text{I} = [0,\text{b}], \\ \\ x_0 = \varphi \in \mathcal{C} = \mathcal{C}([-r,0];X), \end{cases} \tag{0.0.2}$$

where $f : I \times \mathcal{C} \to X$ is a function satisfying some conditions; $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on X; for $t \geq 0$, $B(t)$ is a closed linear operator with domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$. The control u belongs to $L^2(I,U)$ which is a Banach space of admissible controls, where U is a Banach space, and x_t denotes the history function of $\mathcal{C}$ of the state from the time $t-r$ up to the present time t, and is defined by $x_t(\theta) = x(t+\theta)$ for $-r \leq \theta \leq 0$.

- Partial functional integrodifferential equation with infinite delay in a Banach space $(X, \ \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) \ + \displaystyle\int_0^t \gamma(t-s)x(s)ds \ + \ f(t,x_t) + Cu(t), \\ \qquad\qquad \text{for } t \in I = [0,b] \\ \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{0.0.3}$$

where $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space X; for $t \geq 0$, $\gamma(t)$ is a closed linear operator with domain $\mathcal{D}(\gamma(t)) \supset \mathcal{D}(A)$. The control u takes values from another Banach space U. The operator $C(t)$ belongs to $\mathcal{L}(U,X)$ which is the Banach space of bounded linear operators from U into X, and the phase space $\mathcal{B}$ is a linear space of functions mapping $]-\infty,0]$ into X satisfying axioms which will be described later, for every $t \geq 0$, x_t denotes the history function of $\mathcal{B}$ defined by $x_t(\theta) = x(t+\theta)$ for $-\infty \leq \theta \leq 0$, $f : I \times \mathcal{B} \to X$ is a continuous function satisfying some conditions.

We give sufficient conditions that ensure the controllability of the systems without assuming the compactness of the semigroup, by supposing that their linear homogeneous and undelayed parts admit a resolvent operator in the sense of Grimmer and by making use of the Hausdorff measure of noncompactness.
In the second part, we consider equations (0.0.1), (0.0.2) and (0.0.3), in the

case where the operator $C = C(t)$ (time dependent), the function $g \equiv= 0$, the Banach spaces X and U are separable and reflexive. Using techniques of convex optimization, *a priori* estimation, and applying Balder's Theorem, we establish the existence of optimal controls for the following Lagrange optimal control problem associated to each of the equations:

$$(\mathcal{LP}) \begin{cases} \text{Find a control } u^0 \in \mathcal{U}_{ad} \ \text{ such that} \\ \\ \mathcal{J}(u^0) \leq \mathcal{J}(u) \ \text{ for all } \ u \in \mathcal{U}_{ad}, \end{cases}$$

where

$$\mathcal{J}(u) := \int_0^T \mathcal{L}\Big(t,\, x_t^u,\, x^u(t),\, u(t)\Big)\, dt,$$

$\mathcal{L}$ is some functional, x^u denotes the mild solution corresponding to the control $u \in \mathcal{U}_{ad}$, and $\mathcal{U}_{ad}$ denotes the set of admissible controls.

Publications arising from the thesis and other peer-review publications

[A] Published/Accepted Papers from the thesis

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4) Khalil Ezzinbi and **Patrice Ndambomve**, *Partial Functional Integrodifferential Equations and Optimal Controls in Banach Spaces*, (**Accepted to Appear: Communications in Optimization Theory**).

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[B] Other Published/Accepted peer-review publications

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2) C. E. Chidume, **P. Ndambomve**, A. U. Bello, *The split equality fixed point problem for demi-contractive mappings*, **Journal of Nonlinear Analysis and Optimization, Theory & Applications, Vol. 6, No. 1, (2015), 61-69**.

3) C. E. Chidume, A. U. Bello, **P. Ndambomve**, *Strong and* Δ*-convergence theorems for common fixed points of a finite family of multi-valued demicontractive mappings in CAT(0) spaces*, Abstract and Applied Analysis, Volume 2014, Article ID 805168.

4) C.E. Chidume, A.U. Bello, M.E. Okpala, **P. Ndambomve**, *Strong Convergence Theorem for Fixed Poins of Nearly Uniformly L-Lipschitzian Asymptotically generalized Hemicontractive Mappings*, **International Journal of Mathematical Analysis, Vol. 9, 2015, no. 52, 2555-2569**.

5) C.E. Chidume, M.E. Okpala, A.U. Bello and **P. Ndambomve**, *Convergence theorems for finite family of a general class of multi-valued strictly pseudo-contractive mappings*, **Fixed Point Theory and Applications, (2015) 2015:119, DOI 10.1186/s13663-015-0365-7**.

Contents

CHAPTER 1

Introduction

1.1 General Introduction

In various fields of science and engineering such as Electronics, Fluid Dynamics, Physical Sciences, etc..., many problems that are related to linear viscoelasticity, nonlinear elasticity and Newtonian or non- Newtonian fluid mechanics have mathematical models which are described by differential or integral equations or integrodifferential equations which have received considerable attention during the last decades. Control Theory arises in many modern applications in engineering and environmental sciences [2]. It is one of the most interdisciplinary research areas [22][63] and its empirical concept for technology goes back to antiquity with the works of Archimede, Philon, etc..., [73]. A control system is a dynamical system on which one can act by the use of suitable parameters (i.e., the controls) in order to achieve a desired behavior or state of the system. Control systems are usually modeled by mathematical formalism involving mainly ordinary differential equations, partial differential equations or functional differential equations. In condensed expression, they often take the form of differential equation :

$$x'(t) \; = \; F(t, x(t), u(t)) \qquad \text{for} \quad \mathrm{t} \geq 0,$$

where x is the state and u is the control. While studying a control system, two most common problems that appear are the controllability and the optimal controls problems. The controllability problem consists in checking the possibility of steering the control system from an initial state (initial condition)

to a desired terminal one (boundary condition), by an appropriate choice of the control u, while the optimal control problem consists in finding the input function (the control or the command) so as to optimize (maximize or minimize) the objective function. Control Theory of integrodifferential equations with classical initial conditions and with delays, have received considerable attention by researchers during the last decades.
This thesis is a contribution to Control Theory of Partial Functional Integrodifferential equations in Banach spaces. It is made up of two parts:

- **Part I:** Controllability results for some partial functional integrodifferential equations in Banach spaces.
- **Part II:** Optimal controls of some partial functional integrodifferential equations in Banach spaces.

It lies at the interface between Nonlinear Functional Analysis, Optimization Theory and Dynamical Systems. In the first part, we establish the controllability for some partial functional integrodifferential equations, with nonlocal initial conditions, with finite delay and then with infinite delay. The second part deals with the solvability and the existence of optimal controls for these partial functional integrodifferential equations, with Cauchy initial conditions, with finite delay and then with infinite delay. We use fixed point techniques to solve the controllability problem and convex optimization techniques to solve the optimal control problems.

1.1.1 Nonlocal Differential Equations

Many problems arising in engineering and life sciences are modeled mathematically by differential equations. Differential equations are one of the most powerful and frequently used tools in mathematical modeling. Depending on the nature of the problem, these equations may take various forms like ordinary differential equations, partial differential equations or functional differential equations. In condensed expression, they often take the following form:

$$x'(t) \;=\; F(t, x(t)) \qquad \text{for} \quad \mathrm{t} \geq 0,$$

where x is the state. Most often, these problems are subject to some initial conditions. The classical initial condition is that referred to as the Cauchy initial condition, given by $x(0) = x_0$, where x_0 is some initial state of the system at time $t = 0$. However, in many real world contexts such as Engineering, Environmental sciences, Demography, etc..., nonlocal constraints (such as isoperimetric or energy condition, multipoint boundary condition and flux boundary condition) appear and have received considerable attention during the last decades, cf. [14] and [15]. They usually take the following form: $x(0) = x_0 + g(x)$, where g is a function satisfying some conditions, and x is the state or a solution of the differential equation in question. Observe that the initial condition in this case depends on the solution of the system. So, the concept of nonlocal initial condition not only extends that of Cauchy initial condition, but also turns out to have better effects in applications as it may take into account future measurements over a certain period after the initial time t equals 0.

1.1.2 Delay Differential Equations

In the mathematical description of a great number of physical phenomena, one usually suppose that the evolution of the system depends only on its current state. However, there are situations where the evolution of a process depends not only on its current state, but also on past states of the system. Such phenomena arise in many areas, in particular, in population dynamics. Amongst the mathematical models that can describe such situations are delay differential equations whose delays can be of neutral type. One then has equations whose temporal terms are, in a general way, nonlocal terms, involving values of the state at past, discrete or distributed times. These nonlocal terms are either of order 0 (delay equations) or of order 1 (neutral equations). In genera1, the delays appear because of the necessary time for the system to respond to certain evolution, or because a certain threshold (limit value) must be attained before the system can be activated. Delay differential equations lie at the interface between ordinary and partial differential equations. The difference with ordinary differential equations is that the initial data are themselves functions. This requires more elaborate mathematical study than for ordinary differential equations, the nature of the delay (discrete, continuous, infinite, state dependent, $\cdots$) potentially complicating it.

Example.([5]) *Let $N(t)$ denote the density of adults at time t, in a biological population composed of adult and juvenile individuals. Assume that the length of the juvenile period is exactly r units of time for each individual. Assume that adults produce offspring at a per capita rate λ and that their probability per unit of time of dying is μ. Assume that a newborn survives the juvenile period with probability ν and put $t = \lambda\nu$. Then the dynamics of N can be described by the differential equation*

$$\frac{dN}{dt}(t) = -\mu N(t) + kN(t-r)$$

which involves a nonlocal term, $kN(t-r)$ meaning that newborns become adults with some delay. So the time variation of the population density N involves the current as well as the past values of N. Such equations are called delay equations.

In finite dimension, a complete theory has been developed for delay differential equations, while the theory in infinite dimension is far from being complete. Such equations in infinite dimension are represented by differential equations which are nonlocal in time: knowing the solution at a given time requires knowing it on a time interval whose lenght is equal to the delay. These types of equations have lots of applications in physics, chemistry, biology, population dynamics, $\cdots$.
Equations with finite delay genera1ly take the following abstract form:

$$\begin{cases} x'(t) = Ax(t) \ + \ F(t, x_t) \qquad \text{for } t \geq 0 \\ \\ x_0 = \varphi \in \mathcal{C} = \mathcal{C}([-r, 0]; X), \end{cases} \tag{1.1.1}$$

where $F : \mathbb{R}^+ \times \mathcal{C} \to X$ is a function satisfying some conditions; $A : \mathcal{D}(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space X; for $t \geq 0$; $\mathcal{C}([-r, 0], X)$ (the phase space), denotes the Banach space of continuous functions $x : [-r, 0] \to X$ with supremum norm $\|x\|_\infty = \sup_{t\in I} \|x(t)\|_X$, x_t denotes the history function of $\mathcal{C}$ of the state from the time $t - r$ up to the present time t, and is defined by $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$. When $r = \infty$, we say that the delay is infinite, and this class of equations englobes equations known in mechanics, namely Volterra equations. When $A = 0$, and $X = \mathbb{R}^n$, we have delay differential equations in finite dimension. For this type of equations, the solution operator is compact, and this property has made it possible to develop a complete theory for delay differential equations in finite dimension. In infinite dimension, this property is no longer satisfied, and this lack of regularity requires the development of new functional analysis tools to tackle problems related to the qualitative aspect of the solutions.
In the literature devoted to equations with finite delay, the phase space is the space of continuous functions on $[-r, 0]$, for some $r > 0$, endowed with the uniform norm topology. But when the delay is unbounded (i.e., infinite), the phase space denoted by $\mathcal{B}$ is a linear space of functions mapping $]-\infty, 0]$ into X satisfying some axioms. The selection of the phase space $\mathcal{B}$ plays an important role in both qualitative and quantitative theories. A usual choice is a normed space satisfying the following suitable axioms, which was introduced by Hale and Kato [82]:

$(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a normed linear space of functions mapping $]-\infty, 0]$ into X and satisfying the following axioms:

($\mathbf{A_1}$) There exist a positive constant H and functions $K : \mathbb{R}^+ \to \mathbb{R}^+$ continuous and $M : \mathbb{R}^+ \to \mathbb{R}^+$ locally bounded, such that for $a > 0$, if $x :]-\infty, a] \to X$ is continuous on $[0, a]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0, a]$, the following conditions hold:

(i) $x_t \in \mathcal{B}$,

(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$, which is equivalent to $\|\varphi(0)\| \leq H\|\varphi\|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$,

(iii) $\|x_t\|_{\mathcal{B}} \leq K(t) \sup\limits_{0 \leq s \leq t} \|x(s)\| + M(t)\|x_0\|_{\mathcal{B}}$.

($\mathbf{A_2}$) For the function x in ($\mathbf{A_1}$), $t \to x_t$ is a $\mathcal{B}$-valued continuous function for $t \in [0, a]$.

($\mathbf{A_3}$) The space $\mathcal{B}$ is complete.

Example [83] Let the spaces
BC the space of bounded continuous functions defined from $(-\infty, 0]$ to X;
BUC the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to X;
$C^\infty := \Big\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exists}\Big\}$;
$C^0 := \Big\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0\Big\}$, be endowed with the uniform norm

$$\|\phi\| = \sup_{\theta \leq 0} \|\phi(\theta)\|.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions ($\mathbf{A_1}$) – ($\mathbf{A_3}$).
Equations with infinite delay generally take the following abstract form:

$$\begin{cases} x'(t) = Ax(t) \ + \ F(t, x_t) & \text{for } t \geq 0 \\ \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.1.2}$$

where $F : \mathbb{R}^+ \times \mathcal{B} \to X$ is a function satisfying some conditions; $A : \mathcal{D}(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t \geq 0}$ on a Banach space X; for $t \geq 0$; and the phase space $\mathcal{B}$ is a linear space of functions mapping $]-\infty, 0]$ into X satisfying axioms ($\mathbf{A_1}$) – ($\mathbf{A_3}$), for every $t \geq 0$, x_t denotes the history function of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t + \theta) \ \text{ for } \ -\infty \leq \theta \leq 0,$$

1.2 Partial Functional Integrodifferential Equations

In many areas of applications such as Engineering, Electronics, Fluid Dynamics, Physical Sciences, etc..., integrodifferential equations appear and have received considerable attention during the last decades. Consider the following system:

$$\begin{cases} x'(t) = Ax(t) \ + \ \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t) \qquad \text{for t} \in \text{I} = [0, \text{b}] \\ x(0) = x_0, \end{cases} \tag{1.2.1}$$

which is a linear Volterra integrodifferential equation that arises in the analysis of heat conduction in materials with memory and viscoelasticity. Equation (1.2.1) has been studied by many authors under various hypotheses concerning the operators A and B, see for instance the works by Chen and Grimmer [17], Hannsgen [32, 33], Miller [58, 59], Miller and Wheeler [60, 61], and the references contained in them.

It was not until 1982 that Grimmer [85] proved the existence and uniqueness of resolvent operators for this integrodifferential equation that give the variation of parameter formula for the solutions.

In recent years, much work on the existence and regularity of solutions of the nonlinear Volterra integrodifferential equations with delays and with nonlocal conditions have been done by many authors by applying the resolvent operator theory giving by Grimmer in [85], see e.g., [43, 106] and the references therein. The objective of this thesis is to study the controllability and the existence of optimal controls for some class of partial functional integrodifferential equations in Banach spaces.

We motivate the study by giving the occurence of these partial functional integrodifferential equations in different areas of applications in science.

1.2.1 A Model in Viscoelasticity

Integrodifferential equations have applications in many problems arising in physical systems. The following one-dimensional model in viscoelasticity is one of the applications of that theory

$$\begin{cases} \alpha\dfrac{\partial^2\omega}{\partial t^2}(t,\xi)+\beta\dfrac{\partial\omega}{\partial t}(t,\xi)=\dfrac{\partial\varphi}{\partial\xi}(t,\xi)+h(t,\xi), \\ \gamma\dfrac{\partial\omega}{\partial\xi}(t,\xi)+\displaystyle\int_0^t a(t-s)\dfrac{\partial\omega}{\partial\xi}(s,\xi)ds=\varphi(t,\xi),\ (t,\xi)\in\mathbb{R}^+\times[0,1], \\ \omega(t,0)=\omega(t,1)=0, \quad t\in\mathbb{R}^+, \\ \omega(0,\xi)=\omega_0(\xi), \quad \xi\in[0,1], \end{cases}$$

where, ω is the displacement, φ is the stress, h is the external force, $\alpha,\ \gamma>0$ and β are constants. In this model, the first equation describes the linear momentum while the second equation describes the constitutive relation between stress and strain. Setting $\gamma=1,\quad v=\frac{\partial\omega}{\partial t}$, and $u=\frac{\partial\omega}{\partial\xi}$, the above equations can be rewritten as follows

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix}=\begin{bmatrix} 0 & \partial_\xi \\ \frac{\partial_\xi}{\alpha} & 0 \end{bmatrix}\left\{\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}+\int_0^t\begin{bmatrix} a(t-s) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u(s) \\ v(s) \end{bmatrix}ds\right\}$$

$$+\begin{bmatrix} 0 & 0 \\ 0 & -\frac{\beta}{\alpha} \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}+\begin{bmatrix} 0 \\ \frac{h(t)}{\alpha} \end{bmatrix},\ t\geq 0.$$

Setting

$$x(t)=\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \quad A=\begin{bmatrix} 0 & \partial_\xi \\ \frac{\partial_\xi}{\alpha} & 0 \end{bmatrix}, \quad B(t)=\begin{bmatrix} a(t-s) & 0 \\ 0 & 0 \end{bmatrix}$$

$$K=\begin{bmatrix} 0 & 0 \\ 0 & -\frac{\beta}{\alpha} \end{bmatrix}, \quad \text{and} \quad p(t)=\begin{bmatrix} 0 \\ \frac{h(t)}{\alpha} \end{bmatrix},$$

we can rewrite the above equation into the following abstract form:

$$\begin{cases} x'(t)=A\Big[x(t)+\displaystyle\int_0^t B(t-s)x(s)\Big]ds+Kx(t)+p(t) \quad \text{for} \quad \text{t}\geq 0 \\ x(0)=x_0. \end{cases}$$

The operator A here is unbounded, while K and $B(t)$ are bounded operators for $t\geq 0$ on a Banach space X. When $AB(t)=B(t)A$, we obtain the following equation:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)Ax(s)ds + Kx(t) + p(t) \quad \text{for} \quad \mathrm{t} \geq 0 \\ \\ x(0) = x_0. \end{cases} \tag{1.2.2}$$

which has been studied in [23]. We note that in general, the equality $AB(t) = B(t)A$ does not hold.
Setting $f(t, x(t)) = Kx(t) + p(t)$, equation (1.2.2) becomes

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)Ax(s)ds + f(t, x(t)) \quad \text{for} \quad \mathrm{t} \geq 0 \\ \\ x(0) = x_0. \end{cases} \tag{1.2.3}$$

which to the best of our knowledge has not been investigated for controllability problem. This problem is addressed in the first part of this thesis, in the case where equation (1.2.3) admits a nonlocal condition.

1.2.2 A Model in Heat Conduction in Materials with Memory

Consider a heat flow in a rigid body Ω of a material with memory. Let $w(t, \xi)$, $e(t, \xi)$, $q(t, \xi)$ and $s(t, \xi)$ denote, respectively, the temperature, the internal energy, the heat flux, and the external heat supply at time t and position ξ. The balance law for the heat transfer is given by: (see e.g., [53]

$$e_t(t, \xi) + \operatorname{div} q(t, \xi) = s(t, \xi) \tag{1.2.4}$$

and the physical properties of the body suggest the dependence of e and q on w and ∇w, respectively. For instance, assuming the Fourier Law, i.e.,

$$e(t, \xi) = c_1 w(t, \xi) \tag{1.2.5}$$

$$q(t, \xi) = -c_2 \nabla w(t, \xi), \tag{1.2.6}$$

where c_1, c_2 are positive constants, one deduces from (1.2.4) the classical heat equation

$$w_t(t, \xi) = c\Delta w(t, \xi) + f(t, \xi) \tag{1.2.7}$$

with $c = c_1^{-1} c_2$ and $f(t, \xi) = c_1^{-1} s(t, \xi)$.
In many materials, the assumptions (1.2.5) and (1.2.6) are not justified because they take no account of the memory effects: several models have been

proposed to overcome this difficulty (see e.g. [100, 101, 102]): one of them consists in substituting (1.2.6) with

$$q(t,\xi) = -c_2\nabla w(t,\xi) - \int_{-\infty}^{t} h(t-s)\nabla w(s,\xi)ds. \tag{1.2.8}$$

Taking for simplicity $c_1 = c_2 = 1$, we get from (1.2.4), (1.2.5) and (1.2.8) that

$$w_t(t,\xi) = \Delta w(t,\xi) + \int_{-\infty}^{t} h(t-s)\Delta w(s,\xi)ds + s(t,\xi). \tag{1.2.9}$$

If we assume that the thermal history w of the body Ω is known up to $t = 0$ and the temperature of the boundary Γ of Ω is constant (=0) for all t, we are led to the following system:

$$\begin{cases} w_t(t,\xi) = \Delta w(t,\xi) + \displaystyle\int_{0}^{t} h(t-s)\Delta w(s,\xi)ds + f(t,\xi), \quad (t,\xi) \in [0,b]\times\Omega \\ w(0,\xi) = w_0(\xi), \quad \xi \in \overline{\Omega} \\ w(t,\xi) = 0, \quad (t,\xi) \in [0,b]\times\Gamma, \end{cases} \tag{1.2.10}$$

where $b > 0$ is arbitrarily fixed. If we prescribe h (in addition to f) then (1.2.10) is a Cauchy-Dirichlet problem for an integrodifferential equation in the unknown w, which has been studied by several authors in the last decades (see e.g., [103, 104, 105] and references therein).
Now, if we consider that the thermal history of the body Ω is known from the time $t - r$ (for some $r > 0$) up to the present time t, the temperature of the boundary $\partial\Omega$ of Ω is constant ($= 0$) for all t, and the external heat flux depends on the thermal history of the body, then, system (1.2.10) becomes the following integrodifferential equation with finite delay:

$$\begin{cases} w_t(t,\xi) = \Delta w(t,\xi) + \displaystyle\int_{0}^{t} h(t-s)\Delta w(s,\xi)ds + f(t, w(t-r,\xi)) \\ \qquad\qquad \text{for } (t,\xi) \in [0,b]\times\Omega \\ w(t,\xi) = \psi(t,\xi) \text{ for } t \in [-r,0] \text{ and } x \in \overline{\Omega} \end{cases} \tag{1.2.11}$$

where ψ is a given initial function and r is a positive number. In the one dimensional setting, Travis and Webb [75] were the first to consider equation

(1.2.11) with $h = 0$, and studied its existence and stability properties.
Now define

$$\begin{aligned} x(t)(\xi) &= w(t,\xi) \\ Ax &= \Delta x \end{aligned}$$

$$\varphi(\theta)(\xi) = \psi(\theta,\xi) \text{ for } \theta \in [-r,0] \text{ and } \xi \in \Omega.$$

$$\begin{aligned} f(t,\varphi)(\xi) &= f(t,\varphi(-r)(\xi)) \quad \text{for } t \in [0,b] \text{ and } \xi \in \Omega \\ (B(t)x)(\xi) &= h(t)\Delta x(t)(\xi) \text{ for } t \in [0,b] \text{ and } \xi \in \Omega. \end{aligned}$$

Then, equation (1.2.11) can be transformed into the following abstract form:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds + f(t,x_t) \text{ for } t \in I = [0,b], \\ x_0 = \varphi \in \mathcal{C}([-r,0],X), \end{cases} \tag{1.2.12}$$

where X is a Banach space. Equation (1.2.12) has been studied by many authors (see e.g., [83] and the references contained in it). But to the best of our knowledge, this equation has never been considered for controllability and existence of optimal controls and this motivates the work in this thesis. An example of a material with memory is *Shape-memory polymers* (SMPs), which are polymeric smart materials that have the ability to return from a deformed state (temporary shape) to their original (permanent) shape induced by an external stimulus (trigger), such as temperature change.

1.3 Controllability of Dynamical Systems

A dynamical system is a system that evolves in time through the iterated application of an underlying dynamical rule. It is a mathematical model that one usually constructs in order to investigate some physical phenomenon that evolves in time. This model usually involves mainly ordinary differential equations, partial differential equations or functional differential equations, which describe the evolution of the process under study in mathematical terms.
Controllability plays an essential role in the development of modern mathematical control systems. It has many important applications not only in control theory and systems theory, but also in such areas as industrial and chemical process control, reactor control, control of electric bulk power systems, aerospace engineering and recently in quantum systems theory. It is

one of the fundamental concepts in the mathematical control theory. This is a qualitative property of dynamical control systems and is of particular importance in control theory. A systematic study of controllability started at the beginning of sixties in the last century, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out [46]. Roughly speaking, controllability generally means, that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. The notion of controllability was identified by Kalman [41], as one of the central properties determining system behavior. The basic controllability problem in continuous time is formulated as follows: one is interested in steering a control system, whose state $x(t)$ (at time t) defined on a fixed time interval $0 \leq t \leq b$, is modelled by the solution of a differential equation:

$$\frac{dx}{dt} = F(t, x, u) \text{ on } [0, b], \tag{1.3.1}$$

from an initial state $x(0) = x_0$, to a desired state x_1, using a control u from the set of admissible controls, in time b.
For finite dimensional autonomous linear systems, when in (1.3.1), $F(t, x, u) = Ax + Bu$, where A is an $n \times n$ matrix and B is an $n \times m$ matrix, the *Kalman rank condition* [41] gives a necessary and sufficient condition for controllability of the system. It says that system (1.3.1) with $F(t, x, u) = Ax + Bu$ is controllable if and only if the controllability matrix $[A|B] = [B|AB|A^2B|\cdots|A^{n-1}B]$ of size $n \times nm$ has rank n. This is a useful and simple test, and much effort has been spent on trying to generalize it to nonlinear systems in various forms. The systematic study of controllability questions for continuous time nonlinear systems was begun in the early 70's. At that time, the papers [52], [74], and [47], building on previous work ([18], [34]) on partial differential equations, gave a nonlinear analogue of the above Kalman controllability rank condition. This analogue provides only a necessary test, not sufficient. Also in infinite dimensional spaces, even for linear system, one can hardly get a necessary and sufficient condition for controllability, for more details see [37], and the references contained in it.
Consider the following infinite dimensional linear system:

$$\begin{cases} x'(t) = Ax(t) + Bu(t) \text{ on } [0, b], \\ x(0) = x_0 \end{cases} \tag{1.3.2}$$

where A generates a strongly continuous semigroup of bounded linear operators $\big(T(t)\big)_{t\geq 0}$ on a Banach space X and B is a bounded linear operator from a Banach space U into X.

Now if x is a classical solution of (1.3.2), then $x(t) \in \mathcal{D}(A)$, for all $t \in [0, b]$. In the genera1 case, when A is unbounded, $\mathcal{D}(A) \neq X$, which means that the system cannot be steered to all of X. Therefore, only a mild solution of (1.3.2) given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds, \tag{1.3.3}$$

will be considered with the following definition of controllability (exact controllability).

Definition 1.3.1 *The system (1.3.2) is said to be controllable (exactly controllable) on the interval $I = [0, b]$ if for any two states $x_0,\ x_1 \in X$, there exists a control $u \in L^2(I, U)$ such that the mild solution x of (1.3.2) satisfies $x(b) = x_1$.*

Solving the controllability problem in infinite dimension boils down to showing the existence of mild solutions, using fixed point theorems, and finding the appropriate control for which the mild solution satifies the equality condition in the above definition. The appropriate control varies from system to system; it is defined or constructed using the following assumption:
(**H**) The linear operator W from $L^2(I, U)$ into X, defined by

$$Wu = \int_0^b T(b-s)Bu(s)\, ds,$$

induces a bounded inverse operator $\widetilde{W}^{-1}$ defined on $L^2(I, U)/Ker(W)$.
Then by defining the control u by $u(t) = \widetilde{W}^{-1}\Big(x_1 - T(b)x_0\Big)(t)$, and using it in equation (1.3.3), Hypothesis (**H**) yields $x(b) = x_1$, showing that system (1.3.2) is controllable on $I = [0, b]$.
The construction of $\widetilde{W}^{-1}$ is outlined as follows (see [7, 68]):
Let $Y = L^2(I, U)/Ker(W)$. Since $Ker(W)$ is closed, Y is a Banach space under the norm

$$\|[u]\|_Y = \inf_{u \in [u]} \|u\|_{L^2(I,U)} = \inf_{W\hat{u}=0} \|u + \hat{u}\|$$

where $[u]$ are the equivalence classes of u.
Define $\widetilde{W} : Y \to X$ by

$$\widetilde{W}[u] = Wu, \quad u \in [u].$$

Then, $\widetilde{W}$ is one-to-one and $\|\widetilde{W}[u]\| \leq \|W\|\|[u]\|_Y$.
Also, $V = Range(W)$ is a Banach space with the norm $\|v\|_V = \|\widetilde{W}^{-1}v\|_Y$.

To see this, note that this norm is equivalent to the graph norm on $\mathcal{D}(\widetilde{W}^{-1}) = Range(\widetilde{W})$. $\widetilde{W}$ is bounded, and since $\mathcal{D}(\widetilde{W}) = Y$ is closed, $\widetilde{W}^{-1}$ is closed. So, the above norm makes $Range(W) = V$, a Banach space.
Moreover,

$$\|Wu\|_V = \|\widetilde{W}^{-1}Wu\|_Y = \|\widetilde{W}^{-1}\widetilde{W}[u]\| = \|[u]\| = \inf_{u\in[u]} \|u\| \leq \|u\|,$$

So, $W \in \mathcal{B}(L^2(I,U), X)$.
Since, $L^2(I,U)$ is reflexive, and $Ker(W)$ is weakly closed, the infimum is actually achieved. Therefore, for any $v \in V$, a control $u \in L^2(I,U)$ can be chosen so that $u = \widetilde{W}^{-1}v$.

Several authors have studied the controllability of linear and nonlinear systems with various initial conditions, and linear and nonlinear delay systems in infinite dimensional Banach spaces (see [7] and the references therein).
The controllability problem of nonlinear systems described by functional integrodifferential equations with nonlocal conditions in infinite dimensional Banach spaces, has been studied extensively by many authors, see for instance [8]-[16], [40, 55, 56, 69, 70, 83, 71], [77], and the references therein. For example in [77], the authors proved the controllability of an integrodifferential system with nonlocal conditions basing on the measure of noncompactness and the Sadovskii fixed-point theorem, and in [69], R. Atmania and S. Mazouzi have proved the controllability of a semilinear integrodifferential system using Schaefer fixed-point theorem and requiring the compactness of the semigroup.
In [69], the authors assumed the compactness of the operator semigroup and in [9], the authors assumed the compactness of the resolvent operator for integral equations, whereas in [16, 40, 77], the authors managed to drop this condition, in the same way as J. Wang, Z. Fan and Y. Zhou [40] have done for the nonlocal controllability of some semilinear dynamic systems with fractional derivative.
Many authors have also studied the controllability problem of nonlinear systems with delay in inifinite dimensional Banach spaces: see for instance [8], [55], [70], [83], [71], etc and the references therein. For example in [83], the authors proved the controllability of semilinear functional evolution equations with infinite delay using the nonlinear alternative of Leray-Schauder type. In [56], Meili Li, Miansen Wang and Fengqin Zhang proved the controllability of an impulsive functional differential system with finite delay using the Schaefer fixed-point. In [71], S. Selvi and M. Mallika Arjunan proved the controllability for impulsive differential systems with finite delay using

the Mönch fixed-point theorem, and in [8], K. Balachandran and R. Sakthivel studied the controllability of functional semilinear integrodifferential systems in Banach spaces using Schaefer fixed point theorem and assuming the compactness of the operator semigroup.
In this part of the thesis, motivated by the above works, we give sufficient conditions that guarantee the controllability of the following dynamical systems described by the following partial functional integrodifferential models:

- Partial functional integrodifferential equation subject to a nonlocal initial condition in a Banach space $(X, \ \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) \ + \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t,x(t)) + Cu(t), \\ \qquad\qquad \text{for } t \in I = [0,b] \\ \\ x(0) = x_0 + g(x), \end{cases} \tag{1.3.4}$$

  where $x_0 \in X, \ g : \mathcal{C}(I,X) \to X$ and $f : I \times X \to X$ are functions satisfying some conditions; $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on X; for $t \geq 0$, $B(t)$ is a closed linear operator with domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$. The control u belongs to $L^2(I,U)$ which is a Banach space of admissible controls, where U is a Banach space.

- Partial functional integrodifferential equation with finite delay in a Banach space $(X, \ \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) \ + \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t,x_t) + Cu(t), \\ \qquad\qquad \text{for } t \in I = [0,b] \\ \\ x_0 = \varphi \in \mathcal{C} = \mathcal{C}([-r,0];X), \end{cases} \tag{1.3.5}$$

  where $f : I \times \mathcal{C} \to X$ is a function satisfying some conditions; $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on X; for $t \geq 0$, $B(t)$ is a closed linear operator with domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$. The control u belongs to $L^2(I,U)$ which is a Banach space of admissible controls, where U is a Banach space, and x_t denotes the history function of $\mathcal{C}$ of the state from the time $t-r$ up to the present time t, and is defined by $x_t(\theta) = x(t+\theta)$ for $-r \leq \theta \leq 0$.

- Partial functional integrodifferential equation with infinite delay in a Banach space $(X, \ \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) \ + \displaystyle\int_0^t \gamma(t-s)x(s)ds \ + \ f(t, x_t) + Cu(t), \\ \qquad\qquad \text{for } t \in I = [0, b] \\ \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.3.6}$$

where $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space X; for $t \geq 0$, $\gamma(t)$ is a closed linear operator with domain $\mathcal{D}(\gamma(t)) \supset \mathcal{D}(A)$. The control u takes values from another Banach space U. The operator C belongs to $\mathcal{L}(U, X)$ which is the Banach space of bounded linear operators from U into X, and the phase space $\mathcal{B}$ is a linear space of functions mapping $]-\infty, 0]$ into X satisfying axioms which will be described later, for every $t \geq 0$, x_t denotes the history function of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t + \theta) \ \text{ for } -\infty \leq \theta \leq 0,$$

$f : I \times \mathcal{B} \to X$ is a continuous function satisfying some conditions.

To the best of our knowledge, up to now no work has reported neither on the controllability of partial functional integrodifferential equation with nonlocal initial condition (equation (1.3.4)), with finite delay (equation (1.3.5)) and with inifinite delay (equation (1.3.6)) in Banach spaces, using the resolvent operator approach. It has been an untreated topic in the literature, and this fact is the main aim and motivation of the present work.
Our approach constists in transforming the problem into a fixed-point problem of an appropriate operator and to apply the Mönch fixed-point theorem, making use of the Hausdorff measure of noncompactness, and without assuming the compactness of the resolvent operator for the associated linear integral part. The results obtained in this part improve, extend and complement many other important results in the litterature. They are summarized in chapters 3, 4 and 5.
Equations (1.3.4), (1.3.5) and (1.3.6) are models that arise in the analysis of heat conduction in materials with memory [85], and viscoelasticity. The control term $Cu(t)$ is the heating intensity and the integral part $\displaystyle\int_0^t B(t-s)x(s)ds$ is the memory of the system. Materials with memory are interesting because they act adaptively to their environment. They can be shaped easily into different forms at low temperature, but return to their

original forms on heating. Steering such systems from an initial state (initial condition) to a desired terminal one (boundary condition), by an appropriate choice of a control (which could be the heating intensity), is of interest to many scientists and engineers. The questions, whether one can heat the material in such a way that the initial state is transferred onto a desired state in time b and under which constraints on the control parameter u, are of interest.

