ITERATIVE ALGORITHMS FOR SINGLE-VALUED AND MULTI-VALUED NONEXPANSIVE-TYPE MAPPINGS IN REAL LEBESGUE SPACES.

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CERTIFICATE OF APPROVAL

Ph.D. THESIS

This is to certify that

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a graduate student in the DEPARTMENT OF PURE AND APPLIED MATHEMATICS, AFRICAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, with I.D. No: 70058 has satisfactorily completed the requirements for research work for the degree of Doctor of Philosophy in Mathematics. The work embodied in this dissertation is original and has not been submitted in part or full for any other diploma or degree of this or any other university.

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Dedication

This thesis is dedicated to my amiable wife Chisom.

For her endless love, support and encouragement.

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Abstract

Algorithms for single-valued and multi-valued nonexpansive-type mappings have continued to attract a lot of attentions because of their remarkable utility and wide applicability in modern mathematics and other reasearch areas,(most notably medical image reconstruction, game theory and market economy).
The first part of this thesis presents contributions to some crucial new concepts and techniques for a systematic discussion of questions on algorithms for single-valued and multi-valued mappings in real Hilbert spaces. Novel contributions are made on iterative algorithms for fixed points and solutions of the split equality fixed point problems of some single-valued pseudocontractive-type mappings in real Hilbert spaces. Interesting contributions are also made on iterative algorithms for fixed points of a general class of multivalued strictly pseudocontractive mappings in real Hilbert spaces using a new and novel approach and the thorems were gradually extended to a countable family of multi-valued mappings in real Hilbert spaces.It also contains contains original research and important results on iterative approximations of fixed points of multi-valued tempered Lipschitz pseudocontractive mappings in Hilbert spaces.

Apart from using some well known iteration methods and identities, some very new and innovative iteration schemes and identities are constructed. The thesis serves as a basis for unifying existing ideas in this area while also generalizing many existing concepts. In order to demonstrate the wide applicability of the theorems, there are given some nontrivial examples and the technique is demonstrated to be more valuable than other methods currently in the literature.

The second part of the thesis focuses on some related optimization problems in some Banach spaces. Some iterative algorithms are proposed for common

solutions of zeroes of a monotone mapping and a finite family of nonexpansive mappings in Lebesgue spaces.

The thesis presents in a unified manner, most of the recent works of this author in this direction, namely:

- Let H_1, H_2, H_3 be real Hilbert spaces, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ two Lipschitz hemicontractive mappings, and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear mappings. Then the coupled sequence (x_n, y_n) generated by the algorithm

$$\begin{cases} (x_1, y_1) \in H_1 \times H_2, \text{ chosen arbitarily,} \\ (x_{n+1}, y_{n+1}) = (1-\alpha)[(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)] \\ +\alpha G(u_n, v_n), \\ (u_n, v_n) = (1-\alpha)[(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)] \\ +\alpha G(x_n, y_n), \\ \alpha \in (0, L^{-2}(\sqrt{L^2+1} - 1)) \\ \lambda \in (0, \frac{2\alpha}{\lambda(A,B)}), \end{cases}$$

  converges weakly to a solution (x^*, y^*) of the Split Equality Problem.

- Let K be a nonempty, closed, convex subset of a real Hilbert space H. Let $T : K \to CB(K)$ be a mapping satisfying

$$D(Tx, Ty) \leq \|x - y\|^2 + kD(Ax, Ay), \ \ k \in (0, 1), A := I - T.$$

  Assume that $F(T) \neq \emptyset$ and $Tp = \{p\}$ $\forall p \in F(T)$. Then, the sequence $\{x_n\}$ generated by a certain Krasnolselskii type algorithm is an approximate fixed point sequence of T and under appropriate mild conditions, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

- Let K be a nonempty, closed and convex subset of a real Hilbert space H. For $i = 1, 2, ..., m$, let $T_i : K \to CB(K)$ be a family of mappings satisfying

$$D(T_ix, T_iy) \leq \|x - y\|^2 + k_iD(A_ix, A_iy), k_i \in (0, 1), \quad A_i := I - T_i,$$

  for each i. Suppose that $\cap_{i=1}^m F(T_i) \neq \emptyset$ and assume that for $p \in \cap_{i=1}^m F(T_i)$, $T_ip = \{p\}$. Then, the sequence $\{x_n\}$ generated by the algorithm:

$$\begin{cases} x_0 \in K \text{ chosen arbitarily,} \\ x_{n+1} = (\lambda_0)x_n + \sum\limits_{i=1}^{m} \lambda_i y_n^i, \\ y_n^i \in S_n^i := \left\{ z_n^i \in T_i x_n : D^2(\{x_n\}, T_i x_n) \leq \|x_n - z_n^i\|^2 + \frac{1}{n^2} \right\} \\ \lambda_0 \in (k,1), \ \sum\limits_{i=0}^{m} \lambda_i = 1, \ \ and \ \ k := \max\{k_i, i = 1,2,...,m,\}. \end{cases}$$

is an approximate fixed point sequence for the finite family of mappings.

- Let $T_i : K \to CB(K)$ be a countably infinite family of mappings satisfying

  $$D(T_i x, T_i y) \leq \|x-y\|^2 + k_i D(A_i x, A_i y), k_i \in (0,1), \quad A_i := I - T_i.$$

  Assume that $\kappa := \sup\limits_i k_i \in (0,1)$, $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and for $p \in \cap_{i=1}^{\infty} F(T_i)$, $T_i p = \{p\}$. Then, the Krasnoselskii type sequence $\{x_n\}$ generated by the algorithm:

  $$\begin{cases} x_0 \in K, \ \ arbitrary, \\ \zeta_n^i \in \Gamma_n^i := \left\{ z_n^i \in T_i x_n : D^2(\{x_n\}, T_i x_n) \leq \|x_n - z_n^i\|^2 + \frac{1}{n^2} \right\} \\ x_{n+1} = \delta_0 x_n + \sum\limits_{i=1}^{\infty} \delta_i \zeta_n^i, \\ \delta_0 \in (\kappa, 1), \ \ \sum_{i=0}^{\infty} \delta_i = 1, \end{cases}$$

  is an approximate fixed point sequence of the family T_i.

- Let H be a real Hilbert space, $K \subseteq H$ be a nonempty, closed and convex. Let $T : K \to CB(K)$ be a multivalued mapping satisfying $F(T) \neq \emptyset$, $diam(Tx \cup Ty) \leq L\|x-y\|$ for some $L > 0$, and

  $$D^2(Tx, Tp) \leq \|x-p\|^2 + D^2(x, Tx), \ \ \forall x \in H, p \in F(T). \qquad (0.0.1)$$

  Let $\{x_n\}$ be a sequence defined by the algorithm:

  $$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\lambda)x_n + \lambda z_n, \quad \lambda \in (0, L^{-2}[\sqrt{1+L^2} - 1]) \\ z_n \in \Gamma^n := \{u_n \in Ty_n : D(x_n, Ty_n) \leq \|x_n - u_n\|^2 + \theta_n\} \\ y_n = (1-\lambda)x_n + \lambda w_n, \\ w_n \in \Pi^n := \{v_n \in Tx_n : D(x_n, Tx_n) \leq \|x_n - v_n\|^2 + \theta_n\} \\ \theta_n \geq 0, \sum\limits_{n=1}^{\infty} \theta_n < \infty \end{cases}$$

Then $p \in F(T)$, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is an approximate fixed point sequence of T.

- Let $E = L_p, 1 < p \leq 2$, and $E^* = L_q$, $\frac{1}{p} + \frac{1}{q} = 1$. For $k = 1, 2, ..., N$, let $T_k : E \to E$ be a finite family of nonextensive mappings and $A : E \to E^*$ be an $\eta-$strongly monotone mapping which is also $L-$Lipschitzian. Assume that $S := A^{-1}(0) \cap \cap_{k=1}^{N} Fix(T_k) \neq \emptyset$. Then for arbitrary $x_1 \in E$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = j^{-1}\Big(j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)\Big), n \geq 1 \qquad (0.0.2)$$

converges to the common solution of the problem VIP$^*(A, Fix(T_{[n]}))$, where $T_{[n]} := T_{n\, mod\, N}$, and $\lambda \in (0, \frac{\eta}{2L_1^2 L_2})$, L_1, L_2 the Lipschitz constants for the mappings A and j^{-1}, respectively.

- Let $E = L_p$, $2 \leq p < \infty$ and $A : L_p \to L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, be an η-strongly monotone mapping which is also Lipschitzian. For $k = 1, 2, ..., N$, let $T_k : L_p \to L_p$ be a finite family of nonextensive mappings. Assume that $S := A^{-1}(0) \cap \cap_{k=1}^{N} Fix(T_k) \neq \emptyset$. Then for arbitrary $x_1 \in E$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = j^{-1}\Big(j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)\Big), n \geq 1 \qquad (0.0.3)$$

converges strongly to the unique common solution of the problem VIP$^*(A, Fix(T_k))$, where $T_{[n]} := T_{n\, mod\, N}$, and $\lambda_n \in \left(0, \frac{\eta}{2L_1 L_2^{\frac{p}{p-1}}}\right)$ satisfies $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\sum_{n=1}^{\infty} \lambda_n^{\frac{p}{p-1}} < \infty$, L_1, L_2 are the Lipschitz constants for the mappings A and j^{-1}, respectively.

Research articles arising from the thesis and other peer-reviewed publication

[A] **Published/Accepted papers from the thesis.**

1. **M. E. Okpala,** *Split equality fixed point problem for Lipschitz Hemi-contractive mappings*,**(Accepted(2015) Advances in Fixed Point Theory.**

2. **M.E. Okpala,** *A Remark On the Theorem of Ishikawa,* **British Journal of Mathematics and Computer Science** Vol 7 Issue 72015.

3. C. E. Chidume and **M.E. Okpala,** *On a general class of multi-valued strictly pseudocontractive mapping*, **Journal of Nonlinear Analysis and Optimization, Theory & Applications**(Springer-Verlag) Vol 5 No 2. (2014).

4. C. E. Chidume, **M. E. Okpala,** A. U. Bello,and P. Ndambomve, *Convergence theorems for finite family of a general class of Multi-valued Strictly Pseudocontractive Mappings*, **Fixed Point Theory and Applications)** (2015) 2015:119 DOI 10.1186/s13663-015-0365-7

Other peer-reviewed publication.

5. C. E. Chidume, P. Ndambomve, A. U. Bello, **M. E. Okpala**, *The multiple-sets split equality fixed point problem for countable families of multi-valued demi-contractive mappings*, **International Journal of Mathematical Analysis** Vol. 9, 2015, no. 10, 453 - 469.

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1. **M.E. Okpala**, *Fixed Point Iteration for a Countable Family of Multivalued Strictly Pseudocontractive-type Mappings* **(Submitted: June (2015) SpringerPlus)**

CHAPTER 1

General Introduction

Fixed Point Theory is concerned with solutions of the equation

$$x = Tx \tag{1.0.1}$$

where T is a (possibly) nonlinear operator defined on a metric space. Any x that solves (1.0.1) is called a fixed point of T and the collection of all such elements is denoted by $F(T)$. For a multi-valued mapping $T : X \to 2^X$, a fixed point of T is any x in X such that $x \in Tx$.

Fixed Point Theory is inarguably the most powerful and effective tools used in modern nonlinear analysis today. It is still an area of current intensive research as it has vast applicability in establishing existence and uniqueness of solutions of diverse mathematical models like solutions to optimization problems, variational analysis, and ordinary differential equations. These models represent various phenomena arising in different fields, such as steady state temperature distribution, neutron transport theory, economic theories, chemical equations, optimal control of systems, models for population, epidemics and flow of fluids.

For example, given an initial value problem

$$\begin{cases} \frac{dx(t)}{dt} = f(t, x(t)), \\ x(t_0) = x_0. \end{cases} \tag{1.0.2}$$

This system is transformed into the functional equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

To establish existence of solution to system (1.0.2), we consider the operator $T : X \to X(X = C([a, b]))$ defined by

$$Tx = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Then finding a solution to the initial value problem (1.0.2) amounts to finding a fixed point of T.

The existence(and uniqueness) of solution to equation (1.0.1), certainly, depends on the geometry of the space and the nature of the mapping T. Existence theorems are concerned with establishing sufficient conditions under which the equation (1.0.1) will have a solution, but does not neccesarily show how to find them. There are very many existence and uniqueness theorems in the literature(see e.g. Kirk [67], Kato [62], Kōmura [68]).

Though existence theorems do not indicate how to construct a process starting from a nonfixed point and convergent to a fixed point, they nevertheless enhance understanding of conditions under which the existence of such fixed points is guaranteed.

On the other hand, iterative methods of fixed points theory is concerned with approximation or computation of sequences which converge to solutions of (1.0.1). This is part of the problem that is being addressed in this thesis.

The pivot of the iterative methods of fixed point theory is the Banach contraction mapping principle. It states that a self map T on a complete metric space (X, d) satisfying

$$d(Tx, Ty) \leq kd(x, y), \ \ 0 \leq k < 1, \ \ \forall x, y \in X, \tag{1.0.3}$$

neccesarily has a unique fixed point and for any starting point x_1, the sequence $\{T^n x_1\}$ converges strongly to that fixed point.

Many authors, see for example Alber [7], Boyd and Wong [25], have now investigated more general conditions under which a mapping will have a unique fixed point and also developed iterative sequences that converge to such fixed points.

If $k = 1$ in the inequality (1.0.3) above, the mapping T is tagged nonexpansive. There are many examples that show that $x_{n+1} = T^n(x)$ need not converge to a fixed point of a nonexpansive mapping T, even if it has a unique fixed point. We then need to impose additional conditions on T (and/or the space X) and also modify the sequence $T^n(x)$ to ensure convergence to a fixed point of T .

These notable iterative algorithms were introduced for nonexpansive mappings, namely, the Krasnosel'skii sequence presented in [69] as: $x_1 \in X$ and

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n),$$

the Krasnoselskii-Mann algorithm given by: $x_1 \in X$,

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \quad \lambda \in (0,1),$$

the Halpern algorithm given in [59] as: $u \in X$ arbitrary and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n,$$

and the more general Mann sequence presented in [72] as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n.$$

Diverse convergence theorems have been proved for these sequences, depending on the smoothness of the underlying space and/or the compactness of the mapping T.

Efforts to establish convergence theorems for nonexpansive mappings is likely the most rewarding research venture in nonlinear analysis. It has helped in the development of the geometry of Banach spaces and other related class of mappings, namely, monotone and accretive operators.

A mapping $M : X \to X^*$ is called $\eta-$strongly monotone if

$$\langle x-y, Mx - My \rangle \geq \eta \|x-y\|^2, \;\; \forall x, y \in X,$$

and $A : X \to X$ is called $\eta-$strongly accretive if

$$\langle Ax - Ay, j(x-y) \rangle \geq \eta \|x-y\|^2, \;\; \forall x, y \in X,$$

where $\langle ., . \rangle$ is the duality pairing between X and X^*, $j(x-y) \in J(x-y)$ where J is the normalized duality mapping. When $\eta = 0$, these mappings are called monotone and accretive, respectively. If X is Hilbert space, these two notions agree and they are simply refered to as monotone.

Accretive mappings have properties that are similar to those of monotone mappings. However, the use of the strongly nonlinear mapping J make the study of such mappings difficult. In a sense, the duality mapping on a Banach space has all the properties of the Banach space that makes it differ from a Hilbert space and the space can be characterized, almost, exclusive by the mapping.

These two ideas have proved to be very useful in many areas of interest. The idea of accretive operators appear very often in partial differential equation, in the existence theory of nonlinear evolution equations. On the other hand, the idea of monotone operators appear in optimization theory and that, in particular, include the increasingly important set-valued mapping called the subdifferential. Given a convex, lower semicontinous function f, the subdifferential is $\partial f : X \to 2^{X^*}$ given by

$$\partial f(x) := \{x^* \in X^* : f(y) - f(x) \geq \langle y - x, x^* \rangle, \ \ \forall y \in X\}.$$

The subdifferential is a monotone mapping and it is well known that $0 \in \partial f(\bar{x})$ if and only if $f(\bar{x}) = \inf_{x \in X} f(x)$. This motivates the study of the more general problem of finding a zero, i.e $\bar{x}$ such that $0 \in A\bar{x}$, of a monotone operator A.

The question on the existence of zeros is studied under the concept of maximal monotone operators. A monotne mapping A is maximal monotone if the graph $G(A)$ is a maximal element when graphs of monotone operators in $X \times X^*$ are partially ordered by set inclusion. In that case, for any $(x, y) \in X \times X^*$, the inequality

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \forall x_2 \in D(A), y_2 \in Ax_2$$

implies $y_1 \in Ax_2$. Maximal accretive mappings are defined accordingly.

The accretive operators are intimately connected with an important generalization of nonexpansive mappings called the pseudocontractive mappings. A mapping is pseudocontractive in the terminology of Browder and Petryshyn [23] if for x, y in X, and for all $r > 0$,

$$\|x - y\| \leq \|(x - y) + r[(x - Tx) - (y - Ty)]\|, \ \ .$$

By a result of Kato [62], this is equivalent to

$$\langle (I - T)x - (I - T)y, j(x - y) \rangle \geq 0.$$

Thus, a mapping T is pseudocontractive if and only if the complementary operator $A := I - T$ is accretive. Moreover, the zeros of A coincides with the fixed points of T.

Another interesting relationship is that the resolvent of an accretive mapping A always exists(i.e $I + \lambda A$ is invertible) and it is nonexpansive. The resolvent of A is a set valued mapping $J_\lambda : X \to 2^X$ defined by

$$J_\lambda(x) = (I + \lambda A)^{-1}x, \quad \lambda > 0.$$

In this case, $A^{-1}(0) = Fix(J_\lambda)$. More precisely, the mapping J_λ is in fact firmly nonexpansive, i.e

$$\|J_\lambda(x) - J_\lambda(y)\|^2 \le \langle x - y, J_\lambda(x) - J_\lambda(y)\rangle, \;\; \forall x, y \in X.$$

The existence and approximation algorithms for zeros of maximal monotone operators are usually formulated in relation with the corresponding problem for fixed points of firmly nonexpansive mappings. This makes the study of firmly nonexpansive, and the more general pseudocontractive mappings, an important tool for monotone operators and the theory of optimization.

The metric projection operator has become a veritable tool in dealing with variational inequalities problem by iterative-projection method in Hilbert spaces. Variational inequality problem $VIP(A, C)$ involving an accretive operator A and a convex set C can be proved to be equivalent to the fixed point problem involving the nonexpansive mapping

$$T = P_C(I - \lambda A)$$

for arbitrary positive number λ. Conversely, given a differentiable functional f, the $VIP(\nabla f, C)$ is simply the optimality condition for the minimization problem

$$\min_{x \in C} f(x).$$

Metric projection operators in Hilbert spaces are accretive and nonexpansive and gives absolutely best approximations of any element of the closed convex set. However, in the Banach space setting, this operator no longer possess most of those properties that made them so effective in Hilbert spaces.

To study monotone-type mappings and the related pseudocontractive mappings in Banach spaces, some analogues of the Hilbert space type projection operators were introduced. These mappings are natural extentions of the classical projection operators to Banach spaces. They have also helped in the approximation of monotone operator in Banach spaces.

In the last five years or so, intensive effort are invested in developing feasible iterative algorithm for approximating fixed points of multivalued pseudocontractive type mappings and/or, correspondingly, zeros of monotone mappings in Hilbert spaces and in the general Banach spaces. In each case, attempts are made to recover Hilbert space type identities for these mappings. Most of the study aim to derive a generalization of the multi-valued nonexpansive mapping introduced in the classical work of Nadler [80]. Such method depends heavily on the characterisation of the Hausdorf distance defined on closed and bounded sets. The generalizations of existing ideas, on the other hand, should be due to the generalization of some properties of the Hausdorff distance.

In this thesis, we first establish some new characterizations of the Hausdorf metric and use the ideas thereby to define some more general class of multivalued pseudocontractive mappings and prove convergent theorems for the class of mappings defined. Attempts would be made to apply some of the ideas obtained to real problems of interest. An example in this regard include applications to split equality fixed point problems, introduced by Moudafi and Al-Shemas[79] in (2013), which is formulated as finding a point x in a convex set C and y in a convex set Q such that their images Ax and By under some linear transformations A and B satisfy $Ax = By$. It serves as an inverse problem model in which constraints are imposed on the solutions in the domain of a linear mapping as well as in its range.

This thesis gives new insight and direction in the study of a general class of multivalued pseudocontractive mappings. It also studies a new method for finding a common solution of a monotone operator and family of a general class of nonexpansive mappings in some classical Banach spaces using the idea of generalized projections.

The rest of the thesis is organized as follows. Chapter 2 introduces some notions and recalls some basic definitions and ideas which are the bedrocks for the formulation of our theorems and for effective reading of the subsequent chapters.Detailed literature review involving multi-valued nonexpansive and pseudocontractive-type mappings are presented. In Chapter 3, convergence of a coupled iterative algorithm to a solution of some split equality problem is presented. Chapter 4, deals with some contributions to convergence theorems for a general class of multivalued striclty pseudocontractive mappings and Chapter 5 deals with the extension to finite and countable family. Chapter 6 is devoted to convergence theorems for a class of multivalued Lipschitz pseudocontractive mappings. We finally present in Chapter 7, an iterative algorithm for common element of zeros of a monotone mapping and fixed points of a general class of nonexpansive mappings in real Banach spaces.

CHAPTER 2

Theoretical Framework

In this chapter, we aim to highlight some definitions on which the problems are formulated and introduce some concepts and ideas used in the rest of the chapters. This will include an overview of the geometry of some Banach spaces and some well known iterative methods for single valued and multivalued pseudocontractive mappings.

2.1 Notions and Definitions

Unless otherwise specified, X represents a Banach space with norm $\|.\|$. The dual space X^* of X is the Banach space of all bounded linear functionals on X. It is endowed with the norm

$$\|x^*\|_{X^*} := \sup_{\|x\|=1} \langle x, x^* \rangle,$$

where $\langle ., . \rangle$ represent the pairing between the elements of X and X^*. Given any sequence $\{x_n\}$ in X, we take $x_n \to x^*$ to mean $\{x_n\}$ converges strongly to x^* and $x_n \rightharpoonup x^*$ to mean that $\{x_n\}$ converges weakly to x^*. The set of real numbers including $+\infty$ is represented by $\bar{\mathbb{R}}$

2.1.1 Some Well known Definitions

Definition 2.1.1 *A mapping $T : X \to X$ is called $L-$Lipschitzian if there exists $L > 0$ such that*

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y, \in X. \tag{2.1.1}$$

Remark 2.1.1 *If $L = 1$ in the inequality (2.1.1), the mapping is called non-expansive and if $L < 1$, it is called a strict contraction. It is well known that $F(T)$ is closed and convex whenever T is nonexpansive.*

Definition 2.1.2 *A mapping $T : X \to X$ is pseudocontractive in the terminology of Browder and Petryshyn [23] if*

$$\|x - y\| \leq \|(x - y) + r[(x - Tx) - (y - Ty)]\|, \ \ \forall x, y \in X, \ \ r > 0. \quad (2.1.2)$$

Remark: By the result of Kato [62], stated in Lemma (2.1.1) this is equivalent to

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq 0.$$

Thus, a mapping T is pseudocontractive if and only if the the complementary operator $A := I - T$ is accretive.

A well known proper subclass of the class of pseudocontractive mappings is the class of strictly pseudocontractive mapping.

Definition 2.1.3 *Given a real Hilbert space H and a closed convex subset K of H, let $T : K \to K$ be a mapping. Then T is said to be*

- *strictly pseudocontractive if there exists $k \in [0, 1)$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in K. \quad (2.1.3)$$

- *demi-contractive if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that*

$$\|Tx - Tp\|^2 \leq \|x - p\|^2 + k\|x - Tx\|, \forall (x, p) \in K \times F(T),$$

- *hemicontractive if $F(T) \neq \emptyset$ and*

$$\|Tx - Tp\|^2 \leq \|x - p\|^2 + \|x - Tx\|, \quad \forall (x, p) \in K \times F(T).$$

2.1.2 The Notion of Subdifferential

We present in this section a few basic definitions about differentiability of functions and in particular the subdifferential mapping.

Definition 2.1.4 *Given a mapping $f : X \to \bar{\mathbb{R}}$. We say that f is:*

- *proper if*

$$D(f) := \{x \in X : f(x) < \infty\} \neq \emptyset.$$

- *convex if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \quad \forall \lambda \in (0,1), \;\; x, y \in D(f).$$

- *lower semi-continuous (lsc) at* $x_0 \in D(f)$ *if*

$$f(x_0) \leq \liminf_{x \to x_0} f(x).$$

- $G-$*differentiable at* $x_0 \in D(f)$ *if there exists a bounded linear mapping* $f'(x_0) \in X^*$ *such that*

$$\langle h, f'(x_0) \rangle = \lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t}.$$

- $Fre'chet-$*differentiable at* $x_0 \in D(f)$ *if it is* $G-$*differentiable, with derivative* $f'(x_0)$, *and*

$$\lim_{t \to 0} \sup_{\|h\|=1} \left| \frac{f(x_0 + th) - f(x_0)}{t} - \langle h, f'(x_0) \rangle \right| = 0.$$

- *subdifferentiable at* $x_0 \in D(f)$ *if there exists a bounded linear mapping* $x^* \in X^*$, *called a subgradient element, such that*

$$f(x) - f(x_0) \geq \langle x - x_0, x^* \rangle, \quad , \forall x \in X.$$

Remark 2.1.2 *It is known, see for example Cioranescu [46], that every convex lower semicontinous function* f *is subdifferentiable in the interior of its domain. Moreover,* f *is* $G-$*differentiable if and only if the subdifferential* $\partial f(x)$ *contains only one element, namely* $f'(x) = \nabla f(x)$, *for each* $x \in D(f)$.

Definition 2.1.5 *The subdifferential of a functional* f *at* x_0 *is the set valued mapping* $\partial f : X \to 2^{X^*}$ *defined by*

$$\partial f(x_0) := \{x^* \in X^* : f(x) - f(x_0)) \geq \langle x - x_0, x^* \rangle\} \;\; \forall x \in X.$$

The subdifferential is an increasing important multivalued mapping due to its frequent use in the theory of optimization. Many functions of interest, for example, the absolute value function $f(x) = |x|$ on $\mathbb{R}$, are not differentiable. They may however be subdifferentiable. Therefore $0 \in \partial f(\bar{x})$ if and only if $f(x) \geq f(\bar{x})$ holds for all $x \in X$. Finding a minimizer of f therefore is equivalent to finding an $\bar{x} \in X$ with $0 \in \partial f(\bar{x})$. This technique has been applied successfully for example in game theory and market economy, in the existence theory for equilibria.

2.1.3 Duality Mappings and Characterization of Some Banach Spaces

We present some characterizations of spaces according to their duality mappings.

It is a common knowlegde that the domain of a functional f is, almost, never compact in the infinite dimensional spaces and therefore strong convergence is almost never guaranteed. To enforce a form of convergence of a minimizing sequence, one uses some other properties of the functional. In particular, one assumes that f is weakly lower semicontinuous, i.e "if $x_n \rightharpoonup u$, then $f(u) \leq \liminf f(x_n)$". It is known that every convex lower semicontinous function is weakly lower semicontinous.

