CONTRIBUTIONS TO ITERATIVE ALGORITHMS FOR NONLINEAR EQUATIONS IN BANACH SPACES

SHEHU, YEKINI
B.Tech.(Hons.)(Lautech), M.Sc. (UNN)

CONTRIBUTIONS TO ITERATIVE ALGORITHMS FOR NONLINEAR EQUATIONS IN BANACH SPACES

Thesis

Presented in Partial Fulfilment of the
Requirements for the Degree of
Doctor of Philosophy
in Mathematics
of the
African University of Science and Technology
Abuja
Nigeria

By

SHEHU, YEKINI
B.Tech.(Hons.)(Lautech), M.Sc. (UNN)
I.D./70085

African University of Science and Technology

December 2012.

GRADUATE SCHOOL
AFRICAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
ABUJA, NIGERIA

CERTIFICATE OF APPROVAL

Ph.D. THESIS

This is to certify that the Ph.D. thesis of

Shehu Yekini

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics, this day, 19 December, 2012.

Thesis Committee:

Thesis Supervisor

Member

Member

Member

Member

PUBLICATIONS

1. C. E. Chidume and Y. Shehu, Iterative approximation of solutions of equations of Hammerstein type in certain Banach spaces, **Appl. Math. Comput.** (219) (2013), 5657-5667 (Elsevier Journal).

2. C. E. Chidume and Y. Shehu, Strong convergence theorem for approximation of solutions of equations of Hammerstein type, **Nonlinear Anal.** 75 (2012), 5664-5671 (Elsevier Journal).

3. C. E. Chidume and Y. Shehu, Approximation of solutions of generalized equations of Harmmerstein type, **Comp. Math. Appl.** 63 (2012), 966-974 (Elsevier Journal).

4. C. E. Chidume, C. O. Chidume and Y. Shehu, Strong convergence theorems for a Mann type iterative scheme for a countable family of Lipschitzian mappings, **J. Appl. Math. Comput.** 35 (2011), 251-261 (Springer Journal).

5. Y. Shehu, Strong convergence theorems for infinite family of relatively quasi nonexpansive mappings and systems of equilibrium problems, **Appl. Math. Comput.** (218) (2012), 5146-5156 (Elsevier Journal).

6. Y. Shehu, Fixed Point Solutions of Generalized Equilibrium Problems for Nonexpansive Mappings, **J. Comput. Appl. Math.** 234 (2010), 892-898 (Elsevier Journal).

7. Y. Shehu, An Iterative Method for Fixed Point Problems, Variational Inclusions and Generalized Equilibrium Problems, **Math. Comp. Model.** (54) (2011), 1394-1404 (Elsevier Journal).

8. Y. Shehu, An Iterative Method for nonexpansive semigroups, Variational Inclusions and Generalized Equilibrium Problems, **Math. Comp. Model.** (55) (2012), 1301-1314 (Elsevier Journal).

9. Y. Shehu, A new iterative scheme for a countable family of relatively nonexpansive mappings and an equilibrium problem in Banach spaces, **J. Glob Optim.** 54 (2012), 519-535 (Springer Journal).

10. Y. Shehu, Strong Convergence Theorems for infinite family of quasi-nonexpansive mappings and Generalized Equilibrium Problems and variational inequality problems, **Comp. Math. Appl.** 61 (2011), 357-366 (Elsevier Journal).

11. Y. Shehu, Strong convergence theorems for nonlinear mappings, variational inequality problems and system of generalized mixed equilibrium problems, **Math. Comp. Model.** (54) (2011), 2259-2276 (Elsevier Journal).

DEDICATION

To
my wife, Rachel, my son, Chidubem Oluwashina and my late father, Mr. Tijani Shehu Adeniran, who passed away on 3 December, 2012.

List of Figures and Symbols (Optional)

Acknowledgements

I am grateful to God Almighty, my rock, my fortress, my deliverer, my strength, my buckler, the horn of my salvation, and my high tower for His mercies, abundant love and protection over me and over those who have contributed directly or indirectly towards the success of this thesis and my Ph.D. programme.
I wish to express my sincere heartfelt appreciation to my supervisor and the Director of the Mathematics Institute, African University of Science and Technology, Abuja, Professor Charles Ejike Chidume, FAS, for his assistance (moral and financial), devotion, constant motivation, patience, kindness and understanding. I owe him much more than I could possibly express for the academic freedom he gave me which is extremely difficult to give to a student by his supervisor (I stand to be challenged!) For this he has done, I will ever be grateful to him. I thank him for what he has done, what he is doing and what he will still do for me. If I were to do Ph.D. programme again, I would gladly do it under him!
I am also grateful to Prof. F. I. Njoku and Prof. I. U. Asuzu for introducing me to Prof. C. E. Chidume. I thank Dr. Bashir Ali, Dr. Eric Ofoedu, Dr. Guy Degla and Dr. Djitte Ngalla, for their help in various ways and their positive contributions to my success in this programme. My thanks also go to Jeremiah Ezeora, Maaruf Shehu and other Ph.D. students at the Mathematics Institute, African University of Science and Technology, Abuja, for nice stay together and for sharing a lot of mathematical experiences. I wish to thank Dr. Mrs. Ifeoma Chidume, the wife of my supervisor for her motherly role in my life.
My sincere gratitude to the African University of Science and Technology, Abuja, for giving me scholarship to conduct this research work. I am grateful.
My appreciation goes to the entire staff members of Mathematics Department, University of Nigeria, Nsukka for their motivation and encouragement.

My special appreciation goes to my wife, Rachel Ihunanya for her love, patience and understanding. My gratitude also goes to all those who helped me in one way or the other toward the completion of this work whose names are not mentioned here.
May God Almighty bless you all.

Shehu Yekini
December, 2012.

ABSTRACT

- Let H be a real Hilbert space. Let $K, F : H \to H$ be bounded, continuous and monotone mappings. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in E defined iteratively from arbitrary $u_1,\ \ v_1 \in H$ by

$$\begin{cases} u_{n+1} = u_n - \beta_n(Fu_n - v_n) - \beta_n(u_n - u_1), \\ v_{n+1} = v_n - \beta_n(Kv_n + u_n) - \beta_n(v_n - v_1), \quad n \geq 1 \end{cases}$$

  where $\{\beta_n\}_{n=1}^{\infty}$ is a real sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \beta_n^2 < \infty$ and $\sum_{n=1}^{\infty} \beta_n = +\infty$. Then, the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*.

- Let H be a real Hilbert space. For each $i = 1, 2, ...m$, let $F_i,\ \ K_i : H \to H$ be bounded and monotone mappings. Let $\{u_n\}_{n=1}^{\infty},\ \ \{v_{i,n}\}_{n=1}^{\infty},\ \ i = 1, 2, \ldots, m$ be sequences in H defined iteratively from arbitrary $u_1,\ \ v_{i,1} \in H$ by

$$\begin{cases} u_{n+1} = u_n - \lambda_n\alpha_n\Big(u_n + \sum_{i=1}^{m} K_i v_{i,n}\Big) - \lambda_n\theta_n(u_n - u_1), \\ v_{i,n+1} = v_{i,n} - \lambda_n\alpha_n(F_i u_n - v_{i,n}) - \lambda_n\theta_n(v_{i,n} - v_{i,1}), \quad i = 1, 2, \ldots, m \end{cases}$$

  where $\{\lambda_n\}_{n=1}^{\infty},\ \ \{\alpha_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are real sequences in $(0,1)$ such that $\lambda_n = o(\theta_n),\ \ \alpha_n = o(\theta_n)$ and $\sum_{i=1}^{\infty} \lambda_n\theta_n = +\infty$. Suppose that $u + \sum_{i=1}^{m} K_i F_i u = 0$ has a solution in H. Then, there exist real constants $\varepsilon_0,\ \ \varepsilon_1 > 0$ and a set $\Omega \subset W$ such that if $\alpha_n \leq \varepsilon_0\theta_n$ and $\lambda_n \leq \varepsilon_1\theta_n, \forall n \geq n_0$, for some $n_0 \in \mathbb{N}$ and $w^* := (u^*, x_1^*, x_2^*, \ldots, x_m^*) \in \Omega$ (where $x_i^* = F_i u^*,\ \ i = 1, 2, \ldots, m$), the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*.

- Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm. Let K be a nonempty closed convex subset of E and $\{T_n\}_{n=1}^{\infty}$ be a sequence of L_n-Lipschitzian mappings of K into

itself with $L_n \geq 1$, $\sum\limits_{n=1}^{\infty}(L_n - 1) < \infty$. Let $\bigcap\limits_{n=1}^{\infty} F(T_n) \neq \emptyset$. For a fixed $\delta \in (0,1)$ and each $n \in \mathbb{N}$, define $S_n : K \to K$ by $S_n x := (1-\delta)x + \delta T_n x, \quad \forall x \in K$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K defined by $x_1 = u \in K$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n) S_n x_n,$$

for all $n \in \mathbb{N}$. Suppose that (K_{min}) $\bigcap\limits_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\lim\limits_{n\to\infty} ||T_{n+1}x_n - T_n x_n|| = 0$. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to some common fixed points of $\{T_n\}_{n=1}^{\infty}$.

- Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $T : H \to H$ be a nonexpansive mapping such that $\Omega := F(T) \cap I(A,M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction map with constant $\gamma \in (0,1)$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by $x_1 \in H$,

$$\begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in K, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) T\Big[\alpha_n f(x_n) + (1-\alpha_n) J_{M,\lambda}(u_n - \lambda A u_n)\Big], \end{cases}$$

  for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1] and $\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$ satisfying:
  (i) $0 < c \leq \beta_n \leq d < 1$,
  (ii) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum\limits_{n=1}^{\infty} \alpha_n = \infty$,
  (iii) $\lambda \in (0, 2\alpha]$,
  (iv) $0 < a \leq r_n \leq b < 2\mu, \quad \lim\limits_{n\to\infty} |r_{n+1} - r_n| = 0$,
  then $\{x_n\}_{n=1}^{\infty}$ converges strongly to z, where $z := P_\Omega f(z)$ and $P_\Omega f(z)$ is the metric projection of $f(z)$ onto Ω.

- Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $\Im := \{T(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Omega := F(\Im) \cap I(A,M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction mapping with a constant $\gamma \in (0,1)$. Let $\{t_n\} \subset (0,\infty)$ be a real sequence such that $\lim_{n\to\infty} t_n = \infty$.

Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by $x_1 \in H$,

$$\begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in K \\ w_n = J_{M,\lambda}(u_n - \lambda A u_n) \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\Big(\frac{1}{t_n}\int_0^{t_n} T(u)[\alpha_n f(x_n) + (1-\alpha_n)w_n]du\Big), \end{cases}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in (0,1) and $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ satisfying:
(i) $\lim\limits_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
(ii) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum\limits_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iii) $\lambda \in (0, 2\alpha]$,
(iv) $0 < a \leq r_n \leq b < 2\mu, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,
(v) $\lim\limits_{n\to\infty} \frac{|t_n - t_{n-1}|}{t_n} \frac{1}{\alpha_n(1-\beta_n)} = 0$,
then $\{x_n\}_{n=1}^{\infty}$ converges strongly to z, where $z := P_\Omega f(z)$ and $P_\Omega f(z)$ is the metric projection of $f(z)$ onto Ω.

- Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Let F be a bifunction from $C \times C$ satisfying $(A1) - (A4)$. Suppose $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively nonexpansive mappings of C into E such that $\Omega := (\cap_{n=0}^{\infty} F(T_n)) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\}, \ \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $u, \ u_0 \in E$,

$$\begin{cases} x_n = T_{r_n} u_n, \\ u_{n+1} = J^{-1}(\alpha_n J u + \beta_n J x_n + \gamma_n J T_n x_n), \quad n \geq 0, \end{cases}$$

with the conditions
(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum\limits_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < b \leq \beta_n \gamma_n \leq 1$;
(iii) $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$ satisfying $\liminf\limits_{n\to\infty} r_n > 0$.
Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_\Omega u$, where $\Pi_\Omega u$ is the generalized projection of u onto Ω.

- Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $B : C \to E^*$ is an operator satisfying $(B1) - (B3)$ and $\{T_n\}_{n=1}^{\infty}$ is an infinite family of relatively-quasi nonexpansive mappings of C into itself such that $F := VI(C, B) \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \cap \Big(\cap_{n=1}^{\infty} F(T_n)\Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in$

$C,\ \ C_1 = C,\ \ x_1 = \Pi_{C_1} x_0,$

$$\begin{cases} \upsilon_n = \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n) \\ y_n = J^{-1}(\alpha_n J\upsilon_n + (1-\alpha_n)JT_n\upsilon_n) \\ z_n = J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n) \\ u_n = T^{G_m}_{r_{m,n}} T^{G_{m-1}}_{r_{m-1,n}} ... T^{G_2}_{r_{2,n}} T^{G_1}_{r_{1,n}} z_n \\ C_{n+1} = \{w \in C_n : \phi(w,u_n) \le \phi(w,x_n) + \beta_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \ge 1, \end{cases}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\lim_{n\to\infty} \beta_n = 0$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [a,b]$ for some a,b with $0 < a < b < \frac{c^2\alpha}{2}$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_{k,n}\}_{n=1}^{\infty} \subset (0,\infty)$, $(k = 1,2,...,m)$ satisfying $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1,2,...,m)$. Suppose that for each bounded subset D of C, the ordered pair $(\{T_n\}, D)$ satisfies either condition AKTT or condition *AKTT. Let T be the mapping from C into E defined by $Tx := \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that T is closed and $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.

Contents

CHAPTER 1

General Introduction

1.1 Introduction

The contributions of this thesis fall within the general area of nonlinear functional analysis, an area with vast amount of applicability in recent years, as such becoming the object of an increasing amount of study. We devote our attention to *three* important topics within the area.

1. Approximation of solution of nonlinear equations of Hammerstein type.
2. Iterative algorithms for common fixed points of a family of mappings and,
3. Algorithms for common solutions of common fixed point problems for a family of nonlinear maps; variational inequality problems; and equilibrium problems.

1.2 Iterative algorithms for Hammerstein equations

A nonlinear integral equation of Hammerstein type (see, e.g., Hammerstein [102]) is one of the form

$$u(x) + \int_{\Omega} k(x,y)f(y,u(y))dy = h(x) \tag{1.2.1}$$

where dy is a σ-finite measure on the measure space Ω; the real kernel k is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear and h is a given function on Ω. If we now define an operator K by

$$Kv(x) = \int_{\Omega} k(x,y)v(y)dy; \quad x \in \Omega,$$

and the so-called *superposition* or *Nemytskii* operator F by $Fu(y) := f(y, u(y))$ then, the integral equation (1.2.1) can be put in operator theoretic form as follows:

$$u + KFu = 0, \tag{1.2.2}$$

where, without loss of generality, we have taken $h \equiv 0$.

Interest in equation (1.2.2) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's functions can, as a rule, be transformed into the form (1.2.2). Among these, we mention the problem of the forced oscillations of finite amplitude of a pendulum (see, e.g., Pascali and Sburlan [152], Chapter IV).

Example 1.2.1 *The amplitude of oscillation $v(t)$ is a solution of the problem*

$$\begin{cases} \frac{d^2v}{dt^2} + a^2 \sin v(t) = z(t), \quad t \in [0,1] \\ v(0) = v(1) = 0, \end{cases} \tag{1.2.3}$$

where the driving force $z(t)$ is periodical and odd. The constant $a \neq 0$ depends on the length of the pendulum and on gravity. Since the Green's function for the problem

$$v''(t) = 0, \quad v(0) = v(1) = 0,$$

is the triangular function

$$k(t,x) = \begin{cases} t(1-x), & 0 \le t \le x, \\ x(1-t), & x \le t \le 1, \end{cases}$$

problem (1.2.3) *is equivalent to the nonlinear integral equation*

$$v(t) = -\int_0^1 k(t,x)[z(x) - a^2 \sin v(x)]dx. \tag{1.2.4}$$

If

$$\int_0^1 k(t,x)z(x)dx = g(t) \quad \text{and} \quad v(t) + g(t) = u(t),$$

then (1.2.4) *can be written as the Hammerstein equation*

$$u(t) + \int_0^1 k(t,x)f(x,u(x))dx = 0,$$

where $f(x,u(x)) = a^2 \sin[u(x) - g(x)]$.

Equations of Hammerstein type play a crucial role in the theory of optimal control systems and in automation and network theory (see, e.g., Dolezale [90]). Several existence and uniqueness theorems have been proved for equations of the Hammerstein type (see, e.g., Brezis and Browder [20, 21, 22], Browder [29], Browder and De Figueiredo [33], Browder and Gupta [32], Chepanovich [47], De Figueiredo [91]).

Let H be a separable real Hilbert space and C be a closed subspace of H. For a given $f \in C$, consider the Hammertein equation

$$(I + KF)u = h \tag{1.2.5}$$

and its n^{th} Galerkin approximation given by

$$(I + K_nF_n)u_n = P^*h, \tag{1.2.6}$$

where $K_n = P_n^*KP_n : H \to C_n$ and $F_n = P_nFP_n^* : C_n \to H$, where the symbols have their usual meanings (see [152]). Under this setting, Brezis and Browder [22] proved the following approximation theorem.

Theorem 1.2.2 *Let H be a separable real Hilbert space. Let $K : H \to C$ be a bounded continuous monotone operator and $F : C \to H$ be an angle-bounded and weakly compact mapping. Then, for each $n \in \mathbb{N}$, the Galerkin approximation* (1.2.6) *admits a unique solution u_n in C_n and $\{u_n\}_{n=1}^{\infty}$ converges strongly in H to the unique solution $u \in C$ of the equation* (1.2.5).

Prior to 2001, the only known method for approximating solutions of nonlinear Hammerstein equations, as far as we know, was the Galerkin method (1.2.6) of Brezis and Browder. The difficulties associated with using Galerkin method are well known.

In [70], Chidume and Osilike proved the following convergence theorem.

Theorem 1.2.3 *Let E be a real Banach space with a uniformly convex dual E^*. and suppose that:*
(i) F is a nonlinear set-valued accretive map of E into itself with open domain D;
(ii) K is a linear single-valued accretive map of E into itself with domain $D(K)$ such that $Im(F) \subset D(K)$; K^{-1} exists and satisfies

$$\langle K^{-1}x - K^{-1}y, j(x - y)\rangle \geq \beta||x - y||^2$$

for all $x, y \in Im(K)$ and $\beta > 0$. Suppose also that for each $h \in Im(K)$ the equation $h \in x + KFx$ has a solution $x^* \in D$. Define the set-valued map S with domain D by $Sx = K^{-1}h - K^{-1}x - Fx + x,\ \ x \in D$. Let $\{c_n\}$ be a real sequence satisfying:

(iii) $0 \le c_n < 1 \forall n \ge 1$;
(iv) $\sum_{i=1}^{\infty} c_n = \infty$;
(v) $c_n b(c_n) < \infty$.
Then there exist a neighbourhood $B = B_d(x^*) \subset D$ of x^* and a real number $N_0 \ge 0$ such that for any $n \ge N_0$, and any initial guess $x_1 \in B$, the sequence $\{x_n\}$ generated from x_1 by

$$x_{n+1} = (1 - c_n)x_n + c_n\zeta_n, \quad \forall \zeta_n \in Sx_n,$$

remains in D and converges strongly to x^*.

We remark here that the recurrence formula used in Theorem 1.2.3 involves K^{-1} which is also assumed to be strongly monotone and this, apart from limiting the class of mappings to which such iterative scheme is applicable, is also not convenient in applications. Furthermore, convergence is guaranteed if the initial guess is chosen in a neighbourhood of the solution which is not known precisely. This is similar to a result of Bruck [37].

Due to further research, Chidume and Zegeye [75] introduced an iterative algorithm for solutions of nonlinear Hammerstein equations (1.2.2) which involves *Cartesian product* of Banach spaces and proved the following strong convergence theorem.

Theorem 1.2.4 *(Chidume and Zegeye [75]) Let X be a real q-uniformly smooth Banach space. Let $F, K : X \to X$ with $D(K) = F(X) = X$ be bounded maps such that the following conditions hold:*
(i) for each $u_1, u_2 \in X$, there exists a strictly increasing function $\phi_1 : [0, \infty) \to [0, \infty), \phi_1(0) = 0$ such that

$$\langle Fu_1 - Fu_2, j_q(u_1 - u_2)\rangle \ge \phi_1(||u_1 - u_2||)||u_1 - u_2||^{q-1};$$

(ii) for each $u_1, u_2 \in X$, there exists a strictly increasing function $\phi_2 : [0, \infty) \to [0, \infty), \phi_2(0) = 0$ such that

$$\langle Ku_1 - Ku_2, j_q(u_1 - u_2)\rangle \ge \phi_2(||u_1 - u_2||)||u_1 - u_2||^{q-1};$$

(iii) $\phi_i(t) \ge (d + r_i)t$ for all $t \in [0, \infty)$ and $i = 1, 2$ for some $r_i > 0$ and $d := q^{-1}(1 + d_q - c^{-1}2^{q-1})$.
Assume that $0 = u + KFu$ has solution u in X. Let $E := X \times X$ be with norm $||z||_E^q = ||u||_X^q + ||v||_X^q$ for $z = (u, v) \in E$ and define the map $T : E \to E$ by $Tz := T(u, v) = (Fu - v, u + Kv)$. Then there exists a real number $\gamma_0 > 0$ such that, if the real sequence $\{\alpha_n\} \subset [0, \gamma_0]$ satisfies the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{i=1}^{\infty} \alpha_n = \infty$, then for arbitrary $z_0 \in E$, the sequence $\{z_n\}$ defined by

$$z_{n+1} = z_n - \alpha_n T z_n, n \ge 0,$$

converges strongly to $z = [u, v]$ where $v = Fu$ and u is the unique solution of $0 = u + KFu$.

We observe that the operators K and F in Theorem 1.2.4 need not be defined on compact or angle-bounded subset of X and the iterative algorithm does not involve K^{-1}. The initial guess does not have to be chosen in a neighbourhood of the solution. Therefore, Theorem 1.2.4 improves the results of Chidume and Osilike. However, the draw back of Theorem 1.2.4 is that the result cannot be used by non-specialists because the iterative algorithm is in a Cartesian product.

In 2005, Chidume and Zegeye [77] obtained an auxiliary operator, defined in a real Hilbert space in terms of K and F that is monotone whenever K and F are, and constructed a *coupled iterative algorithm* that converges strongly to the solution of equation (1.2.2) in the original Banach space. In particular, they proved the following theorem for approximation of solution of a nonlinear integral equation of Hammerstein type in a real Hilbert space.

Theorem 1.2.5 *(Chidume and Zegeye, [77]) Let H be a real Hilbert space. Let $F, K : H \to H$ be bounded monotone mappings satisfying the range condition. Suppose the equation $0 = u + KFu$ has a solution in H. Let $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1)$ satisfying the following conditions:*
(i) $\lim_{n\to\infty} \theta_n = 0$;
(ii) $\sum \lambda_n\theta_n = \infty$, $\lim_{n\to\infty} \frac{\lambda_n}{\theta_n} = 0$;
(iii) $\lim_{n\to\infty} \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} = 0$;
Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in H defined iteratively from arbitrary $u_1, \ v_1 \in H$ by

$$\begin{cases} u_{n+1} = u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - w), \\ v_{n+1} = v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - w), \end{cases}$$

where $w \in H$ is arbitrary but fixed. Then, there exists $d > 0$ such that if $\lambda_n \leq d$ and $\frac{\lambda_n}{\theta_n} \leq d^2$ for all $n \geq 1$, then the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to u^* and v^* respectively, in H, where u^* is the solution of the equation $0 = u + KFu$ and $v^* = Fu^*$.

In 2009, Chidume and Djitte [61] studied a new iteration method introduced by Chidume and Zegeye [75] which does not involve K^{-1} and which converges strongly to a solution of (1.2.2) when K and F are *Lipschitz and ϕ-strongly accretive.* In particular, they proved the following theorem for approximation of solution of (1.2.2) in a real q-uniformly smooth Banach space.

Theorem 1.2.6 *(Chidume and Djitte [61]) For $q > 1$, let X be a real q-uniformly smooth Banach space and $F, K : X \to X$ be maps with $D(K) = F(X) = X$ such that the following conditions hold:*

(i) F is Lipschitzian with constant L_F and there exists a positive constant $\alpha > 0$ such that

$$\langle Fu_1 - Fu_2, j_q(u_1 - u_2)\rangle \geq \alpha ||u_1 - u_2||_X^2 \quad \forall u_1, u_2 \in X;$$

(ii) K is Lipschitzian with constant L_K and there exists a positive constant $\beta > 0$ such that

$$\langle Ku_1 - Ku_2, j_q(u_1 - u_2)\rangle \geq \beta ||u_1 - u_2||_X^2 \quad \forall u_1, u_2 \in X;$$

(iii) $(1 + c_q)(1 + d_q) \geq 2^q, \gamma > \frac{\gamma_q}{q}$, where

$$\gamma_q := \frac{[(1 + c_q)(1 + d_q) - 2^q]}{1 + c_q},$$

$L := \max(L_F, L_K), \gamma := \min(\alpha, \beta), \delta_q := d_q^2 L + d_q L + \frac{q}{q'} d_q L + d_q$ and q' is the Holder conjugate of q. Let $\{u_n\}$ and $\{v_n\}$ be sequences in X defined as follows:

$$u_0 \in X, u_{n+1} = u_n - \epsilon(Fu_n - v_n), n \geq 1;$$
$$v_0 \in X, v_{n+1} = v_n - \epsilon(Kv_n + u_n), n \geq 1;$$

with

$$0 < \epsilon < \min\left\{\frac{1}{\Lambda_q}, \left(\frac{q\gamma}{\lambda_0 \delta_q}\right)^{\frac{1}{q-1}}\right\}, \Lambda_q := \frac{1}{\frac{q\gamma(\lambda_0 - 1)}{\lambda_0} - \gamma_q},$$

where λ_0 is any number such that$\lambda_0 > \frac{q\gamma}{q\gamma - \gamma_q}$ and the symbols have their usual meanings. Assume that $u + KFu = 0$ has a solution u^*. Then, $\{u_n\}$ converges to u^* in X, $\{v_n\}$ converges to v^* in X and u^* is the unique solution of $u + KFu = 0$ with $v^* = Fu^*$.

Furthermore, it is observed that in L_p spaces, $1 < p < 2$, the condition $(1 + c_q)(1 + d_q) \geq 2^q$ does not necessarily hold, so, Chidume and Djitte [61] used a different tool to obtain the conclusions of Theorem 1.2.6 in these spaces. They proved the following theorem in L_p spaces, $1 < p < 2$.

Theorem 1.2.7 *(Chidume and Djitte [61]) Let $X = L_p(1 + \sqrt{1 - \gamma} < p \leq 2)$ and $F, K : X \to X$ be maps with $D(K) = F(X) = X$ such that the following conditions hold:*
(i) F is Lipschitzian with constant L_F and there exists a positive constant $\alpha > 0$ such that

$$\langle Fu_1 - Fu_2, j_q(u_1 - u_2)\rangle \geq \alpha ||u_1 - u_2||_X^2 \quad \forall u_1, u_2 \in X;$$

(ii) K is Lipschitzian with constant L_K and there exists a positive constant $\beta > 0$ such that

$$\langle Ku_1 - Ku_2, j_q(u_1 - u_2)\rangle \geq \beta ||u_1 - u_2||_X^2 \quad \forall u_1, u_2 \in X;$$

(iii) Assume that $\gamma := \min(\alpha, \beta) > p(2-p)2^q, \gamma > \frac{\gamma_q}{q}$, *where*

$$\gamma_q := \frac{[(1+c_q)(1+d_q) - 2^q]}{1+c_q},$$

$L := \max(L_F, L_K), \gamma := \min(\alpha, \beta), \delta_q := d_q^2 L + d_q L + \frac{q}{q'} d_q L + d_q$ *and* q' *is the Holder conjugate of* q. *Let* $\{u_n\}$ *and* $\{v_n\}$ *be sequences in* X *defined as follows:*

$$u_0 \in X, u_{n+1} = u_n - \epsilon(Fu_n - v_n), n \geq 1;$$
$$v_0 \in X, v_{n+1} = v_n - \epsilon(Kv_n + u_n), n \geq 1;$$

with

$$0 < \epsilon < \min\Big\{\frac{1}{\Lambda_p}, \Big(\frac{p\gamma}{\lambda_0 \delta_p}\Big)^{\frac{1}{p-1}}\Big\}, \Lambda_q := \frac{1}{\frac{p\gamma(\lambda_0 - 1)}{\lambda_0} - p^2(2-p)},$$

where λ_0 *is any number such that*$\lambda_0 > \frac{\gamma}{\gamma - p(2-p)}$ *and* δ_p *is as in Theorem 1.2.6. Assume that* $u + KFu = 0$ *has a solution* u^*. *Then,* $\{u_n\}$ *converges to* u^* *in* X, $\{v_n\}$ *converges to* v^* *in* X *and* u^* *is the unique solution of* $u + KFu = 0$ *with* $v^* = Fu^*$.

Furthermore, Chidume and Djitte [60] extended Theorem 1.2.6 and Theorem 1.2.7 and proved that an explicit coupled iteration process recently introduced by Chidume and Zegeye [75] which does not involve K^{-1} converges strongly to a solution of (1.2.2) when K and F are *bounded and strongly accretive.* In particular, they proved the following theorems.

Theorem 1.2.8 *(Chidume and Djitte, [60]) For* $q > 1$, *let* X *be a real* q*-uniformly smooth Banach space and* $F, K : X \to X$ *be maps with* $D(K) = F(X) = X$ *such that the following conditions hold:*
(i) F *is bounded and there exists a positive constant* $\alpha > 0$ *such that*

$$\langle Fu_1 - Fu_2, j_q(u_1 - u_2)\rangle \geq \alpha||u_1 - u_2||_X^2 \quad \forall u_1, u_2 \in X;$$

(ii) K *is bounded and there exists a positive constant* $\beta > 0$ *such that*

$$\langle Ku_1 - Ku_2, j_q(u_1 - u_2)\rangle \geq \beta||u_1 - u_2||_X^2 \quad \forall u_1, u_2 \in X;$$

(iii) $(1+c_q)(1+d_q) \geq 2^q, \gamma := \min(\alpha, \beta) > \frac{\delta_q}{q}$, *where* $\delta_q := \frac{[(1+c_q)(1+d_q)-2^q]}{1+c_q}$. *Let* $\{u_n\}$ *and* $\{v_n\}$ *be sequences in* X *defined iteratively from arbitrary points* $u_1, v_1 \in X$ *as follows:*

$$u_{n+1} = u_n - \alpha_n(Fu_n - v_n), n \geq 1;$$
$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n), n \geq 1;$$

where $\{\alpha_n\}$ *is a positive sequence satisfying* $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = +\infty$ *and* $\sum_{n=1}^{\infty} \alpha_n^q < \infty$. *Then, there exists a constant* $d_0 > 0$ *such that if* $0 < \alpha_n \leq d_0, \{u_n\}$ *converges to* u^* *in* X, $\{v_n\}$ *converges to* v^* *in* X *and* u^* *is the unique solution of* $u + KFu = 0$ *with* $v^* = Fu^*$.

Theorem 1.2.9 *(Chidume and Djitte [60]) Let* $X = L_p(1 + \sqrt{1-\gamma} < p \leq 2)$ *and* $F, K : X \to X$ *be maps with* $D(K) = F(X) = X$ *such that the following conditions hold:*
(i) F *is bounded and there exists a positive constant* $\alpha > 0$ *such that*

$$\langle Fu_1 - Fu_2, j_q(u_1 - u_2)\rangle \geq \alpha||u_1 - u_2||_X^2 \quad \forall u_1, u_2 \in X;$$

(ii) K *is bounded and there exists a positive constant* $\beta > 0$ *such that*

$$\langle Ku_1 - Ku_2, j_q(u_1 - u_2)\rangle \geq \beta||u_1 - u_2||_X^2 \quad \forall u_1, u_2 \in X;$$

Let $\{u_n\}$ *and* $\{v_n\}$ *be sequences in* X *defined iteratively from arbitrary points* $u_1, v_1 \in X$ *as follows:*

$$u_{n+1} = u_n - \alpha_n(Fu_n - v_n), n \geq 1;$$
$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n), n \geq 1;$$

where $\{\alpha_n\}$ *is a positive sequence satisfying* $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = +\infty$ *and* $\sum_{n=1}^{\infty} \alpha_n^q < \infty$. *Then, there exists a constant* $d_0 > 0$ *such that if* $0 < \alpha_n \leq d_0, \{u_n\}$ *converges to* u^* *in* X, $\{v_n\}$ *converges to* v^* *in* X *and* u^* *is the unique solution of* $u + KFu = 0$ *with* $v^* = Fu^*$.

Chidume and Djitte [60, 61] remarked that Theorem 1.2.6 and Theorem 1.2.8 hold in $X = L_p(2 \leq p \leq \gamma + \sqrt{\gamma^2 + 4})$ and Theorem 1.2.7 and Theorem 1.2.9 hold in $X = L_p(1 + \sqrt{1-\gamma} < p \leq 2)$. Therefore, Chidume and Djitte [60, 61] included these interesting open questions.

Open question 1. Do Theorem 1.2.7 and Theorem 1.2.9 hold in L_p spaces for all p such that $1 < p \leq 2$?
Open question 2. Do Theorem 1.2.6 and Theorem 1.2.8 hold in L_p spaces, $2 \leq p < \infty$?

Recently, Chidume and Ofoedu [68] extended the results of Chidume and Zegeye [77] (Theorem 1.2.5 above). They continued to study modifications of the coupled iterative algorithm introduced in [77]. They proved the following strong convergence theorem for approximation of solution of a nonlinear integral equation of Hammerstein type in 2-uniformly smooth real Banach space.

Theorem 1.2.10 *(Chidume and Ofoedu, [68]) Let E be a 2-uniformly smooth real Banach space. Let $F, K : E \to E$ be bounded and accretive mappings. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in E defined iteratively from arbitrary $u_1,\ \ v_1 \in E$ by*

$$\begin{cases} u_{n+1} = u_n - \lambda_n \alpha_n (Fu_n - v_n) - \lambda_n \theta_n (u_n - u_1), \\ v_{n+1} = v_n - \lambda_n \alpha_n (Kv_n + u_n) - \lambda_n \theta_n (v_n - v_1), \end{cases}$$

where $\{\lambda_n\}_{n=1}^{\infty},\ \ \{\alpha_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are real sequences in $(0,1)$ such that $\lambda_n = o(\theta_n),\ \ \alpha_n = o(\theta_n)$ and $\sum_{i=1}^{\infty} \lambda_n \theta_n = +\infty$. Suppose that $u + KFu = 0$ has a solution in E. Then, there exist real constants $\varepsilon_0,\ \ \varepsilon_1 > 0$ and a set $\Omega \subset W = E \times E$ such that if $\alpha_n \le \varepsilon_0 \theta_n$ and $\lambda_n \le \varepsilon_1 \theta_n, \forall n \ge n_0$, for some $n_0 \in \mathbb{N}$ and $w^* := (u^*, v^*) \in \Omega$ (where $v^* = Fu^*$), the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*.

A prototype of the parameters in Theorems 1.2.5 and 1.2.10 is $\lambda_n = (n+1)^{-a}$ and $\theta_n = (n+1)^{-b}$ with $0 < b < a$ and $a + b < 1$. We verify that these choices satisfy, in particular, condition (iii) of Theorem 1.2.5. In fact, using the fact that $(1+x)^p \le 1 + px$, for $x > -1$ and $0 < p < 1$, we have

$$\begin{aligned} 0 \ \le\ & \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n \theta_n} = \left[\left(1 + \frac{1}{n}\right)^b - 1\right](n+1)^{a+b} \\ \le\ & b\frac{(n+1)^{a+b}}{n} = b\frac{n+1}{n}\frac{1}{(n+1)^{1-(a+b)}} \to 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Quite recently, Chidume and Djitte [64] made a significant improvement on the Galerkin method of Brezis and Browder [22] and proved the following convergence theorem.

Theorem 1.2.11 *(Chidume and Djitte, [64]) Let H be a real Hilbert space. Let $F, K : H \to H$ be bounded monotone mappings satisfying the range condition. Suppose the equation $0 = u + KFu$ has a solution in H. Let $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ be real sequences in $[0,1)$ satisfying the following conditions:*
(i) $\lim\limits_{n\to\infty} \theta_n = 0$;
(ii) $\sum \lambda_n \theta_n = \infty, \quad \lim\limits_{n\to\infty} \frac{\lambda_n}{\theta_n} = 0$;
(iii) $\lim\limits_{n\to\infty} \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n \theta_n} = 0$;
Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in H defined iteratively from arbitrary $u_1,\ \ v_1 \in H$ by

$$\begin{cases} u_{n+1} = u_n - \lambda_n (Fu_n - v_n) - \lambda_n \theta_n (u_n - w), \\ v_{n+1} = v_n - \lambda_n (Kv_n + u_n) - \lambda_n \theta_n (v_n - w), \end{cases}$$

where $w \in H$ is arbitrary but fixed. Then, there exists $d_0 > 0$ such that if $\lambda_n \le d_0 \theta_n$ for all $n \ge n_0$ for some $n_0 \ge 1$, then the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*, a solution of the equation $0 = u + KFu$.

In Chapter 2 of this thesis, following the ideas of Chidume and Zegeye [77] and Chidume and Djitte [64], we construct a new *coupled explicit iterative* procedure that converges strongly to the solution of equation (1.2.2). The parameters of this new scheme studied here admit the canonical choice $\frac{1}{n}$, which is not the case in the theorems of Chidume and Zegeye [77], Chidume and Djitte [64] and Chidume and Ofoedu [68]. Furthermore, the 3(three) iteration parameters of Chidume and Zegeye [77] and 4(four) iteration parameters of Chidume and Ofoedu [68] have been reduced to 1(one) parameter satisfying the conditions: $\sum_{n=1}^{\infty}\beta_n^2 < \infty$ and $\sum_{n=1}^{\infty}\beta_n = +\infty$. This improves significantly the efficiency of the algorithm studied in Chidume and Zegeye [77], Chidume and Djitte [64] and Chidume and Ofoedu [68]. Our results improve on the results of Chidume and Djitte [60, 61, 64].

Furthermore, in Chapter 3, we extend the results of Chapter 2 to generalized equations of Hammerstein type, i.e., equations of the type

$$u + \sum_{i=1}^{m} K_i F_i u = 0 \tag{1.2.7}$$

in *real Hilbert spaces*. This equation is sometimes called equation of Urysohn type (see, e.g., [152], p. 203). Here, we introduce *a new explicit iteration scheme* which converges strongly to a solution of (1.2.7) when $K_i, \ \ F_i, \ \ i = 1, 2, \ldots, m$ are bounded and monotone assuming existence.

1.3 Algorithms for common fixed points

Definition 1.3.1 *Let E be a real linear space. A mapping $T : D(T) \subset E \to E$ is said to be nonexpansive if*

$$||Tx - Ty|| \leq ||x - y|| \ \ \forall x, y \in D(T).$$

A point $x \in D(T)$ is called *a fixed point* of T if $Tx = x$.

Markov [136] (see also Kakutani [111]) showed that if a *commuting family* of bounded *linear* transformations $T_\alpha, \alpha \in \Delta$ (Δ an arbitrary index set) of a normed linear space E into itself leaves some nonempty compact convex subset K of E invariant, then the family has at least one common fixed point. (The actual result of Markov is more general than this but this version is adequate for our purposes). Motivated by this result, De Marr studied the problem of the existence of a common fixed point for a family of *nonlinear* maps, and proved the following theorem.

Theorem 1.3.2 *(De Marr [88], p.1139) Let E be a Banach space and K be a nonempty compact convex subset of E. If F is a nonempty commuting family of nonexpansive mappings of K into itself, then the family F has a common fixed point in K.*

Browder proved the result of De Marr in a uniformly convex Banach space, requiring that K be only nonempty closed bounded and convex.

Theorem 1.3.3 *(Browder [23], Theorem 1) Let E be a uniformly convex Banach space, K a nonempty closed convex and bounded subset of E, $\{T_\lambda\}$ a commuting family of nonexpansive self-mappings of K. Then, the family $\{T_\lambda\}$ has a common fixed point in K.*

Bauschke [9] was the first to introduce a Halpern-type iterative process for approximating a common fixed point for a finite family of r nonexpansive self-mappings. He proved the following theorem.

Theorem 1.3.4 *(Bauschke [9], Theorem 3.1) Let K be a nonempty closed convex subset of a Hilbert space H and $T_1, T_2, \ldots, T_r$ be a finite family of non- expansive mappings of K into itself with $F := \cap_{i=1}^{r} F(T_i) \neq \emptyset$ and $F = F(T_r T_{r-1} \ldots T_1) = F(T_1 T_r \ldots T_2) = \ldots = F(T_{r-1} T_{r-2} \ldots T_1 T_r)$. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ which satisfies $C1: \lim\limits_{n\to\infty} \lambda_n = 0$; $C2: \sum_{n=0}^{\infty} \lambda_n = +\infty$ and $C3: \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+r}| < \infty$. Given points $u, x_0 \in K$; let $\{x_n\}$ be generated by*

$$x_{n+1} = \lambda_{n+1} u + (1 - \lambda_{n+1}) T_{n+1} x_n, n \geq 0, \tag{1.3.1}$$

where $T_n = T_{n \bmod r}$. Then, $\{x_n\}$ converges strongly to $P_F u$, where $P_F : H \to F$ is the metric projection.

Takahashi *et al.* [196] extended this result to uniformly convex Banach spaces. O'Hara *et al.* [149] proved a complementary result to that of Bauschke, still in the framework of Hilbert spaces, replacing $C3$ with the following new condition: $C4: \lim\limits_{n\to\infty} \frac{\lambda_n}{\lambda_{n+r}} = 1$, or equivalently, $\lim\limits_{n\to\infty} \frac{\lambda_n - \lambda_{n+r}}{\lambda_{n+r}} = 0$.

Their main theorems are the following.

Theorem 1.3.5 *(O'Hara et al. [149], Theorem 3.3) Let $\{\lambda_n\} \subset (0,1)$ satisfy $\lim\limits_{n\to\infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = +\infty$. Let K be a nonempty closed and convex subset of a Hilbert space H and let $T_n : K \to K$, $n = 1, 2, \ldots$ be nonexpansive mappings such that $F := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $V_1, V_2, \ldots, V_n : K \to K$ are nonexpansive mappings with the property: for all $k = 1, 2, \ldots, N$ and for any bounded subset C of K, there holds $\lim_{n\to\infty} \sup_{x\in C} ||T_n x - V_k(T_n x)|| = 0$. For $x_0, u \in K$, define*

$$x_{n+1} = \lambda_{n+1} u + (1 - \lambda_{n+1}) T_{n+1} x_n, n \geq 0. \tag{1.3.2}$$

Then, $x_n \to Pu$, where P is the projection from H.

Theorem 1.3.6 *(O'Hara et al. [149], Theorem 4.1) Let K be a nonempty closed convex subset of a Hilbert space H and $T_1, T_2, \ldots, T_N$ be nonexpansive self-mappings of K with $F := \cap_{i=1}^N F(T_i) \neq \emptyset$. Assume that $F = F(T_r T_{r-1} \ldots T_1) = F(T_1 T_r \ldots T_2) = \ldots = F(T_{r-1} T_{r-2} \ldots T_1 T_r)$. Let $\{\lambda_n\} \subset (0,1)$ satisfy the following conditions: (i) $\lim\limits_{n\to\infty} \lambda_n = 0$, (ii) $\sum_{n=0}^{\infty} \lambda_n = +\infty$ and (iii) $\lim\limits_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$. Given points $x_0, u \in K$, the sequence $\{x_n\} \subset K$ is defined by*

$$x_{n+1} = \lambda_{n+1} u + (1 - \lambda_{n+1}) T_{n+1} x_n, n \geq 0. \tag{1.3.3}$$

Then, $x_n \to P_F u$, where P_F is the projection of u onto F.