1.4 Optimal Control of Dynamical Systems

In studying dynamical systems in order to improve the system behavior, one problem that usually surfaces is that of optimal control. In the simplest form, there is a given dynamical system (linear or nonlinear, discrete or continuous), described by an ordinary differential equation, a partial differential equation or a functional differential equation, for which input functions (controls or commands) can be specified. There is also an objective or a cost function whose value is determined by system behavior, and is in some sense a measure of the quality of that behavior. The optimal control problem is that of finding the input function (the control or the command) so as to optimize (maximize or minimize) the objective function. The problem seeks to optimize the objective function subject to the constraints construed by the model describing the evolution of the underlying system.
Before even precising mathematically this vocabulary, it is important to note that we all do (more or less) optimal control without even paying attention to it: to go from one place to another as fast as possible, or using the shortest path, to maximize the income of investments or shares or minimize debts are examples of problems of this type.
For example, the dynamical system might be a space vehicule with inputs corresponding to rocket thrust. The objective might then be to reach the moon with minimum expenditure of fuel. As another example, the system might be the nation's economy, with controls corresponding to governement monetary and fiscal policy. The objective might be to maximize the aggregate deviations of unemployment and interest rates from fixed target values. Finally, as a third example, which is of interest in this thesis, the system might represent the dynamics of heat flow in materials with memory, with controls corresponding to the heating intensities. The objective might be to minimize or maximize the heating intensity so as to obtain a particular form for the material.
The basic optimal control problem in continuous time is formulated as follows: one is interested in a system (a space vehicule, a nation's economy,

heat flow in materials with memory, etc...) whose state $x(t)$ (at time t) defined on a fixed time interval $0 \leq t \leq T$, is modelled by the solution of a differential equation:

$$\frac{dx}{dt} = \dot{x} = F(t, x, u) \text{ on } [0, T], \; x(0) = x_0, \tag{1.4.1}$$

where $x_0 \in \mathbb{R}^n$, and $T \in \mathbb{R}$ are fixed with $0 < T$; F is a given function from $\mathbb{R} \times \mathbb{R}^n \times U$ with values in $\mathbb{R}^n$. As we can see, one of the arguments of F is a function u defined on the interval $[0, T]$ with values in a given set U, which is the set of admissible controls. This function u (the control, or the command) translates mathematically the actions (or decisions) that one can exercise on the evolution of the system; the set U corresponds to the restrictions or the constraints that must be respected by the controls (for example: limited ressources, bounds on the acceleration or the speed for the driving of a vehicule or the heat for the heat flow in a material, etc...).
To formulate an optimal control problem, is to define the state of the system $x(t)$ and the differential system that describes its evolution, the class of admissible controls $u(t)$ and finally an *evolution criterion* or a *cost function*; most often, it is a criterion cumulated in time added to a final cost, whose typical form is:

$$J(u) = \int_0^T g(t, x(t), u(t))\, dt + h(x(T)) \tag{1.4.2}$$

where g and h are given functions, defined respectively in $\mathbb{R} \times \mathbb{R}^n \times U$ and $\mathbb{R}^n$ with values in $\mathbb{R}$. The problem to solve is then to determine the optimal cost and an optimal control, that is to solve the following optimization problem:

$$\begin{cases} \inf_{u(t)} \left(\int_0^T g(t, x(t), u(t))dt + h(x(T)) \right) \\ \frac{dx}{dt} = F(t, x, u) \text{ on } [0, T], \; x(0) = x_0, \end{cases} \tag{1.4.3}$$

Problem (1.4.3) is called *Bolza Problem*, and when $h \equiv 0$, we have a *Lagrange Problem*, which we are interested in in this thesis. The two popular solution techniques of an optimal control problem are Pontryagin's maximum principle and the Hamilton- Jacobi-Bellman equation [50]. In infinite dimension, methods and techniques of convex optimization are used. Optimal control has become a highly established research front in recent years with numerous contributions to the theory, in both deterministic and stochastic contexts. Its application to diverse fields such as biology, economics, ecology,

engineering, finance, management, and medicine cannot be overlooked (see, e.g., [49] and the references contained in it). The associated mathematical models are formulated, for example, as systems of ordinary, partial, delay, or stochastic differential and integrodifferential equations or discrete dynamical systems, for both scalar and multicriteria decision-making contexts.
Problems of existence of optimal controls for nonlinear differential systems have been studied extensively by many authors under various hypotheses see e.g.,[96], [97], [98], [87], [99], [95],[94], [88], [90], [89], and the references contained in them.
In [96], Wang *et al.* studied the existence and continuous dependence of mild solutions and the optimal controls of a Lagrange problem for some fractional integrodifferential equation with infinite delay in Banach spaces using the using the techniques of *a priori* estimation and extension of step by steps. Wand and Zhou [97] discussed the optimal controls of a Lagrange problem for fractional evolution equations. In [98], Wei *et al.* studied the optimal controls for nonlinear impulsive integrodifferential equations of mixed type on Banach spaces. In [99], the authors studied the existence of mild solutions and the optimal controls of a Lagrange problem for some impulsive fractional semilinear differential equations, using the techniques of *a priori* estimation.
In this part of the thesis, motivated by the above works, we establish the solvability and existence of optimal controls of the following dynamical systems described by the following partial functional integrodifferential models:

- Partial functional integrodifferential equation with finite delay in a Banach space $(X, \ \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) \ + \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t,x_t) + C(t)u(t), \\ \qquad\qquad \text{for } t \in I = [0,T], \\ \\ x_0 = \varphi \in \mathcal{C} = \mathcal{C}([-r,0];X), \end{cases} \tag{1.4.4}$$

  where $f : I \times \mathcal{C} \to X$ is a function satisfying some conditions; $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a separable reflexive Banach space X; for $t \geq 0$, $B(t)$ is a closed linear operator with domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$. The control u takes values from another separable reflexive Banach space U. The operator $C(t)$ belongs to $\mathcal{B}(U,X)$ which is the Banach space of bounded linear operators from U into X, and $\mathcal{C}([-r,0],X)$ denotes the Banach space of continuous functions $x : [-r,0] \to X$ with supremum norm $\|x\|_\infty = \sup_{t\in I} \|x(t)\|_X$, x_t denotes the history function of $\mathcal{C}$ of the state from the time $t-r$ up to the present time t, and is defined by

$x_t(\theta) = x(t+\theta)$ for $-r \leq \theta \leq 0$.

- Partial functional integrodifferential equation with infinite delay in a Banach space $(X, \ \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) \ + \displaystyle\int_0^t \gamma(t-s)x(s)ds \ + \ f(t,x_t) + C(t)u(t), \\ \qquad\qquad \text{for } t \in I = [0,b] \\ \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.4.5}$$

where $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a separable reflexive Banach space X; for $t \geq 0$, $\gamma(t)$ is a closed linear operator with domain $\mathcal{D}(\gamma(t)) \supset \mathcal{D}(A)$. The control u takes values from another separable reflexive Banach space U. The operator $C(t)$ belongs to $\mathcal{L}(U,X)$ which is the Banach space of bounded linear operators from U into X, and the phase space $\mathcal{B}$ is a linear space of functions mapping $]-\infty, 0]$ into X satisfying axioms which will be described later, for every $t \geq 0$, x_t denotes the history function of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t+\theta) \ \text{ for } \ -\infty \leq \theta \leq 0,$$

$f : I \times \mathcal{B} \to X$ is a continuous function satisfying some conditions.

- Partial functional integrodifferential equation subject to Cauchy initial condition in a Banach space $(X, \ \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) \ + \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t,x(t)) + C(t)u(t), \\ \qquad\qquad \text{for } t \in I = [0,b] \\ \\ x(0) = x_0 \in X \end{cases} \tag{1.4.6}$$

where $x_0 \in X$ and $f : I \times X \to X$ is a function satisfying some conditions; $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a separable reflexive Banach space X; for $t \geq 0$, $B(t)$ is a closed linear operator with domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$. The control u takes values from another separable reflexive Banach space U. The operator $C(t)$ belongs to $\mathcal{L}(U,X)$ which is the Banach space of bounded linear operators from U into X.

Little is known and done about the existence of optimal controls for integrodifferential equations with delay using the resolvent operator for integral equations approach, especially the problem of existence of optimal controls of a Lagrange problem for equations (1.4.4), (1.4.5) and (1.4.6) have been an untreated topic in the literature, and this motivates the present work. We tackle this problem by using the techniques of *a priori* estimation of mild solutions, giving by the resolvent operator for the associated linear integral part, and techniques of convex optimization. The results obtained in this part complement many other important results in the literature. They are summarized in chapters 6, 7 and 8.

1.5 Methods

In [68], Quinn and Carmichael proved that a controllability problem can be converted into a fixed point problem. So our approach constists in transforming the problems (1.3.4), (1.3.5) and (1.3.6) into fixed-point problems of appropriate operators and to apply the Hausdorff measure of noncompactness and the Mönch fixed-point theorem, without requiring the compactness of the operator semigroup, in the same way as J. Wang, Z. Fan and Y. Zhou [40] have done for the nonlocal controllability of some semilinear dynamic systems with fractional derivative. This method enables us overcome the resolvent operator case considered in this thesis. In contrary to the evolution semigroup case considered in [55, 71], here the semigroup property can not be used because resolvent operators in general do not form semigroups. As for the existence of optimal controls of the associated Lagrange problems to equations (1.4.4), (1.4.5) and (1.4.6), we apply techniques of convex optimization together with Balder's Theorem to obtain our results.

1.5.1 Measures of Noncompactness

Measures of noncompactness are very useful tools in nonlinear analysis. For instance, in metric fixed point theory and in the theory of operator equations in Banach spaces. They are also used in the studies of functional equations, ordinary and partial differential equations, fractional partial differential equations, integral and integrodifferential equations, optimal control theory, and in the characterizations of compact operators between Banach spaces (see e.g., [6, 11, 12, 67]). The concept of measure of noncompactness was first defined and studied by Kuratowski [48] in 1930. In 1955, G. Darbo [21] used it to prove his fixed point theorem. We have the following definition:

Definition 1.5.1 *Let E^+ be the positive cone of an order Banach space $(E, \preceq)$. That is*

$$E^+ = \left\{ x \in E : \quad 0 \preceq x \right\}.$$

And let X be an arbitrary Banach space. A function Φ defined on the set of all bounded subsets of X with values in E^+ is called a measure of noncompactness (MNC) on X if $\Phi(\overline{co}(D)) = \Phi(D)$ for every bounded subset $D \subset X$, where $\overline{co}(D)$ stands for the closed convex hull of D.

Example 1.5.2 *([6], Example 1, p. 19]) Let X be an arbitrary metric space and $\mathcal{M}_X$ denote the set of all bounded subsets of X. Then the map $\Phi_1 : \mathcal{M}_X \to [0, \infty)$ defined by*

$$\Phi_1(D) = \begin{cases} 0, & \text{if } D \text{ is relatively compact} \\ \\ 1, & \text{otherwise.} \end{cases}$$

is a measure of noncompactness, the so-called discrete measure of noncompactness.

Two measures of noncompactness which are frequently used in many branches of nonlinear analysis and its applications, are Kuratowski measure of noncompactness and the Hausdorff measure of noncompactness, which we shall use in this thesis. The frequent use of this latter one is due to the fact that it is defined in a natural way and has several useful properties.

Definition 1.5.3 *Let D be a bounded subset of a normed space Z. The Kuratowski measure of noncompactness of D (shortly MNC) is defined by*

$$\alpha(D) = \inf \Big\{ \epsilon > 0 : D \; has \; a \; finite \; cover \; by \; sets \; of \; diameter \; less \; than \; \epsilon \Big\},$$

$$= \inf \Big\{ \epsilon > 0 : D \subset \bigcup_{k=1}^{n} S_k, \; S_k \subset Z, \; diam(S_k) < \epsilon \; (k = 1, 2, \cdots, n \in \mathbb{N}) \Big\}.$$

Definition 1.5.4 *Let D be a bounded subset of a normed space Z. The Hausdorff measure of noncompactness of D (shortly MNC) is defined by*

$$\beta(D) = \inf \Big\{ \epsilon > 0 : D \; has \; a \; finite \; cover \; by \; balls \; of \; radius \; less \; than \; \epsilon \Big\}$$

$$= \inf \Big\{ \epsilon > 0 : D \subset \bigcup_{k=1}^{n} B(x_k, r_k), \; x_k \in Z, \; r_k < \epsilon \; (k = 1, 2, \cdots, n \in \mathbb{N}) \Big\}.$$

The compactness conditions described by means of measures of noncompactness are useful in showing the existence of solutions for differential and integral equations in Banach spaces. Measures of noncompactness have applications in many fields where loss of compactness arises. For example, integral equations with strongly singular kernels, differential equations over unbounded domains, functional differential equations of neutral type or with deviating argument, linear differential operators with nonempty essential spectrum, nonlinear superposition operators between various function spaces, initial value problems in Banach spaces etc, see e.g.,[1, 11, 4, 25] and the references therein.

1.5.2 Fixed Point Theory

The fixed point technique is one of the useful methods mainly applied in the existence and uniqueness of solutions of differential equations and the controllability of dynamical systems. One of the main branches of fixed point theory deals with the topological properties of the operators involved. With respect to the topological aspect, the two main theorems are Brouwer's theorem and its infinite dimensional version, Schauder's fixed point theorem. In both theorems, compactness plays an essential role. In 1955, Darbo [21] extended Schauder's theorem to the setting of noncompact operators, introducing the notion of k-set contraction which is closely related to the idea of measures of noncompactness. In 1967, Sadovskii [26] gave a fixed point result more general than Darbo's theorem using the concept of condensing map. Thus, the fixed point theory for condensing mappings allows us to obtain a relationship between the two theories. In 1980, Mönch [62] gave a fixed point theorem for maps between Banach spaces, which extends the Schauder and more generally, Sadovskii fixed point theorems.
In this thesis, we use the Mönch fixed point theorem and the Hausdorff measure of noncompactness to prove the controllability results for partial functional integrodifferential equations. The advantage of using this fixed point theorem is to weaken the compactness assumption of the operator semigroup and the resolvent operator for integral equations.

1.5.3 Semigroup Theory

The theory of semigroups of bounded linear operators is part of functional analysis. It is an extensive mathematical subject with substantial applications to many fields of analysis, and has developed quite rapidly since the discovery of the generation theorem by Hille and Yosida in 1948. Semigroups are a powerful tool in solving evolution equations. They give the variation of parameter formula for mild solutions. In [66], Pazy discussed the existence and uniqueness of mild, strong and classical solutions of evolution equations using semigroup theory and fixed point theorems.

1.5.4 Resolvent Operator for Integral Equations

This concept was first introduced in 1982 by Ronald Grimmer [85]. He proved the existence and uniqueness of resolvent operators for integrodifferential equations that give the variation of parameter formula for the solution. Resolvent operators in general do not form semigroup, and are a better tool in solving integrodifferential equations than semigroups. In [24], W. Desch, R. Grimmer and W. Schappacher proved the equivalence of the compactness of the resolvent operator for integral equations and that of the operator semigroup. In this thesis, we show a similar result on the equivalence between the operator-norm continuity of the resolvent operator for integral equations and the C_0-semigroup. This result is important because it allows to drop the compactness assumption on the resolvent operator and assume its operator-norm continuity, in proving our controllability results. In [69], the authors assumed the compactness of the operator semigroup and in [9], the authors assumed the compactness of the resolvent operator whereas in [16, 40, 77], the authors managed to drop this condition which motivates our current work. Our contributions in this direction are summerized in Chapters 3 and 4.

1.6 Organization of the Thesis

The present study in this thesis deals firstly with the fixed point approach via measures of noncompactness for proving controllability results for partial functional integrodifferential systems in Banach spaces.
We note that the partial functional integrodifferential equations (1.3.4), (1.3.5) and (1.3.6) have not been investigated (to the best of our knowledge) for controllability.
The second part of the thesis deals with the existence of optimal controls of the associated Lagrange problems to equations (1.4.4), (1.4.5) and (1.4.6), we apply techniques of convex optimization together with Balder's Theorem.
In chapter 2, we give preliminary results that will be used in proving our main results. In particular, we establish the equivalence between operator-norm continuity of the semigroup $\big(T(t)\big)_{t\geq 0}$ generated by A and the resolvent operator $\big(R(t)\big)_{t\geq 0}$ corresponding to the associated linear equation.

In chapter 3 , we prove the controllability result for some partial functional integrodifferential equation with nonlocal initial conditions in Banach spaces (equation (1.3.4)), by using resolvent operators for integral equations and the Mönch fixed point theorem.

In chapter 4, we establish a controllability result for some partial functional integrodifferential equation with finite delay in Banach spaces (equation (1.3.5)), by using resolvent operators for integral equations and the Mönch fixed point theorem.

In chapter 5, we establish a controllability result for some partial functional integrodifferential equation with infinite delay in Banach spaces (equation (1.3.6)), by using resolvent operators for integral equations and the Mönch fixed point theorem.

In chapter 6, we prove the solvability and the existence of optimal controls for some partial functional integrodifferential equation with finite delay in Banach spaces (equation (1.4.4)), by using the techniques of convex optimization, a-priori estimation and contraction mapping principle.

In Chapter 7 , we prove the solvability and the existence of optimal controls for some partial functional integrodifferential equation with infinite delay in Banach spaces (equation (1.4.5)), by using the techniques of convex optimization, a-priori estimation and contraction mapping principle.

In Chapter 8, we prove the solvability and the existence of optimal controls for some partial functional integrodifferential equation with classical initial conditions in Banach spaces (equation (1.4.6)), by using the techniques of convex optimization, a-priori estimation and contraction mapping principle.

CHAPTER 2

Preliminaries

In this Chapter, we give some definitions and preliminary results that will be needed in the subsequent chapters.

2.1 Semigroups

Definition 2.1.1 *Let X be a Banach space. A family $\big(T(t)\big)_{t\geq 0}$ of bounded linear operators from X to X is called a strongly continuous semigroup of bounded linear operators if the following three conditions are satisfied :*

i) $T(0) = Id_X$

ii) $T(t+s) = T(t)T(s), \quad \forall t, s \geq 0$

iii) $\forall x \in X$, *the map* $t \mapsto T(t)x \in X$ *defined from* $[0, +\infty)$ *into* X *is continuous at the right of* 0.

A strongly continuous semigroup of bounded linear operators on X will be called a C_0-semigroup.

If $\lim_{t\to 0^+} \|T(t) - I\| = 0$, then we say that the *semigroup is uniformly continuous.*
Example of Semigroup:
Let's define

$$\begin{array}{rccl} T: & [0,\infty) & \longrightarrow & \mathcal{B}\big(\mathcal{C}_0(\mathbb{R})\big) \\ & t & \mapsto & T(t) \end{array}$$

which at each $f \in \mathcal{C}_0(\mathbb{R})$ assigns $T(t)f \in \mathcal{C}_0(\mathbb{R})$ defined by

$$T(t)f \;:\; x \;\mapsto\; f(t+x)\ .$$

Then the family of operators $(T(t))_{t\geq 0}$ is a strongly continuous semigroup.
Justification:
Clearly T is well-defined and (i) For all $f \in \mathcal{C}_0(\mathbb{R})$ and for all $x \in \mathbb{R}$, we have

$$(T(0)f)(x) \;=\; f(0+x) \;=\; f(x).$$

Thus $T(0)f = f$ and so $T(0) = I$
(ii) Let $s,\, t \in [0, \infty)$, $f \in \mathcal{C}_0(\mathbb{R})$ and $x \in \mathbb{R}$. Then we have that

$$\begin{aligned}\big((T(s)T(t))f\big)(x) &= \Big[T(s)\big(T(t)f\big)\Big](x)\\ &= \big(T(t)f\big)(s+x)\\ &= f(t+s+x)\\ &= f(s+t+x)\\ &= \big(T(s+t)f\big)(x)\end{aligned}$$

which shows that $T(s)T(t) = T(s+t)$.
(iii) We now show the continuity of $t \mapsto T(t)f$ at the right of 0. Let $f \in \mathcal{C}_0(\mathbb{R})$, then

$$\begin{aligned}\|T(t)f - f\|_{\mathcal{C}_0(\mathbb{R})} &= \sup_{x\in\mathbb{R}}\big|\big(T(t)f\big)(x) - f(x)\big|\\ &= \sup_{x\in\mathbb{R}}|f(t+x) - f(x)| \quad \longrightarrow \quad 0 \quad \text{as} \quad t \longrightarrow 0^+\end{aligned}$$

by uniform continuity of f.
Thus

$$\lim_{t\longrightarrow 0^+} \|T(s)f - f\|_{\mathcal{C}_0(\mathbb{R})} = 0.$$

And therefore $(T(s))_{s\geq 0}$ defined above is a strongly continuous semigroup.

Infinitesimal Generator of a C_0-semigroup

Definition 2.1.2 *The infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ of bounded linear operators of a Banach space X, is the linear operator $A : \mathcal{D}(A) \longrightarrow X$ with domain*

$$\mathcal{D}(A) \;=\; \left\{x \in X :\; \lim_{t\to 0^+} \frac{T(t)x - x}{t} \quad exists\right\}$$

and defined by

$$Ax \;:=\; \lim_{t\to 0^+} \frac{T(t)x - x}{t} \;=\; \frac{d^+T(t)x}{dt}\Big|_{t=0} \qquad for \quad x \in \mathcal{D}(A)\,.$$

2.2 Resolvent Operator for Integral Equations

Let $I = [0, b]$, $b > 0$ and let X be a Banach space. A measurable function $x : I \to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable. We denote by $L^1_B(I, X)$ the Banach space of functions $x : I \to X$ which are Bochner integrable normed by

$$\|x\|_{L^1} = \int_0^b \|x(t)\| dt.$$

Consider the following linear homogeneous equation:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds \quad \text{for} \quad \text{t} \geq 0 \\ x(0) = x_0 \in X. \end{cases} \tag{2.2.1}$$

where A and $B(t)$ are closed linear operators on a Banach space X.
In the sequel, we assume A and $\big(B(t)\big)_{t\geq 0}$ satisfy the following conditions:

$(\mathbf{H_1})$ A is a densely defined closed linear operator in X. Hence $\mathcal{D}(A)$ is a Banach space equipped with the graph norm defined by, $|y| = \|Ay\| + \|y\|$ which will be denoted by $(X_1, |\cdot|)$.

$(\mathbf{H_2})$ $\big(B(t)\big)_{t\geq 0}$ is a family of linear operators on X such that $B(t)$ is continuous when regarded as a linear map from $(X_1, |\cdot|)$ into $(X, \|\cdot\|)$ for almost all $t \geq 0$ and the map $t \mapsto B(t)y$ is measurable for all $y \in X_1$ and $t \geq 0$, and belongs to $W^{1,1}(\mathbb{R}^+, X)$. Moreover there is a locally integrable function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|B(t)y\| \;\leq\; \rho(t)|y| \quad \text{and} \quad \left\|\frac{d}{dt}B(t)y\right\| \;\leq\; \rho(t)|y| \;.$$

Remark 2.2.1 *Note that* $(\mathbf{H_2})$ *is satisfied in the modelling of heat conduction in materials with memory and viscosity. More details can be found in [106].*

Definition 2.2.2 *(see e.g., [83]) A resolvent operator for equation* (2.2.1) *is a family* $\big(R(t)\big)_{t\geq 0}$ *of bounded linear operators valued function*

$$R : \; [0, +\infty) \;\longrightarrow\; \mathcal{B}(X)$$

such that

(i) $R(0) = Id_X$ *and* $\|R(t)\| \leq Ne^{\beta t}$ *for some constants* N *and* β.

(ii) *For all* $x \in X$, *the map* $t \mapsto R(t)x$ *is continuous for* $t \geq 0$.

(iii) *Moreover for* $x \in X_1$, $R(\cdot)x \in \mathcal{C}^1(\mathbb{R}^+; X) \cap \mathcal{C}(\mathbb{R}^+; X_1)$ *and*

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds. \end{aligned}$$

Observe that the map defined on $\mathbb{R}^+$ by $t \mapsto R(t)x_0$ solves equation (2.2.1) for $x_0 \in \mathcal{D}(A)$.

We have the following example of a resolvent operator for equation (2.2.1) in $\mathbb{R}$.

Example([24]) Let $X = \mathbb{R}$, $Ay = 2y$, and $B(t)y = -2y$ in (2.2.1). Then in that case, we have

$$R(t)x_0 = e^t(\cos t + \sin t)x_0 \quad \text{and} \quad T(t)x_0 = e^{2t}x_0.$$

Remark 2.2.3 *The above example also shows that, in general, the resolvent operator* $\big(R(t)\big)_{t\geq 0}$ *for equation* (2.2.1) *does not satisfy the semigroup law, namely,*

$$R(t+s) \neq R(t)R(s) \qquad \textit{for some } t,\, s > 0\,.$$

In this chapter, we establish the equivalence between operator-norm continuity of the semigroup $\big(T(t)\big)_{t\geq 0}$ generated by A and the resolvent operator $\big(R(t)\big)_{t\geq 0}$ corresponding to the linear equation (2.2.1). We shall then assume the operator-norm continuity of $\big(R(t)\big)_{t\geq 0}$, to prove the controllability of equations (1.3.4), (1.3.5) and (1.3.6).

Now consider the following system:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t \left[B_1(t-s) + B_2(t-s)\right] x(s)ds \;\; \text{for t} \geq 0 \\ \\ x(0) = x_0 \in X, \end{cases} \tag{2.2.2}$$

where $B_1(t)$ and $B_2(t)$ are closed linear operators in X and satisfy $(\mathbf{H_2})$. Then we have the next Lemma coming from [24].

Lemma 2.2.4 *(Perturbation result)([24]) Suppose A satisfies $(\mathbf{H_1})$ and $\big(B_1(t)\big)_{t\geq 0}$ and $\big(B_2(t)\big)_{t\geq 0}$ satisfy $(\mathbf{H_2})$. Let $\big(R_{B_1}(t)\big)_{t\geq 0}$ be a resolvent operator of equation (2.2.1) and $\big(R_{B_1+B_2}(t)\big)_{t\geq 0}$ be a resolvent operator of equation (2.2.2). Then*

$$R_{B_1+B_2}(t)x \;-\; R_{B_1}(t)x \;=\; \int_0^t R_{B_1}(t-s)Q(s)x\,ds$$

where the operator Q is defined by

$$Q(t)x \;=\; \int_0^t B_2'(t-s)\int_0^s R_{B_1+B_2}(\tau)x\,d\tau\,ds \;+\; B_2(0)\int_0^t R_{B_1+B_2}(s)x\,ds,$$

is uniformly bounded on bounded intervals, and for each $x \in X$, $Q(\cdot)x$ belongs to $\mathcal{C}([0,\infty),X)$.

Based on this and the following corollary from ([24], p.224), we prove the operator-norm continuity of the resolvent operator $\big(R(t)\big)_{t\geq 0}$ for $t > 0$.

Corollary 2.2.5 *([24]) Let A be a closed, densely defined linear operator in X, $B(t) = 0$ for all $t \geq 0$, and $\big(R(t)\big)_{t\geq 0}$ be a resolvent operator for equation (2.2.1). Then $\big(R(t)\big)_{t\geq 0}$ is a C_0-semigroup with infinitesimal generator A.*

Theorem 2.2.6 *Let A be the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ and let $\big(B(t)\big)_{t\geq 0}$ satisfy $(\mathbf{H_2})$. Then the resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (2.2.1) is operator-norm continuous (or continuous in the uniform operator topology) for $t > 0$ if and only if $\big(T(t)\big)_{t\geq 0}$ is operator-norm continuous for $t > 0$.*

Proof. Since $\big(T(t)\big)_{t\geq 0}$ is a C_0-semigroup, there exist M and ω such that

$$||T(t)|| \;\leq\; Me^{\omega t} \qquad \text{for all } t \geq 0.$$

Let

$$Q(t)x \;=\; B(0)\int_0^t R(s)x\,ds \;+\; \int_0^t B'(t-s)\int_0^s R(\tau)x\,d\tau\,ds.$$

By Lemma 2.2.4 (applied to equation (2.2.1)) we have that the operators $\big(Q(t)\big)_{t\geq 0}$ are uniformly bounded for t in a bounded interval and that

$$R(t)x \;=\; T(t)x \;+\; \int_0^t T(t-s)Q(s)x\,ds.$$

Now set $\alpha = \sup\limits_{0\leq t\leq b} \|Q(t)\|$ and let $x \in X$ be arbitrary and such that $\|x\| \leq 1$. Suppose first that $\big(T(t)\big)_{t\geq 0}$ is operator-norm continuous for $t > 0$. Then on the one hand, we have for every $t \in (0, b)$ and $h \in (0, b-t)$ that :

$$\begin{aligned}\Big\|R(t+h)x - R(t)x\Big\| \quad &\leq \quad \Big\|\Big[T(t+h) - T(t)\Big]x\Big\| \\ &+ \quad \alpha\int_0^t \Big\|\Big[T((t-s)+h) - T(t-s)\Big]x\Big\|\, ds \\ &+ \quad \alpha\int_t^{t+h} Me^{\omega(t-s+h)}\, ds.\end{aligned}$$

It follows that

$$\begin{aligned}\Big\|R(t+h) - R(t)\Big\| \quad &\leq \quad \Big\|T(t+h) - T(t)\Big\| + \alpha\int_0^t \Big\|T((t-s)+h) - T(t-s)\Big\|\, ds \\ &+ \quad \alpha\int_t^{t+h} Me^{\omega(t-s+h)}\, ds.\end{aligned}$$

Therefore by using the operator-norm continuity of $\big(T(t)\big)_{t\geq 0}$ on $(0, +\infty)$ and the Lebesgue dominated convergence theorem, we deduce that

$$\Big\|R(t+h) - R(t)\Big\| \;\to\; 0 \quad \text{as} \quad h \to 0^+.$$

On the other hand, for every $t \in (0, b]$ and $h \in (-t, 0)$, we have that

$$\begin{aligned}\Big\|R(t+h) - R(t)\Big\| \quad &\leq \quad \Big\|T(t+h) - T(t)\Big\| + \alpha\int_0^t \Big\|T((t-s)+h) - T(t-s)\Big\|\, ds \\ &+ \quad \alpha\int_{t+h}^{t} Me^{\omega(t-s)}\, ds.\end{aligned}$$

And using the operator-norm continuity of $\big(T(t)\big)_{t\geq 0}$ and the Lebesgue dominated convergence theorem, we have that

$$\Big\|R(t+h) - R(t)\Big\| \;\to\; 0 \qquad \text{as} \quad h \to 0^-.$$

Hence

$$\Big\|R(t+h) - R(t)\Big\| \;\to\; 0 \qquad \text{when} \quad h \to 0 \quad \text{with} \quad t+h \in [0, b].$$

Thus $\big(R(t)\big)_{t\geq 0}$ is operator-norm continuous when $\big(T(t)\big)_{t\geq 0}$ is operator-norm continuous.

The converse is proved similarly by using the identities

$$T(t)x \;=\; -R(t)x + \int_0^t T(t-s)Q(s)x\,ds \;=\; -R(t)x + \int_0^t T(s)Q(t-s)x\,ds$$

and the continuity of $\big(Q(t)\big)_{t\geq 0}$ that will follow from the property $(\mathbf{H_2})$ of $\big(B(t)\big)_{t\geq 0}$ and the continuity of $\big(R(t)\big)_{t\geq 0}$.
In fact suppose that $\big(R(t)\big)_{t\geq 0}$ is operator-norm continuous for $t > 0$. Then on the one hand, we have for every $t \in (0, b)$ and $h \in (0, b-t)$ that :

$$\begin{aligned}\Big\|T(t+h)x - T(t)x\Big\| &\leq \Big\|\Big[R(t+h) - R(t)\Big]x\Big\| \\ &+ \int_0^t Me^{\omega s}\Big\|\Big[Q((t-s)+h) - Q(t-s)\Big]x\Big\|\,ds \\ &+ \alpha\int_t^{t+h} Me^{\omega s}\,ds.\end{aligned}$$

It follows that

$$\begin{aligned}\Big\|T(t+h) - T(t)\Big\| &\leq \Big\|R(t+h) - R(t)\Big\| \\ &+ \int_0^t Me^{\omega s}\Big\|Q((t-s)+h) - Q(t-s)\Big\|\,ds \\ &+ \alpha\int_t^{t+h} Me^{\omega s}\,ds.\end{aligned}$$

Therefore by using the operator-norm continuity of $\big(R(t)\big)_{t\geq 0}$ on $(0,+\infty)$, the continuity of $\big(Q(t)\big)_{t\geq 0}$ and the Lebesgue dominated convergence theorem, we have that

$$\Big\|T(t+h) - T(t)\Big\| \;\to\; 0 \quad \text{as} \quad h \to 0^+.$$

On the other hand, for every $t \in (0, b]$ and $h \in (-t, 0)$, we have that

$$\begin{aligned}\Big\|T(t+h)x - T(t)x\Big\| &\leq \Big\|\Big[R(t+h) - R(t)\Big]x\Big\| \\ &+ \int_0^t Me^{\omega s}\Big\|\Big[Q((t-s)+h) - Q(t-s)\Big]x\Big\|\,ds \\ &+ \alpha\int_{t+h}^{t} Me^{\omega s}\,ds.\end{aligned}$$

It follows that

$$\begin{aligned}\Big\|T(t+h)-T(t)\Big\| &\leq \Big\|R(t+h)-R(t)\Big\| \\ &+ \int_0^t Me^{\omega s}\Big\|Q((t-s)+h)-Q(t-s)\Big\|\,ds \\ &+ \alpha\int_{t+h}^t Me^{\omega s}\,ds.\end{aligned}$$

Therefore by using the operator-norm continuity of $\big(R(t)\big)_{t\geq 0}$ on $(0,+\infty)$, the continuity of $\big(Q(t)\big)_{t\geq 0}$ and the Lebesgue dominated convergence theorem, we have that

$$\Big\|T(t+h)-T(t)\Big\| \;\to\; 0 \quad \text{as} \quad h\to 0^-.$$

Hence,

$$\Big\|T(t+h)-T(t)\Big\| \;\to\; 0 \quad \text{as} \quad h\to 0 \text{ with } \quad t+h\in[0,b].$$

Thus $\big(T(t)\big)_{t\geq 0}$ is operator-norm continuous when $\big(R(t)\big)_{t\geq 0}$ is.

□

2.3 Measure of Noncompactness

For proving the main results in the first part of the thesis, we recall the notion of measure of noncompactness and the Mönch fixed-point theorem. For more information on this, the reader can see [12]. In order to use the Hausdorff measure of noncompactness, we recall some properties related to this concept.

Definition 2.3.1 *Let D be a bounded subset of a normed space Z. The Hausdorff measure of noncompactness (shortly MNC) is defined by*

$$\beta(D)=\inf\Big\{\epsilon>0: D\ has\ a\ finite\ cover\ by\ balls\ of\ radius\ less\ than\ \epsilon\Big\}.$$

Theorem 2.3.2 *([12])*

Let $D,\ D_1,\ D_2$ be bounded subsets of a Banach space Z. The Hausdorff MNC has the following properties:

(i) *If $D_1\subset D_2$, then $\beta(D_1)\leq\beta(D_2)$, (monotonicity).*

(ii) $\beta(D) = \beta(\overline{D})$.