If the mapping f is differentiable, then the convexity can be characterized exclusively by the derivative as follows

$$\langle u - v, f'(u) - f'(v)\rangle \geq 0 \;\; \forall u, v \in X, \tag{2.1.4}$$

where $\langle ., .\rangle$ is the pairing between the elements of the dual X^* and X. Any mapping $A : X \to X^*$ satisfying the type of inequality (2.1.4), i.e

$$\langle u - v, A(u) - A(v)\rangle \geq 0 \;\; \forall u, v \in X,$$

is called a *monotone* mapping. We have noted that if f is convex and lower semicontinuous but not neccesarily differentiable, we may still obtain the subdifferential of f. The multivalued mapping ∂f satisfies the inequality (2.1.4) in the sense that

$$\langle u^* - v^*, u - v\rangle \geq 0, \;\; \forall u, v \in X, u^* \in \partial f(u), v^* \in \partial f(v). \tag{2.1.5}$$

This sugests that the inequality (2.1.4) is applicable to a wide range of areas including multi-valued mappings.

Definition 2.1.6 *Given a Banach space X with the topological dual X^*. We recall that*

- *the modulus of convexity of X is a mapping $\delta_X : [0, 2] \to \mathbb{R}$ defined by*

$$\delta_X(t) = \inf\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| = \|y\| = 1, \;\; \|x - y\| = t\}, \tag{2.1.6}$$

- *and the modulus of smoothness is a mapping $\rho_X : (0, \infty) \to \mathbb{R}$ defined by*

$$\rho_X(t) = \sup\{\frac{1}{2}\Big(\|x + y\| + \|x - y\|\Big) - 1 : \|x\| = 1, \|y\| = t\}. \tag{2.1.7}$$

The space X is said to be uniformly convex whenever $\delta_X(t) > 0$ for each $t \in (0, 2]$ and uniformly smooth if $\lim\limits_{t \to 0^+} \frac{\rho_X(t)}{t} = 0$. Given real numbers $p, q > 1$, the space X is called $p-$uniformly convex(resp. $q-$uniformly smooth) if for some constant $c > 0$,

$$\delta_X(t) \geq ct^p \quad (resp.\ \ \rho_X(t) \leq ct^q).$$

Moreover, X is uniformly smooth if and only if X^* is uniformly convex and vice versa. Also, if $\frac{1}{p} + \frac{1}{q} = 1$, then X^* is $q-$uniformly smooth if X is $p-$uniformly convex and vice versa.

Common examples of $p-$uniformly convex spaces are the L_p spaces, $1 < p < \infty$. Given a measure space $(\Omega, \mathcal{A}, \mu)$, we define a real Lebesgue space($L_p(\Omega)$), $1 < p < \infty$ as

$$L_p(\Omega) := \{f,\ \ f : \Omega \to \bar{R},\ \ f\ \ \mathcal{A} - measurable, \int\limits_{\Omega} |f|^p d\mu < \infty\}.$$

In this case, $(L_p(\Omega), \|.\|_p)$, where

$$\|f\|_p = \Big(\int\limits_{\Omega} |f|^p d\mu\Big)^{\frac{1}{p}},$$

is a normed linear space.

In the special case that $\Omega = \mathbb{N}$ and μ is the counting measure δ, the space

$$L_p(\mathbb{N}) = \{f,\ \ f : \mathbb{N} \to \mathbb{R}, f(n) = x_n, \int\limits_{\mathbb{N}} |f|^p\ d\delta < \infty\},$$

corresponds to

$$l_p := \{(x_n)_n : \sum_{n=1}^{\infty} |x_n|^p < \infty\}.$$

Let q be the Holder conjugate exponent of p, i.e

$$q := \frac{p}{p-1}, 1 < p < \infty.$$

For $u \in L_q(\Omega)$, we may define the linear functional F_u on $(L_p(\Omega))^*$ by

$$F_u(f) := \int_{\Omega} f.u\ d\mu,\ \ f \in L_p.$$

The Holder inequality

$$|F_u(f)| = \Big|\int_\Omega f.u\ d\mu\Big| \le \|u\|_q \|f\|_p$$

gives $\|F_u\|_* \le \|u\|_q$. Then, the mapping

$$I_p : L_q(\Omega) \to (L_p(\Omega))^*, \quad u \mapsto F_u,$$

is a one to one bounded linear operator with $\|I_p\|_{B(L_q,(L_p)^*} \le 1$.

With the isometry above in mind, we will habitually identify the space $(L_p)^*$ with L_q in the sense that for any $\phi \in (L_p)^*$, there exist $u_\phi \in L_q$, such that $\langle f, \phi\rangle = \int_\Omega f.u_\phi\ d\mu,\ \forall\ f \in L_p$, and $\|\phi\|_* = \|u_\phi\|_q$.

It is well known, see for example Chidume [34], that

(a) X is $p-$uniformly convex if and only if X^* is $q-$uniformly smooth,

(b) X is $q-$uniformly smooth if and only if X^* is $p-$uniformly convex

Let X be a real $p-$uniformly convex and uniformly smooth Banach space with the dual X^* which is $q-$uniformly smooth and uniformly convex. We define the functional $f_p : X \to \mathbb{R}$ by

$$f_p(x) := \frac{1}{p}\|x\|^p, \ \ x \in X.$$

It is obvious that f_p is striclty convex and lower semicontinuous. Then the subdifferential of f_p, which is actually the Fre'tchet derivative, is denoted by J_p where

$$J_p(x) = \{j_p(x) \in X^* : \langle j_p(x), x\rangle = \|x\|^p = \|j_p(x)\|^p\}. \tag{2.1.8}$$

This mapping from X to X^* is in most cases nonlinear and called the generalized duality mapping of X with a guage function $\phi(t) = t^{p-1}$.

Topics on convex analysis and duality principles have expanded considerably and have become increasingly popular in recent times. Among the applications, we may mention the following:

(a) *Game theory, market economy, optimization, convex programming*; see Aubin, [10], [11], Aubin and Ekeland [12], Barbu and Precupanu [14],

(b) *Mechanics*; see Moreau [77], Lions [70], Damlamian [49](for a problem arising in plasma physics),

(c) *Theory of monotone operators and nonlinear semigroups*, see Brezis[18], Browder [20],

(d) *Variational problems involving periodic solutions of Hamiltonian systems and nonlinear vibrating strings*; see Clark and Ekeland [47], Ekeland [55],

(d) *Theory of large deviations in probability*; see Azencott *et al.* [13]

For $p = 2$, in the functional f_p above, the mapping $J_2 := J : X \to 2^{X^*}$ given by

$$J(x) = \{j(x) \in X^* : \langle j(x), x\rangle = \|x\|^2 = \|j(x)\|^2\}, \tag{2.1.9}$$

is called the normalized duality mapping. When it is understood that J is single valued, we may use $J(x)$ and $j(x)$ interchangeably. Some of its very useful properties are:

(a) For any $x \in X$, $J(x) \neq \emptyset$ (due to Hahn Banach theorem).

(b) For any real number α, $J(\alpha x) = \alpha J(x)$, for all $x \in X$.

(c) If X is a reflexive and smooth Banach space, then J is single-valued and onto.

(d) If X is strictly convex, then J is injective.

(e) If X is reflexive and strictly convex and X^* is strictly convex, then $J^* : X^* \rightrightarrows X^{**}(= X)$ is a duality mapping on X^* satisfying $J^{-1} = J^*$.

The normalized duality mapping is in most cases nonlinear and it is not symmetric unless X is a Hilbert space. Thus, in the conjectural formula

$$\langle x, J(y)\rangle = \langle y, J(x)\rangle \quad (?),$$

the left hand side is linear in x, but the right hand side is not, unless J is a linear map.
The restriction of our study to L_p spaces is due to the fact that the exact expression of the duality mapping is known only in L_p spaces.

Concrete examples: in ℓ^4, the duality map $J : \ell^4 \to \ell^{4/3}$ is

$$J(x) = (x_1^3, x_2^3, x_3^3, \dots)$$

Thus,

$$\langle x, J(y)\rangle = \sum_i x_i y_i^3$$

which of course need not be the same as

$$\langle y, J(x)\rangle = \sum_i x_i^3 y_i$$

In an L_p space, $1 < p < \infty$,

$$J(f) := \Big\{\phi_q \in (L_p)^* = Lq : \phi_q(f) = \int_\Omega g(t)f(t)d\mu\Big\},$$

where

$$g(t) := \frac{|f|^{p-1} sgn f(t)}{\|f\|_p^{p-2}}, f \in L_p(\Omega).$$

It is easliy seen that $J_p(x) = \|x\|^{p-2}J(x)$. The following additional properties holds for the generalized duality mapping:

(i) $J_p(\alpha x) = \alpha^{p-1}J(x),\ \ \forall\alpha > 0, x \in X.$

(ii) $\langle x, J_p(x)\rangle = \|x\|^p$ *and* $\|J_p(x)\| = \|x\|^{p-1}$

(iii) $\|x - y\|^p \leq \|x\|^p - p\langle y, J_p(x)\rangle + c_p\|y\|^p,\ \ \forall x, y \in X,\ \ \ c_p > 0.$

(iv) For $p > 1$, J_p is single valued if X^* is strictly convex.

(v) X^* is uniformly convex if and only if J_p is strictly convex and uniformly continuous on bounded subsets of X.

Given an arbitrary real normed linear space $X = (X, \|.\|)$, and a fixed $x \in X$. Consider the functional

$$I_x : X^* \to \mathbb{R}, \quad x^* \mapsto \langle x^*, I_x\rangle = \langle x, x^*\rangle.$$

Clearly, $I_x \in (X^*)^* = X^{**}$. Moreover,

$$\begin{aligned}\|I_x\|_{**} :&= \sup_{\|x^*\|_*=1} \langle x^*, I_x\rangle \\ &= \sup_{\|x^*\|_*=1} \langle x, x^*\rangle \\ &= \|x\|.\end{aligned}$$

Thus, the mapping $I : X \to X^{**}, x \mapsto I_x$ is a linear isometry, with $\|I_x\|_{**} = \|x\|$. So we have a canonical isometric embedding of X into X^{**}. In general, $I(X) \subseteq X^{**}$. The spaces X is called *reflexive* if equality holds.

The normalized duality mapping is strong enough to characterize the reflexivity of a spaces, as stated below

Theorem 2.1.1 (Cioranescu [46]) *Let X be a Banach space and J the normalized duality mapping. Then X is reflexive if and only if*

$$\bigcup_{x\in X} J(x) = X^*.$$

Other basic relationships between the geometric properties of the classes of Banach spaces and its generalized duality, as can be found for example, in Chidume [34], Cioranescu [46], is summarized as follows:

Proposition 2.1.1 *Let X be a Banach space. Then the following assertions hold:*

(a) The space X is smooth if and only if the generalized duality mapping J_p is single valued.

(b) The space X is uniformly smooth if and only if the generalized duality mapping J_p is single valued and norm to norm uniformly continous on bounded subsets of X.

(c) If X has a uniformly $G-$differentiable norm, then J_p is norm to weak* uniformly continuous on bounded subsets of X.

Definition 2.1.7 (Browder [21]) *A Banach space X is said to have a weakly continuous duality mapping if there exists a $p > 1$ such that J_p is single-valued and weak* sequentially continuous, that is,*

$$if \;\; x_n \rightharpoonup x, \quad then \;\; , J_p(x_n) \overset{*}{\rightharpoonup} J_p(x)$$

An example of a space with weakly continuous duality mapping is l_p, $1 < p < \infty$. For spaces having a weakly contiunuous duality mapping, the following holds.

Theorem 2.1.2 (Cioranescu [46], Riech [92]) *Suppose that X has a weakly continuous duality mapping J_p and that the sequence $\{x_n\}$ converges weakly to x. Then*

$$\limsup_{n\to\infty} \|x_n - z\|^p = \limsup_{n\to\infty} \|x_n - x\|^p + \|z - x_n\|^p$$

for all $z \in X$. In particular, X satisfies the Opial's condition; that is,

$$if \;\; x_n \rightharpoonup x, \;\; , then \limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - z\|$$

for all $z \in X, z \neq x$.

One more important property of the duality mapping in Banach spaces is beautifully captured in the theorem of Kato as follows

Lemma 2.1.1 (Kato [62]) *Let $x, y \in X$. Then $\|x\| \leq \|x + \alpha y\|$ for every $\alpha > 0$ if and only if there exists $j(x) \in J(x)$ such that $\langle y, j(x)\rangle \geq 0$.*

2.1.4 Metric Projections in Banach Spaces

Given a nonempty closed and convex subsubset K of a Hilbert space, the metric projection or the proximal mapping on K is a mapping $P_K : H \to K$ such that for each $x \in H$, the uniquely existing element $P_K x \in K$ satisfies

$$\|x - P_k x\| = \min_{y \in K} \|x - y\|.$$

A very important inequality that characterize the metric projection in Hilbert spaces is stated below.

Proposition 2.1.2 *For arbitrary x in H, $z = P_K x$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \forall y \in K.$$

From this proposition we derive that

(i) $\|P_K x - P_K y\|^2 \leq \langle x - y, P_K x - P_K y\rangle$ *for all $x, y \in H$; that is, the metric projection is firmly nonexpansive.*

(ii) $\|x - P_K x\|^2 + \|y - P_K x\|^2 \leq \|x - y\|^2$ *for all $x \in H$ and $y \in K$.*

(iii) If K is a closed subspace, then P_K coincides with the orthogonal projection from H onto K. Thus, for any $y \in K$, $\langle x - P_K x, y\rangle = 0$.

Remark: *The convexity of the set K is very crucial in the existence of the mapping P_K. This can be seen in the example where $K := \{e_1, e_2, ..., e_n, ...\} \subset l_2$, $e_n = (0, 0, ..., \frac{n+2}{n}, ...)$. Certainly, K is closed but not convex. It is easy to see that $P_K 0 = \emptyset$. In some case when the structure of the convex set K is simple, we can easily calculate the metric projection onto such a set.*

Example 2.1.3:*(a) Let $K = \bar{B}(u, r)$. Then*

$$P_K x = \begin{cases} u + r\frac{(x-u)}{\|x-u\|}, & if x \notin K, \\ x, & if x \in K. \end{cases}$$

(b) Given a nonzero mapping $f : H \to \mathbb{R}$ and $K := \{y \in H : f(y) = \alpha\}$ a hyperplane , then

$$P_K x = x - \frac{f(x) - \alpha}{\|f\|^2} f.$$

(c) Given a nonzero mapping $f : H \to \mathbb{R}$ and $K := \{y \in H : f(y) \leq \alpha\}$ a closed half space, then

$$P_K x = \begin{cases} x - \frac{f(x)-\alpha}{\|f\|^2} f, & if f(x) > \alpha \\ x, & if f(x) < \alpha. \end{cases}$$

(d) If K is the image set of an $m \times n$ matrix A with full column rank, then

$$P_K x = A(A^T A)^{-1} A^T x.$$

2.1.5 Generalised Projections in Banach Spaces

Definition 2.1.8 (Bregman [17]) *Let $f : X \to (-\infty, +\infty]$ be a G-differentiable function. The function $\Delta_f : D(f) \times intD(f) \to [0, +\infty)$ defined by*

$$\Delta_f(x, y) := f(y) - f(x) - \langle y - x, \nabla f(y)\rangle \tag{2.1.10}$$

is called the Bregman Distance with respect to f.

This idea was introduced by Bregman [17] while defining a nonexpansive-type mappings in Banach spaces, in the formulation process and analyses of feasibility and optimization problems. This type of mapping is now widely applied in solving variational inequalities problems. The idea has also been employed for convex minimization problems and other related problems in Banach spaces, see for example, Alber [6] and Schȯpfer *et al.*[99], Bregman [17], Reich [89]. Alghamdi *et al.* [8], Ugwunnadi *et al.* [103] . For $f = f_p$ defined above, the Bregman distance on X is

$$\Delta_p(x, y) := f_p(y) - f_p(x) - \langle y - x, J_p(x)\rangle. \tag{2.1.11}$$

With $\phi_p(x, y) := p\Delta_p(x, y)$, this is equivalent to

$$\phi_p(x, y) = \|y\|^p - p\langle y, J_p(x)\rangle + (p - 1)\|x\|^p, \tag{2.1.12}$$

and for $p = 2$ we obtain the Lyapunov functional introduced by Alber [3] and given by

$$\phi(x, y) = \|y\|^2 - 2\langle y, J(x)\rangle + \|x\|^2. \tag{2.1.13}$$

In Hilbert spaces, the mapping J is simply the identity mapping and $\phi(x, y) = \|x - y\|^2$. The following chain of inequalities also holds:

$$\Delta_p(x, y) \leq \langle x - y, J_p(x) - J_p(y)\rangle. \tag{2.1.14}$$

2.1.6 Inequalities in Banach spaces

Among all Banach spaces, the Hilbert spaces generally have the simplest and most clearly discernable geometric structure. Always available in a Hilbert space is the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \tag{2.1.15}$$

which is equivalent to the polarisation identity

$$\|x + y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2 \tag{2.1.16}$$

and then the inequality

$$\|(1-\lambda)x+\lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1.17}$$

Unfortunately, Hilbert spaces are luxulious and most real life problems do not reside in it. We therefore look for other spaces which will be nearest to Hilbert spaces and which will possibly possess analogues of identities (2.1.15), (2.1.16), and (2.1.17). Many of the analogues of these identities have now been found(see e.g Chidume [34], and Xu and Roach [109]). The most widely applicable inequalities in Banach spaces are summarized below as follows:

Theorem 2.1.3 *Let X be a real normed space, and $J_p : X \to 2^{X^*}$, $1 < p < \infty$, be the generalized duality map. Then, for any $x, y \in X$, the following inequality holds:*

$$\|x+y\|^p \le \|x\|^p + p\langle y, j_p(x+y)\rangle$$

for all $j_p(x+y) \in J(x+y)$. If $p = 2$, then

$$\|x+y\|^2 \le \|x\| + 2\langle y, j(x+y)\rangle$$

where j is the normalized duality mapping.

Henceforth, we define $W_p(\lambda) := \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$. where $\lambda \in [0,1]$ and $1 < p < \infty$. Then, then we obtain the following:

Theorem 2.1.4 *Let X be a $p-$uniformly convex space. Then there exist constants $c_p > 0$, $d_p > 0$ such that for every $x, y \in X$, and $j_p(x) \in J_p(x)$, the following inequalities hold:*

$$\|x+y\|^p \ge \|x\|^p + p\langle y, j_p(x)\rangle + d_p\|y\|^p, \tag{2.1.18}$$
$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - c_p W_p(\lambda)\|x-y\|^p, \tag{2.1.19}$$

for all $\lambda \in [0,1]$.

Theorem 2.1.5 *Let X be a real $q-$uniformly smooth Banach space. Then, there exist constants $c_q > 0, d_q > 0$ such that for each $x, y \in X$,*

$$\|x+y\|^q \le \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q, \tag{2.1.20}$$
$$\|\lambda x + (1-\lambda)y\|^q \ge \lambda\|x\|^q + (1-\lambda)\|y\|^q - c_q W_q(\lambda)\|x-y\|^q. \tag{2.1.21}$$

2.1.7 Recurrent inequalities

In drawing inferences about the convergence or otherwise of a given iterative sequence, one or more of these recurrence relations are often used.

Lemma 2.1.2 *[Xu [102]] Let $\{\rho_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$\rho_{n+1} \leq \rho_n + \lambda_n, \;\; n \geq 0,$$

such that $\sum_{n=1}^{\infty} \lambda_n < \infty$. Then, $\lim \rho_n$ exists. If, in addition, $\{\rho_n\}$ has a subsequence that converges to 0, then ρ_n converges to 0 as $n \to \infty$.

Lemma 2.1.3 (Xu and Kim [108]) *Assume that $\{\rho_n\}$ is a sequence of nonnegative real numbers satisfying the conditions*

$$\rho_{n+1} \leq (1 - \alpha_n)\rho_n + \alpha_n \beta_n, \forall n \geq 1 \tag{2.1.22}$$

(i) $\{\alpha_n\} \subseteq [0, 1]$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$. Then, $\lim_{n\to\infty} \rho_n = 0$.

Lemma 2.1.4 (Xu [106]) *Let $\{\rho_n\} \subset \mathbb{R}^+$ be a sequence such that*

$$\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \lambda_n b_n,$$

where $\lambda_n \in (0, 1)$ satisfies (i) $\lim_{n\to 0} \lambda_n = 0$, and (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$. If $\limsup_{n\to\infty} b_n \leq 0$, the $\lim_{n\to\infty} \rho_n = 0$.

Lemma 2.1.5 (Maing'e [71]) *Let $\{\rho_n\}$ be a sequence satisfying*

$$\rho_{n+1} \leq (1 - \lambda_n)\rho_n + b_n + c_n, \forall n \geq 0,$$

where $\{c_n\} \subset \mathbb{R}^+, \{\lambda_n\} \subseteq (0, 1)$ and $\{b_n\} \subseteq \mathbb{R}$. Assume that $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:
(i) If $b_n \leq \lambda_n C$ for some $C \geq 0$, then $\{\rho_n\}$ is bounded. (ii) If $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\lambda_n} \leq 0$, then $\lim \rho_n = 0$.

2.2 Iterative Algorithm for Single Valued Mappings

This section is concerned with established iterative methods for fixed points of nonexpansive-type mappings.

2.2.1 The Contraction Mapping Principle

Iterative methods for fixed points of maps were developed after an elegant application of the method by Picard in the approximation of the fixed point of a strict contraction. It states that in a complete metric space (X, d), a single valued mapping $T : X \to X$ which satisfies

$$d(Tx, Ty) \leq kd(x, y), \quad \forall \ x, y \in X, 0 \leq k < 1, \tag{2.2.1}$$

has a unique fixed point x^*. Moreover, such a fixed point is given by

$$x^* = \lim_{n\to\infty} T^n x_1.$$

This easily follows from the fact that

$$\begin{aligned} d(x, y) &\leq d(x, Tx) + d(Tx, Ty) + d(Ty, y) \\ &\leq d(x, Tx) + kd(x, y) + d(Ty, y), \ \forall x, y \in X, \end{aligned}$$

and as a consequence

$$d(x, y) \leq \frac{1}{1-k}[d(x, Tx) + d(Ty, y)]$$

This is the Fundamental Contraction Inequality [87]. It shows that a contraction mapping can have at most one fixed point. It also gives that

$$\begin{aligned} d(T^n x, T^m x) &\leq \frac{1}{1-k}[d(T^n x, T^n(Tx)) + d(T^m(Tx), T^m x)] \\ &\leq \frac{k^n + k^m}{1-k} d(x, Tx) \end{aligned}$$

This shows that $\{T^n x\}$ is Cauchy and thus converges to some element $x^* \in X$.

Apart from the existence of solution, suppose one is willing to accept an error of ϵ from the actual fixed point x^* of T when your initial guess is x, the inequality easily gives us an integer N such that $x^{*'} = T^N x$ will be a satisfactory answer. Since we seek $d(T^n x, x^*) \leq \epsilon$, we may pick N large enough so that

$$\frac{k^N}{1-k} d(x, Tx) < \epsilon.$$

If we take log on both sides of the inequality above, we obtain(see eg. [78])

Stopping Rule

$$If \quad N > \frac{log(\epsilon) + log(1-k) - logd(x, Tx)}{logk},$$

then $d(T^n x, x^{*'}) < \epsilon$. From the practical programming point of view, for example, this gives a criteria for terminating the algorithm while still ensuring the quality of the approximation of the solution.

2.2.2 Nonexpansive Mappings

It is obvious that a great deal of real life problems of interest do not belong to the class of contractions.The fixed point theory for non-expansive mappings includes the theory for contractive mappings and contain the isometries, in particular, the identity mapping.

The theory of non-expansive mappings is different from that of contraction mappings. For example, consider the mapping that rotates the unit ball in $\mathbb{R}^2$ to the right by 180°. This map is $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $Tx = -x$ is nonexpansive but not a strict contraction. Obviously, T has a unique fixed point, namely $x = 0$, and $d(Tx, Ty) = d(x, y)$. However, T is not a contraction and for any $x \neq 0$, $x_n = T^n x = (-1)^n x$ does not converge. Thus the Picard sequence does not converge to the fixed point of T and to obtain fixed point theory for nonexpansive mapppings therefore, we need to modify the Picard sequence.

The study of the class of nonexpansive mappings is the most productive area of research in nonlinear analysis. It led to the development of the geometry of Banach spaces and has also helped to develop the related theory of monotone and accretive operators.
The Krasnoselski's sequence defined by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n$$

has been established to be an effective iterative scheme for approximating the fixed point of nonexpansive self mappings in real normed spaces. The more general iterative sequence, namely, the Mann iterative scheme given by $x_1 \in X$ and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,$$

where $\alpha_n \in (0,1)$ satisfies $\sum \alpha_n = \infty$ and $\sum \alpha_n^2 < \infty$, has also proved to be successful for approximating fixed points of nonexpansive mappings and infact the slightly more general quasi-nonexpansive mappings i.e mappings T satisfying $F(T) \neq \emptyset$ and $\|Tx - Tp\| \leq \|x - p\|, \;\; \forall x \in D(T)$ and $p \in F(T)$.

2.2.3 Pseudocontractive Mappings

For the more general class of strictly pseudocontractive mappings, the Mann sequence has been developed to succesfully approximate the fixed point. Pseudocontractive mappings are generalizations of nonexpansive mappings and have been studied extensively, for example, by Browder and Petryshn [23], Browder [24], Dhompongsa *et al.* [51], Kirk [52], Martinet [74], Xu [102] and a host of other authors.

For several years, it was a problem of interest to know whether the Mann iteration process would, *always*, converge strongly to a fixed point of an arbitrary pseudocontractive mapping.

In 1974, in the setting where T is Lipschitzian and pseudocontractive with compact domain, Ishikwa [60] introduced a new iteration process,(now known as the Ishikawa process), namely,

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n, \tag{2.2.2}$$

for suitable α_n and β_n, and proved a strong convergence theorems to a fixed point of the map T.

Certainly, the Mann iteration process is less computationally involved than the Ishikawa process. Moreover, the order of convergence of the Mann process is $\frac{1}{n}$ whereas that of Ishikawa is $\frac{1}{\sqrt{n}}$. Thus, if the Mann process converges, then it is more desirable than the Ishikawa process.

Already, strong convergence of the Ishikawa and Mann iteration processes to a fixed point of T have been established(see, for example, Browder [24]), even in normed linear spaces, in the case where T belongs to that proper subclass of Lipschitz pseudocontractive mapping -the strictly pseudocontractive mappings.

In 2001, Chidume and Mutangadura [40] gave an example to show that, for a Lipschitzian pseudocontractive mapping T defined on a real Hilbert space, a Mann iteration process may fail to converge to a fixed point of T, even when the set K is compact and the fixed point of T is unique. Thus the problem was resolved in the negative. However, it is still a problem of interest on one hand, to get a scheme more easily applicable than the Ishikawa, and on the other hand to see if the Ishikawa process will work also for some Banach spaces.

2.3 Some Important Results on Iterative Methods for Multivalued Mappings

Fixed point theory for multi-valued mappings continues to attract a lot of attention because of its numerous real world applications in game theory and market economy, differential inclusions, and constrained optimization. Iterative methods for approximating fixed points of nonexpansive-type mappings constitute the central tools used in signal processing and image reconstruction (see, e.g., Byrne[27]). They are also used in devising critical points in optimal control problems and energy management problems. The applications of fixed point theory for multi-valued mappings on the problem of differential equations (DEs) with discontinuous right-hand sides gave birth to the existence theory of differential inclusions (DIs).

Game theory and market economy is, perhaps, the most socially recognized application of multi-valued mappings.