Jung [109] extended the results of O'Hara *et al.* [149] to uniformly smooth Banach spaces. Furthermore, Jung *et al.* [110] studied the iteration scheme (1.3.1), where the iteration parameter $\{\lambda_n\}$ satisfies the following conditions: $C1 : \lim\limits_{n\to\infty} \lambda_n = 0$; $C2 : \sum_{n=0}^{\infty} \lambda_n = +\infty$ and $C5 : |\lambda_{n+N} - \lambda_n| \leq o(\lambda_{n+N}) + \sigma_n$, where $\sum_{n=1}^{\infty} \sigma_n < \infty$.

They proved the following theorem.

Theorem 1.3.7 *(Jung et al. [110], Theorem 3.1) Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$ and let K be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N$ be nonexpansive mappings from K into itself with $F := \cap_{i=1}^N F(T_i) \neq \emptyset$ and $F = F(T_N T_{N-1} \ldots T_1) = F(T_1 T_N \ldots T_2) = \ldots = F(T_{N-1} T_{N-2} \ldots T_1 T_N)$. Let $\{\lambda_n\} \subset (0,1)$ satisfy the following conditions: (i) $\lim\limits_{n\to\infty} \lambda_n = 0$, (ii) $\sum_{n=0}^{\infty} \lambda_n = +\infty$ and (iii) $|\lambda_{n+N} - \lambda_n| \leq o(\lambda_{n+N}) + \sigma_n$, where $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then, the iterative sequence $\{x_n\}$ defined by (1.3.1) converges strongly to $Q_F u$, where Q_F is a sunny nonexpansive retraction of K onto F.*

Zhou *et al.* [219] proved that the conditions: $(C1)$ and $(C2)$ are indeed sufficient to guarantee the strong convergence of the iteration sequence of (1.3.1) in each of the following situations: (a) E is a Hilbert space; (b) E is a Banach space with weakly sequentially continuous duality map and the sequence $\{x_n\}$ of (1.3.1) is weakly asymptotically regular; (c) E is a reflexive Banach space whose norm is uniformly Gâteaux differentiable and in which every weakly compact convex subset of E has the fixed point property for nonexpansive mappings, and the sequence $\{x_n\}$ of (1.3.1) is asymptotically regular. Their main results are the following theorems.

Theorem 1.3.8 *(Zhou et al. [219], Theorem 6) Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and a weakly continuous duality mapping J_φ for some gauge function φ. Let K be a nonempty closed* *convex subset of* E*. Assume that every weakly compact convex subset of* E *has the fixed point property for nonexpansive mappings. Let* $T_1, T_2, \ldots, T_r$ *be nonexpansive mappings of* K *into itself such that* $F := \cap_{i=1}^r F(T_i) \neq \emptyset$*. Assume also that* $F = F(T_r T_{r-1} \ldots T_1) = F(T_1 T_r \ldots T_2) = \ldots = F(T_{r-1} T_{r-2} \ldots T_1 T_r)$*. Let* $\{\lambda_n\}$ *be a sequence in* $(0, 1)$ *which satisfies* $(C1)$ *and* $(C2)$*. Let* $\{x_n\}$ *be the sequence defined by* (1.3.1) *and assume that* $\{x_n\}$ *is weakly asymptotically regular, then the sequence* $\{x_n\}$ *converges strongly to a common fixed point of* $T_1, T_2, \ldots, T_r$*.*

Theorem 1.3.9 *(Zhou et al. [219], Theorem 10) Let* E *be a reflexive Banach space with a uniformly Gâteaux differentiable norm, and let* K *be a nonempty closed convex subset of* E*. Assume that every weakly compact convex subset of* E *has the fixed point property for nonexpansive mappings. Let* $T_1, T_2, \ldots, T_r$ *be nonexpansive mappings of* K *into itself such that* $F := \cap_{i=1}^r F(T_i) \neq \emptyset$*. Assume also that* $F = F(T_r T_{r-1} \ldots T_1) = F(T_1 T_r \ldots T_2) = \ldots = F(T_{r-1} T_{r-2} \ldots T_1 T_r)$*. Let* $\{\lambda_n\}$ *be a sequence in* $(0, 1)$ *which satisfies* $(C1)$ *and* $(C2)$*. Let* $\{x_n\}$ *be the sequence defined by* (1.3.1) *and assume that* $\{x_n\}$ *is weakly asymptotically regular, then the sequence* $\{x_n\}$ *converges strongly to a common fixed point of* $T_1, T_2, \ldots, T_r$*.*

We remark here that in their proofs of Theorems 1.3.8 and 1.3.9, the authors use the concept of Banach limits, proving in the process two results involving these limits.

In all the above discussion, $T_1, T_2, \ldots, T_N$ remain self-mappings of a nonempty subset of the Banach space E. If, however, the domain of $T_1, T_2, \ldots, T_N$, $D(T_i) \equiv K, i = 1, 2, \ldots, N$, is a proper subset of E and T_i maps K into E for each i, then the recursion formula (1.3.1) may fail to be well defined. To overcome this, an algorithm for non-self mappings was defined for the scheme (1.3.1) by Chidume *et al.* [79]. Using this algorithm, Chidume *et al.* [79] proved the following theorems.

Theorem 1.3.10 *(Chidume et al., [79]) Let* K *be a nonempty closed convex subset of a reflexive Banach space* E *with a uniformly Gâteaux differentiable norm. Assume that* K *is a sunny nonexpansive retract of* E *with* Q *as sunny nonexpansive retraction. Assume that every nonempty closed bounded convex subset of* K *has the fixed point property for nonexpansive mappings. Let* $T_i : K \to E, i = 1, 2, \ldots, r$ *be family of nonexpansive mappings which are weakly inward with* $F := \cap_{i=1}^r F(T_i) \neq \emptyset$ *and* $F = F(QT_r QT_{r-1} \ldots QT_1) = F(QT_1 QT_r \ldots QT_2) = \ldots = F(QT_{r-1} QT_{r-2} \ldots QT_1 QT_r)$*. For a given* $u, x_0 \in K$*, let* $\{x_n\}$ *be generated by the algorithm*

$$x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1}) QT_{n+1} x_n, n \geq 0,$$

where $T_n = T_{n \mod r}$ *and* $\{\alpha_n\}$ *is a real sequence in* $[0, 1]$ *satisfying the following conditions:* (i) $\lim_{n\to\infty} \alpha_n = 0$*,* (ii) $\sum_{n=0}^{\infty} \alpha_n = +\infty$ *and either* *(iii)* $\sum_{n=0}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$ *or (iii)** $\lim_{n\to\infty} \frac{\alpha_{n+r}-\alpha_n}{\alpha_{n+r}} = 0$. *Then,* $\{x_n\}$ *converges strongly to a common fixed point of the family* $\{T_1, T_2, \ldots, T_r\}$*. Further, if* $Px_0 = \lim_{n\to\infty} x_n$ *for each* $x_0 \in K$*, then* P *is sunny nonexpansive retraction of* K *onto* F*.*

Theorem 1.3.11 *(Chidume et al., [79]) Let* K *be a nonempty closed convex subset of a real strictly convex Banach space* E *with a uniformly Gâteaux differentiable norm. Assume that* K *is a sunny nonexpansive retract of* E *with* Q *as sunny nonexpansive retraction. Assume that every nonempty closed bounded convex subset of* K *has the fixed point property for nonexpansive mappings. For each* $i = 1, 2, \ldots, r$*, let* $T_i : K \to E$ *be family of nonexpansive mappings which are weakly inward with* $F := \cap_{i=1}^{r} F(T_i) \neq \emptyset$*. Let* $S_i : K \to E$ *be a family of mappings defined by* $S_i := (1-\lambda_i)I + \lambda_i T_i, 0 < \lambda_i < 1$*, for each* $i = 1, 2, \ldots, r$*. For a given* $u, x_0 \in K$*, let* $\{x_n\}$ *be generated by the algorithm*

$$x_{n+1} = \alpha_{n+1}u + (1-\alpha_{n+1})QS_{n+1}x_n, n \geq 0,$$

where $S_n = S_{n \mod r}$ *and* $\{\alpha_n\}$ *is a real sequence in* $[0,1]$ *satisfying the following conditions:* (i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=0}^{\infty} \alpha_n = +\infty$ *and either* (iii) $\sum_{n=0}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$ *or* $(iii)^*$ $\lim_{n\to\infty} \frac{\alpha_{n+r}-\alpha_n}{\alpha_{n+r}} = 0$. *Then,* $\{x_n\}$ *converges strongly to a common fixed point of the family* $\{T_1, T_2, \ldots, T_r\}$*. Further, if* $Px_0 = \lim_{n\to\infty} x_n$ *for each* $x_0 \in K$*, then* P *is sunny nonexpansive retraction of* K *onto* F*.*

We remark that the requirement that the underlying space E be a Hilbert space, or satisfy Opial's condition, or admit weak sequential continuous duality map imposed in several theorems, in particular, in the theorems of Bauschke [9], in Theorems 1.3.5, 1.3.6, 1.3.7 and 1.3.8 excludes the application of any of these theorems in, for example, L_p spaces, $1 < p < \infty, p \neq 2$ because it is well known that these spaces do not admit weak sequentially continuous duality mappings and do not satisfy Opial's condition.

In 2008, Chidume and Ali [52] introduced *a new iteration scheme* with respect to which these strong conditions on the underlying space are dispensed with, and conditions $C1$ and $C2$ are sufficient to guarantee the strong convergence of the sequence generated by the recursion formula of the iterative scheme to a common fixed point of $T_1, T_2, \ldots, T_r$. In particular, they proved the following theorem.

Theorem 1.3.12 *(Chidume and Ali [52], Theorem 3.1) Let* E *be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let* K *be a nonempty closed convex subset of* E*. Let* $T_1, T_2, \ldots, T_N$ *be a family of nonexpansive self-mappings of* K*, with* $F := \cap_{i=1}^{N} F(T_i) \neq \emptyset$ *and* $F = F(T_N T_{N-1} \ldots T_1) = F(T_1 T_N \ldots T_2) = \ldots = F(T_{N-1} T_{N-2} \ldots T_1 T_N)$*. Let*

$\{\lambda_n\}$ *be a sequence in* $(0,1)$ *satisfying the following conditions:* $C1: \lim_{n\to\infty}\lambda_n = 0$; $C2: \sum_{n=0}^{\infty}\lambda_n = +\infty$. *For a fixed* $\delta \in (0,1)$, *define* $S_n : K \to K$ *by* $S_n x := (1-\delta)x + \delta T_n x \forall x \in K$ *where* $T_n = T_{n \ mod \ N}$. *For arbitrary fixed* $u, x_0 \in K$, *let* $B := \{x \in K : T_N T_{N-1}\dots T_1 x = \gamma x + (1-\gamma)u, \text{ for some } \gamma > 1\}$ *be bounded and let*

$$x_{n+1} = \lambda_{n+1}u + (1-\lambda_{n+1})S_{n+1}x_n, \text{ for } n \geq 0.$$

Assume $\lim_{n\to\infty}||T_n x_n - T_{n+1}x_n|| = 0$. *Then,* $\{x_n\}$ *converges strongly to a common fixed point of the family* $T_1, T_2, \dots, T_N$.

We remark that the iteration process of Chidume and Ali [52] can be used, for example, in the cases when E is a reflexive Banach space with uniformly Gâteaux differentiable norm, and in which every weakly compact convex subset of E has the fixed point property for nonexpansive mappings, and T_n satisfies a mild condition. Moreover, the underlying space will not be required to admit weak sequential continuous duality maps or to satisfy Opial's condition. In addition, the sequence $\{x_n\}$ will not be assumed to be asymptotically regular and the method of proof does not involve the use of Banach limits.

Consequently, Chidume and Ali [52] proved the following convergence theorem for non-self maps which complements the results of Chidume *et al.* [79].

Theorem 1.3.13 *(Chidume and Ali [52], Theorem 4.1) Let* K *be a nonempty closed convex subset of a real reflexive Banach space* E *which has a uniformly Gâteaux differentiable norm. Let* $T_i : K \to E, i = 1, 2, \dots, N$ *be a family of nonexpansive mappings which are weakly inward with* $F := \cap_{i=1}^{N} F(T_i) = F(QT_N QT_{N-1}\dots QT_1) = F(QT_1 QT_N \dots QT_2) = \dots$
$= F(QT_{N-1}QT_{N-2}\dots QT_1 QT_N) \neq \emptyset$. *For a fixed* $\delta \in (0,1)$, *define* $S_n : K \to K$ *by* $S_n x := (1-\delta)x + \delta QT_n x \ \ \forall x \in E$. *For arbitrary fixed* $u, x_0 \in K$, *let* $B := \{x \in K : T_N T_{N-1}\dots T_1 x = \gamma x + (1-\gamma)u, \text{ for some } \gamma > 1\}$ *be bounded and let the sequence* $\{x_n\}$ *be generated iteratively by*

$$x_{n+1} = \lambda_{n+1}u + (1-\lambda_{n+1})S_{n+1}x_n, \text{ for } n \geq 0,$$

where $T_n = T_{n \ mod \ N}$ *and* $\{\lambda_n\}$ *is a real sequence which satisfies* $(C1)$ *and* $(C2)$. *Assume* $\lim_{n\to\infty}||QT_n x_n - QT_{n+1}x_n|| = 0$. *Then,* $\{x_n\}$ *converges strongly to a common fixed point of the family* $T_1, T_2, \dots, T_N$. *Further, if* $Pu = \lim_{n\to\infty} x_n$ *for each* $u \in K$, *then* P *is sunny nonexpansive retraction of* K *onto* F.

We observe that all the theorems proved in Chidume and Ali [52] hold, in particular, in L_p spaces, $1 < p < \infty$.

Another important class of nonlinear mappings more general than the class of nonexpansive mappings called *asymptotically nonexpansive mapping* was introduced in 1972 by Goebel and Kirk [97]. Let K be a nonempty subset of a normed linear space, a mapping $T : K \to K$ is called *asymptotically nonexpansive* if there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $||T^n x - T^n y|| \leq k_n ||x - y||$ for all $x, y \in K$ and $n = 1, 2, \ldots$. It was proved [97] that if K is a nonempty closed, convex and bounded subset of a real *uniformly convex Banach space* and T has a fixed point. It is clear that every nonexpansive mapping is asymptotically nonexpansive. The following example shows that the class of asymptotically nonexpansive mappings properly contains the class of nonexpansive mappings.

Example 1.3.14 *(Goebel and Kirk, [97]) Let B be a unit ball of the real Hilbert space l_2 and let $T : B \to B$ be defined by $T(\{x_1, x_2, \ldots\}) = \{0, x_1^2, a_2x_2, a_3x_3, \ldots\}$ where $\{a_n\}$ is a sequence of numbers such that $0 < a_n < 1$ and $\prod_{n=2}^{\infty} a_n = \frac{1}{2}$. Then $||Tx - Ty|| \leq 2||x - y||$, for all $x, y \in B$ and moreover, $||T^n x - T^n y|| \leq k_n ||x - y||$, with $k_n := 2\prod_{i=2}^{n} a_i$. Observe that T is not nonexpansive and that $\lim_{n\to\infty} k_n = 1$, so that T is asymptotically nonexpansive mapping.*

The averaging iteration process $x_{n+1} := (1 - \alpha_n)x_n + \alpha_n T x_n$ where $T : K \to K$ is asymptotically nonexpansive, K is a closed convex and bounded subset of a Hilbert space was introduced by Schu [172]. He considered the following iteration scheme: $E = H$, a Hilbert space, K is a nonempty closed convex and bounded subset of H, $T : K \to K$ is completely continuous and asymptotically nonexpansive, for each $x_0 \in K, x_{n+1} := (1 - \alpha_n)x_n + \alpha_n T x_n, n \geq 0$, where $\sum_{n=0}^{\infty}(k_n^2 - 1) < 1$ and $\{\alpha_n\}$ is a real sequence satisfying appropriate conditions. He proved that $\{x_n\}$ converges strongly to a fixed point of T. This result has been extended to uniformly convex Banach spaces in the following theorems.

Theorem 1.3.15 *(Rhoades, [169]) Let E be uniformly convex and K be a nonempty closed convex and bounded subset of E. Suppose $T : K \to K$ is completely continuous and asymptotically nonexpansive; for each $x_0 \in K$, let $\{x_n\}$ be defined as follows:*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, n \geq 0$$

where $\sum_{n=0}^{\infty}(k_n^r - 1) < 1$ for some $r > 1, \epsilon \leq 1 - \alpha_n \leq 1 - \epsilon$ for all positive integer n and some $\epsilon > 0$. Then $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$.

Theorem 1.3.16 *(Rhoades, [169]) Let E be uniformly convex and K be a nonempty closed convex and bounded subset of E. Suppose $T : K \to K$ is completely continuous and asymptotically nonexpansive; for each $x_0 \in K$, let $\{x_n\}$ be defined as follows:*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, n \geq 0$$

where $\sum_{n=0}^{\infty}(k_n^r - 1) < 1, r := \max\{2, p\}$ *and* $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon$ *for all positive integer* n *and some* $\epsilon > 0$. *Then,* $\{x_n\}$ *converges strongly to some fixed point of* T.

Let E be a real uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping. Then, for a fixed $u \in K$ and each integer $n \geq 1$, by the Banach Contraction Mapping Principle, there exists a unique $x_n \in K$ such that

$$x_n = \frac{1}{n}u + (1 - \frac{1}{n})Tx_n. \tag{1.3.4}$$

It follows immediately from this equation that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. One of the most useful results concerning algorithms for approximating fixed points of nonexpansive mappings in real uniformly smooth Banach spaces is the celebrated convergence theorem of Reich [168] who proved that the implicit sequence $\{x_n\}$ defined in equation (1.3.4) actually converges strongly to a fixed point of T. Several authors have tried to obtain a result analogous to that of Reich [168] for asymptotically nonexpansive mappings. Suppose K is a nonempty bounded closed convex subset of a real uniformly smooth Banach space E and $T : K \to K$ is an asymptotically nonexpansive mapping with sequence $k_n \geq 1$ for all $n \geq 1$. Fix $u \in K$ and define, for each integer $n \geq 1$, the contraction mapping $S_n : K \to K$ by

$$S_(x) = \Big(1 - \frac{t_n}{k_n}\Big)u + \frac{t_n}{k_n}T^n x, \tag{1.3.5}$$

where $\{t_n\} \in [0, 1)$ is any sequence such that $t_n \to 1$. Then, by the Banach Contraction Mapping Principle, there exists a unique point x_n fixed by S_n, i.e., there exists x_n such that

$$x_n = \Big(1 - \frac{t_n}{k_n}\Big)u + \frac{t_n}{k_n}T^n x_n. \tag{1.3.6}$$

The question now arises as to whether or not this sequence converges to a fixed point of T. A partial answer is given in the following theorem.

Theorem 1.3.17 *(Lim and Xu, [124]) Let* E *be a uniformly smooth Banach space,* K *be a nonempty closed convex and bounded subset of* $E, T : K \to K$ *be an asymptotically nonexpansive mapping with sequence* $k_n \in [1, \infty)$. *Fix* $u \in K$ *and let* $\{t_n\} \subset [0, 1)$ *be chosen such that* $\lim_{n\to\infty} t_n = 1$ *and* $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$. *Then,* (*i*) *for each integer* $n \geq 0$, *there is a unique* $x_n \in K$ *such that*

$$x_n = \Big(1 - \frac{t_n}{k_n}\Big)u + \frac{t_n}{k_n}T^n x_n;$$

suppose in addition that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, *then,* (*ii*) *the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T.

Chidume *et al.* [65] extended Theorem 1.3.17 to reflexive Banach spaces with uniformly Gâteaux differentiable norms. As an application, they proved that the explicit sequence $\{z_n\}$ iteratively generated by

$$z_1 \in K, z_{n+1} = \Big(1 - \frac{t_n}{k_n}\Big)u + \frac{t_n}{k_n}T^n z_n, n \geq 1;$$

converges strongly to a fixed point of the asymptotically nonexpansive mapping T.

We now have the following theorems.

Theorem 1.3.18 *(Chidume et al., [65]) Let E be a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of $E, T : K \to K$ be an asymptotically nonexpansive mapping with sequence $k_n \in [1, \infty)$. Let $u \in K$ be fixed and let $\{t_n\} \subset (0, 1)$ be such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$. Then, (i) for each integer $n \geq 0$, there is a unique $x_n \in K$ such that*

$$x_n = \Big(1 - \frac{t_n}{k_n}\Big)u + \frac{t_n}{k_n}T^n x_n;$$

suppose in addition that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, then, (ii) the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 1.3.19 *(Chidume et al., [65]) Let E be a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of $E, T : K \to K$ be an asymptotically nonexpansive mapping with sequence $k_n \in [1, \infty)$. Let $u \in K$ be fixed and let $\{t_n\} \subset (0, 1)$ be such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} \frac{k_n - 1}{k_n - t_n} = 0$. Define the sequence $\{z_n\}$ iteratively generated by*

$$z_1 \in K, z_{n+1} = \Big(1 - \frac{t_n}{k_n}\Big)u + \frac{t_n}{k_n}T^n z_n, n \geq 1.$$

Then, (i) for each integer $n \geq 0$, there is a unique $x_n \in K$ such that

$$x_n = \Big(1 - \frac{t_n}{k_n}\Big)u + \frac{t_n}{k_n}T^n x_n;$$

suppose in addition that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0, ||z_n - T^n z_n|| = o\Big(1 - \frac{t_n}{k_n}\Big)$, then, (ii) the sequence $\{z_n\}$ converges strongly to a fixed point of T.

The concept of non-self asymptotically nonexpansive mappings was introduced by Chidume *et al.* [71] as an important generalization of asymptotically nonexpan- sive self-mappings.

Using an Ishikawa-like scheme, Takahashi and Tamura [195] proved strong and weak convergence of a sequence dened by $x_{n+1} = \alpha_n S[\beta_n T x_n + (1 - \beta_n)xn] + (1 - \alpha_n)xn$ to a common fxed point of a pair of nonexpansive mappings T and S. Wang [510] used a similar scheme and the definition of Chidume *et al.* [71] to prove strong and weak convergence theorems for a pair of non-self asymptotically nonexpansive mappings. More precisely he proved the following theorems.

Theorem 1.3.20 *(Wang, [202]) Let K be a nonempty closed convex subset of uniformly convex Banach space E. Suppose $T_1, T_2 : K \to E$ are two non-self asymptotically nonexpansive mappings with sequences $\{k_n\}$ and $\{l_n\} \in [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty, \sum_{n=1}^{\infty}(l_n - 1) < \infty, k_n \to 1, l_n \to 1$ as $n \to \infty$, respectively. Let $\{x_n\}$ be generated by*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), n \geq 1, \end{cases} \tag{1.3.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. If one of T_1 and T_2 is completely continuous, and $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

Theorem 1.3.21 *(Wang, [202]) Let $K, T_1, T2, \{k_n\}, \{l_n\}$ and $\{x_n\}$ be as in Theorem 1.3.20. If one of T_1 and T_2 is semi-compact, and $F(T_1) \cap F(T_2) \neq \emptyset$, then, $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Theorem 1.3.22 *(Wang, [202]) Let $K, T_1, T2, \{k_n\}, \{l_n\}$ and $\{x_n\}$ be as in Theorem 1.3.20. If E satisfies Opial's condition, and $F(T_1) \cap F(T_2) \neq \emptyset$, then, $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.*

In 2007, Chidume and Ali [54] introduced an iteration process for approximating common fixed points for finite families of non-self asymptotically nonexpansive mappings. For these families of operators, strong convergence theorems are proved in uniformly convex Banach spaces and weak convergence theorems are proved in real uniformly convex Banach spaces that satisfy Opial's condition, or have Fréchet differentiable norms, or whose dual spaces have the Kadec-Klee property. In particular, they proved the following theorems.

Theorem 1.3.23 *(Chidume and Ali, [54]) Let E be a real uniformly convex Banach space and K be a closed convex nonempty subset of E which is also a nonexpansive retract with retraction P. Let $T_1, T_2 \ldots, T_m : K \to E$ be asymptotically nonexpansive mappings of K into E with sequences $\{k_{in}\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ satisfying $k_{in} \to 1$, as $n \to \infty$ and $\sum_{n=1}^{\infty}(k_{in} - 1) < \infty, i = 1, 2, \ldots, m$. Let $\{\alpha_n\}$ be a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. If one of*

$\{T_i\}_{i=1}^m$ is either completely continuous or semi-compact, then, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_{n+m-2}), \\ y_{n+m-2} = P((1-\alpha_n)x_n + \alpha_n T_2(PT_2)^{n-1}y_{n+m-3}), \\ \vdots \\ y_n = P((1-\alpha_n)x_n + \alpha_n T_m(PT_m)^{n-1}x_n), n \geq 1. \end{cases} \tag{1.3.8}$$

converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$.

Theorem 1.3.24 *(Chidume and Ali, [54]) Let E be a real uniformly convex Banach space and K be a closed convex nonempty subset of E which is also a nonexpansive retract with retraction P. Let $T_1, T_2 \ldots, T_m : K \to E$ be asymptotically nonexpansive mappings of K into E with sequences $\{k_{in}\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$ be as in Theorem 1.3.23. If E satisfies Opial's condition or has a Fréchet differentiable norm, then, the sequence $\{x_n\}$ defined by* (1.3.8) *converges weakly to a common fixed point of $\{T_i\}_{i=1}^m$.*

Theorem 1.3.25 *(Chidume and Ali, [54]) Let E be a real uniformly convex Banach space whose dual E^* satisfies the Kadec-Klee property. Let K be a nonempty closed convex subset of E. Let $T_1, T_2 \ldots, T_m : K \to K$ be nonexpansive mappings. Let the sequence $\{\alpha_n\}_{n=1}^\infty$ be as in Theorem 1.3.23 and $\{x_n\}$ be defined iteratively by* (1.3.8)*. Then, $\{x_n\}$ converges weakly to some common fixed point of $\{T_i\}_{i=1}^m$.*

Recently, Alber *et al.* [2] introduced the class of total asymptotically nonexpansive mappings.

Definition 1.3.26 *A mapping $T : K \to K$ is said to be total asymptotically nonexpansive if there exist nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}, n \geq 1$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in K$,*

$$||T^n x - T^n y|| \leq ||x - y|| + \mu_n \phi(||x - y||) + l_n, n \geq 1. \tag{1.3.9}$$

We remark here that if $\phi(\lambda) = \lambda$, then (1.3.9) reduces to

$$||T^n x - T^n y|| \leq (1 + \mu_n)||x - y|| + l_n, n \geq 1.$$

In addition, if $l_n = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. If $\mu_n = 0$ and $l_n = 0$ for all $n \geq 1$, we obtain from (1.3.9) the class of mappings that includes the class of nonexpansive mappings.

The idea of introducing the class of total asymptotically nonexpansive mappings is to unify various definitions of classes of mappings associated with the class of asymptotically nonexpansive mappings and which are extensions of nonexpansive mappings; and to prove general convergence theorems applicable to all these classes.

Chidume and Ofoedu [69] studied an iterative sequence for the approximation of common fixed points of finite families of total asymptotically nonexpansive mappings, and gave necessary and sufficient conditions for the convergence of the scheme to common fixed points of the mappings in arbitrary real Banach spaces. Furthermore, a sufficient condition for convergence of the iteration process to a common fixed point of these mappings was established in real uniformly convex Banach spaces. They proved the following theorems.

Theorem 1.3.27 *(Chidume and Ofoedu, [69]) Let E be a real Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K, i = 1, 2, \ldots, m$ be m continuous total asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}, \{l_{in}\}, n \geq 1, i = 1, 2, \ldots, m$ such that $F := \cap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be given by*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_m^n x_n, \quad \text{if} \quad m = 1, n \geq 1, \\ \\ x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_{1n} \\ y_{1n} = (1-\alpha_n)x_n + \alpha_n T_2^n y_{2n} \\ \vdots \\ y_{(m-2)n} = (1-\alpha_n)x_n + \alpha_n T_{m-1}^n y_{(m-1)n} \\ y_{(m-1)n} = (1-\alpha_n)x_n + \alpha_n T_m^n x_n, \quad m \geq 2, \quad n \geq 1, \end{cases} \tag{1.3.10}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ bounded away from 0 and 1. Suppose $\sum_{n=1}^{\infty} \mu_{in} < \infty, \sum_{n=1}^{\infty} l_{in} < \infty, i = 1, 2, \ldots, m$ and suppose that there exist $M_i, M_i^* > 0$ such that $\phi_i(\lambda_i) \leq M_i^* \lambda_i$ for all $\lambda_i \geq M_i, i = 1, 2, \ldots, m$. Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of $T_i, i = 1, 2, \ldots, m$ if and only if $\liminf_{n\to\infty} d\Big(x_n, F\Big) = 0$, where $d\Big(x_n, F\Big) = \inf_{y \in F} ||x_n - y||, n \geq 1$.

Theorem 1.3.28 *(Chidume and Ofoedu, [69]) Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K, i = 1, 2, \ldots, m$ be m uniformly continuous total asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}, \{l_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \mu_{in} < \infty, \sum_{n=1}^{\infty} l_{in} < \infty, i = 1, 2, \ldots, m$ and $F := \cap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{\alpha_{in}\} \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. From arbitrary $x_1 \in E$, define* *the sequence* $\{x_n\}$ *by* (1.3.10). $M_i, M_i^* > 0$ *such that* $\phi_i(\lambda_i) \leq M_i^* \lambda_i$ *for all* $\lambda_i \geq M_i, i = 1, 2, \ldots, m$, *and that one of* $T_1, T_2, \ldots, T_m$ *is compact, then* $\{x_n\}$ *converges strongly to some* $p \in F$.

In particular, the results of Chidume and Ofoedu [69] unify, extend and generalize the corresponding results of Alber *et al.* [2] and a host of other results on the approximation of common fixed points of finite families of several classes of nonlinear mappings.

Definition 1.3.29 *Let* $n \in \mathbb{N}$, *a mapping* $T_n : K \to K$ *is said to be* L_n*-Lipschitzian if there exists* $L_n \geq 0$ *such that*

$$||T_n x - T_n y|| \leq L_n ||x - y||, \tag{1.3.11}$$

for all $x, y \in K$. *A single mapping* $T : K \to K$ *is called Lipschitzian or Lipschitz if there exists* $L \geq 0$ *such that* $||Tx - Ty|| \leq L||x - y|| \ \forall x, y \in K$. *If* $L = 1$, *then* T *is called nonexpansive.*

Nilsrakoo and Saejung [145] used Mann's iteration and hybrid method in mathematical programming to obtain *weak* and *strong* convergence of a Mann-type sequence and a sequence generated by the so-called CQ method, respectively, to common fixed points for a countable family of Lipschitzian mappings in *Hilbert spaces*. They proved the following theorems.

Theorem 1.3.30 *(Nilsrakoo and Saejung, [145]) Let* K *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of* L_n*-Lipschitzian mappings from* K *into itself with* $\sum\limits_{n=1}^{\infty}(L_n - 1) < \infty$ *and let* $\bigcap\limits_{n=1}^{\infty} F(T_n) \neq \emptyset$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* K *defined by* $x_1 \in K$ *and* $x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_n x_n$, *for all* $n \in \mathbb{N}$, *where* $\{\alpha_n\}_{n=1}^{\infty}$ *is a sequence in [0,1) with* $\sum\limits_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. *Let* $\sum\limits_{n=1}^{\infty} \sup\{||T_{n+1}x - T_n x|| : x \in B\} < \infty$ *for any bounded subset* B *of* K *and* T *be a mapping of* K *into itself defined by* $Tx := \lim\limits_{n \to \infty} T_n x$ *for all* $x \in K$ *and suppose that* $F(T) = \bigcap\limits_{n=1}^{\infty} F(T_n) \neq \emptyset$, *then* $\{x_n\}_{n=1}^{\infty}$ *converges weakly to* $w \in F(T)$. *Moreover,* $\lim\limits_{n \to \infty} P_{F(T)} x_n = w \in F(T)$.

Theorem 1.3.31 *(Nilsrakoo and Saejung, [145]) Let* K *be a nonempty bounded closed convex subset of a real Hilbert space* H. *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of* L_n*-Lipschitzian mappings from* K *into itself with* $L_n \geq 1$ *and let* $\bigcap\limits_{n=1}^{\infty} F(T_n) \neq \emptyset$. *Assume that* $\{\alpha_n\}_{n=1}^{\infty}$ *is a sequence in [0,1) with*

$\limsup \alpha_n < 1$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* C *defined as follows:*

$$\begin{cases} x_0 \in K, \quad arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_n = \{z \in K : ||y_n - z||^2 \leq ||x_n - z||^2 + \theta_n\}, \\ Q_n = \{z \in K : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{K_n \cap Q_n} x_0, \quad n = 0, 1, 2, ..., \end{cases} \tag{1.3.12}$$

where

$$\theta_n = (1-\alpha_n)(L_n^2 - 1)(diam \quad K)^2 \to 0 \quad \text{as} \quad n \to \infty$$

Let $\sum_{n=1}^{\infty} \sup\{||T_{n+1}x - T_n x|| : x \in B\} < \infty$ *for any bounded subset* B *of* K *and* T *be a mapping of* K *into itself defined by* $Tx := \lim_{n\to\infty} T_n x$ *for all* $x \in C$ *and suppose that* $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, *then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $P_{F(T)}x_0$.

Aoyama *et al.*[5] used *Halpern-type iteration process* (see e.g., Halpern [101]) to obtain strong convergence to common fixed points for a countable family of *nonexpansive* mappings in a uniformly convex Banach space with uniformly Gâteaux differentiable norm. They proved the following theorem.

Theorem 1.3.32 *(Aoyama et al., [5]) Let* E *be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and* K *a nonempty closed convex subset of* E. *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of nonexpansive mappings of* K *into itself such that* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ *and* $\{\alpha_n\}_{n=1}^{\infty}$ *a sequence in [0,1] such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* K *generated by* $x_1 = u \in K$ *and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T_n x_n, \tag{1.3.13}$$

for all $n \in \mathbb{N}$. *Suppose that*

$$\sum_{n=1}^{\infty} \sup\{||T_{n+1}x - T_n x|| : x \in B\} < \infty \tag{1.3.14}$$

for any bounded subset B *of* K. *Let* T *be a mapping of* K *into itself defined by* $Tx := \lim_{n\to\infty} T_n x$ *for all* $x \in K$ *and suppose that* $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. *If either*

$$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \tag{1.3.15}$$

or

$$\alpha_n \in (0,1] \quad \forall n \in \mathbb{N}, \quad \lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1, \tag{1.3.16}$$

then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* Qu*, where* Q *is the sunny nonexpansive retraction of* E *onto* $F(T) := \bigcap_{n=1}^{\infty} F(T_n)$*.*

We make the following remark concerning Theorem 1.3.30, Theorem 1.3.30 and Theorem 1.3.32. In these theorems, *the condition* (1.3.14) *is difficult to check in application.* Furthermore, in [5] page 2356, the authors make the following acknowledgement: "In our main theorem, we assume that $\sum_{n=1}^{\infty} \sup\{||T_{n+1}x - T_n x|| : x \in B\} < \infty$ for any bounded subset B of K. In general, one cannot apply this result for a sequence of nonexpansive mappings."

Maingé [129] studied the Halpern-type scheme for approximation of a common fixed point for a countable infinite family of nonexpansive mappings in a Hilbert space. Define $N_I := \{i \in \mathbb{N} : T_i \neq I\}$ (I being the identity mapping on a real normed space E).

He proved the following theorems.

Theorem 1.3.33 *(Maingé, [129]) Let* K *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $\{T_i\}$ *be a countable family of nonexpansive self- mappings of* $K, \{t_n\}$ *and* $\{\sigma_{i,t_n}\}$ *be sequences in* $(0,1)$ *satisfying the following conditions:* (*i*) $\lim_{n\to\infty} t_n = 0$, (*ii*) $\sum_{i=1}^{\infty} \sigma_{i,t_n} = (1 - t_n)$, (*iii*) $\forall i \in N_I, \lim_{n\to\infty} \frac{t_n}{\sigma_{i,t_n}} = 0$*. Define a fixed point sequence* $\{x_n\}$ *by*

$$x_{t_n} = t_n C x_{t_n} + \sum_{i=1}^{\infty} \sigma_{i,t_n} T_i x_{t_n} \tag{1.3.17}$$

where $C : K \to K$ *is a strict contraction. Assume* $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$*, then* $\{x_{t_n}\}$ *converges strongly to a unique fixed point of the contraction* $P_F o C$*, where* P_F *is the metric projection from* H *onto* F*.*

Theorem 1.3.34 *(Maingé, [129]) Let* K *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $\{T_i\}$ *be a countable family of nonexpansive self- mappings of* $K, \{\alpha_n\}$ *and* $\{\sigma_{i,n}\}$ *be sequences in* $(0,1)$ *satisfying the* *following conditions:* (*i*) $\sum_{i=1}^{\infty}\alpha_n = \infty, \quad \sum_{i=1}^{\infty}\sigma_{i,n} = (1-\alpha_n),$

$$(ii)\begin{cases} \frac{1}{\sigma_{i,n}}\Big|1-\frac{\alpha_{n-1}}{\alpha_n}\Big| \to 0, \quad \text{or} \quad \sum_{n=1}^{\infty}\frac{1}{\sigma_{i,n}}|\alpha_{n-1}-\alpha_n| < \infty, \\ \frac{1}{\alpha_n}\Big|\frac{1}{\sigma_{i,n}}-\frac{1}{\sigma_{i,n-1}}\Big| \to 0, \quad \text{or} \quad \sum_{n=1}^{\infty}\Big|\frac{1}{\sigma_{i,n}}-\frac{1}{\sigma_{i,n-1}}\Big| < \infty, \\ \frac{1}{\sigma_{i,n}\alpha_n}\sum_{k=0}^{\infty}|\sigma_{k,n}-\sigma_{k,n-1}| \to 0, \quad \text{or} \quad \sum_{k=0}^{\infty}\frac{1}{\sigma_{i,n}\alpha_n}\sum_{k=0}^{\infty}|\sigma_{k,n}-\sigma_{k,n-1}| < \infty. \end{cases} \tag{1.3.18}$$

(*iii*) $\forall i \in N_I, \lim_{n\to\infty}\frac{\alpha_n}{\sigma_{i,n}} = 0$. *Then, the sequence* $\{x_n\}$ *defined iteratively by* $x_1 \in K$,

$$x_{n+1} = \alpha_n C x_n + \sum_{i=1}^{\infty}\sigma_{i,n}T_i x_n \tag{1.3.19}$$

converges strongly to a unique fixed point of the contraction $P_F o C$, *where* P_F *is the metric projection from* H *onto* F.

Theorem 1.3.33, from the point of view of applications, seems much better than Theorem 1.3.30, Theorem 1.3.31 and Theorem 1.3.32, which impose the condition (1.3.14), a condition that is clearly very difficult to verify in any application.

Chidume *et al.* [58] proved theorems, with recursion formulas simpler than (1.3.17) and (1.3.19), that extended Theorems 1.3.33 and 1.3.34 to l_p spaces, $1 < p < \infty$. Furthermore, they also proved convergence theorems which are applicable in L_p spaces, $1 < p < \infty$. Moreover, in their more general setting, some of the conditions on the sequences $\{\alpha_n\}$ and $\{\sigma_{i,n}\}$ imposed in Theorem 1.3.34 were dispensed with or weakened.

Furthermore, Chidume and Chidume [57] proved theorems on iterative approximation for common fixed points for a countable infinite family of nonexpansive mappings in real uniformly convex Banach spaces with uniformly Gâteaux differentiable norms. Furthermore, in this more general setting, the recursion formula studied is much simpler than (1.3.12) studied by Nilsrakoo and Saejung [145] in a Hilbert space. In the special case that $= l_p, 1 < p < \infty$, the condition (1.3.14) of Aoyama *et al.* [5] is dispensed with. In particular, they proved the following theorems.

Theorem 1.3.35 *(Chidume and Chidume, [57]) Let* E *be a uniformly convex real Banach space with a uniformly Gâteaux differentiable norm. Let* K *be a closed, convex and nonempty subset of* E. *Let* $\{T_i\}$ *be a family of nonexpansive self-mappings of* K. *For arbitrary fixed* $\delta \in (0,1)$, *define a family of nonexpansive maps* $\{S_i\}$ *by* $S_i := (1-\delta)I + \delta T_i \forall i \in \mathbb{N}$ *where* I *is an identity map of* K. *Assume* $\cap_{i=1}^{\infty}F(T_i) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\sigma_{i,n}\}$ *be sequences* *in* $(0,1)$ *satisfying the following conditions:* (i) $\lim_{n\to\infty}\alpha_n = 0, (ii)$ $\sum_{i=1}^{\infty}\alpha_n = \infty, (iii)$ $\sum_{i=1}^{\infty}\sigma_{i,n} = (1-\alpha_n)$ *and* (iv) $\lim_{n\to\infty}\sum_{i=1}^{\infty}|\sigma_{i,n+1}-\sigma_{i,n}| = 0$. *Define a sequence* $\{x_n\}$ *iteratively by* $x_1, u \in K$,

$$x_{n+1} = \alpha_n u + \sum_{i=1}^{\infty}\sigma_{i,n}S_i x_n, n \geq 1. \qquad (1.3.20)$$

If at least one of the $T_i's$ *is demicompact, then,* $\{x_n\}$ *converges strongly to an element of* F.

Theorem 1.3.36 *(Chidume and Chidume, [57]) Let* E *be a uniformly convex real Banach space with a uniformly Gâteaux differentiable norm. Let* K *be a closed, convex and nonempty sunny nonexpansive retract of* E *with* Q *as the sunny nonexpansive retraction. Let* $T_i : K \to E, i \in \mathbb{N}$ *be a family of nonexpansive mappings of* K *into* E*. For arbitrary fixed* $\delta \in (0,1)$*, define a family of nonexpansive maps* $\{S_i\}$ *by* $S_i := (1-\delta)I + \delta Q T_i \forall i \in \mathbb{N}$ *where* I *is an identity map of* K*. Assume* $\cap_{i=1}^{\infty}F(T_i) \neq \emptyset$*. Let* $\{\alpha_n\}$ *and* $\{\sigma_{i,n}\}$ *be sequences in* $(0,1)$ *satisfying the following conditions:* (i) $\lim_{n\to\infty}\alpha_n = 0, (ii)$ $\sum_{i=1}^{\infty}\alpha_n = \infty, (iii)$ $\sum_{i=1}^{\infty}\sigma_{i,n} = (1-\alpha_n)$ *and* (iv) $\lim_{n\to\infty}\sum_{i=1}^{\infty}|\sigma_{i,n+1} - \sigma_{i,n}| = 0$*. Define a sequence* $\{x_n\}$ *iteratively by* $x_1, u \in K$,

$$x_{n+1} = \alpha_n u + \sum_{i=1}^{\infty}\sigma_{i,n}S_i x_n, n \geq 1. \qquad (1.3.21)$$

If at least one of the $T_i's$ *is demicompact, then,* $\{x_n\}$ *converges strongly to an element of* F.