(iii) $\beta(D) = 0$ *if and only if* D *is relatively compact.*

(iv) $\beta(\lambda D) = |\lambda|\beta(D)$ *for any* $\lambda \in \mathbb{R}$, *(Homogeneity)*

(v) $\beta(D_1 + D_2) \leq \beta(D_1) + \beta(D_2)$, *where* $D_1 + D_2 = \{d_1 + d_2 : \ d_1 \in D_1, \ d_2 \in D_2\}$, *(subadditivity)*

(vi) $\beta(\{a\} \cup D) = \beta(D)$ *for every* $a \in Z$.

(vii) $\beta(D) = \beta(\overline{co}(D))$, *where* $\overline{co}(D)$ *is the closed convex hull of* D.

(viii) *For any map* $G : \mathcal{D}(G) \subseteq X \to Z$ *which is Lipschitz continuous with a Lipschitz constant* k, *we have*

$$\beta(G(D)) \ \leq \ k\beta(D),$$

for any subset $D \subseteq \mathcal{D}(G)$.

(xi) $\beta(D) = \inf\{d_Z(D, E);\ E \subseteq Z$ *is relatively compact*$\} = \inf\{d_Z(D, E);\ E \subseteq Z$ *is finite valued*$\}$, *where* $d_Z(D, E)$ *means the nonsymmetric (or symmetric) Hausdorff distance between* D *and* E *in* Z.

(x) *If* $\{D_n\}_{n=1}^{\infty}$ *is a decreasing sequence of bounded closed nonempty subsets of* Z *and* $\lim_{n\to\infty} \beta(D_n) = 0$, *then,* $\bigcap_{n=1}^{\infty} D_n$ *is nonempty and compact in* Z.

Lemma 2.3.3 *([40, 67]) If* $(u_n)_{n\geq 1}$ *is a sequence of Bochner integrable functions from* $I = [a, b]$ *into a Banach space* Z *with the estimation* $\|u_n(t)\| \leq \mu(t)$ *for almost all* $t \in I$ *and every* $n \geq 1$, *where* $\mu \in L^1(I, \mathbb{R})$, *then the function*

$$\psi(t) \ = \ \beta(\{u_n(t) : n \geq 1\})$$

belongs to $L^1(I, \mathbb{R})$ *and satisfies the following estimation*

$$\beta\left(\left\{\int_a^t u_n(s)ds \ : \ n \geq 1\right\}\right) \ \leq \ 2\int_a^t \psi(s)ds,$$

for all $t \in [a, b]$

Lemma 2.3.4 *([12]) If $W \subseteq \mathcal{C}([a,b],X)$ is bounded and equicontinuous, then $\beta(W(t))$ is continuous and*

$$\beta\left(\int_a^t W(s)\,ds\right) \leq \int_a^t \beta(W(s))\,ds,$$

for all $t \in [a,b]$, where $\displaystyle\int_a^t W(s)\,ds = \left\{\int_a^t w(s)\,ds : w \in W\right\}$

Lemma 2.3.5 *Let Z be a Banach space and $(T_n)_{n\geq 1}$ be a sequence of bounded linear maps on Z converging pointwise to $T \in \mathcal{B}(Z)$. Then for any compact set K in Z, T_n converges to T uniformly in K, namely,*

$$\sup_{x\in K} \|T_n(x) - T(x)\| \longrightarrow 0, \text{ as } n \to +\infty.$$

Proof. By the uniform boundedness principle, we have that $\sup\limits_{n\geq 1} \|T_n\| < \infty$. Let $M = \sup\limits_{n\geq 1} \|T_n\|$ and $\epsilon > 0$ be arbitrarily given. Then there exist $a_1,\ a_2, \cdots,\ a_m$ such that $K \subset \displaystyle\bigcup_{i=1}^m B\Big(a_i, \frac{\epsilon}{2(M+1)}\Big)$.
Also, for any $x \in K$, there exists $i \in \{1, \cdots, m\}$ such that $x \in B\Big(a_i, \frac{\epsilon}{2(M+1)}\Big)$.
Since for $i = 1,\ 2,\ \cdots,\ m$,
$T_n(a_i) \to T(a_i)$, there exists $N_\epsilon > 0$ such that

$$\|T_n(a_i) - T(a_i)\| \leq \frac{\epsilon}{M+1}, \text{ for } n \geq N_\epsilon \text{ and for any } i = 1, \cdots, m.$$

We have that

$$\begin{aligned}
\|T_n(x) - T(x)\| &\leq \|T_n(x) - T_n(a_i)\| + \|T_n(a_i) - T(a_i)\| + \|T(a_i) - T(x)\| \\
&\leq \|T_n\|\|x - a_i\| + \|T\|\|a_i - x\| + \|T_n(a_i) - T(a_i)\| \\
&\leq \frac{\epsilon M}{M+1} + \|T_n(a_i) - T(a_i)\| \\
&\leq \frac{\epsilon M}{M+1} + \frac{\epsilon}{M+1} = \epsilon
\end{aligned}$$

Therefore $\sup\limits_{x\in K} \|T_n(x) - T(x)\| \to 0$ as $n \to +\infty$.

□

2.4 The Mönch Fixed Point Theorem and Balder's Theorem

We recall a nonlinear alternative of Mönch's type for non selfmaps.

Theorem 2.4.1 *[62](Mönch, 1980) Let G be an open neighborhood of the origin in a Banach space Z. Suppose that $F : \overline{G} \to Z$ is a continuous map which satisfies the following conditions:*

(i) $D \subset \overline{G}$ *countable and* $D \subseteq \overline{co}(\{0\} \cup F(D)) \quad \Longrightarrow \quad \overline{D}$ *is compact.*

(ii) $F(x) \neq \lambda x \quad$ *for all* $x \in \partial G$ *and* $\lambda > 1$, *(Leray-Schauder condition).*

Then F has a fixed point.

We now state the following nonlinear alternative of Mönch's type for self-maps, which we shall use in the proof of the controllability results.

Theorem 2.4.2 *[62](Mönch, 1980) Let $\mathcal{K}$ be a closed and convex subset of a Banach space Z and $0 \in \mathcal{K}$. Assume that $F : \mathcal{K} \to \mathcal{K}$ is a continuous map which satisfies Mönch's condition, namely,*
let $D \subseteq \mathcal{K}$ be countable and $D \subseteq \overline{co}(\{0\} \cup F(D))$, then $\overline{D}$ is compact.
Then F has a fixed point.

The following Theorem is needed in the proof of the existence of optimal controls.

Theorem 2.4.3 *(Balder's Theorem, [107]) Let $(\Sigma, \mathcal{F}, \mu)$ be a finite nonatomic measure space, $(X, \|\cdot\|)$ a separable Banach space, and $(V, |\cdot|)$ a separable reflexive Banach space, and V' its dual. Let $\theta : \Sigma \times X \times V \to (-\infty, +\infty]$ be a given $\mathcal{F} \times \mathcal{L}(X \times V)$- measurable function. The associated integral functional $I_\theta : L^1_X \times L^1_V \to [-\infty, +\infty]$ is defined by:*

$$I_\theta(x, v) = \int_\Sigma \theta(t, x(t), v(t))\mu(dt),$$

where L^1_X denotes the set of all absolutely summable functions from Σ to X. The following three conditions

(i) $\theta(t, \cdot, \cdot)$ *is sequencially lower semicontinuous on* $X \times V$, μ*-a.e.,*

(ii) $\theta(t, x, \cdot)$ *is convex on* V *for* $x \in X$, μ*-a.e.,*

(iii) *there exist* $\sigma > 0$ *and* $\varphi \in L^1_{\mathbb{R}}$ *such that*

$$\theta(t, x, v) \geq \varphi(t) - \sigma(\|x\| + |v|), \quad \text{for all} \;\; x \in X, \;\; v \in V, \;\; \mu\text{-a.e.},$$

are sufficient for sequential strong-weak lower semicontinuity of I_θ *on* $L^1_X \times L^1_V$*. Moreover, they are also necessary, provided that* $I_\theta(\overline{x}, \overline{v}) < +\infty$ *for some* $\overline{x} \in L^1_X, \overline{v} \in L^1_V$*.*

Lemma 2.4.4 *(Mazur's Lemma, [108]) Let* Z *be a Banach space and* G *be a convex and closed set in* Z*. Then* G *is weakly closed in* Z*.*

Part I

Controllability of some Partial Functional Integrodifferential Equations in Banach Spaces

CHAPTER 3

Controllability for some Partial Functional Integrodifferential Equations with Nonlocal Conditions in Banach Spaces

[1]

3.1 Introduction

Several authors have studied the controllability problem of nonlinear systems described by functional integrodifferential equations with nonlocal conditions in infinite dimensional Banach spaces: see for instance [69], [9], [16], [40], [77], and the references therein. For example in [77], the authors proved the controllability of an integrodifferential system with nonlocal conditions basing on the measure of noncompactness and the Sadovskii fixed-point theorem, and in [69], R. Atmania and S. Mazouzi have proved the controllability of a semilinear integrodifferential system using Schaefer fixed-point theorem and requiring the compactness of the semigroup.

In [69], the authors assumed the compactness of the operator semigroup and in [9], the authors assumed the compactness of the resolvent operator whereas in [16, 40, 77], the authors managed to drop this condition which motivates our current work.

In this chapter, we study the controllability of some systems that arise in the

[1]The results of this chapter are contents of the following paper
- Khalil Ezzinbi, Guy Degla, Patrice Ndambomve [42]

analysis of heat conduction in materials with memory [85], and viscoelasticity, and take the form of the following model of partial functional integrodifferential equation subject to a nonlocal initial condition in a Banach space $(X, \ \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) \ + \ \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t,x(t)) + Cu(t) \qquad \text{for } t \in I = [0, b] \\ \\ x(0) = x_0 + g(x), \end{cases} \tag{3.1.1}$$

where $x_0 \in X, \ g : \mathcal{C}(I, X) \to X$ and $f : I \times X \to X$ are functions satisfying some conditions; $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on X; for $t \geq 0$, $B(t)$ is a closed linear operator with domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$. The control u belongs to $L^2(I, U)$ which is a Banach space of admissible controls, where U is a Banach space. The operator C belongs to $\mathcal{B}(U, X)$ which is the Banach space of bounded linear operators from U into X, and $\mathcal{C}(I, X)$ denotes the Banach space of continuous functions $x : \ I \to X$ with supremum norm $\|x\|_\infty = \sup_{t\in I} \|x(t)\|_X$.
The particular case $B(t) = 0$ has been considered by Y.K. Chang, J.J. Nieto and W.S. Li [16], where the authors used Sadovskii fixed-point theorem and the operator semigroup to prove their result.
Here our goal is to study equation (3.1.1) where $B(t) \neq 0$ as a generalization of the result by Y.K. Chang, J.J. Nieto and W.S. Li without requiring the compactness of the operator semigroup, in the same way as J. Wang, Z. Fan and Y. Zhou [40] have done for the nonlocal controllability of some semilinear dynamic systems with fractional derivative. Our approach constists in transforming the problem (3.1.1) into a fixed-point problem of an appropriate operator and to apply the Mönch fixed-point theorem.

Definition 3.1.1 *A mild solution of equation* (3.1.1) *is a function* $x \in \mathcal{C}(I, X)$ *such that*

$$x(t) \ = \ R(t)[x_0 + g(x)] + \int_0^t R(t-s)\big[f(s, x(s)) + Cu(s)\big]\, ds \quad \text{for} \quad t \in I.$$

Definition 3.1.2 *Equation* (3.1.1) *is said to be controllable on the interval* I *if for every* $x_0, \ x_1 \in X$*, there exist a control* $u \in L^2(I, U)$ *and a mild solution* x *of equation* (3.1.1) *satisfying the condition* $x(b) = x_1$.

3.2 Main Result

In the sequel, we shall denote by β the Hausdorff measure of noncompactness. Consider the following hypotheses.

($\mathbf{H_3}$) The linear operator $W : L^2(I, U) \to X$ satisfies the following condition:

(i) W defined by

$$Wu = \int_0^b R(b-s)Cu(s)\,ds,$$

is surjective so that it induces an isomorphism between $L^2(I,U)\,/_{\mathrm{Ker}W}$ and X again denoted by W with inverse W^{-1} taking values in $L^2(I,U)\,/_{\mathrm{Ker}W}$, (see e.g., [68]).

(ii) There exists a function $L_W \in L^1(I, \mathbb{R}^+)$ such that for any bounded set $Q \subset X$ we have

$$\beta((W^{-1}Q)(t)) \leq L_W(t)\beta(Q),$$

where β is the Hausdorff MNC.

($\mathbf{H_4}$) The function $f : I \times X \longrightarrow X$ satisfies the following conditions:

(i) $f(\cdot, x)$ is measurable for $x \in X$ and $f(t, \cdot)$ is continuous for a.e $t \in I$.

(ii) There exist a function $L_f \in L^1(I, \mathbb{R}^+)$ and a nondecreasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x)\| \leq L_f(t)\phi(\|x\|) \text{ for } x \in X, \quad t \in I \text{ and } \liminf_{r\to+\infty} \frac{\phi(r)}{r} = 0.$$

(iii) There exists a function $h \in L^1(I, \mathbb{R}^+)$ such that for any bounded set $D \subset X$,

$$\beta(f(t,D)) \leq h(t)\beta(D) \text{ for a.e } t \in I,$$

where β is the Hausdorff MNC.

($\mathbf{H_5}$) $g : \mathcal{C}(I,X) \longrightarrow X$ is continuous, compact and satisfies $\liminf_{r\to+\infty} \frac{g_r}{r} = 0$ where

$$g_r = \sup\Big\{\|g(x)\| : \|x\| \leq r\Big\}.$$

Theorem 3.2.1 *Suppose equation* (2.2.1) *has a resolvent operator* $\Big(R(t)\Big)_{t\geq 0}$ *that is continuous in the operator-norm topology for* $t > 0$ *and hypotheses* $(\mathbf{H_3}) - (\mathbf{H_5})$ *are satisfied. Then equation* (3.1.1) *is controllable on I, provided that*

$$\gamma = \Big(1 + 2M_1M_2\|L_W\|_{L^1}\Big)\Big(2M_1\|h\|_{L^1}\Big) < 1,$$

where $M_1 = \sup_{0\leq t\leq b} \|R(t)\|$ *and* M_2 *is such that* $\|C\| = M_2$.

Proof. Note at once that $M_1 < +\infty$ according to the exponential growth of the resolvent operator $\big(R(t)\big)_{t\geq 0}$. Using $(\mathbf{H_3})$ we define the following control by

$$u_x(t) \;=\; W^{-1}\left(x_1 - R(b)[x_0 + g(x)] - \int_0^b R(b-s)f(s,x(s))\,ds\right)(t) \quad \text{for } t \in I,$$

for an arbitrarily given function $x \in \mathcal{C}(I,X)$.
Using this control, we shall show that the operator $K : \mathcal{C}(I,X) \to \mathcal{C}(I,X)$ defined by

$$(Kx)(t) \;=\; R(t)[x_0 + g(x)] + \int_0^t R(t-s)\Big[f(s,x(s)) + Cu_x(s)\Big]\,ds,$$

has a fixed point x which is just a mild solution of the equation (3.1.1). Observe that $(Kx)(b) = x_1$ and so the control u_x steers the integrodifferential equation from x_0 to x_1 in time b. This means that equation (3.1.1) is controllable on I.

For each positive number r, let $B_r = \{x \in \mathcal{C}(I,X) : \|x\|_\infty \leq r\}$. We shall prove the above theorem through the following steps.

<u>**Step 1.**</u> We claim that there exists $r > 0$ such that $K(B_r) \subset B_r$.
Suppose on the contrary that this is not true. Then for each positive number r, there exists a function $x_r \in B_r$, such that $K(x_r) \notin B_r$, *i.e.*, $\|(Kx_r)(\tau)\| > r$, for some $\tau = \tau(r) \in I$. Now

$$\frac{\|(Kx_r)(\tau)\|}{r} \;>\; 1 \quad \text{that implies that} \qquad \liminf_{r\to+\infty} \frac{\|(Kx_r)(\tau)\|}{r} \;\geq\; 1\,. \quad (*)$$

On the other hand, let $M_3 = \|W^{-1}\|$. We have

$$\begin{aligned}
\Big\|(Kx_r)(\tau)\Big\| &\leq M_1\|x_0\| + M_1\|g(x_r)\| + M_1\int_0^b \Big\|f(s,x_r(s))\Big\|\,ds \\
&+ bM_1M_2M_3\Big(\|x_1\| + M_1\|x_0\|\Big) \\
&+ bM_1M_2M_3\Big(M_1\|g(x_r)\| + M_1\int_0^b \Big\|f(s,x_r(s))\Big\|\,ds\Big) \\
&\leq M_1\|x_0\| + M_1 g_r + M_1\phi(r)\|L_f\|_{L^1} \\
&+ bM_1M_2M_3\Big(\|x_1\| + M_1\|x_0\|\Big) \\
&+ bM_1M_2M_3\Big(M_1 g_r + M_1\phi(r)\|L_f\|_{L^1}\Big) \\
&\leq w_r := \Big(1 + bM_1M_2M_3\Big)M_1\|x_0\| + \Big(1 + bM_1M_2M_3\Big)M_1 g_r \\
&+ \Big(1 + bM_1M_2M_3\Big)M_1\|L_f\|_{L^1}\phi(r) + bM_1M_2M_3\|x_1\|.
\end{aligned}$$

It follows now that

$$\liminf_{r\to+\infty} \frac{\|(Kx_r)(\tau)\|}{r} = 0$$

since

$$\liminf_{r\to+\infty} \frac{w_r}{r} = 0 = \liminf_{r\to+\infty} \frac{g_r}{r}.$$

This is clearly a contradiction to $(*)$. Consequently, there exists $r > 0$ such that $K(B_r) \subset B_r$.

Step 2. The operator K is continuous on B_r.
To this end, let $(x_n)_{n\geq 1} \subset B_r$ such that $x_n \to x$ in B_r. Set

$$(K_1x)(t) := R(t)[x_0 + g(x)] \quad \text{and}$$

$$(K_2x)(t) := \int_0^t R(t-s)\big[f(s,x(s)) + Cu_x(s)\big]ds \quad \text{for} \quad t \in I,$$

then $K = K_1 + K_2$.
Therefore, since g is continuous, we have that

$$\|K_1x_n - K_1x\| \leq M_1\|g(x_n) - g(x)\| \to 0, \text{ as } n \to +\infty.$$

To see the continuity of K_2, we first set

$F_n(s) = f(s, x_n(s))$ for every n and a.e. s, and $F(s) = f(s, x(s))$ for a.e. s.

Therefore by $(\mathbf{H_4}) - (\mathbf{i})$, $F_n(s) \to F(s)$ and by $(\mathbf{H_4}) - (\mathbf{ii})$,

$$\Big\|F_n(s)\Big\| = \Big\|f(s, x_n(s))\Big\| \leq L_f(s)\phi(\|x_n\|) \leq \phi(r)L_f(s)$$

for every n and a.e. s.

It follows from Lebesgue dominated convergence theorem that

$$\int_0^t \Big\|F_n(s) - F(s)\Big\| ds \longrightarrow 0, \; as\ n \to +\infty, \; t \in I.$$

Moreover, we have that

$$\Big\|K_2x_n - K_2x\Big\| \leq M_1 \int_0^t \Big\|F_n(s) - F(s)\Big\|ds + M_1M_2b^{\frac{1}{2}}\Big\|u_{x_n} - u_x\Big\|_{L^2(I,U)},$$

where

$$\Big\|u_{x_n} - u_x\Big\|_{L^2(I,U)} \leq M_3 \left(M_1\Big\|g(x_n) - g(x)\Big\| + M_1 \int_0^b \Big\|F_n(s) - F(s)\Big\|ds\right).$$

Thus it follows that $\Big\|K_2x_n - K_2x\Big\| \longrightarrow 0\ as\ n \to +\infty$, showing that K_2 is continuous on B_r. Hence K is continuous on B_r.

Step 3. The Mönch condition holds.

Suppose that $D \subseteq B_r$ is countable and $D \subseteq \overline{co}\Big(\{0\} \cup K(D)\Big)$. We have to show that D is relatively compact.

To this end, it suffices to show that $\beta(D) = 0$, where β is the Hausdorff MNC. Since D is countable, we can describe it as $D = \{x_n\}_{n\geq 1}$. Therefore $K(D) = \{Kx_n\}_{n\geq 1}(t)$ and its relative compactness implies that D is also relatively compact. So we have to prove that $K(D)$ is equibounded and equicontinuous on I in order to use Ascoli-Arzela's theorem. We show that $K(D)$ is equicontinuous. To this end, let $y \in K(D)$, and $0 \leq t_1 < t_2 \leq b$, there is $x \in D$ such that $y = Kx$ and then

$$\begin{aligned}\Big\|y(t_2) - y(t_1)\Big\| \leq\ & \Big\|R(t_2)x_0 - R(t_1)x_0\Big\| + \Big\|R(t_2)g(x) - R(t_1)g(x)\Big\| \\ & + \Big\|\int_0^{t_2} R(t_2 - s)f(s, x(s))ds - \int_0^{t_1} R(t_1 - s)f(s, x(s))ds\Big\| \\ & + \Big\|\int_0^{t_2} R(t_2 - s)Cu_x(s)ds - \int_0^{t_1} R(t_1 - s)Cu_x(s)ds\Big\|\end{aligned}$$

Firstly for $t_1 > 0$. By (**ii**) of definition 2.2.2, the first term on the right hand side tends to 0 as $|t_2 - t_1| \to 0$. That is

$$\Big\|R(t_2)x_0 - R(t_1)x_0\Big\| \to 0 \quad \text{as} \quad |t_2 - t_1| \to 0.$$

Moreover, we have that

$$\begin{aligned}\Big\|R(t_2)g(x) - R(t_1)g(x)\Big\| &\leq \Big\|R(t_2) - R(t_1)\Big\|\Big\|g(x)\Big\| \\ &\leq \Big\|R(t_2) - R(t_1)\Big\|g_r,\end{aligned}$$

where $g_r = \sup\{\|g(x)\| : \|x\| \leq r\}$.
And so $\Big\|R(t_2) - R(t_1)\Big\| \to 0$ as $|t_2 - t_1| \to 0$, by the continuity of $\big(R(t)\big)_{t\geq 0}$ for $t > 0$ in the operator-norm topology.
Now for $t_1 = 0$, we have that
$\Big\|R(h)g(x) - g(x)\Big\| \leq \sup_{y\in\overline{g(B_r)}} \Big\|R(h)y - y\Big\| \to 0,$ as $h \to 0^+$, by Lemma 2.3.5 since $\overline{g(B_r)}$ is compact. Therefore
$\Big\|R(t_2)g(x) - R(t_1)g(x)\Big\| \to 0$ as $t_2 \to t_1$.

Now we have that

$$\begin{aligned}&\Big\|\int_0^{t_2} R(t_2 - s)f(s, x(s))ds - \int_0^{t_1} R(t_1 - s)f(s, x(s))ds\Big\| \\ &\leq M_1 \int_{t_1}^{t_2} \Big\|f(s, x(s))\Big\|ds \\ &+ \int_0^{t_1} \Big\|R(t_2 - s) - R(t_1 - s)\Big\|\Big\|f(s, x(s))\Big\|ds \\ &\leq M_1\phi(r) \int_{t_1}^{t_2} L_f(s)ds \\ &+\phi(r) \int_0^{t_1} \Big\|R(t_2 - s) - R(t_1 - s)\Big\|L_f(s)ds\end{aligned}$$

The right hand side tends to 0 as $t_2 \to t_1$ by Lebesgue dominated convergence theorem, showing that the family $\left\{\int_0^t R(t - s)f(s, x(s))ds \ , \ x \in D\right\}$ is equicontinuous. Also we have that

$$\Big\|\int_0^{t_2} R(t_2-s)Cu_x(s)ds - \int_0^{t_1} R(t_1-s)Cu_x(s)ds\Big\|$$

$$\leq M_1 \int_{t_1}^{t_2} \Big\|Cu_x(s)\Big\|ds$$

$$+\int_0^{t_1} \Big\|R(t_2-s)-R(t_1-s)\Big\|\Big\|Cu_x(s)\Big\|ds$$

$$\leq M_2M_3\Big[\|x_1\| + M_1\Big(\|x_0\|+g_r\Big) + M_1\phi(r)\|L_f\|_{L^1}\Big]$$

$$\Big[M_1(t_2-t_1) + \int_0^{t_1} \Big\|R(t_2-s)-R(t_1-s)\Big\|ds\Big]$$

and the right hand side tends to 0 as $t_2 \to t_1$ by Lebesgue dominated convergence theorem, showing that the family

$$\Big\{\int_0^t R(t-s)Cu_x(s)\,ds\,;\ x\in D\Big\}$$

is equicontinuous and therefore the set $K(D)$ is equicontinuous on I.
We now show that $K(D)$ is equibounded. To do this, we show that for all $t\in[0,b]$, the set $\Big\{K(x)(t);\ x\in D\Big\}$ is relatively compact. We achieve this by using the measure of noncompactness. We have that for $t=0$, the set

$$\Big\{(Kx)(0)\,;\ x\in D\Big\} \ = \ \Big\{x_0+g(x)\,;\ x\in D\Big\} \ = \ x_0+g(D)$$

is relatively compact in X. Since g is compact, then $\overline{g(D)}$ is compact also.
For $t\in(0,b]$, we have that,

$$\beta\Big(\{(K_1x_n)(t)\}_{n\geq 1}\Big) \ \leq\ \beta\Big(\{R(t)(x_0+g(x_n))\}_{n\geq 1}\Big) \ =\ 0,$$

by the compactness of g. Also we have by $(\mathbf{H_3})-(\mathbf{ii})$ that

$$\beta\Big(\{u_{x_n(t)}\}_{n\geq 1}(t)\Big)$$

$$= \beta\Big(W^{-1}\Big\{x_1 - R(b)\Big(x_0 + g(x_n)\Big) - \int_0^b R(t-s)f(s, x_n(s))\,ds\Big\}_{n\geq 1}(t)\Big)$$

$$\leq L_W(t)\beta\Big(\Big\{x_1 - R(b)\Big(x_0 + g(x_n)\Big) - \int_0^b R(t-s)f(s, x_n(s))\,ds\Big\}_{n\geq 1}(t)\Big)$$

$$\leq L_W(t)\beta\Big(\Big\{x_1 - R(b)\Big(x_0 + g(x_n)\Big)\Big\}_{n\geq 1}(t)\Big)$$

$$+L_W(t)\beta\Big(\Big\{\int_0^b R(t-s)f(s, x_n(s))\,ds\Big\}_{n\geq 1}(t)\Big).$$

By Lemma 2.3.3 and $(\mathbf{H_4}) - (\mathbf{iii})$, we deduce that

$$\begin{aligned}\beta\Big(\{u_{x_n(t)}\}_{n\geq 1}(t)\Big) &\leq 2M_1 L_W(t)\left(\int_0^b h(s)\,ds\right)\beta\Big(\{x_n(s)\}_{n\geq 1}(t)\Big)\\ &\leq 2M_1 L_W(t)\left(\int_0^b h(s)\,ds\right)\beta\Big(D(t)\Big).\end{aligned}$$

It follows that

$$\beta\Big(\{(K_2 x_n)(t)\}_{n\geq 1}\Big)$$

$$= \beta\left(\left\{\int_0^t R(t-s)f(s, x_n(s))\,ds + \int_0^t R(t-s)Cu_{x_n}(s)\,ds\right\}_{n\geq 1}(t)\right)$$

$$\leq 2M_1\left(\int_0^b h(s)\,ds\right)\beta\Big(D(t)\Big)$$

$$+2M_1M_2\left(\int_0^b L_W(s)\,ds\right)2Ma1\left(\int_0^b h(s)\,ds\right)\beta\Big(D(t)\Big)$$

$$\leq 2M_1\|h\|_{L^1}\beta\Big(D(t)\Big) + 2M_1M_2\|L_W\|_{L^1}2M_1\|h\|_{L^1}\beta\Big(D(t)\Big)$$

$$\leq \Big(1 + 2M_1M_2\|L_W\|_{L^1}\Big)\Big(2M_1\|h\|_{L^1}\Big)\beta\Big(D(t)\Big).$$

It follows that

$$\begin{aligned}\beta\Big(K(D)(t)\Big) &\leq \beta\Big(K_1(D)(t)\Big) + \beta\Big(K_2(D)(t)\Big)\\ &\leq \Big(1 + 2M_1M_2\|L_W\|_{L^1}\Big)\Big(2M_1\|h\|_{L^1}\Big)\beta\Big(D(t)\Big).\end{aligned}$$

That is $\beta\Big(K(D(t))\Big) \leq \gamma\beta\Big(D(t)\Big)$. But from Mönch's condition, we have

$$\beta\Big(D(t)\Big) \leq \beta\Big(\overline{co}\Big(\{0\} \cup K(D(t))\Big)\Big) = \beta\Big(K(D(t))\Big) \leq \gamma\beta\Big(D(t)\Big).$$

This implies that $\beta\Big(D(t)\Big) = 0$, since $\gamma < 1$, which implies that $\beta\Big(K(D)(t)\Big) = 0$. This shows that $\overline{K(D)(t)}$ is compact, that is $\overline{\{K(x)(t);\, x \in D\}}$ is compact as desired. So $K(D)$ is equicontinuous and equibounded and therefore by Ascoli-Arzela's Theorem, we have that $K(D)$ is relatively compact.

But

$$\beta\Big(D\Big) \leq \beta\Big(\overline{co}\Big(\{0\} \cup K(D)\Big)\Big) = \beta\Big(K(D)\Big) = 0.$$

This implies that $\beta\Big(D\Big) = 0$, showing that $\overline{D}$ is compact in X as desired. Thus D is relatively compact and the Mönch condition is satisfied. Therefore by Theorem 2.4.2, K has a fixed point x in B_r, which is a mild solution of equation (3.1.1) and satisfies $x(b) = x_1$. And the proof is complete.

□

we now illustrate our main result by the following example.

3.3 Example of Application

Let Ω be bounded domain in $\mathbb{R}^n$ with smooth boundary and consider the following nonlinear integrodifferential equation.

$$\begin{cases} \frac{\partial v(t,\xi)}{\partial t} = \Delta v(t,\xi) + \displaystyle\int_0^t \zeta(t-s)\Delta v(s,\xi)\,ds + \frac{e^{-t}}{k+e^t}\sin(v(t,\xi)) + a\omega(t,\xi) \\ \qquad\qquad \text{for } t \in [0,b] = I \text{ and } \xi \in \Omega \\ v = 0 \text{ on } \partial\Omega \\ v(0,\xi) = v_0(\xi) + \displaystyle\int_\Omega\int_0^b \rho(t,\xi)\log\Big(1 + |v(t,\eta)|^{\frac{1}{2}}\Big)\,dt\,d\eta \quad \text{for } \xi \in \Omega, \end{cases} \tag{3.3.1}$$

where $a > 0$, $k \geq 1$, $\omega : I \times \Omega \to \Omega$ continuous in t and $\omega(t,\xi) = 0$ for all $\xi \in \partial\Omega$, $\rho \in \mathcal{C}(I \times \overline{\Omega})$ and $\rho(t,\xi) = 0$ for all $\xi \in \partial\Omega$, and $\zeta \in W^{1,1}(\mathbb{R}^+, \mathbb{R})$

Let $X = U = \mathcal{C}_0(\overline{\Omega})$, the space of all continuous functions from $\overline{\Omega}$ to $\mathbb{R}$ vanishing on the boundary. We define $A : \mathcal{D}(A) \subset X \to X$ by:

$$\begin{cases} \mathcal{D}(A) = \Big\{ v \in \mathcal{C}_0(\overline{\Omega}) \cap H_0^1(\Omega);\ \Delta v \in \mathcal{C}_0(\overline{\Omega}) \Big\} \\ \\ Av \ = \ \Delta v, \end{cases} \tag{3.3.2}$$

for each $v \in \mathcal{D}(A)$.

Theorem 3.3.1 *(***Theorem 4.1.4, p. 82 of [76]***) If Ω has a $\mathcal{C}^1$-boundary, then the operator A defined as above, is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{C}_0(\overline{\Omega})$.*

By Theorem 8.4.1 above, A generates a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ of contractions on $\mathcal{C}_0(\overline{\Omega})$. Moreover, $\big(T(t)\big)_{t\geq 0}$ generated by A above, is compact for $t > 0$ and operator-norm continuous for $t > 0$. Then by Theorem 2.2.6, the corresponding resolvent operator is operator-norm continuous. Define

$$x(t)(\xi) \ = \ v(t,\xi), \quad x'(t)(\xi) \ = \ \frac{\partial v(t,\xi)}{\partial t}.$$

$$f(t,x)(\xi) \ = \ \frac{e^{-t}}{k+e^t}\sin(x(t)(\xi)) \quad \text{for } t \in I \text{ and } \ \xi \in \Omega$$

$C : X \to X$ be defined by $Cu \ = \ a\omega$.

$$(B(t)x)(\xi) \ = \ \zeta(t)\Delta x(t)(\xi) \ \text{ for } \ t \in I, \quad x \in \mathcal{D}(A) \text{ and } \ \xi \in \Omega.$$

$g : \mathcal{C}(I,X) \to X$ be defined by

$$g(x)(\xi) \ = \ \int_\Omega \int_0^b \rho(t,\xi) \log\Big(1+|x(t)(\eta)|^{\frac{1}{2}}\Big)\, dt\, d\eta \quad \text{for } \xi \in \overline{\Omega} \text{ and } \ x \in \mathcal{C}(I,X).$$

Equation (3.3.1) is then transformed into the following form

$$\begin{cases} x'(t) = Ax(t) \ + \ \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t,x(t)) + Cu(t) \qquad \text{for t} \in \text{I} = [0,\text{b}], \\ \\ x(0) = x_0 + g(x). \end{cases} \tag{3.3.3}$$

f is Lipschitz continuous with respect to the second variable and moreover we have that

$$\Big\|f(t,x)\Big\| \leq \frac{e^{-t}}{k+e^t} \quad \text{for} \quad (t,x) \in I \times \overline{\Omega}.$$

Consequently, f satisfies $(\mathbf{H_4}) - (\mathbf{i})$, $(\mathbf{H_4}) - (\mathbf{ii})$ and $(\mathbf{H_4}) - (\mathbf{iii})$, with $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\phi(x) = 1$.

Moreover,

$$\Big\|g(x)\Big\|_{\mathcal{C}_0(\overline{\Omega})} \leq \Big(b\,\mathrm{mes}(\Omega)\Big) M_\rho \Big(\|x\|\Big)^{\frac{1}{2}},$$

where $M_\rho = \max_{(t,\xi)\in I\times\overline{\Omega}} |\rho(t,\xi)|$.

It is clear that with $g_r = \sup\Big\{\|g(x)\| : \|x\|_\infty \leq r\Big\}$, we have $\lim_{r\to+\infty} \frac{g_r}{r} = 0$.

Lemma 3.3.2 *The map* $g : \mathcal{C}(I, \mathcal{C}_0(\overline{\Omega})) \to \mathcal{C}_0(\overline{\Omega})$ *defined by*

$$g(x)(\xi) \;=\; \int_\Omega \int_0^b \rho(t,\xi) \log\Big(1+|x(t)(\eta)|^{\frac{1}{2}}\Big)\, dt\, d\eta \quad \textit{for}\, \xi \in \overline{\Omega} \;\textit{and}\, x \in \mathcal{C}(I,X),$$

is compact.