Consider, for example, a game $G(x_n, K_n)$ involving N players, namely $n = 1, 2, ..., N$. Here, K_n, a nonempty compact and convex subset of $\mathbb{R}^{m_n}$, is the collection of possible strategies of the n^{th} player. The continuous function $x_n : \Pi_{i=1}^{N} K_n \to \mathbb{R}$, is the gain(payoff) function. Any vector y_n in K_n is the action which is available to the individual n to take. The collective action of all the N players is then $y := (y_1, y_2, ..., y_N) \in K := \Pi_{i=1}^{N} K_n$. Given any n, y and $y_n \in K_n$, we use these standard notations:

$$\begin{aligned} K_{-n} &:= K_1 \times K_2 \times ... \times K_{n-1} \times K_{n+1} \times ... \times K_n \\ y_{-n} &:= (y_1, ..., y_{n-1}, y_{n+1}, ..., y_N) \\ (y_n, y_{-n}) &= (y_1, y_2, ..., y_{n-1}, y_n, y_{n+1}, ..., y_N). \end{aligned}$$

In this regard, the n'^{th} player maximizes his own gain, using a strategy y_n^*, subject to the fact that the other players have choosen their strategies y_{-n} if and only if

$$x_n(y_n^*, y_{-n}) = \max_{y_n \in K_n} x_n(y_n, y_{-n}).$$

Define a multi-valued mapping $T_n : K_{-n} \to 2^{K_n}$ by

$$T_n(y_{-n}) = Arg \max_{y_n \in K_n} x_n(y_n, y_{-n})$$

Then, the collective action $y^* = (y_1^*, y_2^*, ..., y_N^*)$ is called a *Nash equilibrium* point if each y_n^* is the most effective response that the n'^{th} player can make to the actions y_{-n}^* of the other $N - 1$ players. This is stated differently as

$$x_n(y_n^*) = \max_{y_n \in K_n} x_n(y_n, y_{-n}^*),$$

or, in other words,

$$y_n^* \in T_n(y_{-n}^*).$$

Therefore, $y^* = (y_1^*, y_2^*, ..., y_N^*)$ is a fixed point of the multi-valued mapping $T : K \to 2^K$ given by

$$T(y) = [T_1(y_{-1}), T_2(y_{-2}), ..., T_N(y_{-N})]$$

For early results involving fixed points of multi-valued mapping, (see, for example, Brouwer [19], Kakutani [61], Nash [82, 83], Geanakoplos [58], Downing and Kirk [52]). For details on the applications of this type of mappings in Nonsmooth Differential Equations, one may consult, for example, Chang [32], Chidume [35], Deimling [50], Erbe and Krawcewicz [53], Nadler [80], Ofoedu and Zegeye [85], Reich *et al.* [91, 94, 95] and the references therein.

Though many theory for multi-valued mappings in the literature have dealt with the existence of fixed points for such mappings, only very few have dealt with iterative algorithms for computing them.

2.3.1 Iterative Methods for Multivalued Nonexpansive-type Mappings

Given a metric space (X, d), we denote by $CB(X)$ the family of nonempty, closed and bounded subsets of X and $K(X)$ the family of all compact subsets of X. Then, the Hausdorff distance defined by

$$D(A, B) := \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

is a metric on this family $CB(X)$.

Remark 2.3.1 *It is understood that the Hausdorff metric D on $CB(X)$ depends on the pressigned metric d of X and that a distinct metric will yeild a distinct Hausdorff metric.*

Remark 2.3.2 *Elsewhere, the Hausdorff metric is defined as*

$$D(A, B) := \inf\{\epsilon > 0 : B \subseteq A + B(0, \epsilon), A \subseteq B + B(0, \epsilon)\}.$$

Definition 2.3.1 *A mapping $T : X \to CB(X)$ is called a multivalued Lipschitz mapping on X if*

$$D(Tx, Ty) \leq Ld(x, y), \quad \forall x, y \in X, \quad L \geq 0.$$

If $L = 1$ the mapping is called nonexpansive and for $L < 1$, it is called a multivalued contraction mapping.

The first work on iterative methods for fixed points of multi-valued (nonexpansive) mappings by the application of Hausdorff metric was due to Markin [73], and followed by an extensive work by Nadler [80]. Since then, several authors have dealt with the problem of approximating fixed points of multivalued nonexpansive mappings in real Hilbert spaces using the Hausdorff metric(see, e.g., Abbas *et al.* [1],Khan *et al.* [63, 64], Panyanak [88], Sastry *et al.* [98, 100, 101], Zegeye and Shahzad [112] and the references therein) and for their generalizations (see e.g., Chidume *et al.* [35], Chidume and Ezeora [36] and the references therein).

In [80], the author combined an idea of multi-valued mappings and Lipschitz mappings and proved an analogue of the Banach contraction mapping principle for multi-valued contraction mappings. He proved that given a complete metric space X, and a contraction mapping $T : X \to K(X)$ with constant $\kappa(K(X)$ is the set of all compact subsets of X), then the set-valued mapping $T^* : K(X) \to K(X)$ definded by $T^*A := \cup\{Tz : z \in A\}$ satisfies

$$D(T^*A, T^*B) \leq \kappa D(A, B)$$

Thus, since $(K(X), D)$ is a complete metric space, the mapping T^* has a fixed point due to the contraction mapping principle. However, this does not automatically yield a fixed point of the mapping T.

Nevertheless, he proved that if $T : X \to CB(X)$ is a multi-valued contraction mapping, then T has a fixed point. For this, he used the fact that for a contraction with constant $\kappa < 1$, there holds $\sum\limits_{n=1}^{\infty} \kappa^n < \infty$. Given sets A and B in $CB(X)$ and $a \in A$, though there may not be a points b such that $d(a, b) \leq D(A, B)$, we nevertheless have by the definition of the Hausdorff metric, that there exists $b \in B$ satisfying $d(a, b) \leq D(A, B) + \kappa$. This property is now refered to as the Nadler's condition.

Thus, an iterative procedure for the fixed point of T is as follows:
Given any $x_0 \in X$, choose $x_1 \in Tx_0$ and $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq D(Tx_0, Tx_1) + \kappa.$$

Repeat this process such that at the n^{th} stage, x_{n+1} is choosen with

$$d(x_n, x_{n+1}) \leq D(Tx_{n-1}, Tx_n) + \kappa^n.$$

Then, there holds that for arbitrary $k \geq 1$,

$$d(x_n, x_{n+k}) \leq \sum_{j=n}^{n+k-1} \kappa^j d(x_0, x_1) + \sum_{j=n}^{n+k-1} j.\kappa^j.$$

Remark: The choice of the contraction constant κ does not play any essential role in the procedure above. Infact, we could have choosen an arbitrary positive sequence $\{\lambda_n\}$ satisfying $\sum_{n=1}^{\infty} n\lambda_n < \infty$. In Chapters 4, 5, , 6 we will demonstrate this idea using a reverse Nadler's condition and prove convergence thorems for some general classes of multivalued mappings.

Many well known researchers have proved other complementary convergence theorems for the more general class of the multi-valued nonexpansive mappings, in some classical Banach spaces, in recent times. Example includes Browder [22], Chidume *et al.*[35], Chidume and Okpala [39], Halpern[59], Ofoedu and Zegeye [85], Panyanak [88], Reich[90], Reich and Zaslavski [91], Sastry and Babu [98], and Xu [107] .

2.3.2 Iterative Methods for Multivalued Strictly Pseudocontractive Mappings

The study of fixed points for pseudocontractive mappings developed out of a need to generalize certain ideas which are applicable to nonexpansive mappings. This class of mappings have been studied extensively, for example, by Browder and Petryshn [23], Browder [24], Daffer and Kaneko [48], Deimling [50], Downing and Kirk [52], Xu [102] and a host of other authors.

Fixed point problems involving a multi-valued mapping T can be reformulated as a zero problem for a multi-valued mapping A, namely;

$$\text{Find } 0 \in Ax, \qquad \text{where } A = I - T$$

and I is the identity mapping of K.

Many problems in applications can be modeled in the form of $0 \in Ax$, where, for example. $A : H \to 2^H$ is a monotone operator, that is $\langle u-v, x-y\rangle \geq 0$ for all $u \in Ax, v \in Ay, x, y \in H$. Typical examples include the equilibrum state of evolution equations and critical points of some functionals defined on Hilbert spaces. For example, let $f : H \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. It is known (see e.g., Rockafellar [96], Minty [76]) that the multi-valued mapping $T := \partial f$, *the subdifferential* of f, is maximal monotone, where for each $w \in H$,

$$\begin{aligned} w \in \partial f(x) &\Leftrightarrow f(y) - f(x) \geq \langle y - x, w\rangle, \quad \forall y \in H, \\ &\Leftrightarrow x \in Argmin(f - \langle ., w\rangle). \end{aligned}$$

In this case, a solution of the inclusion problem $0 \in \partial f(x)$, if any, is a critical point of f, which is precisely a minimizer of f.

The *proximal point algorithm* introduced by Martinet [74], and studied extensively by Rockafeller [96] Aoyoma *et al.* [9], which has also been studied by a host of other authors, is connected with the iterative algorithm for solutions of $0 \in Ax$ where A is a maximal monotone operator on a Hilbert space.

In studying the equation $Au = 0$, Browder [24], defined an operator $T := I - A$, where I is the identity mapping on H. He called such an operator a *pseudocontractive mapping*. It is that the solutions of $Au = 0$ when A is monotone are precisely the fixed points of pseudocontractive mapping T. Every nonexpansive mapping is pseudocontractive and continuous but a pseudocontractive mapping is not neccesarily continuous. Thus the study of iterative methods for fixed points of pseudontractive mappings is established with additional assumptions of continuity of the mappings (e.g., Lipschitz condition), in general.

While pseudocontractive mappings are generally not continuous, a subclass of pseudocontractive mappings, the *strictly pseudocontractive mappings*, inherits Lipschitz property from their definitions. The study of fixed point theory for strictly pseudocontractive mappings helps in the study of fixed point theory for nonexpansive mappings and for Lipschitz pseudocontractive mappings. Consequently, the study by several authors of iterative methods for fixed points of multi-valued strictly pseudocontractive mappings has motivated our study of a more *general class* of multi-valued strictly pseudocontractive mappings which certainly includes the important class of multi-valued nonexpansive maps.

In this section, we study the notion of multi-valued stricly pseudocontractive mappings (see e.g., [35], [85]), which is a generalization of single-valued strictly pseudocontractive mappings, defined by Browder and Petryshyn [23] on Hilbert spaces.

Definition 2.3.2 *A single-valued mapping $T : K \subseteq H \to H$ is called*

- *pseudocontractive if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in K. \tag{2.3.1}$$

- *monotone if*

$$\langle Tx - Ty, x - y\rangle \geq 0, \;\; \forall x, y \in D(T).$$

Definition 2.3.3 *A map $T : K \to CB(K)$ is said to be hemicompact if, for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence, say, $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.*

Note that if K is compact, then every multi-valued mapping $T : K \to CB(K)$ is hemicompact.

The theory of multi-valued nonexpansive mappings(and, in particular, psuedo-contractive mappings) is much harder than the corresponding theory of single valued nonexpansive mappings(see e.g. Khan and Yildirim[63]). The extension of the notion of single valued pseudocontractive mappings to multi-valued pseudocontractive mappings has some of these challenges:

- Definition of the mapping: There is a problem of getting a right definition for the multi-valued analogue which would be a generalization of the single-valued case. There are several definitions available which will be a generalisation of the single valued case and one has to get the most natural among them to be able to establish some convergence theorems.
- Identities: In multi-valued settings, the metric induced by the norm on X is not applicable and there is the need to develop new identities and other notions of distances which will be applicable. One notion of metric for sets that is readily applicable here is the Hausdorf metric.
- Inference: Many thoerems and lemmas that are developed for single valued mappings cannot be carried over to multi-valued cases and it is always difficult to make conclusions.

In [98], Sastry and Babu proved the following result for multi-valued nonexpansive mappings in Hilbert spaces :

Theorem 2.3.1 *(Sastry and Babu [98]) Let H be real Hilbert space, K be a nonempty, compact and convex subset of H, and $T : K \longrightarrow CB(K)$ be a multi-valued nonexpansive map with a fixed point p. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n\beta_n = \infty$. where α_n and β_n are sequences of real numbers. Then, the sequence defined by*

$$\begin{cases} y_n & = (1-\beta_n)x_n + \beta_n z_n, \;\; z_n \in Tx_n, \;\; \|z_n - x^*\| = d(x^*, Tx_n), \\ \\ x_{n+1} & = (1-\alpha_n)x_n + \alpha_n u_n, \;\; u_n \in Ty_n, \;\; \|u_n - x^*\| = d(y_n, x^*). \end{cases} \tag{2.3.2}$$

converges strongly to a fixed point of T.

In [88], Panyanak extended the result of Sastry and Babu [98] to uniformly convex spaces. He proved the following theorem.

Theorem 2.3.2 *(Panyanak [88]). Let E be a uniformly convex real Banach space, and let K be a nonempty, compact, and convex subset of E and $T : K \to CB(K)$ a multivalued nonexpansive mapping with a fixed point p. Assume that $0 \le \alpha_n, \beta_n < 1$ (ii) $\beta_n \to 0$ (iii) $\sum_{n\to\infty} \alpha_n\beta_n = \infty$. Then, the sequence $\{x_n\}$ given by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n z_n, \quad z_n \in Tx_n, \\ \|z_n - p\| &= d(p, Tx_n), \\ x_{n+1} &= (1-\alpha)x_n + \alpha_n u_n, \quad u_n \in Ty_n, \\ \|u_n - p\| &= d(Ty_n, p). \end{aligned}$$

converges strongly to a fixed point of T .

In [100], Song and Wang pointed out a gap in the theorem of Panyanak as follows:

(i) The sequence $\{x_n\}$ depends obviously on the fixed point p which is unknown and thus some conclusions cannot be reached.

(ii) The sequence is Ishikwa type which has a low rate of convergence and it is therefore difficult to implement.

(iii) The condition $\|z_n - p\| = d(p, Tx_n)$ implies that Tx is proximinal for each $x \in D(T)$ and it therefore further reduces the class of mappings to which the theorem is applicable.

They modified the Ishikawa-type sequence of Panyanak [88] and used Nadler's condition to obtain an Ishikawa-type iterative sequence which is guaranteed to converge to a fixed point of T. However, they still assumed that the multivalued mapping T satisfies the so-called *Condition I*. Moreover, they remarked that their result holds for Mann iteration if $\beta_n \equiv 0$, which leaves one wondering why they used an Ishikawa type sequence in the first place. On the otherhand, it deals only with nonexpansive mappings which is a subclass of strictly pseudocontractive mappings.

In [35], Chidume *et al.*, gave a multivalued analogue of Definition (2.3.2) as follows:

Definition 2.3.4 *Let K be a closed, convex, and nonempty subset of H. A mapping $T : K \to CB(K)$ is called a multivalued $k-$strictly pseudocontractive mapping if for some $k \in (0,1)$ and for all $x, y \in K$, there holds*

$$D^2(Tx, Ty) \le \|x-y\|^2 + k\|(x-u) - (y-v)\|^2, \tag{2.3.3}$$

for all $u \in Tx$, $v \in Ty$.

They used a certain *Krasnoselskii's-type* sequence and proved the following theorem:

Theorem 2.3.3 *(Chidume et al. [35]) Let K be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T : K \to CB(K)$ is a multi-valued k-strictly pseudocontractive mapping such that $F(T) \neq \emptyset$. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is hemicompact and continuous. Let $\{x_n\}$ be a sequence defined iteratively from $x_0 \in K$ by*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \tag{2.3.4}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

The result of Chidume *et al.* is certainly better than most of the results in the literature because it deals with strictly pseudocontractive mappings which is more general than nonexpansive mappings and also the problem of finding $z_n \in Tx_n$ such that $\|z_n - x^*\| = d(x^*, Tx_n)$ as in Sastry and Babu does not arise. However, as remarked in [38] , the inequaulity (2.3.3) is equivalent to

$$D^2(Tx, Ty) \leq \|x-y\|^2 + k \inf_{(u,v)\in(Tx,Ty)} \|(x-u)-(y-v)\|^2. \tag{2.3.5}$$

which is very restrictive and therefore not far ahead of the single-valued case given by ineqaulity (2.3.2).

Nevertheless, Chidume *et al.* [35], extended the result to q-uniformly smooth real Banach spaces and obtained the following result:

Theorem 2.3.4 *(Chidume et al. [35]) Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Let $T : K \to CB(K)$ be a multi-valued k-strictly pseudocontractive mapping with $F(T) \neq \emptyset$ and such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is continuous and hemicompact. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \tag{2.3.6}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

We now pose the following questions of interest:
Question 1: Can an interative scheme be used for multivalued pseudocontractive mappings which is easier than the so-called Ishikawa process?
Question 2: Can convergence theorems be proved for a class of mappings more general than that proved by Chidume *et al?*

In the next chapters, we will provide affirmative and partial answers to these questions as we show the results we have obtained so far.

CHAPTER 3

Contributions on Iterative Algorithms for Some Single-valued Pseudocontractive-type Mappings

Most important iteration procedures for single valued mappings currently in the literature [16], can be summarised as follows:

(1) $x_{n+1} = Tx_n, \;\; n \geq 0$ 1890 Picard

$\Uparrow \lambda = 1$

(2) $x_{n+1} = \dfrac{1}{2}(x_n + Tx_n), n \geq 0 \geq 0$ 1955 Krasnoselski

$\Uparrow \lambda = \dfrac{1}{2}$

(3) $x_{n+1} = (1-\lambda)x_n + \lambda Tx_n,\; n \geq 0,\; 0 \leq \lambda \leq 1,$ 1957 (Krasnoselski-)Shaeffer

$\Uparrow a_n = \lambda(const.)$

(4) $x_{n+1} = (1-a_n)x_n + a_nTx_n, n \geq 0, a_n \in [0,1],$

$\lim\limits_{n\to\infty} a_n = 0, \sum a_n = \infty$ 1953 Mann

$\Uparrow b_n = 0$

(5) $x_{n+1} = (1-a_n)x_n + a_nT[(1-b_n)x_n + b_nTx_n],\; n \geq 0, 0 \leq a_n \leq b_n \leq \;\; 1,$

$\lim\limits_{n\to} b_n = 0, \;\; \sum\limits_{n=0}^{\infty} a_nb_n = \infty$ 1974 Ishikawa

There is a need for an iterative procedure that fills the gap between (4) and (5) above in the sense that here $a_n = b_n = \lambda$ simply for some $\lambda \in (0,1)$. In this chapter, we state a theorem in this regard and demonstrate how such algorithm is applicable in split equality fixed point problems..

3.1 On the Split Equality Fixed Point Problem

The split equality problem was introduced by Moudafi and Al-Shemas[79] in (2013) as a generalization of the split feasibilty problem which appear as inverse problems in phase retrivial, medical image recontruction, intensity modulated radiation therapy(IMRT) and so on (see e.g., Byrne [26], Censor *et al*.[29], Censor *et al*. [28], and Censor and Elfving [30]). It serves as a model for inverse problems in the case where constraints are imposed on the solutions in the domain of a linear transformation and also in its range.

The split equality problem of Moudafi is stated as follows:

$$\text{Find } \ x \in C = F(S) \ \text{ and } \ y \in Q = F(T) \text{ such that } \ Ax = By, \tag{3.1.1}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, H_1, H_2, and H_3 are real Hilbert spaces, while $S : H_1 \to H_1$ and $T : H_2 \to H_2$ firmly quasi-nonexpansive mappings, respectively.

They studied the convergence of a weakly coupled iterative algorithm given by

$$(SEP) \begin{cases} x_{n+1} = S(x_n - \gamma_n A^*(Ax_n - By_n)); \\ y_{n+1} = T(y_n + \gamma_n B^*(Ax_n - By_n)); n \geq 1 \end{cases} \tag{3.1.2}$$

where A^* and B^* are the adjoints of A and B, respectively, while λ is the sum of the spectral radii of A^*A and $\gamma_n \in (0, \frac{2}{\lambda})$.

The iterative algorithm of Moudafi was for firmly quasi-nonexpansive mapping which has very attractive properties that makes the use of this simple iterative algorithm introduced suitable.
The algorithm of Moudafi and Al-shamas has great merits because it is implementable without the use of projections and yet it is a generalization of the split feasibility problem if we set $H_3 = H_2$ and $B = I$. The algorithm was extended by Yuan-Fang *et al*. [56] who introduced the following algorithm for solving problem (3.1.2):

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S(x_n - \gamma_n A^*(Ax_n - By_n)); \\ \\ y_{n+1} = (1 - \alpha_n)y_n + \alpha_n T(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1, \end{cases} \tag{3.1.3}$$

where $S : H_1 \to H_1$, $T : H_2 \to H_2$ are still two *firmly quasi-nonexpansive mappings*, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are bounded linear operators, A^* and B^* are the adjoints of A and B, respectively, $\gamma_n \in (0, \frac{2}{\lambda})$, where λ is the sum of the spectral radii of A^*A and B^*B, respectively, and $\{\alpha_n\} \subset [\alpha, 1]$ (for some $\alpha > 0$). Under suitable conditions, the authors obtained strong and weak convergence results, respectively.

It was therefore natural to investigate if the split equality problem can be extended to a more general class of mappings apart from the class of firmly quasi-nonexpansive mappings studied by Moudafi and Al-Shamas [79], and Yuan-Fang *et al.* [56].

Motiviated by the work of Moudafi and Al-Shamas, Chidume *et al.* [41] studied convergence theorems for split equality problem involving two *demi-contractive* mappings. They introduced the following Krasnoselskii-type iterative algorithm

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = (1-\alpha)\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) + \alpha U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big); \\ y_{n+1} = (1-\alpha)\Big(y_n + \gamma B^*(Ax_n - By_n)\Big) + \alpha T\Big(y_n + \gamma B^*(Ax_n - By_n)\Big), \quad \forall n \geq 1, \end{cases} \tag{3.1.4}$$

where $U : H_1 \to H_1$, $T : H_2 \to H_2$ are two *demi-contractive mappings* defined on Hilbert spaces. The class of demi-contractive mappings properly contains the class of firmly quasi-nonexpansive mappings which was studied by Moudafi and Al-Shemas [79].

The aim of the present study is to extend the split equality problem of Moudafi and Al-Shamas [79], and Chidume *et al.* [41], to Lipschitz hemicontractive mappings. The very important class of hemicontractive mapping contains pseudocontractive mappings with nonempty fixed point sets. The later has been studied extensively, for example, by Browder and Petryshn [23], Browder [24], Chidume [33], Chidume and Zegeye [42], Kirk [52], Maruster[75], Xu [102] and a host of other authors, and is known to properly contain the important class of demicontractive mappings studied by Chidume *et al.* [41]. We will discuss some weak and strong convergence theorem for a mean value sequence introduced.

Our theorems and corollaries extend and generalize the results of Censor and

Segal [31], Chidume it et al. [41], Maruster *et al.* [75], Moudafi and Al-Shemas [79], Xu [105], Yuan-Fang *et al.* [56], and a host of other results.

3.2 Main Results

In this section we present a Krasnoselskii's type algorithm for fixed points of Lipschitz pseudocontractive mappings and propose a coupled iterative algorithm for solving the split equality fixed point problem, involving hemicontractive mappings.

We recall a well know lemma on Hilbert spaces which will be used in the sequel. We will first prove the following theorem for Lipschiptz pseudocontractive mappings:

Theorem 3.2.1 *Let H be a Hilbert space, $K \subseteq H$ be a nonempty, closed and convex. Let T be a Lipschitzian and pseudocontractive self-map of K, with Lipschitz constant $L > 0$, such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by $x_1 \in K$ and*

$$x_{n+1} = (1-\lambda)x_n + \lambda T y_n, \tag{3.2.1}$$

$$y_n = (1-\lambda)x_n + \lambda T x_n, \tag{3.2.2}$$

where $\lambda \in (0, L^{-2}[\sqrt{1+L^2}-1])$. Then, for each $p \in F(T)$, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. 3.2.1 *Let $p \in F(T)$. Using Lemma* **??**, *and following a procedure similar to that of Ishikawa [60], we have*

$$\|x_{n+1} - p\|^2 = \|(1-\lambda)(x_n - p) + \lambda(Ty_n - p)\|^2$$

$$= (1-\lambda)\|x_n - p\|^2 + \lambda\|Ty_n - p\|^2 - \lambda(1-\lambda)\|x_n - Ty_n\|^2, \tag{3.2.3}$$

$$\|Ty_n - p\|^2 = \|Ty_n - Tp\|^2 \leq \|y_n - p\|^2 + \|y_n - Ty_n\|^2, \tag{3.2.4}$$

$$\|y_n - p\|^2 = \|(1-\lambda)(x_n - p) + \lambda(Tx_n - p)\|^2,$$

$$= (1-\lambda)\|x_n - p\|^2 + \lambda\|Tx_n - p\|^2 - \lambda(1-\lambda)\|x_n - Tx_n\|^2, \tag{3.2.5}$$

$$\|y_n - Ty_n\|^2 = \|(1-\lambda)(x_n - Ty_n) + \lambda(Tx_n - Ty_n)\|^2,$$

$$= (1-\lambda)\|x_n - Ty_n\|^2 + \lambda\|Tx_n - Ty_n\|^2 - \lambda(1-\lambda)\|x_n - Tx_n\|^2, \quad (3.2.6)$$

and

$$\|Tx_n - p\|^2 = \|Tx_n - Tp\|^2 \leq \|x_n - p\|^2 + \|x_n - Tx_n\|^2 \quad (3.2.7)$$

Substituting (3.2.4)-(3.2.7) into (3.2.15), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= (1-\lambda)\|x_n - p\|^2 + \lambda\|Ty_n - p\|^2 - \lambda(1-\lambda)\|x_n - Ty_n\|^2 \\ &\leq (1-\lambda)\|x_n - p\|^2 + \lambda[\|y_n - p\|^2 + \|y_n - Ty_n\|^2] - \lambda(1-\lambda)\|x_n - Ty_n\|^2.\end{aligned}$$

Thus

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq{} & (1-\lambda)\|x_n - p\|^2 + \lambda[(1-\lambda)\|x_n - p\|^2 + \lambda\|x_n - p\|^2 + \lambda\|x_n - Tx_n\|^2 \\ & - \lambda(1-\lambda)\|x_n - Tx_n\|^2 + (1-\lambda)\|x_n - Ty_n\|^2 + \lambda\|Tx_n - Ty_n\|^2 \\ & - \lambda(1-\lambda)\|x_n - Tx_n\|^2] - \lambda(1-\lambda)\|x_n - Ty_n\|^2 \\ ={} & \|x_n - p\|^2 + \lambda^2\|x_n - Tx_n\|^2 - 2\lambda^2(1-\lambda)\|x_n - Tx_n\|^2 + \lambda^2\|Tx_n - Ty_n\|^2\end{aligned}$$

$$\leq \|x_n - p\|^2 - \lambda^2(1 - 2\lambda - \lambda^2 L^2)\|x_n - Tx_n\|^2. \quad (3.2.8)$$

From (3.2.8), we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|, \quad (3.2.9)$$

and

$$\lambda^2(1 - 2\lambda - \lambda^2 L^2)\|x_n - Tx_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \quad (3.2.10)$$

Using (3.2.9) and Lemma 2.1.2, we have that

$$\lim_{n\to\infty} \|x_n - p\|$$

exists. Moreover, $1 - 2\lambda - \lambda^2 L^2 > 0 \quad \Leftrightarrow \quad |\lambda + \frac{1}{L^2}| < L^{-2}\sqrt{L^2+1}$. *Therefore, since* $0 < \lambda \in (0, L^{-2}[\sqrt{1+L^2} - 1])$, *we have* $1 - 2\lambda - \lambda^2 L^2 > 0$. *Taking limits on both sides of (3.2.10), we have*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

Thus the theorem is established.