Theorem 1.3.37 *(Chidume and Chidume, [57]) Let* E *be a uniformly convex real Banach space with a uniformly Gâteaux differentiable norm. Let* K *be a compact, convex and nonempty sunny nonexpansive retract of* E *with* Q *as the sunny nonexpansive retraction. Let* $T_i : K \to E, i \in \mathbb{N}$ *be a family of nonexpansive mappings of* K *into* E*. For arbitrary fixed* $\delta \in (0,1)$*, define a family of nonexpansive maps* $\{S_i\}$ *by* $S_i := (1-\delta)I + \delta Q T_i \forall i \in \mathbb{N}$ *where* I *is an identity map of* K*. Assume* $\cap_{i=1}^{\infty}F(T_i) \neq \emptyset$*. Let* $\{\alpha_n\}$ *and* $\{\sigma_{i,n}\}$ *be sequences in* $(0,1)$ *satisfying the following conditions:* (i) $\lim_{n\to\infty}\alpha_n = 0, (ii)$ $\sum_{i=1}^{\infty}\alpha_n = \infty, (iii)$ $\sum_{i=1}^{\infty}\sigma_{i,n} = (1-\alpha_n)$ *and* (iv) $\lim_{n\to\infty}\sum_{i=1}^{\infty}|\sigma_{i,n+1} - \sigma_{i,n}| = 0$*. Define a sequence* $\{x_n\}$ *iteratively by* $x_1, u \in K$,

$$x_{n+1} = \alpha_n u + \sum_{i=1}^{\infty}\sigma_{i,n}S_i x_n, n \geq 1. \qquad (1.3.22)$$

If at least one of the $T_i's$ is demicompact, then, $\{x_n\}$ converges strongly to an element of F.

We remark here that the following conditions in (ii) of Theorem 1.3.34:

$$\begin{cases} \frac{1}{\sigma_{i,n}}\left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right| \to 0, \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{\sigma_{i,n}}|\alpha_{n-1} - \alpha_n| < \infty, \\ \frac{1}{\alpha_n}\left|\frac{1}{\sigma_{i,n}} - \frac{1}{\sigma_{i,n-1}}\right| \to 0, \quad \text{or} \quad \sum_{n=1}^{\infty} \left|\frac{1}{\sigma_{i,n}} - \frac{1}{\sigma_{i,n-1}}\right| < \infty. \end{cases} \tag{1.3.23}$$

are dispensed with even in the more general settings of Chidume and Chidume [57]. Furthermore, the requirement

$$\frac{1}{\sigma_{i,n}\alpha_n}\sum_{k=0}^{\infty}|\sigma_{k,n} - \sigma_{k,n-1}| \to 0, \quad \text{or} \quad \sum_{k=0}^{\infty}\frac{1}{\sigma_{i,n}\alpha_n}\sum_{k=0}^{\infty}|\sigma_{k,n} - \sigma_{k,n-1}| < \infty$$

also imposed in (ii) of Theorem 1.3.34 is weakened to $\lim_{n\to\infty} \sum_{i=1}^{\infty} |\sigma_{i,n+1} - \sigma_{i,n}| = 0$ in the results of Chidume and Chidume [57].

In Chapter 4 of this thesis, we prove strong convergence theorem for common fixed points of L_n-Lipschitzian mappings in *real Banach spaces much more general than uniformly convex real Banach spaces* considered in Aoyama *et al.* (Theorem 1.3.32 above). A corollary of our theorem extends Theorem 1.3.32 from uniformly convex real Banach spaces with uniformly Gâteaux differentiable norm to real Banach spaces with uniformly Gâteaux differentiable norm and possessing uniform normal structure. Furthermore, in this our more general setting, condition

$$\sum_{n=1}^{\infty}\sup\{||T_{n+1}x - T_nx|| : x \in B\} < \infty$$

for any bounded subset B of K, which is not easily satisfied in several situations and which is assumed in Nilsrakoo and Saejung [145] and in Aoyama *et al.*[5] is replaced with a simpler condition which *is satisfied for example, for a sequence of nonexpansive mappings.* Our results improve and extend the results of Nilsrakoo and Saejung [145], Aoyama *et al.* [5] and many other important recent results.

1.4 Algorithm for common solutions of three problems

Let C be nonempty, closed and convex subset of a Banach space E. An operator $A : C \to E^*$ is called *α-inverse-strongly monotone*, if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha||Ax - Ay||^2, \quad \forall x, y \in C, \tag{1.4.1}$$

and A is said to be *monotone* if

$$\langle x-y, Ax-Ay\rangle \geq 0, \quad \forall x,y \in C. \tag{1.4.2}$$

Let A be a monotone operator from C into E^*, the classical variational inequality is to find $x^* \in C$ such that

$$\langle y-x^*, Ax^*\rangle \geq 0, \quad \forall y \in C. \tag{1.4.3}$$

The set of solutions of (1.4.3) is denoted by $VI(C,A)$. Let H be a real Hilbert space. Let $A : H \to H$ be a single-valued nonlinear mapping and let $M : H \to 2^H$ be a set-valued mapping. The variational inclusion problem is to find $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{1.4.4}$$

where θ is a zero vector in H. The set of solutions to the variational inclusion (1.4.4) is denoted by $I(A,M)$. When $A = 0$, then (1.4.4) becomes the inclusion problem introduced by Rockafellar [170]. Variational inequality theory, which was introduced by Stampacchia [183] in 1964, has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences; see [92, 94, 96, 115, 151] and the references therein. The ideas and techniques of the variational inequalities are being applied in a variety of diverse areas of sciences and proved to be productive and innovative. It has been shown that this theory provides a simple, natural and unified framework for a general treatment of unrelated problems. The variational inequality (1.4.3) is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $x^* \in E$ such that $Bx^* = 0$ and so on. An important problem is how to find a solution of (1.4.3) whenever it exists.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$. The equilibrium problem (see, for example [87, 96, 127]) is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.4.5}$$

We shall denote the solutions set of (1.4.5) by $EP(F)$. Numerous problems in Physics, optimization and economics reduce to finding a solution of problem (1.4.5). The equilibrium problem includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium, and game theory as special cases (see, for example, [18, 142, 151]).

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0$ for all $x,y,\in C$;
(A3) for each $x,y\in C$, $\lim_{t\to 0} F(tz+(1-t)x,y)\leq F(x,y)$;
(A4) for each $x\in C$, $y\mapsto F(x,y)$ is convex and lower semicontinuous.

Under these assumptions, the following existence theorems are proved for equilibrium problems in Banach spaces.

Lemma 1.4.1 *(Blum and Oettli [18]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let F be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r>0$ and $x\in E$. Then, there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z, Jz-Jx\rangle\geq 0 \quad \text{for all} \quad y\in C.$$

Lemma 1.4.2 *(Combettes and Hirstoaga, [87], Takahashi and Zembayashi [199]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $F: C\times C\to\mathbb{R}$ satisfies (A1)-(A4). For $r>0$ and $x\in E$, define a mapping $T_r: E\to C$ as follows:*

$$T_r(x)=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z, Jz-Jx\rangle\geq 0, \forall y\in C\}$$

for all $z\in E$. Then, the following assertions:
1. T_r is single-valued;
2. T_r is firmly nonexpansive-type mapping, i.e., for any $x,y\in E$,

$$\langle T_rx-T_ry, JT_rx-JT_ry\rangle\leq\langle T_rx-T_ry, Jx-Jy\rangle;$$

3. $F(T_r)=EP(F)$;
4. $EP(F)$ is closed and convex.

Equilibrium problems which were introduced by Blum and Oettli [18] in 1994 have had a great impact and influence in pure and applied sciences. It has been shown that the equilibrium problems theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization.

Recently, Takahashi and Takahashi [193] introduced an iterative scheme for approximating the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a generalized equilibrium problem in a real Hilbert space. In particular, they proved the following theorem.

Theorem 1.4.3 *Let K be a closed convex nonempty subset of a real Hilbert space H. Let F be a bifunction from $K\times K$ satisfying $(A1)-(A4)$, A be an α-inverse-strongly monotone mapping of K into H and let T be a nonexpansive* *mapping of K into itself. Suppose $F(T)\bigcap EP \neq \emptyset$ and $u \in K$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are generated by $x_1 \in K$,*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \quad \forall y \in K \\ x_{n+1} = \beta_n x_n + (1-\beta_n)T[\alpha_n u + (1-\alpha_n)z_n], \quad n \geq 1; \end{cases} \tag{1.4.6}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \subset [0,1)$ and $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$. If $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are chosen so that $\{r_n\}_{n=1}^{\infty} \subset [a,b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$\lim_{n\to\infty} |r_{n+1} - r_n| = 0, \quad 0 < c \leq \beta_n \leq d < 1$$

then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $z_0 = P_{F(T)\bigcap EP}u$.

In Chapter 5 of this thesis, we introduce a new iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions to a variational inclusion and the set of solutions to a generalized equilibrium problem in a real Hilbert space. Furthermore, using our new iterative scheme, we prove strong convergence theorems which extend many recent important results. Finally, we apply our results to solve a convex minimization problem in a real Hilbert space. In particular, our results are much more applicable than the results of Takahashi and Takahashi [193].

Let H be a real Hilbert space with norm $||.||$ and inner product $\langle ., .\rangle$ and let K be a nonempty, closed and convex subset of H. One parameter family $\Gamma := \{T(t) : 0 \leq t < \infty\}$ is said a *(continuous) Lipschitzian semigroup on K of mappings* from K into K if the following conditions are satisfied:

1. $T(0)x = x$ for all $x \in K$;
2. $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
3. for each $t > 0$, there exists a bounded measurable function $L_t : (0, \infty) \to [0, \infty)$ such that $||T(t)x - T(t)y|| \leq L_t||x - y||, \quad x, y \in K$;
4. for each $x \in K$, the mapping $T(.)x$ from $[0, \infty)$ into K is continuous.

A Lipschitzian semigroup Γ is called *nonexpansive (or contractive)* if $L_t = 1$ for all $t > 0$ and *asymptotically nonexpansive* if $\limsup_{t\to\infty} L_t \leq 1$, respectively. Let $F(\Gamma)$ denote the common fixed point set of the semigroup Γ, i.e., $F(\Gamma) := \{x \in K : T(t)x = x, \quad \forall t > 0\}$. There are many papers concerning the existence of common fixed points of $\{T(t) : 0 \leq t < \infty\}$. As a matter of fact, Browder [23] proved that if K is bounded, then $F(\Gamma)$ is nonempty.

Recently, Cianciaruso *et al.* [85] introduced the following implicit and explicit algorithms for finding a common element of the set of common fixed points of a one-parameter nonexpansive semigroup $\Im := \{T(s) : 0 \leq s < \infty\}$ which is also a solution to an equilibrium problem in a real Hilbert space.

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \ \ \forall y \in H, \\ x_n = (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds + \alpha_n \gamma f(x_n). \end{cases}$$

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \ \ \forall y \in H, \\ x_{n+1} = (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds + \alpha_n \gamma f(x_n). \end{cases}$$

They proved that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to a common fixed point $x^* \in F(\Im)$ which is also a solution to an equilibrium problem in a real Hilbert space.

Motivated by the results of Cianciaruso *et al.* [85], we prove strong convergence theorems for finding a common element of the set of fixed points of a nonexpansive semigroup, the set of solutions to a variational inclusion and the set of solutions to a generalized equilibrium problem in a real Hilbert space. Our results here are much more applicable than the results of Cianciaruso *et al.* [85]. These results are given in Chapter 6 of this thesis.

We denote by ϕ, the functional on $E \times E$ defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, J(y)\rangle + ||y||^2, \ \ \forall x, y \in E. \tag{1.4.7}$$

It is obvious from (1.4.7) that

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \ \ \forall x, y \in E. \tag{1.4.8}$$

Definition 1.4.4 *Let C be a nonempty, closed and convex subset of E and let $\{T_n\}_{n=0}^{\infty}$ be a countable family of mappings from C into E. A point $p \in C$ is said to be an asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - T_n x_n|| = 0$. The set of asymptotic fixed points of $\{T_n\}_{n=0}^{\infty}$ is denoted by $\widehat{F}(\{T_n\}_{n=0}^{\infty})$. We say that $\{T_n\}_{n=0}^{\infty}$ is countable family of relatively nonexpansive mappings (see, for example, [185]) if the following conditions are satisfied:*
(R1) $F(\{T_n\}_{n=0}^{\infty}) \neq \emptyset$;
(R2) $\phi(p, T_n x) \leq \phi(p, x), \ \ \forall x \in C, \ \ p \in F(T_n), \ \ n \geq 0$;
(R3) $\cap_{n=0}^{\infty} F(T_n) = \widehat{F}(\{T_n\}_{n=0}^{\infty})$.

Definition 1.4.5 *A point $p \in C$ is said to be a strong asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges strongly*

to p and $\lim_{n\to\infty} ||x_n - T_n x_n|| = 0$. *The set of strong asymptotic fixed points of* $\{T_n\}_{n=0}^{\infty}$ *is denoted by* $\widetilde{F}(\{T_n\}_{n=0}^{\infty})$. *We say that a family of mappings* $\{T_n\}_{n=0}^{\infty}$ *is countable family of weak relatively nonexpansive mappings (see, for example, [185]) if the following conditions are satisfied:*
(R1) $F(\{T_n\}_{n=0}^{\infty}) \neq \emptyset$;
(R2) $\phi(p, T_n x) \leq \phi(p, x), \quad \forall x \in C, \quad p \in F(T_n), \quad n \geq 0$;
(R3) $\cap_{n=0}^{\infty} F(T_n) = \widetilde{F}(\{T_n\}_{n=0}^{\infty})$.

Definition 1.4.6 *Let* C *be a nonempty, closed and convex subset of* E *and let* T *be a mapping from* C *into* E. *A point* $p \in C$ *is said to be an asymptotic fixed point of* T *if* C *contains a sequence* $\{x_n\}_{n=0}^{\infty}$ *which converges weakly to p and* $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. *The set of asymptotic fixed points of* T *is denoted by* $\widehat{F}(T)$. *We say that a mapping* T *is relatively nonexpansive if the following conditions are satisfied:*
(R1) $F(T) \neq \emptyset$;
(R2) $\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \quad p \in F(T)$;
(R3) $F(T) = \widehat{F}(T)$.

Definition 1.4.7 *A point* $p \in C$ *is said to be a strong asymptotic fixed point of* T *if* C *contains a sequence* $\{x_n\}_{n=0}^{\infty}$ *which converges strongly to p and* $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. *The set of strong asymptotic fixed points of* T *is denoted by* $\widetilde{F}(T)$. *We say that a mapping* T *is weak relatively nonexpansive (see, for example, [113, 215]) if the following conditions are satisfied:*
(R1) $F(T) \neq \emptyset$;
(R2) $\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \quad p \in F(T)$;
(R3) $F(T) = \widetilde{F}(T)$.

If T satisfies (R1) and (R2), then T is said to be *relatively quasi-nonexpansive*. The asymptotic behavior of a relatively nonexpansive mapping was studied in [38, 39]. It is easy to see that the class of relatively quasi-nonexpansive mappings contains the class of relatively nonexpansive mappings. Definition 1.4.6 (Definition 1.4.7) is a special form of Definition 1.4.4 (Definition 1.4.5) as $T_n \equiv T, \ \forall n \geq 0$. Furthermore, Su *et al.* [185] gave an example which is a countable family of weak relatively nonexpansive mappings but not a countable family of relatively nonexpansive mappings. It is obvious that relatively nonexpansive mapping is weak relatively nonexpansive mapping. In fact, for any mapping $T : C \to C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$. Kang *et al.* [113] gave an example of a weak relatively nonexpansive mapping which is not relatively nonexpansive. Clearly, in a Hilbert space H, relatively nonexpansive (relatively quasi-nonexpansive) mappings and nonexpansive (quasi-nonexpansive) mappings are the same, for $\phi(x, y) = ||x - y||^2, \ \forall x, y \in H$. This implies that

$$\phi(p, Tx) \leq \phi(p, x) \Leftrightarrow ||Tx - p|| \leq ||x - p||, \quad \forall x \in C, \quad p \in F(T).$$

Recently, considerable research efforts have been devoted to developing iterative methods for approximating a common fixed point (when it exists) for a family of relatively nonexpansive type mappings.

In 2004, Matsushita and Takahashi [135] introduced an iterative sequence for finding fixed points of relatively nonexpansive mappings in Banach spaces, and then proved weak and strong convergence theorems by using the notion of generalized projection. In particular, they proved the following theorems.

Theorem 1.4.8 *(Matsushita and Takahashi, [135]) Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself such that $F(T)$ is nonempty, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), n \geq 0. \tag{1.4.9}$$

If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to some fixed point of T.

Theorem 1.4.9 *(Matsushita and Takahashi, [135]) Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself such that $F(T)$ is nonempty, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), n \geq 0. \tag{1.4.10}$$

If the interior of $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to some fixed point of T.

Recently, Qin *et al.* [160] introduced the following hybrid projection algorithm for two families of relatively quasi-nonexpansive mappings which are more general than relatively nonexpansive mappings in a Banach space: $x_0 \in C$,

$$\begin{cases} z_{n,i} = J^{-1}(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n) \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 + (1-\alpha_{n,i})Jz_{n,i}) \\ C_{n,i} = \{z \in C : \phi(z, y_{n,i}) \leq \phi(z, x_n) + \alpha_{n,i}(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ C_n = \cap_{i=1}^{\infty} C_{n,i} \\ Q_0 = C \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, Jx_n - Jx_0\rangle \leq 0\} \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{1.4.11}$$

They proved under appropriate conditions on the parameters that the sequence $\{x_n\}$ generated by (1.4.11) converges strongly to a common fixed point of the two families $\{T_i\}$ and $\{S_i\}$.

Quite recently, Su *et al.* [185] proved the following strong convergence theorem by hybrid iterative scheme for approximation of common fixed point of two countable families of weak relatively nonexpansive mappings in uniformly convex and uniformly smooth Banach space.

Theorem 1.4.10 *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $\{T_n\}_{n=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$ are two countable families of weak relatively nonexpansive mappings of C into itself such that $\Omega := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \neq \emptyset$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C$,*

$$\begin{cases} z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ C_n = \{w \in C_{n-1} \cap Q_{n-1} : \phi(w, y_n) \leq \phi(w, x_n)\}, \\ C_0 = \{w \in C : \phi(w, y_0) \leq \phi(w, x_0)\}, \\ Q_n = \{w \in C_{n-1} \cap Q_{n-1} : \langle x_n - w, Jx_0 - Jx_n \rangle \geq 0\}, \\ Q_0 = C, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n \geq 1, \end{cases}$$

with the conditions
(i) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
(iii) $0 \leq \alpha_n \leq \alpha < 1$ *for some* $\alpha \in (0,1)$.
Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_{\Omega} x_0$.

Recently, algorithms and convergence results for common solutions of fixed point problems, variational inequality problems and equilibrium problems have been obtained in Banach spaces. Takahashi and Zembayashi [199] introduced an iterative scheme and proved a strong convergence theorem for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space. They proved the following theorem.

Theorem 1.4.11 *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)(A4)$ and let S be a relatively nonexpansive mapping from C into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let*

$\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases} \tag{1.4.12}$$

for every $n \in \mathbb{N} \cup \{0\}$*, where* J *is the duality mapping on* E*,* $\{\alpha_n\} \subset [0,1]$ *satisfies* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\{r_n\} \subset [a, 1)$ *for some* $a > 0$*. Then,* $\{x_n\}$ *converges strongly to* $\Pi_{F(S)\cap EP(f)}x$*, where* $\Pi_{F(S)\cap EP(f)}$ *is the generalized projection of* E *onto* $F(S) \cap EP(f)$*.*

Furthermore, Cholamjiak [82] introduced a hybrid iterative scheme for approximation of a fixed point of relatively quasi- nonexpansive mapping which is also a solution to equilibrium problem and variational inequality problems in a 2-uniformly convex real Banach space which is also uniformly smooth: $x_0 \in C, \quad C_1 = C, \quad x_1 = \Pi_{C_1} x_0,$

$$\begin{cases} \upsilon_n = \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n) \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JS\upsilon_n) \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C \\ C_{n+1} = \{w \in C_n : \phi(w, u_n) \leq \phi(w, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{cases}$$

Then, he proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$, where $F := F(T) \bigcap F(S) \bigcap VI(C, B) \bigcap EP(F) \neq \emptyset$.

In Chapter 7 of this thesis, we prove strong convergence theorems for approximation of common element of the set of fixed points of a countable family of relatively nonexpansive mappings and set of solutions of an equilibrium problem in uniformly convex and uniformly smooth real Banach spaces. Our iterative scheme in this chapter *appears simpler* than the iterative scheme of Takahashi and Zembayashi [199]. Furthermore, our results are obtained for countable family of relatively nonexpansive mappings. Thus, our results extend and improve on the results of Takahashi and Zembayashi [199], Nilsrakoo and Saejung [147] and many other important recent results.

In Chapter 8 of this thesis, we introduce a new hybrid projection algorithm based on the shrinking projection method and prove strong convergence theorem for approximation of a common element of the set of common fixed point of an infinite family of relatively quasi-nonexpansive mappings, set of solutions to a variational inequality problem and the set of solutions to system of generalized mixed equilibrium problems in a 2-uniformly convex real Banach space which is also uniformly smooth. Consequently, we use our new iterative scheme to prove strong convergence theorem for approximation of a fixed point of a weak relatively nonexpansive mapping which is also a common solution to variational inequality problem and system of generalized mixed equilibrium problems in a Banach space. Our results extend the results of Martinez-Yanes and Xu [131], Plubtieng and Ungchittrakool [157], Cholamjiak [82] and many other recent important results in the literature. Finally, we use our results to obtain several applications in a Banach space.

1.5 Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $||.||$ and let K be a nonempty closed convex subset of H.

For any point $u \in H$, there exists a unique point $P_K u \in K$ such that

$$||u - P_K u|| \leq ||u - y||, \quad \forall y \in K.$$

P_K is called the *metric projection* of H onto K. It is know that P_K is a nonexpansive mapping of H onto K. It is also known that P_K satisfies

$$\langle x - y, P_K x - P_K y \rangle \geq ||P_K x - P_K y||^2, \tag{1.5.1}$$

for all $x, y \in H$. Furthermore, $P_K x$ is characterized by the properties $P_K x \in K$ and

$$\langle x - P_K x, P_K x - y \rangle \geq 0, \tag{1.5.2}$$

for all $y \in K$. In the context of the variational inequality problem, this implies that

$$x^* \in VI(K, A) \Leftrightarrow x^* = P_K(x^* - \lambda A x^*), \quad \forall \lambda > 0.$$

If A is α-inverse-strongly monotone mapping of K into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x, y \in K$ and $\lambda > 0$,

$$\begin{aligned} & ||(I - \lambda A)x - (I - \lambda A)y||^2 = ||x - y - \lambda(Ax - Ay)||^2 \\ = \; & ||x - y||^2 - 2\lambda \langle Ax - Ay, x - y \rangle + \lambda^2 ||Ax - Ay||^2 \\ \leq \; & ||x - y||^2 + \lambda(\lambda - 2\alpha)||Ax - Ay||^2 \end{aligned} \tag{1.5.3}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of K into H.

A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in M(x)$ and $g \in M(y)$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping

M is said to be *maximal* if the graph $G(M)$ is not properly contained in the graph of any other monotone mapping, where $G(M) := \{(x, y) \in H \times H : y \in Mx\}$ is the graph of a set-valued mapping M. It is also known that M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in M(x)$. The resolvent operator $J_{M,\lambda}$ associated with M and λ is the mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad u \in H, \quad \lambda > 0. \tag{1.5.4}$$

It is known that the resolvent operator $J_{M,\lambda}$ is single-valued, nonexpansive and 1-inverse-strongly monotone (see, for example, [19]) and that a solution of (1.4.4) is a fixed point of $J_{M,\lambda}(I - \lambda A)$, $\forall \lambda > 0$ (see, for example, [119]). If $0 < \lambda < 2\alpha$, it is easy to see that $J_{M,\lambda}(I - \lambda A)$ is nonexpansive and $I(A, M)$ is closed and convex.

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to have a *Gâteaux differentiable* norm (and E is called *smooth*) if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$; E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Further, E is said to be *uniformly smooth* if the limit exists uniformly for $(x, y) \in S \times S$. The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := sup\Big\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1,\ \|y\| = \tau\Big\};\ \tau > 0.$$

E is equivalently said to be *smooth* if $\rho_E(\tau) > 0$, $\forall\, \tau > 0$. Let $q > 1$, E is said to be q-uniformly smooth (or to have a modulus of smoothness of power type $q > 1$) if there exists $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. Hilbert spaces, $L_p(or\ \ l_p)$ spaces, $1 < p < \infty$, and the Sobolev spaces, W_m^p, $1 < p < \infty$, are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L_p(or\ \ell_p) \text{ or } W_m^p \text{ is} \begin{cases} p - \text{uniformly smooth if } 1 < p \leq 2 \\ 2 - \text{uniformly smooth if } p \geq 2. \end{cases}$$

Let E be a real normed space and let J_q, $(q > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = ||x||^q \ \text{ and } ||f|| = ||x||^{q-1}\},$$

where E^* denotes the dual space of E and $\langle ., .\rangle$ denotes the generalized duality pairing. It is well known (see, for example, Xu [210]) that $J_q(x) = ||x||^{q-2}J(x)$ if $x \neq 0$. For $q = 2$, the mapping $J = J_2$ from E to 2^{E^*} is called normalized duality mapping. It is well known that if E is uniformly smooth, then J is single-valued (see, e.g., [210, 211]). In the sequel, we shall denote the single-valued normalized duality mapping by j.

We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^2\}.$$

The following properties of J are well known (The reader may consult [49, 191, 192] for more details):

1. If E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E.
2. $J(x) \neq \emptyset, \quad x \in E$.
3. If E is reflexive, then J is a mapping from E onto E^*.
4. If E is smooth, then J is single valued.

It is well known that if the space E is uniformly smooth or E^* is strictly convex, then duality mapping J is single-valued (see, e.g., [210, 211]). Furthermore, if E has a uniformly Gâteaux differentiable norm then the duality mapping is norm-to-weak* uniformly continuous on bounded subsets of E. If $E = H$ is a Hilbert space then the duality mapping becomes the identity map of H.

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E. Following Alber [1], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) := arg\min_{y\in C}\phi(y, x), \quad \forall x \in E.$$

The existence and uniqueness of Π_C follow from the property of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example [1, 3, 86, 112, 192]). If E is a Hilbert space, then Π_C is the metric projection of H onto C.

We denote by ℓ^∞ the Banach space consisting of all bounded functions from $\mathbb{N}$ into $\mathbb{R}$ (i.e., all bounded real sequences) with supremum norm. We recall that $\mu \in (\ell^\infty)^*$ is called a *mean* if $||\mu|| = \mu(I_{\mathbb{N}}) = 1$. It is equivalent to

$$\inf_{n\in\mathbb{N}} a_n \leq \mu(a_n) \leq \sup_{n\in\mathbb{N}} a_n \tag{1.5.5}$$

for all $a_n \in \ell^\infty$. We also know that if $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\mu(a) \leq \mu(b)$ holds. Sometimes, we denote by $\mu_n(a_n)$ the value $\mu(a)$. $\mu \in (\ell^\infty)^*$ is called a *Banach limit* if the following hold:

(i) μ is a mean;
(ii) $\mu(a) = \mu_n(a_{n+1})$ for all $a \in \ell^\infty$. That is, putting $b_n = a_{n+1}$ for $n \in \mathbb{N}$, we have $\mu(a) = \mu(b)$. It is obvious that

$$\mu(a) = \mu_n(a_{n+k})$$

for a Banach limit $\mu, a \in \ell^\infty$, and $k \in \mathbb{N}$. We know that Banach limits exist; see [8]. We also know that

$$\liminf_{n\to\infty} a_n \leq \mu(a) \leq \limsup_{n\to\infty} a_n$$

for all $a \in \ell^\infty$.

A mapping H from E to E^* is said to be
(i) monotone if $\langle Hx - Hy, x - y\rangle \geq 0, \ \ \forall x, y \in E$;
(ii) strictly monotone if H is monotone and $\langle Hx - Hy, x - y\rangle = 0$ if and only if $x = y$;
(iii) β-Lipschitz continuous if there exists a constant $\beta \geq 0$ such that $||Hx - Hy|| \leq \beta||x - y||, \ \ \forall x, y \in E$.

Let M be a set valued mapping from E to E^* with domain $D(M) = \{z \in E : Mz \neq \emptyset\}$ and range $R(M) = \cup\{Mz : z \in D(M)\}$. A set-valued mapping M is said to be
(i) monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(M)$ and $y_i \in Mx_i, \ \ i = 1, 2$;
(ii) r-strongly monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq r||x_1 - x_2||^2$ for each $x_i \in D(M)$ and $y_i \in Mx_i, \ \ i = 1, 2$;
(iii) maximal monotone if M is monotone and its graph $G(M) := \{(x, y) : y \in Mx\}$ is not properly contained in the graph of any other monotone operator;
(iv) a general H-monotone if M is monotone and $(H + \lambda M)E = E^*$ holds for every $\lambda > 0$, where H is a mapping from E to E^*.

We denote the set $\{x \in E : 0 \in Mx\}$ by $M^{-1}0$. From Li *et al.* [121] we know that if $H : E \to E^*$ is strictly monotone and $M : E \to 2^{E^*}$ is a general H-monotone mapping, then $M^{-1}0$ is closed and convex. Furthermore, for every $\lambda > 0$ and $x^* \in E^*$, there exists a unique $x \in D(M)$ such that $x = (H + \lambda M)^{-1}x^*$. Thus, we may define a single-valued mapping $T_\lambda : E \to D(M)$ by $T_\lambda x = (H + \lambda M)^{-1}Hx$. It is obvious that $M^{-1}0 = F(T_\lambda)$ for all $\lambda > 0$.

Let $\{T_n\}$ be a sequence of mappings from C into E. For a subset B of C, we say that

(i) $(\{T_n\}, B)$ satisfies condition AKTT (see, for example, [5, 146]) if

$$\sum_{n=1}^{\infty} \sup\{||T_{n+1}x - T_n x|| : x \in B\} < \infty;$$

(ii) $(\{T_n\}, B)$ satisfies condition *AKTT (see, for example, [146]) if

$$\sum_{n=1}^{\infty} \sup\{||JT_{n+1}x - JT_n x|| : x \in B\} < \infty.$$

Let E be a smooth, strictly convex and reflexive Banach space. We denote a set-valued operator A from E to E^* by $A \subset E \times E^*$. We denote the set $\{x \in E : 0 \in Ax\}$ by $A^{-1}0$. A monotone operator A is said to be *maximal* if its graph $G(A) := \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then $A^{-1}0$ is closed and convex. Let $A \subset E \times E^*$ be a maximal monotone operator. Then for each $r > 0$ and $x \in E$, there corresponds a unique element $x_r \in D(A)$ satisfying

$$J(x) \in J(x_r) + rA(x_r),$$

see, for example, [191]. We define the *resolvent* of A by $J_r x = x_r$. In other words, $J_r = (I + rA)^{-1}J$ for all $r > 0$. We know that J_r is relatively nonexpansive and $A^{-1}0 = F(J_r)$ for all $r > 0$ (see, for example, [191]), where $F(J_r)$ denotes the fixed points set of J_r. Let A be a maximal monotone operator, we define the *Yosida approximation* of A by $A_r := r^{-1}(J - JJ_r),\ \ r > 0$. We know that $(J_r x, A_r x) \in A$ for all $r > 0$ and $x \in E$.

Let E be a reflexive, strictly convex and smooth Banach space. The duality mapping J^* from E^* into $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^*, that is, $J^* = J$. We make use of the following function V as studied by Alber [1]:

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||y||^2 \qquad (1.5.6)$$

for all $x \in E$ and $x^* \in E^*$. Thus $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$.

We shall make use of the following lemmas in the sequel.

Lemma 1.5.1 *Let E be a real normed space. Then*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, j(x + y)\rangle,$$

for all $x, y \in E$ and for all $j(x + y) \in J(x + y)$.

Lemma 1.5.2 *(see, e.g., [16, 200, 209]) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 1,$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\sigma_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ satisfy the conditions:
(i) $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty;$
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0,$
(iii) $\gamma_n \geq 0 (n \geq 1), \quad \sum_{n=1}^{\infty} \gamma_n < \infty$. *Then, $a_n \to 0$ as $n \to \infty$.*

Lemma 1.5.3 *(Chidume and Djitte, [62]) Let H be a real Hilbert space and $A : H \to H$ be a monotone mapping with $D(A) = H$. Suppose that $R(I + s_0A) = H$ for some $s_0 > 0$. Then A satisfies the range condition, i.e., $R(I + sA) = H$ for all $s > 0$.*

Lemma 1.5.4 *(Reich, [164]) Let H be a real Hilbert space and $A : H \to H$ be a monotone mapping with $D(A) = H$. Let $J_t x := (I + tA)^{-1}x, \quad t > 0$ be the resolvent of A and assume that $A^{-1}(0)$ is nonempty. Suppose A satisfies the range condition. Then, for each $x \in H$, $\lim_{t\to\infty} J_t x \in A^{-1}(0)$.*

Lemma 1.5.5 *(Cioranescu, [86]) Let A be a continuous monotone mapping defined on a real Hilbert space H with $D(A) = R(A) = H$. Then A is maximal monotone.*

We emphasize here that Lemma 1.5.5 is a special case of Corollary 2.7 in p. 156 of the book [86].

Lemma 1.5.6 *(Chidume, [49]) For $q > 1$, let X be a real q-uniformly smooth Banach space. Let $E := X \times X$ and with norm*

$$||x||_E := \Big(||x_1||_X^q + ||x_2||_X^q\Big)^{\frac{1}{q}},$$

for arbitrary $x = [x_1, x_2] \in E$. Let $E^* := X^* \times X^*$ denote the dual space of E. For arbitrary $x = [x_1, x_2] \in E$ define the map $j_q^E : E \to E^*$ by

$$j_q^E(x) = j_q^E[x_1, x_2] := [j_q^X(x_1), j_q^X(x_2)],$$

so that for arbitrary $x = [x_1, x_2], \quad y = [y_1, y_2]$ in E the duality pairing $\langle ., . \rangle$ is given by

$$\langle x, j_q^E(y)\rangle = \langle x_1, j_q^X(y_1)\rangle + \langle x_2, j_q^X(y_2)\rangle.$$

Then (a) j_q^E is a duality mapping on E; (b) E is q-uniformly smooth.

Lemma 1.5.7 *(Shioji and Takahashi, [181]) Let $(x_0, x_1, x_2, ...) \in l_\infty$ be such that $\mu_n x_n \leq 0$ for all Banach limits μ. If $\limsup_{n\to\infty}(x_{n+1} - x_n) \leq 0$, then $\limsup_{n\to\infty} x_n \leq 0$.*

Lemma 1.5.8 *(Xu, [210]) Let $q > 1$ and let E be a real Banach space. Then the following are equivalent:*
(1) E is q-uniformly smooth.
(2) There exists a constant $d_q > 0$ such that $\forall x, y \in E$

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x)\rangle + d_q||y||^q.$$

Let E be a 2-uniformly smooth Banach space. Then we know from [210] that there exists a constant $k > 0$ such that the following inequality holds:

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x)\rangle + k||y||^2, \quad \forall x, y \in E.$$

Lemma 1.5.9 *(Suzuki, [186]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}\Big(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||\Big) \leq 0$. Then, $\lim_{n\to\infty}||y_n - x_n|| = 0$.*

Lemma 1.5.10 *(Lim and Xu, [124]) Suppose E is a Banach space with uniform normal structure, K a nonempty bounded subset of E and $T : K \to K$ is a uniformly L-Lipschitzian mapping with $L < N(E)^{\frac{1}{2}}$. Suppose also there exists a nonempty bounded closed convex subset A of K with the following property (P):*

$$x \in A \ \text{ implies } \ \omega_w(x) \in A,$$

(where $\omega_w(x)$ is the ω-limit set of T at x, that is, the set $\{y \in E : y = \text{ weak } \ \omega-\lim T^{n_j}x \ \text{ for some } \ n_j \to \infty\}$). Then T has a fixed point in A.

Lemma 1.5.11 *(Lemaire, [119]) Let $M : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz continuous mapping. Then the mapping $S = M + A : H \to 2^H$ is a maximal monotone mapping.*

Lemma 1.5.12 *(Shimizu and Takahashi, [180]) Let D be a nonempty, bounded, closed and convex subset of a real Hilbert space H and let $\Im := \{T(u) : 0 \leq u < \infty\}$ a nonexpansive semigroup on D, then for any $h \geq 0$,*

$$\lim_{t\to\infty}\sup_{x\in D}\Big|\Big|T(h)\Big(\frac{1}{t}\int_0^t T(u)xdu\Big) - \Big(\frac{1}{t}\int_0^t T(u)xdu\Big)\Big|\Big| = 0.$$

Lemma 1.5.13 *(Alber [1], Kamimura and Takahashi [112]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C, \quad \forall y \in E.$$

Lemma 1.5.14 *(Alber [1], Kamimura and Takahashi [112]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $x \in E$ and $z \in C$. Then*

$$z = \Pi_C x \Leftrightarrow \langle y - z, J(x) - J(z) \rangle \leq 0, \quad \forall y \in C.$$

Lemma 1.5.15 *(Matsushita and Takahashi [134]) Let C be a nonempty, closed and convex subset of a smooth, strictly convex Banach space E. Let T be a relatively nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Lemma 1.5.16 *(Alber [1]) Let E be a real reflexive, strictly convex and Banach space and V be as in* (1.5.6). *Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 1.5.17 *(Kim et al. [45]) Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R}, \quad g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\Big|\Big| \sum_{n=1}^{\infty} \lambda_n x_n \Big|\Big|^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

The following lemma is an analogue of Lemma 1.5.17 with respect to ϕ.

Lemma 1.5.18 *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R}, \quad g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\phi\Big(x, J^{-1}\Big(\sum_{n=1}^{\infty} \lambda_n J x_n\Big)\Big) \leq \sum_{n=1}^{\infty} \lambda_n \phi(x, x_n) - \lambda_i \lambda_j g(||Jx_i - Jx_j||).$$

It is easy to see that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a smooth Banach space E, then $x_n - y_n \to 0, \quad n \to \infty$ implies that $\phi(x_n, y_n) \to 0, \quad n \to \infty$.

Lemma 1.5.19 *(Takahashi and Zembayashi [199]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $F : C \times C \to \mathbb{R}$ satisfies $(A1) - (A4)$ and let $r > 0$. Then for each $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Lemma 1.5.20 *(Kamimura and Takahashi [112]) Let C be a nonempty closed convex subset of a smooth, uniformly convex Banach space E. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be sequences in E such that either $\{x_n\}_{n=1}^\infty$ or $\{y_n\}_{n=1}^\infty$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

Lemma 1.5.21 (Mainge [128]) *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 1.5.22 *(Alber [1]) If E is a uniformly convex and uniformly smooth Banach space, $\delta_E(\epsilon)$ is the modulus of convexity of E and $\rho_E(t)$ is the modulus of smoothness of E, then the inequalities*

$$8d^2\delta_E(\frac{||x-\xi||}{4d}) \leq \phi(x,\xi) \leq 4d^2\rho_E(\frac{4||x-\xi||}{d})$$

hold for all x and ξ in E, where $d = \sqrt{\frac{(||x||^2+||\xi||^2)}{2}}$.

Lemma 1.5.23 *(Xia and Huang [208]) Let E be a Banach space with dual space E^*, $\quad H : E \to E^*$ a strictly monotone mapping and $M : E \to 2^{E^*}$ a general H-monotone mapping. Then*
(i) $(H + \lambda M)^{-1}$ is a single valued mapping;
(ii) if E is reflexive and $M : E \to 2^{E^*}$ is r-strongly monotone, $(H + \lambda M)^{-1}$ is Lipschitz continuous with constant $\frac{1}{\lambda r}$, where $r > 0$.

Lemma 1.5.24 *(Li et al. [121]) Let E be a uniformly convex real Banach space which is also uniformly smooth with $\delta_E(\epsilon) \geq k\epsilon^2$ and $\rho_E(t) \leq ct^2$ for some $k, c > 0$ and let E^* be the dual of E. Let $H : E \to E^*$ be a strictly monotone and β-Lipschitz continuous mapping and let $M : E \to 2^{E^*}$ be a general H-monotone mapping and r-strongly monotone mapping with $r > 0$. If there exists a $\lambda > 0$ such that $64c\beta^2 \leq \frac{1}{2}k\lambda^2 r^2$, then T_λ is a relatively nonexpansive mapping.*

We know that the following lemma holds in a 2-uniformly convex Banach space.

Lemma 1.5.25 *(Beauzamy [17]) Let E be a 2-uniformly convex Banach space, then for all x, y from any bounded set of E and $jx \in Jx, \quad jy \in Jy$, we have*

$$\langle x - y, jx - jy \rangle \geq \frac{c^2}{2} ||x - y||^2,$$

where $\frac{1}{c}$ is the 2-uniformly convexity constant.

The fixed points set $F(T)$ of a relatively quasi-nonexpansive mapping is closed and convex as given in the following lemma.

Lemma 1.5.26 *(Qin et al. [161], Nilsrakoo and Saejung [146]) Let C be a nonempty, closed and convex subset of a smooth, uniformly convex Banach space E. Let T be a closed relatively quasi-nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

We shall also use the following lemmas in our results.

Lemma 1.5.27 *(Baillon and Haddad [7]) Let E be a Banach space, let f be a continuously Freéchet differentiable convex functional on E and let ∇f be the gradient of f. If ∇f is $\frac{1}{\alpha}$-Lipschitz continuous, then ∇f is α-inverse-strongly monotone.*

Lemma 1.5.28 *(Nilsrakoo and Saejung [146], Aoyama et al. [5]) Let C be a nonempty, closed and subset of a Banach space E and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of K into E. Let B be a subset of C with $(\{T_n\}, B)$ satisfying AKTT. Then there exists a mapping $T : B \to E$ such that $Tx := \lim_{n\to\infty} T_n x, \quad \text{for all} \quad x \in B$ and $\lim_{n\to\infty} \sup\{||T_n x - Tx|| : x \in B\} = 0$.*

Lemma 1.5.29 *(Nilsrakoo and Saejung [146]) Let E be reflexive and strictly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty subset of E and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into E. Let B be a subset of C with $(\{T_n\}, B)$ satisfying *AKTT. Then there exists a mapping $T : B \to E$ such that $Tx := \lim_{n\to\infty} T_n x, \quad \text{for all} \quad x \in B$ and $\lim_{n\to\infty} \sup\{||JT_n x - JTx|| : x \in B\} = 0$.*

Lemma 1.5.30 *Optimization Theorem (see, e.g., Chidume [50] for a proof) Let E be a real reflexive Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex proper lower semi-continuous function. Suppose*

$$\lim_{||x||\to\infty} f(x) = +\infty.$$

Then, there exists $\overline{x} \in E$ such that $f(\overline{x}) \leq f(x), \forall x \in E$, i.e.,

$$f(\overline{x}) = \inf_{x\in E} f(x).$$

Part I

Approximation of Solution of Equations of Hammerstein Type

CHAPTER 2

Strong Convergence Theorem for Approximation of Solutions of Equations of Hammerstein Type

2.1 Introduction

Chidume and Zegeye [77] (Theorem 1.2.5 of Chapter 1) introduced an iterative scheme and proved strong convergence theorems for approximation of solution to a nonlinear integral equation of Hammerstein type (1.2.2) in real Hilbert spaces. In this chapter, we introduce a simpler iterative scheme and prove strong convergence theorems for approximation of solution to a nonlinear integral equation of Hammerstein type (1.2.2) in real Hilbert spaces.

We first prove this important lemma in real Hilbert spaces.