Proof. Let $E \subset \mathcal{C}(I, \mathcal{C}_0(\overline{\Omega}))$ be bounded. Then, by computing as above, we have that

$$\Big\|g(x)\Big\|_{\mathcal{C}_0(\overline{\Omega})} \leq \Big(b\,\mathrm{mes}(\Omega)\Big) M_\rho \Big(\|x\|\Big)^{\frac{1}{2}}, \quad \text{for all} \quad x \in E.$$

So $g(E)$ is bounded.
Now since ρ is uniformly continuous on $I \times \overline{\Omega}$, it follows that $g(E)$ is equicontinuous on $\overline{\Omega}$. Therefore, by Ascoli-Arzela's theorem, $g(E)$ is relatively compact in $\mathcal{C}_0(\overline{\Omega})$. Hence, g is compact.

□

We have by Lemma 3.3.2 that g is compact and therefore satisfies $(\mathbf{H_5})$. Now for $\xi \in \Omega$, the operator W is given by

$$(Wu)(\xi) = \int_0^1 R(1-s)\omega u(s,\xi)\, ds.$$

Assuming that W satisfies $(\mathbf{H_3})$, then all the conditions of Theorem 3.2.1 hold and equation (3.3.3) is controllable.

CHAPTER 4

Controllability for some Partial Functional Integrodifferential Equations with Finite Delay in Banach Spaces

[1]

4.1 Introduction

In this chapter, we study the controllability for some systems that arise in the analysis of heat flow in a rigid body of a material with memory [53, 85]. Materials with memory are interesting because they act adaptively to their environment. They can be shaped easily into different forms at low temperature, but return to their original forms on heating. Steering such systems from an initial state (initial condition) to a desired terminal one (boundary condition) by an appropriate choice of a control, is of interest to many scientists and engineers.
These systems take the form of the following abstract model of partial functional integrodifferential equation with finite delay in a Banach space $(X, \|\cdot\|)$

[1] The results of this chapter are contents of the following paper
- Khalil Ezzinbi and Patrice Ndambomve [44]

:

$$\begin{cases} x'(t) = Ax(t) \ + \ \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t,x_t) + Cu(t) \ \text{ for } t \in I = [0,b] \\ \\ x_0 = \varphi \in \mathcal{C} = \mathcal{C}([-r,0];X), \end{cases} \tag{4.1.1}$$

where $f : I \times \mathcal{C} \to X$ is a function satisfying some conditions; $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on X; for $t \geq 0$, $B(t)$ is a closed linear operator with domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$. The control u belongs to $L^2(I,U)$ which is a Banach space of admissible controls, where U is a Banach space. The operator $C \in \mathcal{L}(U,X)$, where $\mathcal{L}(U,X)$ denotes the Banach space of bounded linear operators from U into X and $\mathcal{C}([-r,0],X)$ denotes the Banach space of continuous functions $\varphi : [-r,0] \to X$ with supremum norm $\|\varphi\| = \sup\limits_{\theta\in[-r,0]} \|\varphi(\theta)\|_X$, x_t denotes the history function that is defined by

$$x_t(\theta) = x(t+\theta), \text{for } -r \leq \theta \leq 0.$$

The particular cases in which $B(t) = 0$ and $A = A(t)$ were considered by K. Balachandran and R. Sakthivel; S. Baghli, M. Benchohra and K. Ezzinbi; S. Selvi and M. M. Arjunan and many others. In this work, we extend the work of K. Balachandran and R. Sakthivel by considering some integrodifferential equation when $B(t) \neq 0$, and without any compactness assumption.

The controllability problem of nonlinear systems described by integrodifferential equations in infinite dimensional Banach spaces has been studied by several authors: see for instance [8]-[16], [40, 55, 56, 69, 70, 83, 71] and the references therein. Many authors have also studied the controllability problem of nonlinear systems with delay in inifinite dimensional Banach spaces: see for instance [8], [55], [70], [83], [71], etc and the references therein. For example in [83], the authors proved the controllability of semilinear functional evolution equations with infinite delay using the nonlinear alternative of Leray-Schauder type. In [56], M. Li, M. Wang and F. Zhang proved the controllability of an impulsive functional differential system with finite delay using Schaefer's fixed-point theorem. In [71], S. Selvi and M. M. Arjunan proved the controllability for impulsive differential systems with finite delay using Mönch's fixed-point Theorem, and in [8], K. Balachandran and R. Sakthivel studied the controllability of functional semilinear integrodifferential systems in Banach spaces using Schaefer's fixed point Theorem and the authors assumed that the semigroup generated by the linear part is compact.

In many areas of applications such as Engineering, Electronics, Fluid Dynamics, Physical Sciences, etc..., integrodifferential equations appear and

have received considerable attention during the last decades. In recent years, much work on the existence and regularity of solutions of nonlinear integrodifferential equations with finite delay has been done by many authors by applying the resolvent operator theory, see e.g., [43] and the references therein.
In [85], R. Grimmer proved the existence and uniqueness of resolvent operators for these integrodifferential equations that give the variation of parameters formula for the solution. In [24], W. Desch, R. Grimmer and W. Schappacher proved the equivalence of the compactness of the resolvent operator and that of the operator semigroup. In this work, we use the equivalence between the operator-norm continuity of the associated resolvent operator and that of the operator semigroup. This property allows us to drop the compactness assumption on the operator semigroup, considered by the authors in [8, 83], and prove that the operator solution satisfies the Mönch condition. The variation of parameters formula for the mild solutions of equation (4.1.1) is given by the resolvent operator, and we prove the controllability result using the Mönch's fixed-point Theorem and the measure of noncompactness. This method enables us overcome the resolvent operator case considered in this work. In contrary to the evolution semigroup case considered in [55, 71], here the semigroup property can not be used because resolvent operators in general do not form semigroups.
To the best of our knowledge, up to now no work has reported on controllability of partial functional integrodifferential equation (4.1.1) with finite delay. It has been an untreated topic in the literature, and this fact is the main aim and motivation of the present work.

4.2 Preliminary Results

Definition 4.2.1 *A mild solution of equation* (4.1.1) *is a function* $x \in \mathcal{C}([-r, b], X)$ *satisfying the relation*

$$x(t) = \begin{cases} R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)\big[f(s, x_s) + Cu(s)\big]\, ds & \text{for } t \in I, \\ \varphi(t) \quad \text{for } -r \le t \le 0. \end{cases} \tag{4.2.1}$$

Definition 4.2.2 *Equation* (4.1.1) *is said to be controllable on the interval* I *if for every* $\varphi \in \mathcal{C}$ *and* $x_1 \in X$*, there exists a control* $u \in L^2(I, U)$ *such that a mild solution* x *of equation* (4.1.1) *satisfies the condition* $x(b) = x_1$.

We now state the following useful result for equicontinuous subsets of $\mathcal{C}([a,b];X)$, where X is a Banach space.

Lemma 4.2.3 *[12] Let $M \subset \mathcal{C}([a,b];X)$ be bounded and equicontinuous. Then $\beta(M(t))$ is continuous and*

$$\beta(M) = \sup\{\beta(M(t));\, t \in [a,b]\}, \quad \textit{where } M(t) = \{x(t);\, x \in M\}.$$

Lemma 4.2.4 *[12] Let $M \subset \mathcal{C}([a,b];X)$ be bounded and equicontinuous. Then the set $\overline{co}(M)$ is also bounded and equicontinuous.*

To prove the controllability for equation (4.1.1), we need the following results.

Lemma 4.2.5 *[40] If $(u_n)_{n\geq 1}$ is a sequence of Bochner integrable functions from I into a Banach space Y with the estimation $\|u_n(t)\| \leq \mu(t)$ for almost all $t \in I$ and every $n \geq 1$, where $\mu \in L^1(I,\mathbb{R})$, then the function*

$$\psi(t) \;=\; \beta(\{u_n(t) : n \geq 1\})$$

belongs to $L^1(I,\mathbb{R}^+)$ and satisfies the following estimation

$$\beta\left(\left\{\int_0^t u_n(s)ds \;:\; n \geq 1\right\}\right) \;\leq\; 2\int_0^t \psi(s)ds.$$

4.3 Controllability result

In this section, we give sufficient conditions ensuring the controllability of equation (4.1.1). For that goal, we need to assume that:

$(\mathbf{H_3})$

(i) The following linear operator $W : L^2(I,U) \to X$ defined by

$$Wu \;=\; \int_0^b R(b-s)Cu(s)\,ds,$$

is surjective so that it induces an isomorphism between $L^2(I,U)/_{\mathrm{Ker}W}$ and X again denoted by W with inverse W^{-1} taking values in $L^2(I,U)/_{\mathrm{Ker}W}$, (see e.g.,[68]).

(ii) There exists a function $L_W \in L^1(I, \mathbb{R}^+)$ such that for any bounded set $Q \subset X$ we have

$$\beta((W^{-1}Q)(t)) \leq L_W(t)\beta(Q),$$

where β is the Hausdorff MNC.

($\mathbf{H_4}$) The function $f : I \times \mathcal{C} \longrightarrow X$ satisfies the following two conditions:

(i) $f(\cdot, \varphi)$ is measurable for $\varphi \in \mathcal{C}$ and $f(t, \cdot)$ is continuous for a.e $t \in I$,

(ii) for every positive integer q, there exists a function $l_q \in L^1(I, \mathbb{R}^+)$ such that

$$\sup_{\|\varphi\|_{\mathcal{C}} \leq q} \|f(t, \varphi)\| \leq l_q(t) \text{ for a.e. } t \in I \text{ and } \liminf_{q \to +\infty} \int_0^b \frac{l_q(t)}{q} dt = l < +\infty,$$

(iii) there exists a function $h \in L^1(I, \mathbb{R}^+)$ such that for any bounded set $D \subset \mathcal{C}([-r, 0], X)$,

$$\beta(f(t, D)) \leq h(t)\beta(D) \quad \text{for} \quad \text{a.e } t \in I.$$

($\mathbf{H_5}$)

$$\gamma = \Big(1 + 2M_b M_2 \|L_W\|_{L^1}\Big)\Big(2M_b\|h\|_{L^1}\Big) < 1,$$

where $M_b = \sup_{0 \leq t \leq b} \|R(t)\|$ and M_2 is such that $M_2 = \|C\|$.

Theorem 4.3.1 *Suppose that hypotheses* ($\mathbf{H_3}$) $-$ ($\mathbf{H_5}$) *hold and equation (2.2.1) has a resolvent operator* $\big(R(t)\big)_{t \geq 0}$ *that is continuous in the operator-norm topology for* $t > 0$*. Then equation* (4.1.1) *is controllable on* I *provided that*

$$M_b(1 + M_b M_2 M_3 b)l < 1, \tag{4.3.1}$$

where M_3 *is such that* $M_3 = \|W^{-1}\|$.

Proof. Using ($\mathbf{H_3}$) and given an arbitrary function $x \in \mathcal{C}([-r, b], X)$, we define the control as usual by the following formula:

$$u_x(t) = W^{-1}\left\{x_1 - R(b)\varphi(0) - \int_0^b R(b-s)f(s, x_s)\,ds\right\}(t) \qquad \text{for } t \in I.$$

We define the following space

$$E_b = \Big\{ x : [0,b] \to X \text{ continuous such that } x(0) = \varphi(0) \Big\}.$$

For each $x \in E_b$ we define its continuous extension $\widetilde{x}$ from $[-r,b]$ to X as follows

$$\widetilde{x}(t) = \begin{cases} x(t) & \text{if } t \in [0,b] \\ \varphi(t) & \text{if } t \in [-r,0] \end{cases}$$

We show using this control that the operator $P : E_b \to E_b$ defined by

$$(Px)(t) = R(t)\varphi(0) + \int_0^t R(t-s)\big[f(s,\widetilde{x}_s) + Cu_x(s)\big]\,ds \quad \text{for } t \in I = [0,b]$$

has a fixed-point. This fixed point is then a mild solution of equation (4.1.1). Observe that $(Px)(b) = x_1$. This means that the control u_x steers the integrodifferential equation from φ to x_1 in time b which implies the equation (4.1.1) is controllable on I.

For each $\varphi \in \mathcal{C}$, we define the function $y \in \mathcal{C}([0,b],X)$ by $y(t) = R(t)\varphi(0)$ and its extension $\widetilde{y}$ in $\mathcal{C}([-r,b],X)$ by

$$\widetilde{y}(t) = \begin{cases} y(t) & \text{if } t \in [0,b] \\ \varphi(t) & \text{if } t \in [-r,0] \end{cases}$$

For each $z \in \mathcal{C}([0,b],X)$, set $\widetilde{x}(t) = \widetilde{z}(t) + \widetilde{y}(t)$, where $\widetilde{z}$ is the extension by zero of the function z on $[-r,0]$. Observe that x satifies (4.2.1) if and only if $z(0) = 0$ and

$$z(t) = \int_0^t R(t-s)\big[f(s,\widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)\big]\,ds \quad \text{for t} \in [0,\text{b}],$$

where

$$u_z(t) \;=\; W^{-1}\left\{ x_1 - R(b)\varphi(0) - \int_0^b R(b-s)f(s,\widetilde{z}_s + \widetilde{y}_s)\,ds \right\}(t).$$

Now let

$$E_b^0 = \Big\{ z : [0,b] \to X \text{ continuous such that } z(0) = 0 \Big\}.$$

Thus E_b^0 is a Banach space provided with the supremum norm. Define the operator $K : E_b^0 \to E_b^0$ by

$$(Kz)(t) = \int_0^t R(t-s)\big[f(s, \widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)\big]\, ds \quad \text{for} \quad t \in [0, b]$$

Note that the operator P has a fixed point if and only if K has one. So to prove that P has a fixed point, we only need to prove that K has one.
For each positive number q, let $B_q = \{z \in E_b^0 : \|z\| \leq q\}$. We shall prove the theorem in the following steps.

Step1. We claim that there exists $q > 0$ such that $K(B_q) \subset B_q$. We proceed by contradiction. Assume that it is not true. Then for each positive number q, there exists a function $z^q \in B_q$, such that $K(z^q) \notin B_q$, *i.e.*, $\|(Kz^q)(\tau)\| > q$ for some $\tau \in [0, b]$. Now we have that

$$\begin{aligned}
q &< \Big\|(Kz^q)(\tau)\Big\| \\
&\leq M_b \int_0^b \Big\|f(s, \tilde{z}_s^q + \widetilde{y}_s)\Big\|\, ds + M_b \int_0^b \|Cu_{z^q}(s)\|\, ds \\
&\leq M_b \int_0^b \Big\|f(s, \tilde{z}_s^q + \widetilde{y}_s)\Big\|\, ds + M_b \int_0^b \Big\|BW^{-1}\Big[x_1 - R(b)\varphi(0) - \int_0^b R(b-s) f(s, \tilde{z}_s^q)\, ds\Big]\Big\|\, ds \\
&\leq bM_bM_2M_3 \left(\|x_1\| + M_b\|\varphi\| + M_b \int_0^b \|f(s, \tilde{z}_s^q)\|\, ds\right) + M_b \int_0^b \Big\|f(s, \tilde{z}_s^q + \widetilde{y}_s)\Big\|\, ds \\
&\leq bM_bM_2M_3 \left(\|x_1\| + M_b\|\varphi\| + M_b \int_0^b l_{q'}(s)\, ds\right) + M_b \int_0^b l_{q'}(s)\, ds,
\end{aligned}$$

where $q' = q + q_0$, with $q_0 = \sup_{s \in [-r, b]} \|\widetilde{y}(s)\|$.
Hence

$$q \leq \Big(1 + M_bM_2M_3b\Big) M_b \int_0^b l_{q'}(s)\, ds + M_bM_2M_3b\Big(\|x_1\| + M_b\|\varphi\|_{\mathcal{C}}\Big).$$

Dividing both sides by q and noting that $q' = q + q_0 \to +\infty$ as $q \to +\infty$, we obtain that

$$1 \leq \Big(1 + M_bM_2M_3b\Big) M_b \left(\frac{\displaystyle\int_0^b l_{q'}(s)\, ds}{q}\right) + \frac{M_bM_2M_3b\Big(\|x_1\| + M_b\|\varphi\|_{\mathcal{C}}\Big)}{q}$$

and

$$\liminf_{q\to+\infty}\left(\frac{\displaystyle\int_0^b l_{q'}(s)\,ds}{q}\right)=\liminf_{q\to+\infty}\left(\frac{\displaystyle\int_0^b l_{q'}(s)\,ds}{q'}\frac{q'}{q}\right)=l.$$

Thus we have, $1\le\Big(1+M_bM_2M_3b\Big)M_bl$, and this contradicts (4.3.1). Hence for some positive number q, we must have $K(B_q)\subset B_q$.

Step2. $K:\,E_b^0\to E_b^0$ is continuous. In fact let $K:=K_1+K_2$, where

$$(K_1z)(t)=\int_0^t R(t-s)f(s,\widetilde{z}_s+\widetilde{y}_s)\,ds\quad\text{and}\quad(K_2z)(t)=\int_0^t R(t-s)Cu_z(s)\,ds.$$

Let $\{z^n\}_{n\ge1}\subset E_b^0$ with $z^n\to z$ in E_b^0. Then there exists a number $q>1$ such that $\|z^n(t)\|\le q$ for all n and *a.e.* $t\in I$. So $z^n,\,z\in B_q$. By $(\mathbf{H_4})-(\mathbf{i})$, $f(t,\widetilde{z}_t^n+\widetilde{y}_t)\to f(t,\widetilde{z}_t+\widetilde{y}_t)$ for each $t\in[0,b]$. And by $(\mathbf{H_4})-(\mathbf{ii})$,

$$\|f(t,\widetilde{z}_t^n+\widetilde{y}_t)-f(t,\widetilde{z}_t+\widetilde{y}_t)\|\le 2l_{q'}(t).$$

Then we have

$$\|K_1z^n-K_1z\|_{\mathcal{C}}\le M_b\int_0^b\|f(s,\widetilde{z}_s^n+\widetilde{y}_s)-f(s,\widetilde{z}_s+\widetilde{y}_s)\|\,ds\longrightarrow0,\ as\ n\to+\infty$$

by dominated convergence Theorem. Also we have that

$$\|K_2z^n-K_2z\|_{\mathcal{C}}\le M_b^2M_2M_3b\int_0^b\|f(s,\widetilde{z}_s^n)-f(s,\widetilde{z}_s)\|\,ds\longrightarrow0,\ as\ n\to+\infty$$

by dominated convergence Theorem. Thus

$$\|Kz^n-Kz\|\le\|K_1z^n-K_1z\|+\|K_2z^n-K_2z\|\longrightarrow0,\ as\ n\to+\infty.$$

Hence K is continuous on E_b^0.

Step3. $K(B_q)$ is equicontinuous on $[0, b]$. In fact let $t_1,\, t_2 \in I,\;\; t_1 < t_2$ and $z \in B_q$, we have

$$\begin{aligned}
\|(Kz)(t_2) - (Kz)(t_1)\| \;\leq\; & \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\| \|f(s, \widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)\| \, ds \\
+ & \int_{t_1}^{t_2} \|R(t_2 - s)\| \|f(s, \widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)\| \, ds \\
\leq & \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\| l_{q'}(s) \, ds \\
+ & \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\| M_2 M_3 \times \\
& \left[\|x_1\| + M_b \|\varphi\| + M_b \int_0^b l_{q'}(\tau) \, d\tau \right] ds \\
+ & \int_{t_1}^{t_2} \|R(t_2 - s)\| l_{q'}(s) \, ds \\
+ & \int_{t_1}^{t_2} \|R(t_2 - s)\| M_2 M_3 \left(\|x_1\| + M_b \|\varphi\| + M_b \int_0^b l_{q'}(\tau) \, d\tau \right) ds
\end{aligned}$$

By the continuity of $\big(R(t)\big)_{t \geq 0}$ in the operator-norm toplogy, the dominated convergence Theorem, we conclude that the right hand side of the above inequality tends to zero and independent of z as $t_2 \to t_1$. Hence $K(B_q)$ is equicontinuous on I.

Step4. We show that the Mönch's condition holds.

Suppose that $D \subseteq B_q$ is countable and $D \subseteq \overline{co}(\{0\} \cup K(D))$. We shall show that $\beta(D) = 0$, where β is the Hausdorff MNC. Without loss of generality, we may assume that $D = \{z^n\}_{n \geq 1}$. Since K maps B_q into an equicontinuous family, $K(D)$ is also equicontinuous on I.

By $(\mathbf{H_3})-(\mathbf{ii})$, $(\mathbf{H_4})-(\mathbf{iii})$ and Lemma 4.2.5, we have that

$$\begin{aligned}
\beta\Big(\{u_{z^n}(t)\}_{n\geq 1}\Big) &= \beta\left(W^{-1}\left\{x_1 - R(b)\varphi(0) - \int_0^b R(t-b)f\Big(s,\{\tilde{z}_s^n+\widetilde{y}_s\}_{n\geq 1}\Big)\,ds\right\}_{n\geq 1}(t)\right)\\
&\leq L_W(t)\beta\left(\{x_1-R(b)\varphi(0)\}\right)\\
&\quad + L_W(t)\beta\left(\left\{\int_0^b R(t-b)f\Big(s,\{\tilde{z}_s^n+\widetilde{y}_s\}_{n\geq 1}\Big)\,ds\right\}_{n\geq 1}\right)\\
&\leq 2M_bL_W(t)\left(\int_0^b h(s)\beta\left(\{\tilde{z}_s^n\}_{n\geq 1}+\{\widetilde{y}_s\}\right)ds\right)\\
&\leq 2M_bL_W(t)\left(\int_0^b h(s)\Big[\beta\left(\{\tilde{z}_s^n\}_{n\geq 1}\right)+\beta\Big(\{\widetilde{y}_s\}\Big)\Big]\,ds\right)\\
&\leq 2M_bL_W(t)\left(\int_0^b h(s)\beta\left(\{\tilde{z}_s^n\}_{n\geq 1}\right)\,ds\right),\ \text{since}\ \Big\{\widetilde{y}_s:\ s\in[0,b]\Big\}\ \text{is compact}\\
&\leq 2M_bL_W(t)\left(\int_0^b h(s)\sup_{-r\leq\theta\leq 0}\beta\left(\{\tilde{z}_s^n(\theta)\}_{n\geq 1}\right)\,ds\right)\\
&\quad \Big(\text{by Lemma 4.2.3, since}\ D=\{z^n\}_{n\geq 1}\ \text{is equicontinuous}\Big)\\
&\leq 2M_bL_W(t)\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)
\end{aligned}$$

This implies that

$$\begin{aligned}
\beta\Big(\{(Kz^n)(t)\}_{n\geq 1}\Big) &\leq \beta\left(\left\{\int_0^t R(t-s)f(s,\{\tilde{z}_s^n+\widetilde{y}_s\}_{n\geq 1})\,ds\right\}_{n\geq 1}\right)\\
&\quad +\beta\left(\left\{\int_0^t R(t-s)u_{z^n}(s)\,ds\right\}_{n\geq 1}\right)\\
&\leq 2M_b\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)\\
&+ 2M_bM_2\left(\int_0^b L_W(s)\,ds\right)2M_b\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)\\
&\leq 2M_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)\\
&+ 2M_bM_2\|L_W\|_{L^1}2M_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\beta\Big(K(D)(t)\Big) &\leq 2M_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)+2M_bM_2\|L_W\|_{L^1}2M_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)\\
&\leq \Big(1+2M_bM_2\|L_W\|_{L^1}\Big)2M_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)\\
&= \gamma\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)
\end{aligned}$$

Since D and $K(D)$ are equicontinuous on $[0,b]$ and D is bounded, it follows by Lemma 4.2.3 that $\beta\Big(K(D)\Big)\leq\gamma\beta\Big(D\Big)$, where γ is as defined in ($\mathbf{H_5}$). Thus from the Mönch condition, we get that

$$\beta\Big(D\Big)\leq\beta\Big(\overline{co}(\{0\}\cup K(D)\Big)=\beta\Big(K(D)\Big)\leq\gamma\beta\Big(D\Big),$$

and since $\gamma<1$, this implies $\beta\Big(D\Big)=0$, which implies that D is relatively compact as desired in B_q and the Mönch condition is satisfied. We conclude by Theorem 2.4.2, that K has a fixed point z in B_q. Then $x=z+y$ is a fixed point of P in E_b and thus equation (4.1.1) is controllable on $[0,b]$.

we now illustrate our main result by the following example.

4.4 Example

Consider the following nonlinear integrodifferential equation.

$$\begin{cases}
\dfrac{\partial v}{\partial t}(t,\xi)=\dfrac{\partial v}{\partial \xi}(t,\xi)+\displaystyle\int_0^t\zeta(t-s)\frac{\partial v}{\partial\xi}(s,\xi)\,ds+\alpha(t)\sin(v(t-r,\xi))+\omega u(t,\xi)\\
\qquad\qquad\text{for } t\in[0,1]=I \text{ and } \xi\in[0,\pi]\\
v(t,0)=v(t,\pi)=0 \quad\text{for } t\in[0,1]\\
v(\theta,\xi)=\psi(\theta,\xi) \text{ for } \theta\in[-r,0] \text{ and } \xi\in[0,\pi].
\end{cases}\tag{4.4.1}$$

where $\omega>0$, $\alpha\in\mathcal{C}([0,1];\mathbb{R})$, $u:I\times[0,\pi]\to[0,\pi]$ continuous in t and $u(t,0)=u(t,\pi)=0$ and $\zeta\in W^{1,1}(\mathbb{R}^+,\mathbb{R}^+)$

Let $X=U=\mathcal{C}_0([0,\pi])$ be the space of all continuous functions from $[0,\pi]$ to $\mathbb{R}$ vanishing at 0 and at π.

We define $A : \mathcal{D}(A) \subset X \to X$ by:

$$\begin{cases} \mathcal{D}(A) = \Big\{ y \in X : y' \text{ exists and } y' \in X \Big\} \\ \\ Ay \;=\; y'. \end{cases}$$

Then, A is the infinitesimal generator of a C_0-semigroup. Moreover, the C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ generated by A above and defined by

$$T(t)y(s) = y(t+s) \quad \text{for } y \in X,$$

is not compact but is operator-norm continuous for $t > 0$. Then by Theorem 2.2.6, the corresponding resolvent operator is operator-norm continuous. Define

$$x(t)(\xi) \;=\; v(t,\xi)$$

$$\varphi(\theta)(\xi) = \psi(\theta,\xi), \quad \theta \in [-r,0], \quad \xi \in [0,\pi].$$

$$f(t,\varphi)(\xi) \;=\; \alpha(t)\sin(\varphi(-r)(\xi)) \quad \text{for } t \in I \quad \text{and } \xi \in [0,\pi]$$

$C : X \to X$ be defined by $Cu \;=\; \omega u$.

$$(B(t)x)(\xi) \;=\; \zeta(t)\Delta v(t,\xi) \;\text{ for }\; t \in I, \quad x \in \mathcal{D}(A) \text{ and } \;\xi \in [0,\pi].$$

Equation (4.4.1) is then transformed into the following form

$$\begin{cases} x'(t) = Ax(t) \;+\; \displaystyle\int_0^t B(t-s)x(s)ds \;+\; f(t,x_t) + Cu(t) \;\text{ for } t \in I, \\ \\ x_0 = \varphi. \end{cases} \tag{4.4.2}$$

f is Lipschitz continuous with respect to the second variable, moreover we have

$$\sup_{\|\varphi\|\leq q} \Big\| f(t,\varphi) \Big\| \leq q\,|\alpha(t)|.$$

So f satisfies $(\mathbf{H_4}) - (\mathbf{i})$ and $(\mathbf{H_4}) - (\mathbf{ii})$ with $l_q(t) \;=\; q\,|\alpha(t)|$. Also f satisfies $(\mathbf{H_4}) - (\mathbf{iii})$, since f is Lipschitz. Now for $\xi \in$, the operator W is given by

$$(Wu)(\xi) = \omega \int_0^1 R(1-s)u(s,\xi)\,ds.$$

Assuming that W satisfies $(\mathbf{H_3})$, then all the conditions of Theorem 4.3.1 hold and equation (4.4.2) is controllable.

CHAPTER 5

Controllability Results for some Partial Functional Integrodifferential Equations with Infinite Delay in Banach Spaces

[1]

5.1 Introduction

In this chapter, we study the controllability of the following partial functional integrodifferential equation with infinite delay in a Banach space $(X, \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t \gamma(t-s)x(s)ds + f(t, x_t) + Cu(t) & \text{for } t \in I = [0, b] \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{5.1.1}$$

where $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space X; for $t \geq 0$, $\gamma(t)$ is a closed linear operator with domain $\mathcal{D}(\gamma(t)) \supset \mathcal{D}(A)$. The control u belongs to $L^2(I, U)$ which is a Banach space of admissible controls, where U is a Banach space. The operator $C \in \mathcal{L}(U, X)$, where $\mathcal{L}(U, X)$ denotes the Banach space of bounded

[1]The results of this chapter are contents of the following paper
- Khalil Ezzinbi and Patrice Ndambomve [45]

linear operators from U into X, and the phase space $\mathcal{B}$ is a linear space of functions mapping $]-\infty, 0]$ into X satisfying axioms which will be described later, for every $t \geq 0$, x_t denotes the history function of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t+\theta) \ \text{ for } \ -\infty \leq \theta \leq 0,$$

$f : I \times \mathcal{B} \to X$ is a continuous function satisfying some conditions. In the literature devoted to equations with finite delay, the phase space is the space of continuous functions on $[-r, 0]$, for some $r > 0$, endowed with the uniform norm topology. But when the delay is unbounded, the selection of the phase space $\mathcal{B}$ plays an important role in both qualitative and quantitative theories. A usual choice is a normed space satisfying some suitable axioms, which was introduced by Hale and Kato [82].
The particular cases in which $\gamma(t) = 0$ and $A = A(t)$ were considered by K. Balachandran and R. Sakthivel; S. Baghli, M. Benchohra and K. Ezzinbi; S. Selvi and M. M. Arjunan and many others. In this work, we extend the work of K. Balachandran and R. Sakthivel by considering some integrodifferential equation when $\gamma(t) \neq 0$, and without any compactness assumption.
The controllability problem of nonlinear systems described by integrodifferential equations in infinite dimensional Banach spaces has been studied by several authors: see for instance [8]-[16], [40, 55, 56, 69, 70, 83, 71] and the references therein. Many authors have also studied the controllability problem of nonlinear systems with delay in inifinite dimensional Banach spaces: see for instance [8], [55], [70], [83], [71], etc and the references therein. For example in [83], the authors proved the controllability of semilinear functional evolution equations with infinite delay using the nonlinear alternative of Leray-Schauder type. In [56], M. Li, M. Wang and F. Zhang proved the controllability of an impulsive functional differential system with finite delay using Schaefer's fixed-point theorem. In [71], S. Selvi and M. M. Arjunan proved the controllability for impulsive differential systems with finite delay using Mönch's fixed-point Theorem, and in [8], K. Balachandran and R. Sakthivel studied the controllability of functional semilinear integrodifferential systems in Banach spaces using Schaefer's fixed point Theorem and the authors assumed that the semigroup generated by the linear part is compact.
In many areas of applications such as Engineering, Electronics, Fluid Dynamics, Physical Sciences, etc..., integrodifferential equations appear and have received considerable attention during the last decades. In recent years, much work on the existence and regularity of solutions of nonlinear integrodifferential equations with finite delay has been done by many authors by applying the resolvent operator theory, see e.g., [43] and the references therein.

In [85], R. Grimmer proved the existence and uniqueness of resolvent operators for these integrodifferential equations that give the variation of parameters formula for the solution. In [24], W. Desch, R. Grimmer and W. Schappacher proved the equivalence of the compactness of the resolvent operator and that of the operator semigroup. In this work, we use the equivalence between the operator-norm continuity of the associated resolvent operator and that of the operator semigroup. This property allows us to drop the compactness assumption on the operator semigroup, considered by the authors in [8, 83], and prove that the operator solution satisfies the Mönch condition. The variation of parameters formula for the mild solutions of equation (5.1.1) is given by the resolvent operator, and we prove the controllability result using the Mönch's fixed-point Theorem and the measure of noncompactness. This method enables us overcome the resolvent operator case considered in this work. In contrary to the evolution semigroup case considered in [55, 71], here the semigroup property can not be used because resolvent operators in general do not form semigroups.
To the best of our knowledge, up to now no work has reported on controllability of partial functional integrodifferential equation (5.1.1) with infinite delay. It has been an untreated topic in the literature, and this fact is the main aim and motivation of the present work.

5.2 Preliminary Results

In this section we introduce some definitions and Lemmas that will be used throughout the paper.
In this work, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [82]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a normed linear space of functions mapping $]-\infty, 0]$ into X and satisfying the following axioms:

($\mathbf{A_1}$) There exist positive constant H and functions $K : \mathbb{R}^+ \to \mathbb{R}^+$ continuous and $M : \mathbb{R}^+ \to \mathbb{R}^+$ locally bounded, such that for $a > 0$, if $x :]-\infty, a] \to X$ is continuous on $[0, a]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0, a]$, the following conditions hold:

(i) $x_t \in \mathcal{B}$,

(ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$, which is equivalent to $\|\varphi(0)\| \le H\|\varphi\|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$,

(iii) $\|x_t\|_{\mathcal{B}} \le K(t) \sup_{0\le s\le t} \|x(s)\| + M(t)\|x_0\|_{\mathcal{B}}$.

($\mathbf{A_2}$) For the function x in ($\mathbf{A_1}$), $t \to x_t$ is a $\mathcal{B}$-valued continuous function for $t \in [0, a]$.

($\mathbf{A_3}$) The space $\mathcal{B}$ is complete.

Example [43] Let the spaces
BC the space of bounded continuous functions defined from $(-\infty, 0]$ to X;
BUC the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to X;
$C^\infty := \Big\{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta) \text{ exists}\Big\}$;
$C^0 := \Big\{\phi \in BC : \lim_{\theta\to-\infty} \phi(\theta) = 0\Big\}$, be endowed with the uniform norm

$$\|\phi\| = \sup_{\theta\le 0} \|\phi(\theta)\|.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions ($\mathbf{A_1}$) $-$ ($\mathbf{A_3}$).

Definition 5.2.1 *Let $u \in L^2(I, U)$ and $\varphi \in \mathcal{B}$. A function $x :]-\infty, b] \to X$ is called a mild solution of equation (5.1.1) if $x \in \mathcal{C}([0, b]; X)$ and satisfies the following integral equation*

$$x(t) = \begin{cases} R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)\left[f(s, x_s) + Cu(s)\right] ds \quad \textit{for } t \in I \\[2ex] \varphi(t) \quad \textit{for} \quad -\infty \le t \le 0. \end{cases} \tag{5.2.1}$$

Definition 5.2.2 *Equation* (5.1.1) *is said to be controllable on the interval I if for every $\varphi \in \mathcal{B}$ and $x_1 \in X$, there exists a control $u \in L^2(I, U)$ such that a mild solution x of equation* (5.1.1) *satisfies the condition $x(b) = x_1$.*

Let

$$R_b = \sup_{t\in[0,b]} \|R(t)\|, \quad K_b = \sup_{t\in[0,b]} \|K(t)\|, \quad M_b = \sup_{t\in[0,b]} \|M(t)\|.$$

We now state the following useful result for equicontinuous subsets of $\mathcal{C}([a, b]; X)$, where X is a Banach space.

Lemma 5.2.3 *[12] Let $M \subset \mathcal{C}([a, b]; X)$ be bounded and equicontinuous. Then $\beta(M(t))$ is continuous and*

$$\beta(M) = \sup\{\beta(M(t));\, t \in [a, b]\}, \quad \textit{where } M(t) = \{x(t);\, x \in M\}.$$

Lemma 5.2.4 *[12] Let $M \subset \mathcal{C}([a,b];X)$ be bounded and equicontinuous. Then the set $\overline{co}(M)$ is also bounded and equicontinuous.*

To prove the controllability for equation (5.1.1), we need the following results.