We recall the following definition.

Definition 3.2.1 (Demiclosedness principle) *Let* $T : K \to K$ *be a mapping. Then* $I - T$ *is called demiclosed at zero if for any sequence* $\{x_n\}$ *in* H *such that* $x_n \rightharpoonup x$, *and* $\|x_n - Tx_n\| \to 0$, *then* $Tx = x$.

Lemma 3.2.1 *[Opial's Lemma [86]] Let H be a real Hilbert space and $\{x_n\}$ be a sequence in H for which there exists a nonempty set $\Gamma \subseteq H$ such that for every $x \in \Gamma$, $\lim_{n\to\infty} \|x_n - x\|$ exists and any weak-cluster point of the sequence belongs to Γ. Then, there exists $x^* \in \Gamma$ such that $\{x_n\}$ converges weakly to x^*.*

Lemma 3.2.2 *Let H_1 and H_2 be two real Hilbert spaces. Then, the product $H_1 \times H_2$ is a Hilbert with inner product $\langle (x_1, x_2), (y_1, y_2)\rangle_* := \langle x_1, y_1\rangle_1 + \langle x_2, y_2\rangle_2$ where $\langle ., .\rangle_1$, $\langle ., .\rangle_2$ are the inner products on H_1 and H_2 respectively.*

The Split Equality Problem for hemicontractive mappings is stated as:

$$\text{Find } \ x \in C = F(S) \ \text{ and } \ y \in Q = F(T) \text{ such that } \ Ax = By, \tag{3.2.11}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, H_1, H_2, and H_3 are real Hilbert spaces, while $S : H_1 \to H_1$ and $T : H_2 \to H_2$ hemicontractive mappings, respectively.

Henceforth, given two Lipschitz hemicontractive mappings S and T, we define the set

$$\Gamma := \{(p, q) \in H_1 \times H_2 : Sp = p, Tq = q\}, \tag{3.2.12}$$

and a mapping $G : H_1 \times H_2 \to H_1 \times H_2$ by

$$G(x, y) := (S(x - \lambda A^*(Ax - By)), T(y + \lambda B^*(Ax - By)). \tag{3.2.13}$$

It is easy to see that G is Lipschitz. Moreover, for $(p, q) \in \Gamma$, $G(p, q) = (p, q)$.

Now consider the coupled iterative algorithm given below

$$\begin{cases} (x_1, y_1) \in H_1 \times H_2, \ \text{chosen arbitarily}, \\ (x_{n+1}, y_{n+1}) = (1-\alpha)[(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)] + \alpha G(u_n, v_n), \\ (u_n, v_n) = (1-\alpha)[(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)] + \alpha G(x_n, y_n), \\ \alpha \in (0, L^{-2}(\sqrt{L^2+1} - 1)) \\ \lambda \in (0, \frac{2\alpha}{\bar{\lambda}(A,B)}), \end{cases} \tag{3.2.14}$$

where $\bar{\lambda}(A, B)$ is the sum of the spectral radii of A^*A and B^*B and L the Lipschitz constant of G. We show in what follows that the iterative sequence generated by the algorithm above converges weakly to a solution of the split equalty problem (3.2.11).

Theorem 3.2.2 *Let H_1, H_2, H_3 be real Hilbert spaces, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ two Lipschitz hemicontractive mappings, and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear mappings. Then the coupled sequence (x_n, y_n) generated by the algorithm (3.2.14) converges weakly to a solution (x^*, y^*) of problem (3.2.11).*

Proof: Define $\|(x,y)\|_*^2 = \|x\|_1^2 + \|y\|_2^2$. Taking $(p,q) \in \Gamma$ and using Lemma 3.2.2, we obtain

$$\begin{aligned}
&\|(x_{n+1}, y_{n+1}) - (p,q)\|_*^2 = \|(1-\alpha)((x_n - \lambda A^*(Ax_n - By_n), y_n \\
&+ \lambda B^*(Ax_n - By_n)) - (p,q)) + \alpha(G(u_n, v_n) - (p,q))\|_*^2 \\
&\leq (1-\alpha)\Big[\|(x_n, y_n) - (p,q)\|_*^2 - 2\lambda\|Ax_n - By_n\|_*^2 + \lambda^2(\bar{\lambda}(A,B))\|Ax_n - By_n\|^2\Big] \\
&+ \alpha\|G(u_n, v_n) - (p,q)\|_*^2 \\
&- \alpha(1-\alpha)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)) - G(u_n, v_n)\|_*^2,
\end{aligned}$$

It follows from the definition of the mapping G and the hemicontractive properties of S and T we get

$$\begin{aligned}
\|G(u_n, v_n) - (p,q)\|_*^2 &= \|G(u_n, v_n) - G(p,q)\|_*^2 \\
&\leq \|(u_n - \lambda A^*(Au_n - Bv_n), v_n + \lambda B^*(Au_n - Bv_n)) - (p,q)\|_*^2 \\
&+ \|(u_n - \lambda A^*(Au_n - Bv_n), v_n + \lambda B^*(Au_n - Bv_n)) - G(u_n, v_n)\|_*^2 \\
&\leq \|(u_n, v_n) - (p,q)\|_*^2 - \lambda(2 - \lambda(\bar{\lambda}(A,B)))\|Au_n - Bv_n\|^2 \\
&+ \|(u_n - \lambda A^*(Au_n - Bv_n), v_n + \lambda B^*(Au_n - Bv_n)) - G(u_n, v_n)\|_*^2.
\end{aligned}$$

In view of the inequalities above, we obtain

$$\|(x_{n+1}, y_{n+1}) - (p,q)\|_*^2 \leq (1-\alpha)\Big[\|(x_n, y_n) - (p,q)\|_*^2 \tag{3.2.15}$$

$$- \lambda(2 - \lambda(\bar{\lambda}(A,B))\|Ax_n - By_n\|^2\Big] \tag{3.2.16}$$

$$+ \alpha\Big[\|(u_n, v_n) - (p,q)\|_*^2 - \lambda(2 - \lambda(\bar{\lambda}(A,B)))\|Au_n - Bv_n\|^2 \tag{3.2.17}$$

$$+ \|(u_n - \lambda A^*(Au_n - Bv_n), v_n + \lambda B^*(Au_n - Bv_n)) - G(u_n, v_n)\|_*^2.\Big] \tag{3.2.18}$$

$$- \alpha(1-\alpha)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) - G(u_n, v_n)\|_*^2, \tag{3.2.19}$$

Using the definition of u_n and v_n, we have the folowing chain of inequalities:

$$\begin{aligned}
&\|(u_n, v_n) - (p,q)\|_*^2 = \|(1-\alpha)[(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)) \\
&\quad - (p,q) + \alpha[G(x_n, y_n) - (p,q)]\|_*^2 \\
&\quad \leq (1-\alpha)\Big[\|(x_n, y_n) - (p,q)\|_*^2 - \lambda(2 - \lambda(\bar{\lambda}(A,B)))\|Ax_n - By_n\|^2\Big] \\
&\quad + \alpha\Big[\|(x_n, y_n) - (p,q)\|_*^2 - \lambda(2 - \lambda(\bar{\lambda}(A,B)))\|Ax_n - By_n\|^2 \\
&\quad + \|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) - G(x_n, y_n)\|_*^2\Big] \\
&\quad - \alpha(1-\alpha)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) - G(x_n, y_n)\|_*^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|(u_n - \lambda A^*(Au_n - Bv_n), v_n + \lambda B^*(Au_n - Bv_n)) - G(u_n, v_n)\|_*^2. \\
&\qquad = \|(1-\alpha)[(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)) - G(u_n, v_n)] \\
&\qquad + \alpha[G(x_n, y_n) - G(u_n, v_n)\|^2, \\
&\qquad \leq (1-\alpha)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)) - G(u_n, v_n)\|^2 \\
&\qquad + \alpha\|G(x_n, y_n) - G(u_n, v_n)\|^2 \\
&\qquad - \alpha(1-\alpha)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)) - G(x_n, y_n)\|^2,
\end{aligned}$$

If we substitute these inequalities into their rightful positions in the inequality (3.2.15), we get the following:

$$\begin{aligned}
&\|(x_{n+1}, y_{n+1}) - (p, q)\|_*^2 \leq (1-\alpha)\Big[\|(x_n, y_n) - (p, q)\|_*^2 - \lambda(2 - \lambda(\bar{\lambda}(A, B))\|Ax_n - By_n\|^2\Big] \\
&\qquad + \alpha\Big[(1-\alpha)\Big[\|(x_n, y_n) - (p, q)\|_*^2 - \lambda(2 - \lambda(\bar{\lambda}(A, B)))\|Ax_n - By_n\|^2\Big] \\
&\qquad + \alpha\Big[\|(x_n, y_n) - (p, q)\|_*^2 - \lambda(2 - \lambda(\bar{\lambda}(A, B)))\|Ax_n - By_n\|^2 \\
&\qquad + \|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) - G(x_n, y_n)\|_*^2\Big] \\
&\qquad - \alpha(1-\alpha)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) - G(x_n, y_n)\|_*^2. \\
&\qquad - \lambda(2 - \lambda(\bar{\lambda}(A, B)))\|Au_n - Bv_n\|^2\Big] \\
&\qquad + (1-\alpha)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)) - G(u_n, v_n)\|^2 \\
&\qquad + \alpha\|G(x_n, y_n) - G(u_n, v_n)\|^2 \\
&\qquad - \alpha(1-\alpha)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n)) - G(x_n, y_n)\|^2,\Big] \\
&\qquad - \alpha(1-\alpha)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) - G(u_n, v_n)\|_*^2.
\end{aligned}$$

Gathering all the similar terms together, we obtain

$$\begin{aligned}
&\|(x_{n+1}, y_{n+1}) - (p, q)\|_*^2 \leq \|(x_n, y_n) - (p, q)\|_*^2 - \lambda(2 - \lambda(\bar{\lambda}(A, B)))\|Ax_n - By_n\|^2 \\
&\qquad - (\alpha^2 - 2\alpha^3)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) - G(x_n, y_n)\|_*^2 \\
&\qquad - \alpha\lambda(2 - \lambda(\bar{\lambda}(A, B)))\|Au_n - Bv_n\|^2\Big] \\
&\qquad + \alpha^2\|G(x_n, y_n) - G(u_n, v_n)\|_*^2
\end{aligned}$$

Again since S and T are Lipschiptz with Lipschitz constant, say, L_s and L_t

respectively. Set $L = \max\{L_s, L_t\}$. Then,

$$\begin{aligned}
\|G(x_n, y_n) - G(u_n, v_n)\|^2 &= \|S(x_n - \lambda A^*(Ax_n - By_n) - S(u_n - \lambda A^*(Au_n - Bv_n)\|_1^2 \\
&+ \|T(y_n + \lambda B^*(Ax_n - By_n) - T(v_n + \lambda B^*(Au_n - Bv_n)\|_2^2 \\
&\le L_s^2\|(x_n - \lambda A^*(Ax_n - By_n)) - (u_n - \lambda A^*(Au_n - Bv_n)\|_1^2 \\
&+ L_t^2\|(y_n + \lambda B^*(Ax_n - By_n) - (v_n + \lambda B^*(Au_n - Bv_n)\|_2^2 \\
&\le L^2\Big[\|(x_n - \lambda A^*(Ax_n - By_n)) - u_n\|_1^2 \\
&\quad + \|(y_n - \lambda B^*(Ax_n - By_n)) - v_n\|_2^2, \\
&\quad + 2\lambda\langle Ax_n - Au_n - \lambda(Ax_n - By_n), Au_n - Bv_n\rangle, \\
&\quad - 2\lambda\langle By_n - Bv_n - \lambda(Ax_n - By_n), Au_n - Bv_n\rangle, \\
&+ \lambda^2(\bar{\lambda}(A, B))\|Au_n - Bv_n\|^2\Big]. \\
&\le L^2\Big[\alpha^2\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) - G(x_n, y_n)\|_*^2 \\
&+ 2\lambda\langle Ax_n - By_n, Au_n - Bv_n\rangle - \lambda(2 - \lambda\bar{\lambda}(A, B))\|Au_n - Bv_n\|^2\Big]
\end{aligned}$$

Since $2\lambda\langle Ax_n - By_n, Au_n - Bv_n\rangle \le 2\lambda\|Ax_n - By_n\|^2 + 2\lambda\|Au_n - Bv_n\|^2$, we conclude that

$$\begin{aligned}
\|G(x_n, y_n) - G(u_n, v_n)\|^2 \le L^2\Big[&\alpha^2\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) \\
&- G(x_n, y_n)\|_*^2 + 2\lambda\|Ax_n - By_n\|^2 + \lambda^2\bar{\lambda}(A, B))\|Au_n - Bv_n\|^2\Big]
\end{aligned}$$

Substituting this in its rightful place gives

$$\begin{aligned}
\|(x_{n+1}, y_{n+1}) - (p, q)\|_*^2 &\le \|(x_n, y_n) - (p, q)\|_*^2 - \lambda(2 - \lambda(\bar{\lambda}(A, B)))\|Ax_n - By_n\|^2 \\
&- (\alpha^2 - 2\alpha^3)\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) - G(x_n, y_n)\|_*^2 \\
&- \alpha\lambda(2 - \lambda(\bar{\lambda}(A, B)))\|Au_n - Bv_n\|^2\Big] \\
&+ \alpha^2 L^2\Big[\alpha^2\|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) - G(x_n, y_n)\|_*^2 \\
&2\lambda\|Ax_n - By_n\|^2 + \lambda^2\bar{\lambda}(A, B))\|Au_n - Bv_n\|^2\Big] \\
&= \|(x_n, y_n) - (p, q)\|_*^2 \\
&+ [-2\lambda + 2\lambda\alpha^2 L^2 + \lambda^2(\bar{\lambda}(A, B))]\|Ax_n - By_n\|^2 \\
&- \alpha^2(1 - 2\alpha - \alpha^2 L^2) \\
&\times \|(x_n - \lambda A^*(Ax_n - By_n), y_n + \lambda B^*(Ax_n - By_n) - G(x_n, y_n)\|_*^2 \\
&+ [-2\alpha\lambda + \alpha\lambda^2\bar{\lambda}(A, B)) + \alpha^2 L^2\lambda^2\bar{\lambda}(A, B))]\|Au_n - Bv_n\|^2
\end{aligned}$$

Finally, if we observe that $1-2\alpha-\alpha^2L^2>0$ is the same as $|\alpha+\frac{1}{L^2}|<L^{-2}\sqrt{L^2+1}$, then, since $\alpha\in(0,L^{-2}[\sqrt{1+L^2}-1])$, we have $1-2\alpha-\alpha^2L^2>0$. Therefore, we have $\alpha^2L^2<1-2\alpha$ and $2-2\alpha^2L^2>0$. Certainly, $\alpha<\min\{\frac{1}{2},\frac{1}{L}\}$, and $-2\lambda+2\lambda\alpha^2L^2+\lambda^2(\bar{\lambda}(A,B))<-2\lambda+2\lambda(1-2\alpha)+\lambda^2(\bar{\lambda}(A,B))=-4\alpha\lambda+\lambda^2(\bar{\lambda}(A,B))<0$ since $\lambda<\frac{2\alpha}{\bar{\lambda}(A,B)}$. Finally, $-2\alpha\lambda+\alpha\lambda^2\bar{\lambda}(A,B))+\alpha^2L^2\lambda^2\bar{\lambda}(A,B)<-2\alpha\lambda+\alpha\lambda^2\bar{\lambda}(A,B))+\lambda^2\bar{\lambda}(A,B)(1-2\alpha)<-2\alpha\lambda+\lambda^2\bar{\lambda}(A,B)<0$.From the previous chain of inequalities we may now conclude the following,

$$\|(x_{n+1},y_{n+1})-(p,q)\|_*^2\le\|(x_n,y_n)-(p,q)\|_*^2, \tag{3.2.20}$$

$$\begin{aligned}&[2\lambda-2\lambda\alpha^2L^2-\lambda^2(\bar{\lambda}(A,B))]\|Ax_n-By_n\|^2 &(3.2.21)\\ &\quad\le\|(x_n,y_n)-(p,q)\|_*^2-\|(x_{n+1},\quad y_{n+1})-(p,q)\|_*^2 &(3.2.22)\end{aligned}$$

and

$$[\alpha^2(1-2\alpha-\alpha^2L^2)\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2$$

$$\le\|(x_n,y_n)-(p,q)\|_*^2-\|(x_{n+1},y_{n+1})-(p,q)\|_*^2. \tag{3.2.23}$$

Using Lemma (2.1.2) we have by (3.2.20) that $\|(x_n,y_n)-(p,q)\|_*^2$ has a limit. Therefore, taking limits on both sides of (3.2.21), and (3.2.23) respectively, we have that

$$\lim_{n\to\infty}\|Ax_n-By_n\|^2=0, \tag{3.2.24}$$

$$\lim_{n\to\infty}\|(x_n-\lambda A^*(Ax_n-By_n),y_n+\lambda B^*(Ax_n-By_n)-G(x_n,y_n)\|_*^2=0. \tag{3.2.25}$$

Next, we show that $\lim_{n\to\infty}\|x_n-S(x_n)\|_1=0$ and $\lim_{n\to\infty}\|y_n-S(y_n)\|_2=0$. The fact that $\|(x_n,y_n)-(p,q)\|_*^2$ has a limit shows that both $\{x_n\}$ and $\{y_n\}$ are bounded. Suppose that x^* and y^* are weak cluster points of the sequences $\{x_n\}$ and $\{y_n\}$ such that $x_{n_k}\rightharpoonup x^*$ and $y_{n_k}\rightharpoonup y^*$ repectively. Then

$$\lim_{k\to\infty}\|S(x_{n_k}-\lambda A^*(Ax_{n_k}-By_{n_k}))-Sx_{n_k}\|\le L_s\bar{\lambda}(A,B)\lim_{k\to\infty}\|Ax_{n_k}-By_{n_k}\|=0,$$

and similarly,

$$\lim_{k\to\infty}\|T(y_{n_k}+\lambda B^*(Ax_{n_k}-By_{n_k}))-Ty_{n_k}\|\le L_t\bar{\lambda}(A,B)\lim_{k\to\infty}\|Ax_{n_k}-By_{n_k}\|=0.$$

Therefore we have

$$\begin{aligned}\|x_{n_k} - S(x_{n_k})\| &\leq \|x_{n_k} - (x_{n_k} - \lambda A^*(Ax_{n_k} - By_{n_k})\| \\ &\quad + \|(x_{n_k} - \lambda A^*(Ax_{n_k} - By_{n_k})) - S(x_{n_k} - \lambda A^*(Ax_{n_k} - By_{n_k}))\| \\ &\quad L_s\bar{\lambda}(A,B)\|Ax_{n_k} - By_{n_k}\| \to 0 \ \ as \ \ k \to \infty.\end{aligned}$$

A similar computation gives that $\lim_{k\to\infty} \|y_{n_k} - T(y_{n_k})\| = 0$. Since S and T are demiclosed at zero, we conclude that $x^* = S(x^*)$ and $y^* = T(y^*)$. Again, since $x_{n_k} \rightharpoonup x^*$ and $y_{n_k} \rightharpoonup y^*$, we have that

$$Ax_{n_k} - By_{n_k} \rightharpoonup Ax^* - By^*,$$

and by the weak lower semi-continuity of norm,

$$\|Ax^* - By^*\| \leq \lim_{n\to\infty} \inf \|Ax_{n_k} - By_{n_k}\| = 0,$$

So, $Ax^* = By^*$ and thus $(x^*, y^*) \in \Gamma$. In conclusion, we have obtain thus far that for each $(p,q) \in \Gamma$, the sequence $\|(x_n, y_n) - (p,q)\|_*^2$ has a limit. Moreover, each weak cluster point of the sequence (x_n, y_n) is an element of Γ. We may now invoke the celebrated Opial's Lemma 3.2.1 to conclude that there exist $(x^*, y^*) \in \Gamma$ such that (x_n, y_n) converges weakly to (x^*, y^*). Hence the iterative sequence (x_n, y_n) converges weakly to a solution of the spit equality problem (3.2.11). The proof is complete.

We may stregthen the conditions of the theorem and obtrain strong convergence of the sequence as follows:

Theorem 3.2.3 *Suppose that the assupmtions of Theorem (3.2.2) are fulfilled. Assume, in addition, that the mappings S and T are also hemicompact. Then, for any initial point (x_1, y_1), the coupled iterative sequence (x_n, y_n) derived from the algorithm converges strongly to a solution of problem (SEP).*

Proof: We have obtained fromTheorem 3.2.2 that (x_n, y_n) is bounded, and that $\lim_{n\to\infty} \|x_n - S(x_n)\| = 0$, and $\lim_{n\to\infty} \|y_n - T(y_n)\| = 0$. On the other hand, since S and T are hemicompact, we have some subsequence $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that $x_{n_k} \to x^*$ and $y_{n_k} \to y^*$. The subsequence also converge weakly and therefore $Ax_{n_k} - By_{n_k} \rightharpoonup Ax^* - By^*$. As we have shown above, this yields $Ax^* = By^*$ and $(x^*, y^*) \in \Gamma$. Going back to the proof of Theorem (3.2.2), we have that $\lim_{n\to\infty} \|(x_n, y_n) - (x^*, y^*)\|_*^2$ exists and then $\lim_{k\to\infty} \|(x_{n_k}, y_{n_k}) - (x^*, y^*)\|_*^2$. We may conclude by Lemma (2.1.2) that $(x_n, y_n) \to (x^*, y^*) \in \Gamma$. So our iterative algorithm converges to a solution of (SEP) and the proof is complete.

Corollary 3.2.1 *Suppose that the mappings S and T in Theorem (3.2.3) are hemicompact and demicontractive. Then, for any initial point (x_1, y_1), the coupled iterative sequence (x_n, y_n) derived from the algorithm converges strongly to a solution of problem (SEP).*

In conclusion, our theorems extend and complement the results of Chidume *et al.* [41], Xu[105], Moudafi and Al-Shamas [79] and many other authors to the more general class of Lipschitz hemicontractive mappings.

Remark: The main theorems of this chapter namely, Theorem 3.2.1 and Theorem 3.2.2 are contents of the journal articles

- M.E. Okpala, *A Remark On the Theorem of Ishikawa,* **British Journal of Mathematics and Computer Science** Vol 7 Issue 7 2015
- M. E. Okpala *Split equality fixed point problem for Lipschitz Hemi-contractive mappings*,**(Accepted(2015) Advances in Fixed Point Theory).**

CHAPTER 4

Contributions on Iterative Algorithms for a General Class of Multivalued Strictly Pseudocontractive mappings

In this chapter we will survey some techniques for approximating fixed points of a more general class of multivalued pseudocontractive mappings which we will define shortly.

First, we recall the single valued definition of strictly pseudocontractive mapping due to Browder and Petryshin [23] as follows:

Definition 4.0.2 *Let K be a nonempty subset of a Hilbert space H. A map $T : K \to H$ is called strictly pseudocontractive if there exists $k \in [0, 1)$ such that*

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in K. \tag{4.0.1}$$

The following definition of multivalued strictly pseudocontractive mappings was introduced in Chidume *et al.* [35]:

Definition 4.0.3 *Let H be a real Hilbert space and let D be a nonempty, open and convex subset of H. Let $T : \overline{D} \to CB(\overline{D})$ be a mapping. Then, T is called a multi-valued $k-$strictly pseudocontractive mapping if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$, we have*

$$D^2(Tx, Ty) \le \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \tag{4.0.2}$$

for all $u \in Tx, \ v \in Ty$.

Remark 4.0.1 *It is easy to see that inequality (4.0.2) is equivalent to*

$$D^2(Tx,Ty) \le \|x-y\|^2 + k \inf_{(u,v)} \|(x-u)-(y-v)\|^2.$$

Remark 4.0.2
- *For $k=1$, in definition (4.0.3), the mapping T was called multi-valued pseudocontractive and $k=0$, is the multi-valued non-expansive introduced in Nadler [80].*
- *The definition is an extension of the definition of single-valued strictly pseudocontractive mappings to multi-valued maps.*

Definition 4.0.4 *A map $T : K \to CB(K)$ is said to be hemicompact if, for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence, say, $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.*

Note that if K is compact, then every multi-valued mapping $T : K \to CB(K)$ is hemicompact.

Definition 4.0.5 *Let H be a real Hilbert space and let T be a multi-valued mapping. The multi-valued mapping $I-T$ is said to be strongly demiclosed at 0 (see, e.g., [57]) if for any sequence $\{x_n\} \subseteq D(T)$ such that $x_n \to p$ and $d(x_n, Tx_n)$ converges strongly to 0, then $d(p,Tp)=0$.*

We will recall the following important characterization of the metric projection in Hilbert spaces which is also stated in proposition 2.1.2.

Lemma 4.0.3 *Let H be a Hilbert space, $K \subset H$ be nonempty, closed and convex, $z \in H$ and $x \in K$. Then $x = P_K z$ if and only if*

$$\langle z-x, w-x\rangle \le 0 \quad \forall w \in K.$$

4.1 Main Results

We first prove the following important preliminary results.

Lemma 4.1.1 *Let E be a normed linear space , $A, B \in CB(E)$ and $x_0, y_0 \in E$ arbitrary. The following hold;*

(a) $D(A,B) = D(x_0 + A, x_0 + B)$.

(b) $D(A,B) = D(-A,-B)$.

(c) $D(x_0 + A, y_0 + B) \leq \|x_0 - y_0\| + D(A, B)$.

(d) $D(\{x_0\}, A) = \sup_{a \in A} \|x_0 - a\|$.

(e) $D(\{x_0\}, A) = D(\{0\}, x_0 - A)$.