Lemma 2.1.1 *Let H be a real Hilbert space and $F, K : H \to H$ be monotone mappings with $D(F) = H = D(K)$. Suppose F and K satisfy the range condition (i.e., F and K are monotone mappings, the range $(I + rF) = H$ for all $r > 0$ and the range $(I + rK) = H$ for all $r > 0$.) Let $W := H \times H$ and $A : W \to W$ be a mapping defined by*

$$Aw = (Fu - v, Kv + u), \quad \forall w = (u, v) \in W.$$

Then, A is monotone and also satisfies the range condition.

Proof We know that

$$||x||_W := \Big(||x_1||_H^2 + ||x_2||_H^2\Big)^{\frac{1}{2}},$$

for arbitrary $x = [x_1, x_2] \in W$ and for arbitrary $x = [x_1, x_2],\ \ y = [y_1, y_2]$ in W the inner product $\langle ., . \rangle_W$ is given by

$$\langle x, y \rangle_W = \langle x_1, y_1 \rangle_H + \langle x_2, y_2 \rangle_H.$$

We first prove that A is monotone. Now let $w_1 = (u_1, v_1),\ \ w_2 = (u_2, v_2) \in W$. Then, we have $Aw_1 = (Fu_1 - v_1, Kv_1 + u_1)$ and $Aw_2 = (Fu_2 - v_2, Kv_2 + u_2)$. From this, we have $Aw_1 - Aw_2 = (Fu_1 - Fu_2 + v_2 - v_1, Kv_1 - Kv_2 + u_1 - u_2)$. Using the fact that F and K are monotone, we obtain

$$\begin{aligned} & \langle Aw_1 - Aw_2, w_1 - w_2 \rangle_W = \langle Fu_1 - Fu_2 + v_2 - v_1, u_1 - u_2 \rangle_H \\ & + \langle Kv_1 - Kv_2 + u_1 - u_2, v_1 - v_2 \rangle_H \\ = \ & \langle Fu_1 - Fu_2, u_1 - u_2 \rangle_H + \langle Kv_1 - Kv_2, (v_1 - v_2 \rangle_H \geq 0. \end{aligned}$$

So, A is monotone.
Next, we show that A satisfies the range condition. It suffices to show that $R(I_W + rA) = W$ for some $r \in (0, 1)$. In fact, let r_0 be such that $0 < r_0 < 1$. Since F and K are monotone and satisfy the range condition, then it is known that $(I + r_0 F)$ and $(I + r_0 K)$ are bijective and moreover, the resolvent $J_{r_0}^F := (I + r_0 F)^{-1}$ of F and the resolvent $J_{r_0}^K := (I + r_0 K)^{-1}$ of K are nonexpansive. Let $h := (h_1, h_2) \in W$. Define $G := W \to W$ by

$$Gw = \Big(J_{r_0}^F (h_1 + r_0 v), J_{r_0}^K (h_2 - r_0 u) \Big), \ \ \forall w = (u, v) \in W.$$

Using the fact that $J_{r_0}^F$ and $J_{r_0}^K$ are nonexpansive, we have

$$||Gw_1 - Gw_2||_W \leq r_0 ||w_1 - w_2||, \ \ \forall w_1, w_2 \in W.$$

Therefore, G is a contraction. So, by the Banach contraction mapping principle, G has a unique fixed point $w^* = (u^*, v^*) \in W$, that is, $Gw^* = w^*$ or equivalently,

$$u^* = J_{r_0}^F (h_1 + r_0 v^*) \ \text{ and } \ v^* = J_{r_0}^K (h_2 - r_0 u^*).$$

This implies that $(I_W + r_0 A)w = h$. Therefore, $R(I_W + r_0 A) = W$. By Lemma 1.5.3, it follows that A satisfies the range condition. This completes the proof. ■

Remark 2.1.2 *Suppose $u + KFu = 0$ in H and $A : H \times H \to H \times H$ is defined by $Aw = (Fu - v, Kv + u),\ \ \forall w = (u, v) \in H \times H$. Observe that u^* in H is a solution of $u + KFu = 0$ if and only if $w^* = (u^*, v^*)$ is a solution of $Aw = 0$ in $H \times H$ for $v^* = Fu^*$.*

2.2 Main Results

Theorem 2.2.1 *Let H be a real Hilbert space. Let $K, F : H \to H$ be bounded, continuous and monotone mappings. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in E defined iteratively from arbitrary $u_1, \ \ v_1 \in E$ by*

$$\begin{cases} u_{n+1} = u_n - \beta_n(Fu_n - v_n) - \beta_n(u_n - u_1), \\ v_{n+1} = v_n - \beta_n(Kv_n + u_n) - \beta_n(v_n - v_1), \quad n \geq 1 \end{cases} \tag{2.2.1}$$

where $\{\beta_n\}_{n=1}^{\infty}$ is a real sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\beta_n^2 < \infty$ and $\sum_{n=1}^{\infty}\beta_n = +\infty$. Then, there exists a constant $\gamma_0 > 0$ such that if $\beta_n \leq \gamma_0, \forall n \geq n_0$, for some $n_0 \in \mathbb{N}$, the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to a solution u^* of the equation $u + KFu = 0$.

Proof Let $W := H \times H$ with the norm $||w||_W := \Big(||u||^2 + ||v||^2\Big)^{\frac{1}{2}}$. Define the sequence $\{w_n\}_{n=1}^{\infty}$ in W by $w_n := (u_n, v_n)$. Let u^* be a solution of $u + KFu = 0, \ \ v^* = Fu^*$ and $w^* = (u^*, v^*)$. We observe that $u^* = -Kv^*$. We first show that the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are bounded. It suffices to show that $\{w_n\}_{n=1}^{\infty}$ is bounded. For this, let $n_0 \in \mathbb{N}$, then there exists $r > 0$ sufficiently large such that $w_1 \in B(w^*, \frac{r}{2}), \ \ w_{n_0} \in B(w^*, r)$. Define $B := \overline{B(w^*, r)}$. Since F and K are bounded, then

$$M_1 = \sup\Big\{||Fx - y||_H^2 + 2r^2 : (x, y) \in B\Big\} < +\infty$$

$$M_2 = \sup\{||Ky + x||_H^2 + 2r^2 : (x, y) \in B\} < +\infty.$$

Let $M = M_1 + M_2$. If we choose $\gamma_0 = \frac{r^2}{32(M+1)}$ and use the fact that $\beta_n \leq \gamma_0, \ \ \forall n \geq n_0$, then we have $\beta_n \leq \frac{r^2}{32(M+1)}, \ \ \forall n \geq n_0$. We show that $w_n \in B$ for all $n > n_0$. We do this by induction. By construction, $w_{n_0} \in B$. Suppose $w_n \in B$ for $n > n_0$. We prove that $w_{n+1} \in B$. Observe that

$$||w_{n+1} - w^*||^2 = ||u_{n+1} - u^*||^2 + ||v_{n+1} - v^*||^2.$$

By Lemma 1.5.1, we obtain

$$\begin{aligned} ||u_{n+1} - u^*||^2 &= ||u_n - u^* - \beta_n(Fu_n - v_n) - \beta_n(u_n - u_1)||^2 \\ &= ||u_n - u^*||^2 - 2\langle \beta_n(Fu_n - v_n) + \beta_n(u_n - u_1), u_n - u^*\rangle \\ &\quad +\beta_n^2||(Fu_n - v_n) + (u_n - u_1)||^2 \\ &\leq ||u_n - u^*||^2 - 2\langle \beta_n(Fu_n - v_n) + \beta_n(u_n - u_1), u_n - u^*\rangle \\ &\quad +4\beta_n^2 M_1 \\ &\leq ||u_n - u^*||^2 - 2\beta_n\langle Fu_n - v_n, u_n - u^*\rangle \\ &\quad -2\beta_n\langle u_n - u_1, u_n - u^*\rangle + 4\beta_n^2 M_1. \end{aligned} \tag{2.2.2}$$

But since F is monotone, we obtain

$$\begin{aligned}\langle Fu_n - v_n, u_n - u^*\rangle &= \langle Fu_n - Fu^*, u_n - u^*\rangle + \langle v^* - v_n, u_n - u^*\rangle \\ &\geq \langle v^* - v_n, u_n - u^*\rangle.\end{aligned}$$

Also,

$$\langle u_n - u_1, u_n - u^*\rangle = ||u_n - u^*||^2 + \langle u^* - u_1, u_n - u^*\rangle.$$

Substituting in (3.2.2), we obtain

$$\begin{aligned}||u_{n+1} - u^*||^2 &\leq (1 - 2\beta_n)||u_n - u^*||^2 - 2\beta_n\langle v^* - v_n, u_n - u^*\rangle \\ &\quad -2\beta_n\langle u^* - u_1, u_n - u^*\rangle + 4\beta_n^2 M_1. \qquad (2.2.3)\end{aligned}$$

Similarly,

$$\begin{aligned}||v_{n+1} - v^*||^2 &\leq (1 - 2\beta_n)||v_n - v^*||^2 - 2\beta_n\langle u_n - u^*, v_n - v^*\rangle \\ &\quad -2\beta_n\langle v^* - v_1, v_n - v^*\rangle + 4\beta_n^2 M_2. \qquad (2.2.4)\end{aligned}$$

Since $\langle v^* - v_n, u_n - u^*\rangle + \langle u_n - u^*, v_n - v^*\rangle = 0$, we obtain from (3.2.3) and (3.2.4) that

$$\begin{aligned}||w_{n+1} - w^*||^2 &\leq (1 - 2\beta_n)||w_n - w^*||^2 - 2\beta_n\Big[\langle v^* - v_n, u_n - u^*\rangle \\ &\quad +\langle u_n - u^*, v_n - v^*\rangle\Big] - 2\beta_n\Big[\langle u^* - u_1, u_n - u^*\rangle \\ &\quad +\langle v^* - v_1, v_n - v^*\rangle\Big] + 4\beta_n^2 M \qquad (2.2.5) \\ &\leq (1 - 2\beta_n)||w_n - w^*||^2 + \beta_n\Big[||w_1 - w^*||^2 + ||w_n - w^*||^2\Big] \\ &\quad +4\beta_n^2 M \\ &= (1 - \beta_n)||w_n - w^*||^2 + \beta_n||w_1 - w^*||^2 + 4\beta_n^2 M \\ &< (1 - \beta_n)r^2 + \beta_n\frac{r^2}{4} + 4\beta_n^2 M \\ &\leq (1 - \beta_n)r^2 + \frac{\beta_n r^2}{4} + \frac{\beta_n r^2 M}{8(M+1)} \\ &\leq (1 - \beta_n)r^2 + \frac{3\beta_n r^2}{8} \\ &= r^2 - \frac{5\beta_n r^2}{8} < r^2.\end{aligned}$$

Hence, $w_{n+1} \in B$. Thus, by induction, $\{w_n\}_{n=1}^\infty$ is bounded and so are $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$.

Next, we show that the sequence $\{u_n\}_{n=1}^\infty$ converges strongly to u^*. Since $\{w_n\}_{n=1}^\infty$ is bounded, we have that $\{Aw_n\}_{n=1}^\infty$ is bounded. Furthermore, since K and F are bounded, continuous and monotone mappings, we have that the mapping A is a bounded, continuous and monotone mapping and by Theorem 2 of [138], A satisfies the range condition. Observe that if for all $\gamma > 0$, we define

$$A_\gamma : W \to W \quad \text{by} \quad A_\gamma w = \gamma A w, \quad \forall w \in W = H \times H,$$

then we easily see that A_γ satisfies the range condition and

$$A^{-1}(0) = A_\gamma^{-1}(0) = F(J_s^{A_\gamma}) = \{w \in W : J_s^{A_\gamma} w = w\},$$

where $J_s^{A_\gamma}$ is the resolvent of the operator A_γ, $\forall \gamma > 0$. Observe that

$$||A_\gamma w_n|| = \gamma ||Aw_n|| \leq \gamma \sup_{w \in B'} ||Aw||, \quad \forall n \geq 1$$

(where $B' = B \cup \{w_1, w_2, \ldots, w_{n_0 - 1}\}$). This implies that $\lim_{\gamma \to 0} ||A_\gamma w_n|| = 0$. Furthermore, we obtain from Lemma 1.5.6 that $\lim_{s \to \infty} J_s^{A_\gamma} w_1 = w^* \in A^{-1}(0)$. We show that if

$$\xi_n := \max\{\langle w_1 - w^*, w_n - w^* \rangle, 0\}, \quad \forall n \geq 1,$$

then $\lim_{n \to \infty} \xi_n = 0$. We further observe that since $J_s^{A_\gamma} = (I + sA_\gamma)^{-1}$, then $(I + sA_\gamma) J_s^{A_\gamma} w_1 = w_1$. This implies that the composition $A_\gamma o J_s^{A_\gamma} w_1 = \frac{1}{s}\Big(w_1 - J_s^{A_\gamma} w_1\Big)$ and thus since A_γ is monotone, we have that

$$\Big\langle A_\gamma w_n - \frac{1}{s}\Big(w_1 - J_s^{A_\gamma} w_1\Big), w_n - J_s^{A_\gamma} w_1 \Big\rangle \geq 0, \quad \forall s > 0 \text{ and } \gamma > 0.$$

This implies that for some positive constant $M > 0$,

$$\begin{aligned} \Big\langle w_1 - J_s^{A_\gamma} w_1, w_n - J_s^{A_\gamma} w_1 \Big\rangle &\leq s\Big\langle A_\gamma w_n, w_n - J_s^{A_\gamma} w_1 \Big\rangle \\ &\leq ||A_\gamma w_n|| sM. \end{aligned}$$

Thus, $\limsup_{\gamma \to 0} \Big\langle w_1 - J_s^{A_\gamma} w_1, w_n - J_s^{A_\gamma} w_1 \Big\rangle \leq 0$, $\forall n \geq 1$. Therefore, given $\epsilon > 0$, there exists $\delta := \delta(\epsilon) > 0$ such that for all $\gamma \in (0, \delta]$,

$$\Big\langle w_1 - J_s^{A_\gamma} w_1, w_n - J_s^{A_\gamma} w_1 \Big\rangle < \epsilon.$$

Moreover, we have (in particular, for $\gamma = \delta$) that for some constant $M_0 > 0$,

$$\begin{aligned} \langle w_1 - w^*, w_n - w^* \rangle &= \Big\langle w_1 - w^*, (w_n - w^*) - (w_n - J_s^{A_\delta} w_1) \Big\rangle \\ &\quad + \Big\langle w_1 - J_s^{A_\delta} w_1, w_n - J_s^{A_\delta} w_1 \Big\rangle \\ &\quad + \Big\langle J_s^{A_\delta} w_1 - w^*, w_n - J_s^{A_\delta} w_1 \Big\rangle \\ &< \Big\langle w_1 - w^*, (w_n - w^*) - (w_n - J_s^{A_\delta} w_1) \Big\rangle \\ &\quad + \Big|\Big| J_s^{A_\delta} w_1 - w^* \Big|\Big| M_0 + \epsilon. \end{aligned} \tag{2.2.6}$$

Observe that

$$\lim_{s\to\infty}\Big\langle w_1 - w^*, (w_n - w^*) - (w_n - J_s^{A_\delta} w_1)\Big\rangle = 0.$$

Thus, as $s \to \infty$, we obtain from (2.2.6) that $\langle w_1 - w^*, w_n - w^*\rangle \le \epsilon$, so that

$$\limsup_{n\to\infty}\langle w_1 - w^*, w_n - w^*\rangle \le \epsilon \tag{2.2.7}$$

and since $\epsilon > 0$ is arbitrary, (2.2.7) gives

$$\limsup_{n\to\infty}\langle w_1 - w^*, w_n - w^*\rangle \le 0$$

from which we can deduce that $\lim\limits_{n\to\infty}\xi_n = 0$. From (3.2.7), we have

$$\begin{aligned}
||w_{n+1} - w^*||^2 &\le (1-2\beta_n)||w_n - w^*||^2 - 2\beta_n\Big[\langle v^* - v_n, u_n - u^*\rangle \\
&\quad +\langle u_n - u^*, v_n - v^*\rangle\Big] - 2\beta_n\Big[\langle u^* - u_1, u_n - u^*\rangle \\
&\quad +\langle v^* - v_1, v_n - v^*\rangle\Big] + 4\beta_n^2 M \\
&= (1-2\beta_n)||w_n - w^*||^2 + 2\beta_n\langle w_1 - w^*, w_n - w^*\rangle + \beta_n^2 M \\
&\le (1-2\beta_n)||w_n - w^*||^2 + 2\beta_n\xi_n + \beta_n^2 M.
\end{aligned}$$

Hence, by Lemma 1.5.2, we have that $w_n \to w^*$ as $n \to \infty$. But $w_n = (u_n, v_n)$ and $w^* = (u^*, v^*)$. This implies that $u_n \to u^*$. This completes the proof. ■

Corollary 2.2.2 *Let H be a real Hilbert space. Let $K, F : H \to H$ be bounded and maximal monotone mappings. Let $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ be sequences in H defined iteratively by (2.2.1), where $\{\beta_n\}_{n=1}^\infty$ is a real sequence in $(0,1)$ such that $\sum_{n=1}^\infty \beta_n^2 < \infty$ and $\sum_{n=1}^\infty \beta_n = +\infty$. Suppose that $u^* \in H$ is a solution to $u + KFu = 0$. Then, there exists a constant $\gamma_0 > 0$ such that if $\beta_n \le \gamma_0, \forall n \ge n_0$, for some $n_0 \in \mathbb{N}$, the sequence $\{u_n\}_{n=1}^\infty$ converges strongly to u^*.*

Proof Our desired result follows from Theorem 2.2.1 and Lemma 1.5.5. ■

Corollary 2.2.3 *Let H be a real Hilbert space. Let $K, F : H \to H$ be bounded monotone mappings satisfying the range condition. Let $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ be sequences in H defined iteratively by (2.2.1), where $\{\beta_n\}_{n=1}^\infty$ is a real sequence in $(0,1)$ such that $\sum_{n=1}^\infty \beta_n^2 < \infty$ and $\sum_{n=1}^\infty \beta_n = +\infty$. Suppose that $u^* \in H$ is a solution to $u + KFu = 0$. Then, there exists a constant $\gamma_0 > 0$ such that if $\beta_n \le \gamma_0, \forall n \ge n_0$, for some $n_0 \in \mathbb{N}$, the sequence $\{u_n\}_{n=1}^\infty$ converges strongly to u^*.*

Proof Our desired result follows from the same method of proof of Theorem 2.2.1 and Lemma 2.1.1. ∎

Clearly, every generalized Lipschitz map is bounded. So the following corollary is a special case of the main Theorem 2.2.1.

Corollary 2.2.4 *Let H be a real Hilbert space. Let $K, F : H \to H$ be a generalized Lipschitz maximal monotone mappings. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in H defined iteratively by* (2.2.1)*, where $\{\beta_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\beta_n^2 < \infty$ and $\sum_{n=1}^{\infty}\beta_n = +\infty$. Suppose that $u^* \in H$ is a solution to $u + KFu = 0$. Then, there exists a constant $\gamma_0 > 0$ such that if $\beta_n \leq \gamma_0, \forall n \geq n_0$, for some $n_0 \in \mathbb{N}$, the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*.*

We now give an example where the boundedness assumption on K and F can be dropped.

Corollary 2.2.5 *Let H be a real Hilbert space. Let $K, F : H \to H$ be uniformly continuous and maximal monotone mappings. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in H defined iteratively by* (2.2.1)*, where $\{\beta_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\beta_n^2 < \infty$ and $\sum_{n=1}^{\infty}\beta_n = +\infty$. Suppose that $u^* \in H$ is a solution to $u + KFu = 0$. Then, there exists a constant $\gamma_0 > 0$ such that if $\beta_n \leq \gamma_0, \forall n \geq n_0$, for some $n_0 \in \mathbb{N}$, the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*.*

Proof Our desired result follows from Theorem 2.2.1 and the result of [55]. ∎

The following example illustrates the conditions under which the Nemitskyi operator F is well defined, bounded, continuous and monotone. The reader can consult Pascali and Sburlan [152], Chapter IV, pages 166-167 for more details.

Example 2.2.6 *Let Ω be a domain of σ-finite measure in $\mathbb{R}^N$, let $M > 0$ be an integer and let $f : \Omega \times \mathbb{R}^M \to \mathbb{R}$ be a function satisfying the Caratheodory conditions:*
(i) $f(.,r) : \Omega \to \mathbb{R}$ is measurable for all fixed $r \in \mathbb{R}^M$;
(ii) $f(x,.) : \mathbb{R}^M \to \mathbb{R}$ is continuous for almost all $x \in \Omega$.
Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. To such a function we associate the superposition or Nemitskyi operator

$$Fu(x) = f(x, u(x)),$$

$u : \mathbb{R} \to \mathbb{R}^m$ is in L^p. Suppose that f verifies the conditions $(i) - (ii)$ and an inequality of the form

$$|f(x,u)| \leq g_1(x) + c\sum_{i=1}^{M} |u_i|^{p-1},$$

where $g_1 \in L^q(\Omega)$ *and* $c > 0$. *Then* F *is a well-defined bounded continuous operator from* $\Big[L^p(\Omega)\Big]^M$ *into* $L^q(\Omega)$.
If moreover, $f(x,r)$ *is nondecreasing (monotone) in* r *for each fixed* $x \in \Omega$, *i.e.,*

$$f(x,r) \leq f(x,s) \quad \text{for} \quad r \leq s,$$

then F *is a monotone operator from* $L^p(\Omega)$ *into* $L^q(\Omega)$.

Remark 2.2.7 *It is easy to see that the iterative scheme studied in this chapter seems far simpler than the iterative scheme used by Chidume and Zegeye [77] (Theorem 1.2.5 above) and Chidume and Ofoedu [68] (Theorem 1.2.10 above) in the sense that the conditions* $\sum_{n=1}^{\infty}\beta_n^2 < \infty$ *and* $\sum_{n=1}^{\infty}\beta_n = +\infty$ *on our iteration parameter are natural and not too strong compared to the conditions imposed on iteration parameters by these authors. Prototype for our iteration parameter is, for example,* $\beta_n = \frac{1}{n+1}, \quad n \geq 1$.

Remark 2.2.8 *All the results in this chapter are due to Chidume and Shehu [73].*

CHAPTER 3

Approximation of Solutions of Generalized Equations of Hammerstein Type

3.1 Introduction

In this chapter, we introduce an iterative scheme for approximating a solution of a generalized equation of Hammerstein type and prove strong convergence theorems of our scheme for approximation of solution of generalized equation of Hammerstein type in real Hilbert spaces.

We first prove an important lemma which concerns the Cartesian product of q-uniformly smooth Banach spaces.

Lemma 3.1.1 *For $q > 1$, let X be a real q-uniformly smooth Banach space. Let $E := X \times X \times \ldots \times X$ with N factors and with norm*

$$||x||_E := \Big(||x_1||_X^q + ||x_2||_X^q + \ldots + ||x_N||_X^q\Big)^{\frac{1}{q}},$$

for arbitrary $x = [x_1, x_2, \ldots, x_N] \in E$. Let $E^* := X^* \times X^* \times \ldots \times X^*$ with N factors denote the dual space of E. For arbitrary $x = [x_1, x_2, \ldots, x_N] \in E$ define the map $j_q^E : E \to E^*$ by

$$j_q^E(x) = j_q^E[x_1, x_2, \ldots, x_N] := [j_q^X(x_1), j_q^X(x_2), \ldots, j_q^X(x_N)],$$

so that for arbitrary $x = [x_1, x_2, \ldots, x_N]$, $y = [y_1, y_2, \ldots, y_N]$ in E the duality pairing $\langle ., . \rangle$ is given by

$$\langle x, j_q^E(y)\rangle = \langle x_1, j_q^X(y_1)\rangle + \langle x_2, j_q^X(y_2)\rangle + \ldots + \langle x_N, j_q^X(y_N)\rangle.$$

Then (a) j_q^E is a duality mapping on E; (b) E is q-uniformly smooth.

Proof (a) For arbitrary $x = [x_1, x_2, \ldots, x_N] \in E$, let $j_q^E(x) = j_q^E[x_1, x_2, \ldots, x_N] = \psi_q$. Then
$\psi_q = [j_q^X(x_1), j_q^X(x_2), \ldots, j_q^X(x_N)]$ in E^*. Observe that for $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}
||\psi_q||_{E^*} &= \Big(||[j_q^X(x_1), j_q^X(x_2), \ldots, j_q^X(x_N)]||\Big)^{\frac{1}{p}} \\
&= \Big(||j_q^X(x_1)||_{X^*}^p + ||j_q^X(x_2)||_{X^*}^p + \ldots + ||j_q^X(x_N)||_{X^*}^p\Big)^{\frac{1}{p}} \\
&= \Big(||x_1||_X^{(q-1)p} + ||x_2||_X^{(q-1)p} + \ldots + ||x_N||_X^{(q-1)p}\Big)^{\frac{1}{p}} \\
&= \Big(||x_1||_X^q + ||x_2||_X^q + \ldots + ||x_N||_X^q\Big)^{\frac{q-1}{q}} \\
&= ||x||_E^{q-1}.
\end{aligned}$$

Hence, $||\psi_q||_{E^*} = ||x||_E^{q-1}$. Furthermore,

$$\begin{aligned}
\langle x, \psi_q\rangle &= \langle [x_1, x_2, \ldots, x_N], [j_q^X(x_1), j_q^X(x_2), \ldots, j_q^X(x_N)]\rangle \\
&= \langle x_1, j_q^X(x_1)\rangle + \langle x_2, j_q^X(x_2)\rangle + \ldots + \langle x_N, j_q^X(x_N)\rangle \\
&= ||x_1||_X^q + ||x_2||_X^q + \ldots + ||x_N||_X^q \\
&= \Big(||x_1||_X^q + ||x_2||_X^q + \ldots + ||x_N||_X^q\Big)^{\frac{1}{q}} \Big(||x_1||_X^q + ||x_2||_X^q + \ldots + ||x_N||_X^q\Big)^{\frac{q-1}{q}} \\
&= ||x||_E . ||\psi_q||_{E^*}.
\end{aligned}$$

Hence, j_q^E is a single-valued duality mapping on E.
(b) Let $x = [x_1, x_2, \ldots, x_N]$, $\quad y = [y_1, y_2, \ldots, y_N]$ be arbitrary elements in E. We compute as follows:

$$\begin{aligned}
||x + y||_E^q &= ||[x_1 + y_1, x_2 + y_2, \ldots, x_N + y_N]||_E^q \\
&= ||x_1 + y_1||_X^q + ||x_2 + y_2||_X^q + \ldots + ||x_N + y_N||_E^q \\
&\leq ||x_1||_X^q + ||x_2||_X^q + \ldots + ||x_N||_X^q \\
&\quad + q\Big\{\langle y_1, j_q^X(x_1)\rangle + \langle y_2, j_q^X(x_2)\rangle + \ldots + \langle y_N, j_q^X(x_N)\rangle\Big\} \\
&\quad + d_q\Big(||y_1||_X^q + ||y_2||_X^q + \ldots + ||y_N||_X^q\Big),
\end{aligned}$$

for some constant $d_q > 0$ (since X is a q-uniformly smooth Banach space). It follows that

$$||x + y||_E^q \leq ||x||_E^q + q\langle y, j_q^E(x)\rangle + d_q||y||_E^q.$$

Hence, E is a q-uniformly smooth Banach space. ∎

3.2 Main Results

Theorem 3.2.1 *Let H be a real Hilbert space. For each $i = 1, 2, ...m$, let $F_i,\ K_i : H \to H$ be bounded and monotone mappings. Let $\{u_n\}_{n=1}^{\infty},\ \{v_{i,n}\}_{n=1}^{\infty},\ i = 1, 2, \ldots, m$ be sequences in H defined iteratively from arbitrary $u_1,\ v_{i,1} \in H$ by*

$$\begin{cases} u_{n+1} = u_n - \lambda_n\alpha_n\Big(u_n + \sum_{i=1}^{m} K_i v_{i,n}\Big) - \lambda_n\theta_n(u_n - u_1), \\ v_{i,n+1} = v_{i,n} - \lambda_n\alpha_n(F_i u_n - v_{i,n}) - \lambda_n\theta_n(v_{i,n} - v_{i,1}),\quad i = 1, 2, \ldots, m, \end{cases} \tag{3.2.1}$$

where $\{\lambda_n\}_{n=1}^{\infty},\ \{\alpha_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$ such that $\lambda_n = o(\theta_n),\ \alpha_n = o(\theta_n)$ and $\sum_{i=1}^{\infty}\lambda_n\theta_n = +\infty$. Suppose that $u+\sum_{i=1}^{m} K_iF_iu = 0$ has a solution in H. Then, there exist constants $\varepsilon_0,\ \varepsilon_1 > 0$ such that if $\alpha_n \le \varepsilon_0\theta_n$ and $\lambda_n \le \varepsilon_1\theta_n, \forall n \ge n_0$, for some $n_0 \in \mathbb{N}$, the sequences $\{u_n\}_{n=1}^{\infty},\ \{v_{i,n}\}_{n=1}^{\infty},\ i = 1, 2, \ldots, m$ are bounded.

Proof Let $W := H \times H \times \ldots \times H$ be the Cartesian product with $(m + 1)$ factors and the norm $||w||_W := \Big(||u||^2 + \sum_{i=1}^{m} ||x_i||^2\Big)^{\frac{1}{2}}$. Define the sequence $\{w_n\}_{n=1}^{\infty}$ in W by $w_n := (u_n, v_{1,n}, v_{2,n}, \ldots, v_{m,n})$. Let u^* be a solution of $u + \sum_{i=1}^{m} K_iF_iu = 0,\ \ x_1^* = F_1u^*,\ \ x_2^* = F_2u^*, \ldots, x_m^* = F_mu^*$ and $w^* = (u^*, x_1^*, x_2^*, \ldots, x_m^*)$. We observe that $u^* = -\sum_{i=1}^{m} K_ix_i^*$. It suffices to show that $\{w_n\}_{n=1}^{\infty}$ is bounded. For this, let $n_0 \in \mathbb{N}$, then there exists $r > 0$ sufficiently large such that $w_1 \in B(w^*, \frac{r}{2}),\ \ w_{n_0} \in B(w^*, r)$. Define $B := \overline{B(w^*, r)}$. Since $F_i,\ \ K_i$ are bounded, then

$$M_0 = \sup\Big\{\Big|\Big|u + \sum_{i=1}^{m} K_ix_i\Big|\Big|_H^2 + 2r^2 : (u, x_1, x_2, \ldots, x_m) \in B\Big\} < +\infty$$

$M_i = \sup\{||F_iu - x_i||_H^2 + 2r^2 : (u, x_1, x_2, \ldots, x_m) \in B\} < +\infty,\ i = 1, 2, \ldots, m.$

Let $M = \sum_{i=0}^{m} M_i$. We show that $w_n \in B$ for all $n > n_0$. We do this by induction. By construction, $w_{n_0} \in B$. Suppose $w_n \in B$ for $n > n_0$. We prove that $w_{n+1} \in B$. Observe that

$$||w_{n+1} - w^*||^2 = ||u_{n+1} - u^*||^2 + \sum_{i=1}^{m} ||v_{i,n+1} - x_i^*||^2.$$

Then we obtain

$$
\begin{aligned}
&||u_{n+1}-u^*||^2 = ||u_n-u^*-\lambda_n\alpha_n\Big(u_n+\sum_{i=1}^{m}K_iv_{i,n}\Big)-\lambda_n\theta_n(u_n-u_1)||^2\\
= \;& ||u_n-u^*||^2-2\lambda_n\Big\langle\alpha_n\Big(u_n+\sum_{i=1}^{m}K_iv_{i,n}\Big)+\theta_n(u_n-u_1),u_n-u^*\Big\rangle\\
& +\lambda_n^2\Big|\Big|\alpha_n\Big(u_n+\sum_{i=1}^{m}K_iv_{i,n}\Big)+\theta_n(u_n-u_1)\Big|\Big|^2\\
\leq \;& ||u_n-u^*||^2-2\lambda_n\alpha_n\Big\langle u_n+\sum_{i=1}^{m}K_iv_{i,n},u_n-u^*\Big\rangle\\
& -2\lambda_n\theta_n\langle u_n-u_1,u_n-u^*\rangle+4\lambda_n^2M_0. \qquad (3.2.2)
\end{aligned}
$$

But since K_i is monotone, we obtain

$$
\begin{aligned}
&\Big\langle u_n+\sum_{i=1}^{m}K_iv_{i,n},u_n-u^*\Big\rangle = \Big\langle u_n+\sum_{i=1}^{m}K_iv_{i,n}-u^*-\sum_{i=1}^{m}K_ix_i^*,u_n-u^*\Big\rangle\\
= \;& ||u_n-u^*||^2+\langle K_1v_{1,n}-K_1x_1^*,(u_n-u^*)+(v_{1,n}-x_1^*)-(v_{1,n}-x_1^*)\rangle\\
& +\langle K_2v_{2,n}-K_2x_2^*,(u_n-u^*)+(v_{2,n}-x_2^*)-(v_{2,n}-x_2^*)\rangle\\
& +\ldots+\langle K_mv_{m,n}-K_mx_m^*,(u_n-u^*)+(v_{m,n}-x_m^*)-(v_{m,n}-x_m^*)\rangle\\
\geq \;& ||u_n-u^*||^2+\langle K_1v_{1,n}-K_1x_1^*,(u_n-u^*)-(v_{1,n}-x_1^*)\rangle\\
& +\langle K_2v_{2,n}-K_2x_2^*,(u_n-u^*)-(v_{2,n}-x_2^*)\rangle\\
& +\ldots+\langle K_mv_{m,n}-K_mx_m^*,(u_n-u^*)-(v_{m,n}-x_m^*)\rangle
\end{aligned}
$$

and

$$\langle u_n-u_1,u_n-u^*\rangle = ||u_n-u^*||^2+\langle u^*-u_1,u_n-u^*\rangle.$$

Thus, substituting in (3.2.2), we obtain

$$
\begin{aligned}
||u_{n+1}-u^*||^2 \;\leq\; & (1-2\lambda_n\theta_n)||u_n-u^*||^2+2\lambda_n\alpha_nA_0\\
& -2\lambda_n\theta_n\langle u^*-u_1,u_n-u^*\rangle+4\lambda_n^2M_0 \qquad (3.2.3)
\end{aligned}
$$

where

$$
\begin{aligned}
A_0 \;=\; & \sup_{n\geq n_0}|\langle K_1v_{1,n}-K_1x_1^*,(u_n-u^*)-(v_{1,n}-x_1^*)\rangle\\
& +\langle K_2v_{2,n}-K_2x_2^*,(u_n-u^*)-(v_{2,n}-x_2^*)\rangle\\
& +\ldots+\langle K_mv_{m,n}-K_mx_m^*,(u_n-u^*)-(v_{m,n}-x_m^*)\rangle|.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&||v_{1,n+1}-x_1^*||^2 = ||v_{1,n}-x_1^*-\lambda_n\alpha_n(F_1u_n-v_{1,n})-\lambda_n\theta_n(v_{1,n}-v_{1,1})||^2\\
\leq\ & ||v_{1,n}-x_1^*||^2-2\lambda_n\Big\langle\alpha_n(F_1u_n-v_{1,n})+\theta_n(v_{1,n}-v_{1,1}),v_{1,n}-x_1^*\Big\rangle\\
&+\lambda_n^2||\alpha_n(F_1u_n-v_{1,n})+\theta_n(v_{1,n}-v_{1,1})||^2\\
\leq\ & ||v_{1,n}-x_1^*||^2-2\lambda_n\alpha_n\langle F_1u_n-v_{1,n},v_{1,n}-x_1^*\rangle\\
&-2\lambda_n\theta_n||v_{1,n}-x_1^*||^2-2\lambda_n\theta_n\langle x_1^*-v_{1,1},v_{1,n}-x_1^*\rangle+4\lambda_n^2M_1. \qquad (3.2.4)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\langle F_1u_n-v_{1,n},v_{1,n}-x_1^*\rangle = \langle F_1u_n-F_1v_{1,n}+F_1v_{1,n}-F_1x_1^*\\
&+F_1x_1^*-v_{1,n},v_{1,n}-x_1^*\rangle\\
=\ & \langle F_1v_{1,n}-F_1x_1^*,v_{1,n}-x_1^*\rangle+\langle F_1u_n-F_1v_{1,n}+F_1x_1^*-v_{1,n},v_{1,n}-x_1^*\rangle\\
\geq\ & \langle F_1u_n-F_1v_{1,n}+F_1x_1^*-v_{1,n},v_{1,n}-x_1^*\rangle.
\end{aligned}
$$

Substituting this into (3.2.4), we have

$$
\begin{aligned}
||v_{1,n+1}-x_1^*||^2 \ \leq\ & (1-2\lambda_n\theta_n)||v_{1,n}-x_1^*||^2+2\lambda_n\alpha_nA_1\\
&-2\lambda_n\theta_n\langle x_1^*-v_{1,1},v_{1,n}-x_1^*\rangle+4\lambda_n^2M_1 \quad (3.2.5)
\end{aligned}
$$

where $A_1=\sup_{n\geq n_0}|\langle F_1u_n-F_1v_{1,n}+F_1x_1^*-v_{1,n},v_{1,n}-x_1^*\rangle|$. Continuing, we have for each $i=2,3,\dots,m$ that

$$
\begin{aligned}
||v_{i,n+1}-x_1^*||^2 \ \leq\ & (1-2\lambda_n\theta_n)||v_{i,n}-x_i^*||^2+2\lambda_n\alpha_nA_i\\
&-2\lambda_n\theta_n\langle x_i^*-v_{i,1},v_{i,n}-x_i^*\rangle+4\lambda_n^2M_i \quad (3.2.6)
\end{aligned}
$$

where $A_i=\sup_{n\geq n_0}|\langle F_iu_n-F_iv_{1,n}+F_ix_i^*-v_{1,n},v_{i,n}-x_i^*\rangle|$. From (3.2.3), (3.2.5) and (3.2.6), we obtain

$$
\begin{aligned}
&||w_{n+1}-w^*||^2\leq(1-2\lambda_n\theta_n)||w_n-w^*||^2+2\lambda_n\alpha_nA\\
&-2\lambda_n\theta_n\Big[\langle u^*-u_1,u_n-u^*\rangle+\sum_{i=1}^m\langle x_i^*-v_{i,1},v_{i,n}-x_i^*\rangle\Big]\\
&+4\lambda_n^2M \qquad (3.2.7)\\
\leq\ & (1-2\lambda_n\theta_n)||w_n-w^*||^2+2\lambda_n\alpha_nA+4\lambda_n^2M\\
&+\lambda_n\theta_n\Big[||w_1-w^*||^2+||w_n-w^*||^2\Big]\\
=\ & (1-\lambda_n\theta_n)||w_n-w^*||^2+\lambda_n\theta_n||w_1-w^*||^2+2\lambda_n\alpha_nA+4\lambda_n^2M\\
<\ & (1-\lambda_n\theta_n)r^2+\lambda_n\theta_n\frac{r^2}{4}+2\lambda_n\alpha_nA+4\lambda_n^2M,
\end{aligned}
$$

where $A=\sum_{i=0}^m A_i$. If we choose $\varepsilon_0=\frac{r^2}{16(A+1)}$ and $\varepsilon_1=\frac{r^2}{64(M+1)}$ and use the fact that $\alpha_n\leq\varepsilon_0\theta_n$ and $\lambda_n\leq\varepsilon_1\theta_n$, we obtain

$$
||w_{n+1}-w^*||^2<r^2-\frac{9\lambda_n\theta_nr^2}{16}<r^2.
$$

Hence, $w_{n+1} \in B$. Thus, by induction, $\{w_n\}_{n=1}^{\infty}$ is bounded and so are $\{u_n\}_{n=1}^{\infty}$, $\{v_{i,n}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, m$. This completes the proof. ■

Remark 3.2.2 *Examples of* $\{\lambda_n\}, \{\theta_n\}$ *and* $\{\alpha_n\}$ *in Theorem 3.2.1 are* $\lambda_n = \frac{1}{(n+1)^a}, \theta_n = \frac{1}{(n+1)^b}$ *and* $\alpha_n = \frac{1}{n+1}$ *where* $0 < b < a$ *and* $a + b < 1$. *Furthermore, if* $\lambda_n = \frac{1}{n+1} = \theta_n$, *then* $\sum_{i=1}^{\infty} \lambda_n \theta_n < +\infty$.

Theorem 3.2.3 *Let* H *be a real Hilbert space. For each* $i = 1, 2, ...m$, *let* F_i, $K_i : H \to H$ *be bounded and monotone mappings. Let* $\{u_n\}_{n=1}^{\infty}, \{v_{i,n}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, m$ *be sequences in* H *defined iteratively by* (3.2.1), *where* $\{\lambda_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\theta_n\}_{n=1}^{\infty}$ *are sequences in* $(0, 1)$ *such that* $\lambda_n = o(\theta_n), \alpha_n = o(\theta_n)$ *and* $\sum_{i=1}^{\infty} \lambda_n \theta_n = +\infty$. *Suppose that* $u + \sum_{i=1}^{m} K_i F_i u = 0$ *has a solution* u^* *in* H. *Then, there exist constants* $\varepsilon_0, \varepsilon_1 > 0$ *and a set* $\Omega \subset W$ *such that if* $\alpha_n \leq \varepsilon_0 \theta_n$ *and* $\lambda_n \leq \varepsilon_1 \theta_n, \forall n \geq n_0$, *for some* $n_0 \in \mathbb{N}$ *and* $w^* := (u^*, x_1^*, x_2^*, \ldots, x_m^*) \in \Omega$ *(where* $x_i^* = F_i u^*$, $i = 1, 2, \ldots, m$*), the sequence* $\{u_n\}_{n=1}^{\infty}$ *converges strongly to* u^*.

Proof Since, by Theorem 3.2.1, we have that $\{u_n\}_{n=1}^{\infty}$, $\{v_{i,n}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, m$ are bounded, there exists $R > 0$ sufficiently large such that $u_n \in \overline{B_H(u^*, R)}$, $v_{i,n} \in \overline{B_H(x_i^*, R)}$, $\forall n \geq 1$, $i = 1, 2, \ldots, m$. Furthermore, the sets $\overline{B_H(u^*, R)}$ and $\overline{B_H(x_i^*, R)}$ are bounded, closed, convex and nonempty subsets of H. For each $i = 0, 1, \ldots, m$, define the maps $\varphi_i : H \to \mathbb{R}$ by

$$\varphi_0(x) := \mu_n ||u_n - x||_H^2, \quad \varphi_i(y) := \mu_n ||v_{i,n} - y||_H^2, \quad i = 1, \ldots, m$$

(where μ is a Banach limit). Then for each $i = 0, 1, \ldots, m$, φ_i is continuous, convex and coercive. Since H is reflexive, there exist $x^* \in \overline{B_H(u^*, R)}$ and $y_i^* \in \overline{B_H(x_i^*, R)} (i = 1, \ldots, m)$ such that

$$\varphi_0(x^*) = \min\{\varphi_0(x) : x \in \overline{B_H(u^*, R)}\}$$

and

$$\varphi_i(y_i^*) = \min\{\varphi_i(y) : y \in \overline{B_H(x_i^*, R)}\}.$$

So, the sets

$$\Omega_0 := \Big\{u \in \overline{B_H(u^*, R)} : \varphi_0(u) = \min_{x \in \overline{B_H(u^*, R)}} \varphi_0(x)\Big\}$$

and

$$\Omega_i := \Big\{x_i \in \overline{B_H(x_i^*, R)} : \varphi_i(x_i) = \min_{y \in \overline{B_H(x_i^*, R)}} \varphi_i(y)\Big\}$$

are nonempty sets.