Lemma 5.2.5 *[40] If $(u_n)_{n\geq 1}$ is a sequence of Bochner integrable functions from I into a Banach space Y with the estimation $\|u_n(t)\| \leq \mu(t)$ for almost all $t \in I$ and every $n \geq 1$, where $\mu \in L^1(I,\mathbb{R})$, then the function*

$$\psi(t) \;=\; \beta(\{u_n(t) : n \geq 1\})$$

belongs to $L^1(I,\mathbb{R}^+)$ and satisfies the following estimation

$$\beta\left(\left\{\int_0^t u_n(s)ds \;:\; n \geq 1\right\}\right) \;\leq\; 2\int_0^t \psi(s)ds.$$

5.3 Controllability result

In this section, we give sufficient conditions ensuring the controllability of equation (5.1.1). For that goal, we need to assume that:

($\mathbf{H_3}$)

(i) The following linear operator $W \,:\, L^2(I,U) \to X$ defined by

$$Wu \;=\; \int_0^b R(b-s)Cu(s)\,ds,$$

is surjective so that it induces an isomorphism between $L^2(I,U)\,/_{\mathrm{Ker}W}$ and X again denoted by W with inverse W^{-1} taking values in $L^2(I,U)\,/_{\mathrm{Ker}W}$, (see e.g.,[68]).

(ii) There exists a function $L_W \in L^1(I,\mathbb{R}^+)$ such that for any bounded set $Q \subset X$ we have

$$\beta((W^{-1}Q)(t)) \leq L_W(t)\beta(Q),$$

where β is the Hausdorff MNC.

($\mathbf{H_4}$) The function $f \,:\, I \times \mathcal{B} \longrightarrow X$ satisfies the following two conditions:

(i) $f(\cdot,\varphi)$ is measurable for $\varphi \in \mathcal{B}$ and $f(t,\cdot)$ is continuous for a.e $t \in I$,

(ii) for every positive integer q, there exists a function $l_q \in L^1(I,\mathbb{R}^+)$ such that

$$\sup_{\|\varphi\|_{\mathcal{B}}\leq q} \|f(t,\varphi)\| \leq l_q(t) \text{ for a.e. } t \in I \text{ and } \liminf_{q\to+\infty} \int_0^b \frac{l_q(t)}{q} dt = l < +\infty,$$

(iii) there exists a function $h \in L^1(I,\mathbb{R}^+)$ such that for any bounded set $D \subset \mathcal{B}$,

$$\beta(f(t,D)) \leq h(t) \sup_{-\infty<\theta\leq 0} \beta(D(\theta)) \text{ for a.e } t \in I,$$

where

$$D(\theta) = \{\phi(\theta) : \phi \in D\}.$$

$(\mathbf{H_5})$

$$\tau = \Big(1 + 2R_b M_2 \|L_W\|_{L^1}\Big)\Big(2R_b\|h\|_{L^1}\Big) < 1,$$

where $R_b = \sup_{0\leq t\leq b} \|R(t)\|$ and M_2 is such that $M_2 = \|C\|$.

Theorem 5.3.1 *Suppose that hypotheses* $(\mathbf{H_3}) - (\mathbf{H_5})$ *hold and equation (2.2.1) has a resolvent operator* $\big(R(t)\big)_{t\geq 0}$ *that is continuous in the operator-norm topology for* $t > 0$. *Then equation* (5.1.1) *is controllable on* I *provided that*

$$R_b(1 + R_b M_2 M_3 b)K_b l < 1, \tag{5.3.1}$$

where M_3 *is such that* $M_3 = \|W^{-1}\|$.

Proof. Using $(\mathbf{H_3})$ and given an arbitrary function x, we define the control as usual by the following formula:

$$u_x(t) \;=\; W^{-1}\left\{x_1 - R(b)\varphi(0) - \int_0^b R(b-s)f(s,x_s)\,ds\right\}(t) \qquad \text{for } t \in I.$$

For each $x \in \mathcal{C}([0,b],X)$ such that $x(0) = \varphi(0)$, we define its extension $\widetilde{x}$ from $]-\infty,b]$ to X as follows

$$\widetilde{x}(t) = \begin{cases} x(t) & \text{if } t \in [0,b] \\ \varphi(t) & \text{if } t \in]-\infty,0] \end{cases}$$

We define the following space

$$E_b = \Big\{ x :]-\infty, b] \to X \text{ such that } x|_I \in \mathcal{C}([0,b], X) \text{ and } x_0 \in \mathcal{B} \Big\}.$$

where $x|_I$ is the restriction of x to I.
We show using this control that the operator $P:\, E_b \to E_b$ defined by

$$(Px)(t) = R(t)\varphi(0) + \int_0^t R(t-s)\big[f(s,\widetilde{x}_s) + Cu_x(s)\big]\,ds \quad \text{for} \quad t \in I = [0,b]$$

has a fixed-point. This fixed point is then a mild solution of equation (5.1.1). Observe that $(Px)(b) = x_1$. This means that the control u_x steers the integrodifferential equation from φ to x_1 in time b which implies the equation (5.1.1) is controllable on I.
For each $\varphi \in \mathcal{B}$, we define the function $y \in \mathcal{C}([0,b], X)$ by $y(t) = R(t)\varphi(0)$ and its extension $\widetilde{y}$ on $]-\infty, 0]$ by

$$\widetilde{y}(t) = \begin{cases} y(t) & \text{if} \quad t \in [0,b] \\ \\ \varphi(t) & \text{if} \quad t \in]-\infty, 0] \end{cases}$$

For each $z \in \mathcal{C}([0,b], X)$, set $\widetilde{x}(t) = \widetilde{z}(t) + \widetilde{y}(t)$, where $\widetilde{z}$ is the extension by zero of the function z on $]-\infty, 0]$. Observe that x satifies (5.2.1) if and only if $z(0) = 0$ and

$$z(t) = \int_0^t R(t-s)\big[f(s,\widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)\big]\,ds \quad \text{for t} \in [0,\text{b}],$$

where

$$u_z(t) \;=\; W^{-1}\Big\{ x_1 - R(b)\varphi(0) - \int_0^b R(b-s)f(s,\widetilde{z}_s + \widetilde{y}_s)\,ds \Big\}(t).$$

Now let

$$E_b^0 = \Big\{ z \in E_b \text{ such that } z_0 = 0 \Big\}.$$

Thus E_b^0 is a Banach space provided with the supremum norm. Define the operator $\Gamma \,:\, E_b^0 \to E_b^0$ by

$$(\Gamma z)(t) = \int_0^t R(t-s)\big[f(s,\widetilde{z}_s + \widetilde{y}_s) + Cu_z(s)\big]\,ds \quad \text{for} \quad t \in [0,b]$$

Note that the operator P has a fixed point if and only if Γ has one. So to prove that P has a fixed point, we only need to prove that Γ has one.

For each positive number q, let $B_q = \{z \in E_b^0 : \|z\| \leq q\}$. Then, for any $z \in B_q$, we have by axiom $(\mathbf{A_1})$ that

$$\begin{aligned}
\|z_s + y_s\| &\leq \|z_s\|_{\mathcal{B}} + \|y_s\|_{\mathcal{B}} \\
&\leq K(s)\|z(s)\| + M(s)\|z_0\|_{\mathcal{B}} + K(s)\|y(s)\| + M(s)\|y_0\|_{\mathcal{B}} \\
&\leq K_b\|z(s)\| + K_b\|R(t)\|\|\varphi(0)\| + M_b\|\varphi\|_{\mathcal{B}} \\
&\leq K_b\|z(s)\| + K_bR_bH\|\varphi\|_{\mathcal{B}} + M_b\|\varphi\|_{\mathcal{B}} \\
&\leq K_b\|z(s)\| + \Big(K_bR_bH + M_b\Big)\|\varphi\|_{\mathcal{B}} \\
&\leq K_b\, q + \Big(K_bR_bH + M_b\Big)\|\varphi\|_{\mathcal{B}}
\end{aligned}$$

Thus,

$$\|z_s + y_s\| \leq K_b\, q + \Big(K_bR_bH + M_b\Big)\|\varphi\|_{\mathcal{B}} =: q'.$$

We shall prove the theorem in the following steps.

Step1. We claim that there exists $q > 0$ such that $\Gamma(B_q) \subset B_q$. We proceed by contradiction. Assume that it is not true. Then for each positive number q, there exists a function $z^q \in B_q$, such that $\Gamma(z^q) \notin B_q$, *i.e.*, $\|(\Gamma z^q)(t)\| > q$ for some $t \in [0, b]$. Now we have that

$$\begin{aligned}
q &< \Big\|(\Gamma z^q)(t)\Big\| \\
&\leq R_b \int_0^b \Big\|f(s, \tilde{z}_s^q + \widetilde{y}_s)\Big\|\, ds + R_b \int_0^b \|Cu_{z^q}(s)\|\, ds \\
&\leq R_b \int_0^b \Big\|f(s, \tilde{z}_s^q + \widetilde{y}_s)\Big\|\, ds + R_b \int_0^b \Big\|CW^{-1}\Big[x_1 - R(b)\varphi(0) - \int_0^b R(b-s)f(s, \tilde{z}_s^q)\, ds\Big]\Big\|\, ds \\
&\leq bR_bM_2M_3\left(\|x_1\| + R_b\|\varphi(0)\| + R_b\int_0^b \|f(s, \tilde{z}_s^q)\|\, ds\right) + R_b\int_0^b \Big\|f(s, \tilde{z}_s^q + \widetilde{y}_s)\Big\|\, ds \\
&\leq bR_bM_2M_3\left(\|x_1\| + R_bH\|\varphi\|_{\mathcal{B}} + R_b\int_0^b l_{q'}(s)\, ds\right) + R_b\int_0^b l_{q'}(s)\, ds,
\end{aligned}$$

where $q' := K_b\, q + q_0$, with $q_0 := \Big(K_bR_bH + M_b\Big)\|\varphi\|_{\mathcal{B}}$.
Hence

$$q \leq \Big(1 + R_bM_2M_3b\Big)R_b\int_0^b l_{q'}(s)\, ds + R_bM_2M_3b\Big(\|x_1\| + R_bH\|\varphi\|_{\mathcal{B}}\Big).$$

Dividing both sides by q and noting that $q' = K_bq + q_0 \to +\infty$ as $q \to +\infty$, we obtain that

$$1 \leq \Big(1 + R_b M_2 M_3 b\Big) R_b \left(\frac{\displaystyle\int_0^b l_{q'}(s)\, ds}{q} \right) + \frac{R_b M_2 M_3 b \Big(\|x_1\| + R_b H \|\varphi\|_{\mathcal{B}} \Big)}{q}$$

and

$$\liminf_{q \to +\infty} \left(\frac{\displaystyle\int_0^b l_{q'}(s)\, ds}{q} \right) = \liminf_{q \to +\infty} \left(\frac{\displaystyle\int_0^b l_{q'}(s)\, ds}{q'} \frac{q'}{q} \right) = l K_b.$$

Thus we have, $1 \leq \Big(1 + R_b M_2 M_3 b\Big) R_b K_b l$, and this contradicts (5.3.1). Hence for some positive number q, we must have $\Gamma(B_q) \subset B_q$.

Step2. $\Gamma : E_b^0 \to E_b^0$ is continuous. In fact let $\Gamma := \Gamma_1 + \Gamma_2$, where

$$(\Gamma_1 z)(t) = \int_0^t R(t-s) f(s, \widetilde{z}_s + \widetilde{y}_s)\, ds \quad \text{and} \quad (\Gamma_2 z)(t) = \int_0^t R(t-s) C u_z(s)\, ds.$$

Let $\{z^n\}_{n\geq 1} \subset E_b^0$ with $z^n \to z$ in E_b^0. Then there exists a number $q > 1$ such that $\|z^n(t)\| \leq q$ for all n and *a.e.* $t \in I$. So $z^n, z \in B_q$. By $(\mathbf{H_4}) - (\mathbf{i})$, $f(t, \widetilde{z}_t^n + \widetilde{y}_t) \to f(t, \widetilde{z}_t + \widetilde{y}_t)$ for each $t \in [0, b]$. And by $(\mathbf{H_4}) - (\mathbf{ii})$,

$$\|f(t, \widetilde{z}_t^n + \widetilde{y}_t) - f(t, \widetilde{z}_t + \widetilde{y}_t)\| \leq 2 l_{q'}(t).$$

Then we have

$$\|\Gamma_1 z^n - \Gamma_1 z\|_{\mathcal{C}} \leq R_b \int_0^b \|f(s, \widetilde{z}_s^n + \widetilde{y}_s) - f(s, \widetilde{z}_s + \widetilde{y}_s)\|\, ds \longrightarrow 0, \ \text{as } n \to +\infty$$

by dominated convergence Theorem. Also we have that

$$\|\Gamma_2 z^n - \Gamma_2 z\|_{\mathcal{C}} \leq R_b^2 M_2 M_3 b \int_0^b \|f(s, \widetilde{z}_s^n) - f(s, \widetilde{z}_s)\|\, ds \longrightarrow 0, \ \text{as } n \to +\infty$$

by dominated convergence Theorem. Thus

$$\|\Gamma z^n - \Gamma z\| \leq \|\Gamma_1 z^n - \Gamma_1 z\| + \|\Gamma_2 z^n - \Gamma_2 z\| \longrightarrow 0, \ \text{as } n \to +\infty.$$

Hence Γ is continuous on E_b^0.

Step3. $\Gamma(B_q)$ is equicontinuous on $[0, b]$. In fact let $t_1,\, t_2 \in I, \quad t_1 < t_2$ and $z \in B_q$, we have

$$\begin{aligned}
\|(\Gamma z)(t_2) - (\Gamma z)(t_1)\| \;\leq\; & \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\| \|f(s, \widetilde{z}_s + \widetilde{y}_s) + C u_z(s)\| \, ds \\
+ & \int_{t_1}^{t_2} \|R(t_2 - s)\| \|f(s, \widetilde{z}_s + \widetilde{y}_s) + C u_z(s)\| \, ds \\
\leq & \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\| l_{q'}(s) \, ds \\
+ & \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\| M_2 M_3 \times \\
& \left[\|x_1\| + R_b H \|\varphi\|_{\mathcal{B}} + R_b \int_0^b l_{q'}(\tau) \, d\tau \right] ds \\
+ & \int_{t_1}^{t_2} \|R(t_2 - s)\| l_{q'}(s) \, ds \\
+ & \int_{t_1}^{t_2} \|R(t_2 - s)\| M_2 M_3 \left(\|x_1\| + R_b H \|\varphi\|_{\mathcal{B}} + R_b \int_0^b l_{q'}(\tau) \, d\tau \right) ds
\end{aligned}$$

By the continuity of $\big(R(t)\big)_{t \geq 0}$ in the operator-norm toplogy, the dominated convergence Theorem, we conclude that the right hand side of the above inequality tends to zero and independent of z as $t_2 \to t_1$. Hence $\Gamma(B_q)$ is equicontinuous on I.

Step4. We show that the Mönch's condition holds.

Suppose that $D \subseteq B_q$ is countable and $D \subseteq \overline{co}(\{0\} \cup \Gamma(D))$. We shall show that $\beta(D) = 0$, where β is the Hausdorff MNC. Without loss of generality, we may assume that $D = \{z^n\}_{n \geq 1}$. Since Γ maps B_q into an equicontinuous family, $\Gamma(D)$ is also equicontinuous on I.

By $(\mathbf{H_3})-(\mathbf{ii})$, $(\mathbf{H_4})-(\mathbf{iii})$ and Lemma 5.2.5, we have that

$$\begin{aligned}
\beta\Big(\{u_{z^n}(t)\}_{n\geq 1}\Big) &= \beta\left(W^{-1}\left\{x_1 - R(b)\varphi(0) - \int_0^b R(t-b) f\Big(s, \{\tilde{z}_s^n + \widetilde{y}_s\}_{n\geq 1}\Big)\, ds\right\}_{n\geq 1}(t)\right) \\
&\leq L_W(t)\beta\left(\{x_1 - R(b)\varphi(0)\}\right) \\
&\quad + L_W(t)\beta\left(\left\{\int_0^b R(t-b) f\Big(s, \{\tilde{z}_s^n + \widetilde{y}_s\}_{n\geq 1}\Big)\, ds\right\}_{n\geq 1}\right) \\
&\leq 2R_b L_W(t)\left(\int_0^b h(s)\beta\left(\{\tilde{z}_s^n\}_{n\geq 1} + \{\widetilde{y}_s\}\right) ds\right) \\
&\leq 2R_b L_W(t)\left(\int_0^b h(s)\Big[\beta\left(\{\tilde{z}_s^n\}_{n\geq 1}\right) + \beta\Big(\{\widetilde{y}_s\}\Big)\Big]\, ds\right) \\
&\leq 2R_b L_W(t)\left(\int_0^b h(s)\beta\left(\{\tilde{z}_s^n\}_{n\geq 1}\right)\, ds\right), \text{ since } \Big\{\widetilde{y}_s :\ s\in[0,b]\Big\} \text{ is compact} \\
&\leq 2R_b L_W(t)\left(\int_0^b h(s) \sup_{-\infty<\theta\leq 0}\beta\left(\{\tilde{z}_s^n(\theta)\}_{n\geq 1}\right)\, ds\right) \\
&\quad \Big(\text{by Lemma 5.2.3, since } D = \{z^n\}_{n\geq 1} \text{ is equicontinuous}\Big) \\
&\leq 2R_b L_W(t)\left(\int_0^b h(s)\, ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)
\end{aligned}$$

This implies that

$$\begin{aligned}
\beta\Big(\{(\Gamma z^n)(t)\}_{n\geq 1}\Big) &\leq \beta\left(\left\{\int_0^t R(t-s) f(s, \{\tilde{z}_s^n + \widetilde{y}_s\}_{n\geq 1})\, ds\right\}_{n\geq 1}\right) \\
&\quad + \beta\left(\left\{\int_0^t R(t-s) u_{z^n}(s)\, ds\right\}_{n\geq 1}\right) \\
&\leq 2R_b\left(\int_0^b h(s)\, ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right) \\
&+ 2R_b M_2\left(\int_0^b L_W(s)\, ds\right) 2R_b\left(\int_0^b h(s)\, ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right) \\
&\leq 2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right) \\
&+ 2R_b M_2\|L_W\|_{L^1} 2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}\beta\Big(\Gamma(D)(t)\Big) &\leq 2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)+2R_bM_2\|L_W\|_{L^1}2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)\\ &\leq \Big(1+2R_bM_2\|L_W\|_{L^1}\Big)2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)\\ &= \tau\sup_{0\leq t\leq b}\beta\Big(D(t)\Big)\end{aligned}$$

Since D and $\Gamma(D)$ are equicontinuous on $[0,b]$ and D is bounded, it follows by Lemma 5.2.3 that $\beta\Big(\Gamma(D)\Big)\leq\tau\beta\Big(D\Big)$, where τ is as defined in $(\mathbf{H_5})$. Thus from the Mönch condition, we get that

$$\beta\Big(D\Big)\leq\beta\Big(\overline{co}(\{0\}\cup\Gamma(D)\Big)=\beta\Big(\Gamma(D)\Big)\leq\tau\beta\Big(D\Big),$$

and since $\tau<1$, this implies $\beta\Big(D\Big)=0$, which implies that D is relatively compact as desired in B_q and the Mönch condition is satisfied. We conclude by Theorem 2.4.2, that Γ has a fixed point z in B_q. Then $x=z+y$ is a fixed point of P in E_b and thus equation (5.1.1) is controllable on $[0,b]$.
we now illustrate our main result by the following example.

5.4 Example

Consider the following nonlinear integrodifferential equation.

$$\begin{cases}\dfrac{\partial v}{\partial t}(t,\xi)=\dfrac{\partial v}{\partial\xi}(t,\xi)+\displaystyle\int_0^t\zeta(t-s)\frac{\partial v}{\partial\xi}(s,\xi)\,ds+\int_{-\infty}^0\alpha(\theta)g(t,v(t+\theta,\xi))\,d\theta+\eta\omega(t,\xi)\\ \qquad\qquad\text{for } t\in I=[0,1] \text{ and } \xi\in[0,\pi]\\ v(t,0)=v(t,\pi)=0\quad\text{for } t\in[0,1]\\ v(\theta,\xi)=\phi(\theta,\xi)\quad\text{for } \theta\in]-\infty,0] \text{ and } \xi\in[0,\pi],\end{cases}\tag{5.4.1}$$

where $\eta>0$, $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and Lipschitzian with respect to the second variable, the initial data function $\phi:\mathbb{R}^-\times[0,\pi]\to\mathbb{R}$ is a given function, $\omega:[0,1]\times[0,\pi]\to\mathbb{R}$ continuous in t and $\omega(t,0)=\omega(t,\pi)=0$, $\alpha:\mathbb{R}^-\to\mathbb{R}$ is continuous, $\alpha\in L^1(\mathbb{R}^-,\mathbb{R})$ and $\zeta\in W^{1,1}(\mathbb{R}^+,\mathbb{R}^+)$.
Let $X=U=\mathcal{C}_0([0,\pi],\mathbb{R})$, the space of all continuous functions from $[0,\pi]$ to $\mathbb{R}$ vanishing at 0 and π equipped with the uniform norm topology, and

the phase space $\mathcal{B} = BUC(\mathbb{R}^-, X)$, the the space of uniformly bounded continuous functions endowed with the following norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta \leq 0} \|\varphi(\theta)\|.$$

Then, the space $BUC(\mathbb{R}^-, X)$ satisfies axioms $(\mathbf{A_1})$, $(\mathbf{A_2})$ and $(\mathbf{A_3})$.

We define $A : \mathcal{D}(A) \subset X \to X$ by:

$$\begin{cases} \mathcal{D}(A) = \Big\{ y \in X : \, y' \text{ exists and } y' \in X \Big\} \\ \\ Ay \; = \; y'. \end{cases}$$

Then, A is the infinitesimal generator of a C_0-semigroup. Moreover, the C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ generated by A above and defined by

$$T(t)y(s) = y(t+s) \;\; \text{for } y \in X,$$

is operator-norm continuous for $t > 0$. Thus by Theorem 2.2.6, the corresponding resolvent operator is operator-norm continuous for $t > 0$.
Now define

$$x(t)(\xi) \; = \; v(t,\xi), \quad x'(t)(\xi) \; = \; \frac{\partial v(t,\xi)}{\partial t}, \quad \omega(t,\xi) = u(t)(\xi).$$

$$\varphi(\theta)(\xi) = \phi(\theta,\xi) \;\; \text{for } \theta \in]-\infty, 0] \text{ and } \xi \in [0,\pi].$$

$$f(t,\psi)(\xi) \; = \; \int_{-\infty}^{0} \alpha(\theta) g(t, \psi(\theta)(\xi))\, d\theta \;\; \text{for } \theta \in]-\infty, 0] \text{ and } \xi \in [0,\pi].$$

$C : X \to X$ be defined by $\Big(Cu(t)\Big)(\xi) \; = \; Cu(t)(\xi) \; = \; \eta\omega(t,\xi).$

$$(\gamma(t)x)(\xi) \; = \; \zeta(t)\Delta v(t,\xi) \;\; \text{for} \;\; t \in [0,1], \;\; x \in \mathcal{D}(A) \text{ and } \;\; \xi \in [0,\pi].$$

We suppose that $\varphi \in BUC(\mathbb{R}^-, X)$. Then, equation (5.4.1) is then transformed into the following form:

$$\begin{cases} x'(t) = Ax(t) \; + \; \displaystyle\int_0^t \gamma(t-s)x(s)ds \; + \; f(t,x_t) + Cu(t) \;\; \text{for } t \in I = [0,1], \\ \\ x_0 = \varphi \in \mathcal{B}. \end{cases} \tag{5.4.2}$$

Suppose there exists a continuous function $p \in L^1(I;\mathbb{R}^+)$ such that

$$|g(t,y_1) - g(t,y_2)| \leq p(t)|y_1 - y_2| \;\; \text{for } t \in I \text{ and } y_1, \, y_2 \in \mathbb{R}.$$

and

$$g(t, 0) = 0 \quad \text{for } t \in I.$$

One can see that f is Lipschitz continuous with respect to the second variable and moreover for $\varphi \in \mathcal{B}$, we have we have

$$\sup_{\|\varphi\|_{\mathcal{B}} \leq q} \left\| f(t, \varphi) \right\| \leq q \|\alpha\| \, p(t).$$

So f satisfies $(\mathbf{H_4}) - (\mathbf{i})$ and $(\mathbf{H_4}) - (\mathbf{ii})$ with $l_q(t) = q \|\alpha\| \, p(t)$. Also f satisfies $(\mathbf{H_4}) - (\mathbf{iii})$ by condition $(\mathbf{viii})$ of Theorem 2.3.2, since f is Lipschitz. Now for $\xi \in [0, \pi]$, the operator W is given by

$$(Wu)(\xi) = \eta \int_0^1 R(1 - s)\omega(s, \xi) \, ds.$$

Assuming that W satisfies $(\mathbf{H_3})$, then all the conditions of Theorem 5.3.1 hold and equation (5.4.2) is controllable.

Part II

Optimal Controls of some Partial Functional Integrodifferential Equations in Banach Spaces

CHAPTER 6

Solvability and Optimal Control for some Partial Functional Integrodifferential Equations with Finite Delay

[1]

6.1 Introduction

The aim of this chapter is to study the existence of mild solutions and the optimal controls of some systems that take the form of the following partial functional integrodifferential equation with finite delay in a Banach space $(X, \ \|\cdot\|)$.

$$\begin{cases} x'(t) = Ax(t) \ + \ \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t, x_t) + C(t)u(t) \ \ \text{for } t \in I = [0, b] \\ x_0 = \varphi \in \mathcal{C} = \mathcal{C}([-r, 0]; X), \end{cases} \tag{6.1.1}$$

where $f : I \times \mathcal{C} \to X$ is a function satisfying some conditions; $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a separable reflexive Banach space X; for $t \geq 0$, $B(t)$ is a closed linear operator with

[1]The results of this chapter are contents of the following paper
- Khalil Ezzinbi and Patrice Ndambomve [91]

domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$. The control $u(t)$ takes values from another separable reflexive Banach space U. The operator $C(t)$ belongs to $\mathcal{L}(U, X)$, the Banach space of bounded linear operators from U into X, and $\mathcal{C}([-r, 0], X)$ denotes the Banach space of continuous functions $\varphi : [-r, 0] \to X$ with supremum norm $\|\varphi\| = \sup_{\theta \in [-r,0]} \|\varphi(\theta)\|$, x_t denotes the history function of $\mathcal{C}$ defined by

$$x_t(\theta) = x(t + \theta) \quad \text{for } \theta \in [-r, 0].$$

In many areas of applications such as engineering, electronics, fluid dynamics, physical sciences, etc..., integrodifferential equations appear and have received considerable attention during the last decades. In [85], R. Grimmer has proved the existence and uniqueness of resolvent operators for these integrodifferential equations that give the variation of parameter formula for the solution. In recent years, much work has been done on the existence and regularity of solutions of nonlinear integrodifferential equations with finite delay by many authors by applying the resolvent operator theory, for integral equations see e.g., [86] and the references therein. Problems of controllability and existence of optimal controls for nonlinear differential equations have been studied extensively by many authors under various hypotheses (see e.g., [87], [88],[89],[90],[94] [95]), but little is known and done about the existence of optimal controls for integrodifferential equations with delay using the resolvent operator theory. In [96], the authors studied the existence and continuous dependence of mild solutions and the optimal controls of a Lagrange problem for some fractional integrodifferential equation with infinite delay in Banach spaces using the using the techniques of *a priori* estimation and extension of step by steps. Wand and Zhou [97] discussed the optimal controls of a Lagrange problem for fractional evolution equations. In [98], Wei *et al.* studied the optimal controls for nonlinear impulsive integrodifferential equations of mixed type on Banach spaces. In [99], the authors studied the existence of mild solutions and the optimal controls of a Lagrange problem for some impulsive fractional semilinear differential equations, using the techniques of *a priori* estimation. Motivated by these works, we investigate the solvability and the existence of optimal controls of a Lagrange problem for equation (6.1.1), using the techniques of *a priori* estimation of mild solutions. The existence and uniqueness of mild solutions is obtained using the theory of resolvent operator for integral equations. Furthermore, to the best of our knowledge, the optimal controls for partial functional integrodifferential equation (6.1.1) with finite delay are untreated in the literature, and this fact motivates us to extend the existing ones and make new development of the present work on this issue.

6.2 Existence of mild solutions for equation (6.1.1)

We make the following assumptions.

($\mathbf{H_3}$) The function $f : I \times \mathcal{C} \to X$ satisfies the following conditions:

(i) $f(\cdot, \psi)$ is measurable for $\psi \in \mathcal{C}$,

(ii) for any $\rho > 0$, there exists $L_f(\rho) > 0$ such that

$$\|f(t,\psi_1)-f(t,\psi_2)\| \leq L_f(\rho)\|\psi_1-\psi_2\| \quad \text{for } \|\psi_1\| \leq \rho,\ \|\psi_2\| \leq \rho \text{ and } t \in [0,b],$$

(iii) there exists $a_f > 0$ such that

$$\|f(t,\psi)\| \leq a_f(1+\|\psi\|) \quad \text{for all} \quad \psi \in \mathcal{C} \quad \text{and} \quad t \in [0,b].$$

($\mathbf{H_4}$) Let U be the separable reflexive Banach space from which the control u takes values and assume $C \in L^\infty(I; \mathcal{L}(U,X))$.

($\mathbf{H_5}$) The multivalued map $\Gamma : I \to 2^U \setminus \{\emptyset\}$ has closed, convex, and bounded values, Γ is graph measurable, and $\Gamma(\cdot) \subseteq \Omega$ where Ω is a bounded set in U.

We denote by $\mathcal{U}_{ad}$ the set of admissible controls defined by:

$$\mathcal{U}_{ad} = \Big\{ u : I \to U \text{ such that } u \text{ is measurable and } u(t) \in \Gamma(t),\ a.e. \Big\}.$$

Then, we have the following:

Theorem 6.2.1 *[96] $\mathcal{U}_{ad} \neq \emptyset$ and $\mathcal{U}_{ad} \subset L^2(I,U)$ is bounded, closed and convex. Also, $Cu \in L^2(I,U)$ for all $u \in \mathcal{U}_{ad}$.*

Definition 6.2.2 *Let $u \in \mathcal{U}_{ad}$ and $\varphi \in \mathcal{C}$. A function $x \in \mathcal{C}([-r,b],X)$ is called a mild solution of equation (6.1.1) if*

$$x(t) = \begin{cases} R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)\left[f(s,x_s) + C(s)u(s)\right] ds & \text{for } t \in I \\ \varphi(t) \quad \text{for} \quad -r \leq t \leq 0. \end{cases} \tag{6.2.1}$$

We have the following Theorem on existence of mild solutions to equation (6.1.1) with respect to a given control $u \in \mathcal{U}_{ad}$.

Theorem 6.2.3 *Assume that* $(\mathbf{H_1}) - (\mathbf{H_5})$ *hold. Then for each* $u \in \mathcal{U}_{ad}$, *equation* (6.1.1) *has a unique mild solution on* $[-r, b]$.

Proof. Let $b_1 \leq b,\ \rho > 0$, and $\psi \in \mathcal{C}$ such that $\|\psi\| \leq \rho$. For $t \in [0, b_1]$, we have by the local Lipschitz condition on f that

$$\|f(t,\psi)\| \leq L_f(\rho)\|\psi\| + \|f(t,0)\| \leq L_f(\rho)\rho + \sup_{s\in[0,b_1]} \|f(s,0)\|.$$

b_1 will be chosen sufficiently small enough to get the local existence of mild solutions.
Let $\varphi \in \mathcal{C},\ \rho = \|\varphi\| + 1$ and $\rho^* = L_f(\rho)\rho + \sup_{s\in[0,b_1]} \|f(s,0)\|$.
We define the following space

$$E_\varphi = \Big\{ x \in \mathcal{C}([-r,b_1];X) \text{ such that } x(\theta) = \varphi(\theta) \text{ for } \theta \in [-r,0] \text{ and } \sup_{s\in[0,b_1]} \|x(s) - \varphi(0)\| \leq 1 \Big\}.$$

For $x \in E_\varphi$, one can see that $\|x_t\| \leq 1 + \|\varphi\| = \rho$.
Then, E_φ is a closed subset of $\mathcal{C}([-r,b_1];X)$ which is endowed with the uniform norm topology. Let

$$M_b = \sup_{t\in[0,b]} \|R(t)\|.$$

Define the operator $K :\ E_\varphi \to \mathcal{C}([-r,b_1];X)$ by

$$(Kx)(t) = \begin{cases} R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)\big[f(s,x_s) + C(s)u(s)\big]\, ds & \text{for } t \in [0,b_1] \\ \varphi(t) \quad \text{for} \quad t \in [-r,0] \end{cases}$$

We claim that $K\Big(E_\varphi\Big) \subset E_\varphi$. In fact let $x \in E_\varphi$ and $t \in [0,b_1]$. Then,

$$\begin{aligned} \|(Kx)(t) - \varphi(0)\| &\leq \|R(t)\varphi(0) - \varphi(0)\| \\ &\quad + \int_0^t \Big\| R(t-s)\big[f(s,x_s) + C(s)u(s)\big] \Big\|\, ds \\ &\leq \|R(t)\varphi(0) - \varphi(0)\| + M_b\, \rho^* t + M_b\, \|C\|\, \|u\|_{L^2} \sqrt{t}. \end{aligned}$$

Now, choose b_1 sufficiently small such that

$$\sup_{s\in[0,b_1]} \Big\{ \|R(s)\varphi(0) - \varphi(0)\| + M_b\, \rho^* s + M_b\, \|C\|\, \|u\|_{L^2} \sqrt{s} \Big\} < 1. \qquad (6.2.2)$$

Consequently,

$$\|(Kx)(t)-\varphi(0)\| \leq \|R(t)\varphi(0)-\varphi(0)\|+M_b\,\rho^* t+M_b\,\|C\|\,\|u\|_{L^2}\sqrt{t} < 1 \text{ for } t \in [0,b_1].$$

Hence, $K\Big(E_\varphi\Big) \subset E_\varphi$.
Let x, $y \in E_\varphi$ and $t \in [0,b_1]$. Then, $\|x_s\|$, $\|y_s\| \leq \rho$ for $s \in [0,b_1]$ and we have

$$\begin{aligned}\|(Kx)(t)-(Ky)(t)\| &\leq M_b \int_0^t \|f(s,x_s)-f(s,y_s)\|\,ds \\ &\leq M_b\,L_f(\rho)\int_0^t \|x_s-y_s\|\,ds \\ &\leq M_b\,L_f(\rho)\int_0^t \sup_{\tau\in[0,s]}\|x(\tau)-y(\tau)\|\,d\tau \\ &\leq M_b\,L_f(\rho)\,b_1\,\|x-y\|\end{aligned}$$

Now, since

$$M_b\,L_f(\rho)\,b_1 \leq M_b\,\rho^*\,b_1 < \sup_{s\in[0,b_1]}\Big\{\|R(s)\varphi(0)-\varphi(0)\|+M_b\,\rho^* s+M_b\,\|C\|\,\|u\|_{L^2}\sqrt{s}\Big\}.$$

Condition (6.2.2) implies that

$$M_b\,L_f(\rho)\,b_1 < 1.$$

Thus, K is a strict contraction on E_φ. It follows from the contraction mapping principle that K has a unique fixed point $x \in E_\varphi$, which is the unique mild solution of equation (6.1.1) with respect to u on $[-r,b_1]$.
Using the same arguments, we can show that x can be extended to a maximal interval of existence $[0,t_{\max}[$.