Proof: (a) By definition, we have

$$\begin{aligned} D(x_0 + A, x_0 + B) &= \max\left\{\sup_{a \in A} d(x_0 + a, x_0 + B); \sup_{b \in B} d(x_0 + b, x_0 + B)\right\} \\ &= \max\{\sup_{a \in A} d(a, B); \sup_{b \in B} d(b; A)\} \\ &= D(A, B). \end{aligned}$$

(b) We have

$$\begin{aligned} D(-A, -B) &= \max\{\sup_{-a \in -A} d(-a; -B); \sup_{-b \in -B} d(-b; -A)\} \\ &= \max\{\sup_{a \in A} d(a; B); \sup_{b \in B} d(b; A)\} \\ &= D(A, B). \end{aligned}$$

(c) It is known that for any set $B \subseteq E$, $x, y \in E$ arbitrary, the inequality

$$d(x, B) \leq \|x - y\| + d(y, B)$$

holds. Using this inequality we have

$$\begin{aligned} d(x_0 + a, y_0 + B) &\leq \|(x_0 + a) - (y_0 + a)\| + d(y_0 + a, y_0 + B) \\ &= \|x_0 - y_0\| + d(a, B), \end{aligned}$$

and similarly

$$d(y_0 + b, x_0 + A) \leq \|x_0 - y_0\| + d(b, A).$$

Therefore, taking sup over A and B respectively, we have

$$\sup_{a \in A} d(x_0 + a, y_0 + B) \leq \|x_0 - y_0\| + \sup_{a \in A} d(a, B),$$

and

$$\sup_{b \in B} d(y_0 + b, x_0 + A) \leq \|x_0 - y_0\| + \sup_{b \in B} d(b, A).$$

Thus $D(x_0 + A, y_0 + B) \leq \|x_0 - y_0\| + D(A, B)$.
(d) It is obvious that $d(x_0; A) = \sup_{x_0 \in \{x_0\}} d(x_0, A)$. On the otherhand, for any $a \in A$, we have

$$d(a; \{x_0\}) = \|a - x_0\| \geq d(x_0; A).$$

Taking sup over A we have

$$\sup_{a\in A} d(a,\{x_0\}) \geq d(x_0, A),$$

and therefore

$$D(\{x_0\}, A) := \max\{\sup_{a\in A} d(a;\{x_0\}); \sup_{x_0\in\{x_0\}} d(x_0; A)\} = \sup_{a\in A} d(a,\{x_0\}).$$

(e)

$$\begin{aligned} D(\{x_0\}, A) &:= \max\{\sup_{a\in A} d(a,\{x_0\}), d(x_0, A)\} \\ &= \max\{\sup_{a\in A} \|x_0 - a\|), \inf_{a\in A} \|x_0 - a\|\} \\ &= \max\{\sup_{a\in A} d(0, x_0 - A), d(0, x_0 - A)\} \\ &= D(\{0\}, x_0 - A). \end{aligned}$$

This lemma has many intersting implications. For example, recall the Nadler's condition, that is,

Lemma 4.1.2 (Nadler [80]) *Given A and $B \in CB(H)$ and $a \in A$. For every $\gamma > 0$, there exists $b \in B$ such that*

$$\|a - b\| \leq D(A, B) + \gamma.$$

From the property (d), we obtain the following lemma which gives the reverse of the Nadler's lemma as follows.

Lemma 4.1.3 *Let $B \in CB(H)$ and $\gamma > 0$ be given. Then for any $a \in H$, there exist $b \in B$ such that*

$$D(\{a\}, B) \leq \|a - b\| + \gamma.$$

We now introduce the following class of *generalized $k-$ strictly pseudocontractive multi-valued mappings.*

Definition 4.1.1 *Let H be a real Hilbert space and let K be a nonempty subset of H. Let $T : K \to CB(K)$ be a multi-valued mapping. Then T is called* **generalized $k-$strictly pseudocontractive multi-valued mapping** *if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$, we have*

$$D^2(Tx, Ty) \leq \|x - y\|^2 + kD^2(Ax, Ay), \quad A := I - T, \tag{4.1.1}$$

and I is the identity operator on K.

Remark 4.1.1 *Definition (4.1.1) seems to be a more natural generalization of the single-valued definition (2.1.3) given by Browder and Petryshin [23] than the definition (4.0.3) given by Chidume et al. [35].*

We now prove that the class of generalized $k-$strictly pseudocontractive mappings properly contains the class introduced in the Definition (4.0.3).

Proposition 4.1.1 *Let $T : K \to CB(K)$ be a multi-valued $k-$strictly pseudocontractive mapping, then T is a generalized $k-$strictly pseudocontractive multi-valued mapping.*

Proof Given that T is a multi-valued $k-$strictly pseudocontractive mapping, we have

$$D^2(Tx, Ty) \leq \|x-y\|^2 + k \inf_{(u,v)\in(Tx,Ty)} \|(x-u)-(y-v)\|^2. \tag{4.1.2}$$

We now show that inequality (2.3.5) implies inequality (4.1.1).

$$\begin{aligned} D(x-Tx, y-Ty) :&= \max\Big\{ \sup_{u\in Tx} d(x-u; y-Ty); \sup_{v\in Ty} d(y-v; x-Tx)\Big\} \\ &\geq \sup_{u\in Tx} d(x-u; y-Ty) \\ &\geq d(x-u_0; y-Ty), \qquad u_0 \in Tx. \end{aligned}$$

Now, given $\epsilon > 0$, there exist $v_\epsilon \in Ty$ such that

$$\begin{aligned} d(x-u_0; y-Ty) &\geq \|(x-u_0)-(y-v_\epsilon)\| - \epsilon \\ &\geq \inf_{(u,v)\in(Tx,Ty)} \|(x-u)-(y-v)\| - \epsilon. \end{aligned}$$

Thus, for arbitrary $\epsilon > 0$, we have

$$\inf_{(u,v)\in(Tx,Ty)} \|(x-u)-(y-v)\| \leq D(x-Tx, y-Ty) + \epsilon,$$

and therefore, since $\epsilon > 0$ is arbitrary, we have:

$$\inf_{(u,v)\in(Tx,Ty)} \|(x-u)-(y-v)\| \leq D(x-Tx, y-Ty). \tag{4.1.3}$$

We therefore obtain from (4.1.2) and (4.1.3) that:

$$D^2(Tx, Ty) \leq \|x-y\|^2 + kD^2(x-Tx, y-Ty).$$

Thus, every multi-valued $k-$strictly pseudocontractive mapping is also a *generalized* $k-$strictly pseudocontractive multi-valued mapping.

We now give an example to show that this inclusion is proper.

For the example, we shall need the following lemma which is trivially proved.

Lemma 4.1.4 *Let a, b be real numbers such that $0 \leq a \leq 4b$. Then,*

$$(a-b)^2 \leq b^2 + \frac{1}{2}a^2. \tag{4.1.4}$$

Example 4.1.1 *Let H be a real Hilbert space. Define a mapping*

$$T : H \to CB(H) \quad by$$

$$Tx := \begin{cases} \overline{B}(-x, \|x\|), & \|x\| > 0 \\ \{0\}, & x = 0, \end{cases}$$

where

$$\overline{B}(-x, \|x\|) = \{u \in H : \|u + x\| \leq \|x\|\}.$$

Then, for distinct nonzero x and y, we have the following identities which follow from the definition of T:

$$\begin{aligned} x - Tx &= \overline{B}(2x, \|x\|), \\ y - Ty &= \overline{B}(2y, \|y\|), \\ Tx &= \{w \in H : \|w + x\| \leq \|x\|\}, \\ Ty \setminus Tx &= \{z \in H : \|z + y\| \leq \|y\|, \|z + x\| > \|x\|\}. \end{aligned}$$

We now establish the following equation:

$$D(Tx, Ty) = \|x - y\| + \Big|\|y\| - \|x\|\Big|. \tag{4.1.5}$$

First, we assume without loss of generality that $\|y\| \geq \|x\|$. Then we proceed as follows:

Claim 1: $\forall z \in Ty \setminus Tx$, *$d(z, Tx) = \|z - P_{(Tx)}z\|$, where*

$$P_{(Tx)}z := -x + \frac{\|x\|}{\|x+z\|}\Big(z + x\Big)(\textit{see also Example } 2.1.3(a)). \tag{4.1.6}$$

Proof of Claim 1. *Let $w \in Tx$. Then, $\|w + x\| \leq \|x\|$. Furthermore,*

$$\begin{aligned}
\langle z - P_{(Tx)}z, w - P_{(Tx)}z\rangle &= \Big\langle z + x - \frac{\|x\|}{\|x+z\|}(z+x), w + x - \frac{\|x\|}{\|x+z\|}(z+x)\Big\rangle \\
&= \frac{\|x+z\| - \|x\|}{\|x+z\|}\Big(\langle z+x, w+x\rangle - \frac{\|x\|}{\|x+z\|}\Big\langle z+x, z+x\Big\rangle\Big) \\
&= \frac{\|x+z\| - \|x\|}{\|x+z\|}\Big(\langle z+x, w+x\rangle - \|x\|\|z+x\|\Big) \\
&\leq \frac{\|x+z\| - \|x\|}{\|x+z\|}\Big(\|z+x\|\|w+x\| - \|x\|\|z+x\|\Big) \\
&\leq (\|x+z\| - \|x\|)\Big(\|x\| - \|x\|\Big) \\
&= 0.
\end{aligned}$$

Thus, it follows that,

$$\langle z - P_{(Tx)}z, w - P_{(Tx)}z\rangle \leq 0,$$

and applying Lemma(4.0.3), the claim is proved. Now, set

$$z_0 := -x + \Big(1 + \frac{\|y\|}{\|x-y\|}\Big)(x-y). \tag{4.1.7}$$

Clearly $z_0 \in Ty \setminus Tx$ since

$$\|z_0 + y\| = \Big\|\frac{\|y\|}{\|x-y\|}(x-y)\Big\| = \|y\|$$

and,

$$\begin{aligned}
\|z_0 + x\| = \Big(1 + \frac{\|y\|}{\|x-y\|}\Big)\|x-y\| &= \|x-y\| + \|y\| \\
&\geq \|x-y\| + \|x\| \\
&> \|x\|.
\end{aligned}$$

Moreover, from equation (4.1.6),

$$\begin{aligned}
P_{(Tx)}z_0 &= -x + \frac{\|x\|}{\|x+z_0\|}\Big(z_0 + x\Big) \\
&= -x + \frac{\|x\|}{\|x-y\| + \|y\|}\Big[\frac{\|x-y\| + \|y\|}{\|x-y\|}(x-y)\Big] \\
&= -x + \frac{\|x\|}{\|x-y\|}\Big(x-y\Big).
\end{aligned} \tag{4.1.8}$$

Therefore by (4.1.8) and (4.1.7) we obtain that

$$d(z_0, Tx) = \|z_0 - P_{(Tx)}z_0\| = \|x-y\| + \|y\| - \|x\|, \tag{4.1.9}$$

establishing Claim 1.

Claim 2: $d(z_0, Tx) = \sup_{v\in Ty} d(v, Tx)$.

Proof of Claim 2. *Let $z \in Ty \setminus Tx$ be arbitrary. We have,*

$$\|z+y\| \leq \|y\|, \ \ \|z+x\| > \|x\|,$$

and so, using equation (4.1.9),

$$\begin{aligned}\|z - P_{(Tx)}z\| &= \left\|z + x - \frac{\|x\|}{\|x+z\|}\Big(z+x\Big)\right\| \\ &= \|z+x\|\Big(1 - \frac{\|x\|}{\|x+z\|}\Big) \\ &= \|x+z\| - \|x\| \\ &\leq \|x-y\| + \|z+y\| - \|x\| \\ &\leq \|x-y\| + \|y\| - \|x\| \\ &= \|z_0 - P_{(Tx)}z_0\|.\end{aligned}$$

For $z \in (Ty \cap Tx)$, we have $d(z, Tx) = 0$. Thus, we obtain that,

$$d(z, Tx) \leq \|z_0 - P_{(Tx)}z_0\| \quad \forall z \in Ty.$$

Using the fact that $z_0 \in Ty$, we obtain

$$\sup_{v\in Ty} d(v, Tx) = \|z_0 - P_{(Tx)}z_0\| = \|x-y\| + \|y\| - \|x\|.$$

Thus, Claim 2 is established.

We now consider the case $\|x\| \geq \|y\|$.
For $\|x\| \geq \|y\|$, we have by interchanging the roles of x and y,

$$\sup_{u\in Tx} d(u, Ty) = \|x-y\| + \|x\| - \|y\|.$$

Therefore,

$$\max\Big\{\sup_{y\in Ty} d(y, Tx), \sup_{x\in Tx} d(x, Ty)\Big\} = \|x-y\| + \Big|\|y\| - \|x\|\Big|. \tag{4.1.10}$$

For $x = y$, $Tx = Ty$, and $D(Tx, Ty) = 0$. Moreover, for $x = 0$, $y \neq 0$, a straightforward computation gives

$$D(0, Ty) = 2\|y\| = \|0 - y\| + \Big|0 - \|y\|\Big|.$$

Thus, the identity (4.1.5) is fully established for arbitrary $x, y \in H$.

Following similar procedure, we obtain

$$D(x - Tx, y - Ty) = 2\|x - y\| + \Big|\|y\| - \|x\|\Big| \quad \forall x, y \in H. \tag{4.1.11}$$

We set

$$\begin{aligned} a &:= D(x - Tx, y - Ty). \\ b &:= \|x - y\|. \end{aligned}$$

Then, using equations (4.1.11) and (4.1.5), we obtain that,

$$a - b = D(Tx, Ty).$$

Clearly, by equation (4.1.11),

$$a = 2\|x - y\| + \Big|\|y\| - \|x\|\Big| \leq 4\|x - y\| = 4b.$$

Therefore, by Lemma (4.1.4),

$$D^2(Tx, Ty) \leq \|x - y\|^2 + \frac{1}{2}\Big(D(x - Tx, y - Ty)\Big)^2 \qquad \forall x, y \in H.$$

Therefore, T is a generalized $k-$strictly pseudocontractive multi-valued mapping with $k = \frac{1}{2}$.

We now show that T is not a multi-valued $k-$strictly pseudocontractive mapping in the sense of definition (4.0.3).
We establish this by contradiction. So, assume that there exists $k \in [0, 1)$ such that inequlity (4.0.2) holds. Choose $x \in H \setminus \{0\}$. Set $y = 2x$, $u = v = 0 \in (Tx \cap Ty)$. Then,

$$\|x - y\| = \|x\|,$$

$$D(Tx, Ty) = \|x - y\| + \Big|\|y\| - \|x\|\Big| = 2\|x\|,$$

and

$$\|(x - u) - (y - v)\| = \|x - y\| = \|x\|.$$

Thus,

$$4\|x\|^2 = D^2(Tx,Ty) \le \|x-y\|^2 + k\|(x-u)-(y-v)\|^2 \le 2\|x\|^2.$$

This is a contradiction to $x \in H \setminus \{0\}$. Therefore, T is not a multi-valued $k-$strictly pseudocontractive mapping for any $k \in (0,1)$.

To prove our main theorem, we first prove the following important propositions.

Proposition 4.1.2 *Let K be a nonempty subset of a real Hilbert space H and $T : K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping. Then T is Lipschitzian.*

Proof: Let $x, y \in D(T)$. Then,

$$\begin{aligned} D^2(Tx,Ty) &\le \|x-y\|^2 + kD^2(x-Tx, y-Ty) \\ &\le \|x-y\|^2 + k\Big(\|x-y\| + D(Tx,Ty)\Big)^2, \text{by Lemma (4.1.1), (c), (b).} \\ &\le \Big(\|x-y\| + \sqrt{k}\|x-y\| + \sqrt{k}D(Tx,Ty)\Big)^2. \end{aligned}$$

Thus,

$$D(Tx,Ty) \le (1+\sqrt{k})\|x-y\| + \sqrt{k}D(Tx,Ty),$$

and hence,

$$D(Tx,Ty) \le \frac{1+\sqrt{k}}{1-\sqrt{k}}\Big\|x-y\Big\|,$$

as proposed.

Remark 4.1.2 *Propostion (4.1.2) is an improvement of Proposition 8 of [35] because it does not assume that Tx is weakly closed for each $x \in K$.*

Proposition 4.1.3 *Let K be a nonempty and closed subset of a real Hilbert space H and let $T : K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping. Then, $(I - T)$ is strongly demiclosed at zero.*

Proof: Let $\{x_n\}$ be a sequence in K such that $x_n \to x$ and $d(x_n, Tx_n) \to 0$. For each $n \in \mathbb{N}$, take $y_n \in Tx_n$ such that $\|x_n - y_n\| \le d(x_n, Tx_n) + \frac{1}{n}$.
Then,

$$\begin{aligned} d(x,Tx) &\le \|x-x_n\| + \|x_n-y_n\| + d(y_n,Tx) \\ &\le \|x-x_n\| + d(x_n,Tx_n) + \frac{1}{n} + D(Tx_n,Tx) \\ &\le \|x-x_n\| + d(x_n,Tx_n) + \frac{1}{n} + \frac{1+\sqrt{k}}{1-\sqrt{k}}\Big\|x_n-x\Big\|. \end{aligned}$$

Thus, taking limits on both sides as $n \to \infty$, we have $d(x, Tx) = 0$. Since Tx is closed, $x \in Tx$.

Observe that for any given sequence $\{x_n\} \subseteq K$, the set

$$U^n := \Big\{y_n \in Tx_n : D^2(\{x_n\}, Tx_n) \leq \|x_n - y_n\|^2 + \frac{1}{n^2}\Big\},$$

is nonempty for each $n \in \mathbb{N}$ due to Lemma 4.1.3.

We now prove the following theorem.

Theorem 4.1.1 *Let K be a nonempty, closed, convex subset of a real Hilbert space H. Let $T : K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping such that $F(T) \neq \emptyset$. Assume $Tp = \{p\}$ $\forall p \in F(T)$. Define a sequence $\{x_n\}$ by the algorithm*

$$\begin{cases} x_0 \in K \text{ chosen arbitarily,} \\ x_{n+1} = (1-\lambda)x_n + \lambda y_n \\ y_n \in U^n := \Big\{z_n \in Tx_n : D^2(\{x_n\}, Tx_n) \leq \|x_n - z_n\|^2 + \frac{1}{n^2}\Big\}, \\ \lambda \in (0, 1-k), \end{cases}$$

Then, $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

Proof: Let $p \in F(T)$. Then, using Lemma4.1.1, (d) and (e), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 = \|(1-\lambda)(x_n - p) + \lambda(y_n - p)\|^2 \\
&= (1-\lambda)\|x_n - p\|^2 + \lambda\|y_n - p\|^2 - \lambda(1-\lambda)\|x_n - y_n\|^2 \\
&\leq (1-\lambda)\|x_n - p\|^2 + \lambda D^2(Tx_n, Tp) - \lambda(1-\lambda)\|x_n - y_n\|^2 \\
&\leq (1-\lambda)\|x_n - p\|^2 + \lambda\Big(\|x_n - p\|^2 + kD^2(x_n - Tx_n, 0)\Big) - \lambda(1-\lambda)\|x_n - y_n\|^2 \\
&= (1-\lambda)\|x_n - p\|^2 + \lambda\|x_n - p\|^2 + \lambda k D^2(\{x_n\}, Tx_n) - \lambda(1-\lambda)\|x_n - y_n\|^2 \\
&\leq (1-\lambda)\|x_n - p\|^2 + \lambda\|x_n - p\|^2 + \lambda k\Big(\|x_n - y_n\|^2 + \frac{1}{n^2}\Big) - \lambda(1-\lambda)\|x_n - y_n\|^2. \\
&= \|x_n - p\|^2 + \frac{\lambda k}{n^2} - \lambda(1-\lambda-k)\|x_n - y_n\|^2.
\end{aligned}$$

Thus,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + \frac{\lambda k}{n^2} - \lambda(1-\lambda-k)\|x_n - y_n\|^2. \tag{4.1.12}$$

By Lemma 2.1.2, the sequence $\Big\{\|x_n - p\|\Big\}$ has a limit and therefore, $\{x_n\}$ is bounded. Moreover, we have from inequality (4.1.12) that

$$\lambda(1-\lambda-k)\|x_n - y_n\|^2 \leq \|x_n - p\|^2 + \frac{\lambda k}{n^2} - \|x_{n+1} - p\|^2.$$

Taking $\limsup$ on both sides, we get that

$$\lambda(1-\lambda-k)\limsup_{n\to\infty}\|x_n - y_n\| \leq 0,$$

and therefore $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Since $d(x_n, Tx_n) \leq \|x_n - y_n\|$, it follows that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Remark 4.1.3 *Theorem (4.1.1) is quite interesting because it dealt with a much larger class of multi-valued mappings and yet did not face the problem of computing $z_n \in Tx_n$ such that $\|z_n - x^*\| = d(x^*, Tx_n)$ as it is, for example, in Sastry and Babu [98] and a host of other articles.*

Corollary 4.1.1 *Let K be a nonmempty, closed and convex subset of a real Hilbert space H, and let $T : K \to CB(K)$ be a generalized $k-$strictly pseudo-contractive multi-valued mapping, with $F(T) \neq \emptyset$ and assume $Tp = \{p\}$ for each $p \in F(T)$. Suppose that T is hemicompact. Then, the sequence $\{x_n\}$ defined in Theorem 4.1.1 converges strongly to a fixed point of T.*

Proof: By Theorem 4.1.1, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T is hemi-compact, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \to q$ as $n \to \infty$ and let $y_{n_k} \in Tx_{n_k}$ such that
$\|x_{n_k} - y_{n_k}\| \leq d(x_{n_k}, Tx_{n_k}) + \frac{1}{k}$.. Then

$$\begin{aligned} d(q, Tq) &\leq \|q - x_{n_k}\| + \|x_{n_k} - y_{n_k}\| + d(y_{n_k}, Tq) \\ &\leq \|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + \frac{1}{k} + D(Tx_{n_k}, Tq) \\ &\leq \|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + \frac{1}{k} + \frac{1+\sqrt{k}}{1-\sqrt{k}}\Big\|x_{n_k} - q\Big\|. \end{aligned}$$

Thus, taking limits on the righthand side as $k \to \infty$, we have $d(q, Tq) = 0$. Since Tq is closed, $q \in Tq$. Moreover, $x_{n_k} \to q$ as $n \to \infty$ gives $\|x_{n_k} - q\| \to 0$ as $n \to \infty$. Thus, using inequality 4.1.12 and Lemma 2.1.2, $\lim_{n\to\infty}\|x_n - q\| = 0$. Therefore $\{x_n\}$ converges strongly to a fixed point q of T as claimed.

Remark 4.1.4 *Observe that we did not assume that Tx is proximinal for each $x \in K$ neither did we require any continuity assumption on T nor any compactness assumption on K. Consequenctly, Corollary (4.1.1) is a significant improvement Chidume etal. [35].*

Corollary 4.1.2 *Let K be a nonmepty, compact and convex subset of a real Hilbert space H, and let $T : K \to CB(K)$ be a generalized $k-$strictly pseudo-contractive multi-valued mapping, with $F(T) \neq \emptyset$ and assume $Tp = \{p\}$ for each $p \in F(T)$. Then, the sequence $\{x_n\}$ defined in Theorem 4.1.1 converges strongly to a fixed point of T.*

Proof: Since K is compact, every map $T : K \to CB(K)$ is hemicompact. Thus, by Corollary 4.1.1, we have that $\{x_n\}$ converges strongly to some $p \in F(T)$.

Remark 4.1.5 *Our theorem and corollaries in this section improve and generalize convergence theorems for multi-valued nonexpansive mappings in [1], [35], [63, 64],[88, 98, 100], in the following sense:*

(i) *The class of mappings considered in this secction contains the class of multi-valued $k-$ strictly pseudocontractive mappings as special case, which itself properly contain the class of multi-valued nonexpansive maps.*

(ii) *The algorithm here is Krasnoselkii type, which is known to have a geometric order of convergence, and the theorem is proved for the much larger class of generalized multi-valued strict pseudocontractive mappings.*

(iii) *Inequality (4.1.1) of definition (4.1.1) is a more natural generalisation of the single-valued psudo-contractive mappings as given by inequality(2.1.3).*

(iv) *The condition that Tx be weakly closed for each $x \in K$ imposed in [35] is dispensed with here.*

We conclude, by saying that the condition $T(p) = \{p\}$ for all $p \in F(P)$, which is imposed in our theorem and corollaries is not crucial. Certainly our example (4.1.1) satisfies the condition since $T0 = \{0\}$ is the unique fixed point of T. However, some work in the litrature shows that this condition can be replaced with other conditions which does not assume that the multi-valued mapping is single-valued on the nonempty fixed point set. Details of this can be found, for example, in [101] and [112].

Remark: The main theorems and examples of this chapter appeared in

- C. E. Chidume and M.E. Okpala *On a general class of multi-valued strictly pseudocontractive mapping*, **Journal of Nonlinear Analysis and Optimization, Theory & Applications** Vol 5 No 2. (2014).

CHAPTER 5

Contribution on Countable Family of Multi-valued Strictly Pseudocontractive Mappings

In this Chapter, we discuss the extension of the main theorem of the last chapter to finite family and then countable family of generalized $k-$strictly pseudocontractive multivalued mappings in Hilbert spaces.

The extension of the main theorem of the last chapter to a finite family is quite straight forward. It makes use of the following identity valid in Hilbert spaces.

Lemma 5.0.5 *([36]) Let H be a real Hilbert space and let $\{x_i, i = 1, 2, ..., m\} \subseteq H$. For $\alpha_i \in (0, 1)$, $i = 1, 2, ..., m$ such that $\sum_{i=1}^{m} \alpha_i = 1$, the following identity holds:*

$$\Big\| \sum_{i=1}^{m} \alpha_i x_i \Big\|^2 = \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \sum_{1 \le i < j \le m} \alpha_i \alpha_j \|x_i - x_j\|^2,$$

5.1 Theorems for a Finite Family of Multi-valued Strictly Pseudocontractive Maps

Given a finite family $\{T_i, i = 1, ..., m\}$ of generalized k_i-strictly pseudocontractive multi-valued mappings and arbitrary sequence $\{x_n\} \subseteq K$, let

$$S_n^i := \Big\{ y_n^i \in T_i x_n : D^2(\{x_n\}, T_i x_n) \le \|x_n - y_n^i\|^2 + \frac{1}{n^2} \Big\}.$$

Certainly, S_n^i is not empty for each $n \ge 1$ by Lemma 4.1.3. We now state and prove our main theorem of this section.

Theorem 5.1.1 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. For $i = 1, 2, ..., m$, let $T_i : K \to CB(K)$ be a family of generalized k_i-strictly pseudocontractive multi-valued mappings with $k_i \in (0, 1)$. Suppose that $\cap_{i=1}^m F(T_i) \neq \emptyset$ and assume that for $p \in \cap_{i=1}^m F(T_i)$, $T_i p = \{p\}$. Define a sequence $\{x_n\}$ by $x_0 \in K$ arbitrary and,*

$$x_{n+1} = (\lambda_0)x_n + \sum_{i=1}^{m} \lambda_i y_n^i, \tag{5.1.1}$$

where $y_n^i \in S_n^i$, $\lambda_0 \in (k, 1)$, $\sum\limits_{i=0}^{m} \lambda_i = 1$, and $k := \max\{k_i, i = 1, 2, ..., m\}$. Then, for each $i = 1, 2, ..., m$, $\lim\limits_{n\to\infty} d(x_n, T_i x_n) = 0$.