Let $t \in (0, 1)$, $\Omega := \Omega_0 \times \Omega_1 \times \ldots \times \Omega_m$ and let $w^* := (u^*, x_1^*, x_2^*, \ldots, x_m^*) \in \Omega$. Then, by convexity of $\overline{B_H(u^*, R)}$, we have that $(1 - t)u^* + tu_1 \in \overline{B_H(u^*, R)}$.

Thus, $\mu_n||u_n - u^*||^2 \leq \mu_n||u_n - (1-t)u^* - tu_1||^2$. Moreover, we have, by Lemma 1.5.1 that

$$||u_n - u^* - t(u_1 - u^*)||^2 \leq ||u_n - u^*||^2 - 2t\langle u_1 - u^*, u_n - u^* - t(u_1 - u^*)\rangle.$$

This implies that $\mu_n\langle u_1 - u^*, u_n - u^* - t(u_1 - u^*)\rangle \leq 0$. Furthermore, we obtain that

$$\lim_{t\to 0}\Big(\langle u_1 - u^*, u_n - u^*\rangle - \langle u_1 - u^*, u_n - u^* - t(u_1 - u^*)\rangle\Big) = 0.$$

Thus, given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $t \in (0, \delta_\epsilon)$ and $\forall n \in \mathbb{N}$,

$$\langle u_1 - u^*, u_n - u^*\rangle < \epsilon + \langle u_1 - u^*, u_n - u^* - t(u_1 - u^*)\rangle.$$

Taking Banach limit on both sides of this inequality, we obtain

$$\mu_n\langle u_1 - u^*, u_n - u^*\rangle \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\mu_n\langle u_1 - u^*, u_n - u^*\rangle \leq 0.$$

Furthermore, since $\{u_n\}_{n=1}^\infty$, $\{v_{i,n}\}_{n=1}^\infty$, F_i, K_i, $i = 1, 2, \dots, m$ are all bounded, we have that

$$||u_{n+1} - u_n|| \leq \lambda_n\Big[\alpha_n(||u_n|| + \sum_{i=1}^m ||K_i v_{i,n}||) + \theta_n||u_n - u_1||\Big] \leq \lambda_n K_0,$$

for some constant $K_0 > 0$. Thus, $\lim_{n\to\infty}||u_{n+1} - u_n|| = 0$. Again, we have

$$\lim_{n\to\infty}\Big(\langle u_1 - u^*, u_{n+1} - u^*\rangle - \langle u_1 - u^*, u_n - u^*\rangle\Big) = 0.$$

Thus, the sequence $\{\langle u_1 - u^*, u_n - u^*\rangle\}$ satisfies the conditions of Lemma 1.5.7. Hence, we obtain that

$$\limsup_{n\to\infty}\langle u_1 - u^*, u_n - u^*\rangle \leq 0.$$

Following the same line of arguments, we obtain

$$\limsup_{n\to\infty}\langle v_{i,1} - x_i^*, v_{i,n} - x_i^*\rangle \leq 0, \quad i = 1, 2, \dots, m.$$

Now, define

$$\sigma_n := \max\{\langle u_1 - u^*, u_n - u^*\rangle, 0\} \text{ and } \xi_{i,n} := \max\{\langle v_{i,1} - x_i^*, v_{i,n} - x_i^*\rangle, 0\}, \ (i = 1, \dots, m)$$

then, $\lim_{n\to\infty}\sigma_n = 0 = \lim_{n\to\infty}\xi_{i,n}$, $i = 1, 2, \dots, m$. Furthermore,

$$\langle u_1 - u^*, u_n - u^*\rangle \leq \sigma_n$$

$$\langle v_{i,1} - x_i^*, v_{i,n} - x_i^* \rangle \le \xi_{i,n}, \quad i = 1, 2, \ldots, m.$$

From (3.2.7), we have

$$\begin{aligned}
&||w_{n+1} - w^*||^2 \le (1 - 2\lambda_n\theta_n)||w_n - w^*||^2 + 2\lambda_n\alpha_n A \\
&-2\lambda_n\theta_n\Big[\langle u^* - u_1, u_n - u^* \rangle + \sum_{i=1}^{m} \langle x_i^* - v_{i,1}, v_{i,n} - x_i^* \rangle\Big] + 4\lambda_n^2 M \\
\le\ & (1 - 2\lambda_n\theta_n)||w_n - w^*||^2 + 2(\lambda_n\alpha_n + \lambda_n^2)c + 2\lambda_n\theta_n\Big(\sigma_n + \sum_{i=1}^{m} \xi_{i,n}\Big) \\
=\ & (1 - 2\lambda_n\theta_n)||w_n - w^*||^2 + \gamma_n,
\end{aligned}$$

where $\gamma_n = 2(\lambda_n\alpha_n + \lambda_n^2)c + 2\lambda_n\theta_n\Big(\sigma_n + \sum_{i=1}^{m} \xi_{i,n}\Big) = o(2\lambda_n\theta_n)$ for some $c > 0$. Hence, by Lemma 1.5.2, we have that $w_n \to w^*$ as $n \to \infty$. But $w_n = (u_n, v_{1,n}, v_{2,n}, \ldots, v_{m,n})$ and $w^* = (u^*, x_1^*, x_2^*, \ldots, x_m^*)$. This implies that $u_n \to u^*$. This completes the proof. ∎

Clearly, every generalized Lipschitz map is bounded. So, we obtain the following corollaries.

Corollary 3.2.4 *Let H be a real Hilbert space. For each $i = 1, 2, ...m$, let F_i, $K_i : H \to H$ be generalized Lipschitz and monotone mappings. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_{i,n}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, m$ be sequences in H defined iteratively by (3.2.1), where $\{\lambda_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$ such that $\lambda_n = o(\theta_n)$, $\alpha_n = o(\theta_n)$ and $\sum_{i=1}^{\infty} \lambda_n\theta_n = +\infty$. Suppose that $u + \sum_{i=1}^{m} K_iF_iu = 0$ has a solution in H. Then, there exist constants ε_0, $\varepsilon_1 > 0$ and a set $\Omega \subset W$ such that if $\alpha_n \le \varepsilon_0\theta_n$ and $\lambda_n \le \varepsilon_1\theta_n, \forall n \ge n_0$, for some $n_0 \in \mathbb{N}$ and $w^* := (u^*, x_1^*, x_2^*, \ldots, x_m^*) \in \Omega$ (where $x_i^* = F_iu^*$, $i = 1, 2, \ldots, m$), the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*.*

Corollary 3.2.5 *Let H be a real Hilbert space. For each $i = 1, 2, ...m$, let $F_i : H \to H$ be generalized Lipschitz, monotone mapping and $K_i : H \to H$ bounded, monotone mapping. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_{i,n}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, m$ be sequences in H defined iteratively by (3.2.1), where $\{\lambda_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$ such that $\lambda_n = o(\theta_n)$, $\alpha_n = o(\theta_n)$ and $\sum_{i=1}^{\infty} \lambda_n\theta_n = +\infty$. Suppose that $u + \sum_{i=1}^{m} K_iF_iu = 0$ has a solution in H. Then, there exist constants ε_0, $\varepsilon_1 > 0$ and a set $\Omega \subset W$ such that if $\alpha_n \le \varepsilon_0\theta_n$ and $\lambda_n \le \varepsilon_1\theta_n, \forall n \ge n_0$, for some $n_0 \in \mathbb{N}$ and $w^* := (u^*, x_1^*, x_2^*, \ldots, x_m^*) \in \Omega$ (where $x_i^* = F_iu^*$, $i = 1, 2, \ldots, m$), the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*.*

Remark 3.2.6 *All the results in this chapter are due to Chidume and Shehu [74].*

Remark 3.2.7 *Our Theorem 3.2.1 has recently been proved in q-uniformly smooth Banach spaces which include L_p spaces with $1 < p < \infty$ and this* *result has just been published (see [72]). This our new result extends the result of Chidume and Ofoedu [68].*

Part II

Iterative Algorithm for Common Fixed Points of a Family of Mappings

CHAPTER 4

Strong Convergence Theorems for a Mann-Type Iterative Scheme for a Family of Lipschitzian Mappings

4.1 Introduction

Aoyama *et al.*[5] used Halpern-type iteration process (see, e.g., Halpern [101]) to obtain strong convergence to common fixed points for a countable family of nonexpansive mappings in a uniformly convex Banach space with uniformly Gâteaux differentiable norm. It is our purpose in this chapter to prove strong convergence theorems for common fixed points of L_n-Lipschitzian mappings in real Banach spaces much more general than uniformly convex real Banach spaces considered in Aoyama *et al.* [5].

4.2 Main Results

Lemma 4.2.1 *Let K be a nonempty closed convex subset of a real Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of L_n-Lipschitzian mappings of K into itself with $L_n \geq 1$, $\sum\limits_{n=1}^{\infty}(L_n - 1) < \infty$. Let $\bigcap\limits_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K defined by $x_1 = u \in K$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T_n x_n, \tag{4.2.1}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}_{n=1}^{\infty}$ is bounded.

Proof Let $x^* \in \bigcap\limits_{n=1}^{\infty} F(T_n)$. Then

$$\begin{aligned}
||x_{n+1} - x^*|| &\leq \alpha_n||u - x^*|| + (1-\alpha_n)||T_n x_n - x^*|| \\
&\leq \alpha_n||u - x^*|| + (1-\alpha_n)L_n||x_n - x^*|| \\
&\leq \alpha_n||u - x^*|| + (1-\alpha_n)e^{(L_n-1)}||x_n - x^*|| \\
&\leq e^{(L_n-1)}\Big[\alpha_n||u - x^*|| + (1-\alpha_n)||x_n - x^*||\Big] \\
&\leq 2e^{(L_n-1)}\max\{||u - x^*||, ||x_n - x^*||\} \\
&\vdots \\
&\leq 2e^{\sum\limits_{n=1}^{\infty}(L_n-1)}\max\{||u - x^*||, ||x_1 - x^*||\} < \infty. \quad (4.2.2)
\end{aligned}$$

Hence, $\{x_n\}_{n=1}^{\infty}$ is bounded and so, $\{T_n x_n\}_{n=1}^{\infty}$ is also bounded. ■

Remark 4.2.2 *Since $\{x_n\}_{n=1}^{\infty}$ is bounded, there exists $R > 0$ sufficiently large such that $x_n \in B^* := \overline{B_R(x^*)}$, $\forall n \in \mathbb{N}$. Furthermore, the set $K \bigcap B^*$ is a bounded closed and convex nonempty subset of E. If we define a map $\varphi : E \to \mathbb{R}$ by $\varphi(y) = \mu_n||x_n - y||^2$, where μ_n denotes a Banach limit, then, φ is continuous, convex and $\varphi(y) \to +\infty$ as $||y|| \to +\infty$. Thus, if E is a reflexive Banach space, there exists $x_0 \in K \bigcap B^*$ such that $\varphi(x_0) = \min\limits_{y \in K \bigcap B^*} \varphi(y)$. So,*

$$K_{min} := \Big\{x \in K \bigcap B^* : \varphi(x) = \min_{y \in K \bigcap B^*} \varphi(y)\Big\} \neq \emptyset.$$

We now prove the following theorems.

Theorem 4.2.3 *Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm. Let K be a nonempty closed convex subset of E and $\{T_n\}_{n=1}^{\infty}$ be a sequence of L_n-Lipschitzian mappings of K into itself with $L_n \geq 1$, $\sum\limits_{n=1}^{\infty}(L_n - 1) < \infty$. Let $\bigcap\limits_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K defined by $x_1 = u \in K$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T_n x_n, \quad (4.2.3)$$

for all $n \in \mathbb{N}$. Suppose that $\lim\limits_{n\to\infty}||T_{n+1}x_{n+1} - T_n x_n|| = 0$ and $(K_{min}) \bigcap\limits_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to some common fixed point of $\{T_n\}_{n=1}^{\infty}$.

Proof Let $x^* \in (K_{min}) \bigcap_{n=1}^{\infty} F(T_n)$ and $t \in (0,1)$. Then by the convexity of $K \bigcap B^*$ we have that $(1-t)x^* + tu \in K \bigcap B^*$. It then follows from Remark 4.2.2 that $\varphi(x^*) \leq ((1-t)x^* + tu)$. Using Lemma 1.5.1, we have that

$$||x_n - x^* - t(u - x^*)||^2 \leq ||x_n - x^*||^2 - 2t\langle u - x^*, j(x_n - x^* - t(u - x^*)\rangle.$$

Thus, taking Banach limits over $n \geq 1$ gives

$$\begin{aligned} &\mu_n||x_n - x^* - t(u - x^*)||^2 \leq \mu_n||x_n - x^*||^2 \\ &-2t\mu_n\langle u - x^*, j(x_n - x^* - t(u - x^*)\rangle. \end{aligned} \tag{4.2.4}$$

This implies, $2t\mu_n\langle u-x^*, j(x_n-x^*-t(u-x^*)\rangle \leq \varphi(x^*) \leq \varphi((1-t)x^*+tu) \leq 0$, so that $\mu_n\langle u-x^*, j(x_n-x^*-t(u-x^*)\rangle \leq 0, \ \ \forall n \in \mathbb{N}$. Since the normalized duality mapping is norm-to-weak* uniformly continuous on bounded subsets of E, we obtain, as $t \to 0$, that

$$\langle u - x^*, j(x_n - x^*)\rangle - \langle u - x^*, j(x_n - x^* - t(u - x^*))\rangle \to 0.$$

Hence, for all $\epsilon > 0$, there exists $\delta > 0$ such that $\forall t \in (0, \delta)$ and for all $n \geq 1$,

$$\langle u - x^*, j(x_n - x^*)\rangle < \langle u - x^*, j(x_n - x^* - t(u - x^*))\rangle + \epsilon.$$

Consequently,

$$\mu_n\langle u - x^*, j(x_n - x^*)\rangle < \mu_n\langle u - x^*, j(x_n - x^* - t(u - x^*))\rangle + \epsilon \leq \epsilon.$$

Since ϵ is arbitrary, we have $\mu_n\langle u-x^*, j(x_n-x^*)\rangle \leq 0$. Let $M := \sup_{n\in\mathbb{N}}||T_n x_n||$. We next show that $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$. Now

$$\begin{aligned} & ||x_{n+1} - x_n|| = ||\alpha_n u + (1-\alpha_n)T_n x_n - (\alpha_{n-1}u + (1-\alpha_{n-1})T_{n-1}x_{n-1})|| \\ = \ & ||(\alpha_n - \alpha_{n-1})u + (1-\alpha_n)(T_n x_n - T_{n-1}x_{n-1}) + (\alpha_{n-1} - \alpha_n)T_{n-1}x_{n-1}|| \\ \leq \ & |\alpha_n - \alpha_{n-1}|||u|| + (1-\alpha_n)||T_n x_n - T_{n-1}x_{n-1}|| + |\alpha_n - \alpha_{n-1}|||T_{n-1}x_{n-1}|| \\ \leq \ & |\alpha_n - \alpha_{n-1}|(||u|| + M) + (1-\alpha_n)||T_n x_n - T_{n-1}x_{n-1}||, \quad \text{for each } n \in \mathbb{N}. \end{aligned}$$

The last inequality implies that $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$. Therefore, from the norm-to-weak* uniform continuity on j on bounded sets, we obtain that

$$\lim_{n\to\infty}(\langle u - x^*, j(x_{n+1} - x^*)\rangle - \langle u - x^*, j(x_n - x^*)\rangle) = 0.$$

Thus, the sequence $\{\langle u - x^*, j(x_n - x^*)\rangle\}$ satisfies the conditions of Lemma 1.5.7. Hence, we obtain that $\limsup_{n\to\infty}\langle u - x^*, j(x_n - x^*)\rangle \leq 0$. Now, using

Lemma 1.5.1 and the recursion formula (4.2.3), we obtain

$$\begin{aligned}
||x_{n+1}-x^*||^2 &\leq ||\alpha_n(u-x^*)+(1-\alpha_n)(T_nx_n-x^*)||^2\\
&\leq (1-\alpha_n)^2||T_nx_n-x^*||^2+2\alpha_n\langle u-x^*,j(x_{n+1}-x^*)\rangle\\
&\leq (1-\alpha_n)^2L_n^2||x_n-x^*||^2+2\alpha_n\langle u-x^*,j(x_{n+1}-x^*)\rangle\\
&\leq (1-\alpha_n)L_n^2||x_n-x^*||^2+2\alpha_n\langle u-x^*,j(x_{n+1}-x^*)\rangle\\
&= (1-\alpha_n)||x_n-x^*||^2+(1-\alpha_n)(L_n^2-1)||x_n-x^*||^2\\
&\quad +2\alpha_n\langle u-x^*,j(x_{n+1}-x^*)\rangle\\
&\leq (1-\alpha_n)||x_n-x^*||^2+2\alpha_n\langle u-x^*,j(x_{n+1}-x^*)\rangle\\
&\quad +M'(L_n-1),
\end{aligned}$$

for some $M' > 0$. Hence, by Lemma 1.5.2, we have that the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$. This completes the proof. ■

Theorem 4.2.4 *Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm. Let K be a nonempty closed convex subset of E and $\{T_n\}_{n=1}^{\infty}$ be a sequence of L_n-Lipschitzian mappings of K into itself with $L_n \geq 1$, $\sum_{n=1}^{\infty}(L_n-1)<\infty$. Let $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For a fixed $\delta \in (0,1)$ and each $n \in \mathbb{N}$, define $S_n : K \to K$ by $S_n := (1-\delta)x+\delta T_nx, \ \ \forall x \in K$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K defined by $x_1 = u \in K$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)S_nx_n, \tag{4.2.5}$$

for all $n \in \mathbb{N}$. Suppose that (K_{min}) $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\lim_{n\to\infty}||T_{n+1}x_n - T_nx_n|| = 0$. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to some common fixed points of $\{T_n\}_{n=1}^{\infty}$.

Proof Observe that S_n is κ_n-Lipschitzian with $\kappa_n = 1+\delta(L_n-1) \geq 1$ and has the same set of fixed points as T_n. Define

$$\beta_n := (1-\delta)\alpha_n+\delta, \ \ n \geq 0; \ \ y_n := \frac{x_{n+1}-x_n+\beta_nx_n}{\beta_n}, \ \ n \geq 0.$$

Observe also that $\beta_n \to \delta$ as $n \to \infty$, and if $\{x_n\}_{n=1}^{\infty}$ is bounded, then $\{y_n\}_{n=1}^{\infty}$ is bounded. Let $x^* \in \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(S_n)$. One easily shows by Lemma 4.2.1 that $||x_n-x^*|| \leq 2e^{\sum_{n=1}^{\infty}(L_n-1)}\max\{||u-x^*||,||x_1-x^*||\} < \infty$ for all integers $n \geq 1$, and so, $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{T_nx_n\}_{n=1}^{\infty}$ and $\{S_nx_n\}_{n=1}^{\infty}$

are all bounded. Observe also that from the definitions of β_n and S_n, we obtain that $y_n = \frac{\alpha_n u+(1-\alpha_n)\delta T_n x_n}{\beta_n}$ so that

$$\begin{aligned}||y_{n+1}-y_n||-||x_{n+1}-x_n|| &\leq \Big|\frac{\alpha_{n+1}}{\beta_{n+1}}-\frac{\alpha_n}{\beta_n}\Big|||u|| \\ &+\frac{(1-\alpha_{n+1})}{\beta_{n+1}}\delta||T_{n+1}x_{n+1}-T_nx_n|| \\ &+\Big|\frac{(1-\alpha_{n+1})}{\beta_{n+1}}-\frac{(1-\alpha_n)}{\beta_n}\delta\Big|||T_nx_n||.\end{aligned}$$

This implies that $\limsup_{n\to\infty}(||y_{n+1}-y_n||-||x_{n+1}-x_n||)\leq 0$. Hence, by Lemma 1.5.9, $||y_n-x_n||\to 0$ as $n\to\infty$. Consequently, $\lim_{n\to\infty}||x_{n+1}-x_n|| = \lim_{n\to\infty}\beta_n||y_n-x_n|| = 0$. Following the line of proof of Theorem 4.2.3, we have that $\limsup_{n\to\infty}\langle u-x^*, j(x_{n+1}-x^*)\rangle\leq 0$. Furthermore,

$$\begin{aligned}||x_{n+1}-x^*||^2 &\leq ||\alpha_n(u-x^*)+(1-\alpha_n)(S_nx_n-x^*)||^2 \\ &\leq (1-\alpha_n)^2||S_nx_n-x^*||^2+2\alpha_n\langle u-x^*, j(x_{n+1}-x^*)\rangle \\ &\leq (1-\alpha_n)^2\kappa_n^2||x_n-x^*||^2+2\alpha_n\langle u-x^*, j(x_{n+1}-x^*)\rangle \\ &\leq (1-\alpha_n)\kappa_n^2||x_n-x^*||^2+2\alpha_n\langle u-x^*, j(x_{n+1}-x^*)\rangle \\ &= (1-\alpha_n)||x_n-x^*||^2+(1-\alpha_n)(\kappa_n^2-1)||x_n-x^*||^2 \\ &\quad +2\alpha_n\langle u-x^*, j(x_{n+1}-x^*)\rangle \\ &\leq (1-\alpha_n)||x_n-x^*||^2+2\alpha_n\langle u-x^*, j(x_{n+1}-x^*)\rangle \\ &\quad +M_3(\kappa_n-1),\end{aligned}$$

for some constant $M_3>0$. Hence, by Lemma 1.5.2, we have that the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to $x^*\in\bigcap_{n=1}^\infty F(T_n)$. This completes the proof. ■

We now give an example in which our condition (K_{min}) $\bigcap_{n=1}^\infty F(T_n)\neq\emptyset$ is easily satisfied.

Corollary 4.2.5 *Let E be a real Banach space with uniformly Gâteaux differentiable norm. Assume that E has uniform normal structure. Let K be a nonempty closed convex subset of E and $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive mappings of K into itself. Let $\bigcap_{n=1}^\infty F(T_n)\neq\emptyset$, $\lim_{n\to\infty}||T_{n+1}x_n-T_nx_n|| = 0$ and $\{\alpha_n\}_{n=1}^\infty$ be a sequence in $[0,1]$ such that $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^\infty\alpha_n=\infty$ and $\sum_{n=1}^\infty|\alpha_{n+1}-\alpha_n|<\infty$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in K defined by*

$x_1 = u \in K$ *and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T_n x_n, \tag{4.2.6}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to some common fixed point of* $\{T_n\}_{n=1}^{\infty}$.

Proof Since E has uniform normal structure, it is reflexive. $K_{min} := \left\{x \in K\bigcap B^* : \varphi(x) = \min_{y\in K\bigcap B^*} \varphi(y)\right\}$ is a nonempty, closed and convex subset of K that has property (P) (see, Lemma 1.5.10). This follows as in [124](see also, [49]) (since every nonexpansive mapping is asymptotically nonexpansive with $k_n \equiv 1 < N(E)^{\frac{1}{2}}, \ \ \forall n \in \mathbb{N}$). Hence, $(K_{min}) \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. The result therefore follows. This completes the proof. ∎

Corollary 4.2.6 *Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm. Let K be a nonempty closed convex subset of E and $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of K into itself. Let $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For a fixed $\delta \in (0,1)$ and each $n \in \mathbb{N}$, define $S_n : K \to K$ by $S_n := (1-\delta)x + \delta T_n x, \ \ \forall x \in K$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K defined by $x_1 = u \in K$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)S_n x_n, \tag{4.2.7}$$

for all $n \in \mathbb{N}$. Suppose that $(K_{min}) \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\lim_{n\to\infty} ||T_{n+1}x_n - T_n x_n|| = 0$. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to some common fixed points of $\{T_n\}_{n=1}^{\infty}$.

Remark 4.2.7 *Let K be a nonempty closed convex subset of a real Banach space E with uniformly Gâteaux differentiable norm and $T : K \to K$ a nonexpansive map. For $x_1 = u \in K$, let $\{x_n\}_{n=1}^{\infty}$ be defined by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Sx_n, \tag{4.2.8}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0,1) satisfying some conditions and $S := (1-\delta)I + \delta T$ for some fixed $\delta \in (0,1)$. It is proved in Chidume and Chidume [56] that $\{x_n\}_{n=1}^{\infty}$ is an approximate fixed point sequence of the nonexpansive map S (i.e., $||x_n - Sx_n|| \to 0, \ \ n \to \infty$).

An example of a finite family of maps satisfying the limiting condition $\lim_{n\to\infty} ||T_{n+1}x_n - T_n x_n|| = 0$ in Theorem 4.2.3, Theorem 4.2.4, Corollary 4.2.5 and Corollary 4.2.6 is given as follows

Example 4.2.8 *(Chidume and Ali, [52]) Let K be a nonempty closed convex subset of a real Banach space E with uniformly Gâteaux differentiable norm and $S : K \to K$ a nonexpansive map. Let $\{x_n\}_{n=1}^{\infty}$ be defined by* (4.2.8). *From [56], $\{x_n\}_{n=1}^{\infty}$ is an approximate fixed point sequence of S. Now, let $\{S_i\}_{i=1}^{m}$ be a finite family of maps defined by $S_n := S^n, \quad n = n \mod m$. Write $S^m = S^{m-1}oS$. Then, it is easy to see that*

$$||S_{n+1}x_{n+1} - S_nx_n|| = \begin{cases} ||x_n - Sx_n||, & n \neq km, \quad k \in \mathbb{N} \\ (m-1)||x_n - Sx_n||, & n = km, \quad k \in \mathbb{N}. \end{cases}$$

so that, in all cases,

$$||S_{n+1}x_{n+1} - S_nx_n|| \to 0, \quad n \to \infty.$$

Remark 4.2.9 *As stated in the preliminaries, all uniformly convex real Banach spaces and all uniformly smooth real Banach spaces have uniform normal structure and are reflexive. Consequently, our theorems apply to uniformly smooth real Banach spaces with uniformly Gâteaux differentiable norms. Also, in particular, Corollary 4.2.5 extends Theorem 1.3.32 to real Banach spaces with uniformly Gâteaux differentiable norms possessing uniform normal structure, and with condition* (1.3.14) *replaced with* $\lim_{n\to\infty} ||T_{n+1}x_{n+1} - T_nx_n|| = 0$. *Furthermore, Corollary 4.2.6, using a different Mann-type iteration process, yields the conclusion of Theorem 1.3.32 in the same more general Banach spaces, again with condition* (1.3.14) *replaced with* $\lim_{n\to\infty} ||T_{n+1}x_{n+1} - T_nx_n|| = 0$ *and at the same time dispensing with* (1.3.15) *or* (1.3.16). *Finally, while the authors of [5] admit that their condition* (1.3.14) *cannot, in general, be applied for a sequence of nonexpansive mappings, Example* (4.2.8) *shows that Corollary 4.2.5 and Corollary 4.2.6 are applicable for this class of mappings even in our more general setting.*

Remark 4.2.10 *All the results in this chapter are due to Chidume et al.[59].*

Part III

Algorithms for Common Solutions of Common Fixed Point Problems for a Family of Nonlinear Maps; Variational Inequality Problems and Equilibrium Problems

CHAPTER 5

An Iterative Method for Fixed Point Problems, Variational Inclusions and Generalized Equilibrium Problems

5.1 Introduction

In this chapter, we introduce a new iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions to a generalized equilibrium problem and the set of solutions to variational inclusion in a real Hilbert space. Furthermore, we show that our iterative scheme converges strongly to a common element of the three aforementioned sets. Finally, we apply our results to solve an optimization problem. Our results extend many important recent results.

5.2 Main Results

We prove the following strong convergence theorem.

Lemma 5.2.1 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $T : H \to H$ be a nonexpansive mapping such that $\Omega := F(T) \cap I(A, M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction map with constant $\gamma \in (0, 1)$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by $x_1 \in H$,*

$$\begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in K, \\ w_n = J_{M,\lambda}(u_n - \lambda A u_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) T\Big[\alpha_n f(x_n) + (1 - \alpha_n) w_n\Big], \end{cases} \tag{5.2.1}$$

for all $n \geq 1$*, where* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are sequences in [0,1] and* $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ *satisfying:*
(i) $0 < c \leq \beta_n \leq d < 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\lambda \in (0, 2\alpha]$,
(iv) $0 < a \leq r_n \leq b < 2\mu, \quad \lim_{n\to\infty} |r_{n+1} - r_n| = 0$,
then $\{x_n\}_{n=1}^{\infty}$ *is bounded.*

Proof We first show that $I - \lambda A$ is nonexpansive.
For all $x, y \in K$ and $\lambda \in (0, 2\alpha]$, we obtain

$$\begin{aligned}
&||(I - \lambda A)x - (I - \lambda A)y||^2 = ||(x - y) - \lambda(Ax - Ay)||^2 \\
= \;& ||x - y||^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2||Ax - Ay||^2 \\
\leq \;& ||x - y||^2 + \lambda(\lambda - 2\alpha)||Ax - Ay||^2 \\
\leq \;& ||x - y||^2. \qquad (5.2.2)
\end{aligned}$$

Hence, $I - \lambda A$ is nonexpansive.

Next, we show that $\{x_n\}_{n=1}^{\infty}$ is bounded. Observe that u_n can be re-written as $u_n = T_{r_n}(x_n - r_n\psi x_n), \quad n \geq 1$. Let x^* be an element of $\Omega = F(T) \cap I(A, M) \cap EP \neq \emptyset$. Then

$$\begin{aligned}
||w_n - x^*||^2 &= ||J_{M,\lambda}(u_n - \lambda Au_n) - J_{M,\lambda}(x^* - \lambda Ax^*)||^2 \\
&\leq ||(u_n - \lambda Au_n) - (x^* - \lambda Ax^*)||^2 \\
&\leq ||u_n - x^*||^2 \\
&= ||T_{r_n}(x_n - r_n\psi x_n) - x^*||^2 \\
&= ||T_{r_n}(x_n - r_n\psi x_n) - T_{r_n}(x^* - r_n\psi x^*)||^2 \\
&\leq ||(I - r_n\psi)x_n - (I - r_n\psi)x^*||^2 \\
&\leq ||x_n - x^*||^2 + r_n(r_n - 2\mu)||\psi x_n - \psi x^*||^2 \qquad (5.2.3) \\
&\leq ||x_n - x^*||^2 \quad (\text{since } \; r_n < 2\mu, \;\; \forall n \geq 1).
\end{aligned}$$

Now, put $y_n := \alpha_n f(x_n) + (1 - \alpha_n)w_n, \quad n \geq 1$. So,

$$\begin{aligned}
||y_n - x^*|| &= ||\alpha_n(f(x_n) - x^*) + (1 - \alpha_n)(w_n - x^*)|| \\
&\leq \alpha_n||f(x_n) - x^*|| + (1 - \alpha_n)||w_n - x^*|| \\
&\leq \alpha_n(||f(x_n) - f(x^*)|| + ||f(x^*) - x^*||) + (1 - \alpha_n)||x_n - x^*|| \\
&\leq \alpha_n\gamma||x_n - x^*|| + \alpha_n||f(x^*) - x^*|| + (1 - \alpha_n)||x_n - x^*|| \\
&= (1 - \alpha_n(1 - \gamma))||x_n - x^*|| + \alpha_n(1 - \gamma)\frac{1}{(1 - \gamma)}||f(x^*) - x^*||.
\end{aligned}$$

From (5.2.1), we obtain

$$\begin{aligned}
||x_{n+1} - x^*|| &= ||\beta_n(x_n - x^*) + (1-\beta_n)(Ty_n - x^*)|| \\
&\leq \beta_n||x_n - x^*|| + (1-\beta_n)||y_n - x^*|| \\
&\leq \beta_n||x_n - x^*|| + (1-\beta_n)[(1-\alpha_n(1-\gamma))||x_n - x^*|| \\
&\quad +\alpha_n(1-\gamma)\frac{1}{(1-\gamma)}||f(x^*) - x^*||] \\
&= (1-\alpha_n(1-\gamma)(1-\beta_n))||x_n - x^*|| \\
&\quad +\alpha_n(1-\gamma)(1-\beta_n)\frac{1}{(1-\gamma)}||f(x^*) - x^*|| \\
&\leq \max\{||x_n - x^*||, \frac{1}{(1-\gamma)}||f(x^*) - x^*||\} \\
&\vdots \\
&\leq \max\{||x_1 - x^*||, \frac{1}{(1-\gamma)}||f(x^*) - x^*||\}.
\end{aligned}$$

So, $\{x_n\}_{n=1}^{\infty}$ is bounded. Hence, $\{Au_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{Ty_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are bounded.

■

Lemma 5.2.2 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $T : H \to H$ be a nonexpansive mapping such that $\Omega := F(T) \cap I(A,M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction map with constant $\gamma \in (0,1)$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (5.2.1). Then, We next show that $\lim_{n\to\infty}||u_{n+1}-u_n|| = 0$ and $\lim_{n\to\infty}||w_{n+1}-w_n|| = 0$.*

Proof From (5.2.2), we have

$$\begin{aligned}
||w_{n+1} - w_n|| &= ||J_{M,\lambda}(u_{n+1} - \lambda Au_{n+1}) - J_{M,\lambda}(u_n - \lambda Au_n)|| \\
&\leq ||(u_{n+1} - \lambda Au_{n+1}) - (u_n - \lambda Au_n)|| \\
&\leq ||u_{n+1} - u_n||. \qquad (5.2.4)
\end{aligned}$$

On the other hand, from $u_n = T_{r_n}(x_n - r_n\psi x_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1}\psi x_{n+1})$, we obtain

$$F(u_n, y) + \langle \psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in K \qquad (5.2.5)$$

and for each $y \in K$

$$F(u_{n+1}, y) + \langle \psi x_{n+1}, y - u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 (5.2.6)$$

Substituting $y = u_{n+1}$ in (5.2.5) and $y = u_n$ in (5.2.6), we have

$$F(u_n, u_{n+1}) + \langle \psi x_n, u_{n+1} - u_n \rangle + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \langle \psi x_{n+1}, u_n - u_{n+1} \rangle + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.$$

So, from (A2), we have

$$\langle \psi x_{n+1} - \psi x_n, u_n - u_{n+1} \rangle + \langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0$$

and hence,

$$\begin{aligned}
0 &\leq \langle u_n - u_{n+1}, r_n(\psi x_{n+1} - \psi x_n) + \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) - (u_n - x_n) \rangle \\
&= \langle u_{n+1} - u_n, u_n - u_{n+1} + (1 - \frac{r_n}{r_{n+1}})u_{n+1} + (x_{n+1} - r_n \psi x_{n+1}) \\
&\quad -(x_n - r_n \psi x_n) - x_{n+1} + \frac{r_n}{r_{n+1}} x_{n+1} \rangle \\
&= \langle u_{n+1} - u_n, u_n - u_{n+1} + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) + (x_{n+1} - r_n \psi x_{n+1}) \\
&\quad -(x_n - r_n \psi x_n) \rangle.
\end{aligned}$$

It then follows that

$$||u_{n+1} - u_n||^2 \leq ||u_{n+1} - u_n|| \Big\{ \Big|1 - \frac{r_n}{r_{n+1}}\Big| ||u_{n+1} - x_{n+1}|| + ||x_{n+1} - x_n|| \Big\}$$

and so we have

$$||u_{n+1} - u_n|| \leq \Big|1 - \frac{r_n}{r_{n+1}}\Big| ||u_{n+1} - x_{n+1}|| + ||x_{n+1} - x_n||.$$

Without loss of generality, we assume that there exists $d_1 \in \mathbb{R}$ such that $r_n > d_1 > 0, \quad \forall n \geq 1$. Then

$$\begin{aligned}
||u_{n+1} - u_n|| &\leq ||x_{n+1} - x_n|| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| ||u_{n+1} - x_{n+1}|| \\
&\leq ||x_{n+1} - x_n|| + \frac{1}{d_1}|r_{n+1} - r_n| M_1, \qquad (5.2.7)
\end{aligned}$$

where $M_1 := \sup_{n \geq 1} ||u_n - x_n||$. Hence,

$$||w_{n+1} - w_n|| \leq ||x_{n+1} - x_n|| + \frac{1}{d_1}|r_{n+1} - r_n| M_1. \qquad (5.2.8)$$

Now,

$$\begin{aligned}
&||y_{n+1}-y_n|| = ||(\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})w_{n+1})\\
&-(\alpha_n f(x_n)+(1-\alpha_n)w_n)||\\
= \;& ||(\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})w_{n+1})-(\alpha_n f(x_n)+(1-\alpha_{n+1})w_n)\\
&+(\alpha_n-\alpha_{n+1})w_n||\\
\leq \;& \alpha_{n+1}||f(x_{n+1})||+\alpha_n||f(x_n)||+(1-\alpha_{n+1})||w_{n+1}-w_n||\\
&+|\alpha_{n+1}-\alpha_n|||w_n||\\
\leq \;& \alpha_{n+1}||f(x_{n+1})||+\alpha_n|||f(x_n)||+(1-\alpha_{n+1})[||x_{n+1}-x_n||\\
&+\frac{1}{d_1}|r_{n+1}-r_n|M_1]+|\alpha_{n+1}-\alpha_n|||w_n||\\
\leq \;& \alpha_{n+1}||f(x_{n+1})||+\alpha_n|||f(x_n)||+||x_{n+1}-x_n||\\
&+\frac{1}{d_1}|r_{n+1}-r_n|M_1+|\alpha_{n+1}-\alpha_n|||w_n||.
\end{aligned}$$

So,

$$\begin{aligned}
&||Ty_{n+1}-Ty_n|| \leq ||y_{n+1}-y_n||\\
\leq \;& \alpha_{n+1}||f(x_{n+1})||+\alpha_n|||f(x_n)||+||x_{n+1}-x_n||+\frac{1}{d}|r_{n+1}-r_n|M_1\\
&+|\alpha_{n+1}-\alpha_n|||w_n||. \qquad (5.2.9)
\end{aligned}$$

Using assumptions (ii) and (iv) in (5.2.9), we obtain

$$\limsup_{n\to\infty}(||Ty_{n+1}-Ty_n||-||x_{n+1}-x_n||)\leq 0.$$

From Lemma 1.5.9, we have

$$\lim_{n\to\infty}||Ty_n-x_n||=0. \qquad (5.2.10)$$

Consequently, we obtain that

$$\lim_{n\to\infty}||x_{n+1}-x_n||=\lim_{n\to\infty}(1-\beta_n)||Ty_n-x_n||=0.$$

Since $\lim_{n\to\infty}|r_n-r_{n+1}|=0$, we have from (5.2.7) and (5.2.8) that $\lim_{n\to\infty}||u_{n+1}-u_n||=0$ and $\lim_{n\to\infty}||w_{n+1}-w_n||=0$. ■

Lemma 5.2.3 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K\times K$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M: H\to 2^H$ a maximal monotone mapping. Let $T: H\to H$ be a nonexpansive mapping such that $\Omega := F(T)\cap I(A,M)\cap EP\neq\emptyset$ and suppose $f: H\to H$ is a contraction map with constant $\gamma\in(0,1)$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (5.2.1). Then, $\lim_{n\to\infty}||u_n-w_n||=0$ and $\lim_{n\to\infty}||y_n-w_n||=0$.*

Proof Now, by convexity of $||.||^2$ and (5.2.3), we have

$$\begin{aligned}
& ||x_{n+1}-x^*||^2 = ||\beta_n(x_n-x^*)+(1-\beta_n)(Ty_n-x^*)||^2 \\
\leq\ & \beta_n||x_n-x^*||^2+(1-\beta_n)||Ty_n-x^*||^2 \\
\leq\ & \beta_n||x_n-x^*||^2+(1-\beta_n)||y_n-x^*||^2 \\
=\ & \beta_n||x_n-x^*||^2+(1-\beta_n)||\alpha_n(f(x_n)-x^*)+(1-\alpha_n)(w_n-x^*)||^2 \\
\leq\ & \beta_n||x_n-x^*||^2+(1-\beta_n)[\alpha_n||f(x_n)-x^*||^2+(1-\alpha_n)||w_n-x^*||^2] \\
\leq\ & \beta_n||x_n-x^*||^2+(1-\beta_n)[\alpha_n||f(x_n)-x^*||^2+(1-\alpha_n)(||x_n-x^*||^2 \\
& +r_n(r_n-2\mu)||\psi x_n-\psi x^*||^2)] \\
\leq\ & ||x_n-x^*||^2+(1-\beta_n)\alpha_n||f(x_n)-x^*||^2 \\
& +r_n(r_n-2\mu)||\psi x_n-\psi x^*||^2. \qquad (5.2.11)
\end{aligned}$$

Hence,

$$r_n(2\mu-r_n)||\psi x_n-\psi x^*||^2 \leq \alpha_n||f(x_n)-x^*||^2+||x_n-x^*||^2-||x_{n+1}-x^*||^2.$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}||x_{n+1}-x_n||=0$, we have

$$\lim_{n\to\infty}||\psi x_n-\psi x^*||=0.$$

Furthermore, from (5.2.1), (5.2.2) and (5.2.3), we obtain

$$\begin{aligned}
||x_{n+1}-x^*||^2 &= ||\beta_n(x_n-x^*)+(1-\beta_n)(Ty_n-x^*)||^2 \\
&\leq \beta_n||x_n-x^*||^2+(1-\beta_n)||y_n-x^*||^2 \\
&\leq \beta_n||x_n-x^*||^2+(1-\beta_n)\Big[\alpha_n||f(x_n)-x^*||^2 \\
&\quad +(1-\alpha_n)||w_n-x^*||^2\Big] \\
&\leq \beta_n||x_n-x^*||^2+(1-\beta_n)\Big[\alpha_n||f(x_n)-x^*||^2 \\
&\quad +(1-\alpha_n)(||u_n-x^*||^2+\lambda(\lambda-2\alpha)||Au_n-Ax^*||^2)\Big] \\
&\leq \beta_n||x_n-x^*||^2+(1-\beta_n)\Big[\alpha_n||f(x_n)-x^*||^2 \\
&\quad +(1-\alpha_n)(||x_n-x^*||^2+\lambda(\lambda-2\alpha)||Au_n-Ax^*||^2)\Big] \\
&\leq ||x_n-x^*||^2+\alpha_n||f(x_n)-x^*||^2+\lambda(\lambda-2\alpha)||Au_n-Ax^*||^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-\lambda(\lambda-2\alpha)||Au_n-Ax^*||^2 &\leq ||x_n-x^*||^2-||x_{n+1}-x^*||^2 \\
&\quad +\alpha_n||f(x_n)-x^*||^2 \\
&\leq ||x_n-x_{n+1}||(||x_n-x^*||+||x_{n+1}-x^*||) \\
&\quad +\alpha_n||f(x_n)-x^*||^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$, we obtain

$$\lim_{n\to\infty} ||Au_n - Ax^*|| = 0.$$

Now, since $J_{M,\lambda}$ is 1-inverse-strongly monotone, we obtain

$$\begin{aligned}
& ||w_n - x^*||^2 = ||J_{M,\lambda}(u_n - \lambda Au_n) - J_{M,\lambda}(x^* - \lambda Ax^*)||^2 \\
\leq\ & \langle (u_n - \lambda Au_n) - (x^* - \lambda Ax^*), w_n - x^* \rangle \\
=\ & \frac{1}{2}\Big[||(u_n - \lambda Au_n) - (x^* - \lambda Ax^*)||^2 + ||w_n - x^*||^2 \\
& -||(u_n - \lambda Au_n) - (x^* - \lambda Ax^*) - (w_n - x^*)||^2 \Big] \\
\leq\ & \frac{1}{2}\Big[||u_n - x^*||^2 + ||w_n - x^*||^2 - ||(u_n - w_n) - \lambda(Au_n - Ax^*)||^2 \Big] \\
=\ & \frac{1}{2}\Big[||u_n - x^*||^2 + ||w_n - x^*||^2 - ||u_n - w_n||^2 \\
& +2\lambda\langle u_n - w_n, Au_n - Ax^* \rangle - \lambda^2 ||Au_n - Ax^*||^2 \Big].
\end{aligned}$$