Lemma 6.2.4 *[86] If $t_{\max} < b$, then,* $\limsup_{t\to t_{\max}}\|x(t)\| = \infty$.

We show that $t_{\max} = b$. Assume on the contrary that $t_{\max} < b$. Then for $t \in [0,t_{\max}]$ we have that

$$x(t) = R(t)\varphi(0) + \int_0^t R(t-s)\big[f(s,x_s)+C(s)u(s)\big]\,ds.$$

It follows that

$$\begin{aligned}\|x(t)\| &\leq M_b\|\varphi(0)\| + M_b\int_0^t \|f(s,x_s)\|\,ds + M_b\int_0^t \|C(s)u(s)\|\,ds\\ &\leq M_b\|\varphi\| + M_b\,t_{\max}\,a_f + M_b a_f\int_0^t \|x_s\|\,ds + M_b\|C\|\int_0^t \|u(s)\|\,ds\\ &\leq M_b\|\varphi\| + M_b\,t_{\max}\,a_f + M_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2} + M_b\,a_f\int_0^t \|x_s\|\,ds\\ &\leq M_b\|\varphi\| + M_b\,t_{\max}\,a_f + M_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2} + M_b\,a_f\int_0^t \sup_{\tau\in[-r,s]}\|x(\tau)\|\,d\tau\end{aligned}$$

This implies that

$$\sup_{s\in[-r,t]}\|x(s)\| \leq M_b\|\varphi\|+M_b\,t_{\max}\,a_f+M_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2}+M_b\,a_f\int_0^t \sup_{\tau\in[-r,s]}\|x(\tau)\|\,d\tau$$

It follows by Gronwall's inequality that

$$\|x(t)\| \leq \beta^* e^{M_b\,a_f\,t} \quad \text{for } t\in[0,t_{\max}],$$

where $\beta^* = M_b\|\varphi\| + M_b\,t_{\max}\,a_f + M_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2}$.
Thus

$$\lim_{t\to t_{\max}}\|x(t)\| \leq \beta^* e^{M_b\,a_f\,t_{\max}} < \infty.$$

This contradicts Lemma 6.2.4. Therefore, $t_{\max} = b$ and hence, equation (6.1.1) has a unique mild solution on $[-r,b]$.

□

6.3 Continuous Dependence

In this section, we discuss the continuous dependence of the mild solutions of equation (6.1.1) on the controls and initial states.
We have the following a priori estimation.

Lemma 6.3.1 *Suppose* $(\mathbf{H_1})-(\mathbf{H_3})$ *hold and assume that equation (6.1.1) has a mild solution* x_u *on* $[-r,b]$ *with respect to* $u\in\mathcal{U}_{ad}$. *Then, there exists a constant* $\rho>0$ *independent of* u *such that* $\|x_u(t)\|\leq\rho$ *for* $t\in[0,b]$, *(*ρ *depends only on* $\mathcal{U}_{ad}$ *and* φ*).*

Proof. Let $\varphi \in \mathcal{C}$. We define the following space

$$E_b = \Big\{x : [0,b] \to X \text{ continuous such that } x(0) = \varphi(0)\Big\}.$$

For $x \in E_b$, we define its continuous extension $\widetilde{x}$ over $[-r,b]$ by

$$\widetilde{x}(t) = \begin{cases} x(t) & \text{for } \ t \in [0,b] \\ \\ \varphi(t) & \text{for } \ t \in [-r,0] \end{cases}$$

For $\varphi \in \mathcal{C}$, we define the function $y \in \mathcal{C}([0,b],X)$ by $y(t) = R(t)\varphi(0)$ and its extension $\widetilde{y}$ in $\mathcal{C}([-r,b],X)$ by

$$\widetilde{y}(t) = \begin{cases} y(t) & \text{for } \ t \in [0,b] \\ \\ \varphi(t) & \text{for } \ t \in [-r,0] \end{cases}$$

For each $z \in \mathcal{C}([0,b],X)$, set $\widetilde{x}(t) = \widetilde{z}(t) + \widetilde{y}(t)$, where $\widetilde{z}$ is the extension by zero of the function z on $[-r,0]$. Observe that x satifies (6.2.1) if and only if $z(0) = 0$ and

$$z(t) = \int_0^t R(t-s)\big[f(s,\widetilde{z}_s + \widetilde{y}_s) + C(s)u(s)\big]\,ds \ \ \text{for } t \in [0,b],$$

Let

$$M_b = \sup_{t\in[0,b]} \|R(t)\| \ \ \text{and} \ \ \|C\| = \sup_{t\in I} \|C(t)\|_{\mathcal{L}(U,X)}.$$

Since $\mathcal{U}_{ad}$ is bounded, let $\widetilde{K} > 0$ be such that $\|u\|_{L^2} \le \widetilde{K}$ for all $u \in \mathcal{U}_{ad}$.

$$\begin{aligned}
\|z(t)\| &\le M_b \int_0^t \|f(s,\widetilde{z}_s + \widetilde{y}_s)\|\,ds + M_b \int_0^t \|C(s)u(s)\|\,ds \\
&\le M_b\, b\, a_f + M_b a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\|\,ds + M_b\|C\| \int_0^b \|u(s)\|\,ds \\
&\le M_b\, b\, a_f + M_b\sqrt{b}\,\|C\|\|u\|_{L^2} + M_b\, a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\|\,ds \\
&\le M_b\, b\, a_f + M_b\sqrt{b}\,\|C\|\widetilde{K} + M_b\, a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\|\,ds,
\end{aligned}$$

Thus

$$\|z(t)\| \le M_b\, b\, a_f + M_b\sqrt{b}\,\|C\|\widetilde{K} + M_b\, a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\|\,ds. \qquad (6.3.1)$$

$$
\begin{aligned}
\|\widetilde{z}_s + \widetilde{y}_s\| &\leq \|\widetilde{z}_s\| + \|\widetilde{y}_s\| \\
&\leq \|\widetilde{z}_s\| + M_b\|\varphi\| \\
&\leq \sup_{\tau\in[0,s]} \|z(\tau)\| + M_b\|\varphi\|
\end{aligned}
$$

This implies that (6.3.1) can be rewritten as follows

$$
\begin{aligned}
\sup_{s\in[0,t]} \|z(s)\| &\leq M_b\, b\, a_f + M_b\sqrt{b}\,\|C\|\widetilde{K} + M_b^2\, a_f\, b\,\|\varphi\| + M_b\, a_f \int_0^t \sup_{\tau\in[0,s]} \|z(\tau)\|\, d\tau \\
&= M + M_b\, a_f \int_0^t \sup_{\tau\in[0,s]} \|z(\tau)\|\, d\tau.
\end{aligned}
$$

with $M = M_b\, b\, a_f + M_b\sqrt{b}\,\|C\|\widetilde{K} + M_b^2\, a_f\, b\,\|\varphi\|$.

It follows by Gronwall's inequality that

$$\|z(t)\| \leq M e^{b\, a_f\, M_b} =: \widetilde{M}.$$

As a result, for $t \in I$, we have

$$
\begin{aligned}
\|x_u(t)\| &\leq \|z(t)\| + \|R(t)\varphi(0)\| \\
&\leq \widetilde{M} + M_b\|\varphi\| := \rho
\end{aligned}
$$

That is $\|x_u(t)\| \leq \rho$ for all $t \in I$. This completes the proof of the Lemma.

□

We have the following theorem on continuous dependence of the mild solutions of equation (6.1.1) on the controls and initial states.

Theorem 6.3.2 *For all* $\lambda > 0$, *there exists* $\gamma^*(\lambda) > 0$ *such that for all* $\varphi^1,\ \varphi^2 \in B(0,\lambda)$,

$$\|x_t^1 - x_t^2\| \leq \gamma^*(\lambda)\Big(\|\varphi^1 - \varphi^2\| + \|u^1 - u^2\|_{L^2}\Big) \quad \textit{for} \quad t \in [0,b],$$

where

$$x^i(t) = \begin{cases} R(t)\varphi^i(0) + \displaystyle\int_0^t R(t-s)\left[f(s,x_s^i) + C(s)u^i(s)\right]\,ds \quad \textit{for } t \in I \\ \\ \varphi^i(t) \quad \textit{for} \quad -r \le t \le 0, \end{cases} \tag{6.3.2}$$

and $u^i \in \mathcal{U}_{ad}$*, for* $i = 1, 2$.

Proof. Let x^i, for $i = 1, 2$, be two mild solutions of equation (6.1.1), corresponding to the controls $u^i \in \mathcal{U}_{ad}$ and $\lambda > 0$ such that $\varphi^1,\ \varphi^2 \in B(0,\lambda)$.

$$x^i(t) = \begin{cases} R(t)\varphi^i(0) + \displaystyle\int_0^t R(t-s)\left[f(s,x_s^i) + C(s)u^i(s)\right]\,ds \quad \text{for } t \in I \\ \\ \varphi^i(t) \quad \text{for} \quad -r \le t \le 0, \end{cases}$$

From the proof of Lemma 6.3.1, one can see that for $\rho_\lambda = \widetilde{M} + M_b\lambda > 0$, we have $\|x_s^i\| \le \rho_\lambda,\ i = 1,\,2.$
Now, for $t \in [0,b]$, we have

$$\begin{aligned} \|x^1(t) - x^2(t)\| &\le M_b\|\varphi^1(0) - \varphi^2(0)\| + M_b \int_0^t \|f(s,x_s^1) - f(s,x_s^2)\|\,ds \\ &+ M_b \int_0^t \|C(s)u^1(s) - C(s)u^2(s)\|\,ds \\ &\le M_b\|\varphi^1 - \varphi^2\| + M_b L_f(\rho_\lambda) \int_0^t \Big(\|x_s^1 - x_s^2\|\Big)\,ds \\ &+ M_b \int_0^t \|C(s)u^1(s) - C(s)u^2(s)\|\,ds \\ &\le M_b\|\varphi^1 - \varphi^2\| + M_b L_f(\rho_\lambda) \int_0^t \|x_s^1 - x_s^2\|\,ds \\ &+ M_b L_f(\rho_\lambda)\sqrt{b}\,\|C\|\,\|u^1 - u^2\|_{L^2} \end{aligned}$$

That is

$$\|x^1(t) - x^2(t)\| \le M_b\|\varphi^1 - \varphi^2\| + M_b L_f(\rho_\lambda) \int_0^t \|x_s^1 - x_s^2\|\,ds + M_b L_f(\rho_\lambda)\sqrt{b}\,\|C\|\,\|u^1 - u^2\|_{L^2} \tag{6.3.3}$$

But we have that

$$\|x_s^1 - x_s^2\| \le \sup_{\tau\in[-r,s]} \|x^1(\tau) - x^2(\tau)\|.$$

It follows that

$$\begin{aligned}\sup_{s\in[-r,t]} \|x^1(s) - x^2(s)\| \;&\le\; M_b\|\varphi^1 - \varphi^2\| + M_b L_f(\rho)\sqrt{b}\,\|C\|\,\|u^1 - u^2\|_{L^2}\\ &+\; M_b L_f(\rho_\lambda)\int_0^t \sup_{\tau\in[-r,s]} \|x^1(\tau) - x^2(\tau)\|\,d\tau.\end{aligned}$$

By Gronwall's inequality, we have that

$$\sup_{s\in[-r,t]} \|x^1(s) - x^2(s)\| \;\le\; \Big[M_b\|\varphi^1 - \varphi^2\| + M_b L_f(\rho_\lambda)\sqrt{b}\,\|C\|\,\|u^1 - u^2\|_{L^2}\Big] e^{M_b L_f(\rho_\lambda) b}.$$

This implies that

$$\|x_t^1 - x_t^2\| \;\le\; \Big[M_b\|\varphi^1 - \varphi^2\| + M_b L_f(\rho_\lambda)\sqrt{b}\,\|C\|\,\|u^1 - u^2\|_{L^2}\Big] e^{M_b L_f(\rho_\lambda) b},$$

Let

$$\gamma^*(\lambda) := \max\Big\{M_b\, e^{M_b L_f(\rho_\lambda) b},\; M_b L_f(\rho_\lambda)\sqrt{b}\,\|C\|\, e^{M_b L_f(\rho_\lambda) b}\Big\}.$$

Then, we have that

$$\|x_t^1 - x_t^2\| \le \gamma^*(\lambda)\Big(\|\varphi^1 - \varphi^2\| + \|u^1 - u^2\|_{L^2}\Big) \;\text{ for } t \in [0, b].$$

And the proof is complete.

□

6.4 Existence of the Optimal Controls

Now, we study the existence of solutions to the following Lagrange problem

$$(\mathcal{LP})\begin{cases}\text{Find a control } u^0 \in \mathcal{U}_{ad} \text{ such that}\\[2ex] \mathcal{J}(u^0) \le \mathcal{J}(u) \text{ for all } u \in \mathcal{U}_{ad},\end{cases}$$

where

$$\mathcal{J}(u) := \int_0^b \mathcal{L}\Big(t,\, x_t^u,\, x^u(t),\, u(t)\Big)\,dt,$$

and x^u denotes the mild solution of (6.1.1) corresponding to the control $u \in \mathcal{U}_{ad}$ and the initial data φ.
For the existence of solutions to problem $(\mathcal{LP})$, we make the following assumptions.

$(\mathbf{H_L})$

(i) The functional $\mathcal{L} : I \times \mathcal{C} \times X \times U \to \mathbb{R} \cup \{\infty\}$ is Borel measurable.

(ii) $\mathcal{L}(t, \cdot, \cdot, \cdot)$ is sequencially lower semicontinuous on $\mathcal{C} \times X \times U$ for almost all $t \in I$.

(iii) $\mathcal{L}(t, \psi, y, \cdot)$ is convex on U for each $\psi \in \mathcal{C}, \quad y \in X$ and almost all $t \in I$.

(iv) There exist constants $\nu, \ \beta \geq 0, \ \gamma > 0$, and $\mu \in L^1(I)$ nonnegative such that

$$\mathcal{L}(t, \psi, y, u) \geq \mu(t) + \nu\|\psi\| + \beta\|y\| + \gamma\|u\|.$$

We have the following result on the existence of optimal controls for problem $(\mathcal{LP})$.

Theorem 6.4.1 *Assume that hypotheses* $(\mathbf{H_1}) - (\mathbf{H_5})$ *and* $(\mathbf{H_L})$ *hold. Then the Lagrange problem* $(\mathcal{LP})$ *admits at least one optimal pair, that is there exists an admissible control pair* $(x^0, u^0) \in \mathcal{C}([-r, b], X) \times \mathcal{U}_{ad}$ *such that*

$$\mathcal{J}(u^0) = \int_0^b \mathcal{L}\Big(t, x_t^0, x^0(t), u^0(t)\Big)\, dt \leq \int_0^b \mathcal{L}\Big(t, x_t^u, x^u(t), u(t)\Big)\, dt = \mathcal{J}(u) \ \textit{ for } u \in \mathcal{U}_{ad}.$$

Proof. If $\inf\Big\{\mathcal{J}(u) \ : \ u \in \mathcal{U}_{ad}\Big\} = \infty$, we are done.
Without loss of generality, assume that $\inf\Big\{\mathcal{J}(u) \ : \ u \in \mathcal{U}_{ad}\Big\} = \delta < \infty$.
Suppose that $\delta = -\infty$, then for each $n \in \mathbb{N}$, there exists $(u^n)_{n\geq 1} \subset \mathcal{U}_{ad}$ such that

$$\mathcal{J}(u^n) < -n \tag{$*$}$$

Boundedness of $\mathcal{U}_{ad}$ implies that $(u^n)_{n\geq 1}$ is bounded and so there exists a subsequence $(u^{n_k})_{k\geq 1}$ of $(u^n)_{n\geq 1}$ that converges weakly to some u^0 in $L^2(I, U)$, since $L^2(I, U)$ is reflexive. But $\mathcal{U}_{ad}$ is closed and convex, so by Mazur's Lemma, it is weakly closed and therefore, $u^0 \in \mathcal{U}_{ad}$. By hypothesis $(\mathbf{H_L})$, $\mathcal{L}(t, \psi, y, \cdot)$ is weakly lower semicontinuous, so we have that

$$\mathcal{L}(t, \psi, y, u^0) \leq \liminf_{k\to\infty} \mathcal{L}(t, \psi, y, u^{n_k}) < -\infty,$$

which implies that $\mathcal{J}(u^0) < -\infty$ using $(*)$. And this is a contradiction since $\mathcal{J}(u^0) \in \mathbb{R} \cup \{\infty\}$. Hence $\delta \in \mathbb{R}$.

Now by the definition of δ, there exists a minimizing sequence, a feasible pair $((x^n, u^n))_{n\geq 1} \subset \mathcal{S}_{ad}$ such that

$$\int_0^b \mathcal{L}\Big(t,\, x_t^n,\, x^n(t),\, u^n(t)\Big)\, dt \longrightarrow \delta \quad \text{as } n \to \infty,$$

where

$$\mathcal{S}_{ad} := \Big\{(x,u)\ :\ x \,\text{is a mild solution of equation (6.1.1) corresponding to the control}\, u \in \mathcal{U}_{ad}\Big\}.$$

Boundedness of $\mathcal{U}_{ad}$ and the fact that $L^2(I,U)$ is reflexive imply that $(u^n)_{n\geq 1}$ has a subsequence denoted for simplicity by $(u^k)_{k\geq 1}$, that converges weakly to some u^0 in $L^2(I,U)$. But $\mathcal{U}_{ad}$ is closed and convex, so by Mazur's Lemma, it is weakly closed and therefore, $u^0 \in \mathcal{U}_{ad}$.
Let

$$x^k(t) = \begin{cases} R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)\left[f(s,x_s^k) + C(s)u^k(s)\right]\, ds \quad \text{for } t \in I \\[2ex] \varphi(t) \quad \text{for} \quad -r \leq t \leq 0. \end{cases}$$

denote the subsequence of $(x^n)_{n\geq 1}$ corresponding to the control sequence $(u^k)_{k\geq 1}$ and x^0 be the mild solution corresponding to the control $u^0 \in \mathcal{U}_{ad}$. We show that $x^k \to x^0$.
For $t \in [0,b]$, we have

$$\begin{aligned} \Big\|x^k(t) - x^0(t)\Big\| &\leq \int_0^t \Big\|R(t-s)\Big[f(s,x_s^k) - f(s,x_s^0)\Big]\Big\|\, ds \\ &+ \int_0^t \Big\|R(t-s)\Big[C(s)u^k(s) - C(s)u^0(s)\Big]\Big\|\, ds \\ &\leq M_b L_f(\rho)\int_0^t \Big\|x_s^k - x_s^0\Big\|\, ds + M_b \int_0^t \Big\|C(s)u^k(s) - C(s)u^0(s)\Big\|\, ds \\ &\leq M_b L_f(\rho)\int_0^t \Big\|x_s^k - x_s^0\Big\|\, ds + M_b\, \sqrt{b}\, \left(\int_0^t \Big\|C(s)u^k(s) - C(s)u^0(s)\Big\|^2\, ds\right)^{\frac{1}{2}} \\ &\leq M_b L_f(\rho)\int_0^t \Big\|x_s^k - x_s^0\Big\|\, ds + M_b\, \sqrt{b}\, \Big\|Cu^k - Cu^0\Big\|_{L^2(I,U)} \end{aligned}$$

That is

$$\left\|x^k(t) - x^0(t)\right\| \leq M_b L_f(\rho) \int_0^t \left\|x_s^k - x_s^0\right\| ds + M_b \sqrt{b} \left\|Cu^k - Cu^0\right\|_{L^2(I,U)} \tag{6.4.1}$$

But we have that

$$\|x_s^k - x_s^0\| \leq \sup_{\tau \in [0,s]} \|x^k(\tau) - x^0(\tau)\|.$$

This implies that

$$\sup_{s\in[0,t]} \left\|x^k(s) - x^0(s)\right\| \leq M_b \sqrt{b} \left\|Cu^k - Cu^0\right\|_{L^2(I,U)} + M_b L_f(\rho) \int_0^t \sup_{\tau\in[0,s]} \left\|x^k(\tau) - x^0(\tau)\right\| d\tau.$$

It follows from Gronwall's inequality that

$$\left\|x^k(t) - x^0(t)\right\| \leq M^{**} \left\|Cu^k - Cu^0\right\|_{L^2(I,U)}, \quad \text{where} \quad M^{**} = M_b \sqrt{b}\, e^{M_b\, b L_f(\rho)}. \tag{6.4.2}$$

We have the following Lemma.

Lemma 6.4.2 *[96] Let $(u^n)_{n\geq 1} \subset \mathcal{U}_{ad}$ and $u^0 \in \mathcal{U}_{ad}$ such that $(u^n)_{n\geq 1}$ converges weakly to u^0. Then,*

$$\left\|Cu^k - Cu^0\right\|_{L^2(I,U)} \longrightarrow 0 \quad \text{as } k \to \infty, \quad \text{if} \quad C \in L^\infty(I; \mathcal{L}(U, X)).$$

We have by (6.4.2) that

$$\left\|x^k - x^0\right\| \leq M^{**} \left\|Cu^k - Cu^0\right\|_{L^2(I,U)},$$

and therefore, it follows by Lemma 6.4.2 that

$$x^k \longrightarrow x^0 \quad \text{as } k \to \infty.$$

We note that $(\mathbf{H_L})$ implies the assumptions of Balder's Theorem. Hence by using Balder's Theorem, we can conclude that $(x_t,\, x,\, u) \mapsto \int_0^b \mathcal{L}\Big(t,\, x_t,\, x(t),\, u(t)\Big)\, dt$ is sequencially lower semicontinuous in the strong topology of $\mathcal{C}([-r, 0], X) \times L^1(I, X) \times L^1(I, U)$.
Now, since $\mathcal{C}([-r, 0], X) \times L^2(I, X) \times L^2(I, U) \subset \mathcal{C}([-r, 0], X) \times L^1(I, X) \times L^1(I, U)$, $\mathcal{J}$ is also sequencially lower semicontinuous on $\mathcal{C}([-r, 0], X) \times$

$L^2(I,X)\times L^2(I,U)$, and in the strong topology of $L^1(I,E_\varphi\times X\times U)$. Hence, $\mathcal{J}$ is weakly lower semicontinuous on $L^2(I,U)$, and since by $(\mathbf{H_L})-(\mathbf{iv})$, $\mathcal{J}>-\infty$, $\mathcal{J}$ attains its infimum at $u^0\in\mathcal{U}_{ad}$, that is

$$\delta=\lim_{k\to\infty}\int_0^b \mathcal{L}\Big(t,\,x_t^k,\,x^k(t),\,u^k(t)\Big)\,dt\geq\int_0^b \mathcal{L}\Big(t,\,x_t^0,\,x^0(t),\,u^0(t)\Big)\,dt=\mathcal{J}(u^0)\geq\delta.$$

Thus, $\delta=\mathcal{J}(u^0)$, and hence there exists an admissible control $u^0\in\mathcal{U}_{ad}$ such that

$$\mathcal{J}(u^0)\leq\mathcal{J}(u)\ \text{ for all }\ u\in\mathcal{U}_{ad}.$$

This completes the proof.

□

We now illustrate our main result by the following example. We observe that in [96], the Langrangian function $\mathcal{L}$ defined by the authors in the example does not satisfy condition $(\mathbf{H_L})-(\mathbf{iv})$, as they claimed. We correct that here.

6.5 Example

Let Ω be bounded domain in $\mathbb{R}^n$ with smooth boundary and consider the following nonlinear integrodifferential equation.

$$\begin{cases}\dfrac{\partial v(t,\xi)}{\partial t}=\Delta v(t,\xi)+\displaystyle\int_0^t\zeta(t-s)\Delta v(s,\xi)\,ds+\alpha(t)\sin(v^2(t-r,\xi))+\beta(t)\omega(t,\xi)\\ \qquad\qquad\text{for } t\in I=[0,1] \text{ and } \xi\in\Omega\\ v(t,\xi)=0 \text{ for } t\in[0,1] \text{ and } \xi\in\partial\Omega\\ v(\theta,\xi)=\phi(\theta,\xi) \text{ for } \theta\in[-r,0] \text{ and } \xi\in\Omega,\end{cases}\tag{6.5.1}$$

where $\alpha,\ \beta\in\mathcal{C}([0,1];\mathbb{R})$, $\omega: I\times\Omega\to\mathbb{R}$ continuous in t and $\zeta\in W^{1,1}(\mathbb{R}^+,\mathbb{R}^+)$

Let $X=U=L^2(\Omega)$. For $\eta>0$, we define the set of admissible controls $\mathcal{U}_{ad}$ by

$$\mathcal{U}_{ad}:=\Big\{u:I\to U \text{ such that } u \text{ is measurable and } \|u\|_{L^2(I,U)}\leq\eta\Big\},$$

where

$$\|u\|^2_{L^2(I,U)} = \int_0^1 \left(\int_\Omega u^2(s)(\xi)\, d\xi \right) ds.$$

We define $A : \mathcal{D}(A) \subset X \to X$ by:

$$\begin{cases} \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \\ \\ Av \;=\; \Delta v \;\text{ for } v \in \mathcal{D}(A). \end{cases}$$

Theorem 6.5.1 (**Theorem 4.1.2, p. 79 of [76]**) *The linear operator A defined above, is the infinitesimal generator of a C_0-semigroup on $L^2(\Omega)$.*

A generates a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on $L^2(\Omega)$.
Define

$$x(t)(\xi) \;=\; v(t,\xi), \quad x'(t)(\xi) \;=\; \frac{\partial v(t,\xi)}{\partial t}, \quad \omega(t,\xi) = u(t)(\xi).$$

$$\varphi(\theta)(\xi) = \phi(\theta,\xi) \;\text{ for } \theta \in [-r,0] \text{ and } \xi \in \Omega.$$

$$f(t,\varphi)(\xi) \;=\; \alpha(t) \sin\big((\varphi^2(-r)(\xi))\big) \;\text{ for } t \in [0,1] \text{ and } \xi \in \Omega.$$

$C(t) : X \to X$ be defined by $\Big(C(t)u(t)\Big)(\xi) \;=\; C(t)u(t)(\xi) = \;\beta(t)\omega(t,\xi).$

$$(B(t)x)(\xi) \;=\; \zeta(t)\Delta v(t,\xi) \;\text{ for }\; t \in [0,1], \;\; x \in \mathcal{D}(A) \text{ and } \;\xi \in \Omega.$$

Equation (6.5.1) is then transformed into the following form

$$\begin{cases} x'(t) = Ax(t) \;+\; \displaystyle\int_0^t B(t-s)x(s)ds \;+\; f(t,x_t) + C(t)u(t) \;\text{ for } t \in I = [0,1], \\ \\ x_0 = \varphi. \end{cases} \tag{6.5.2}$$

One can see that, f satisfies $(\mathbf{H_3})$. Now we consider the following cost function:

$$\mathcal{J}(u) := \int_0^1 \mathcal{L}\Big(t,\, x_t^u,\, x^u(t),\, u(t)\Big)\, dt,$$

where
$\mathcal{L} \,:\, [0,1] \times \mathcal{C}([-r,0], L^2(\Omega)) \times L^2(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$ is defined by:

$$\mathcal{L}\Big(t,\, \psi,\, x,\, u\Big) \;=\; \|\psi\| + \|x\| + \|u\|.$$

$\mathcal{L}$ satisfies all the conditions of hypothesis $(\mathbf{H_L})$. Then,

$$\mathcal{J}(u) = \int_0^1 \Big(\|x_t^u\| + \|x^u(t)\| + \|u(t)\|\Big)\, dt.$$

Hence, all the conditions of Theorem 6.4.1 are satisfied, and therefore, equation (6.5.2) has at least one optimal pair.

CHAPTER 7

On the Solvability and Optimal Control of some Partial Functional Integrodifferential Equations with Infinite Delay in Banach Spaces

[1]

7.1 Introduction

In this chapter, we study the existence of mild solutions and the optimal controls of some systems that take the form of the following partial functional integrodifferential equation with infinite delay in a Banach space $(X, \ \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) \ + \ \displaystyle\int_0^t \gamma(t-s)x(s)ds \ + \ f(t,x_t) + C(t)u(t) \ \text{ for } t \in I = [0,b] \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{7.1.1}$$

where $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a separable reflexive Banach space X; for $t \geq 0$, $\gamma(t)$ is a closed linear operator with domain $\mathcal{D}(\gamma(t)) \supset \mathcal{D}(A)$. The control u takes

[1]The results of this chapter are contents of the following paper
- Khalil Ezzinbi and Patrice Ndambomve [93]

values from another separable reflexive Banach space U. The operator $C(t)$ belongs to $\mathcal{L}(U, X)$ which is the Banach space of bounded linear operators from U into X, and the phase space $\mathcal{B}$ is a linear space of functions mapping $]-\infty, 0]$ into X satisfying axioms which will be described later, for every $t \geq 0$, x_t denotes the history function of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t+\theta) \ \text{ for } -\infty \leq \theta \leq 0,$$

$f : I \times \mathcal{B} \to X$ is a continuous function satisfying some conditions. In the literature devoted to equations with finite delay, the phase space is the space of continuous functions on $[-r, 0]$, for some $r > 0$, endowed with the uniform norm topology. But when the delay is unbounded, the selection of the phase space $\mathcal{B}$ plays an important role in both qualitative and quantitative theories. A usual choice is a normed space satisfying some suitable axioms, which was introduced by Hale and Kato [82].

Problems of controllability and existence of optimal controls for nonlinear differential equations have been studied extensively by many authors under various hypotheses (see e.g., [87], [88],[89],[90],[94] [95]), but little is known and done about the existence of optimal controls for integrodifferential equations with delay using the resolvent operator theory.

In [96], the authors considered the following fractional integrodifferential equation with infinite delay in Banach spaces

$$\begin{cases} {}^C D_t^q x(t) = Ax(t) \ + \ f\left(t,\, x_t,\, \displaystyle\int_0^t g(t,\, s, x_s)ds\right) \ + C(t)u(t) \ \text{ for } t \in I = [0, b] \\ \\ x_0 = \varphi \in \mathcal{B}, \end{cases}$$

where ${}^C D_t^q$ denotes the Caputo fractional derivative of order $q \in (0, 1)$. Using the techniques of *a priori* estimation and extension of step by steps, they studied the existence and continuous dependence of mild solutions and the optimal controls of the associated Lagrange problem. Wand and Zhou [97] discussed the optimal controls of a Lagrange problem for fractional evolution equations. In [98], the authors studied the optimal controls for nonlinear impulsive integrodifferential equations of mixed type on Banach spaces. In [99], the authors studied the existence of mild solutions and the optimal controls of a Lagrange problem for some impulsive fractional semilinear differential equations, using the techniques of *a priori* estimation. Motivated by these works, we investigate the solvability and the existence of optimal controls of a Lagrange problem for equation (7.1.1), using the techniques of *a priori* estimation of mild solutions. The existence and uniqueness of mild solutions is obtained using the theory of resolvent operator for integral equations. Furthermore, to the best of our knowledge, existence results for optimal controls of partial functional integrodifferential equation (7.1.1) with infinite delay are untreated in the literature, and this fact motivates us to extend the existing ones and make new development of the present work on this issue.

7.2 Existence of mild solutions for equation (7.1.1)

In this work, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [82]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a normed linear space of functions mapping $]-\infty, 0]$ into X and satisfying the following axioms:

($\mathbf{A_1}$) There exist positive constant H and functions $K : \mathbb{R}^+ \to \mathbb{R}^+$ continuous and $M : \mathbb{R}^+ \to \mathbb{R}^+$ locally bounded, such that for $a > 0$, if $x :]-\infty, a] \to X$ is continuous on $[0, a]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0, a]$, the following conditions hold:

(i) $x_t \in \mathcal{B}$,

(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$, which is equivalent to $\|\varphi(0)\| \leq H\|\varphi\|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$,

(iii) $\|x_t\|_{\mathcal{B}} \leq K(t) \sup\limits_{0 \leq s \leq t} \|x(s)\| + M(t)\|x_0\|_{\mathcal{B}}$.

($\mathbf{A_2}$) For the function x in ($\mathbf{A_1}$), $t \to x_t$ is a $\mathcal{B}$-valued continuous function for $t \in [0, a]$.

($\mathbf{A_3}$) The space $\mathcal{B}$ is complete.

We assume the following hypotheses.

($\mathbf{H_3}$) The function $f : I \times \mathcal{B} \to X$ satisfies the following conditions:

(i) $f(\cdot, \psi)$ is measurable for $\psi \in \mathcal{B}$,

(ii) for any $\rho > 0$, there exists $L_f(\rho) > 0$ such that

$$\|f(t, \psi_1) - f(t, \psi_2)\| \leq L_f(\rho)\|\psi_1 - \psi_2\|_{\mathcal{B}} \ \text{ for } \|\psi_1\|_{\mathcal{B}} \leq \rho,\ \|\psi_2\|_{\mathcal{B}} \leq \rho \text{ and } t \in [0, b],$$

(iii) there exists $a_f > 0$ such that

$$\|f(t,\psi)\| \le a_f(1 + \|\psi\|_{\mathcal{B}}) \quad \text{for all} \;\; \psi \in \mathcal{B} \;\; \text{and} \quad t \in [0,b].$$

($\mathbf{H_4}$) Let U be the separable reflexive Banach space from which the control u takes values and assume $C \in L^\infty(I; \mathcal{L}(U,X))$.

($\mathbf{H_5}$) The multivalued map $\Gamma : I \to 2^U \setminus \{\emptyset\}$ has closed, convex, and bounded values, Γ is graph measurable, and $\Gamma(\cdot) \subseteq \Omega$ where Ω is a bounded set in U.

We denote by $\mathcal{U}_{ad}$ the set of admissible controls defined by:

$$\mathcal{U}_{ad} = \Big\{ u : I \to U \, ; \, u \text{ is measurable and } u(t) \in \Gamma(t), \; a.e. \Big\}.$$

Then, we have the following:

Theorem 7.2.1 *[96]* $\mathcal{U}_{ad} \neq \emptyset$ *and* $\mathcal{U}_{ad} \subset L^2(I,U)$ *is bounded, closed and convex. Also,* $Cu \in L^2(I,U)$ *for all* $u \in \mathcal{U}_{ad}$.

Definition 7.2.2 *Let* $u \in \mathcal{U}_{ad}$ *and* $\varphi \in \mathcal{B}$. *A function* $x :]-\infty, b] \to X$ *is called a mild solution of equation (7.1.1) if* $x \in \mathcal{C}([0,b];X)$ *and*

$$x(t) = \begin{cases} R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)\left[f(s,x_s) + C(s)u(s)\right] ds & \textit{for } t \in I \\ \varphi(t) \;\; \textit{for} \;\; -\infty \le t \le 0. \end{cases} \tag{7.2.1}$$

Let

$$R_b = \sup_{t\in[0,b]} \|R(t)\|, \;\; K_b = \sup_{t\in[0,b]} \|K(t)\|, \;\; M_b = \sup_{t\in[0,b]} \|M(t)\|, \;\; \|C\| = \sup_{t\in[0,b]} \|C(t)\|_{\mathcal{L}(U,X)}.$$

We have the following theorem on existence of mild solutions of equation (7.1.1) with respect to a given control $u \in \mathcal{U}_{ad}$.

Theorem 7.2.3 *Assume that* $(\mathbf{H_1}) - (\mathbf{H_5})$ *hold. Then for each* $u \in \mathcal{U}_{ad}$, *equation* (7.1.1) *has a unique mild solution on* $]-\infty, b]$.