Proof. 5.1.1 *Let $p \in \cap_{i=1}^m F(T_i)$. Then, using Lemma 5.0.5 together with Lemma 4.1.1, (d) and (e), we have*

$$\begin{aligned}
&\|x_{n+1} - p\|^2 = \|\lambda_0(x_n - p) + \sum_{i=1}^{m} \lambda_i(y_n^i - p)\|^2, \\
&= \lambda_0\|x_n - p\|^2 + \sum_{i=1}^{m} \lambda_i\|y_n^i - p\|^2 - \sum_{i=1}^{m} \lambda_0\lambda_i\|x_n - y_n^i\|^2 - \sum_{1\le i\le j\le m} \lambda_i\lambda_j\|y_n^i - y_n^j\|^2, \\
&\le \lambda_0\|x_n - p\|^2 + \sum_{i=1}^{m} \lambda_i D^2(T_i x_n, T_i p) - \sum_{i=1}^{m} \lambda_0\lambda_i\|x_n - y_n^i\|^2, \\
&\le \lambda_0\|x_n - p\|^2 + \sum_{i=1}^{m} \lambda_i(\|x_n - p\|^2 + k_i D^2(x_n - T_i x_n, \{0\})) - \sum_{i=1}^{m} \lambda_0\lambda_i\|x_n - y_n^i\|^2, \\
&= \sum_{i=0}^{m} \lambda_i\|x_n - p\|^2 + \sum_{i=1}^{m} \lambda_i k_i D^2(\{x_n\}, T_i x_n) - \sum_{i=1}^{m} \lambda_0\lambda_i\|x_n - y_n^i\|^2, \\
&\le \sum_{i=0}^{m} \lambda_i\|x_n - p\|^2 + \sum_{i=1}^{m} \lambda_i k(\|x_n - y_n^i\|^2 + \frac{1}{n^2}) - \sum_{i=1}^{m} \lambda_0\lambda_i\|x_n - y_n^i\|^2, \text{ since } y_n^i \in S_n^i, \\
&\le \|x_n - p\|^2 + \frac{k}{n^2} - \sum_{i=1}^{m} \lambda_i(\lambda_0 - k)\|x_n - y_n^i\|^2.
\end{aligned}$$

Therefore,

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 + \frac{k}{n^2} - \sum_{i=1}^{m} \lambda_i(\lambda_0 - k)\|x_n - y_n^i\|^2, \tag{5.1.2}$$

and then,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 + \frac{k}{n^2}. \tag{5.1.3}$$

Inequality (5.1.3) and Lemma 2.1.2 then give that the sequence $\Big\{\|x_n - p\|\Big\}$ *has a limit and therefore,* $\{x_n\}$ *is bounded. Moreover, we have from inequality (5.1.3) that*

$$\sum_{i=1}^{m} \lambda_i(\lambda_0 - k)\|x_n - y_n^i\|^2 \leq \frac{k}{n^2} + \|x_n - p\|^2 - \|x_{n+1} - p\|^2,$$

and so,

$$\lambda_i(\lambda_0 - k)\|x_n - y_n^i\|^2 \leq \frac{k}{n^2} + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0 \ \ , (as\ n \to \infty),$$

for each $i = 1, 2, ..., m$. *Thus, for each* $i = 1, 2, ..., m$, $\lim\limits_{n\to\infty} \|x_n - y_n^i\| = 0$ *and using the fact* $d(x_n, T_i x_n) \leq \|x_n - y_n^i\|$, *it follows that* $\lim\limits_{n\to\infty} d(x_n, T_i x_n) = 0$.

Thus the thorem is proved.

We also obtain the following corollary, namely:

Corollary 5.1.1 *Let* K *be nonempty, closed and convex subset of a real Hilbert space* H. *For* $i = 1, 2, ..., m$, *let* $T_i : K \to CB(K)$ *be a family of generalized* k_i-*strictly pseudocontractive multi-valued mapping with* $\cap_{i=1}^m F(T_i) \neq \emptyset$. *Assume that for* $p \in \cap_{i=1}^m F(T_i)$, $T_i p = \{p\}$ *and that* T_{i_0} *is hemicompact for some* i_0. *Then, the sequence* $\{x_n\}$ *defined in Theorem 4.1.1 converges strongly to a common fixed point of* $\{T_i, i = 1, 2, ..., m\}$.

Proof. 5.1.2 *By Theorem (4.1.1),* $\lim\limits_{n\to\infty} d(x_n, T_i x_n) = 0$ *for each* i *and in particular* $\lim\limits_{n\to\infty} d(x_n, T_{i_0} x_n) = 0$. *Since* T_{i_0} *is hemicompact, let* $\{x_{n_j}\}$ *be a subsequence of* $\{x_n\}$ *such that* $x_{n_j} \to q$ *as* $j \to \infty$. *For each* $i = 1, 2, ..., m$, *choose* $y_{n_j}^i \in T_i x_{n_j}$ *such that* $\|x_{n_j} - y_{n_j}^i\| \leq d(x_{n_j}, T_i x_{n_j}) + \frac{1}{j}$. *Then,*

$$\begin{aligned} d(q, T_i q) &\leq \|q - x_{n_j}\| + \|x_{n_j} - y_{n_j}^i\| + d(y_{n_j}^i, T_i q) \\ &\leq \|q - x_{n_j}\| + d(x_{n_j}, T_i x_{n_j}) + \frac{1}{j} + D(T_i x_{n_j}, T_i q) \\ &\leq \|q - x_{n_j}\| + d(x_{n_j}, T_i x_{n_j}) + \frac{1}{j} + \frac{1 + \sqrt{k}}{1 - \sqrt{k}}\Big\|x_{n_j} - q\Big\|. \end{aligned}$$

Thus, taking limits on the right hand side as $j \to \infty$, *we have* $d(q, T_i q) = 0$. *Since* $T_i q$ *is closed,* $q \in T_i q$ *for each* i *and therefore* $q \in \cap_{i=1}^m T_i q$. *Moreover,* $x_{n_j} \to q$ *as* $j \to \infty$ *gives* $\|x_{n_j} - q\| \to 0$ *as* $j \to \infty$. *Thus, using inequality (5.1.3) and Lemma (2.1.2),* $\lim\limits_{n\to\infty} \|x_n - q\| = 0$. *Therefore* $\{x_n\}$ *converges strongly to a common fixed point* q *of the maps* T_i, *as claimed.*

Remark 5.1.1 *Collorary 5.1.1 is a significant improvement and generalization of Theorem 2.4 of [36] in the following sense:*

(i) The theorem is proved for the much larger class of generalized $k-$strictly psuedo-contractive multi-valued mappings.

(ii) No continuity assuption is imposed on our maps.

(iii) Only one arbitrary map is required to be hemicompact.

(iv) The condition $\lambda_i \in (k, 1)$, for all i is replaced by the the weaker condition $\lambda_0 \in (k, 1)$.

(v) In the case where we have only one map, $m = 1$, we recover the main theorem of the last chapter.

Furthermore, that condition $y_n^i \in S_n^i$ is more readily applicable than requiring that Tx is proximinal and weakly closed for each x, and then, finding $y_n \in Tx_n$ such that $\|y_n - x_n\| = d(x_n, Tx_n)$ at each iterative step, as it is in [88] and in many others results.

5.2 Fixed point iteration for a countable family of multi-valued strictly pseudocontractive-type mappings

The extension of the theorem to a countably infinite family is not as straight forward as the finite family case. First, there is no known analogue of Lemma (5.0.5) for a countable family. Again, given a countably infite family $\{T_i\}$ of generalized k_i-strictly pseudocontractive multi-valued mappings, it may happent that $\sup\limits_{i\geq 1} k_i = 1$ and the techniques we have applied thus far may not be applicable anymore. Therefore in this section, we will assume that this is not the case. Precisely, we assume that $\sup\limits_{i\geq 1} k_i \in (0, 1)$. We present an example of such a countable family below. For that, we need the following lemma.

Lemma 5.2.1 *Let a, b, c be real numbers such that $0 \leq a \leq bc$, $c > 0$. Then*

$$(a - b)^2 \leq b^2 + \Big(\frac{c-2}{c}\Big)a^2. \tag{5.2.1}$$

Proof. 5.2.1 *The proof is trivially established as follows:*

$$\begin{aligned}
& 0 \le a \le bc, \quad c > 0 \\
\Rightarrow \quad & a^2 \le abc \\
\Rightarrow \quad & \frac{a^2}{c} \le ab \\
\Rightarrow \quad & -2ab \le -\frac{2a^2}{c} \\
\Rightarrow \quad & a^2 - 2ab + b^2 \le a^2 - \frac{2a^2}{c} + b^2 \\
\Rightarrow \quad & (a-b)^2 \le b^2 + \Big(\frac{c-2}{c}\Big)a^2
\end{aligned}$$

Remark 5.2.1 *If we take $c = 4$ in this lemma, we recover Lemma 4.1.4.*

Example 5.2.1 *Define a multi-valued mapping $T_i : l_2(\mathbb{R}) \to CB(l_2(\mathbb{R}))$ by*

$$T_i x := \begin{cases} \{y \in l_2 : \|x + y\| \le \alpha_i \|x\|\}, & x \neq 0 \\ \{0\}, & x = 0, \end{cases} \tag{5.2.2}$$

where $\alpha_i = \frac{7i}{3i-1}$, $i = 1, 2, ...,$. We obtain that

$$x - T_i x := \begin{cases} \{y \in l_2 : \|y - 2x\| \le \alpha_i \|x\|\}, & x \neq 0 \\ \{0\}, & x = 0 \end{cases}$$

Then, for arbitrary $x, y \in l_2(\mathbb{R})$, we compute as follows:

$$D(T_i x, T_i y) = \|x - y\| + \alpha_i \Big| \|x\| - \|y\| \Big|,$$

and

$$D(x - T_i x, y - T_i y) = 2\|x - y\| + \alpha_i \Big| \|x\| - \|y\| \Big|.$$

Now, set

$$a := D(x - T_i x, y - T_i y); \quad b := \|x - y\|.$$

Then, $a - b = D(T_i x, T_i y)$ and

$$\begin{aligned} a =& 2\|x - y\| + \alpha_i \Big| \|x\| - \|y\| \Big| \\ \le& (2 + \alpha_i)\|x - y\|. \end{aligned}$$

Now, for each i, set $2 + \alpha_i = c_i = c$ in Lemma (5.2.1) above. We obtain the identity $\frac{c_i - 2}{c_i} = \frac{\alpha_i}{2+\alpha_i}$, and by the same lemma, we have

$$D^2(T_i x, T_i y) \le \|x - y\|^2 + \frac{\alpha_i}{2 + \alpha_i} D(x - T_i x, y - T_i y).$$

Thus, each $T_i, i = 1, 2, ...,$ is a generalized κ_i-strictly pseudocontractive multi-valued mapping with $\kappa_i = \frac{\alpha_i}{2+\alpha_i} \in (0, 1)$ and each $\kappa_i \le \kappa := \frac{7}{13}$. Moreover, we have $p \in T_i p$ if and only if $p = 0$. Thus, for $p \in \cap_{i=1}^{\infty} F(T_i p)$, $T_i p = \{p\}$.

We present an analogue of Lemma 5.0.5 as follows:

Lemma 5.2.2 *Let H be a real Hilbert space and let $\{x_i\}_{i\in\mathbb{N}}$ be a bounded sequence in H. For $\delta_i \in (0,1)$, such that $\sum_{i=1}^{\infty} \delta_i = 1$, the following identity holds:*

$$\Big\| \sum_{i=1}^{\infty} \delta_i x_i \Big\|^2 = \sum_{i=1}^{\infty} \delta_i \|x_i\|^2 - \sum_{1\le i<j<\infty} \delta_i \delta_j \|x_i - x_j\|^2. \tag{5.2.3}$$

Proof. 5.2.2 *Define $\delta_i(n) := (1 - \sum_{n+1}^{\infty} \delta_i)^{-1}\delta_i$ for each n. It is easy to see that $\sum_{i=1}^{n} \delta_i(n) = 1$ and that $\delta_i(n) \to \delta_i$ as $n \to \infty$. Moreover, by Lemma 5.0.5, we obtain that*

$$\Big\| \sum_{i=1}^{n} \delta_i((n) x_i \Big\|^2 = \sum_{i=1}^{n} \delta_i(n) \|x_i\|^2 - \sum_{1\le i<j<\le n} \delta_i(n)\delta_j(n) \|x_i - x_j\|^2.$$

Since the inequality is true for all natural numbers n, we pass to the limit on both sides and obtain the identity (5.2.3) as proposed.

Next, given a countably infinite family $\{T_i\}_{i\ge 1}$ of generalized κ_i-strictly pseudocontractive multi-valued mappings and an arbitrary sequence $\{x_n\}$ of K, denote by Γ_n^i the set of inexact distal points of x_n with respect to the set $T_i x_n$, i.e

$$\Gamma_n^i := \Big\{ \zeta_n^i \in T_i x_n : D^2(\{x_n\}, T_i x_n) \le \|x_n - \zeta_n^i\|^2 + \frac{1}{n^2} \Big\}.$$

Obviously, Γ_n^i is closed, convex and nonempty for each $n \ge 1$ due to Lemma 4.1.1(d).

In particular, if $T_i x$ is assumed to be proximinal and bounded for each $x \in K$, then $T_i x_n$ has a vector, say η_n^i, of maximum norm, i.e.

$$\|x_n - \eta_n^i\| = \sup_{\zeta_n^i \in T_i x_n} \|x_n - \zeta_n^i\| =: D(\{x_n\}, T_i x_n).$$

In that case, it is certain that $\eta_n^i \in \Gamma_n^i$.

Based upon these analyses, we now prove our main theorem. We will assume henceforth that K is a nonempty, closed and convex subset of a real Hilbert space H. We now state a theorem for a countable family of the mapping.

Theorem 5.2.1 *Let $T_i : K \to CB(K)$ be a countably infinite family of generalized κ_i-strictly pseudocontractive multi-valued mappings such that for some $\kappa \in (0,1)$, $\kappa_i \in (0,\kappa]$. Assume that $\cap_{i=1}^{\infty}F(T_i) \neq \emptyset$ and for $p \in \cap_{i=1}^{\infty}F(T_i)$, $T_ip = \{p\}$. Define the sequence $\{x_n\}$ recursively by*

$$\begin{cases} x_0 \in K, \ \ arbitrary, \\ \zeta_n^i \in \Gamma_n^i, \\ x_{n+1} = \delta_0 x_n + \sum\limits_{i=1}^{\infty} \delta_i \zeta_n^i, \\ \delta_0 \in (\kappa, 1), \ \ \sum_{i=0}^{\infty} \delta_i = 1. \end{cases} \tag{5.2.4}$$

Then, for each i, $\lim\limits_{n\to\infty} d(x_n, T_ix_n) = 0$.

Proof. 5.2.3 *We will first of all establish that the recursion formula $x_{n+1} := \delta_0 x_n + \sum\limits_{i=1}^{\infty} \delta_i \zeta_n^i$ in the algorithm (5.2.4) is well defined. Take $p \in \cap_{i=1}^{\infty}F(T_i)$ arbitrary. We have*

$$\begin{aligned} \|x_n - \zeta_n^i\| &\leq D(x_n, T_ix_n), \\ &= D(x_n + p, p + T_ix_n). \end{aligned}$$

Therefore, we obtain by Lemma 4.1.1(c) that

$$\begin{aligned} \|x_n - \zeta_n^i\| &\leq \|x_n - p\| + D(Tp, T_ix_n), \\ &\leq \|x_n - p\| + \frac{1+\sqrt{\kappa}}{1-\sqrt{\kappa}}\|x_n - p\|. \end{aligned}$$

As a matter of fact, we may apply the triangle inequality and take limits to obtain

$$\|\zeta_n^i\| \leq K_n := \|x_n\| + \frac{2}{1-\sqrt{\kappa}} \inf_{p\in F(T)} \|x_n - p\|.$$

It follows then that

$$\|x_{n+1}\| \leq \delta_0\|x_n\| + \sum_{i=1}^{\infty} \delta_i \|\zeta_n^i\|,$$

and therefore

$$\|x_{n+1}\| \leq \delta_0\|x_n\| + \sum_{i=1}^{\infty} \delta_i K_n \leq K_n.$$

which shows that x_{n+1} is well defined. We show the convergence of $\{x_n\}$ as follows:

$$\|x_{n+1}-p\|^2 = \|\delta_0(x_n-p)+\sum_{i=1}^{\infty}\delta_i(\zeta_n^i-p)\|^2$$

$$= \delta_0\|x_n-p\|^2+\sum_{i=1}^{\infty}\delta_i\|\zeta_n^i-p\|^2-\sum_{i=1}^{\infty}\delta_0\delta_i\|x_n-\zeta_n^i\|^2-\sum_{1\leq i\leq j\leq\infty}\delta_i\delta_j\|\zeta_n^i-\zeta_n^j\|^2$$

$$\leq \delta_0\|x_n-p\|^2+\sum_{i=1}^{\infty}\delta_i D^2(T_ix_n,Tp)-\sum_{i=1}^{\infty}\delta_0\delta_i\|x_n-\zeta_n^i\|^2$$

$$\leq \delta_0\|x_n-p\|^2+\sum_{i=1}^{\infty}\delta_i(\|x_n-p\|^2+\kappa_i D^2(\{0\},x_n-T_ix_n))-\sum_{i=1}^{\infty}\delta_0\delta_i\|x_n-\zeta_n^i\|^2$$

$$= \sum_{i=0}^{\infty}\delta_i\|x_n-p\|^2+\sum_{i=1}^{\infty}\delta_i\kappa_i D^2(\{x_n\},T_ix_n)-\sum_{i=1}^{\infty}\delta_0\delta_i\|x_n-\zeta_n^i\|^2$$

Since $\zeta_n^i\in\Gamma^i{}_n$, we obtain that

$$\|x_{n+1}-p\| \leq \sum_{i=0}^{\infty}\delta_i\|x_n-p\|^2+\sum_{i=1}^{\infty}\delta_i\kappa(\|x_n-\zeta_n^i\|^2+\frac{1}{n^2})-\sum_{i=1}^{\infty}\delta_0\delta_i\|x_n-\zeta_n^i\|^2$$
$$\leq \|x_n-p\|^2+\frac{\kappa}{n^2}-\sum_{i=1}^{\infty}\delta_i(\delta_0-k)(\|x_n-\zeta_n^i\|^2).$$

This is summarised as:

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2+\frac{\kappa}{n^2}-\sum_{i=1}^{\infty}\delta_i(\delta_0-\kappa)\|x_n-\zeta_n^i\|^2, \tag{5.2.5}$$

and therefore

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2+\frac{\kappa}{n^2}. \tag{5.2.6}$$

In accordance with Lemma 2.1.2, $\|x_n-p\|$ has a limit and thus $\{x_n\}$ is bounded. Also, from inequality (5.2.5), there holds:

$$\sum_{i=1}^{\infty}\delta_i(\delta_0-\kappa)\|x_n-\zeta_n^i\|^2 \leq \|x_n-p\|^2+\frac{\kappa}{n^2}-\|x_{n+1}-p\|^2$$

and so for each $i\geq 1$,

$$\delta_i(\delta_0-\kappa)\|x_n-\zeta_n^i\|^2 \leq \|x_n-p\|^2+\frac{\kappa}{n^2}-\|x_{n+1}-p\|^2, \to 0(as\ n\to\infty),$$

Taking limits on both sides as $n \to \infty$, we conclude that $\lim_{n\to\infty} \|x_n - \zeta_n^i\| = 0$. Using the fact that $d(x_n, T_i x_n) \le \|x_n - \zeta_n^i\|$, we get $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$.

Corollary 5.2.1 *Let $T_i : K \to CB(K)$ be a countably infinite family of generalized κ_i-strictly pseudocontractive multi-valued mappings such that for some $\kappa \in (0,1)$, $\kappa_i \in (0,\kappa]$. Assume that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and suppose that for $p \in \cap_{i=1}^{\infty} F(T_i)$, $T_i p = \{p\}$. Assume T_{i_0} is hemicompact for some $i_0 \in \mathbb{N}$. Then, the sequence $\{x_n\}$ defined by algorithm (5.2.4) converges strongly to a fixed point of T.*

Proof. 5.2.4 *We already have that $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ due to Theorem (5.2.1). The mapping T_{i_0} being hemicompact guarantees the existence of some subsequence, say $\{x_{n_k}\}$, of $\{x_n\}$ such that $x_{n_k} \to q$ as $k \to \infty$. Let $\zeta_{n_k}^i \in T_i x_{n_k}$ be such that $\|x_{n_k} - \zeta_{n_k}^i\| \le d(x_{n_k}, T_i x_{n_k}) + \frac{1}{k}$. We estimate that*

$$\begin{aligned} d(q, T_i q) &\le \|q - x_{n_k}\| + \|x_{n_k} - \zeta_{n_k}^i\| + d(\zeta_{n_k}^i, T_i q) \\ &\le \|q - x_{n_k}\| + d(x_{n_k}, T_i x_{n_k}) + \frac{1}{k} + D(T_i x_{n_k}, T_i q) \\ &\le \|q - x_{n_k}\| + d(x_{n_k}, T_i x_{n_k}) + \frac{1}{k} + \frac{1+\sqrt{\kappa}}{1-\sqrt{\kappa}} \Big\|x_{n_k} - q\Big\|. \end{aligned}$$

If we take limits on both sides when $k \to \infty$, we have $d(q, T_i q) = 0$. Using the fact that each $T_i q$ is closed, we obtain that $q \in T_i q$ for each i, and therefore conclude that $q \in \cap_{i=1}^{\infty} T_i q$. Moreover, $x_{n_k} \to q$ as $n \to \infty$ gives $\|x_{n_k} - q\| \to 0$ as $n \to \infty$. Thus, by Lemma (2.1.2) and inequality (5.2.6) ,we get $\lim_{n\to\infty} \|x_n - q\| = 0$. Thus $\{x_n\}$ converges strongly to a fixed point q of T as claimed.

Corollary 5.2.2 *Let $T_i : K \to CB(K)$ be a countably infinite family of generalized κ_i-strictly pseudocontractive multi-valued mapping, with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and assume that for $p \in \cap_{i=1}^{\infty} F(T_i)$, $T_i p = \{p\}$. Then, the sequence $\{x_n\}$ defined above converges strongly to a fixed point of T.*

Proof. 5.2.5 *Since K is compact, the mappings $T_i : K \to CB(K)$ is hemicompact. Thus, by Corollary 5.2.1, we have that $\{x_n\}$ converges strongly to some $p \in F(T)$.*

Remark 5.2.2 *In comparism with Theorem 2.4 of [36], Corollary 5.2.1 has these merits.*

(i) *We proved the theorem for a countably infinite family of a much larger class of mapping which is the generalized $k-$strictly psuedo-contractive multi-valued mappings.*

(ii) *We only needed just one of the maps to be hemicompact and not all of them.*

(iii) *We replaced the 'strong condition'* $\delta_i \in (k, 1)$ *by a weaker condition* $\delta_0 \in (k, 1)$.

(iv) *The condition* $\zeta_n^i \in \Gamma_n^i$ *is more readily applicable than requiring that* Tx *is proximinal and weakly closed for each* x, *and then, computing* $\zeta_n = P_{Tx_n}x_n$ *at each iterative step.*

Remark 5.2.3 *Our theorem and corollaries improve the convergence theorems for multi-valued nonexpansive mappings in [1], [35], [36], [40], [63], [85], [88], [98], [100], in the following sense:*

(i) *The class of mappings considered in this section contains the class of multi-valued* $k-$ *strictly pseudocontractive mappings as a special case, which itself properly contain the class of multi-valued nonexpansive maps.*

(ii) *The algorithm here is of Krasnoselkii type, which is known to have a geometric order of convergence.*

(iii) *The condition that* Tx *be weakly closed for each* $x \in K$ *as can be found, for example, in [35] and [36] is dispensed with here.*

Remark 5.2.4 *The main theorems of this chapter are also contents of the following journal articles:*

1. C. E. Chidume, M. E. Okpala, A. U. Bello,and P. Ndambomve, *Convergence theorems for finite family of a general class of Multi-valued Strictly Pseudocontractive Mappings*, **Fixed Point Theory and Applications)** (2015) 2015:119 DOI 10.1186/s13663-015-0365-7.

2. M.E. Okpala, Fixed Point Iteration for a Countable Family of Multi-valued Strictly Pseudocontractive-type Mappings **(Submitted: (2015) SpringerPlus)**

CHAPTER 6

Contribution on Iterative Method for Multivalued Tempered Lipschitz Pseudocontractive mappings

6.1 Introduction

In this section, we will improve on the algorithm of Chidume and Okpala [39] and develop an iterative algorithm for a much larger class of a lipschitz pseudocontractive mapping. We will show that our iterative sequence is an approximating fixed point sequence for the mapping. Furthermore, under some mild assumption like *hemicompact* (or, in particular, compact), we will prove strong convergence of the sequence.

We will demonstrate with examples that our theorems have some edge over other results like those of Chidume et al. [35], Chidume and Ezeora [36], Panyanak [88], Song and Wang [100], among others. It also complement several known results in the literature.

Few iterative algorithms have been developed for single valued Lipschitz pseudocntractive-type mappings in real Hilbert spaces. However, till now, there is no known algorithm that have been developed for the Multivalued analogue. It is natural, therefore, for us to try to develop a theory for the multi-valued analogues of these mappings. This is the purpose of this Chapter. More precisely, we propose a theory for the class of *tempered Lipschitz* pseudocontractive mappings as a multi-valued analogue for the class of Lipschitchz pseudocontractive mappings.

Recall that a single-valued mapping $T : H \to H$ is said to be

- *pseudocontractive* if
$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(x - y) - (Tx - Ty)\|^2$$
for all $x, y \in H$ and

- *Lipschitzian* if there exists $L \geq 0$ such that
$$\|Tx - Ty\| \leq L\|x - y\|. \qquad (6.1.1)$$

We may employ the idea of Hausdorff metric to define the multivalued analogues of these mappings as follows:

Definition 6.1.1 *A multivalued mapping $T : H \to CB(H)$ is called*

- *pseudocontractive if*
$$D^2(Tx, Ty) \leq \|x - y\|^2 + D^2(Ax, Ay), \quad A := I - T. \qquad (6.1.2)$$

- *Lipschitz(Nadler [80]) if and only if there exists a fixed real number $L \geq 0$ such that*
$$D(Tx, Ty) \leq L\|x - y\|, \quad \forall x, y \in H, \qquad (6.1.3)$$

Other possible way of defining a multi-valued pseudonctractive mapping which will also be a generalization of the single valued case abound. An example is
$$D^2(Tx, Ty) \leq \|x - y\|^2 + \|x - y - (u - v)\|^2, \ \ \forall u \in Tx, \ \ v \in Ty, \qquad (6.1.4)$$
which is equivalent to
$$D^2(Tx, Ty) \leq \|x - y\|^2 + \inf_{(u,v) \in Tx \times Ty} \|x - y - (u - v)\|^2. \qquad (6.1.5)$$
However, we have demonstrated in the previous chapters that method of definition is narrow and quite restrictive. But the definition by the inequality (6.1.2) is very natural and also allows us to accomodate a large class of mappings.

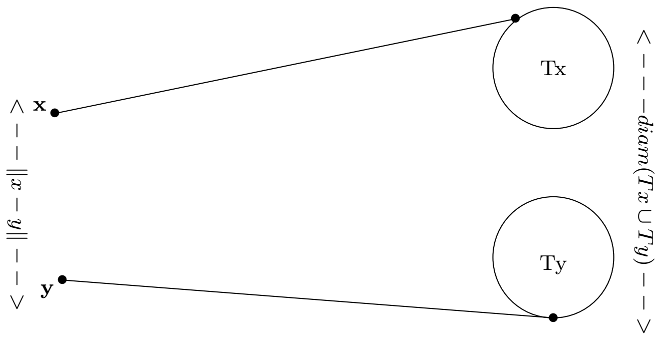
It was remarked in Nadler [80] that requiring a multi-valued mapping to be Lipschitz is a restriction on the mapping. In other words, Lipschitz condition for multi-valued mappings as given by inequality (6.1.3) is not convenient for use in application. There has been a search for some easily applicable Lipschitz conditions for multi-valued mappings, see for example, Elderstein [54].

6.2 Main Results

We propose a new type of Lipschitz condition which is also a natural generalization of the single valued Lipschitz as given by the inequality (6.1.1).