So, we have

$$\begin{aligned}
||w_n - x^*||^2 \leq\ & ||u_n - x^*||^2 - ||u_n - w_n||^2 \\
& +2\lambda\langle u_n - w_n, Au_n - Ax^* \rangle \\
& -\lambda^2 ||Au_n - Ax^*||^2. \qquad (5.2.12)
\end{aligned}$$

From (5.2.1) and (5.2.12), we have

$$\begin{aligned}
& ||x_{n+1} - x^*||^2 = ||\beta_n(x_n - x^*) + (1 - \beta_n)(Ty_n - x^*)||^2 \\
\leq\ & \beta_n ||x_n - x^*||^2 + (1 - \beta_n)(\alpha_n ||f(x_n) - x^*||^2 \\
& +(1 - \alpha_n)||w_n - x^*||^2) \\
\leq\ & \beta_n ||x_n - x^*||^2 + \alpha_n ||f(x_n) - x^*||^2 + (1 - \beta_n)||w_n - x^*||^2 \\
\leq\ & \beta_n ||x_n - x^*||^2 + \alpha_n ||f(x_n) - x^*||^2 + (1 - \beta_n)(||u_n - x^*||^2 \\
& -||u_n - w_n||^2 + 2\lambda\langle u_n - w_n, Au_n - Ax^* \rangle - \lambda^2 ||Au_n - Ax^*||^2) \\
\leq\ & \beta_n ||x_n - x^*||^2 + \alpha_n ||f(x_n) - x^*||^2 + (1 - \beta_n)(||x_n - x^*||^2 \\
& -||u_n - w_n||^2 + 2\lambda\langle u_n - w_n, Au_n - Ax^* \rangle - \lambda^2 ||Au_n - Ax^*||^2) \\
\leq\ & ||x_n - x^*||^2 + \alpha_n ||f(x_n) - x^*||^2 - (1 - \beta_n)||u_n - w_n||^2 \\
& +2(1 - \beta_n)\lambda ||u_n - w_n|| ||Au_n - Ax^*||. \qquad (5.2.13)
\end{aligned}$$

Using assumption (i) in (5.2.13), we have

$$\begin{aligned}
(1 - d)||u_n - w_n||^2 \leq\ & (1 - \beta_n)||u_n - w_n||^2 \\
\leq\ & ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + \alpha_n ||f(x_n) - x^*||^2 \\
& +2\lambda ||x_n - w_n|| ||Au_n - Ax^*||.
\end{aligned}$$

Since $\lim_{n\to\infty}||x_n - x_{n+1}|| = 0$, $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}||Au_n - Ax^*|| = 0$, we obtain

$$\lim_{n\to\infty}||u_n - w_n|| = 0. \tag{5.2.14}$$

From $y_n = \alpha_n f(x_n) + (1-\alpha_n)w_n$, we obtain $y_n - w_n = \alpha_n(f(x_n) - w_n)$. So,

$$||y_n - w_n|| = \alpha_n||f(x_n) - w_n|| \to 0. \tag{5.2.15}$$

■

Lemma 5.2.4 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $T : H \to H$ be a nonexpansive mapping such that $\Omega := F(T) \cap I(A,M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction map with constant $\gamma \in (0,1)$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (5.2.1). Then, $\lim_{n\to\infty}||x_n - u_n|| = 0$.*

Proof From Lemma 1.4.2, we have

$$\begin{aligned}
||u_n - x^*||^2 &= ||T_{r_n}(x_n - r_n\psi x_n) - T_{r_n}(x^* - r_n\psi x^*)||^2 \\
&\leq \langle (x_n - r_n\psi x_n) - (x^* - r_n\psi x^*), u_n - x^*\rangle \\
&= \frac{1}{2}\Big[||(x_n - r_n\psi x_n) - (x^* - r_n\psi x^*)||^2 + ||u_n - x^*||^2 \\
&\quad -||(x_n - r_n\psi x_n) - (x^* - r_n\psi x^*) - (u_n - x^*)||^2\Big] \\
&\leq \frac{1}{2}\Big[||x_n - x^*||^2 + ||u_n - x^*||^2 \\
&\quad -||(x_n - r_n\psi x_n) - (x^* - r_n\psi x^*) - (u_n - x^*)||^2\Big] \\
&= \frac{1}{2}\Big[||x_n - x^*||^2 + ||u_n - x^*||^2 - ||u_n - x_n||^2 \\
&\quad +2r_n\langle x_n - u_n, \psi x_n - \psi x^*\rangle - r_n^2||\psi x_n - \psi x^*||^2\Big]
\end{aligned}$$

and hence

$$\begin{aligned}
||u_n - x^*||^2 &\leq ||x_n - x^*||^2 - ||u_n - x_n||^2 \\
&\quad +2r_n\langle x_n - u_n, \psi x_n - \psi x^*\rangle - r_n^2||\psi x_n - \psi x^*||^2 \\
&\leq ||x_n - x^*||^2 - ||u_n - x_n||^2 \\
&\quad +2r_n||x_n - u_n||||\psi x_n - \psi x^*||.
\end{aligned} \tag{5.2.16}$$

By convexity of $||.||^2$, we have

$$\begin{aligned} &||x_{n+1} - x^*||^2 = ||\beta_n(x_n - x^*) + (1-\beta_n)(Ty_n - x^*)||^2 \\ \leq\ & \beta_n||x_n - x^*||^2 + (1-\beta_n)(\alpha_n||f(x_n) - x^*||^2 \\ & +(1-\alpha_n)||w_n - x^*||^2) \\ \leq\ & \beta_n||x_n - x^*||^2 + (1-\beta_n)(\alpha_n||f(x_n) - x^*||^2 \\ & +(1-\alpha_n)||u_n - x^*||^2) \\ \leq\ & \beta_n||x_n - x^*||^2 + (1-\beta_n)\alpha_n||f(x_n) - x^*||^2 \\ & +(1-\alpha_n)(1-\beta_n)[||x_n - x^*||^2 - ||x_n - u_n||^2] \\ & +2r_n||x_n - u_n||||\psi x_n - \psi x^*||. \end{aligned}$$

Hence

$$\begin{aligned} &(1-\alpha_n)(1-\beta_n)||x_n - u_n||^2 \leq ||x_n - x^*||^2 \\ &-||x_{n+1} - x^*||^2 + (1-\beta_n)\alpha_n||f(x_n) - x^*||^2 \\ &+2r_n||x_n - u_n||||\psi x_n - \psi x^*||. \end{aligned} \tag{5.2.17}$$

Using assumption (i) in (5.2.17), we have

$$\begin{aligned} &(1-d)(1-\alpha_n)||x_n - u_n||^2 \leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 \\ &+(1-\beta_n)\alpha_n||f(x_n) - x^*||^2 + 2r_n||x_n - u_n||||\psi x_n - \psi x^*|| \\ \leq\ & \alpha_n||f(x_n) - x^*||^2 + ||x_{n+1} - x_n||(||x_n - x^*|| + ||x_{n+1} - x^*||) \\ &+2r_n||x_n - u_n||||\psi x_n - \psi x^*||. \end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} ||\psi x_n - \psi x^*|| = 0$ and $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$, then

$$\lim_{n\to\infty} ||x_n - u_n|| = 0.$$

■

Lemma 5.2.5 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ satisfying $(A1) - (A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $T : H \to H$ be a nonexpansive mapping such that $\Omega := F(T) \cap I(A, M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction map with constant $\gamma \in (0,1)$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (5.2.1). Then*

$$\limsup_{n\to\infty} \langle f(z) - z, y_n - z\rangle \leq 0,$$

where $z := P_\Omega f(z)$ and $P_\Omega f(z)$ is the metric projection of $f(z)$ onto Ω.

Proof Furthermore, from (5.2.14) and (5.2.15), we have

$$||Ty_n - y_n|| \leq ||Ty_n - x_n|| + ||x_n - w_n|| + ||w_n - y_n|| \to 0. \quad (5.2.18)$$

Next, we show that

$$\limsup_{n\to\infty} \langle f(z) - z, y_n - z\rangle \leq 0,$$

where $z = P_\Omega f(z)$. The existence of z is justified since P_Ω is nonexpansive and f is a contraction, then $P_\Omega o f$ is a contraction so it has a fixed point. To do this, we choose a subsequence $\{y_{n_j}\}_{j=1}^\infty$ of $\{y_n\}_{n=1}^\infty$ such that

$$\limsup_{n\to\infty} \langle f(z) - z, y_n - z\rangle = \lim_{j\to\infty} \langle f(z) - z, y_{n_j} - z\rangle. \quad (5.2.19)$$

As $\{y_n\}_{n=1}^\infty$ is bounded, there exists a subsequence $\{y_{n_j}\}_{j=1}^\infty$ of $\{y_n\}_{n=1}^\infty$ that converges weakly to w.
We first show that $w \in I(A, M)$. Since A is an $\frac{1}{\alpha}$-Lipschitz monotone mapping and $D(A) = H$, we obtain from Lemma 1.5.11 that $M + A$ is maximal monotone. Let $(v, g) \in G(M + A)$, that is, $g - Av \in M(v)$. Since $w_{n_j} = J_{M,\lambda}(I - \lambda A)u_{n_j}$, we get $(I - \lambda A)u_{n_j} \in (I + \lambda M)w_{n_j}$, that is,

$$\frac{1}{\lambda}(u_{n_j} - \lambda A u_{n_j} - w_{n_j}) \in M(w_{n_j}).$$

Using the maximal monotonicity of $M + A$, we obtain

$$\left\langle v - w_{n_j}, g - Av - \frac{1}{\lambda}(u_{n_j} - \lambda A u_{n_j} - w_{n_j})\right\rangle \geq 0$$

and so

$$\begin{aligned}
\langle v - w_{n_j}, g\rangle &\geq \left\langle v - w_{n_j}, Av + \frac{1}{\lambda}(u_{n_j} - \lambda A u_{n_j} - w_{n_j})\right\rangle \\
&= \left\langle v - w_{n_j}, Av - Aw_{n_j} + Aw_{n_j} - Au_{n_j} + \frac{1}{\lambda}(u_{n_j} - w_{n_j})\right\rangle \\
&\geq 0 + \langle v - w_{n_j}, Aw_{n_j} - Au_{n_j}\rangle + \left\langle v - w_{n_j}, \frac{1}{\lambda}(u_{n_j} - w_{n_j})\right\rangle.
\end{aligned}$$

It follows from $\lim_{j\to\infty} ||w_{n_j} - u_{n_j}|| = 0$, $\lim_{j\to\infty} ||Aw_{n_j} - Au_{n_j}|| = 0$ and $\lim_{j\to\infty} w_{n_j} = w$ (since $\lim_{j\to\infty} ||y_{n_j} - w_{n_j}|| = 0$ and $\lim_{j\to\infty} y_{n_j} = w$) that

$$\lim_{j\to\infty} \langle v - w_{n_j}, g\rangle = \langle v - w, g\rangle \geq 0.$$

Using the maximal monotonicity of $M + A$, we obtain $\theta \in (M + A)(w)$ and this implies that $w \in I(A, M)$.

We next show that $w \in EP$. Since $u_n = T_{r_n}(x_n - r_n \psi x_n), \quad n \geq 1$, we have for any $y \in K$ that

$$F(u_n, y) + \langle \psi x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0.$$

Furthermore, replacing n by n_j in the last inequality and using $(A2)$, we obtain

$$\langle \psi x_{n_j}, y - u_{n_j} \rangle + \frac{1}{r_{n_j}} \langle y - u_{n_j}, u_{n_j} - x_{n_j} \rangle \geq F(y, u_{n_j}). \tag{5.2.20}$$

Let $z_t := ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in K$. This implies that $z_t \in K$. Then, by (5.2.20), we have

$$\begin{aligned} \langle z_t - u_{n_j}, \psi z_t \rangle &\geq \langle z_t - u_{n_j}, \psi z_t \rangle - \langle z_t - u_{n_j}, \psi x_{n_j} \rangle \\ &\quad - \Big\langle z_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}} \Big\rangle + F(z_t, u_{n_j}) \\ &= \langle z_t - u_{n_j}, \psi z_t - \psi u_{n_j} \rangle + \langle z_t - u_{n_j}, \psi u_{n_j} - \psi x_{n_j} \rangle \\ &\quad - \Big\langle z_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}} \Big\rangle + F(z_t, u_{n_j}). \end{aligned} \tag{5.2.21}$$

Since $||x_{n_j} - u_{n_j}|| \to 0, \quad j \to \infty$, we obtain $||\psi x_{n_j} - \psi u_{n_j}|| \to 0, \quad j \to \infty$. Furthermore, by the monotonicity of ψ, we obtain $\langle z_t - u_{n_j}, \psi z_t - \psi u_{n_j} \rangle \geq 0$. From (5.2.14) and (5.2.15), we obtain that

$$||y_n - u_n|| \leq ||u_n - w_n|| + ||y_n - w_n|| \to 0, \quad n \to \infty.$$

Since $\lim\limits_{j \to \infty} ||y_{n_j} - u_{n_j}|| = 0$ and $\lim\limits_{j \to \infty} y_{n_j} = w$, we obtain that $\lim\limits_{j \to \infty} u_{n_j} = w$. Then, using assumption $(A4)$ in (5.2.21), we obtain

$$\langle z_t - w, \psi z_t \rangle \geq F(z_t, w), \quad j \to \infty. \tag{5.2.22}$$

Using $(A1)$, $(A4)$ and (5.2.22) we also obtain

$$\begin{aligned} 0 &= F(z_t, z_t) \leq tF(z_t, y) + (1-t)F(z_t, w) \\ &\leq tF(z_t, y) + (1-t)\langle z_t - w, \psi z_t \rangle \\ &= tF(z_t, y) + (1-t)t\langle y - w, \psi z_t \rangle \end{aligned}$$

and hence

$$0 \leq F(z_t, y) + (1-t)\langle y - w, \psi z_t \rangle.$$

Letting $t \to 0$ and using assumption $(A3)$, we have, for each $y \in K$,

$$0 \leq F(w, y) + \langle y - w, \psi w \rangle. \tag{5.2.23}$$

This implies that $w \in EP$.

Finally, we show that $w \in F(T)$. Assume the contrary that $w \neq Tw$. Then by Opial's condition, we obtain from (5.2.18) that

$$\begin{aligned}\liminf_{j\to\infty}||y_{n_j} - w|| &< \liminf_{j\to\infty}||y_{n_j} - Tw|| \\ &\leq \liminf_{j\to\infty}(||y_{n_j} - Ty_{n_j}|| + ||Ty_{n_j} - Tw||) \\ &\leq \liminf_{j\to\infty}||y_{n_j} - w||.\end{aligned}$$

This is a contradiction. Hence, $w \in F(T)$. Furthermore $w \in \Omega = F(T) \bigcap I(A, M) \bigcap EP$. By (5.2.19) and property of metric projection, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z) - z, y_n - z\rangle &= \lim_{j\to\infty}\langle f(z) - z, y_{n_j} - z\rangle \\ &= \langle f(z) - z, w - z\rangle \leq 0.\end{aligned}$$

■

Theorem 5.2.6 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ satisfying $(A1) - (A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $T : H \to H$ be a nonexpansive mapping such that $\Omega := F(T) \cap I(A, M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction map with constant $\gamma \in (0, 1)$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (5.2.1). Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to z, where $z := P_\Omega f(z)$ and $P_\Omega f(z)$ is the metric projection of $f(z)$ onto Ω.*

Proof Since $y_n - z = \alpha_n(f(x_n) - z) + (1 - \alpha_n)(w_n - z)$, we have

$$\begin{aligned}||y_n - z||^2 &\leq ||\alpha_n(f(x_n) - z) + (1 - \alpha_n)(w_n - z)||^2 \\ &\leq (1 - \alpha_n)^2||w_n - z||^2 + 2\alpha_n\langle f(x_n) - z, y_n - z\rangle \\ &\leq (1 - \alpha_n)^2||x_n - z||^2 + 2\alpha_n\langle f(x_n) - z, y_n - z\rangle \\ &= (1 - \alpha_n)^2||x_n - z||^2 + 2\alpha_n\langle f(x_n) - f(z), y_n - z\rangle \\ &\quad +2\alpha_n\langle f(z) - z, y_n - z\rangle \\ &\leq (1 - \alpha_n)^2||x_n - z||^2 + 2\alpha_n\gamma||x_n - z||||y_n - z|| \\ &\quad +2\alpha_n\langle f(z) - z, y_n - z\rangle \\ &\leq (1 - \alpha_n)^2||x_n - z||^2 + \alpha_n\gamma(||x_n - z||^2 + ||y_n - z||^2) \\ &\quad +2\alpha_n\langle f(z) - z, y_n - z\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}
||y_n - z||^2 &\leq \frac{1-2\alpha_n+\alpha_n^2+\alpha_n\gamma}{1-\alpha_n\gamma}||x_n-z||^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma}\langle f(z)-z, y_n-z\rangle \\
&= \Big[1-\frac{2(1-\gamma)\alpha_n}{1-\alpha_n\gamma}\Big]||x_n-z||^2 + \frac{\alpha_n^2}{1-\alpha_n\gamma}||x_n-z||^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma}\langle f(z)-z, y_n-z\rangle. \qquad (5.2.24)
\end{aligned}$$

Using (5.2.24) in (5.2.1), we obtain

$$\begin{aligned}
& ||x_{n+1}-z||^2 \leq \beta_n||x_n-z||^2 + (1-\beta_n)||Ty_n-z||^2 \\
\leq\; & \beta_n||x_n-z||^2 + (1-\beta_n)||y_n-z||^2 \\
\leq\; & \beta_n||x_n-z||^2 + (1-\beta_n)\Big(\Big[1-\frac{2(1-\gamma)\alpha_n}{1-\alpha_n\gamma}\Big]||x_n-z||^2 \\
& + \frac{\alpha_n^2}{1-\alpha_n\gamma}||x_n-z||^2 + \frac{2\alpha_n}{1-\alpha_n\gamma}\langle f(z)-z, y_n-z\rangle\Big) \\
=\; & \Big[1-(1-\beta_n)\frac{2(1-\gamma)\alpha_n}{1-\alpha_n\gamma}\Big]||x_n-z||^2 + \frac{\alpha_n(1-\beta_n)}{1-\alpha_n\gamma}\Big[\alpha_n||x_n-z||^2 \\
& + 2\langle f(z)-z, y_n-z\rangle\Big] \\
=\; & (1-\delta_n)||x_n-z||^2 + \sigma_n, \qquad (5.2.25)
\end{aligned}$$

where $\delta_n := (1-\beta_n)\frac{2(1-\gamma)\alpha_n}{1-\alpha_n\gamma}$ and $\sigma_n := \frac{\alpha_n(1-\beta_n)}{1-\alpha_n\gamma}\Big[\alpha_n||x_n-z||^2 + 2\langle f(z)- z, y_n-z\rangle\Big]$. It is easily verified that $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\frac{\sigma_n}{\delta_n} \leq 0$, then by Lemma 1.5.2, we obtain $\lim_{n\to\infty}||x_n-z|| = 0$. ■

Corollary 5.2.7 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K\times K$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H and A an α-inverse-strongly monotone mapping of K into H. Let $T: K\to K$ be a nonexpansive mapping such that $\Omega := F(T)\bigcap VI(K,A)\bigcap EP \neq \emptyset$ and suppose $f: K\to K$ is a contraction map with constant $\gamma\in(0,1)$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by $x_1\in K$ and*

$$\begin{cases}
F(u_n,y) + \langle\psi x_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0 \quad \forall y\in K,\\
w_n = P_K(u_n-\lambda Au_n),\\
x_{n+1} = \beta_n x_n + (1-\beta_n)T\Big[\alpha_n f(x_n) + (1-\alpha_n)w_n\Big],
\end{cases}$$

for all $n\geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1] and $\{r_n\}_{n=1}^{\infty}\subset(0,\infty)$ satisfying:

(i) $0 < c \leq \beta_n \leq d < 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\lambda \in (0, 2\alpha]$,
(iv) $0 < a \leq r_n \leq b < 2\mu, \quad \lim_{n\to\infty} |r_{n+1} - r_n| = 0$,
then $\{x_n\}$ *converges strongly to* z, *where* $z := P_\Omega f(z)$.

Proof Take $M = \partial\delta_K : H \to 2^H$, where $\delta_K : H \to [0, \infty)$ is the indicator function of K. It is well known that the sub-differential $\partial\delta_K$ is a maximal monotone operator. Then problem (1.4.4) is equivalent to problem (1.4.3) and $J_{M,\lambda} = P_K$. This completes the proof. ■

Corollary 5.2.8 *Let* K *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* F *be a bi-function from* $K \times K$ *satisfying* $(A1) - (A4)$, ψ *a* μ*-inverse-strongly monotone mapping of* K *into* H, A *an* α*-inverse-strongly monotone mapping of* K *into* H *and* $M : H \to 2^H$ *a maximal monotone mapping. Let* $\Omega := I(A, M) \bigcap EP \neq \emptyset$ *and suppose* $f : H \to H$ *is a contraction map with constant* $\gamma \in (0, 1)$. *Suppose* $\{x_n\}$ *and* $\{u_n\}$ *are generated by* $x_1 \in H$ *and*

$$\begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in K, \\ J_{M,\lambda}(u_n - \lambda A u_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)\Big[\alpha_n f(x_n) + (1 - \alpha_n) w_n\Big], \end{cases}$$

for all $n \geq 1$, *where* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are sequences in [0,1] and* $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ *satisfying:*
(i) $0 < c \leq \beta_n \leq d < 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\lambda \in (0, 2\alpha]$,
(iv) $0 < a \leq r_n \leq b < 2\mu, \quad \lim_{n\to\infty} |r_{n+1} - r_n| = 0$,
then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* z, *where* $z := P_\Omega f(z)$.

Proof Put $T = I$ in Theorem 5.2.6. Then, by Theorem 5.2.6, we obtain the desired result. ■

5.3 Application to optimization problem

We now study the following optimization problem:

$$\min_{x\in K} h(x), \tag{5.3.1}$$

where K is a nonempty, closed and convex subset of a real Hilbert space H and $h : K \to \mathbb{R}$ is a convex and lower semi-continuous functional. Let us denote the set of solutions to (5.3.1) by Ω_1. Let $F : K \times K \to \mathbb{R}$ defined by $F(x, y) := h(y) - h(x)$. Let us now find the following equilibrium problem: find $x \in K$ such that

$$F(x, y) \geq 0, \tag{5.3.2}$$

for all $y \in K$. It is obvious that F satisfies conditions $(A1) - (A4)$ and $EP(F) = \Omega_1$, where $EP(F)$ is the set of solutions to (5.3.2). By Theorem 5.2.6, we have the following

Corollary 5.3.1 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $h : K \to \mathbb{R}$ be a convex and lower semi-continuous functional such that $\Omega_1 \neq \emptyset$. Suppose $f : H \to H$ is a contraction map with constant $\gamma \in (0, 1)$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be generated by $x_1 \in H$ and*

$$\begin{cases} h(y) - h(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in K, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n f(x_n) + (1 - \alpha_n)u_n), \end{cases} \tag{5.3.3}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1] and $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ satisfying:
(i) $0 < c \leq \beta_n \leq d < 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\liminf_{n\to\infty} r_n > 0, \quad \lim_{n\to\infty} |r_{n+1} - r_n| = 0$,
then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a solution z of optimization problem (5.3.1), *where $z := P_{\Omega_1} f(z)$.*

Proof Taking $T = I, \quad \psi \equiv 0$ and $A \equiv 0$ in Theorem 5.2.6, we arrive at the desired conclusion. ∎

Remark 5.3.2 *Lemma 2.3 of Takahashi and Takahashi [193] was not used in the proof process of our Theorem 5.2.6.*

Remark 5.3.3 *All the results in this chapter are due to Shehu [173].*

CHAPTER 6

An Iterative Method for Nonexpansive Semigroups, Variational Inclusions and Generalized Equilibrium Problems

6.1 Introduction

In this chapter, we prove strong convergence theorems for finding a common element of the set of common fixed points of a nonexpansive semigroup, the set of solutions to a generalized equilibrium problem and the set of solutions to variational inclusion in a real Hilbert space. Using our results, we obtain mean ergodic theorem for nonexpansive mappings in Hilbert spaces. Our results extend the results of Takahashi and Takahashi [193] and many important recent results.

6.2 Main Results

Lemma 6.2.1 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $\Im := \{T(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Omega := F(\Im) \cap I(A, M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction mapping with a constant $\gamma \in (0, 1)$. Let $\{t_n\} \subset (0, \infty)$ be a real sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by $x_1 \in H$,*

$$\begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in K \\ w_n = J_{M,\lambda}(u_n - \lambda A u_n) \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)\Big(\frac{1}{t_n}\int_0^{t_n} T(u)[\alpha_n f(x_n) + (1 - \alpha_n)w_n]du\Big), \end{cases} \tag{6.2.1}$$

for all $n \geq 1$*, where* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are sequences in (0,1) and* $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ *satisfying:*
(i) $\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$
(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$
(iii) $\lambda \in (0, 2\alpha],$
(iv) $0 < a \leq r_n \leq b < 2\mu, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$
(v) $\lim_{n\to\infty} \frac{|t_n - t_{n-1}|}{t_n} \frac{1}{\alpha_n(1-\beta_n)} = 0,$
then $\{x_n\}_{n=1}^{\infty}$ *is bounded.*

Proof We first show that $I - \lambda A$ is nonexpansive.
For all $x, y \in K$ and $\lambda \in (0, 2\alpha]$, we obtain

$$\begin{aligned}
& ||(I-\lambda A)x - (I-\lambda A)y||^2 = ||(x-y) - \lambda(Ax - Ay)||^2 \\
= \; & ||x-y||^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2||Ax-Ay||^2 \\
\leq \; & ||x-y||^2 + \lambda(\lambda - 2\alpha)||Ax-Ay||^2 \\
\leq \; & ||x-y||^2. \qquad (6.2.2)
\end{aligned}$$

Hence, $I - \lambda A$ is nonexpansive.

Next, we show that $\{x_n\}_{n=1}^{\infty}$ is bounded. Observe that u_n can be re-written as $u_n = T_{r_n}(x_n - r_n\psi x_n), \quad n \geq 1$. Let x^* be an element of $\Omega = F(\Im) \cap I(A, M) \cap EP \neq \emptyset$. Then

$$\begin{aligned}
||w_n - x^*||^2 &= ||J_{M,\lambda}(u_n - \lambda A u_n) - J_{M,\lambda}(x^* - \lambda A x^*)||^2 \\
&\leq ||(u_n - \lambda A u_n) - (x^* - \lambda A x^*)||^2 \\
&\leq ||u_n - x^*||^2 \\
&= ||T_{r_n}(x_n - r_n\psi x_n) - x^*||^2 \\
&= ||T_{r_n}(x_n - r_n\psi x_n) - T_{r_n}(x^* - r_n\psi x^*)||^2 \\
&\leq ||(I - r_n\psi)x_n - (I - r_n\psi)x^*||^2 \\
&\leq ||x_n - x^*||^2 + r_n(r_n - 2\mu)||\psi x_n - \psi x^*||^2 \qquad (6.2.3) \\
&\leq ||x_n - x^*||^2 \quad (\text{since } r_n < 2\mu, \ \forall n \geq 1).
\end{aligned}$$

Now, put $y_n := \alpha_n f(x_n) + (1 - \alpha_n)w_n, \quad n \geq 1$. So,

$$\begin{aligned}
& ||y_n - x^*|| = ||\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(w_n - x^*)|| \\
\leq \; & \alpha_n||f(x_n) - x^*|| + (1-\alpha_n)||w_n - x^*|| \\
\leq \; & \alpha_n(||f(x_n) - f(x^*)|| + ||f(x^*) - x^*||) + (1-\alpha_n)||x_n - x^*|| \\
\leq \; & \alpha_n\gamma||x_n - x^*|| + \alpha_n||f(x^*) - x^*|| + (1-\alpha_n)||x_n - x^*|| \\
= \; & (1 - \alpha_n(1-\gamma))||x_n - x^*|| \\
& + \alpha_n(1-\gamma)\frac{1}{(1-\gamma)}||f(x^*) - x^*||. \qquad (6.2.4)
\end{aligned}$$

From (6.2.1), we obtain

$$\begin{aligned}
||x_{n+1}-x^*|| &= \Big|\Big|\beta_n(x_n-x^*)+(1-\beta_n)\Big(\frac{1}{t_n}\int_0^{t_n}[T(u)y_n-T(u)x^*]du\Big)\Big|\Big| \\
&\le \beta_n||x_n-x^*||+(1-\beta_n)||y_n-x^*|| \\
&\le \beta_n||x_n-x^*||+(1-\beta_n)[(1-\alpha_n(1-\gamma))||x_n-x^*|| \\
&\quad +\alpha_n(1-\gamma)\frac{1}{(1-\gamma)}||f(x^*)-x^*||] \\
&= (1-\alpha_n(1-\gamma)(1-\beta_n))||x_n-x^*|| \\
&\quad +\alpha_n(1-\gamma)(1-\beta_n)\frac{1}{(1-\gamma)}||f(x^*)-x^*|| \\
&\le \max\{||x_n-x^*||,\frac{1}{(1-\gamma)}||f(x^*)-x^*||\} \\
&\vdots \\
&\le \max\{||x_1-x^*||,\frac{1}{(1-\gamma)}||f(x^*)-x^*||\}.
\end{aligned}$$

So, $\{x_n\}_{n=1}^{\infty}$ is bounded. Hence, $\{Au_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\Big\{\frac{1}{t_n}\int_0^{t_n}T(u)y_ndu\Big\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are bounded. ■

Lemma 6.2.2 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K\times K$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M: H\to 2^H$ a maximal monotone mapping. Let $\Im := \{T(u): 0\le u<\infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Omega := F(\Im)\cap I(A,M)\cap EP\neq\emptyset$ and suppose $f: H\to H$ is a contraction mapping with a constant $\gamma\in(0,1)$. Let $\{t_n\}\subset(0,\infty)$ be a real sequence such that $\lim_{n\to\infty}t_n=\infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (6.2.1). Then, $\lim\limits_{n\to\infty}||u_{n+1}-u_n||=0$ and $\lim\limits_{n\to\infty}||w_{n+1}-w_n||=0$.*

Proof From (6.2.4), we have

$$\begin{aligned}
||y_n-x^*|| &\le (1-\alpha_n(1-\gamma))||x_n-x^*||+\alpha_n(1-\gamma)\frac{1}{(1-\gamma)}||f(x^*)-x^*|| \\
&\le ||x_n-x^*||+\frac{1}{(1-\gamma)}||f(x^*)-x^*|| \\
&\le \max\{||x_1-x^*||,\frac{1}{(1-\gamma)}||f(x^*)-x^*||\}+\frac{1}{(1-\gamma)}||f(x^*)-x^*|| \\
&\le ||x_1-x^*||+\frac{2}{(1-\gamma)}||f(x^*)-x^*||.
\end{aligned}$$

Put $D = \{w \in H : ||w - x^*|| \leq ||x_1 - x^*|| + \frac{2}{(1-\gamma)}||f(x^*) - x^*||\}$. Then D is a nonempty, bounded, closed and convex subset of H. Since $T(u)$ is nonexpansive for any $u \in [0, \infty)$, D is $T(u)-$ invariant for each $u \in [0, \infty)$ and contains $\{y_n\}$. Without loss of generality, we may assume that $\Im := \{T(u) : 0 \leq u < \infty\}$ is a nonexpansive semigroup on D. By Lemma 1.5.12, we get

$$\lim_{n\to\infty}\Big|\Big|\Big(\frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big) - T(h)\Big(\frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big)\Big|\Big| = 0, \qquad (6.2.5)$$

for every $h \in [0, \infty)$. Furthermore, observe that

$$\begin{aligned}
||x_{n+1} - T(h)x_{n+1}|| &\leq \Big|\Big|x_{n+1} - \frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big|\Big| \\
&+\Big|\Big|\Big(\frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big) - T(h)\Big(\frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big)\Big|\Big| \\
&+\Big|\Big|T(h)\Big(\frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big) - T(h)x_{n+1}\Big|\Big| \\
\leq\ & 2\Big|\Big|x_{n+1} - \frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big|\Big| \\
&+\Big|\Big|\Big(\frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big) - T(h)\Big(\frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big)\Big|\Big| \\
=\ & 2\beta_n\Big|\Big|x_n - \frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big|\Big| \\
&+\Big|\Big|\Big(\frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big) - T(h)\Big(\frac{1}{t_n}\int_0^{t_n} T(u)y_n du\Big)\Big|\Big|.
\end{aligned}$$

From $\lim_{n\to\infty}\beta_n = 0$ and (6.2.5), we get $\lim_{n\to\infty}||x_{n+1} - T(h)x_{n+1}|| = 0$ and hence

$$\lim_{n\to\infty}||x_n - T(h)x_n|| = 0. \qquad (6.2.6)$$

From (6.2.2), we have

$$\begin{aligned}
||w_{n+1} - w_n|| &= ||J_{M,\lambda}(u_{n+1} - \lambda Au_{n+1}) - J_{M,\lambda}(u_n - \lambda Au_n)|| \\
&\leq ||(u_{n+1} - \lambda Au_{n+1}) - (u_n - \lambda Au_n)|| \\
&\leq ||u_{n+1} - u_n||. \qquad (6.2.7)
\end{aligned}$$

On the other hand, from $u_n = T_{r_n}(x_n - r_n\psi x_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1} - r_{n+1}\psi x_{n+1})$, we obtain

$$F(u_n, y) + \langle\psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in K \qquad (6.2.8)$$

and for each $y \in K$,

$$F(u_{n+1}, y) + \langle\psi x_{n+1}, y - u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \qquad (6.2.9)$$

Substituting $y = u_{n+1}$ in (6.2.8) and $y = u_n$ in (6.2.9), we have

$$F(u_n, u_{n+1}) + \langle \psi x_n, u_{n+1} - u_n \rangle + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \langle \psi x_{n+1}, u_n - u_{n+1} \rangle + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.$$

So, from (A2), we have

$$\langle \psi x_{n+1} - \psi x_n, u_n - u_{n+1} \rangle + \langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0$$

and hence,

$$\begin{aligned} 0 &\leq \langle u_n - u_{n+1}, r_n(\psi x_{n+1} - \psi x_n) + \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) - (u_n - x_n) \rangle \\ &= \langle u_{n+1} - u_n, u_n - u_{n+1} + (1 - \frac{r_n}{r_{n+1}})u_{n+1} + (x_{n+1} - r_n \psi x_{n+1}) \\ &\quad -(x_n - r_n \psi x_n) - x_{n+1} + \frac{r_n}{r_{n+1}} x_{n+1} \rangle \\ &= \langle u_{n+1} - u_n, u_n - u_{n+1} + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1}) + (x_{n+1} - r_n \psi x_{n+1}) \\ &\quad -(x_n - r_n \psi x_n) \rangle. \end{aligned}$$

It then follows that

$$||u_{n+1} - u_n||^2 \leq ||u_{n+1} - u_n|| \left\{ \left| 1 - \frac{r_n}{r_{n+1}} \right| ||u_{n+1} - x_{n+1}|| + ||x_{n+1} - x_n|| \right\}$$

and so we have

$$||u_{n+1} - u_n|| \leq \left| 1 - \frac{r_n}{r_{n+1}} \right| ||u_{n+1} - x_{n+1}|| + ||x_{n+1} - x_n||.$$

Without loss of generality, we assume that there exists $d_1 \in \mathbb{R}$ such that $r_n > d_1 > 0, \quad \forall n \geq 1$. Then

$$\begin{aligned} ||u_{n+1} - u_n|| &\leq ||x_{n+1} - x_n|| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| ||u_{n+1} - x_{n+1}|| \\ &\leq ||x_{n+1} - x_n|| + \frac{1}{d_1} |r_{n+1} - r_n| M_1, \qquad (6.2.10) \end{aligned}$$

where $M_1 := \sup_{n \geq 1} ||u_n - x_n||$. Hence,

$$||w_{n+1} - w_n|| \leq ||x_{n+1} - x_n|| + \frac{1}{d_1} |r_{n+1} - r_n| M_1. \qquad (6.2.11)$$

Now,

$$\begin{aligned}
&||y_{n+1}-y_n|| = ||(\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})w_{n+1})\\
&-(\alpha_n f(x_n)+(1-\alpha_n)w_n)||\\
= \quad & ||\alpha_{n+1}f(x_{n+1})-\alpha_{n+1}f(x_n)+\alpha_{n+1}f(x_n)-\alpha_n f(x_n)\\
&+(1-\alpha_{n+1})w_{n+1}-(1-\alpha_{n+1})w_n\\
&+(1-\alpha_{n+1})w_n-(1-\alpha_n)w_n||\\
\leq \quad & \alpha_{n+1}\gamma||x_{n+1}-x_n||+|\alpha_{n+1}-\alpha_n|(||f(x_n)||+||w_n||)\\
&+(1-\alpha_{n+1})||w_{n+1}-w_n||\\
\leq \quad & \alpha_{n+1}\gamma||x_{n+1}-x_n||+|\alpha_{n+1}-\alpha_n|(||f(x_n)||+||w_n||)\\
&+(1-\alpha_{n+1})||x_{n+1}-x_n||+\frac{1}{d_1}|r_{n+1}-r_n|M_1. \qquad (6.2.12)
\end{aligned}$$

Let $z_n := \frac{1}{t_n}\int_0^{t_n} T(u)y_n du, \quad n \geq 1$. Then, we have

$$\begin{aligned}
&||z_n-z_{n-1}|| = \Big|\Big|\frac{1}{t_n}\int_0^{t_n}[T(u)y_n-T(u)y_{n-1}]du\\
&+\Big(\frac{1}{t_n}-\frac{1}{t_{n-1}}\Big)\int_0^{t_{n-1}}T(u)y_{n-1}du+\frac{1}{t_n}\int_{t_{n-1}}^{t_n}T(u)y_{n-1}du\Big|\Big|.
\end{aligned}$$

Given that

$$\Big(\frac{1}{v}-\frac{1}{w}\Big)w = -\frac{v-w}{v}, \quad v,w \neq 0;$$

if $x^* \in \Omega$, we can write

$$\begin{aligned}
&||z_n-z_{n-1}|| = \Big|\Big|\frac{1}{t_n}\int_0^{t_n}[T(u)y_n-T(u)y_{n-1}]du\\
&+\Big(\frac{1}{t_n}-\frac{1}{t_{n-1}}\Big)\int_0^{t_{n-1}}[T(u)y_{n-1}-T(u)x^*]du\\
&+\frac{1}{t_n}\int_{t_{n-1}}^{t_n}[T(u)y_{n-1}-T(u)x^*]du\Big|\Big|.
\end{aligned}$$

Thus,

$$||z_n-z_{n-1}|| \leq ||y_n-y_{n-1}||+\Big(\frac{2|t_n-t_{n-1}|}{t_n}\Big)||y_{n-1}-x^*||. \qquad (6.2.13)$$

Substituting (6.2.12) into (6.2.13), we obtain

$$\begin{aligned}
&||z_n-z_{n-1}|| \leq \alpha_n\gamma||x_n-x_{n-1}||+|\alpha_n-\alpha_{n-1}|(||f(x_{n-1})||\\
&+||w_{n-1}||)+(1-\alpha_n)||x_n-x_{n-1}||+\frac{1}{d_1}|r_n-r_{n-1}|M_1\\
&+\Big(\frac{2|t_n-t_{n-1}|}{t_n}\Big)||y_{n-1}-x^*||. \qquad (6.2.14)
\end{aligned}$$

From (6.2.1), we have $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$ and this implies that

$$\begin{aligned}
& ||x_{n+1} - x_n|| = ||\beta_n x_n + (1-\beta_n)z_n - \beta_{n-1}x_{n-1} - (1-\beta_{n-1})z_{n-1}|| \\
= \; & ||\beta_n x_n - \beta_{n-1}x_{n-1} + \beta_n x_{n-1} - \beta_n x_{n-1} + (1-\beta_n)z_n - (1-\beta_{n-1})z_{n-1} \\
& +(1-\beta_n)z_{n-1} - (1-\beta_n)z_{n-1}|| \\
\leq \; & \beta_n||x_n - x_{n-1}|| + (1-\beta_n)||z_n - z_{n-1}|| \\
& +|\beta_n - \beta_{n-1}|(||x_{n-1}|| + ||z_{n-1}||). \qquad (6.2.15)
\end{aligned}$$

Using (6.2.14) in (6.2.15), we obtain

$$\begin{aligned}
& ||x_{n+1} - x_n|| \leq [1-(1-\gamma)\alpha_n(1-\beta_n)]||x_n - x_{n-1}|| \\
& +|\alpha_n - \alpha_{n-1}|(||f(x_{n-1})|| + ||w_{n-1}||) \\
& +\frac{1}{d_1}|r_n - r_{n-1}|M_1 + |\beta_n - \beta_{n-1}|(||x_{n-1}|| + ||z_{n-1}||) \\
& +\Big(\frac{2|t_n - t_{n-1}|}{t_n}\Big)||y_{n-1} - x^*|| \\
\leq \; & [1-(1-\gamma)\alpha_n(1-\beta_n)]||x_n - x_{n-1}|| \\
& +D\Big[|\alpha_n - \alpha_{n-1}| + |r_n - r_{n-1}| + |\beta_n - \beta_{n-1}| \\
& +\frac{2|t_n - t_{n-1}|}{t_n}\Big], \qquad (6.2.16)
\end{aligned}$$

where $D := \max\{\sup_{n\geq 1}(||f(x_n)||+||w_n||), \sup_{n\geq 1}(||x_n||+||z_n||), \frac{M_1}{d_1}, \sup_{n\geq 1}||y_n - x^*||\}$. From Lemma 1.5.2 and conditions (i), (ii), (iv) and (v), it follows that

$$\lim_{n\to\infty}||x_{n+1} - x_n|| = 0.$$

Since $\lim_{n\to\infty}|r_n - r_{n+1}| = 0$, we have from (6.2.10) and (6.2.11) that $\lim_{n\to\infty}||u_{n+1} - u_n|| = 0$ and $\lim_{n\to\infty}||w_{n+1} - w_n|| = 0$. ■

Lemma 6.2.3 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K\times K$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $\Im := \{T(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Omega := F(\Im)\cap I(A,M)\cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction mapping with a constant $\gamma \in (0,1)$. Let $\{t_n\} \subset (0,\infty)$ be a real sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by* (6.2.1). *Then, $\lim_{n\to\infty}||u_n - w_n|| = 0$ and $\lim_{n\to\infty}||y_n - w_n|| = 0$.*