Proof. Let $b_1 \le b$, $\rho > 0$, and $\psi \in \mathcal{B}$ such that $\|\psi\|_{\mathcal{B}} \le \rho$. For $t \in [0,b_1]$, we have by the local Lipschitz condition on f that

$$\|f(t,\psi)\| \le L_f(\rho)\|\psi\|_{\mathcal{B}} + \|f(t,0)\| \le L_f(\rho)\rho + \sup_{s\in[0,b_1]} \|f(s,0)\|.$$

b_1 will be chosen sufficiently small enough to get the local existence of mild solutions.
Let $\varphi \in \mathcal{B}$, $\rho = \|\varphi\|_{\mathcal{B}} + 1$ and $\rho^* = L_f(\rho)\rho + \sup_{s\in[0,b_1]} \|f(s,0)\|$.
Define the function

$$y(t) = \begin{cases} R(t)\varphi(0) & \text{for } \ t \in [0,b] \\ \\ \varphi(t) & \text{for } \ t \in]-\infty,0]. \end{cases}$$

By axioms $(\mathbf{A_1}) - (\mathbf{i})$ and $(\mathbf{A_2})$, we deduce that $y_t \in \mathcal{B}$ and $t \mapsto y_t$ is continuous for $t \in [0,b]$. Then, for $\tau_1 \in]0,1[$, there exists $\tau_2 \in]0,1[$ such that

$$\|y_t - \varphi\|_{\mathcal{B}} \leq \tau_1 \ \text{ for } t \in [0,\tau_2].$$

Let $b_1 \in [0,\tau_2]$ be such that

$$K_b R_b\Big(\rho^* b_1 + \|C\| \, \|u\|_{L^2}\sqrt{b_1}\Big) < 1 - \tau_1. \tag{7.2.2}$$

For $x \in \mathcal{C}([0,b],X)$ such that $x(0) = \varphi(0)$, we define its extension $\widetilde{x}$ on $]-\infty,b]$ by

$$\widetilde{x}(t) = \begin{cases} x(t) & \text{for } \ t \in [0,b] \\ \\ \varphi(t) & \text{for } \ t \in]-\infty,0] \end{cases}$$

We introduce the following space

$$\mathcal{E}_\varphi = \Big\{x \in \mathcal{C}([0,b_1];X) \text{ such that } x(0) = \varphi(0) \text{ and } \sup_{s\in[0,b_1]} \|\widetilde{x}_s - \varphi\|_{\mathcal{B}} \leq 1\Big\},$$

provided with the uniform norm topology.
Define the operator $P : \mathcal{E}_\varphi \to \mathcal{C}([0,b_1];X)$ by

$$(Px)(t) = R(t)\varphi(0) + \int_0^t R(t-s)\big[f(s,\widetilde{x}_s) + C(s)u(s)\big]\,ds \ \text{ for } t \in [0,b_1]$$

We claim that $P\Big(\mathcal{E}_\varphi\Big) \subset \mathcal{E}_\varphi$.
In fact let $x \in \mathcal{E}_\varphi$. By $(\mathbf{H_3})$ and axiom $(\mathbf{A_1})$, the function $s \mapsto f(s,\widetilde{x}_s)$ is continuous on $[0,b_1]$. Then, Definition 2.2.2 implies that

$$s \mapsto \int_0^t R(t-s)\big[f(s,\widetilde{x}_s) + C(s)u(s)\big]\,ds$$

is continuous on $[0, b_1]$, consequently, $v = Px$ is continuous on $[0, b_1]$. We claim that $v \in \mathcal{E}_\varphi$. In fact, for any $t \in [0, b_1]$, we have

$$\|\widetilde{v}_t - \varphi\|_{\mathcal{B}} \leq \|\widetilde{v}_t - y_t\|_{\mathcal{B}} + \|y_t - \varphi\|_{\mathcal{B}} \leq \|\widetilde{v}_t - y_t\|_{\mathcal{B}} + \tau_1.$$

Since $\|\widetilde{x}_s - \varphi\|_{\mathcal{B}} \leq 1$ for $s \in [0, b_1]$, then $\|\widetilde{x}_s\|_{\mathcal{B}} \leq \rho$ for $s \in [0, b_1]$. Then,

$$\|f(s, \widetilde{x}_s)\| \leq L_f(\rho)\|\widetilde{x}_s\|_{\mathcal{B}} + \|f(s, 0)\| \leq \rho^*.$$

By axiom $(\mathbf{A_3}) - (\mathbf{iii})$, for any $t \in [0, b_1]$, we have

$$\|\widetilde{v}_t - y_t\|_{\mathcal{B}} \leq K_b \sup_{s \in [0,t]} \|v(s) - y(s)\|$$

and

$$\begin{aligned} \|v(t) - y(t)\| &\leq \int_0^t \Big\| R(t-s)\big[f(s, \widetilde{x}_s) + C(s)u(s)\big] \Big\| \, ds \\ &\leq R_b \, \rho^* b_1 + R_b \, \|C\| \, \|u\|_{L^2} \sqrt{b_1} \\ &< \frac{1}{K_b}(1 - \tau_1). \end{aligned}$$

Consequently, $\|\widetilde{v}_t - y_t\|_{\mathcal{B}} < 1 - \tau_1$. Thus, we deduce that $\|\widetilde{v}_t - \varphi\|_{\mathcal{B}} < 1$ for any $t \in [0, b_1]$, which implies that $v \in \mathcal{E}_\varphi$. Therefore $P\Big(\mathcal{E}_\varphi\Big) \subset \mathcal{E}_\varphi$.

We now show that P is a strict contraction on $[0, b_1]$. In fact, let x^1, $x^2 \in \mathcal{E}_\varphi$ and $t \in [0, b_1]$.

Then, $\|\widetilde{x}^i_s\| \leq \rho$ for $i = 1$, 2. We have that

$$\begin{aligned} \|(Px^1)(t) - (Px^2)(t)\| &\leq R_b \int_0^t \|f(s, \widetilde{x}^1_s) - f(s, \widetilde{x}^2_s)\| \, ds \\ &\leq R_b \, L_f(\rho) \int_0^t \|\widetilde{x}^1_s - \widetilde{x}^2_s\| \, ds \\ &\leq R_b \, L_f(\rho) K_b \int_0^t \sup_{\sigma \in [0,s]} \|x^1(\sigma) - x^2(\sigma)\| \, d\sigma \\ &\leq K_b R_b \, L_f(\rho) \, b_1 \, \|x^1 - x^2\| \end{aligned}$$

Then,

$$\|(Px^1)(t) - (Px^2)(t)\| \leq K_b R_b \, L_f(\rho) \, b_1 \, \|x^1 - x^2\|.$$

Now, since

$$K_b R_b \, L_f(\rho) \, b_1 \leq K_b R_b \, \rho^* \, b_1 < K_b R_b \, \rho^* b_1 + K_b R_b \, \|C\| \, \|u\|_{L^2} \sqrt{b_1},$$

condition (7.2.2) implies that

$$K_b R_b\, L_f(\rho)\, b_1 < 1.$$

Thus, P is a strict contraction on $\mathcal{E}_\varphi$.
It follows from the contraction mapping principle that P has a unique fixed point $x \in \mathcal{E}_\varphi$, which is the unique mild solution of equation (7.1.1) with respect to u on $]-\infty, b_1]$.
Using the same arguments, we can show that x can be extended to a maximal interval of existence $[0, t_{\max}[$.

Lemma 7.2.4 *[109] If $t_{\max} < b$, then, $\limsup_{t\to t_{\max}} \|x(t)\| = \infty$.*

We show that $t_{\max} = b$.
Assume on the contrary that $t_{\max} < b$. Then for $t \in [0, t_{\max}]$ we have that

$$x(t) = R(t)\varphi(0) + \int_0^t R(t-s)\big[f(s, x_s) + C(s)u(s)\big]\, ds.$$

It follows that

$$\begin{aligned}
\|x(t)\| &\le R_b\|\varphi(0)\| + R_b \int_0^t \|f(s, x_s)\|\, ds + R_b \int_0^t \|C(s)u(s)\|\, ds \\
&\le R_b\|\varphi(0)\| + R_b\, t_{\max}\, a_f + R_b a_f \int_0^t \|x_s\|\, ds + R_b\|C\| \int_0^t \|u(s)\|\, ds \\
&\le R_b\|\varphi(0)\| + R_b\, t_{\max}\, a_f + R_b\sqrt{t_{\max}}\, \|C\|\|u\|_{L^2} + R_b\, a_f \int_0^t \|x_s\|\, ds \\
&\le R_b\|\varphi(0)\| + R_b\, t_{\max}\, a_f + R_b\sqrt{t_{\max}}\, \|C\|\|u\|_{L^2} + R_b\, M_{t_{\max}}\, a_f\, t_{\max}\|\varphi(0)\| \\
&\quad + R_b\, K_{t_{\max}}\, a_f \int_0^t \sup_{\tau\in[0,s]} \|x(\tau)\|\, d\tau.
\end{aligned}$$

This implies that

$$\begin{aligned}
\sup_{s\in[0,t]} \|x(s)\| &\le R_b\|\varphi(0)\| + R_b\, t_{\max}\, a_f + R_b\sqrt{t_{\max}}\, \|C\|\|u\|_{L^2} + R_b\, M_{t_{\max}}\, a_f\, t_{\max}\|\varphi(0)\| \\
&\quad + R_b\, K_{t_{\max}}\, a_f \int_0^t \sup_{\tau\in[0,s]} \|x(\tau)\|\, d\tau,
\end{aligned}$$

where

$$K_{t_{\max}} = \sup_{s\in[0,t_{\max}]} K(s) \text{ and } M_{t_{\max}} = \sup_{s\in[0,t_{\max}]} M(s).$$

It follows by Gronwall's inequality that

$$\|x(t)\| \le \beta^* e^{(R_b\, K_{t_{\max}}\, a_f\, t)} \text{ for } t\in[0,t_{\max}],$$

where $\beta^* = R_b\|\varphi(0)\| + R_b\, t_{\max}\, a_f + R_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2} + R_b\, M_{t_{\max}}\, a_f\, t_{\max}\|\varphi(0)\|$. Thus

$$\lim_{t\to t_{\max}} \|x(t)\| \le \beta^* e^{(R_b\, K_{t_{\max}}\, a_f\, t_{\max})} < \infty.$$

This contradicts Lemma 7.2.4. Therefore, $t_{\max} = b$ and hence, equation (7.1.1) has a unique mild solution on $(-\infty, b]$.

□

7.3 Continuous Dependence and Existence of the Optimal Control

In this section, we discuss the continuous dependence of the mild solutions of equation (7.1.1) on the controls and initial states, and the existence of solutions of the Lagrange problem associated to equation (7.1.1).
We have the following a priori estimation.

Lemma 7.3.1 *Suppose* $(\mathbf{H_3})$ *holds and assume that equation (7.1.1) has a mild solution* x_u *on* $]-\infty,b]$ *with respect to* $u\in\mathcal{U}_{ad}$. *Then, there exists a constant* $\rho>0$ *independent of* u *such that* $\|x_u(t)\|\le\rho$ *for* $t\in[0,b]$, *(*ρ *depends only on* $\mathcal{U}_{ad}$ *and* φ*).*

Proof. Let $\varphi\in\mathcal{B}$. For $x\in\mathcal{C}([0,b],X)$ such that $x(0)=\varphi(0)$, we define its extension $\widetilde{x}$ on $]-\infty,b]$ by

$$\widetilde{x}(t) = \begin{cases} x(t) & \text{for } t\in[0,b] \\ \varphi(t) & \text{for } t\in]-\infty,0] \end{cases}$$

Also, we define the function $y\in\mathcal{C}([0,b],X)$ by $y(t)=R(t)\varphi(0)$ and its extension $\widetilde{y}$ on $]-\infty,b]$ by

$$\widetilde{y}(t) = \begin{cases} y(t) & \text{for } t\in[0,b] \\ \varphi(t) & \text{for } t\in]-\infty,0] \end{cases}$$

For each $z \in \mathcal{C}([0,b],X)$, set $\widetilde{x}(t) = \widetilde{z}(t) + \widetilde{y}(t)$, where $\widetilde{z}$ is the extension by zero of the function z on $]-\infty,0]$. Observe that x satifies (7.2.1) if and only if $z(0) = 0$ and

$$z(t) = \int_0^t R(t-s)\big[f(s,\widetilde{z}_s+\widetilde{y}_s) + C(s)u(s)\big]\,ds \;\; \text{for } t \in [0,b].$$

Since $\mathcal{U}_{ad}$ is bounded, let $\widetilde{N} > 0$ be such that $\|u\|_{L^2} \leq \widetilde{K}$ for all $u \in \mathcal{U}_{ad}$.

$$\begin{aligned}
\|z(t)\| &\leq R_b \int_0^t \|f(s,\widetilde{z}_s+\widetilde{y}_s)\|\,ds + R_b \int_0^t \|C(s)u(s)\|\,ds \\
&\leq R_b\, b\, a_f + R_b a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\|\,ds + R_b\|C\| \int_0^b \|u(s)\|\,ds \\
&\leq R_b\, b\, a_f + R_b\sqrt{b}\,\|C\|\|u\|_{L^2} + R_b\, a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\|_{\mathcal{B}}\,ds \\
&\leq R_b\, b\, a_f + R_b\sqrt{b}\,\|C\|\widetilde{N} + R_b\, a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\|_{\mathcal{B}}\,ds,
\end{aligned}$$

Thus

$$\|z(t)\| \leq R_b\, b\, a_f + R_b\sqrt{b}\,\|C\|\widetilde{N} + R_b\, a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\|_{\mathcal{B}}\,ds. \tag{7.3.1}$$

$$\begin{aligned}
\|\widetilde{z}_s + \widetilde{y}_s\|_{\mathcal{B}} &\leq \|\widetilde{z}_s\|_{\mathcal{B}} + \|\widetilde{y}_s\|_{\mathcal{B}} \\
&\leq K(s) \sup_{0\leq\tau\leq s} \|z(\tau)\| + M(s)\|z_0\|_{\mathcal{B}} + K(s) \sup_{0\leq\tau\leq s} \|\widetilde{y}(\tau)\| + M(s)\|\widetilde{y}_0\|_{\mathcal{B}} \\
&\leq K_b \sup_{\tau\in[0,s]} \|z(\tau)\| + K_b R_b\|\varphi\|_{\mathcal{B}} + M_b\|\varphi(0)\|.
\end{aligned}$$

This implies that (7.3.1) can be rewritten as follows

$$\begin{aligned}
\sup_{s\in[0,t]} \|z(s)\| &\leq R_b\, b\, a_f + R_b\sqrt{b}\,\|C\|\widetilde{N} + R_b\, a_f\, b\Big(K_b R_b\|\varphi\|_{\mathcal{B}} + M_b\|\varphi(0)\|\Big) \\
&\quad + K_b R_b\, a_f \int_0^t \sup_{\tau\in[0,s]} \|z(\tau)\|\,d\tau \\
&= N^* + K_b R_b\, a_f \int_0^t \sup_{\tau\in[0,s]} \|z(\tau)\|\,d\tau.
\end{aligned}$$

with $N^* = R_b\, b\, a_f + R_b \sqrt{b}\, \|C\| \widetilde{N} + R_b\, a_f\, b\Big(K_b R_b \|\varphi\|_{\mathcal{B}} + M_b \|\varphi(0)\|\Big)$.

It follows by Gronwall's inequality that

$$\|z(t)\| \leq N^* e^{(b\, a_f\, R_b\, K_b)} =: \widetilde{N^*}.$$

As a result, for $t \in I$, we have

$$\begin{aligned} \|x_u(t)\| &\leq \|z(t)\| + \|R(t)\varphi(0)\| \\ &\leq \widetilde{N^*} + R_b \|\varphi(0)\| := \rho \end{aligned}$$

That is $\|x_u(t)\| \leq \rho$ for all $t \in I$. This completes the proof of the Lemma.

□

We have the following continuous dependence Theorem.

Theorem 7.3.2 *For all* $r > 0$, *there exists* $\lambda^*(r) > 0$ *such that for all* $\varphi^1,\ \varphi^2 \in B(0,r)$,

$$\|x^1(t) - x^2(t)\| \leq \lambda^*(r)\Big(\|\varphi^1(0) - \varphi^2(0)\| + \|\varphi^1 - \varphi^2\|_{\mathcal{B}} + \|u^1 - u^2\|_{L^2}\Big) \ \textit{for}\ t \in [0,b],$$

where

$$x^i(t) = \begin{cases} R(t)\varphi^i(0) + \displaystyle\int_0^t R(t-s)\left[f(s, x_s^i) + C(s)u^i(s)\right] ds & \textit{for}\ t \in I \\ \varphi^i(t) \quad \textit{for} \quad -\infty \leq t \leq 0, \end{cases} \tag{7.3.2}$$

and $u^i \in \mathcal{U}_{ad}$, *for* $i = 1, 2$.

Proof. Let x^i, for $i = 1, 2$, be two mild solutions of equation (7.1.1), corresponding to the controls $u^i \in \mathcal{U}_{ad}$ and initial conditions $\varphi^i \in B(0,r)$.

$$x^i(t) = \begin{cases} R(t)\varphi^i(0) + \displaystyle\int_0^t R(t-s)\left[f(s, x_s^i) + C(s)u^i(s)\right] ds & \text{for}\ t \in I \\ \varphi^i(t) \quad \text{for} \quad -\infty \leq t \leq 0, \end{cases}$$

By Lemma 7.3.1, we have that, there exists a constant $\rho_r = \widetilde{N^*} + R_b\, r > 0$ such that $\|x_s^i\| \leq \rho_r,\ i = 1,\ 2$.

Now, for $t \in [0, b]$, we have

$$\begin{aligned}
\|x^1(t) - x^2(t)\| &\leq R_b\|\varphi^1(0) - \varphi^2(0)\| + R_b \int_0^t \|f(s, x_s^1) - f(s, x_s^2)\| \, ds \\
&+ R_b \int_0^t \|C(s)u^1(s) - C(s)u^2(s)\| \, ds \\
&\leq R_b\|\varphi^1(0) - \varphi^2(0)\| + R_b L_f(\rho_r) \int_0^t \Big(\|x_s^1 - x_s^2\|_{\mathcal{B}} \Big) \, ds \\
&+ R_b \int_0^t \|C(s)u^1(s) - C(s)u^2(s)\| \, ds \\
&\leq R_b\|\varphi^1(0) - \varphi^2(0)\| + R_b L_f(\rho_r) \int_0^t \|x_s^1 - x_s^2\|_{\mathcal{B}} \, ds \\
&+ R_b L_f(\rho_r) \sqrt{b} \, \|C\| \, \|u^1 - u^2\|_{L^2} \\
&\leq R_b\|\varphi^1(0) - \varphi^2(0)\| + R_b L_f(\rho_r) K_b \int_0^t \sup_{\tau \in [0,s]} \|x^1(\tau) - x^2(\tau)\| \, d\tau \\
&+ R_b L_f(\rho_r) M_b \, b \, \|x_0^1 - x_0^2\|_{\mathcal{B}} + R_b L_f(\rho) \sqrt{b} \, \|C\| \, \|u^1 - u^2\|_{L^2} \\
&\leq R_b\|\varphi^1(0) - \varphi^2(0)\| + R_b L_f(\rho_r) K_b \int_0^t \sup_{\tau \in [0,s]} \|x^1(\tau) - x^2(\tau)\| \, d\tau \\
&+ R_b L_f(\rho_r) M_b \, b \, \|\varphi^1 - \varphi^2\|_{\mathcal{B}} + R_b L_f(\rho_r) \sqrt{b} \, \|C\| \, \|u^1 - u^2\|_{L^2}
\end{aligned}$$

It follows that

$$\begin{aligned}
\sup_{s \in [0,t]} \|x^1(s) - x^2(s)\| &\leq R_b\|\varphi^1(0) - \varphi^2(0)\| + R_b L_f(\rho_r) \sqrt{b} \, \|C\| \, \|u^1 - u^2\|_{L^2} \\
&+ R_b L_f(\rho) M_b \, b \, \|\varphi^1 - \varphi^2\|_{\mathcal{B}} + R_b L_f(\rho_r) K_b \int_0^t \sup_{\tau \in [0,s]} \|x^1(\tau) - x^2(\tau)\| \, d\tau.
\end{aligned}$$

By Gronwall's inequality, we have that

$$\sup_{s \in [0,t]} \|x^1(s) - x^2(s)\| \leq N^{**} e^{R_b L_f(\rho_r) K_b \, b},$$

where

$$N^{**} = \Big[R_b\|\varphi^1(0) - \varphi^2(0)\| + R_b L_f(\rho_r) M_b \, b \, \|\varphi^1 - \varphi^2\|_{\mathcal{B}} + R_b L_f(\rho_r) \sqrt{b} \, \|C\| \, \|u^1 - u^2\|_{L^2} \Big].$$

This implies that

$$\|x^1(t) - x^2(t)\| \;\leq\; \Big[R_b\|\varphi^1(0) - \varphi^2(0)\| + R_b L_f(\rho_r) M_b\, b\,\|\varphi^1 - \varphi^2\|_{\mathcal{B}} + R_b L_f(\rho_r)\,\sqrt{b}\,\|C\|\,\|u^1 - u^2\|_{L^2}\Big] e^{R_b L_f(\rho_r) K_b\, b}$$

Let

$$\lambda^*(r) := \max\Big\{R_b\, e^{R_b L_f(\rho_r) K_b\, b},\; R_b L_f(\rho_r) M_b\, b\, e^{R_b L_f(\rho) K_b\, b},\; R_b L_f(\rho_r)\,\sqrt{b}\,\|C\|\, e^{R_b L_f(\rho_r) K_b\, b}\Big\}.$$

Then, we have that

$$\|x^1(t) - x^2(t)\| \leq \lambda^*(r)\Big(\|\varphi^1(0) - \varphi^2(0)\| + \|\varphi^1 - \varphi^2\|_{\mathcal{B}} + \|u^1 - u^2\|_{L^2}\Big) \text{ for } t \in [0, b].$$

And the proof is complete.

□

We now study the existence of solutions to the following Lagrange problem

$$(\mathcal{LP})\begin{cases} \text{Find a control } u^0 \in \mathcal{U}_{ad} \text{ such that} \\ \mathcal{J}(u^0) \leq \mathcal{J}(u) \text{ for all } u \in \mathcal{U}_{ad}, \end{cases}$$

where

$$\mathcal{J}(u) := \int_0^b \mathcal{L}\Big(t,\, x_t^u,\, x^u(t),\, u(t)\Big)\, dt,$$

and x^u denotes the mild solution of (7.1.1) corresponding to the control $u \in \mathcal{U}_{ad}$.
For the existence of solutions to problem $(\mathcal{LP})$, we make the following assumptions.

$(\mathbf{H_L})$

(i) The functional $\mathcal{L} : I \times \mathcal{B} \times X \times U \to \mathbb{R} \cup \{\infty\}$ is Borel measurable.

(ii) $\mathcal{L}(t, \cdot, \cdot, \cdot)$ is sequencially lower semicontinuous on $\mathcal{B} \times X \times U$ for almost all $t \in I$.

(iii) $\mathcal{L}(t, \psi, y, \cdot)$ is convex on U for each $\psi \in \mathcal{B}$, $y \in X$ and almost all $t \in I$.

(iv) There exist constants $\nu,\ \beta \geq 0,\ \gamma > 0$, and $\mu \in L^1(I)$ nonnegative such that

$$\mathcal{L}(t, \psi, y, u) \geq \mu(t) + \nu\|\psi\|_{\mathcal{B}} + \beta\|y\| + \gamma\|u\|_U.$$

We have the following result on the existence of optimal controls for problem $(\mathcal{LP})$.

Theorem 7.3.3 *Assume that hypotheses* $(\mathbf{H_1}) - (\mathbf{H_5})$ *and* $(\mathbf{H_L})$ *hold. Then the Lagrange problem* $(\mathcal{LP})$ *admits at least one optimal pair, that is there exists an admissible control pair* $(x^0, u^0) \in \mathcal{C}([0,b], X) \times \mathcal{U}_{ad}$ *such that*

$$\mathcal{J}(u^0) = \int_0^b \mathcal{L}\Big(t,\, x_t^0,\, x^0(t),\, u^0(t)\Big)\, dt \leq \int_0^b \mathcal{L}\Big(t,\, x_t^u,\, x^u(t),\, u(t)\Big)\, dt = \mathcal{J}(u) \ \ \forall u \in \mathcal{U}_{ad}.$$

Proof. If $\inf \Big\{ \mathcal{J}(u) \,:\, u \in \mathcal{U}_{ad} \Big\} = \infty$, we are done.
Without loss of generality, assume that $\inf \Big\{ \mathcal{J}(u) \,:\, u \in \mathcal{U}_{ad} \Big\} = \delta < \infty$.
Suppose that $\delta = -\infty$, then for each $n \in \mathbb{N}$, there exists $(u^n)_{n\geq 1} \subset \mathcal{U}_{ad}$ such that

$$\mathcal{J}(u^n) < -n \qquad (*)$$

Boundedness of $\mathcal{U}_{ad}$ implies that $(u^n)_{n\geq 1}$ is bounded and so there exists a subsequence $(u^{n_k})_{k\geq 1}$ of $(u^n)_{n\geq 1}$ that converges weakly to some u^0 in $L^2(I, U)$, since $L^2(I, U)$ is reflexive. But $\mathcal{U}_{ad}$ is closed and convex, so by Mazur's Lemma, it is weakly closed and therefore, $u^0 \in \mathcal{U}_{ad}$. By hypothesis $(\mathbf{H_L})$, $\mathcal{L}(t,\, x,\, y,\, \cdot)$ is weakly lower semicontinuous, so we have that

$$\mathcal{L}(t,\, \psi,\, y,\, u^0) \leq \liminf_{k\to\infty} \mathcal{L}(t,\, \psi,\, y,\, u^{n_k}) < -\infty,$$

which implies that $\mathcal{J}(u^0) < -\infty$ using $(*)$. And this is a contradiction since $\mathcal{J}(u^0) \in \mathbb{R} \cup \{\infty\}$. Hence $\delta \in \mathbb{R}$.
Now by the definition of δ, there exists a minimizing sequence, a feasible pair $((x^n, u^n))_{n\geq 1} \subset \mathcal{S}_{ad}$ such that

$$\int_0^b \mathcal{L}\Big(t,\, x_t^n,\, x^n(t),\, u^n(t)\Big)\, dt \longrightarrow \delta \ \text{ as } n \to \infty,$$

where

$$\mathcal{S}_{ad} := \Big\{ (x, u) \,:\, x \text{ is a mild solution of equation (7.1.1) corresponding to the control } u \in \mathcal{U}_{ad} \Big\}.$$

Boundedness of $\mathcal{U}_{ad}$ and the fact that $L^2(I, U)$ is reflexive imply that $(u^n)_{n\geq 1}$ has a subsequence denoted for simplicity by $(u^k)_{k\geq 1}$, that converges weakly to some u^0 in $L^2(I, U)$. But $\mathcal{U}_{ad}$ is closed and convex, so by Mazur's lemma,

it is weakly closed and therefore, $u^0 \in \mathcal{U}_{ad}$.
Let

$$x^k(t) = \begin{cases} R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)\left[f(s,x_s^k) + C(s)u^k(s)\right]\, ds \;\; \text{for } t \in I \\ \\ \varphi(t) \;\; \text{for } \;\; -r \le t \le 0. \end{cases}$$

denote the subsequence of $(x^n)_{n\geq 1}$ corresponding to the control sequence $(u^k)_{k\geq 1}$ and x^0 be the mild solution corresponding to the control $u^0 \in \mathcal{U}_{ad}$. We show that $x^k \to x^0$.
For $t \in [0,b]$, we have

$$\begin{aligned} \left\|x^k(t) - x^0(t)\right\| &\le \int_0^t \left\|R(t-s)\Big[f(s,x_s^k) - f(s,x_s^0)\Big]\right\| ds \\ &+ \int_0^t \left\|R(t-s)\Big[C(s)u^k(s) - C(s)u^0(s)\Big]\right\| ds \\ &\le R_b L_f(\rho) \int_0^t \left\|x_s^k - x_s^0\right\| ds + R_b \int_0^t \left\|C(s)u^k(s) - C(s)u^0(s)\right\| ds \\ &\le R_b L_f(\rho) \int_0^t \left\|x_s^k - x_s^0\right\| ds + R_b\sqrt{b}\left(\int_0^t \left\|C(s)u^k(s) - C(s)u^0(s)\right\|^2 ds\right)^{\frac{1}{2}} \\ &\le R_b L_f(\rho)\, K_b \int_0^t \sup_{\tau\in[0,s]} \left\|x^k(\tau) - x^0(\tau)\right\| d\tau + R_b\sqrt{b}\left\|Cu^k - Cu^0\right\|_{L^2(I,U)} \end{aligned}$$

This implies that

$$\sup_{s\in[0,t]} \left\|x^k(s) - x^0(s)\right\| \le R_b\sqrt{b}\left\|Cu^k - Cu^0\right\|_{L^2(I,U)} + R_b L_f(\rho)\, K_b \int_0^t \sup_{\tau\in[0,s]} \left\|x^k(\tau) - x^0(\tau)\right\| d\tau.$$

It follows from Gronwall's inequality that

$$\left\|x^k(t) - x^0(t)\right\| \le \lambda^{**}\left\|Cu^k - Cu^0\right\|_{L^2(I,U)}, \quad \text{where } \; \lambda^{**} = R_b\sqrt{b}\, e^{R_b\, b\, L_f(\rho)\, K_b}. \tag{7.3.3}$$

We have the following Lemma.

Lemma 7.3.4 *[96] Let $(u^n)_{n\geq 1} \subset \mathcal{U}_{ad}$ and $u^0 \in \mathcal{U}_{ad}$ such that $(u^n)_{n\geq 1}$ converges weakly to u^0. Then,*

$$\left\|Cu^k - Cu^0\right\|_{L^2(I,U)} \longrightarrow 0 \;\; \text{as } k \to \infty, \;\; \text{if } C \in L^\infty(I;\mathcal{L}(U,X)).$$

We have by (7.3.3) that

$$\left\|x^k - x^0\right\| \leq \lambda^{**}\left\|Cu^k - Cu^0\right\|_{L^2(I,U)},$$

and therefore, it follows by Lemma 7.3.4 that

$$x^k \longrightarrow x^0 \quad \text{as } k \to \infty.$$

We note that $(\mathbf{H_L})$ implies the assumptions of Balder's Theorem (see Theorem 2.4.3). Hence by Balder's Theorem, we can conclude that $(x_t,\, x,\, u) \mapsto \int_0^b \mathcal{L}\Big(t,\, x_t,\, x(t),\, u(t)\Big)\, dt$ is sequencially lower semicontinuous in the strong topology of $\mathcal{B} \times L^1(I,X) \times L^1(I,U)$.
Now, since $\mathcal{B} \times L^2(I,X) \times L^2(I,U) \subset \mathcal{B} \times L^1(I,X) \times L^1(I,U)$, $\mathcal{J}$ is also sequencially lower semicontinuous on $\mathcal{B} \times L^2(I,X) \times L^2(I,U)$, and in the strong topology of $L^1(I, \mathcal{B} \times X \times U)$.
Hence, $\mathcal{J}$ is weakly lower semicontinuous on $L^2(I,U)$, and since by $(\mathbf{H_L}) - (\mathbf{iv})$, $\mathcal{J} > -\infty$, $\mathcal{J}$ attains its infimum at $u^0 \in \mathcal{U}_{ad}$, that is

$$\delta = \lim_{k\to\infty} \int_0^b \mathcal{L}\Big(t,\, x_t^k,\, x^k(t),\, u^k(t)\Big)\, dt \geq \int_0^b \mathcal{L}\Big(t,\, x_t^0,\, x^0(t),\, u^0(t)\Big)\, dt = \mathcal{J}(u^0) \geq \delta.$$

Thus, $\delta = \mathcal{J}(u^0)$, and hence there exists an admissible control $u^0 \in \mathcal{U}_{ad}$ such that

$$\mathcal{J}(u^0) \leq \mathcal{J}(u) \quad \text{for all} \quad u \in \mathcal{U}_{ad}.$$

This completes the proof.

□

We now illustrate our main result by the following example.

7.4 Example

Let Ω be bounded domain in $\mathbb{R}^n$ with smooth boundary and consider the following nonlinear integrodifferential equation.

$$\begin{cases} \frac{\partial v(t,\xi)}{\partial t} = \Delta v(t,\xi) + \int_0^t \zeta(t-s)\Delta v(s,\xi)\,ds + \int_{-\infty}^0 \alpha(\theta)g(t,v(t+\theta,\xi))\,d\theta + \beta(t)\omega(t,\xi) \\ \qquad\qquad \text{for } t \in I = [0,1] \text{ and } \xi \in \Omega \\ v(t,\xi) = 0 \text{ for } t \in [0,1] \text{ and } \xi \in \partial\Omega \\ v(\theta,\xi) = \phi(\theta,\xi) \text{ for } \theta \in]-\infty,0] \text{ and } \xi \in \Omega, \end{cases} \tag{7.4.1}$$

where $\beta \in \mathcal{C}([0,1];\mathbb{R})$, $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous and Lipschitzian with respect to the second variable, the initial data function $\phi : \mathbb{R}^- \times \Omega \to \mathbb{R}$ is a given function, $\omega : [0,1]\times\Omega \to \mathbb{R}$ continuous in t, $\alpha : \mathbb{R}^- \to \mathbb{R}$ is continuous, $\alpha \in L^1(\mathbb{R}^-,\mathbb{R})$ and $\zeta \in W^{1,1}(\mathbb{R}^+,\mathbb{R}^+)$.
Let $X = U = L^2(\Omega)$ and the phase space $\mathcal{B} = BUC(\mathbb{R}^-, X)$, the the space of uniformly bounded continuous functions endowed with the following norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta\le 0} \|\varphi(\theta)\|.$$

Then, the space $BUC(\mathbb{R}^-, X)$ satisfies axioms $(\mathbf{A_1})$, $(\mathbf{A_2})$ and $(\mathbf{A_3})$.
For $\eta > 0$, we define the set of admissible controls $\mathcal{U}_{ad}$ by

$$\mathcal{U}_{ad} := \Big\{ u : I \to U : \ u \text{ is measurable and } \|u\|_{L^2(I,U)} \le \eta \Big\}.$$

where

$$\|u\|^2_{L^2(I,U)} = \int_0^1 \left(\int_\Omega u^2(s)(\xi)\,d\xi \right) ds.$$

We define $A : \mathcal{D}(A) \subset X \to X$ by:

$$\begin{cases} \mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega) \\ Av \ = \ \Delta v \ \text{ for } v \in \mathcal{D}(A). \end{cases}$$

Theorem 7.4.1 *(***Theorem 4.1.2, p. 79 of [76]***) A is the infinitesimal generator of a C_0-semigroup on $L^2(\Omega)$.*

Define

$$x(t)(\xi) \ = \ v(t,\xi), \quad x'(t)(\xi) \ = \ \frac{\partial v(t,\xi)}{\partial t}, \quad \omega(t,\xi) = u(t)(\xi).$$

$$\varphi(\theta)(\xi) = \phi(\theta,\xi) \ \text{ for } \theta \in]-\infty,0] \text{ and } \xi \in \Omega.$$

$$f(t,\psi)(\xi) \;=\; \int_{-\infty}^{0} \alpha(\theta) g(t,\psi(\theta)(\xi))\,d\theta \;\text{ for } \theta \in]-\infty,0] \text{ and } \xi \in \Omega.$$

$C(t) : X \to X$ be defined by $\Big(C(t)u(t)\Big)(\xi) \;=\; C(t)u(t)(\xi) = \beta(t)\omega(t,\xi).$

$$(\gamma(t)x)(\xi) \;=\; \zeta(t)\Delta v(t,\xi) \;\text{ for }\; t \in [0,1], \;\; x \in \mathcal{D}(A) \text{ and } \;\xi \in \Omega.$$

We suppose that $\varphi \in BUC(\mathbb{R}^-, X)$. Then, equation (7.4.1) is then transformed into the following form

$$\begin{cases} x'(t) = Ax(t) \;+\; \displaystyle\int_0^t \gamma(t-s)x(s)ds \;+\; f(t,x_t) + C(t)u(t) \;\text{ for } t \in I = [0,1], \\ \\ x_0 = \varphi. \end{cases} \tag{7.4.2}$$

Suppose there exist a continuous function $p \in L^1(I;\mathbb{R}^+)$ such that

$$|g(t,y)| \leq p(t)|y| \;\text{ for } t \in I \text{ and } y \in \mathbb{R}.$$

One can see that, f satisfies $(\mathbf{H_3})$. Now we consider the following cost function:

$$\mathcal{J}(u) := \int_0^1 \mathcal{L}\Big(t,\, x_t^u,\, x^u(t),\, u(t)\Big)\,dt,$$

where
$\mathcal{L} \,:\, [0,1] \times \mathcal{B} \times L^2(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$ is defined by:

$$\mathcal{L}\Big(t,\, \psi,\, x,\, u\Big) \;=\; \|\psi\|_{\mathcal{B}} + \|x\| + \|u\|.$$

$\mathcal{L}$ satisfies all the conditions of hypothesis $(\mathbf{H_L})$. Then,

$$\mathcal{J}(u) = \int_0^1 \Big(\|x_t^u\| + \|x^u(t)\| + \|u(t)\|\Big)\,dt.$$

Hence, all the conditions of Theorem 7.3.3 are satisfied, and therefore, equation (7.4.2) has at least one optimal pair.