Definition 6.2.1 *A multi-valued mapping $T : H \to CB(H)$ is called tempered Lipschitz if there exists $L \geq 0$ such that*

$$diam(Tx \cup Ty) \leq L\|x - y\|, \ \ \forall x, y \in X. \tag{6.2.1}$$



A Tempered Lipschitz Mapping

Definition 6.2.2 *A multivalued mapping $T : H \to CB(H)$ is called a multivalued hemicontractive mapping if $F(T) \neq \emptyset$ and*

$$D^2(Tx, Tp) \leq \|x - p\|^2 + D^2(x, Tx), \ \ \forall x \in H, p \in F(T). \tag{6.2.2}$$

Remark 6.2.1 *The class of hemicontractive mapping, properly contains the class of psuedocontractive mappings with nonempty fixed point set.*

Lemma 6.2.1 *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T : K \to CB(K)$ be a multivalued mapping. Then the iterative algorithm given below,*

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\lambda)x_n + \lambda z_n, \\ z_n \in \Gamma^n := \{u_n \in Ty_n : D(x_n, Ty_n) \leq \|x_n - u_n\|^2 + \theta_n\} \\ y_n = (1-\lambda)x_n + \lambda w_n, \\ w_n \in \Pi^n := \{v_n \in Tx_n : D(x_n, Tx_n) \leq \|x_n - v_n\|^2 + \theta_n\} \\ \theta_n \geq 0, \sum\limits_{n=1}^{\infty} \theta_n < \infty \end{cases} \tag{6.2.3}$$

is well defined.

This is established from Lemma 4.1.1(d) by the fact that $D^2(x_n, Ty_n) = \sup_{u_n \in Ty_n} \|x_n - u_n\|^2$. Thus for any θ_n positive, we can always find a u_n such that $D^2(x_n, Ty_n) \leq \|x_n - u_n\|^2 + \theta_n$.The same is true for Π^n. Thus Γ^n and Π^n are nonempty and the algorithm is well defined.

We now prove the following theorems.

Theorem 6.2.1 *Let H be a real Hilbert space, $K \subseteq H$ be a nonempty, closed and convex. Let $T : K \to CB(K)$ be a tempered Lipschitz hemicontractive mapping. Let $\{x_n\}$ be a sequence defined by the algorithm (6.2.3). Suppose that $\lambda \in (0, L^{-2}[\sqrt{1+L^2} - 1])$. Then, for each $p \in F(T)$, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\lim_{n\to\infty} D(x_n, Tx_n) = 0$.*

Proof. 6.2.1 *Let $p \in F(T)$. We have the following inequality*

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\lambda)(x_n - p) + \lambda(z_n - p)\|^2,\\ &= (1-\lambda)\|x_n - p\|^2 + \lambda\|z_n - p\|^2 - \lambda(1-\lambda)\|x_n - z_n\|^2,\\ &\leq (1-\lambda)\|x_n - p\|^2 + \lambda D^2(Ty_n, Tp) - \lambda(1-\lambda)\|x_n - z_n\|^2.\end{aligned}$$

Using the hemicontractive property of T, we obtain the following:

$$\begin{aligned}D^2(Ty_n, Tp) &\leq \|y_n - p\|^2 + D^2(y_n, Ty_n),\\ &\leq \|y_n - p\|^2 + \|y_n - z_n\|^2 + \theta_n,\\ &\leq (1-\lambda)\|x_n - p\|^2 + \lambda\|w_n - p\|^2 - \lambda(1-\lambda)\|w_n - x_n\|^2\\ &+ \|y - z_n\|^2 + \theta_n\end{aligned}$$

Moreover, since T is tempered Lipschitz, for any $w_n \in Tx_n$ and $z_n \in Ty_n$, we have

$$\|w_n - z_n\| \leq diam(Tx_n \cup Ty_n) \leq L\|x_n - y_n\|.$$

We therefore obtain the following chain of inequalities:

$$\begin{aligned}\|y_n - z_n\|^2 &= \|(1-\lambda)(x_n - z_n) + \lambda(w_n - z_n)\|^2,\\ &= (1-\lambda)\|x_n - z_n\|^2 + \lambda\|w_n - z_n\|^2 - \lambda(1-\lambda)\|x_n - w_n\|^2,\\ &\leq (1-\lambda)\|x_n - z_n\|^2 + \lambda L^2\|x_n - y_n\|^2 - \lambda(1-\lambda)\|x_n - w_n\|^2,\end{aligned}$$

and

$$\|w_n - p\|^2 \leq D^2(Tp, Tx_n) \leq \|x_n - p\|^2 + D^2(x_n, Tx_n).$$

Substituting these inequalities into the first inequality yields the following:

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\lambda)\|x_n-p\|^2+\lambda D^2(Tp,Ty_n)-\lambda(1-\lambda)\|x_n-z_n\|^2\\
&\le (1-\lambda)\|x_n-p\|^2+\lambda\Big[(1-\lambda)\|x_n-p\|^2+\lambda\|w_n-p\|^2\\
&+\|y_n-z_n\|^2+\theta_n-\lambda(1-\lambda)\|w_n-x_n\|^2\Big]-\lambda(1-\lambda)\|x_n-z_n\|^2\\
&\le (1-\lambda)\|x_n-p\|^2+\lambda\Big[(1-\lambda)\|x_n-p\|^2+\lambda\Big[\|x_n-p\|^2+D^2(x_n,Tx_n)\Big]\\
&+\Big[(1-\lambda)\|x_n-z_n\|^2+\lambda L^2\|x_n-y_n\|^2-\lambda(1-\lambda)\|x_n-w_n\|^2\Big]+\theta_n\\
&-\lambda(1-\lambda)\|x_n-w_n\|^2\Big]-\lambda(1-\lambda)\|x_n-z_n\|^2
\end{aligned}
$$

$$
\begin{aligned}
&=\|x_n-p\|^2+\lambda^2D^2(x_n,Tx_n)+\lambda(1-\lambda)\|x_n-z_n\|^2+\lambda^4L^2\|x_n-w_n\|^2\\
&-2\lambda(1-\lambda)\|x_n-w_n\|^2+\lambda\theta_n-\lambda(1-\lambda)\|x_n-z_n\|^2,\Big(y_n-x_n=-\lambda(x_n-w_n)\Big).\\
&\le \|x_n-p\|^2+\lambda^2D^2(x_n,Tx_n)+\lambda^4L^2D^2(x_n,Tx_n)\\
&-2\lambda^2(1-\lambda)\|x_n-w_n\|^2+\lambda\theta_n
\end{aligned}
$$

Since $w_n\in\Pi^n$, *we have that* $\|x_n-w_n\|^2\le D^2(x_n,Tx_n)\le\|x_n-w_n\|^2+\theta_n$ *and therefore*

$$-2\lambda(1-\lambda)\|x_n-w_n\|^2\le -2\lambda(1-\lambda)D^2(x_n,Tx_n)+2\lambda(1-\lambda)\theta.$$

This yields

$$\|x_{n+1}-p\|^2\le\|x_n-p\|^2-\lambda^2(1-2\lambda-\lambda^2L^2)D^2(x_n,Tx_n)+(3\lambda-2\lambda^2)\theta_n.$$

We may now conclude that the following inequalities hold:

$$\|x_{n+1}-p\|\le\|x_n-p\|+(3\lambda-2\lambda^2)\theta_n, \tag{6.2.4}$$

and

$$\lambda^2(1-2\lambda-\lambda^2L^2)D^2(x_n,Tx_n)\le\|x_n-p\|^2+(3\lambda-2\lambda^2)\theta_n-\|x_{n+1}-p\|^2. \tag{6.2.5}$$

Applying Lemma 2.1.2 on (6.2.4) gives us that

$$\lim_{n\to\infty}\|x_n-p\|$$

exists. Obviously, $1-2\lambda-\lambda^2L^2>0 \iff |\lambda+\frac{1}{L^2}|<L^{-2}\sqrt{L^2+1}$. *Moreover, since* $\lambda\in(0,L^{-2}[\sqrt{1+L^2}-1])$, *the inequality* $1-2\lambda-\lambda^2L^2>0$ *is satisfied. Taking limits on both sides of (6.2.5), we have*

$$\lim_{n\to\infty}D(x_n,Tx_n)=0,$$

and the theorem is proved.

We now prove the following corollaries

Corollary 6.2.1 *Let H be a real Hilbert space, $K \subseteq H$ be a nonempty, closed and convex. Let $T : K \to CB(K)$ be a tempered Lipschitz hemicontractive mapping. Assume that T is hemicompact. Let $\{x_n\}$ be a sequence defined by the algorithm* (6.2.3) *where $\lambda \in (0, L^{-2}[\sqrt{1+L^2} - 1])$. Then, $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. 6.2.2 *We have obtained from Theorem 6.2.1, that $\lim_{n\to\infty} D(x_n, Tx_n) = 0$ and therefore $\lim_{n\to\infty} d(x_n, Tx_n)$. Since T is hemicompact, we have a subsequence say $\{x_{n_k}\}$ of $\{x_n\}$, which converges strongly to some $q \in K$. Using the inequality (6.2.4) we obtain that $x_{n_k} \to q$ as $k \to \infty$. Then for any $w_k \in Tx_{n_k}$, we have by Lemma 4.1.1* (c) *that*

$$\begin{aligned} D(q, Tq) &\le \|q - x_{n_k}\| + \|x_{n_k} - w_{n_k}\| + D(w_{n_k}, Tq) \\ &\le \|q - x_{n_k}\| + D(x_{n_k}, Tx_{n_k}) + L\|x_{n_k} - q\| \to 0, \ \text{as } k \to \infty. \end{aligned}$$

Thus $q \in F(T)$. By lemma (2.1.2), we conclude that $\{x_n\}$ converges strongly to $q \in F(T)$.

Definition 6.2.3 (Beer and Concilio [15]) *A metric space (X, d) is called boundedly compact provided each closed and bounded subset of X is compact.*

Elsewhere, boundedly compact spaces are called $m-$compact spaces. Certainly every finite dimensional metric spaces is boundedly compact. This notion of compactness can be extended to sets.

Definition 6.2.4 *A set K is boundedly compact if each bounded sequence in K has a subsequence that converges to a point in K.*

Every boundedly compact set K in a real Hilbert space has a vector x_0 of maximum norm. Indeed, let m denote the supremum of the norms of the vectors in K. Choose a sequence (x_n) in K such that limit of $\|x_n\| = m$. Then the limit of the subsequence of (x_n) that converges to a point $x_0 \in K$ has the property that $\|x_0\| = m$. That is, x_0 has maximal norm.

If we assume that the set K is boundedly compact, we may dispense with the sets Γ^n, Π^n and then θ_n and get the following theorem.

Corollary 6.2.2 *Let H be a real Hilbert space, $K \subseteq H$ be a nonempty, closed, convex and boundedly compact. Let $T : K \to CB(K)$ be a tempered Lipschitz* *hemicontractive mapping. Suppose that* $\lambda \in (0, L^{-2}[\sqrt{1+L^2}-1])$*. Define a sequence* $\{x_n\}$ *iteratively by*

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\lambda)x_n + \lambda z_n^*, \\ y_n = (1-\lambda)x_n + \lambda w_n^*, \end{cases} \tag{6.2.6}$$

where $z_n^* \in Ty_n : \|x_n - z_n^*\| = D(x_n, Ty_n)$, $w_n^* \in Tx_n : \|x_n - w_n^*\| = D(x_n, Tx_n)$ *and* $\lambda \in (0, L^{-2}[\sqrt{1+L^2}-1])$*. Then the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Proof. 6.2.3 *Since the set* K *is boundedly compact, we have by Lemma 4.1.1*(d)*, that* z_n^* *and* w_n^* *both exist and are well defined. Moreover for any* $\theta_n \geq 0$*,* $z_n^* \in \Gamma^n$ *and* $w_n^* \in \Pi^n$*. Thus by Theorem 3.2.1, we obtain* $\lim_{n\to\infty} D(x_n, Tx_n) = 0$*. Moreover,* T *is hemi-compact. We have shown already in Corrolary 6.2.1 that* $q \in F(T)$ *and the theorem is proved.*

Remark 6.2.2 *Our theorem and corollaries improve and generalize convergence theorems for multi-valued nonexpansive mappings in [1], [35], [36], [39], [63], [85], [88], [98], [100], in the following sense:*

(i) *The class of mappings(tempered Lipschitz pseudocontactive mappings) considered in this section properly contains the class of multi-valued* $k-$ *strictly pseudocontractive mappings as a special case, and the later properly contain the class of multi-valued nonexpansive maps.It aslo contain the class of single valued Lipschitz pseudocontractive mappings.*

(ii) *The algorithm presented here is of Krasnoselkii type, which is known to have a geometric rate of convergence and also computationally inexpensive.*

(iii) *The condition that* Tx *be weakly closed for each* $x \in K$ *as can be found, for example, in [35] and [36] and other similar assumptions is not neccesary as was shown here.*

Remark 6.2.3 *The main theorems of this chapter are contents of the following article:*

1. M.E.Okpala, *An Iterative Method for Multivalued Tempered Lipschitz Pseudocontractive Mappings* **(Submitted 2015)** Afrika Matimatika

CHAPTER 7

Iterative Method for Convex Optimization Problems in Real Lebesgue Spaces

7.1 Introduction

Let X be a real Banach space,$f : X \to \mathbb{R}$ be a convex functional, and $T : X \to X$ be a nonexpansive self mapping on a Hilbert space H. Given a (possibly nonlinear) monotone mapping $A : T(H) \to H$, the variational inequality problem $\text{VIP}(A, F(T))$ over $F(T)$ is stated as:

$$\text{Find} \quad x^* \in F(T) \text{ such that } \langle y - x^*, Ax^* \rangle \geq 0, \quad \forall y \in F(T). \tag{7.1.1}$$

It is known that an element x^*, of a closed and convex set K, solves $VIP(A, Fix(P_K))$ if and only if $x^* = P_K(x^* - \lambda Ax^*)$ for some positive number λ.

This result is very important because it gives a basis for constructing iterative methods of approximating solutions of variational inequalities in Hilbert spaces.
The method previously used in solving the variational inequality problem in the late 1960's and later was the gradient projection method

$$x_{n+1} := P_K(x_n - \lambda_{n+1} \nabla f(x_n)), n \geq 1, \tag{7.1.2}$$

where λ_n is a suitably defined sequence of real numbers. This algorithm has been employed widely in applications because it has a good rate of convergence. Under suitable conditions, the sequence generated from this algorithm converges to a solution of the smooth convex optimization problem posed in the

Hilbert space H as:

$$(SCOP)\begin{cases}\text{Minimize } f : H \to \mathbb{R}, \text{(G-differentiable convex functional)}\\ \text{subject to } x \in K(\subseteq H)\text{(closed convex set).}\end{cases} \tag{7.1.3}$$

It is well know that x^* in K solves problem $(SCOP)$ if and only if it satisfies $\langle y - x^*, \nabla f(x^*)\rangle \geq 0, \forall y \in K$. The gradient projection method relies on the fact that for any closed convex subset K of a Hilbert space, $Fix(P_K) = K$ and $P_K : H \to K \subset H$ is a nonexpansive mapping with a nonempty fixed point set. However, the computation of the projection mapping P_K is difficult(except when the convex set K has simple structures) in application.
Based on the fact above, replacing the projection mapping P_K by an arbitrary nonexpansive mapping T, Yamada [110] introduced the steepest descent method given by

$$x_{n+1} := Tx_n - \lambda_{n+1}A(Tx_n), n \geq 1 \;\; (A := \nabla f). \tag{7.1.4}$$

This choice is because $y_n := Tx_n$ is generated by $y_{n+1} := T(y_n - \lambda_{n+1}\nabla f(y_n))$ (the gradient projection method) and for $x^* \in F(T)$, if $x^* = \lim x_n$, then $x^* = \lim y_n$. Thus the method can solve the problem $(SCOP)$ over $K = F(T)$ where T is a nonexpansive self map of H and $\{\lambda_n\}_{n=1}^{\infty}$ is suitably defined as stated below.

Theorem 7.1.1 (Hybrid steepest descent method for VIP(A,F(T)) *Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that a mapping $A : H \to H$ is $L-$Lipschitzian and $\eta-$ strongly monotone over $T(H)$. Then for any $x_0 \in H$, and $\mu \in (0, \frac{2\eta}{L^2})$, and any sequence satisfying*

$$(A1) \quad \lim \lambda_n = 0, \quad (A2) \sum_{n=1}^{\infty} \lambda_n = \infty, \quad and \;\; (A3) \quad \lim(\lambda_n - \lambda_{n+1})\lambda_{n+1}^{-2} = 0,$$

the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (7.1.4) converges strongly to the uniquely existing solution of the problem (7.1.1).

If $K = \cap_{n=1}^{r} Fix(T_i) \neq \emptyset$, where $\{T_i\}_{i=1}^{r}$ is a finite family of nonexpansive mappings, Yamada [110] studied the following algorithm

$$x_{n+1} = T_{[n]}x_n - \lambda_n\mu A(T_{[n]}x_n), \quad n \geq 1, \tag{7.1.5}$$

where $T_{[k]} = T_{k\,mod\,r}$, for $k \geq 1$ and the sequence $\{\lambda_n\}$ satisfies condition $(A1)$, $(A2)$, and $(A4)$: $\sum |\lambda_n - \lambda_{n+N}| < \infty$, and proved the strong convergence of $\{x_n\}$ to the unique solution of problem (7.1.1).

In the case where $A := \nabla f$, we obtain that $x_n \to x^* \in \arg \inf_{x \in F(T)} f(x)$, where $\nabla f : H \to H^*(= H)$ is the gradient of the convex functional f. However, most problems of practical significance are not posed in Hilbert spaces. But, for an arbitary real Banach space $X^* \neq X$. Besides, the exact expression of the duality mapping $J_q : X \to 2^{X^*}$ defined by $J_q(x) = \{x^* \in X^* : \|x\|^q = \|x^*\|^q = \langle x, x^* \rangle\}$ is known only in L_p spaces, $1 < p < \infty$. Therefore, it makes sense if we limit our study to L_p spaces, $1 < p < \infty$ where it is practically possible to compute the duality mapping.

Based on these assertions, an ideal extension of the problem to a Banach spaces and which would solve the problem (SCOP) in Banach spaces ought to be:

$$VIP^*(A, F(T)) \begin{cases} \text{Given a } \textit{nonexpansive} \text{ mapping } T : X \to X \text{ and a strongly} \\ \text{monotone } L\text{−Lipschitzian mapping } A : X \to X^*, \\ \text{find } x^* \in F(T) \ : \ \langle y - x^*, Ax^* \rangle \geq 0, \forall y \in F(T). \end{cases}$$

Considerable research efforts have been devoted to this problem in Hilbert spaces. For example, Xu and Kim [108] replaced the condition $(A3)$ by the less restrictive condition $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}} = 0$ and the condition $(A4)$ replaced by $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+r}}{\lambda_{n+r}} = 0$. The theorems of Xu and Kim [108] are improvements of the results of Yamada because the canonical choice sequence $\lambda_n = \frac{1}{n+1}$ is applicable there but it is not applicable in the result of Yamada [110] with condition $(C3)$. Other significant extensions of the theorems in Hilbert spaces can be found in Wang [104], Zeng and Yao [113], and Yamada et al. [111].
Some of the extensions of the theorem to the more general Banach spaces include Chidume *et al.* [43, 44], Sahu et al. [97].

Most of the extensions of the theorem of Yamada [110] to more general Banach spaces have focused on the problem

$$VIP(A, F(T)) \begin{cases} \text{Given a } \textit{nonexpansive} \text{ mapping } T : X \to X \text{ and a strongly} \\ \text{accretive } L\text{−Lipschitzian mapping } A : X \to X, \\ \text{find } x^* \in F(T) \ : \ \langle y - x^*, j_q(Ax^*) \rangle \geq 0, \forall y \in F(T). \end{cases}$$

This problem certainly has a lot of applications in evolution equation and other area of interest , but it does not neccesarily solve the optimization problem (SCOP). The problem (SCOP) arise in diverse disciplines as differential equations, convex optimization problems, time-optimal control, mathematical programming, demand problems, transport and network problems and so on. Details about these problems can be found, for example, in Kindelehrer and Stampacchia [66], Nagurney [81], and Noor [84].
Though there has been significant progress in solving problem VIP(A, F(T)), the successes achieved so far in using many geometric properties of spaces,

developed in the last two centuries or so, in approximating zeros of accretive-type operators in Banach spaces have not been acchieved in approximating zeros of monotone mappings. The major difficulty in any attempt in this direction is that A goes from E to E^* and most iterative algorithm involving x_n and Ax_n are not suitably defined.
In some case, attempts are made to construct the algorithm by introducing the duality mapping. However the exact values of the duality mapping is unknown outside L_p spaces, for $1 < p < \infty$. Thus, the sequence obtained thereby are usually not possible to implement for practical uses.

Motivated by Chidume et al. [45], we propose an algorithm for the problem $\mathrm{VIP}^*(A, F(T))$ in L_p, spaces for $1 < p < \infty$. Our theorems complements the results of Chidume *et al.* [43, 44], extends to L_p spaces the result of Yamada [110], and generalize the results of Chidume *et al.* [45].

Lemma 7.1.1 (Alber and Ryanzantseva [5], p.48) . *Let $X = L_p$, $p \geq 2$. Then, the inverse of the normalized duality mapping $j^{-1} : X^* \to X$ is Holder continuous on balls. i.e. $\forall u, v \in X^*$ such that $\|u\| \leq R$, $\|v\| \leq R$, then*

$$\|j^{-1}(u) - j^{-1}(v)\| \leq m_p \|u - v\|^{\frac{1}{p-1}},$$

where $m_p := (2^{p+1} L_p c_2^p)^{\frac{1}{p-1}} > 0$ for some $c_2 > 0$.

Definition 7.1.1 *Let E be a smooth real Banach space. The Lyapunov's function is a distance function $\phi : E \times E \to \mathbb{R}$ given by*

$$\phi(x, y) := \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2.$$

In recent times, this type of functional has been studied extensively by many authors including Alber [2], Alber and Guerre-Delabriere [4], Kamimura and Takahashi [65], Reich [93]. It has proved to be a very useful tool for the study of nonlinear mappings in the general Banach spaces.
It is known that on a Hilbert space H, there holds $\phi(x, y) = \|x - y\|^2$. Moreover, by the fact that the normalized duality mapping is the subdifferential of the functional defined by $f(x) = \frac{1}{2}\|x\|^2$, we have that $\phi(x, y) \geq 0$ for all x, y in E.

We define a parallel function $V : E \times E^* \to \mathbb{R}$ by

$$V(x, x^*) = \phi(x, j^{-1}(x^*)), \quad \forall x \in X, x^* \in X^*.$$

The functional is characterized by the following

Lemma 7.1.2 (Alber [2]) *Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Then,*

$$V(x, x^*) \leq V(x, x^* + y^*) - 2\langle j^{-1}x^* - x, y^*\rangle, \quad \forall\, x \in X, x^*, y^* \in X^*. \tag{7.1.6}$$

Following the terminology of Alber and Guerre-Delabriere [4], as can be found also in Chidume *et al.*,[37], we present the following definitions.

Definition 7.1.2 *Let K be a nonempty subset of a Banach space E. A map $T : K \to E$ is called:*

- *strongly suppresive on K if there exist $0 < q < 1$ such that*
$$\phi(Tx, Ty) \leq q\phi(x, y) \ \forall \ x, y \in K, \text{and} \tag{7.1.7}$$
- *nonextensive if*
$$\phi(Tx, Ty) \leq \phi(x, y) \ \forall \ x, y \in K. \tag{7.1.8}$$

It follow from inequalities (7.1.7) and (7.1.8) above that in Hilbert spaces, nonextensive mappings are precisely the nonexpansive mapping and the strongly suppresive mappings are the strict contractions.

7.2 Convergence Theorems in Real Lebesgue(L_p) Spaces

7.2.1 L_p spaces $1 < p \leq 2$.

Theorem 7.2.1 *Let $E = L_p, 1 < p \leq 2$, and $E^* = L_q$, $\frac{1}{p} + \frac{1}{q} = 1$. For $k = 1, 2, ..., N$, let $T_k : E \to E$ be a finite family of nonextensive mappings and $A : E \to E^*$ be an $\eta-$strongly monotone mapping which is also $L-$Lipschitzian. Assume that $S := A^{-1}(0) \cap \cap_{k=1}^{N} Fix(T_k) \neq \emptyset$. Then for arbitrary $x_1 \in E$, the sequence $\{x_n\}$ defined by*
$$x_{n+1} = j^{-1}\Big(j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)\Big), n \geq 1 \tag{7.2.1}$$
converges to the common solution of the problem VIP$^*(A, Fix(T_{[n]}))$, where $T_{[n]} := T_{n \, mod \, N}$, and $\lambda \in (0, \frac{\eta}{2L_1^2 L_2})$, L_1, L_2 the Lipschitz constants for the mappings A and j^{-1}, respectively.

Proof. 7.2.1 *Let $x^* \in S$. Then the sequence $\{x_n\}$ satisfies*

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= V(x^*, j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)) \\
&\leq V(x^*, j(T_{[n]}x_n)) - 2\lambda\Big\langle j^{-1}\Big(j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)\Big) - x^*, AT_{[n]}x_n - Ax^*\Big\rangle \\
&= \phi(x^*, T_{[n]}x_n) - 2\lambda\Big\langle T_{[n]}x_n - x^*, A(T_{[n]}x_n) - Ax^*\Big\rangle \\
&+ 2\lambda\Big\langle T_{[n]}x_n - x^*, A(T_{[n]}x_n) - Ax^*\Big\rangle \\
&- 2\lambda\Big\langle j^{-1}((j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)) - x^*, A(T_{[n]}x_n) - Ax^*\Big\rangle
\end{aligned}$$

$$= \phi(x^*, T_{[n]}x_n) - 2\lambda\Big\langle Tx_n - x^*, AT_{[n]}x_n - Ax^*\Big\rangle$$
$$- 2\lambda\Big\langle j^{-1}((j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)) - j^{-1}(j(T_{[n]}x_n)), AT_{[n]}x_n - Ax^*\Big\rangle.$$
$$\leq \phi(x^*, T_{[n]}x_n) - 2\lambda\eta\|T_{[n]}x_n - x^*\|^2$$
$$+ 2\lambda\|j^{-1}\Big((j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)\Big) - j^{-1}\Big(j(T_{[n]}x_n)\Big)\|\|AT_{[n]}x_n - Ax^*\|$$

By the $\eta-$strong monotonicity of A, we obtain that

$$\langle T_{[n]}x_n - x^*, AT_{[n]}x_n - Ax^*\rangle \geq \eta\|T_{[n]}x_n - x^*\|^2.$$

On the other hand, using the fact that each of the mappings T_k are nonextensive, we have that

$$\phi(x^*, T_{[n]}x_n) = (T_{[n]}x^*, T_{[n]}x_n) \leq \phi(x^*, x_n)$$

Therefore, substituting these relations into the chain of inequalities above, and using the fact that $\lambda \in (0, \frac{\eta}{2L_1^2 L_2})$, we obtain:

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq \phi(x^*, T_{[n]}x_n) - 2\lambda\eta\|T_{[n]}x_n - x^*\|^2\\ &+ 2\lambda^2 L_1^2 L_2\|T_{[n]}x_n - x^*\|^2\\ &\leq \phi(x^*, T_{[n]}x_n) - \lambda\eta\|T_{[n]}x_n - x^*\|^2\\ &\leq \phi(x^*, x_n) - \lambda\eta\|T_{[n]}x_n - x^*\|^2.\end{aligned}$$

Thus $\phi(x^*, x_n)$ is a monotone non-increasing sequence of real numbers that is bounded below, and therefore converges. On the otherhand the same inequality yields

$$\lambda\eta\|T_{[n]}x_n - x^*\|^2 \leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}). \tag{7.2.2}$$

Taking limits on both sides of the inequality (7.2.2), we have that $\lim_{n\to\infty} T_{[n]}x_n = x^*$. But we have that

$$\begin{aligned}\|x_{n+1} - T_{[n]}x_n\| &= \|j^{-1}(j(T_{[n]}x_n) - \lambda A(T_{[n]}x_n)) - j^{-1}(j(T_{[n]}x_n)\|\\ &\leq \lambda L_2\|A(T_{[n]}x_n - A(x^*)\|\\ &\leq \lambda L_2 L_1^2\|T_{[n]}x_n - x^*\| \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \|x_{n+1} - T_{[n]}x_n\| + \|T_{[n]}x_n - x^*\|\\ &\leq (1 + \lambda L_2 L_1^2)\|T_{[n]}x_n - x^*\|\end{aligned}$$

and thus $\lim_{n\to\infty} x_n = x^*$. The uniqueness of x^* follows from the strong monotonicity of the mapping A.
Iin the special case when $T_k = I$ the identity mapping for each k, we have the following result of Chidume *et al.* [45]:

Corollary 7.2.1 *Let $E = L_p, 1 < p \leq 2$, and $E^* = L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, and $A : E \to E^*$ be an $\eta-$strongly monotone mapping which is also $L-$Lipschitzian. Assume that $A^{-1}(0) \neq \emptyset$. Then for arbitrary $x_1 \in E$, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = j^{-1}\Big(j(x_n) - \lambda A(x_n)\Big), n \geq 1 \tag{7.2.3}$$

converges to the uniquely existing $x^* \in A^{-1}(0)$, where $\lambda \in (0, \frac{\eta}{2L_1^2 L_2})$, L_1, L_2 the Lipschitz constants for the mappings A and j^{-1}, respectively.