Proof Now, by the convexity of $||.||^2$ and (6.2.3), we have

$$\begin{aligned}
||x_{n+1}-x^*||^2 &= \Big|\Big|\beta_n(x_n-x^*)+(1-\beta_n)\Big(\frac{1}{t_n}\int_0^{t_n}[T(u)y_n-T(u)x^*]du\Big)\Big|\Big|^2\\
&\le \beta_n||x_n-x^*||^2+(1-\beta_n)\Big|\Big|\frac{1}{t_n}\int_0^{t_n}[T(u)y_n-T(u)x^*]du\Big)\Big|\Big|^2\\
&\le \beta_n||x_n-x^*||^2+(1-\beta_n)||y_n-x^*||^2\\
&= \beta_n||x_n-x^*||^2+(1-\beta_n)||\alpha_n(f(x_n)-x^*)+(1-\alpha_n)(w_n-x^*)||^2\\
&\le \beta_n||x_n-x^*||^2+(1-\beta_n)[\alpha_n||f(x_n)-x^*||^2+(1-\alpha_n)||w_n-x^*||^2]\\
&\le \beta_n||x_n-x^*||^2+(1-\beta_n)[\alpha_n||f(x_n)-x^*||^2+(1-\alpha_n)(||x_n-x^*||^2\\
&\quad +r_n(r_n-2\mu)||\psi x_n-\psi x^*||^2)]\\
&\le ||x_n-x^*||^2+(1-\beta_n)\alpha_n||f(x_n)-x^*||^2\\
&\quad +r_n(r_n-2\mu)||\psi x_n-\psi x^*||^2. \qquad (6.2.17)
\end{aligned}$$

Hence,

$$r_n(2\mu-r_n)||\psi x_n-\psi x^*||^2 \le \alpha_n||f(x_n)-x^*||^2+||x_n-x^*||^2-||x_{n+1}-x^*||^2.$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}||x_{n+1}-x_n||=0$, we have

$$\lim_{n\to\infty}||\psi x_n-\psi x^*||=0.$$

Furthermore, from (6.2.1), (6.2.2) and (6.2.3), we obtain

$$\begin{aligned}
||x_{n+1}-x^*||^2 &= \Big|\Big|\beta_n(x_n-x^*)+(1-\beta_n)\Big(\frac{1}{t_n}\int_0^{t_n}[T(u)y_n-T(u)x^*]du\Big)\Big|\Big|^2\\
&\le \beta_n||x_n-x^*||^2+(1-\beta_n)||y_n-x^*||^2\\
&\le \beta_n||x_n-x^*||^2+(1-\beta_n)\Big[\alpha_n||f(x_n)-x^*||^2\\
&\quad +(1-\alpha_n)||w_n-x^*||^2\Big]\\
&\le \beta_n||x_n-x^*||^2+(1-\beta_n)\Big[\alpha_n||f(x_n)-x^*||^2\\
&\quad +(1-\alpha_n)(||u_n-x^*||^2+\lambda(\lambda-2\alpha)||Au_n-Ax^*||^2)\Big]\\
&\le \beta_n||x_n-x^*||^2+(1-\beta_n)\Big[\alpha_n||f(x_n)-x^*||^2\\
&\quad +(1-\alpha_n)(||x_n-x^*||^2+\lambda(\lambda-2\alpha)||Au_n-Ax^*||^2)\Big]\\
&\le ||x_n-x^*||^2+\alpha_n||f(x_n)-x^*||^2+\lambda(\lambda-2\alpha)||Au_n-Ax^*||^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-\lambda(\lambda-2\alpha)||Au_n-Ax^*||^2 &\le ||x_n-x^*||^2-||x_{n+1}-x^*||^2\\
&\quad +\alpha_n||f(x_n)-x^*||^2\\
&\le ||x_n-x_{n+1}||(||x_n-x^*||+||x_{n+1}-x^*||)\\
&\quad +\alpha_n||f(x_n)-x^*||^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$, we obtain

$$\lim_{n\to\infty} ||Au_n - Ax^*|| = 0.$$

Now, since $J_{M,\lambda}$ is 1-inverse-strongly monotone, we obtain

$$\begin{aligned}
&||w_n - x^*||^2 = ||J_{M,\lambda}(u_n - \lambda Au_n) - J_{M,\lambda}(x^* - \lambda Ax^*)||^2 \\
\leq\ & \langle (u_n - \lambda Au_n) - (x^* - \lambda Ax^*), w_n - x^* \rangle \\
=\ & \frac{1}{2}\Big[||(u_n - \lambda Au_n) - (x^* - \lambda Ax^*)||^2 + ||w_n - x^*||^2 \\
& -||(u_n - \lambda Au_n) - (x^* - \lambda Ax^*) - (w_n - x^*)||^2\Big] \\
\leq\ & \frac{1}{2}\Big[||u_n - x^*||^2 + ||w_n - x^*||^2 - ||(u_n - w_n) - \lambda(Au_n - Ax^*)||^2\Big] \\
=\ & \frac{1}{2}\Big[||u_n - x^*||^2 + ||w_n - x^*||^2 - ||u_n - w_n||^2 + 2\lambda\langle u_n - w_n, Au_n - Ax^* \rangle \\
& -\lambda^2||Au_n - Ax^*||^2\Big].
\end{aligned}$$

So, we have

$$\begin{aligned}
||w_n - x^*||^2 \quad \leq \quad & ||u_n - x^*||^2 - ||u_n - w_n||^2 + 2\lambda\langle u_n - w_n, Au_n - Ax^* \rangle \\
& -\lambda^2||Au_n - Ax^*||^2. \qquad (6.2.18)
\end{aligned}$$

From (6.2.1) and (6.2.18), we have

$$\begin{aligned}
&||x_{n+1} - x^*||^2 = \Big|\Big|\beta_n(x_n - x^*) + (1 - \beta_n)\Big(\frac{1}{t_n}\int_0^{t_n}[T(u)y_n - T(u)x^*]du\Big)\Big|\Big|^2 \\
\leq\ & \beta_n||x_n - x^*||^2 + (1 - \beta_n)(\alpha_n||f(x_n) - x^*||^2 + (1 - \alpha_n)||w_n - x^*||^2) \\
\leq\ & \beta_n||x_n - x^*||^2 + \alpha_n||f(x_n) - x^*||^2 + (1 - \beta_n)||w_n - x^*||^2 \\
\leq\ & \beta_n||x_n - x^*||^2 + \alpha_n||f(x_n) - x^*||^2 + (1 - \beta_n)(||u_n - x^*||^2 \\
& -||u_n - w_n||^2 + 2\lambda\langle u_n - w_n, Au_n - Ax^* \rangle - \lambda^2||Au_n - Ax^*||^2) \\
\leq\ & \beta_n||x_n - x^*||^2 + \alpha_n||f(x_n) - x^*||^2 + (1 - \beta_n)(||x_n - x^*||^2 \\
& -||u_n - w_n||^2 + 2\lambda\langle u_n - w_n, Au_n - Ax^* \rangle - \lambda^2||Au_n - Ax^*||^2) \\
\leq\ & ||x_n - x^*||^2 + \alpha_n||f(x_n) - x^*||^2 - (1 - \beta_n)||u_n - w_n||^2 \\
& +2(1 - \beta_n)\lambda||u_n - w_n||||Au_n - Ax^*||. \qquad (6.2.19)
\end{aligned}$$

From (6.2.19), we have

$$\begin{aligned}
(1 - \beta_n)||u_n - w_n||^2 \quad \leq \quad & ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + \alpha_n||f(x_n) - x^*||^2 \\
& +2\lambda||x_n - w_n||||Au_n - Ax^*||.
\end{aligned}$$

Since $\lim_{n\to\infty} ||x_n - x_{n+1}|| = 0$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} ||Au_n - Ax^*|| = 0$, we obtain

$$\lim_{n\to\infty} (1 - \beta_n)||u_n - w_n||^2 = 0.$$

Since $\lim_{n\to\infty} \beta_n = 0$, we obtain

$$\lim_{n\to\infty} ||u_n - w_n|| = 0. \tag{6.2.20}$$

From $y_n = \alpha_n f(x_n) + (1-\alpha_n)w_n$, we obtain $y_n - w_n = \alpha_n(f(x_n) - w_n)$. So,

$$||y_n - w_n|| = \alpha_n ||f(x_n) - w_n|| \to 0. \tag{6.2.21}$$

■

Lemma 6.2.4 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $\Im := \{T(u) : 0 \le u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Omega := F(\Im) \cap I(A, M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction mapping with a constant $\gamma \in (0, 1)$. Let $\{t_n\} \subset (0, \infty)$ be a real sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (6.2.1). Then, $\lim_{n\to\infty} ||x_n - u_n|| = 0$.*

Proof From Lemma 1.4.2, we have

$$\begin{aligned}
& ||u_n - x^*||^2 = ||T_{r_n}(x_n - r_n\psi x_n) - T_{r_n}(x^* - r_n\psi x^*)||^2 \\
\le\ & \langle (x_n - r_n\psi x_n) - (x^* - r_n\psi x^*), u_n - x^* \rangle \\
=\ & \frac{1}{2}\Big[||(x_n - r_n\psi x_n) - (x^* - r_n\psi x^*)||^2 + ||u_n - x^*||^2 \\
& -||(x_n - r_n\psi x_n) - (x^* - r_n\psi x^*) - (u_n - x^*)||^2\Big] \\
\le\ & \frac{1}{2}\Big[||x_n - x^*||^2 - ||(x_n - r_n\psi x_n) \\
& -(x^* - r_n\psi x^*) - (u_n - x^*)||^2 + ||u_n - x^*||^2\Big] \\
=\ & \frac{1}{2}\Big[||x_n - x^*||^2 + ||u_n - x^*||^2 - ||u_n - x_n||^2 \\
& +2r_n\langle x_n - u_n, \psi x_n - \psi x^* \rangle - r_n^2||\psi x_n - \psi x^*||^2\Big]
\end{aligned}$$

and hence

$$\begin{aligned}
& ||u_n - x^*||^2 \le ||x_n - x^*||^2 - ||u_n - x_n||^2 \\
& +2r_n\langle x_n - u_n, \psi x_n - \psi x^* \rangle - r_n^2||\psi x_n - \psi x^*||^2 \\
\le\ & ||x_n - x^*||^2 - ||u_n - x_n||^2 \\
& +2r_n||x_n - u_n||||\psi x_n - \psi x^*||.
\end{aligned} \tag{6.2.22}$$

By the convexity of $||.||^2$, we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= \Big|\Big|(1-\beta_n)\Big(\frac{1}{t_n}\int_0^{t_n}[T(u)y_n - T(u)x^*]du\Big) \\
&\quad +\beta_n(x_n - x^*)\Big|\Big|^2 \\
&\leq \beta_n||x_n - x^*||^2 + (1-\beta_n)(\alpha_n||f(x_n) - x^*||^2 \\
&\quad +(1-\alpha_n)||w_n - x^*||^2) \\
&\leq \beta_n||x_n - x^*||^2 + (1-\beta_n)(\alpha_n||f(x_n) - x^*||^2 \\
&\quad +(1-\alpha_n)||u_n - x^*||^2) \\
&\leq \beta_n||x_n - x^*||^2 + (1-\beta_n)\alpha_n||f(x_n) - x^*||^2 \\
&\quad +2r_n||x_n - u_n||||\psi x_n - \psi x^*|| \\
&\quad +(1-\alpha_n)(1-\beta_n)[||x_n - x^*||^2 - ||x_n - u_n||^2].
\end{aligned}$$

Hence

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)||x_n - u_n||^2 \leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 \\
&+(1-\beta_n)\alpha_n||f(x_n) - x^*||^2 + 2r_n||x_n - u_n||||\psi x_n - \psi x^*|| \\
\leq\ &\alpha_n||f(x_n) - x^*||^2 + ||x_{n+1} - x_n||(||x_n - x^*|| + ||x_{n+1} - x^*||) \\
&+2r_n||x_n - u_n||||\psi x_n - \psi x^*||.
\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} ||\psi x_n - \psi x^*|| = 0$ and $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$, then

$$\lim_{n\to\infty} (1-\beta_n)(1-\alpha_n)||x_n - u_n||^2 = 0.$$

Since $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$, we obtain

$$\lim_{n\to\infty} ||x_n - u_n|| = 0. \tag{6.2.23}$$

Furthermore, from (6.2.20), (6.2.21) and (6.2.23), we have for every $h \in [0,\infty)$ that

$$\begin{aligned}
&||T(h)y_n - T(h)x_n|| \leq ||y_n - x_n|| \\
\leq\ &||x_n - u_n|| + ||u_n - w_n|| + ||w_n - y_n|| \to 0. \qquad (6.2.24)
\end{aligned}$$

So, we obtain from (6.2.6) and (6.2.24) that

$$||T(h)y_n - x_n|| \leq ||T(h)y_n - T(h)x_n|| + ||T(h)x_n - x_n|| \to 0.$$

Hence, we have for every $h \in [0,\infty)$ that

$$||T(h)y_n - y_n|| \leq ||T(h)y_n - x_n|| + ||x_n - w_n|| + ||w_n - y_n||$$

and this implies that

$$\lim_{n\to\infty} ||T(h)y_n - y_n|| = 0. \tag{6.2.25}$$

■

Lemma 6.2.5 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $\Im := \{T(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Omega := F(\Im) \cap I(A, M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction mapping with a constant $\gamma \in (0, 1)$. Let $\{t_n\} \subset (0, \infty)$ be a real sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (6.2.1). Then,*

$$\limsup_{n\to\infty} \langle f(z) - z, y_n - z\rangle \leq 0,$$

where $z = P_\Omega f(z)$.

Proof The existence of z is justified since P_Ω is nonexpansive and f is a contraction, then $P_\Omega of$ is a contraction so it has a fixed point. To do this, we choose a subsequence $\{y_{n_j}\}_{j=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that

$$\limsup_{n\to\infty} \langle f(z) - z, y_n - z\rangle = \lim_{j\to\infty} \langle f(z) - z, y_{n_j} - z\rangle. \tag{6.2.26}$$

As $\{y_n\}_{n=1}^{\infty}$ is bounded, there exists a subsequence $\{y_{n_j}\}_{j=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ that converges weakly to w.
We first show that $w \in I(A, M)$. Since A is an $\frac{1}{\alpha}$-Lipschitz monotone mapping and $D(A) = H$, we obtain from Lemma 1.5.11 that $M + A$ is maximal monotone. Let $(v, g) \in G(M + A)$, that is, $g - Av \in M(v)$. Since $w_{n_j} = J_{M,\lambda}(I - \lambda A)u_{n_j}$, we get $(I - \lambda A)u_{n_j} \in (I + \lambda M)w_{n_j}$, that is,

$$\frac{1}{\lambda}(u_{n_j} - \lambda A u_{n_j} - w_{n_j}) \in M(w_{n_j}).$$

Using the maximal monotonicity of $M + A$, we obtain

$$\Big\langle v - w_{n_j}, g - Av - \frac{1}{\lambda}(u_{n_j} - \lambda A u_{n_j} - w_{n_j})\Big\rangle \geq 0$$

and so

$$\begin{aligned} \langle v - w_{n_j}, g\rangle &\geq \Big\langle v - w_{n_j}, Av + \frac{1}{\lambda}(u_{n_j} - \lambda A u_{n_j} - w_{n_j})\Big\rangle \\ &= \Big\langle v - w_{n_j}, Av - Aw_{n_j} + Aw_{n_j} - Au_{n_j} + \frac{1}{\lambda}(u_{n_j} - w_{n_j})\Big\rangle \\ &\geq 0 + \langle v - w_{n_j}, Aw_{n_j} - Au_{n_j}\rangle + \Big\langle v - w_{n_j}, \frac{1}{\lambda}(u_{n_j} - w_{n_j})\Big\rangle. \end{aligned}$$

It follows from $\lim_{j\to\infty} ||w_{n_j} - u_{n_j}|| = 0$, $\lim_{j\to\infty} ||Aw_{n_j} - Au_{n_j}|| = 0$ and $\lim_{j\to\infty} w_{n_j} = w$ (since $\lim_{j\to\infty} ||y_{n_j} - w_{n_j}|| = 0$ and $\lim_{j\to\infty} y_{n_j} = w$) that

$$\lim_{j\to\infty} \langle v - w_{n_j}, g\rangle = \langle v - w, g\rangle \geq 0.$$

Using the maximal monotonicity of $M + A$, we obtain $\theta \in (M + A)(w)$ and this implies that $w \in I(A, M)$.

We next show that $w \in EP$. Since $u_n = T_{r_n}(x_n - r_n\psi x_n), \quad n \geq 1$, we have for any $y \in K$ that

$$F(u_n, y) + \langle \psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0.$$

Furthermore, replacing n by n_j in the last inequality and using $(A2)$, we obtain

$$\langle \psi x_{n_j}, y - u_{n_j}\rangle + \frac{1}{r_{n_j}}\langle y - u_{n_j}, u_{n_j} - x_{n_j}\rangle \geq F(y, u_{n_j}). \tag{6.2.27}$$

Let $z_t := ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in K$. This implies that $z_t \in K$. Then, by (8.2.20), we have

$$\begin{aligned} \langle z_t - u_{n_j}, \psi z_t\rangle &\geq \langle z_t - u_{n_j}, \psi z_t\rangle - \langle z_t - u_{n_j}, \psi x_{n_j}\rangle \\ &\quad -\big\langle z_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\big\rangle + F(z_t, u_{n_j}) \\ &= \langle z_t - u_{n_j}, \psi z_t - \psi u_{n_j}\rangle + \langle z_t - u_{n_j}, \psi u_{n_j} - \psi x_{n_j}\rangle \\ &\quad -\big\langle z_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\big\rangle + F(z_t, u_{n_j}). \end{aligned} \tag{6.2.28}$$

Since $||x_{n_j} - u_{n_j}|| \to 0, \quad j \to \infty$, we obtain $||\psi x_{n_j} - \psi u_{n_j}|| \to 0, \quad j \to \infty$. Furthermore, by the monotonicity of ψ, we obtain $\langle z_t - u_{n_j}, \psi z_t - \psi u_{n_j}\rangle \geq 0$. From (6.2.20) and (6.2.21), we obtain that

$$||y_n - u_n|| \leq ||u_n - w_n|| + ||y_n - w_n|| \to 0, \quad n \to \infty.$$

Since $\lim_{j\to\infty} ||y_{n_j} - u_{n_j}|| = 0$ and $\lim_{j\to\infty} y_{n_j} = w$, we obtain that $\lim_{j\to\infty} u_{n_j} = w$. Then, using assumption $(A4)$ in (6.2.28), we obtain

$$\langle z_t - w, \psi z_t\rangle \geq F(z_t, w), \quad j \to \infty. \tag{6.2.29}$$

Using $(A1)$, $(A4)$ and (6.2.29) we also obtain

$$\begin{aligned} 0 &= F(z_t, z_t) \leq tF(z_t, y) + (1-t)F(z_t, w) \\ &\leq tF(z_t, y) + (1-t)\langle z_t - w, \psi z_t\rangle \\ &= tF(z_t, y) + (1-t)t\langle y - w, \psi z_t\rangle \end{aligned}$$

and hence

$$0 \leq F(z_t, y) + (1-t)\langle y - w, \psi z_t\rangle.$$

Letting $t \to 0$ and using assumption $(A3)$, we have, for each $y \in K$,

$$0 \leq F(w, y) + \langle y - w, \psi w\rangle. \tag{6.2.30}$$

That is, $w \in EP$.

Next, we show that $w \in F(\Im)$. Assume the contrary that $w \neq T(h)w$ for some $h \in [0,\infty)$. Then by Opial's condition, we obtain from (6.2.25) that

$$\begin{aligned}\liminf_{j\to\infty}||y_{n_j} - w|| &< \liminf_{j\to\infty}||y_{n_j} - T(h)w||\\ &\leq \liminf_{j\to\infty}(||y_{n_j} - T(h)y_{n_j}|| + ||T(h)y_{n_j} - T(h)w||)\\ &\leq \liminf_{j\to\infty}||y_{n_j} - w||.\end{aligned}$$

This is a contradiction. Hence, $w \in F(\Im)$. Thus $w \in \Omega = F(\Im)\bigcap I(A,M)\bigcap EP$. By (6.2.26) and property of metric projection, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z) - z, y_n - z\rangle &= \lim_{j\to\infty}\langle f(z) - z, y_{n_j} - z\rangle\\ &= \langle f(z) - z, w - z\rangle \leq 0.\end{aligned}$$

■

Theorem 6.2.6 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let $\Im := \{T(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Omega := F(\Im) \cap I(A,M) \cap EP \neq \emptyset$ and suppose $f : H \to H$ is a contraction mapping with a constant $\gamma \in (0,1)$. Let $\{t_n\} \subset (0,\infty)$ be a real sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (6.2.1). Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to z, where $z := P_\Omega f(z)$.*

Proof Since $y_n - z = \alpha_n(f(x_n) - z) + (1-\alpha_n)(w_n - z)$, we have

$$\begin{aligned}&||y_n - z||^2 \leq ||\alpha_n(f(x_n) - z) + (1-\alpha_n)(w_n - z)||^2\\ \leq\ &(1-\alpha_n)^2||w_n - z||^2 + 2\alpha_n\langle f(x_n) - z, y_n - z\rangle\\ \leq\ &(1-\alpha_n)^2||x_n - z||^2 + 2\alpha_n\langle f(x_n) - z, y_n - z\rangle\\ =\ &(1-\alpha_n)^2||x_n - z||^2 + 2\alpha_n\langle f(x_n) - f(z), y_n - z\rangle\\ &+2\alpha_n\langle f(z) - z, y_n - z\rangle\\ \leq\ &(1-\alpha_n)^2||x_n - z||^2 + 2\alpha_n\gamma||x_n - z||||y_n - z||\\ &+2\alpha_n\langle f(z) - z, y_n - z\rangle\\ \leq\ &(1-\alpha_n)^2||x_n - z||^2 + \alpha_n\gamma(||x_n - z||^2 + ||y_n - z||^2)\\ &+2\alpha_n\langle f(z) - z, y_n - z\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}
||y_n - z||^2 &\le \frac{1 - 2\alpha_n + \alpha_n^2 + \alpha_n\gamma}{1 - \alpha_n\gamma}||x_n - z||^2 \\
&+ \frac{2\alpha_n}{1 - \alpha_n\gamma}\langle f(z) - z, y_n - z\rangle \\
= &\Big[1 - \frac{2(1-\gamma)\alpha_n}{1 - \alpha_n\gamma}\Big]||x_n - z||^2 + \frac{\alpha_n^2}{1 - \alpha_n\gamma}||x_n - z||^2 \\
&+ \frac{2\alpha_n}{1 - \alpha_n\gamma}\langle f(z) - z, y_n - z\rangle. \qquad (6.2.31)
\end{aligned}$$

Using (6.2.31) in (6.2.1), we obtain

$$\begin{aligned}
&||x_{n+1} - z||^2 \le \beta_n||x_n - z||^2 + (1 - \beta_n)\Big|\Big|\frac{1}{t_n}\int_0^{t_n}[T(u)y_n - T(u)z]du\Big|\Big|^2 \\
\le\ & \beta_n||x_n - z||^2 + (1 - \beta_n)||y_n - z||^2 \\
\le\ & \beta_n||x_n - z||^2 + (1 - \beta_n)\Big(\Big[1 - \frac{2(1-\gamma)\alpha_n}{1 - \alpha_n\gamma}\Big]||x_n - z||^2 \\
&+ \frac{\alpha_n^2}{1 - \alpha_n\gamma}||x_n - z||^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma}\langle f(z) - z, y_n - z\rangle\Big) \\
=\ & \Big[1 - (1 - \beta_n)\frac{2(1-\gamma)\alpha_n}{1 - \alpha_n\gamma}\Big]||x_n - z||^2 + \frac{\alpha_n(1 - \beta_n)}{1 - \alpha_n\gamma}\Big[\alpha_n||x_n - z||^2 \\
&+ 2\langle f(z) - z, y_n - z\rangle\Big] \\
=\ & (1 - \delta_n)||x_n - z||^2 + \sigma_n, \qquad (6.2.32)
\end{aligned}$$

where $\delta_n := (1 - \beta_n)\frac{2(1-\gamma)\alpha_n}{1-\alpha_n\gamma}$ and $b_n := \frac{\alpha_n(1-\beta_n)}{1-\alpha_n\gamma}\Big[\alpha_n||x_n - z||^2 + 2\langle f(z) - z, y_n - z\rangle\Big]$. It is easily verified that $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\frac{b_n}{\delta_n} \le 0$, then by Lemma 1.5.2, we obtain $\lim_{n\to\infty}||x_n - z|| = 0$. ■

A strong mean convergence theorem for nonexpansive mappings was first established for odd mappings by Baillon [6] and it was later generalized to that of nonlinear semigroups by Simeon Reich [167]. It follows from the above proof that Theorem 6.2.6 is valid for nonexpansive mappings. Thus, we have the following mean ergodic theorem for nonexpansive mappings in Hilbert spaces.

Corollary 6.2.7 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, ψ a μ-inverse-strongly monotone mapping of K into H, A an α-inverse-strongly monotone mapping of K into H and $M : H \to 2^H$ a maximal monotone mapping. Let T be a nonexpansive mapping on H such that*

$\Omega := F(T) \cap I(A, M) \cap EP \neq \emptyset$ *and suppose* $f : H \to H$ *is a contraction mapping with a constant* $\gamma \in (0, 1)$*. Suppose* $\{x_n\}_{n=1}^{\infty}$ *and* $\{u_n\}_{n=1}^{\infty}$ *are generated by* $x_1 \in H$*,*

$$\begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0 \quad \forall y \in K, \\ w_n = J_{M,\lambda}(u_n - \lambda A u_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)\Big(\frac{1}{n+1}\sum_{j=0}^{n} T^j[\alpha_n f(x_n) + (1 - \alpha_n) w_n] du\Big), \end{cases}$$

for all $n \geq 1$*, where* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are sequences in [0,1] and* $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ *satisfying:*
(i) $\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$
(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$
(iii) $\lambda \in (0, 2\alpha]$,
(iv) $0 < a \leq r_n \leq b < 2\mu, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,
then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* z*, where* $z := P_{\Omega} f(z)$*.*

Remark 6.2.8 *Examples of sequences* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{t_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *in Theorem 6.2.6 are*

$$\alpha_n = \frac{1}{n^{\frac{1}{4}}}, \quad \beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}, \quad n \geq 1,$$

$$t_n = n, \quad r_n = \frac{\mu n}{n+1}, \quad n \geq 1.$$

Remark 6.2.9 *All the results in this chapter are due to Shehu [174].*

CHAPTER 7

A New Iterative Scheme for a Countable Family of Relatively Nonexpansive Mappings and an Equilibrium Problem in Banach Spaces

7.1 Introduction

Motivated by the results of Nilsrakoo and Saejung [147], our purpose in this chapter is to introduce a new iterative scheme and prove a strong convergence theorem for a countable family of relatively nonexpansive mappings, which is also a solution to an equilibrium problem in a uniformly convex and uniformly smooth real Banach space. Furthermore, we study strong convergence concerning general H-monotone operators. Our results extend the results of Nilsrakoo and Saejung [147] and many other recent known results in the literature.

7.2 Main Results

Lemma 7.2.1 *Let E be a reflexive, strictly convex and smooth real Banach space. Let C be a nonempty, closed and convex subset of E. Suppose $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively nonexpansive mappings of C into E such that $\cap_{n=0}^{\infty} F(T_n) \neq \emptyset$. If $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence such that $x_n - T_n x_n \to 0$ and $z := \Pi_{\Omega} u$, then*

$$\limsup_{n\to\infty} \langle x_n - z, Ju - Jz \rangle \leq 0.$$

Proof From $(R3)$ of Definition 1.4.4, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup y \in \cap_{n=0}^{\infty} F(T_n)$ and

$$\limsup_{n\to\infty} \langle x_n - z, Ju - Jz\rangle = \lim_{j\to\infty} \langle x_{n_j} - z, Ju - Jz\rangle.$$

By Lemma 1.5.14, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \langle x_n - z, Ju - Jz\rangle &= \lim_{j\to\infty} \langle x_{n_j} - z, Ju - Jz\rangle \\ &= \langle p - z, Ju - Jz\rangle \leq 0. \end{aligned} \qquad (7.2.1)$$

■

Lemma 7.2.2 *Let E be a reflexive, strictly convex and smooth real Banach space. Let C be a nonempty, closed and convex subset of E. Let F be a bifunction from $C \times C$ satisfying $(A1) - (A4)$. Suppose $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively nonexpansive mappings of C into E such that $\Omega := (\cap_{n=0}^{\infty} F(T_n)) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by u, $u_0 \in E$,*

$$\begin{cases} x_n = T_{r_n} u_n, \\ u_{n+1} = J^{-1}(\alpha_n Ju + \beta_n Jx_n + \gamma_n JT_n x_n), \quad n \geq 0, \end{cases} \qquad (7.2.2)$$

Then, $\{x_n\}_{n=0}^{\infty}$ is well defined.

Proof Let $r > 0$ and $z \in E$. Then by Lemma 1.4.1 and Lemma 1.5.19, there exists a single-valued $x \in C$ such that

$$x = T_r(z) = F(x,y) + \frac{1}{r}\langle y - x, Jx - Jz\rangle \geq 0 \quad \text{for all} \quad y \in C.$$

It then follows from (7.2.2) that $\{x_n\}_{n=0}^{\infty}$ is well defined. ■

Lemma 7.2.3 *Let E be a reflexive, strictly convex and smooth real Banach space. Let C be a nonempty, closed and convex subset of E. Let F be a bifunction from $C \times C$ satisfying $(A1) - (A4)$. Suppose $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively nonexpansive mappings of C into E such that $\Omega := (\cap_{n=0}^{\infty} F(T_n)) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by (7.2.2). Then, $\{x_n\}_{n=0}^{\infty}$ is bounded.*

Proof Let $x^* \in \Omega$, then by using (7.2.2) and the fact that T_{r_n} is relatively quasinonexpansive, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, T_{r_{n+1}} u_{n+1}) \leq \phi(x^*, u_{n+1}) \\
&= \phi(x^*, J^{-1}(\alpha_n Ju + \beta_n Jx_n + \gamma_n JT_n x_n)) \\
&\leq \alpha_n \phi(x^*, u) + \beta_n \phi(x^*, x_n) + \gamma_n \phi(x^*, T_n x_n)) \\
&\leq \alpha_n \phi(x^*, u) + \beta_n \phi(x^*, x_n) + \gamma_n \phi(x^*, x_n) \\
&= \alpha_n \phi(x^*, u) + (1 - \alpha_n)\phi(x^*, x_n) \\
&\leq \max\{\phi(x^*, u), \phi(x^*, x_n)\} \\
&\vdots \\
&\leq \max\{\phi(x^*, u), \phi(x^*, x_0)\}. \qquad (7.2.3)
\end{aligned}$$

Hence, $\{x_n\}_{n=0}^{\infty}$ is bounded. ■

Lemma 7.2.4 *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Let F be a bifunction from $C \times C$ satisfying $(A1) - (A4)$. Suppose $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively nonexpansive mappings of C into E such that $\Omega := (\cap_{n=0}^{\infty} F(T_n)) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by (7.2.2) with the conditions*

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < b \leq \beta_n \gamma_n \leq 1$;

(iii) $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$ *satisfying* $\liminf_{n\to\infty} r_n > 0$.

Then $\lim_{n\to\infty} ||x_n - T_n x_n|| = 0$.

Proof By Lemma 7.2.3, we have that $\{x_n\}_{n=0}^{\infty}$ is bounded and this implies that $\{T_n x_n\}_{n=0}^{\infty}$ is also bounded. Since E is uniformly smooth, E^* is uniformly convex. Then from Lemma 1.5.18, we have for some $M > 0$ that

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\leq \phi(x^*, u_{n+1}) \leq \alpha_n \phi(x^*, u) + \beta_n \phi(x^*, x_n) + \gamma_n \phi(x^*, T_n x_n) \\
&\quad - \beta_n \gamma_n g(||Jx_n - JT_n x_n||) \\
&\leq \alpha_n \phi(x^*, u) + (1 - \alpha_n)\phi(x^*, x_n) - \beta_n \gamma_n g(||Jx_n - JT_n x_n||) \\
&\leq \alpha_n M + \phi(x^*, x_n) - \beta_n \gamma_n g(||Jx_n - JT_n x_n||). \qquad (7.2.4)
\end{aligned}$$

This implies that

$$\begin{aligned}
0 < bg(||Jx_n - JT_n x_n||) &\leq \beta_n \gamma_n g(||Jx_n - JT_n x_n||) \\
&\leq \alpha_n M + \phi(x^*, x_n) - \phi(x^*, x_{n+1}). \quad (7.2.5)
\end{aligned}$$

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(x^*, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{\phi(x^*, x_n)\}_{n=0}^{\infty}$ converges and $\phi(x^*, x_n) - \phi(x^*, x_{n+1}) \to 0, \quad n \to \infty$. This implies from (7.2.5) and condition (i) that

$$g(||Jx_n - JT_n x_n||) \to 0, \quad n \to \infty.$$

Since g is a continuous function with $g(0) = 0$, we have

$$||Jx_n - JT_n x_n|| \to 0, \quad n \to \infty.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$||x_n - T_n x_n|| \to 0, \quad n \to \infty. \tag{7.2.6}$$

Case 2. Suppose there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(x^*, x_{n_i}) < \phi(x^*, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then by Lemma 1.5.21, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\phi(x^*, x_{m_k}) \le \phi(x^*, x_{m_k+1}) \quad \text{and} \quad \phi(x^*, x_k) \le \phi(x^*, x_{m_k+1})$$

for all $k \in \mathbb{N}$. Combined with (7.2.5), this gives

$$\begin{aligned} 0 &< bg(||Jx_{m_k} - JT_{m_k} x_{m_k}||) \le \alpha_{m_k} M + \phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1}) \\ &\le \alpha_{m_k} M \end{aligned}$$

for all $k \in \mathbb{N}$. It then follows that

$$g(||Jx_{m_k} - JT_{m_k} x_{m_k}||) \to 0, \quad k \to \infty.$$

By the same arguments as in Case 1, we obtain that

$$||x_{m_k} - T_{m_k} x_{m_k}|| \to 0, \quad k \to \infty.$$

■

Lemma 7.2.5 *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Let F be a bifunction from $C \times C$ satisfying $(A1)-(A4)$. Suppose $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively nonexpansive mappings of C into E such that $\Omega := (\cap_{n=0}^{\infty} F(T_n)) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\}, \ \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by (7.2.2) with the conditions given in Lemma 7.2.4. Let $z_n := J^{-1}(\alpha_n Ju + \beta_n Jx_n + \gamma_n JT_n x_n), \ n \ge 0$. Then, $\limsup_{n\to\infty} \langle z_n - z, Ju - Jz\rangle \le 0$, where $z := \Pi_\Omega u$.*

Proof To show the inequality $\limsup_{n\to\infty}\langle z_n - z, Ju - Jz\rangle \le 0$, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - z, Ju - Jz\rangle = \lim_{j\to\infty}\langle x_{n_j} - z, Ju - Jz\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to p.
The rest of the proof will be divided into two parts.
Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(x^*, x_n)\}_{n=n_0}^{\infty}$ is non-increasing. Then $\{\phi(x^*, x_n)\}_{n=0}^{\infty}$ converges and $\phi(x^*, x_n) - \phi(x^*, x_{n+1}) \to 0, \ \ n \to \infty$. From Lemma 7.2.4, we have

$$\phi(x_n, T_n x_n) \to 0, \quad n \to \infty.$$

Since $x_{n_j} \rightharpoonup p$ and $\{T_n\}_{n=0}^{\infty}$ are uniformly closed, we have $p \in (\cap_{n=0}^{\infty} F(T_n))$.

Next, we show that $p \in EP(F)$. Now, by Lemma 1.5.19, (7.2.4) and condition (i), we obtain

$$\begin{aligned}
\phi(x_n, u_n) &= \phi(T_{r_n} u_n, u_n)\\
&\le \phi(x^*, u_n) - \phi(x^*, x_n)\\
&\le \alpha_{n-1} M + \phi(x^*, x_{n-1}) - \phi(x^*, x_n) \to 0, \quad n \to \infty.
\end{aligned}$$

Using Lemma 1.5.20, we have $\lim_{n\to\infty}||x_n - u_n|| = 0$. Now, since $x_{n_j} \rightharpoonup p$ and $\lim_{n\to\infty}||x_n - u_n|| = 0$, we obtain that $u_{n_j} \rightharpoonup p$. Also, since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n\to\infty}||x_n - u_n|| = 0$, we obtain

$$\lim_{n\to\infty}||Jx_n - Ju_n|| = 0.$$

Since $\liminf_{n\to\infty} r_n > 0$,

$$\lim_{n\to\infty}\frac{||Jx_n - Ju_n||}{r_n} = 0. \tag{7.2.7}$$

Since $x_n = T_{r_n} u_n, \ \ n \ge 0$, by Lemma 1.4.2, we have

$$F(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \ge 0, \quad \forall y \in C.$$

Furthermore, replacing n by n_j in the last inequality and using $(A2)$, we obtain

$$\frac{1}{r_{n_j}}\langle y - x_{n_j}, Jx_{n_j} - Ju_{n_j}\rangle \ge F(y, x_{n_j}). \tag{7.2.8}$$

By $(A4)$, (8.2.19) and $x_{n_j} \rightharpoonup p$, we have

$$F(y,p) \leq 0, \quad \forall y \in C.$$

For fixed $y \in C$, let $z_{t,y} := ty + (1-t)p$ for all $t \in (0,1]$. This implies that $z_t \in C$. This yields that $F(z_t, p) \leq 0$. It follows from $(A1)$ and $(A4)$ that

$$\begin{array}{rcl} 0 & = & F(z_t, z_t) \leq tF(z_t, y) + (1-t)F(z_t, p) \\ & \leq & tF(z_t, y) \end{array}$$

and hence

$$0 \leq F(z_t, y).$$

From condition $(A3)$, we obtain

$$F(p, y) \geq 0, \quad \forall y \in C.$$

This implies that $p \in EP(F)$. Hence, we have $p \in \Omega = EP(F) \cap (\cap_{n=0}^{\infty} F(T_n))$.

Let $y_n := J^{-1}\Big(\frac{\beta_n}{1-\alpha_n} Jx_n + \frac{\gamma_n}{1-\alpha_n} JT_n x_n\Big), \quad n \geq 0$, then

$$\phi(x_n, y_n) \leq \frac{\beta_n}{1-\alpha_n}\phi(x_n, x_n) + \frac{\gamma_n}{1-\alpha_n}\phi(x_n, T_n x_n) \to 0, \quad n \to \infty. \quad (7.2.9)$$

By Lemma 1.5.20, it follows that $||x_n - y_n|| \to 0, \quad n \to \infty$. Furthermore,

$$\begin{array}{rcl} \phi(y_n, z_n) & = & \phi(y_n, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jy_n)) \\ & \leq & \alpha_n \phi(y_n, u) + (1-\alpha_n)\phi(y_n, y_n) \\ & = & \alpha_n \phi(y_n, u) \to 0, \quad n \to \infty. \end{array} \quad (7.2.10)$$

Again, by Lemma 1.5.20, it follows that $||y_n - z_n|| \to 0, \quad n \to \infty$. Then

$$||x_n - z_n|| \leq ||y_n - z_n|| + ||x_n - y_n|| \to 0, \quad n \to \infty. \quad (7.2.11)$$

By (7.2.11), and Lemma 7.2.4, we obtain

$$\limsup_{n\to\infty} \langle z_n - z, Ju - Jz\rangle = \limsup_{n\to\infty} \langle x_n - z, Ju - Jz\rangle \leq 0. \quad (7.2.12)$$

<u>Case 2</u>. Suppose there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(x^*, x_{n_i}) < \phi(x^*, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then, by Lemma 1.5.21, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k+1}) \text{ and } \phi(x^*, x_k) \leq \phi(x^*, x_{m_k+1})$$

for all $k \in \mathbb{N}$. Combined with (7.2.5), this gives

$$\begin{aligned} 0 &< bg(||Jx_{m_k} - JT_{m_k}x_{m_k}||) \leq \alpha_{m_k}M + \phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1}) \\ &\leq \alpha_{m_k}M \end{aligned}$$

for all $k \in \mathbb{N}$. It then follows that

$$g(||Jx_{m_k} - JT_{m_k}x_{m_k}||) \to 0, \quad k \to \infty.$$

By the same arguments as in Case 1, we obtain that

$$\limsup_{k\to\infty} \langle z_{m_k} - z, Ju - Jz\rangle \leq 0. \tag{7.2.13}$$

■

Theorem 7.2.6 *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Let F be a bifunction from $C \times C$ satisfying (A1)−(A4). Suppose $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively nonexpansive mappings of C into E such that $\Omega := (\cap_{n=0}^{\infty} F(T_n)) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by (7.2.2) with the conditions given in Lemma 7.2.4. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_\Omega u$.*

Proof The proof will be divided into two parts.
Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(x^*, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{\phi(x^*, x_n)\}_{n=0}^{\infty}$ converges and $\phi(x^*, x_n) - \phi(x^*, x_{n+1}) \to 0, \quad n \to \infty$. Therefore,

$$\begin{aligned} \phi(z, x_{n+1}) &\leq \phi(z, J^{-1}(\alpha_n Ju + \beta_n Jx_n + \gamma_n JT_n x_n)) \\ &= V(z, \alpha_n Ju + \beta_n Jx_n + \gamma_n JT_n x_n) \\ &\leq V(z, \alpha_n Ju + \beta_n Jx_n + \gamma_n JT_n x_n - \alpha_n(Ju - Jz)) \\ &\quad -2\langle J^{-1}(\alpha_n Ju + \beta_n Jx_n + \gamma_n JT_n x_n) - z, -\alpha_n(Ju - Jz)\rangle \\ &= V(z, \alpha_n Jz + \beta_n Jx_n + \gamma_n JT_n x_n) \\ &\quad +2\alpha_n\langle z_n - z, Ju - Jz\rangle \\ &= \phi(z, J^{-1}(\alpha_n Jz + \beta_n Jx_n + \gamma_n JT_n x_n)) \\ &\quad +2\alpha_n\langle z_n - z, Ju - Jz\rangle \\ &\leq \alpha_n\phi(z, z) + \beta_n\phi(z, x_n) + \gamma_n\phi(z, T_n x_n) \\ &\quad +2\alpha_n\langle z_n - p, Ju - Jz\rangle \\ &\leq (1 - \alpha_n)\phi(z, x_n) + 2\alpha_n\langle z_n - z, Ju - Jz\rangle. \end{aligned} \tag{7.2.14}$$

Now, using (7.2.12), (7.2.14) and Lemma 1.5.2, we obtain $\phi(z, x_n) \to 0, \ n \to \infty$. Hence, $x_n \to z, \quad n \to \infty$.

<u>Case 2</u>. Suppose there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(x^*, x_{n_i}) < \phi(x^*, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then, by Lemma 1.5.21, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k+1}) \quad \text{and} \quad \phi(x^*, x_k) \leq \phi(x^*, x_{m_k+1})$$

for all $k \in \mathbb{N}$. From (7.2.14), we have

$$\phi(z, x_{m_k+1}) \leq (1 - \alpha_{m_k})\phi(z, x_{m_k}) + 2\alpha_{m_k}\langle z_{m_k} - z, Ju - Jz\rangle. \quad (7.2.15)$$

Since $\phi(z, x_{m_k}) \leq \phi(z, x_{m_k+1})$, we have

$$\begin{aligned} \alpha_{m_k}\phi(z, x_{m_k}) &\leq \phi(z, x_{m_k}) - \phi(z, x_{m_k+1}) + 2\alpha_{m_k}\langle z_{m_k} - z, Ju - Jz\rangle \\ &\leq 2\alpha_{m_k}\langle z_{m_k} - z, Ju - Jz\rangle. \end{aligned}$$

In particular, since $\alpha_{m_k} > 0$, we get

$$\phi(z, x_{m_k}) \leq 2\langle z_{m_k} - z, Ju - Jz\rangle. \quad (7.2.16)$$

It then follows from (7.2.13) that $\phi(z, x_{m_k}) \to 0, \quad k \to \infty$. From (7.2.16) and (7.2.15), we have

$$\phi(z, x_{m_k+1}) \to 0, \quad k \to \infty.$$

Since $\phi(z, x_k) \leq \phi(z, x_{m_k+1})$ for all $k \in \mathbb{N}$, we conclude that $x_k \to z, \quad k \to \infty$. This implies that $x_n \to z, \quad n \to \infty$ which completes the proof. ∎

7.3 Applications

(I) Nonexpansive Mappings

Corollary 7.3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ satisfying (A1)−(A4). Let T be a nonexpansive mapping of C into H such that $\Omega := F(T) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by u, $u_0 \in H$,*

$$\begin{cases} x_n = T_{r_n} u_n, \\ u_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad n \geq 0, \end{cases}$$

with the conditions
(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < b \leq \beta_n \gamma_n \leq 1$;
(iii) $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$ *satisfying* $\liminf_{n\to\infty} r_n > 0$.
Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_\Omega u$.