CHAPTER 8

Solvability and Optimal Controls for some Partial Functional Integrodifferential Equations with Classical Initial Conditions in Banach Spaces

[1]

8.1 Introduction

The aim of this chapter is to study the existence of mild solutions and the optimal controls of some systems that take the form of the following partial functional integrodifferential equation in a Banach space $(X, \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) \;+\; \displaystyle\int_0^t B(t-s)x(s)ds \;+\; f(t,x(t)) + C(t)u(t) \;\; \text{for } t \in I = [0,b] \\ x(0) = x_0 \in X, \end{cases} \tag{8.1.1}$$

where $f : I \times X \to X$ is a function satisfying some conditions; $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a separable reflexive Banach space X; for $t \geq 0$, $B(t)$ is a closed linear operator with

[1]The results of this chapter are contents of the following paper
- Khalil Ezzinbi and Patrice Ndambomve [92]

domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$. The control $u(t)$ takes values from another separable reflexive Banach space U. The operator $C(t)$ belongs to $\mathcal{L}(U, X)$, the Banach space of bounded linear operators from U into X.
In many areas of applications such as engineering, electronics, fluid dynamics, physical sciences, etc..., integrodifferential equations appear and have received considerable attention during the last decades. In [85], R. Grimmer proved the existence and uniqueness of resolvent operators for these integrodifferential equations that give the variation of parameter formula for the solution. In recent years, much work has been done on the existence and regularity of solutions of nonlinear integrodifferential equations with various initial conditions by many authors by applying the resolvent operator theory, for integral equations see e.g., [86] and the references therein. Problems of controllability and existence of optimal controls for nonlinear differential equations have been studied extensively by many authors under various hypotheses (see e.g., [87], [88],[89],[90],[94] [95]), but little is known and done about the existence of optimal controls for integrodifferential equations using the resolvent operator theory. Wang and Zhou [97] discussed the optimal controls of a Lagrange problem for the following fractional evolution equations:

$$\begin{cases} D^q x(t) = -Ax(t) \ + f(t, x(t)) + C(t)u(t) \ \text{ for } t \in [0, b] \\ \\ x(0) = x_0 \in X, \end{cases}$$

where D^q denotes the Caputo fractional derivative of order $q \in (0, 1)$ and $-A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators. In [99], the authors studied the existence of mild solutions and the optimal controls of a Lagrange problem for the following impulsive fractional semilinear differential equations,

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) \ + f\left(t, x(t), \displaystyle\int_0^t g(t, s, x(s))ds\right) + C(t)u(t) \ \text{ for } t \in [0, b], \ t \neq t_k \\ \\ \Delta x(t_k) = I_k(x(t_k^-)), \ k = 1, 2, \cdots m \\ \\ x(0) = x_0 \in X, \end{cases}$$

where ${}^C D_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0, 1]$ with lower limit zero and $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup. They used the techniques of *a priori* estimation. In [98], the authors studied the optimal controls for nonlinear impulsive integrodifferential equations of mixed type on Banach spaces.

Motivated by these works, we investigate the solvability and the existence of optimal controls of a Lagrange problem for equation (8.1.1), using the techniques of *a priori* estimation of mild solutions. The existence and uniqueness of mild solutions is obtained using the theory of resolvent operator for integral equations. Furthermore, to the best of our knowledge, the optimal controls for partial functional integrodifferential equation (8.1.1) with classical Cauchy initial conditions are untreated in the literature, and this fact motivates us to extend the existing ones and make new development of the present work on this issue.

8.2 Existence of mild solutions for equation (8.1.1)

We make the following assumptions.

($\mathbf{H_3}$) The function $f : I \times X \to X$ satisfies the following conditions:

- **(i)** $f(\cdot, x)$ is measurable for $x \in X$,
- **(ii)** for any $\rho > 0$, there exists $L_f(\rho) > 0$ such that

$$\|f(t,x) - f(t,y)\| \le L_f(\rho)\|x - y\| \quad \text{for } \|x\| \le \rho,\ \|y\| \le \rho \text{ and } t \in [0,b],$$

- **(iii)** there exists $a_f > 0$ such that

$$\|f(t,x\| \le a_f(1 + \|x\|) \quad \text{for all } \ x \in X \ \text{ and } \quad t \in [0,b].$$

($\mathbf{H_4}$) Let U be the separable reflexive Banach space from which the control u takes values and assume $C \in L^\infty(I; \mathcal{L}(U,X))$.

($\mathbf{H_5}$) The multivalued map $\Gamma : I \to 2^U \setminus \{\emptyset\}$ has closed, convex, and bounded values, Γ is graph measurable, and $\Gamma(\cdot) \subseteq \Omega$ where Ω is a bounded set in U.

We denote by $\mathcal{U}_{ad}$ the set of admissible controls defined by:

$$\mathcal{U}_{ad} = \Big\{ u : I \to U \,\text{such that}\, u \text{ is measurable and } u(t) \in \Gamma(t),\ a.e. \Big\}.$$

Then, we have the following:

Theorem 8.2.1 *[96]* $\mathcal{U}_{ad} \neq \emptyset$ *and* $\mathcal{U}_{ad} \subset L^2(I,U)$ *is bounded, closed and convex. Also,* $Cu \in L^2(I,U)$ *for all* $u \in \mathcal{U}_{ad}$.

Definition 8.2.2 *Let $u \in \mathcal{U}_{ad}$. A function $x \in \mathcal{C}(I;X)$ is called a mild solution of equation (8.1.1) if*

$$x(t) = R(t)x_0 + \int_0^t R(t-s)\,[f(s,x(s)) + C(s)u(s)]\,ds \quad \text{for } t \in I \qquad (8.2.1)$$

We have the following theorem on existence of mild solutions to equation (8.1.1) with respect to a given control $u \in \mathcal{U}_{ad}$.

Theorem 8.2.3 *Assume that* $(\mathbf{H_1}) - (\mathbf{H_5})$ *hold. Then for each* $u \in \mathcal{U}_{ad}$, *equation* (8.1.1) *has a unique mild solution on* $[0,b]$.

Proof. Let $b_1,\ \rho > 0$, and $x \in X$ such that $\|x\| \leq \rho$. For $t \in [0,b_1]$, we have by the local Lipschitz condition on f that

$$\|f(t,x)\| \leq L_f(\rho)\|x\| + \|f(t,0)\| \leq L_f(\rho)\rho + \sup_{s\in[0,b_1]} \|f(s,0)\|.$$

b_1 will be chosen sufficiently small enough to get the local existence of mild solutions.
Let $x \in X,\ \rho = \|x\| + 1$ and $\rho^* = L_f(\rho)\rho + \sup_{s\in[0,b_1]} \|f(s,0)\|$.
We define the following space

$$E_0 = \Big\{ x \in \mathcal{C}([0,b_1];X) \text{ such that } \sup_{s\in[0,b_1]} \|x(s) - x(0)\| \leq 1 \Big\}.$$

Then, E_0 is a closed subset of $\mathcal{C}([0,b_1];X)$ which is endowed with the uniform norm topology. Let

$$M_b = \sup_{t\in[0,b]} \|R(t)\|.$$

Define the operator $K:\ E_0 \to \mathcal{C}([0,b_1];X)$ by

$$(Kx)(t) = R(t)x_0 + \int_0^t R(t-s)\big[f(s,x(s)) + C(s)u(s)\big]\,ds \quad \text{for } t \in [0,b_1]$$

We claim that $K\Big(E_0\Big) \subset E_0$. In fact let $x \in E_0$ and $t \in [0,b_1]$. Then,

$$\begin{aligned} \|(Kx)(t) - x(0)\| &\leq \|R(t)x(0) - x(0)\| \\ &\quad + \int_0^t \Big\| R(t-s)\big[f(s,x(s)) + C(s)u(s)\big] \Big\|\,ds \\ &\leq \|R(t)x(0) - x(0)\| + M_b\,\rho^* t + M_b\,\|C\|\,\|u\|_{L^2}\sqrt{t}. \end{aligned}$$

Now, choose b_1 sufficiently small such that

$$\sup_{s\in[0,b_1]} \Big\{ \|R(s)x(0) - x(0)\| + M_b\,\rho^* s + M_b\,\|C\|\,\|u\|_{L^2}\sqrt{s} \Big\} < 1. \qquad (8.2.2)$$

Consequently,

$$\|(Kx)(t)-x(0)\| \le \|R(t)x(0)-x(0)\|+M_b\,\rho^* t+M_b\,\|C\|\,\|u\|_{L^2}\sqrt{t} < 1 \ \text{ for } t\in[0,b_1].$$

Hence, $K\Big(E_0\Big) \subset E_0$.
Let $x,\, y \in E_0$ and $t \in [0, b_1]$. Then, there exists $\rho > 0$ such that $\|x\|,\ \|y\| \le \rho$. We have that

$$\begin{aligned}
\|(Kx)(t) - (Ky)(t)\| &\le M_b \int_0^t \|f(s,x(s)) - f(s,y(s))\|\,ds \\
&\le M_b\,L_f(\rho)\int_0^t \|x(s)-y(s)\|\,ds \\
&\le M_b\,L_f(\rho)\int_0^t \sup_{\tau\in[0,s]}\|x(\tau)-y(\tau)\|\,d\tau \\
&\le M_b\,L_f(\rho)\,b_1\,\|x-y\|
\end{aligned}$$

Now, since

$$M_b\,L_f(\rho)\,b_1 \le M_b\,\rho^*\,b_1 < \sup_{s\in[0,b_1]} \Big\{ \|R(s)x(0)-x(0)\|+M_b\,\rho^* s+M_b\,\|C\|\,\|u\|_{L^2}\sqrt{s} \Big\}.$$

Condition (8.2.2) implies that

$$M_b\,L_f(\rho)\,b_1 < 1.$$

Thus, K is a strict contraction on E_0. It follows from the contraction mapping principle that K has a unique fixed point $x \in E_0$, which is the unique mild solution of equation (8.1.1) with respect to u on $[0, b_1]$.
Using the same arguments, we can show that x can be extended to a maximal interval of existence $[0, t_{\max}[$.

Lemma 8.2.4 *[86] If $t_{\max} < b$, then, $\limsup_{t\to t_{\max}} \|x(t)\| = \infty$.*

We show that $t_{\max} = b$.
Assume on the contrary that $t_{\max} < b$. Then for $t \in [0, t_{\max}]$ we have that

$$x(t) = R(t)x_0 + \int_0^t R(t-s)\big[f(s,x(s)) + C(s)u(s)\big]\,ds.$$

It follows that

$$\begin{aligned}
\|x(t)\| &\leq M_b\|x_0\| + M_b\int_0^t \|f(s,x(s))\|\,ds + M_b\int_0^t \|C(s)u(s)\|\,ds \\
&\leq M_b\|x_0\| + M_b\,t_{\max}\,a_f + M_b a_f\int_0^t \|x(s)\|\,ds + M_b\|C\|\int_0^t \|u(s)\|\,ds \\
&\leq M_b\|x_0\| + M_b\,t_{\max}\,a_f + M_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2} + M_b\,a_f\int_0^t \|x(s)\|\,ds
\end{aligned}$$

This implies that

$$\|x(t)\| \leq M_b\|x_0\| + M_b\,t_{\max}\,a_f + M_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2} + M_b\,a_f\int_0^t \|x(s)\|\,ds$$

It follows by Gronwall's inequality that

$$\|x(t)\| \leq \beta^* e^{M_b\,a_f\,t} \quad \text{for } t\in[0,t_{\max}],$$

where $\beta^* = M_b\|x_0\| + M_b\,t_{\max}\,a_f + M_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2}$.
Thus

$$\lim_{t\to t_{\max}} \|x(t)\| \leq \beta^* e^{M_b\,a_f\,t_{\max}} < \infty.$$

This contradicts Lemma 8.2.4. Therefore, $t_{\max} = b$ and hence, equation (8.1.1) has a unique mild solution on $[0,b]$.

□

8.3 Continuous Dependence and Existence of the Optimal Control

In this section, we discuss the continuous dependence of the mild solutions of equation (8.1.1) on the controls and initial states, and the existence of solutions of the Lagrange problem associated to equation (8.1.1).
We have the following a priori estimation.

Lemma 8.3.1 *Suppose* $(\mathbf{H_1})-(\mathbf{H_3})$ *holds and assume that equation (8.1.1) has a mild solution* x_u *on* $[0,b]$ *with respect to* $u\in\mathcal{U}_{ad}$*. Then, there exists a constant* $\rho>0$ *independent of* u *such that* $\|x_u(t)\|\leq\rho$ *for* $t\in[0,b]$*, (*ρ *depends only on* $\mathcal{U}_{ad}$ *and* x_0*).*

Proof. Let

$$t \in [0,b],\ M_b = \sup_{t\in[0,b]} \|R(t)\| \ \text{ and } \ \|C\| = \sup_{t\in I} \|C(t)\|_{\mathcal{L}(U,X)}.$$

Since $\mathcal{U}_{ad}$ is bounded, let $\widetilde{K} > 0$ be such that $\|u\|_{L^2} \le \widetilde{K}$ for all $u \in \mathcal{U}_{ad}$. Then, we have that

$$\begin{aligned}
\|x(t)\| &\le M_b\|x_0\| + M_b \int_0^t \|f(s,x(s))\|\,ds + M_b \int_0^t \|C(s)u(s)\|\,ds \\
&\le M_b\|x_0\| + M_b\, b\, a_f + M_b a_f \int_0^t \|x(s)\|\,ds + M_b\|C\| \int_0^b \|u(s)\|\,ds \\
&\le M_b\|x_0\| + M_b\, b\, a_f + M_b\sqrt{b}\,\|C\|\|u\|_{L^2} + M_b\, a_f \int_0^t \|x(s)\|\,ds \\
&\le M_b\|x_0\| + M_b\, b\, a_f + M_b\sqrt{b}\,\|C\|\widetilde{K} + M_b\, a_f \int_0^t \|x(s)\|\,ds.
\end{aligned}$$

Thus

$$\|x(t)\| \le M_b\|x_0\| + M_b\, b\, a_f + M_b\sqrt{b}\,\|C\|\widetilde{K} + M_b\, a_f \int_0^t \|x(s)\|\,ds. \tag{8.3.1}$$

It follows by Gronwall's inequality that

$$\|x(t)\| \le M e^{b\,a_f\,M_b} =: \widetilde{M},$$

with $M = M_b\|x_0\| + M_b\, b\, a_f + M_b\sqrt{b}\,\|C\|\widetilde{K}$.
As a result, for $t \in I$, we have

$$\|x_u(t)\| \le \widetilde{M} := \rho$$

That is $\|x_u(t)\| \le \rho$ for all $t \in I$. This completes the proof of the Lemma.

□

We have the following theorem on continuous dependence of the mild solutions of equation (8.1.1) on the controls and initial states.

Theorem 8.3.2 *For all $\lambda > 0$, there exists $\gamma^*(\lambda) > 0$ such that for all $x_0^1,\ x_0^2 \in B(0,\lambda)$,*

$$\|x^1(t) - x^2(t)\| \le \gamma^*(\lambda)\Big(\|x_0^1 - x_0^2\| + \|u^1 - u^2\|_{L^2}\Big) \ \ \textit{for} \ \ t \in [0,b],$$

where

$$x^i(t) = R(t)x_0^i + \int_0^t R(t-s)\left[f(s, x^i(s)) + C(s)u^i(s)\right] ds \quad \textit{for } t \in I \quad (8.3.2)$$

and $u^i \in \mathcal{U}_{ad}$, *for* $i = 1, 2$.

Proof. Let x^i, for $i = 1, 2$, be two mild solutions of equation (8.1.1), corresponding to the controls $u^i \in \mathcal{U}_{ad}$ and $\lambda > 0$ such that $x_0^1,\ x_0^2 \in B(0, \lambda)$.

$$x^i(t) = R(t)x_0^i + \int_0^t R(t-s)\left[f(s, x^i(s)) + C(s)u^i(s)\right] ds \ \text{ for } t \in I$$

By Lemma 8.3.1, we have that, there exists a constant $\rho_\lambda = \widetilde{M} > 0$ such that $\|x^i(s)\| \leq \rho_\lambda,\ i = 1,\ 2$.
Now, for $t \in [0, b]$, we have

$$\begin{aligned}
\|x^1(t) - x^2(t)\| &\leq M_b\|x_0^1 - x_0^2\| + M_b \int_0^t \|f(s, x^1(s)) - f(s, x^2(s))\| \, ds \\
&+ M_b \int_0^t \|C(s)u^1(s) - C(s)u^2(s)\| \, ds \\
&\leq M_b\|x_0^1 - x_0^2\| + M_b L_f(\rho_\lambda) \int_0^t \Big(\|x^1(s) - x^2(s)\|\Big) \, ds \\
&+ M_b \int_0^t \|C(s)u^1(s) - C(s)u^2(s)\| \, ds \\
&\leq M_b\|x_0^1 - x_0^2\| + M_b L_f(\rho_\lambda) \int_0^t \|x^1(s) - x^2(s)\| \, ds \\
&+ M_b L_f(\rho_\lambda) \sqrt{b}\, \|C\| \, \|u^1 - u^2\|_{L^2}
\end{aligned}$$

That is

$$\|x^1(t)-x^2(t)\| \leq M_b\|x_0^1-x_0^2\| + M_b L_f(\rho_\lambda)\sqrt{b}\,\|C\|\,\|u^1-u^2\|_{L^2} + M_b L_f(\rho_\lambda) \int_0^t \|x^1(s)-x^2(s)\| \, ds. \quad (8.3.3)$$

By Gronwall's inequality, we have that

$$\|x^1(t) - x^2(t)\| \leq \Big[M_b\|x_0^1 - x_0^2\| + M_b L_f(\rho_\lambda)\sqrt{b}\,\|C\|\,\|u^1 - u^2\|_{L^2}\Big] e^{M_b L_f(\rho_\lambda)\, b}.$$

This implies that

$$\|x^1(t) - x^2(t)\| \;\leq\; \Big[M_b\|x_0^1 - x_0^2\| + M_b L_f(\rho_\lambda)\,\sqrt{b}\,\|C\|\,\|u^1 - u^2\|_{L^2}\Big] e^{M_b L_f(\rho_\lambda)\,b},$$

Let

$$\gamma^*(\lambda) := \max\Big\{M_b\,e^{M_b L_f(\rho_\lambda) b},\; M_b L_f(\rho_\lambda)\,\sqrt{b}\,\|C\|\,e^{M_b L_f(\rho_\lambda)\,b}\Big\}.$$

Then, we have that

$$\|x^1(t) - x^2(t)\| \leq \gamma^*(\lambda)\Big(\|x_0^1 - x_0^2\| + \|u^1 - u^2\|_{L^2}\Big) \;\text{ for } t \in [0, b].$$

And the proof is complete.

□

Now, we study the existence of solutions to the following Lagrange problem

$$(\mathcal{LP})\begin{cases} \text{Find a control } u^0 \in \mathcal{U}_{ad} \;\text{ such that} \\ \\ \mathcal{J}(u^0) \leq \mathcal{J}(u) \;\text{ for all }\; u \in \mathcal{U}_{ad}, \end{cases}$$

where

$$\mathcal{J}(u) := \int_0^b \mathcal{L}\Big(t,\, x^u(t),\, u(t)\Big)\,dt,$$

and x^u denotes the mild solution of (8.1.1) corresponding to the control $u \in \mathcal{U}_{ad}$ and the initial data x_0.
For the existence of solutions to problem $(\mathcal{LP})$, we make the following assumptions.

$(\mathbf{H_L})$

(i) The functional $\mathcal{L} : I \times X \times U \to \mathbb{R} \cup \{\infty\}$ is Borel measurable.

(ii) $\mathcal{L}(t,\cdot,\cdot)$ is sequencially lower semicontinuous on $X \times U$ for almost all $t \in I$.

(iii) $\mathcal{L}(t,\, y,\, \cdot)$ is convex on U for each $y \in X$ and almost all $t \in I$.

(iv) There exist constants $\beta \geq 0,\ \gamma > 0$, and $\mu \in L^1(I)$ nonnegative such that

$$\mathcal{L}(t,\, y,\, u) \geq \mu(t) + \beta\|y\| + \gamma\|u\|.$$

We have the following result on the existence of optimal controls for problem $(\mathcal{LP})$.

Theorem 8.3.3 *Assume that hypotheses* $(\mathbf{H_1}) - (\mathbf{H_5})$ *and* $(\mathbf{H_L})$ *hold. Then the Lagrange problem* $(\mathcal{LP})$ *admits at least one optimal pair, that is there exists an admissible control pair* $(x^0, u^0) \in \mathcal{C}([0,b], X) \times \mathcal{U}_{ad}$ *such that*

$$\mathcal{J}(u^0) = \int_0^b \mathcal{L}\Big(t,\, x^0(t),\, u^0(t)\Big)\, dt \le \int_0^b \mathcal{L}\Big(t,\, x^u(t),\, u(t)\Big)\, dt = \mathcal{J}(u) \ \ \textit{for}\, u \in \mathcal{U}_{ad}.$$

Proof. If $\inf\Big\{\mathcal{J}(u)\,:\, u \in \mathcal{U}_{ad}\Big\} = \infty$, we are done.
Without loss of generality, assume that $\inf\Big\{\mathcal{J}(u)\,:\, u \in \mathcal{U}_{ad}\Big\} = \delta < \infty$.
Suppose that $\delta = -\infty$, then for each $n \in \mathbb{N}$, there exists $(u^n)_{n\ge 1} \subset \mathcal{U}_{ad}$ such that

$$\mathcal{J}(u^n) < -n \qquad (*)$$

Boundedness of $\mathcal{U}_{ad}$ implies that $(u^n)_{n\ge1}$ is bounded and so there exists a subsequence $(u^{n_k})_{k\ge1}$ of $(u^n)_{n\ge1}$ that converges weakly to some u^0 in $L^2(I, U)$, since $L^2(I,U)$ is reflexive. But $\mathcal{U}_{ad}$ is closed and convex, so by Mazur's Lemma, it is weakly closed and therefore, $u^0 \in \mathcal{U}_{ad}$. By hypothesis $(\mathbf{H_L})$, $\mathcal{L}(t,\, y,\, \cdot)$ is weakly lower semicontinuous, so we have that

$$\mathcal{L}(t,\, y,\, u^0) \le \liminf_{k\to\infty} \mathcal{L}(t,\, y,\, u^{n_k}) < -\infty,$$

which implies that $\mathcal{J}(u^0) < -\infty$ using $(*)$. And this is a contradiction since $\mathcal{J}(u^0) \in \mathbb{R} \cup \{\infty\}$. Hence $\delta \in \mathbb{R}$.
Now by the definition of δ, there exists a minimizing sequence, a feasible pair $((x^n, u^n))_{n\ge1} \subset \mathcal{S}_{ad}$ such that

$$\int_0^b \mathcal{L}\Big(t,\, x^n(t),\, u^n(t)\Big)\, dt \longrightarrow \delta \ \ \text{as } n \to \infty,$$

where

$$\mathcal{S}_{ad} := \Big\{(x,u)\,:\, x \text{ is a mild solution of equation (8.1.1) corresponding to the control } u \in \mathcal{U}_{ad}\Big\}.$$

Boundedness of $\mathcal{U}_{ad}$ and the fact that $L^2(I,U)$ is reflexive imply that $(u^n)_{n\ge1}$ has a subsequence denoted for simplicity by $(u^k)_{k\ge1}$, that converges weakly to some u^0 in $L^2(I,U)$. But $\mathcal{U}_{ad}$ is closed and convex, so by Mazur's Lemma, it is weakly closed and therefore, $u^0 \in \mathcal{U}_{ad}$.
Let

$$x^k(t) = R(t)x_0 + \int_0^t R(t-s)\left[f(s, x^k(s)) + C(s)u^k(s)\right]\, ds \ \ \text{for } t \in I$$

denote the subsequence of $(x^n)_{n\geq 1}$ corresponding to the control sequence $(u^k)_{k\geq 1}$ and x^0 be the mild solution corresponding to the control $u^0 \in \mathcal{U}_{ad}$. We show that $x^k \to x^0$.
For $t \in [0, b]$, we have

$$\begin{aligned}
\Big\|x^k(t) - x^0(t)\Big\| &\leq \int_0^t \Big\|R(t-s)\Big[f(s, x^k(s)) - f(s, x^0(s))\Big]\Big\|\, ds \\
&+ \int_0^t \Big\|R(t-s)\Big[C(s)u^k(s) - C(s)u^0(s)\Big]\Big\|\, ds \\
&\leq M_b L_f(\rho) \int_0^t \Big\|x^k(s) - x^0(s)\Big\|\, ds + M_b \int_0^t \Big\|C(s)u^k(s) - C(s)u^0(s)\Big\|\, ds \\
&\leq M_b L_f(\rho) \int_0^t \Big\|x^k(s) - x^0(s)\Big\|\, ds \\
&\quad + M_b\sqrt{b}\left(\int_0^t \Big\|C(s)u^k(s) - C(s)u^0(s)\Big\|^2 ds\right)^{\frac{1}{2}} \\
&\leq M_b L_f(\rho) \int_0^t \Big\|x^k(s) - x^0(s)\Big\|\, ds + M_b\sqrt{b}\Big\|Cu^k - Cu^0\Big\|_{L^2(I,U)}
\end{aligned}$$

That is

$$\Big\|x^k(t) - x^0(t)\Big\| \leq M_b L_f(\rho) \int_0^t \Big\|x^k(s) - x^0(s)\Big\|\, ds + M_b\sqrt{b}\Big\|Cu^k - Cu^0\Big\|_{L^2(I,U)} \tag{8.3.4}$$

By Gronwall's inequality, we have that

$$\Big\|x^k(t) - x^0(t)\Big\| \leq M^{**}\Big\|Cu^k - Cu^0\Big\|_{L^2(I,U)}, \quad \text{where } M^{**} = M_b\sqrt{b}\, e^{M_b\, b L_f(\rho)}. \tag{8.3.5}$$

We have the following Lemma.

Lemma 8.3.4 *[96] Let $(u^n)_{n\geq 1} \subset \mathcal{U}_{ad}$ and $u^0 \in \mathcal{U}_{ad}$ such that $(u^n)_{n\geq 1}$ converges weakly to u^0. Then,*

$$\Big\|Cu^k - Cu^0\Big\|_{L^2(I,U)} \longrightarrow 0 \quad \text{as } k \to \infty, \quad \text{if } C \in L^\infty(I; \mathcal{L}(U, X)).$$

We have by (8.3.5) that

$$\left\|x^k - x^0\right\| \leq M^{**}\left\|Cu^k - Cu^0\right\|_{L^2(I,U)},$$

and therefore, it follows by Lemma 8.3.4 that

$$x^k \longrightarrow x^0 \;\text{ as } k \to \infty.$$

We note that $(\mathbf{H_L})$ implies the assumptions of Balder's Theorem. Hence by using Balder's Theorem, we can conclude that $(x,\, u) \mapsto \int_0^b \mathcal{L}\Big(t,\, x(t),\, u(t)\Big)\, dt$ is sequencially lower semicontinuous in the strong topology of $L^1(I,X) \times L^1(I,U)$.
Now, since $L^2(I,X) \times L^2(I,U) \subset L^1(I,X) \times L^1(I,U)$, $\mathcal{J}$ is also sequencially lower semicontinuous on $L^2(I,X) \times L^2(I,U)$, and in the strong topology of $L^1(I,X \times U)$.
Hence, $\mathcal{J}$ is weakly lower semicontinuous on $L^2(I,U)$, and since by $(\mathbf{H_L}) - (\mathbf{iv})$, $\mathcal{J} > -\infty$, $\mathcal{J}$ attains its infimum at $u^0 \in \mathcal{U}_{ad}$, that is

$$\delta = \lim_{k\to\infty} \int_0^b \mathcal{L}\Big(t,\, x^k(t),\, u^k(t)\Big)\, dt \geq \int_0^b \mathcal{L}\Big(t,\, x^0(t),\, u^0(t)\Big)\, dt = \mathcal{J}(u^0) \geq \delta.$$

Thus, $\delta = \mathcal{J}(u^0)$, and hence there exists an admissible control $u^0 \in \mathcal{U}_{ad}$ such that

$$\mathcal{J}(u^0) \leq \mathcal{J}(u) \;\text{ for all }\; u \in \mathcal{U}_{ad}.$$

This completes the proof.

□

we now illustrate our main result by the following example.

8.4 Example

Let Ω be bounded domain in $\mathbb{R}^n$ with smooth boundary and consider the following nonlinear integrodifferential equation.

$$\begin{cases} \dfrac{\partial v(t,\xi)}{\partial t} = \Delta v(t,\xi) + \displaystyle\int_0^t \zeta(t-s)\Delta v(s,\xi)\,ds + \alpha(t)\sin(v^2(t,\xi)) + \beta(t)\omega(t,\xi) \\ \qquad\qquad \text{for } t \in [0,1] = I \text{ and } \xi \in \Omega \\ v(t,\xi) = 0 \text{ for } t \in [0,1] \text{ and } \xi \in \partial\Omega \\ v(0,\xi) = v_0(\xi), \end{cases} \tag{8.4.1}$$

where $\alpha,\ \beta \in \mathcal{C}(I,\mathbb{R})$, $\omega : I \times \Omega \to \mathbb{R}$ continuous in t, and $\zeta \in W^{1,1}(\mathbb{R}^+,\mathbb{R})$.

Let $X = U = L^2(\Omega)$. For $\eta > 0$, we define the set of admissible controls $\mathcal{U}_{ad}$ by

$$\mathcal{U}_{ad} := \Big\{ u : I \to U \text{ such that } u \text{ is measurable and } \|u\|_{L^2(I,U)} \le \eta \Big\},$$

where

$$\|u\|^2_{L^2(I,U)} = \int_0^1 \left(\int_\Omega u^2(s)(\xi)\,d\xi \right) ds.$$

We define $A : \mathcal{D}(A) \subset X \to X$ by:

$$\begin{cases} \mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega) \\ Av \;=\; \Delta v \ \text{ for } v \in \mathcal{D}(A). \end{cases} \tag{**}$$

Theorem 8.4.1 *(***Theorem 4.1.2, p. 79 of [76]***) A is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\ge 0}$ on $L^2(\Omega)$.*

Define

$$x(t)(\xi) \;=\; v(t,\xi), \quad x'(t)(\xi) \;=\; \frac{\partial v(t,\xi)}{\partial t}, \quad \omega(t,\xi) = u(t)(\xi).$$

$$f(t,x(t))(\xi) \;=\; \alpha(t)\sin\big(x^2(t)(\xi)\big) \ \text{ for } t \in [0,1] \text{ and } \xi \in \Omega.$$

$C(t) : X \to X$ be defined by $\Big(C(t)u(t)\Big)(\xi) \;=\; C(t)u(t)(\xi) = \beta(t)\omega(t,\xi)$.

$$(B(t)x)(\xi) \;=\; \zeta(t)\Delta v(t,\xi) \ \text{ for } \ t \in [0,1], \ \ x \in \mathcal{D}(A) \text{ and } \ \xi \in \Omega.$$

Equation (8.4.1) is then transformed into the following form

$$\begin{cases} x'(t) = Ax(t) \ + \ \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t,x(t)) + C(t)u(t) \ \text{ for } t \in I = [0,1], \\ \\ x(0) = x_0. \end{cases} \tag{8.4.2}$$

The operator A defined in (**) is closed, linear and densely defined and so satisfies hypothesis $(\mathbf{H_1})$. Also, the family of operators $\big(B(t)\big)_{t\geq 0}$ defined by $B(t)y = \zeta(t)\Delta y$ is linear and belongs to $W^{1,1}(\mathbb{R}^+,\mathbb{R})$ since $\zeta \in W^{1,1}(\mathbb{R}^+,\mathbb{R})$, and it follows that hypothesis $(\mathbf{H_2})$ is satisfied. Therefore, equation (8.4.2) satisfies hypotheses $(\mathbf{H_1})$ and $(\mathbf{H_2})$, which guarantees the existence of a unique resolvent operator.
One can see that, f satisfies $(\mathbf{H_3})$. Now we consider the following cost function:

$$\mathcal{J}(u) := \int_0^1 \mathcal{L}\Big(t,\, x^u(t),\, u(t)\Big)\, dt,$$

where
$\mathcal{L} \,:\, [0,1] \times L^2(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$ is defined by:

$$\mathcal{L}\Big(t,\, x,\, u\Big) \quad = \quad \|x\| + \|u\|.$$

$\mathcal{L}$ satisfies all the conditions of hypothesis $(\mathbf{H_L})$. Then,

$$\mathcal{J}(u) = \int_0^1 \Big(\|x^u(t)\| + \|u(t)\|\Big)\, dt.$$

Hence, all the conditions of Theorem 8.3.3 are satisfied, and therefore, equation (8.4.2) has at least one optimal pair.

Conclusion and Perspectives

In this work, we have been able to make the following contributions. In the first part, we investigated the controllability results for some partial functional integrodifferential equations with nonlocal initial conditions (equation (1.3.4)), with finite delay (equation (1.3.5)) and with infinite delay (equation (1.3.6)) in Banach spaces, using fixed point techniques and measure of noncompactness. In the second part, we established the existence of optimal pairs for the Lagrange problems associated with some partial functional integrodifferential equations with bounded delay, with unbounded delay and with classical Cauchy initial condition in Banach spaces, using techniques of convex optimization. The main results are obtained by supposing that the linear homogeneous and undelayed parts of these equations admit a resolvent operator in the sense of Grimmer. We note that this class of integrodifferential equations have not been investigated (to the best of our knowledge) for controllability and optimal controls.

The importance of the problems considered in this work is that they have opened doors for new deep research problems such as approximate controllability which is a weaker notion than the axact controllability treated in this thesis, for these partial functional integrodifferential equations. For future work, we shall consider equations (1.3.4), (1.3.5) and (1.3.6) for approximate controllability without compactness assumption on the resolvent operator, we shall also extend them by a Wiener process in the stochastic frame work and use iterative methods for existence of solutions and possibly also for existence of optimal controls.

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