7.2.2 L_p spaces, $2 \leq p < \infty$.

Theorem 7.2.2 *Let $E = L_p$, $2 \leq p < \infty$ and $A : L_p \to L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, be an η-strongly monotone mapping which is also Lipschitzian. For $k = 1, 2, ..., N$, let $T_k : L_p \to L_p$ be a finite family of nonextensive mappings. Assume that $S := A^{-1}(0) \cap \cap_{k=1}^{N} Fix(T_k) \neq \emptyset$. Then for arbitrary $x_1 \in E$, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = j^{-1}\Big(j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)\Big), n \geq 1 \tag{7.2.4}$$

converges strongly to the unique common solution of the problem $VIP^*(A, Fix(T_k))$, where $T_{[n]} := T_{n\, mod\, N}$, and $\lambda_n \in \Big(0, \frac{\eta}{2L_1 L_2^{\frac{p}{p-1}}}\Big)$ satisfies $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$, $\sum\limits_{n=1}^{\infty} \lambda_n^{\frac{p}{p-1}} < \infty$, L_1, L_2 are the Lipschitz constants for the mappings A and j^{-1}, respectively.

Proof. 7.2.2 *Let $x^* \in S$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated satisfies*

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= V(x^*, j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)) \\
&\leq V(x^*, j(T_{[n]}x_n)) - 2\lambda_n \Big\langle j^{-1}\Big(j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)\Big) - x^*, Ax_n - Ax^* \Big\rangle \\
&= \phi(x^*, T_{[n]}x_n) - 2\lambda_n \Big\langle T_{[n]}x_n - x^*, A(T_{[n]}x_n) - Ax^* \Big\rangle \\
&+ 2\lambda_n \Big\langle T_{[n]}x_n - x^*, A(T_{[n]}x_n) - Ax^* \Big\rangle \\
&- 2\lambda_n \Big\langle j^{-1}\Big((j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)\Big) - x^*, A(T_{[n]}x_n) - Ax^* \Big\rangle \qquad = \phi(x^*, T_{[} \\
&+ 2\lambda_n \Big\langle j^{-1}(jT_{[n]}x_n) - j^{-1}\Big((j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)\Big), A(T_{[n]}x_n) - Ax^* \Big\rangle \\
&\leq \phi(x^*, T_{[n]}x_n) - 2\lambda_n \Big\langle T_{[n]}x_n - x^*, AT_{[n]}x_n - Ax^* \Big\rangle \\
&+ 2\lambda_n \|j^{-1}\Big((j(T_{[n]}x_n) - \lambda_n A(T_{[n]}x_n)\Big) - j^{-1}\Big(j(T_{[n]}x_n)\Big)\| \|AT_{[n]}x_n - Ax^*\|
\end{aligned}$$

By the strong monotonicity of A, and the Holder continuity of j^{-1}, we have

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq \phi(x^*, T_{[n]}x_n) - 2\lambda_n\eta\|T_{[n]}x_n - x^*\| \\ &+ 2\lambda_n^{\frac{p}{p-1}} m_p\|AT_{[n]}x_n - Ax^*\|^{\frac{p}{p-1}}, \\ &\leq \phi(x^*, T_{[n]}x_n) - 2\lambda_n\eta\|T_{[n]}x_n - x^*\| \\ &+ 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}}\|T_{[n]}x_n - x^*\|^{\frac{p}{p-1}}.\end{aligned}$$

Now, for $p \geq 2$, if $\|T_{[n]}x_n - x^*\| \geq 1$, then, $\|T_{[n]}x_n - x^*\|^{\frac{p}{p-1}} \leq \|T_{[n]}x_n - x^*\|^2$. So $2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}}\|T_{[n]}x_n - x^*\|^{\frac{p}{p-1}} \leq \lambda\eta\|T_{[n]}x_n - x^*\|^2$ Therefore, we have for this case

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) - \lambda_n\eta\|T_{[n]}x_n - x^*\|^2.$$

Otherwise $\|T_{[n]}x_n - x^*\| < 1$ and thus $2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}}\|T_{[n]}x_n - x^*\|^{\frac{p}{p-1}} \leq 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}}$. Thus, in any case,

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq \phi(x^*, T_{[n]}x_n) - \lambda_n\eta\|T_{[n]}x_n - x^*\|^2 \\ &+ 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}}. \\ &\leq \phi(x^*, T_{[n]}x_n) - \lambda_n\eta\phi(T_{[n]}x_n, x^*) \\ &+ 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}}.\end{aligned}$$

Using the fact that the mapping T_k are nonextensive we conclude that

$$\begin{aligned}\phi(x^*, x_{n+1}) &\leq (1 - \lambda_n\eta)\phi(x^*, T_{[n]}x_n) + 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}} \\ &\leq (1 - \lambda_n\eta)\phi(x^*, x_n) + 2\lambda_n^{\frac{p}{p-1}} m_p L_1^{\frac{p}{p-1}}\end{aligned}$$

.

Therefore we may conclude by Lemma (2.1.3) that $x_n \to x^*$.

Remark 7.2.1 *The canonical choice for the sequence λ_n is $\lambda_n := \frac{1}{n}$.*

Bibliography

[1] M. Abbas, S. H. Khan, A. R. Khan, and R. P. Agarwal, “Common fixed points of two multi-valued nonexpansive mappings by one step iterative scheme”, *Applied Mathematics Letters*, Vol. 24, no.2, (2011) pp. 97-102.

[2] Ya. Alber, “Metric and generalized projection operators in Banach spaces: properties and applications". *In Theory and Applications of Nonlinear Operators of Accretive and Monotone Type* (A. G. Kartsatos, Ed.), Marcel Dekker, New York (1996), pp. 15-50.

[3] Ya. I. Alber and S. Reich, “An iterative method for solving a class of nonlinear operator equations in Banach spaces", *Panamer. Math. J.*, 4(2) (1994), 39-54.

[4] Ya. Alber and S, Guerre-Delabriere, “On the projection methods for fixed point problems", Analysis (Munich), vol. 21 (2001), no. 1, pp. 17-39.

[5] Ya. Alber and I. Ryazantseva,“ Nonlinear Ill Posed Problems of Monotone Type", *Springer, London, UK*, (2006).

[6] Y. Alber Y. and Butnariu D. “Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces" *J. Optim. Theory Appl.* 92 (1997) 33-61.

[7] Y. Alber and S, Guerre-Delabriere, “Principles of weakly contractive maps in Hilbert spaces" in New results in Operator Theory and Its Applications, I. Gohberg and Y. Lyubich, Eds, vol 98 of *Operator Theory: Advances and Applications*, pp. 7-22, Birkhauser, Basel, Switzerland(1997).

[8] M.A. Alghamdi, N. Shahzad, and H. Zegeye, “Strong Convergence Theorems for Quasi-Bregman Nonexpansive Mappings in Reflexive Banach Spaces", *Journal of Applied Mathematics* Vol 2014,(2014) Article ID 58068

[9] K. Aoyama, F. Kohsaka, and W. Takahashi, “Proximal point methods for monotone operators in Banach spaces”, *Taiwanese Journal of Mathematics*, vol. 15, no. 1,(2011) pp. 259-281.

[10] J.P. Aubin, “Mathematical Methods of Game and Economic Theory", *Studies in Mathematics and its Applications 7*. Amsterdam-New York, North-Holland Publ. Co. (1979).

[11] J.P. Aubin, “Optima and Equilibria", *Springer-verlag,* Berlin, (1993).

[12] J.P Aubin and I. Ekeland, Applied Nonlinear Analysis, John Wiley and Son, (1984).

[13] R. Azencott, Y. Guivarch́,Y., and R.F. Gundy,(Hennequin, P. L., ed.), “ Ecole détéde probabilités" de *Saint-Flour VIII*-1978, Springer, (1980).

[14] V. Barbu and I. Precupanu, Convexity and Optimization in Banach Spaces, Noordhoff, (1978).

[15] G. Beer and A Concilio., “A generalization of boundedly compact metric spaces" *Commentationes Mathematicae Universitatis Carolinae*, Vol. 32 (1991), No. 2, 361–367

[16] Berinde, V., “Iterative approximation of fixed points for pseudocontractive operators” *Seminar on Fixed Point Theory*, Volume 3,(2002) 209-216

[17] L. M. Bregman, “A relaxation method of finding a common point of convex sets and its application to the solution of problems in convex programming", *USSR Computational Mathematics and Mathematical Physics*, vol. 7, pp. 620-631, (1967).

[18] H. Brezis, “Opérateurs maximaux monotones et semi-groupes de, contractions" *dans les espaces de Hilbert North-Holland*, (1973).

[19] L. E. J. Brouwer; *“Uber Abbildung von Mannigfaltigkeiten”,* Mathematische Annalen, 71, (1912), no. 4, 598.

[20] F.E. Browder, “Nonlinear operators and nonlinear equations of evolution", *Proc. Sympos. Pure Math.*, Vol. 18, Part 2, American Mathematical Society, (1976).

[21] F. E. Browder, “Convergence theorems for sequences of nonlinear operators in Banach spaces", *Math. Zeitschr.* 100 (1967), 201-225.

[22] F. E. Browder, “Convergence of approximates to Fixed points of nonexpansive nonlinear mappings in Banach spaces", *Arch. Ration. Mech. Ana*l. 24 (1967), 82-90.

[23] F. E. Browder and W. E. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space", *J. Math. Anal. Appl.*, 20 (1967), 197-228.

[24] F. E. Browder, "Nonlinear mappings of nonexpansive and accretive type in Banach Spaces", *Bull. Amer. Math. Soc.,* 73, (1967), 875-882.

[25] D.W. Boyd and J.S.W Wong, "On nonlinear contractions" *Proceeding of the American Mathematical Society*, vol 131, no 12, pp. 3647-3656, 2003.

[26] C. Byrne, "Iterative oblique projection onto convex subsets and the split feasibility problem", *Inverse Problem*, 18 (2002), 441-453.

[27] C. Byrne,"A unified treatment of some iterative algorithms in signal processing and image reconstruction", *Inverse Problems*, vol. 20 no. 1 (2004), pp. 103-120.

[28] Y. Censor, T. Elfving, N. Kopf, T. Bortfeld, "The multiple-sets split feasibility problem and its applications", *Inverse Problem*, 21 (2005), 2071-2084.

[29] Y. Censor, T. Bortfeld, N. Martin, A. Trofimov, "A unified approach for inversion problem in intensity-modulated radiation therapy",*Phys. Med. Bio*l., 51 (2006), 2353-2365.

[30] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space", *Numerical Algorithms*, Vol. 8, No. 2 (1994), 221-239.

[31] Y. Censor and A. Segal, "The split common fixed-point problem for directed operators",*J. Convex Anal.* 16 (2009), 587-600.

[32] K. C. Chang, "The obstacle problem and partial differential equations with discontinuous nonlinearities", *Communications on Pure and Applied Mathematics*, vol. 33(1980), no. 2, pp. 117-146.

[33] C.E. Chidume, " Iterative approximation of fixed point of Lipschitz pseudocontractive maps" *Proceedings of the American Mathematical Society* **129**, (2001) 2245-2251, MR2002e:47071

[34] Charles Chidume, "Geometric properties of Banach spaces and nonlinear iterations" *Springer-Verlag London*, (2009).

[35] C. E. Chidume, C. O. Chidume, N. Djitte, and M. S.Minjibir, "Convergence theorems for fixed points of multi-valued strictly pseudocontractive mappings in Hilbert spaces", *Abstract and Applied Analysis*, (2013) Article ID 629468.

[36] C. E. Chidume and J.N. Ezeora, "Krasnoselskii-type algorithm for family of multi-valued strictly pseudocontractive mappings" , *Fixed Point Theory and Applications*, 2014, 2014:111

[37] C.E. Chidume, M. Khumalo, and H. Zegeye, "Generalized projection and approximation of fixed points of nonself maps", *Journal of Approximation Theory* 120 (2003) 242-252.

[38] C. E. Chidume, M. E. Okpala, A. U. Bello,and P. Ndambomve, "Convergence theorems for finite family of a general class of Multi-valued Strictly Pseudocontractive Mappings", *Journal of Fixed Point Theory and Applications*) (2015) 2015:119 DOI 10.1186/s13663-015-0365-7

[39] C. E. Chidume and M. E. Okpala, "On a general class of multi-valued strictly pseudocontractive mappings" ,*Journal of Nonlinear Analysis and Optimization : Theory & Applications* ,Vol5, No 2 (2015).

[40] C.E. Chidume and S.A. Mutangadura " An Example on Mann Iteration Method For Lipschitz Pseudocontractions" *Proceedings of the American Mathematical Societ*y Vol 129, No 8, pp.2359-2363 (2001).

[41] C. E. Chidume, P. Ndambomve, A. U. Bello, "The split equality fixed point problem for demi-contractive mappings",*Journal of Nonlinear Analysis and Optimization, Theory & Applications* (2014).

[42] C.E. Chidume and H. Zegeye, "Approximate fixed point sequences and Convergence theorems for Lipschitz pseudocontractive maps " *Proceedings of the American Mathematical Society* Vol 132, No 3, pp.831-840 (2003).

[43] C.E Chidume, C. O. Chidume, and B. Ali, "Approximation of fixed points of nonexpansive mappings and solutions of variational inequalities", *J. Ineq. Appl.* (2008), Art. ID 284345, 12 pages.

[44] C.E Chidume, C. O. Chidume, and B. Ali, "Convergence of Hybrid steepest descent method for variational inequalities in Banach spaces", *(Appl. Math.Comput.)*.(2008).

[45] C.E. Chidume, A.U. Bello, and B. Usman, "Krasnoselskii-Type Algorithm for Zeroes of Strongly Monotone Lipschitz Maps in Classical Banach Spaces" *Springerplus* (2015) 4:297 DOI 10.1186/s40064-015-1044-1

[46] I. Cioranescu, "Geometry of Banach spaces, duality mappings and nonlinear problems",*Kluwer Academic Publishers*, (1990).

[47] F. Clarke, and I. Ekeland,"Hamiltonian trajectories having prescribed minimal period",*Comm. Pure Appl. Math.* 33 (1980), pp. 103-116.

[48] P. Z. Daffer and H. Kaneko, " Fixed points of generalized contractive multi-valued mappings", *Journal of Mathematical Analysis and Applications*, vol. 192 no. 2 (1995), pp. 655-666.

[49] A. Damlamian, "Application de la dualité non convexe á un probléme non linéaire á frontiére libre (équilibre dún plasma confiné)", *C. R. Acad. Sci. Paris 28*6 (1978), pp. 153-155.

[50] K. Deimling "Multi-valued differential equations", vol. 1, *Walter de Gruyter & Co., Berlin, Germany*, (1992).

[51] S. Dhompongsa, W.A. Kirk, B. Panyanak, *"Nonexpansive set-valued mappings in metric and Banach spaces", J. Nonlinear Convex Anal.*, 8(2007), 35-45.

[52] D. Downing and W. A. Kirk; "Fixed point theorems for set-valued mappings in metric and Banach spaces", *Math. Japon*, 22 no. 1 (1977), 99-112.

[53] L. Erbe and W. Krawcewicz, *"Existence of solutions to boundary value problems for impulsive second order differential inclusions"*, *The Rocky Mountain Journal of Mathematics*, vol. 22 no.2(1992), pp. 519-539.

[54] M. Elderstein, "An extension of Banach's contraction principle" *proc. Amer. Math. Soc.* 12 (1961), 7-10.

[55] I. Ekeland, "Convexity Methods in Hamiltonian mechanics", *Springer*, (1990).

[56] Y. Fang, Lin Wang, Xue-Jiao Zi, *Strong and weak convergence theorems for a new split feasibility problem*, *International Mathematical Forum*, Vol. 8, (2013), No. 33, 1621-1627.

[57] J. Garcí a-Falset, E. Lorens-Fuster, and T. Suzuki, *"Fixed point theory for a class of generalized nonexpansive mappings"*, *Journal of Mathematical Analysis and Applications*, vol. 375 no. 1(2011), pp. 185-195.

[58] J. Geanakoplos, *"Nash and Walras equilibrium via Brouwer"*, *Economic Theory*, 21 (2003), 585-603.

[59] B. Halpern, "Fixed points of nonexpanding maps", *Bull. Amer. Math. Soc.* 73 (1967), 591-597.

[60] S.Ishikawa, *"Fixed Point by a New Iteration Method"*, *Proceedings of the American Mathematical Society* Vol 44, No 1, (1974).

[61] S. Kakutani, "A generalization of Brouwer's fixed point theorem", *Duke Math. J.* 8(3), 457-459 (1941).

[62] T. Kato,*" Nonlinear semigroups and evolution equations"*, *J. Math. Soc. Japan* 19 (1967), 508-520.

[63] S. H. Khan and I. Yildirim, "Fixed points of multi-valued nonexpansive mappings in Banach spaces", *Fixed Point Theory and Applications*, vol. 2012 article 73, (2012).

[64] S. H. Khan, I. Yildirim, and B. E. Rhoades, "A one-step iterative process for two multi-valued nonexpansive mappings in Banach spaces", *Computers & Mathematics with Applications*, vol. 61 no. 10(2011) pp. 3172-3178.

[65] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space", *SIAMJ. Optim.*, vol. 13 (2002), no. 3, pp. 938-945.

[66] D. Kinderlehrer, and G. Stampacchia, "An introduction to variational inequalities and their applications",*Academic press, Inc.* (1980).

[67] W.A. Kirk, "A fixed point thoerem for mappings which do not increase distance" *Amer. Math. Soc.* vol 72(1965) 1004-1006.

[68] Y. Kōmura, "Nonlinear semi-groups in Hilbert space",*J. Math. Soc. Japan,* 19 (1967), 493-507.

[69] M.A. Krasnosel'skii, "Two observations about the method of successive approximations", *Uspehi Math. Nauk* 10 (1955), 123-127.

[70] J.L. Lions, "Problémes aux limites dans les équations aux dérivés partielles", *Presses de l'Université de Montreal*, 1965

[71] P. E. Maing'e, "Approximation method for common fixed points of nonexpansive mappings in Hilbert spaces",*J. Math. Anal. Appl.* 325 (2007), 469-479.

[72] W. R. Mann, "Mean value methods in iteration", *Proceedings of American Mathematical Society* 4 (1953), no. 3, 506-510.

[73] J. T. Markin, "Continuous dependence of fixed point sets",*Proceedings of the American Mathematical Society*, vol. 38(1973), pp. 545-547.

[74] B. Martinet, *"Régularisation dínéquations variationnelles par approximations successives", Revue Francaise d'informatique et de Recherche operationelle*, vol. 4 (1970), pp. 154-159.

[75] S. Maruster and C. Popirlan, *On the Mann-type iteration and convex feasibility problem*, *J. Comput. Appl. Math.* 212 (2008), 390-396.

[76] G. J. Minty *"Monotone (nonlinear) operators in Hilbert space"*, *Duke Mathematical Journal*, vol. 29 (1962), pp. 341-346.

[77] J.J. Moreau, "Applications of convex analysis to the treatment of elasto-plastic systems, in Applications of methods of Functional Analysis to Problems in Mechanics", *Sympos. IUTAM/IMU (Germain, P. and Nayroles*, B., eds.), Springer, 1976.

[78] D.P. Morton, , *Stopping rules for a class of sampling-based stochastic programming algorithms Operation Research*, **46**, 5 (1998).

[79] A. Moudafi, Eman Al-Shemas, *Simultaneous iterative methods for split equality problem*, *Transactions on Mathematical Programming and Applications*, Vol. 1, No. 2 (2013), 1-11.

[80] S. B. Nadler Jr., "Multi-valued contraction mappings", *Pacific Journal of Mathematics*, vol. 30(1969), pp. 475- 488.

[81] A. Nagurney, "Mathematical models of transportation and networks", *Mathematical Models in Economics*, (2001).

[82] J. F. Nash, *"Equilibrium points in n-person games"*, *Proceedings of the National Academy of Sciences of the United States of America*, vol 36 no. 1 (1950), pp. 48-49.

[83] J. F. Nash, *"Non-coperative games, Annals of Mathematics"*, *Second series* vol 54(1951), pp. 286-295.

[84] M. A. Noor, "General variational inequalities and nonexpansive mappings", *J. Math. Anal. Appl.* 331 (2007) 810-822.

[85] E .U. Ofoedu and H. Zegeye, "Iterative algorithm for multi-valued pseudocontractive mappings in Banach spaces", *J. Math. Anal. Appl.*, vol 372 (2010) pp. 68-76.

[86] Z. Opial, "Weak convergence of the sequence of succesive approximations for nonexpansive mappings", *Bull. Am. Math. Soc.* 73(1967), 591-597

[87] Palais, R.S., *A simple proof of the Banach contraction principle*, *Journal of Fixed Point Theory and Applications*, **2**(2007).

[88] B. Panyanak, "Mann and Ishikawa iterative processes for multi-valued mappings in Banach spaces", *Computers& Mathematics with Applications*, vol. 54 no. 6(2007), pp. 872-877.

[89] S. Reich, “A weak convergence theorem for the alternating method with Bregman distances",*in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, pp. 313-318, Marcel Dekker, New York, NY, USA, 1996.

[90] S. Reich; “Strong convergence theorems for resolvents of accretive operators in Banach spaces", *J. Math. Anal. Appl.* 75 (1980), 287-292.

[91] S. Reich and A. J. Zaslavski, “Convergence of iterates of nonexpansive set-valued mappings in Set Valued Mappings with Applications in Nonlinear Analysis”, *Mathematical Analysis and Applications, Taylor and Francis, London, UK* vol. 4(2002), pp. 411-420.

[92] S. Reich, “Book Review: Geometry of Banach spaces, duality mappings and nonlinear problems", *Bull. Amer. Math.* Soc. 26 (1992), 367-370.

[93] S. Reich, “A weak convergence theorem for the alternating methods with Bergman distance",in:*A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in Lecture notes in pure and Appl. Math.*, vol. 178 (1996), Dekker, New York. pp. 313-318.

[94] S. Reich and A. J. Zaslavski, *“Generic existence of fixed points for set-valued mappings”*, *Set-Valued Analysis*, vol. 10 no. 4(2002), pp. 287-296.

[95] S. Reich and A. J. Zaslavski, *“Two results on fixed points of set-valued nonexpansive mappings”*, *Revue Roumaine de MathÂ ´ematiques Pures et AppliquÂ ´ees*, vol. 51, no. 1(2006), pp. 89-94.

[96] R. T. Rockafellar, *“On the maximality of sums of nonlinear monotone operators”*, *Transactions of the American Mathematical Society*, vol. 149 (1970), pp. 75-88.

[97] D. R. Sahu, N. C. Wong, and J. C. Yao “A Generalized Hybrid Steepest-Descent Method for Variational Inequalities in Banach Spaces", *Fixed point Theory and Applications*, (2011).

[98] K. P. R. Sastry and G. V. R. Babu, “Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point”, *Czechoslovak Mathematical Journal*, vol. 55, no. 4(2005), pp. 817-826.

[99] F. Schȯpfer ,T. Schuster and A.K. Louis “An iterative regularization method for the solution of the split feasibility problem in Banach spaces" *Inverse Problems* 24 (2008).

[100] Y. Song and H. Wang, “Erratum to; Mann and Ishikawa iterative processes for multi-valued mappings in Banach Spaces”*[Comput. Math. Appl.*

54 (2007), 872-877]*Computers & Mathematics With Applications* , vol. 55(2008), pp. 2999-3002.

[101] Y. Song and Y. J. Cho, *"Some notes on Ishikawa iteration for multi-valued mappings", Bulletin of the Korean Mathematical Society*, vol. 48 no. 3(2011), pp. 575-584.

[102] K. K. Tan and H. K. Xu, "Approximating Fixed Points of Nonexpansive mappings by the Ishikawa Iteration Process", *J. Math. Anal. Appl.* 178 no. 2(1993), 301-308.

[103] G.C. Ugwunnadi, B. Ali, I. Idris, and M. S Minjibir, "Strong convergence theorem for quasi-Bregman strictly pseudocontractive mappings and equilibrium problems in Banach spaces" *Fixed Point Theory and Applications 2014*, 2014:231

[104] L. Wang, "An iteration method for nonexpansive mappings in Hilbert spaces", *Fixed Point Theory Appl.*", (2007), Art 28619, 8.

[105] H. K. Xu, *Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces*, *Inverse Problems* 26 (2010) 105018.

[106] H. K. Xu, "Iterative algorithms for nonlinear operators",*Journal of London Mathematical Society* 66 (2002), 240-256.

[107] H. K. Xu, "Inequalities in Banach spaces with applications", *Nonlinear Anal.* 16 (1991), no. 12, 1127-1138

[108] H.K. Xu and T.H. Kim, "Convergence of hybrid steepests-descent methods for variational inequalities",*J. Optimization Theory & Appl.* 119 (2003), 185-201.

[109] Z.B Xu, and G.F. Roach, "Charateristic inequalities of uniformly convex and uniformly smooth Banach spaces". *Journal of Mathematical Analysis and Applications*, **157**, 189-210(1991).

[110] I. Yamada, "The Hybrid Steepest-Descent Method for Variational Inequality Problems over the Intersection of the Fixed-Point Sets of Nonexpansive Mappings", *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, Edited by D. Butnariu, Y. Censor, and S. Reich, North-Holland, Amsterdam, Holland, pp. 473-504, 2001.

[111] I. Yamada, N. Ogura and N. Shirakawa: "A numerically robust hybrid steepest descent method for the convexly constrained generalized inverse problems", *in Inverse Problems, Image Analysis, and Medical Imaging, Contemporary Mathematics*, 313,*Amer. Math. Soc.*, (2002).

[112] H. Zegeye, N. Shahzad, “Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings", *Nonlinear Analysis*, 70 (2009) 2707-2716.

[113] L.C. Zeng and J.C Yao, “Implicit iteration scheme with perturbed mapping for common fixed points of finite family of nonexpansive mappings", *Nonlinear Anal.* 64 (2006), 2507-2515.