(II) General H-Monotone Mappings

Using Lemma 1.5.24 and Theorem 7.2.6, we obtain the following result.

Corollary 7.3.2 *Let E be a uniformly convex real Banach space which is also uniformly smooth with $\delta_E(\epsilon) \geq k\epsilon^2$ and $\rho_E(t) \leq ct^2$ for some $k, c > 0$. Suppose $H : E \to E^*$ is a strictly monotone and β-Lipschitz continuous mapping and $M : E \to 2^{E^*}$ is a general H-monotone mapping and r-strongly monotone mapping with $r > 0$. Let $T_\lambda^M = (H + \lambda M)^{-1}H$. For each $n \geq 0$, suppose there exists a $\lambda_n > 0$ such that $64c\beta^2 \leq \frac{1}{2}k\lambda_n^2 r^2$. Assume that $M^{-1}0 \neq \emptyset$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by u, $u_0 \in E$*

$$x_{n+1} = J^{-1}(\alpha_n Ju + \beta_n Jx_n + \gamma_n JT_{\lambda_n}^M x_n), \quad n \geq 0.$$

with the conditions
(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty;$
(ii) $0 < b \leq \beta_n \gamma_n \leq 1.$
Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{M^{-1}0}u$.

Remark 7.3.3 *Our Theorem 7.2.6 extends the results of Nilsrakoo and Saejung [147] from approximation of fixed point of a single relatively nonexpansive mapping to approximation of common fixed point of a countable family of relatively nonexpansive mappings which is also a solution to an equilibrium problem.*

Remark 7.3.4 *Our iterative process* (7.2.2) *appears simpler than the hybrid algorithm iterative process used by many authors; see [121, 134, 146, 157, 161, 162, 185, 198, 199, 204].*

Now, we give an explicit example validating the claims we made in Theorem 7.2.6, that is, $(\cap_{n=0}^{\infty} F(T_n)) \cap EP(F) \neq \emptyset$.

Example 7.3.5 *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E and $h : C \to \mathbb{R}$ be a convex and a lower semi-continuous functional. Let $F : C \times C \to \mathbb{R}$ be defined by $F(x, y) := h(y) - h(x)$. Let us now find the following equilibrium problem: find $x \in C$ such that*

$$F(x, y) \geq 0, \tag{7.3.1}$$

for all $y \in C$. It is obvious that F satisfies conditions $(A1) - (A4)$. It is known; see [18, 199] that $EP(F) \neq \emptyset$, where $EP(F)$ is the set of solutions to (7.3.1).

Now, for each $n = 0, 1, \ldots,$ *take* $T_n = \Pi_C$*, where* Π_C *is the generalized projection map from a uniformly smooth and uniformly convex real Banach space* E *onto a nonempty, closed and convex subset* C *of* E*. It is known; see [161] that* Π_C *is a relatively nonexpansive mapping from* E *onto* C *and* $F(T_n) = F(\Pi_C) = C$*.*

It then follows that $(\cap_{n=0}^{\infty} F(T_n)) \cap EP(F) = C \cap EP(F) = EP(F) \neq \emptyset$*.*

Remark 7.3.6 *All the results in this chapter are due to Shehu [175, 176].*

CHAPTER 8

Strong Convergence Theorems for Nonlinear Mappings, Variational Inequality Problems and System of Generalized Mixed Equilibrium Problems

8.1 Introduction

In this chapter, we introduce a new hybrid projection algorithm based on the shrinking projection method and prove strong convergence theorem for approximation of a common element of the set of common fixed point of an infinite family of relatively quasi-nonexpansive mappings, set of solutions to a variational inequality problem and the set of solutions to system of generalized mixed equilibrium problems in a 2-uniformly convex real Banach space which is also uniformly smooth. Consequently, we use our new iterative scheme to prove strong convergence theorem for approximation of a fixed point of a weak relatively nonexpansive mapping which is also a common solution to variational inequality problem and system of generalized mixed equilibrium problems in a Banach space. Our results extend the results of Martinez-Yanes and Xu [131], Plubtieng and Ungchittrakool [157] and many other recent and important results in the literature. Finally, we use our results to obtain several applications in a Banach space.

For each $k = 1, 2, ..., m$, let F_k be a bifunction from $C \times C$ satisfying $(A1)-(A4)$, $\varphi_k : C \to \mathbb{R}$ be a lower semicontinuous and convex functional and $A_k : C \to E^*$ be a continuous and monotone mapping, we define

$$T_{r_k}^{G_k}(x) = \{z \in C : G_k(z, y) + \frac{1}{r_k}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\},$$

where $G_k(z, y) = F_k(z, y) + \varphi_k(y) - \varphi_k(z) + \langle A_k z, y - z\rangle, \quad z, y \in C.$

Observe that an operator T in a Banach space E is said to be *closed* if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.
In this chapter, we shall assume that
$(B1)$ B is α-inverse-strongly monotone
$(B2)$ $||By|| \leq ||By - Bu||$ for all $y \in C$ and $u \in VI(C, B)$
$(B3)$ $VI(C, B) \neq \emptyset$.

8.2 Main Results

Lemma 8.2.1 *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $B : C \to E^*$ is an operator satisfying $(B1) - (B3)$ and $\{T_n\}_{n=1}^{\infty}$ is an infinite family of relatively-quasi nonexpansive mappings of C into itself such that $F := VI(C, B) \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \cap \Big(\cap_{n=1}^{\infty} F(T_n)\Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1} x_0$,*

$$\begin{cases} \upsilon_n = \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n) \\ y_n = J^{-1}(\alpha_n J\upsilon_n + (1 - \alpha_n)JT_n\upsilon_n) \\ z_n = J^{-1}(\beta_n Jx_0 + (1 - \beta_n)Jy_n) \\ u_n = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} ... T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} z_n \\ C_{n+1} = \{w \in C_n : \phi(w, u_n) \leq \phi(w, x_n) + \beta_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1, \end{cases} \tag{8.2.1}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\lim_{n\to\infty} \beta_n = 0$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $(k = 1, 2, ..., m)$ satisfying $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1, 2, ..., m)$. Suppose that for each bounded subset D of C, the ordered pair $(\{T_n\}, D)$ satisfies either condition AKTT or condition *AKTT. Let T be the mapping from C into E defined by $Tx := \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that T is closed and $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then, $\Pi_{C_{n+1}} x_0$ is well defined for all $n \geq 0$.

Proof We first show that C_n, $\forall n \geq 1$ is closed and convex. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_n is closed convex for some $n > 1$. From the definition of C_{n+1}, we have that $z \in C_{n+1}$ implies $\phi(z, u_n) \leq \phi(z, x_n) + \beta_n(||x_0||^2 + 2\langle z, Jx_n - Jx_0\rangle)$. This is equivalent to

$$2\langle z, Jx_n - Ju_n\rangle - 2\beta_n\langle z, Jx_n - Jx_0\rangle \leq ||x_n||^2 - ||u_n||^2 + \beta_n||x_0||^2$$

This implies that C_{n+1} is closed and convex for the same $n > 1$. Hence, C_n is closed and convex $\forall n \geq 1$. This shows that $\Pi_{C_{n+1}} x_0$ is well defined for all $n \geq 0$. ■

Lemma 8.2.2 *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $B : C \to E^*$ is an operator satisfying $(B1)-(B3)$ and $\{T_n\}_{n=1}^{\infty}$ is an infinite family of relatively-quasi nonexpansive mappings of C into itself such that $F := VI(C,B) \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \cap \Big(\cap_{n=1}^{\infty} F(T_n)\Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by (8.2.1). Then, $\{x_n\}_{n=0}^{\infty}$ is well defined.*

Proof By taking $\theta_n^k = T_{r_k,n}^{G_k} T_{r_{k-1},n}^{G_{k-1}} ... T_{r_2,n}^{G_2} T_{r_1,n}^{G_1}, \quad k = 1, 2, ..., m$ and $\theta_n^0 = I$ for all $n \geq 1$, we obtain $u_n = \theta_n^m z_n$.

We next show that $F \subset C_n, \quad \forall n \geq 1$. From Lemma 1.4.2, one has that $T_{r_k,n}^{F_k}, \quad k = 1, 2, ..., m$ is relatively quasi-nonexpansive mapping. For $n = 1$, we have $F \subset C = C_1$. Then for each $x^* \in F$, we obtain

$$\begin{aligned}
\phi(x^*, y_n) &= \phi(x^*, J^{-1}(\alpha_n J\upsilon_n + (1-\alpha_n) JT_n \upsilon_n)) \\
&= ||x^*||^2 - 2\alpha_n \langle x^*, J\upsilon_n \rangle - 2(1-\alpha_n)\langle x^*, JT_n\upsilon_n\rangle \\
&\quad + ||\alpha_n J\upsilon_n + (1-\alpha_n)JT_n\upsilon_n||^2 \\
&\leq ||x^*||^2 - 2\alpha_n \langle x^*, J\upsilon_n \rangle - 2(1-\alpha_n)\langle x^*, JT_n\upsilon_n\rangle \\
&\quad + \alpha_n ||J\upsilon_n||^2 + (1-\alpha_n)||JT_n\upsilon_n||^2 \\
&= \alpha_n \phi(x^*, \upsilon_n) + (1-\alpha_n)\phi(x^*, T_n\upsilon_n) \\
&\leq \phi(x^*, \upsilon_n). \qquad (8.2.2)
\end{aligned}$$

Now, by Lemma 1.5.13 and Lemma 1.5.16, we obtain

$$\begin{aligned}
& \phi(x^*, \upsilon_n) = \phi(x^*, \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n)) \\
\leq\ & \phi(x^*, J^{-1}(Jx_n - \lambda_n Bx_n)) \\
=\ & V(x^*, Jx_n - \lambda_n Bx_n) \\
\leq\ & V(x^*, (Jx_n - \lambda_n Bx_n) + \lambda_n Bx_n) \\
& -2\langle J^{-1}(Jx_n - \lambda_n Bx_n) - x^*, \lambda_n Bx_n \rangle \\
=\ & V(x^*, Jx_n) - 2\lambda_n \langle J^{-1}(Jx_n - \lambda_n Bx_n) - x^*, Bx_n \rangle \\
=\ & \phi(x^*, x_n) - 2\lambda_n \langle x_n - x^*, Bx_n \rangle \\
& +2\langle J^{-1}(Jx_n - \lambda_n Bx_n) - x_n, -\lambda_n Bx_n \rangle. \qquad (8.2.3)
\end{aligned}$$

From condition $(B1)$ and $x^* \in VI(C, B)$, we obtain

$$\begin{aligned}
-2\lambda_n \langle x_n - x^*, Bx_n \rangle &= -2\lambda_n \langle x_n - x^*, Bx_n - Bx^* \rangle \\
&\quad -2\lambda_n \langle x_n - x^*, Bx^* \rangle \\
&\leq -2\alpha\lambda_n ||Bx_n - Bx^*||^2. \qquad (8.2.4)
\end{aligned}$$

By Lemma 1.5.25 and condition $(B2)$, we also obtain

$$\begin{aligned}
& 2\langle J^{-1}(Jx_n - \lambda_n Bx_n) - x_n, -\lambda_n Bx_n\rangle \\
= \; & 2\langle J^{-1}(Jx_n - \lambda_n Bx_n) - J^{-1}(Jx_n), -\lambda_n Bx_n\rangle \\
\leq \; & 2||J^{-1}(Jx_n - \lambda_n Bx_n) - J^{-1}(Jx_n)||||\lambda_n Bx_n|| \\
\leq \; & \frac{4}{c^2}||(Jx_n - \lambda_n Bx_n) - (Jx_n)||||\lambda_n Bx_n|| \\
= \; & \frac{4}{c^2}\lambda_n^2||Bx_n||^2 \\
\leq \; & \frac{4}{c^2}\lambda_n^2||Bx_n - Bx^*||^2. \qquad (8.2.5)
\end{aligned}$$

Combining (8.2.3), (8.2.4) and (8.2.5) and $0 < a \leq \lambda_n \leq b < \frac{c^2\alpha}{2}$, we obtain

$$\begin{aligned}
\phi(x^*, \upsilon_n) & \leq \phi(x^*, x_n) - 2\alpha\lambda_n||Bx_n - Bx^*||^2 + \frac{4}{c^2}\lambda_n^2||Bx_n - Bx^*||^2 \\
& = \phi(x^*, x_n) + 2\lambda_n\Big(\frac{2}{c^2}\lambda_n - \alpha\Big)||Bx_n - Bx^*||^2 \\
& \leq \phi(x^*, x_n). \qquad (8.2.6)
\end{aligned}$$

Combining (8.2.2) and (8.2.6), we have

$$\phi(x^*, y_n) \leq \phi(x^*, x_n).$$

Now

$$\begin{aligned}
& \phi(x^*, u_n) \leq \phi(x^*, \theta_n^m z_n) \leq \phi(x^*, z_n) \\
= \; & \phi(x^*, J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n)) \\
= \; & ||x^*||^2 - 2\langle x^*, \beta_n Jx_0 + (1-\beta_n)Jy_n\rangle + ||\beta_n Jx_0 + (1-\beta_n)Jy_n||^2 \\
\leq \; & ||x^*||^2 - 2\beta_n\langle x^*, Jx_0\rangle - 2(1-\beta_n)\langle x^*, Jy_n\rangle \\
& +\beta_n||x_0||^2 + (1-\beta_n)||y_n||^2 \\
\leq \; & \beta_n\phi(x^*, x_0) + (1-\beta_n)\phi(x^*, y_n) \\
\leq \; & \beta_n\phi(x^*, x_0) + (1-\beta_n)\phi(x^*, x_n) \\
= \; & \phi(x^*, x_n) + \beta_n[\phi(x^*, x_0) - \phi(x^*, x_n)] \\
\leq \; & \phi(x^*, x_n) + \beta_n(||x_0||^2 + 2\langle x^*, Jx_n - Jx_0\rangle). \qquad (8.2.7)
\end{aligned}$$

So, $x^* \in C_n$. This implies that $F \subset C_n, \quad \forall n \geq 1$ and the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (8.2.1) is well defined.

■

Lemma 8.2.3 *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $B : C \to E^*$ is an operator satisfying $(B1) - (B3)$ and $\{T_n\}_{n=1}^{\infty}$ is an infinite family of relatively-quasi nonexpansive mappings of C into itself* *such that* $F := VI(C,B) \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \cap \Big(\cap_{n=1}^{\infty} F(T_n)\Big) \neq \emptyset$*. Let* $\{x_n\}_{n=0}^{\infty}$ *be iteratively generated by* (8.2.1)*. Then, the limit of* $\{\phi(x_n, x_0)\}_{n=0}^{\infty}$ *exists.*

Proof We now show that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. From (8.2.1), we have $x_n = \Pi_{C_n} x_0$ which implies that

$$\langle x_n - z, Jx_0 - Jx_n \rangle \geq 0, \quad \forall z \in C_n \tag{8.2.8}$$

and

$$\langle x_n - u, Jx_0 - Jx_n \rangle \geq 0, \quad \forall u \in F. \tag{8.2.9}$$

By Lemma 1.5.13, we have

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, x_n) \\ &\leq \phi(u, x_0) \end{aligned}$$

for each $u \in F \subset C_n, \quad n \geq 1$. Hence, the sequence $\{\phi(x_n, x_0)\}_{n=0}^{\infty}$ is bounded. Since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0.$$

Therefore, $\{\phi(x_n, x_0)\}_{n=0}^{\infty}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}_{n=0}^{\infty}$ exists.

■

Lemma 8.2.4 *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $B : C \to E^*$ is an operator satisfying $(B1) - (B3)$ and $\{T_n\}_{n=1}^{\infty}$ is an infinite family of relatively-quasi nonexpansive mappings of C into itself such that* $F := VI(C,B) \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \cap \Big(\cap_{n=1}^{\infty} F(T_n)\Big) \neq \emptyset$*. Let* $\{x_n\}_{n=0}^{\infty}$ *be iteratively generated by* (8.2.1)*. Then,* $\{x_n\}_{n=0}^{\infty}$ *is Cauchy.*

Proof By the construction of C_n, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It then follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0) \to 0 \ \text{ as } \ m, n \to \infty. \end{aligned}$$

It then follows from Lemma 1.5.20 that $||x_m - x_n|| \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}_{n=0}^{\infty}$ is Cauchy. Since E is a Banach space and C is closed convex, then there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$.

■

Lemma 8.2.5 *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $B : C \to E^*$ is an operator satisfying $(B1) - (B3)$ and $\{T_n\}_{n=1}^{\infty}$ is an infinite family of relatively-quasi nonexpansive mappings of C into itself such that $F := VI(C, B) \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \cap \Big(\cap_{n=1}^{\infty} F(T_n)\Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by (8.2.1). Then, $\lim_{n\to\infty} \phi(y_n, z_n) = 0$.*

Proof Now, since $\phi(x_m, x_n) \to 0$ as $m, n \to \infty$, we have in particular that $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$ and this further implies that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$. Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \beta_n(||x_0||^2 + 2\langle x_{n+1}, Jx_n - Jx_0\rangle), \quad \forall n \geq 0.$$

Since $\lim_{n\to\infty} \beta_n = 0$ and $\{Jx_n\}$ is bounded, it follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

Since E is uniformly convex and smooth, we have from Lemma 1.5.20 that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0 = \lim_{n\to\infty} ||x_{n+1} - u_n||.$$

Therefore,

$$\lim_{n\to\infty} ||x_n - u_n|| = 0.$$

Hence,

$$\lim_{n\to\infty} ||x_n - u_n|| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n\to\infty} ||x_n - u_n|| = 0$, we obtain

$$\lim_{n\to\infty} ||Jx_n - Ju_n|| = 0.$$

From the properties of ϕ, we obtain that

$$\begin{aligned}
&\phi(y_n, x_n) = \phi(y_n, z_n) + \phi(z_n, x_n) + 2\langle y_n - z_n, Jz_n - Jx_n\rangle \\
\leq\ & \phi(y_n, z_n) + \phi(z_n, x_n) + 2||y_n - z_n||||Jz_n - Jx_n|| \\
\leq\ & \phi(y_n, z_n) + \phi(z_n, u_n) + \phi(u_n, x_n) \\
& +2\langle z_n - u_n, Ju_n - Jx_n\rangle \\
& +2||y_n - z_n||||Jz_n - Jx_n|| \\
\leq\ & \phi(y_n, z_n) + \phi(z_n, u_n) + \phi(u_n, x_n) \\
& +2||z_n - u_n||||Ju_n - Jx_n|| + 2||y_n - z_n||||Jz_n - Jx_n||. \quad (8.2.10)
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}\phi(y_n, z_n) &= ||y_n||^2 - 2\langle y_n, \beta_n Jx_0 + (1-\beta_n)Jy_n\rangle \\ &\quad +||\beta_n Jx_0 + (1-\beta_n)Jy_n||^2 \\ &\leq ||y_n||^2 - 2\beta_n\langle y_n, Jx_0\rangle - 2(1-\beta_n)\langle y_n, Jy_n\rangle \\ &\quad +\beta_n||x_0||^2 + (1-\beta_n)||y_n||^2 \\ &= \beta_n(||y_n||^2 - 2\langle y_n, Jx_0\rangle + ||x_0||^2) = \beta_n\phi(y_n, x_0).\end{aligned}$$

Since $\lim_{n\to\infty} \beta_n = 0$, we have that

$$\lim_{n\to\infty} \phi(y_n, z_n) = 0.$$

■

Lemma 8.2.6 *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $B : C \to E^*$ is an operator satisfying $(B1) - (B3)$ and $\{T_n\}_{n=1}^{\infty}$ is an infinite family of relatively-quasi nonexpansive mappings of C into itself such that $F := VI(C, B) \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \cap \Big(\cap_{n=1}^{\infty} F(T_n)\Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by (8.2.1). Then, $\lim_{n\to\infty} ||Jy_n - Jx_n|| = 0$.*

Proof Observe that

$$\begin{aligned}\phi(u_n, x_n) &= ||u_n||^2 - 2\langle u_n, Jx_n\rangle + ||x_n||^2 \\ &= ||u_n||^2 - 2\langle u_n, Jx_n\rangle + ||x_n||^2 + ||x_{n+1}||^2 - ||x_{n+1}||^2 \\ &\quad -2\langle x_{n+1}, Ju_n\rangle + 2\langle x_{n+1}, Ju_n\rangle \\ &= \phi(x_{n+1}, u_n) - 2\langle u_n, Jx_n\rangle + ||x_n||^2 \\ &\quad -||x_{n+1}||^2 + 2\langle x_{n+1}, Ju_n\rangle \\ &= \phi(x_{n+1}, u_n) + (||x_n - x_{n+1}||)(||x_n|| + ||x_{n+1}||) \\ &\quad -2\langle u_n, Jx_n - Ju_n\rangle - 2\langle u_n, Ju_n\rangle + 2\langle x_{n+1}, Ju_n\rangle \\ &= \phi(x_{n+1}, u_n) + (||x_n - x_{n+1}||)(||x_n|| + ||x_{n+1}||) \\ &\quad +2\langle u_n, Ju_n - Jx_n\rangle + 2\langle x_{n+1} - u_n, Ju_n\rangle \\ &\leq \phi(x_{n+1}, u_n) + (||x_n - x_{n+1}||)(||x_n|| + ||x_{n+1}||) \\ &\quad +2||u_n||||Ju_n - Jx_n|| + 2||x_{n+1} - u_n||||Ju_n||.\end{aligned}$$

Since $\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0$, $\lim_{n\to\infty} ||x_n - x_{n+1}|| = 0$, $\lim_{n\to\infty} ||Jx_n - Ju_n|| = 0$ and $\lim_{n\to\infty} ||x_{n+1} - u_n|| = 0$, we obtain that

$$\lim_{n\to\infty} \phi(u_n, x_n) = 0.$$

Similarly, by Lemma 1.5.19 we have

$$\begin{aligned}\phi(u_n, z_n) &= \phi(\theta_n^m z_n, z_n)\\ &\leq \phi(x^*, z_n) - \phi(x^*, \theta_n^m z_n)\\ &\leq \phi(x^*, x_n) - \phi(x^*, u_n). \qquad (8.2.11)\end{aligned}$$

We know that

$$\begin{aligned}\phi(x^*, x_n) - \phi(x^*, u_n) &= ||x_n||^2 - ||u_n||^2 - 2\langle x^*, Jx_n - Ju_n\rangle\\ &\leq \Big|||x_n||^2 - ||u_n||^2\Big| + 2\Big|\langle x^*, Jx_n - Ju_n\rangle\Big|\\ &\leq \Big|||x_n|| - ||u_n||\Big|(||x_n|| + ||u_n||) + 2||x^*||||Jx_n - Ju_n||\\ &\leq ||x_n - u_n||(||x_n|| + ||u_n||) + 2||x^*||||Jx_n - Ju_n||.\end{aligned}$$

Since $\lim_{n\to\infty} ||x_n - u_n|| = 0$ and $\lim_{n\to\infty} ||Jx_n - Ju_n|| = 0$, we obtain

$$\phi(x^*, x_n) - \phi(x^*, u_n) \to 0, \quad n \to \infty.$$

It then follows from (8.2.11) that

$$\phi(u_n, z_n) \to 0, \quad n \to \infty.$$

Since E is uniformly convex and uniformly smooth, we have from Lemma 1.5.20 that

$$\lim_{n\to\infty} ||u_n - z_n|| = 0$$

and since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} ||Ju_n - Jz_n|| = 0.$$

Hence, we have

$$\begin{aligned}\phi(z_n, u_n) &= 2\langle u_n - z_n, Ju_n - Jz_n\rangle - \phi(u_n, z_n)\\ &\leq 2||u_n - z_n||||Ju_n - Jz_n|| - \phi(u_n, z_n). \qquad (8.2.12)\end{aligned}$$

Thus, $\lim_{n\to\infty} \phi(z_n, u_n) = 0$. Now, by $\lim_{n\to\infty} \phi(y_n, z_n) = 0$, $\lim_{n\to\infty} \phi(u_n, x_n) = 0$ and $\lim_{n\to\infty} \phi(z_n, u_n) = 0$, we obtain from (8.2.10) that

$$\lim_{n\to\infty} \phi(y_n, x_n) = 0.$$

Since E is uniformly convex and uniformly smooth, we have from Lemma 1.5.20 that

$$\lim_{n\to\infty} ||y_n - x_n|| = 0$$

and since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} ||Jy_n - Jx_n|| = 0.$$

■

Theorem 8.2.7 *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $B : C \to E^*$ is an operator satisfying $(B1) - (B3)$ and $\{T_n\}_{n=1}^{\infty}$ is an infinite family of relatively-quasi nonexpansive mappings of C into itself such that $F := VI(C, B) \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \cap \Big(\cap_{n=1}^{\infty} F(T_n)\Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by (8.2.1). Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.*

Proof Let $r := \sup_{n\geq 1}\{||\upsilon_n||, ||T_n\upsilon_n||\}$. Since E is uniformly smooth, we know that E^* is uniformly convex. Then from Lemma 1.5.17, we have

$$\begin{aligned}
\phi(x^*, y_n) &= \phi(x^*, J^{-1}(\alpha_n J\upsilon_n + (1-\alpha_n)JT_n\upsilon_n)) \\
&= ||x^*||^2 - 2\alpha_n\langle x^*, J\upsilon_n\rangle - 2(1-\alpha_n)\langle x^*, JT_n\upsilon_n\rangle \\
&\quad +||\alpha_n J\upsilon_n + (1-\alpha_n)JT_n\upsilon_n||^2 \\
&\leq ||x^*||^2 - 2\alpha_n\langle x^*, J\upsilon_n\rangle - 2(1-\alpha_n)\langle x^*, JT_n\upsilon_n\rangle \\
&\quad +\alpha_n||J\upsilon_n||^2 + (1-\alpha_n)||JT_n\upsilon_n||^2 \\
&\quad -\alpha_n(1-\alpha_n)g(||J\upsilon_n - JT_n\upsilon_n||) \\
&= \alpha_n\phi(x^*, \upsilon_n) + (1-\alpha_n)\phi(x^*, T_n\upsilon_n) \\
&\quad -\alpha_n(1-\alpha_n)g(||J\upsilon_n - JT_n\upsilon_n||) \\
&\leq \phi(x^*, x_n) - \alpha_n(1-\alpha_n)g(||J\upsilon_n - JT_n\upsilon_n||).
\end{aligned}$$

It then follows that

$$\alpha_n(1-\alpha_n)g(||J\upsilon_n - JT_n\upsilon_n||) \leq \phi(x^*, x_n) - \phi(x^*, y_n).$$

But

$$\begin{aligned}
&\phi(x^*, x_n) - \phi(x^*, y_n) = ||x_n||^2 - ||y_n||^2 - 2\langle x^*, Jx_n - Jy_n\rangle \\
\leq\ & \Big|||x_n||^2 - ||y_n||^2\Big| + 2\Big|\langle x^*, Jx_n - Jy_n\rangle\Big| \\
\leq\ & \Big|||x_n|| - ||y_n||\Big|(||x_n|| + ||y_n||) + 2||x^*||||Jx_n - Jy_n|| \\
\leq\ & ||x_n - y_n||(||x_n|| + ||y_n||) + 2||x^*||||Jx_n - Jy_n||.
\end{aligned}$$

From $\lim_{n\to\infty}||x_n - y_n|| = 0$ and $\lim_{n\to\infty}||Jx_n - Jy_n|| = 0$, we obtain

$$\phi(x^*, x_n) - \phi(x^*, y_n) \to 0, \quad n \to \infty.$$

Using the condition $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} g(||J\upsilon_n - JT_n\upsilon_n||) = 0.$$

By property of g, we have $\lim_{n\to\infty}||J\upsilon_n - JT_n\upsilon_n|| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}||\upsilon_n - T_n\upsilon_n|| = 0. \tag{8.2.13}$$

We know that

$$\begin{aligned}\phi(x^*, \upsilon_n) &\leq \phi(x^*, x_n) - 2\alpha\lambda_n||Bx_n - Bx^*||^2 \\ &\quad + \frac{4}{c^2}\lambda_n^2||Bx_n - Bx^*||^2 \\ &= \phi(x^*, x_n) + 2\lambda_n\Big(\frac{2}{c^2}\lambda_n - \alpha\Big)||Bx_n - Bx^*||^2. \quad (8.2.14)\end{aligned}$$

By (8.2.2) and (8.2.14), we obtain

$$-2\lambda_n\Big(\frac{2}{c^2}\lambda_n - \alpha\Big)||Bx_n - Bx^*||^2 \leq \phi(x^*, x_n) - \phi(x^*, u_n) \to 0, \quad n \to \infty.$$

Hence,

$$\lim_{n\to\infty}||Bx_n - Bx^*|| = 0.$$

By Lemma 1.5.13, Lemma 1.5.16 and (8.2.5), we have

$$\begin{aligned}\phi(x_n, \upsilon_n) &= \phi(x_n, \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n)) \\ &\leq \phi(x_n, J^{-1}(Jx_n - \lambda_n Bx_n)) \\ &= V(x_n, Jx_n - \lambda_n Bx_n) \\ &\leq V(x_n, (Jx_n - \lambda_n Bx_n) + \lambda_n Bx_n) \\ &\quad -2\Big\langle J^{-1}(Jx_n - \lambda_n Bx_n) - x_n, \lambda_n Bx_n\Big\rangle \\ &= \phi(x_n, x_n) + 2\langle J^{-1}(Jx_n - \lambda_n Bx_n) - x_n, -\lambda_n Bx_n\rangle \\ &= 2\langle J^{-1}(Jx_n - \lambda_n Bx_n) - x_n, -\lambda_n Bx_n\rangle \\ &\leq \frac{4}{c^2}b^2||Bx_n - Bx^*||^2 \to 0, \quad n \to \infty. \quad (8.2.15)\end{aligned}$$

It then follows from Lemma 1.5.20 that $\lim_{n\to\infty}||x_n - \upsilon_n|| = 0$. Since $x_n \to p$, we obtain that $\upsilon_n \to p, \quad n \to \infty$.

<u>Case 1</u>: $(\{T_n\}, \{\upsilon_n\})$ satisfies condition AKTT. We apply Lemma 1.5.28 to get

$$\begin{aligned}||\upsilon_n - T\upsilon_n|| &\leq ||\upsilon_n - T_n\upsilon_n|| + ||T_n\upsilon_n - T\upsilon_n|| \\ &\leq ||\upsilon_n - T_n\upsilon_n|| + \sup\{||T_n x - Tx|| : x \in \{\upsilon_n\}\} \to 0.\end{aligned}$$

<u>Case 2</u>: $(\{T_n\}, \{\upsilon_n\})$ satisfies condition *AKTT. We apply Lemma 1.5.29 to get

$$\begin{aligned}||J\upsilon_n - JT\upsilon_n|| &\leq ||J\upsilon_n - JT_n\upsilon_n|| + ||JT_n\upsilon_n - JT\upsilon_n|| \\ &\leq ||J\upsilon_n - JT_n\upsilon_n|| + \sup\{||JT_n x - JTx|| : x \in \{\upsilon_n\}\} \to 0.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}||\upsilon_n - T\upsilon_n|| = \lim_{n\to\infty}||J^{-1}(J\upsilon_n) - J^{-1}(JT\upsilon_n)|| = 0.$$

From both cases, we obtain

$$\lim_{n\to\infty}||\upsilon_n - T\upsilon_n|| = 0.$$

Since T is closed and $\upsilon_n \to p$, we have $p \in F(T) = \cap_{n=1}^{\infty} F(T_n)$.

By following the same line of arguments as that in the proof of Theorem 3.1 of [84], we have $p \in VI(C,B)$.

Finally, we show that $p \in \cap_{k=1}^{m} GMEP(F_k, \varphi_k)$. Now, by Lemma 1.5.19 we obtain

$$\begin{aligned}
\phi(u_n, y_n) &= \phi(\theta_n^m y_n, y_n) \\
&\leq \phi(x^*, y_n) - \phi(x^*, \theta_n^m y_n) \\
&\leq \phi(x^*, x_n) - \phi(x^*, u_n) \to 0, \quad n \to \infty.
\end{aligned}$$

Using Lemma 1.5.20, we have $\lim_{n\to\infty}||u_n - y_n|| = 0$. Furthermore,

$$||x_n - y_n|| \leq ||x_n - u_n|| + ||u_n - y_n|| \to 0, \quad n \to \infty.$$

Since $x_n \to p, \quad n \to \infty$ and $||x_n - y_n|| \to 0, \quad n \to \infty$, then $y_n \to p, \quad n \to \infty$. By the fact that $\theta_n^k, \quad k = 1, 2, ..., m$ is relatively nonexpansive and using Lemma 1.5.19 again, we have that

$$\begin{aligned}
\phi(\theta_n^k y_n, y_n) &\leq \phi(x^*, y_n) - \phi(x^*, \theta_n^k y_n) \\
&\leq \phi(x^*, x_n) - \phi(x^*, \theta_n^k y_n).
\end{aligned} \tag{8.2.16}$$

Observe that

$$\begin{aligned}
\phi(x^*, u_n) &= \phi(x^*, \theta_n^m y_n) \\
&= \phi(x^*, T_{r_m,n}^{G_m} T_{r_{m-1},n}^{G_{m-1}} ... T_{r_k,n}^{G_k} T_{r_{k-1},n}^{G_{k-1}} ... T_{r_2,n}^{G_2} T_{r_1,n}^{G_1} y_n) \\
&= \phi(x^*, T_{r_m,n}^{G_m} T_{r_{m-1},n}^{G_{m-1}} ... \theta_n^k y_n) \\
&\leq \phi(x^*, \theta_n^k y_n)).
\end{aligned} \tag{8.2.17}$$

Using (8.2.17) in (8.2.16), we obtain

$$\phi(\theta_n^k y_n, y_n) \leq \phi(x^*, x_n) - \phi(x^*, u_n) \to 0, \quad n \to \infty.$$

Then Lemma 1.5.20 implies that $\lim_{n\to\infty}||y_n - \theta_n^k y_n|| = 0, \quad k = 1, 2, ..., m$. Now

$$||p - \theta_n^k y_n|| \leq ||y_n - \theta_n^k y_n|| + ||y_n - p|| \to 0, \quad n \to \infty, \quad k = 1, 2, ..., m.$$

Similarly, $\lim_{n\to\infty}||p - \theta_n^{k-1} y_n|| = 0, \quad k = 1, 2, ..., m$. This further implies that

$$\lim_{n\to\infty}||\theta_n^k y_n - \theta_n^{k-1} y_n|| = 0. \tag{8.2.18}$$

Also, since J is uniformly norm-to-norm continuous on bounded sets and using (8.2.18), we obtain

$$\lim_{n\to\infty} ||J\theta_n^k y_n - J\theta_n^{k-1} y_n|| = 0.$$

Since $\liminf_{n\to\infty} r_{k,n} > 0, \quad (k = 1, 2, ..., m)$, then

$$\lim_{n\to\infty} \frac{||J\theta_n^k y_n - J\theta_n^{k-1} y_n||}{r_{k,n}} = 0. \tag{8.2.19}$$

By Lemma 1.4.2, we have that for each $k = 1, 2, ..., m$

$$G_k(\theta_n^k y_n, y) + \frac{1}{r_{k,n}} \langle y - \theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1} y_n \rangle \geq 0, \quad \forall y \in C.$$

Furthermore, replacing n by n_j in the last inequality and using $(A2)$, we obtain

$$\frac{1}{r_{k,n_j}} \langle y - \theta_{n_j}^k y_{n_j}, J\theta_{n_j}^k y_{n_j} - J\theta_{n_j}^{k-1} y_{n_j} \rangle \geq G_k(y, \theta_{n_j}^k y_{n_j}). \tag{8.2.20}$$

By $(A4)$, (8.2.19) and $\theta_{n_j}^k y_{n_j} \to p$, we have for each $k = 1, 2, ..., m$

$$G_k(y, p) \leq 0, \quad \forall y \in C.$$

Let $z_t := ty + (1-t)p$ for all $t \in (0, 1]$ and $y \in K$. This implies that $z_t \in K$. This yields that $G_k(z_t, p) \leq 0$. It follows from $(A1)$ that

$$\begin{aligned} 0 &= G_k(z_t, z_t) \leq tG_k(z_t, y) + (1-t)G_k(z_t, p) \\ &\leq tG_k(z_t, y) \end{aligned}$$

and hence

$$0 \leq G_k(z_t, y).$$

From condition $(A3)$, we obtain

$$G_k(p, y) \geq 0, \quad \forall y \in C.$$

This implies that $p \in GMEP(F_k, \varphi_k), \quad k = 1, 2, ..., m$. Thus, $p \in \cap_{k=1}^m GMEP(F_k, \varphi_k)$. Hence, we have $p \in F$.

Finally, we show that $p = \Pi_F x_0$. Now by taking the limit in (8.2.8), we have

$$\langle p - z, Jx_0 - Jp \rangle \geq 0, \quad \forall z \in F.$$

By Lemma 1.5.14, we have $p = \Pi_F x_0$. ■

Corollary 8.2.8 *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $\{T_n\}_{n=1}^{\infty}$ is an infinite family of relatively-quasi nonexpansive mappings of C into itself such that $F := \Big(\cap_{k=1}^{m} GMEP(F_k,\varphi_k)\Big) \cap \Big(\cap_{n=1}^{\infty} F(T_n)\Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}x_0$,*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_nx_n) \\ z_n = J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n) \\ u_n = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \ldots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} z_n \\ C_{n+1} = \{w \in C_n : \phi(w,u_n) \leq \phi(w,x_n) + \beta_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad n \geq 1, \end{cases}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$ such that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ and $\lim_{n\to\infty}\beta_n = 0$ and $\{r_{k,n}\}_{n=1}^{\infty} \subset (0,\infty)$, $(k = 1,2,...,m)$ satisfying $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1,2,...,m)$. Suppose that for each bounded subset D of C, the ordered pair $(\{T_n\}, D)$ satisfies either condition AKTT or condition *AKTT. Let T be the mapping from C into E defined by $Tx := \lim_{n\to\infty} T_nx$ for all $x \in C$ and suppose that T is closed and $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.

Corollary 8.2.9 *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $B : C \to E^*$ is an operator satisfying $(B1)-(B3)$ such that $F := VI(C,B) \cap \Big(\cap_{k=1}^{m} GMEP(F_k,\varphi_k)\Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}x_0$,*

$$\begin{cases} \upsilon_n = \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n) \\ y_n = J^{-1}(\alpha_n J\upsilon_n + (1-\alpha_n)J\upsilon_n) \\ z_n = J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n) \\ u_n = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \ldots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} z_n \\ C_{n+1} = \{w \in C_n : \phi(w,u_n) \leq \phi(w,x_n) + \beta_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad n \geq 1, \end{cases}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$ such that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, $\lim_{n\to\infty}\beta_n = 0$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [a,b]$ for some a,b with $0 < a < b < \frac{c^2\alpha}{2}$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_{k,n}\}_{n=1}^{\infty} \subset (0,\infty)$, $(k = 1,2,...,m)$ satisfying $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1,2,...,m)$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.

We now prove the following strong convergence theorem for a weak relatively nonexpansive mapping, system of generalized mixed equilibrium problems and variational inequality problem.

Theorem 8.2.10 *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $B : C \to E^*$ is an operator satisfying $(B1)-(B3)$ and T is a weak relatively nonexpansive mapping of C into itself such that $F := VI(C,B) \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \cap F(T) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1} x_0$,*

$$\begin{cases} \upsilon_n = \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n) \\ y_n = J^{-1}(\alpha_n J\upsilon_n + (1-\alpha_n)JT\upsilon_n) \\ z_n = J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n) \\ u_n = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} \dots T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} z_n \\ C_{n+1} = \{w \in C_n : \phi(w,u_n) \leq \phi(w,x_n) + \beta_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1, \end{cases}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\lim_{n\to\infty} \beta_n = 0$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [a,b]$ for some a,b with $0 < a < b < \frac{c^2\alpha}{2}$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_{k,n}\}_{n=1}^{\infty} \subset (0,\infty)$, $(k = 1,2,...,m)$ satisfying $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1,2,...,m)$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.

Proof Following the line of proof of Theorem 8.2.7, we can show that $\upsilon_n \to p \in C$, $n \to \infty$ and $\lim_{n\to\infty} ||\upsilon_n - T\upsilon_n|| = 0$. From the definition of weak relatively nonexpansive mapping, we have that $p \in \widetilde{F}(T) = F(T)$. The rest of the proof follows from Theorem 8.2.7. ∎

8.3 Applications

Theorem 8.3.1 *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth. Suppose $B : E \to E^*$ is an operator satisfying $(B1)-(B3)$ and let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of closed relatively-quasi nonexpansive mappings of E into itself such that $F := \cap_{n=1}^{\infty} F(T_n) \cap B^{-1}(0) \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in$*

$E, \quad C_1 = E, \quad x_1 = \Pi_{C_1} x_0,$

$$\begin{cases} \upsilon_n = J^{-1}(Jx_n - \lambda_n Bx_n) \\ y_n = J^{-1}(\alpha_n J\upsilon_n + (1-\alpha_n)JT_n\upsilon_n) \\ z_n = J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n) \\ u_n = T_{r_{m,n}}^{G_m} T_{r_{m-1,n}}^{G_{m-1}} ... T_{r_{2,n}}^{G_2} T_{r_{1,n}}^{G_1} z_n \\ C_{n+1} = \{w \in C_n : \phi(w, u_n) \leq \phi(w, x_n) + \beta_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1, \end{cases}$$

*where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\lim_{n\to\infty} \beta_n = 0$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $(k = 1, 2, ..., m)$ satisfying $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1, 2, ..., m)$. Suppose that for each bounded subset D of E, the ordered pair $(\{T_n\}, D)$ satisfies either condition AKTT or condition *AKTT. Let T be the mapping from E into itself defined by $Tx := \lim_{n\to\infty} T_n x$ for all $x \in E$ and suppose that T is closed and $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.*

Proof Putting $C = E$ in Theorem 8.2.7, we obtain that $\Pi_E = I$, $VI(E, B) = B^{-1}(0)$ and the condition $(B2)$ of Theorem 8.2.7 holds for all $y \in E$ and $u \in B^{-1}(0)$. Hence, we obtain the desired result. ∎

Let K be a nonempty, closed and convex cone in E and B an operator of K into E^*. We define the *polar* of K in E^* to be the set

$$K^* := \{y^* \in E^* : \langle x, y^*\rangle \geq 0, \quad \forall x \in K\}.$$

Then the element $u \in K$ is called a solution of the *complementarity problem* if

$$Bu \in K^*, \quad \langle u, Bu\rangle = 0.$$

The set of solutions of the complementarity problem is denoted by $C(K, B)$. We shall assume that B satisfies the following conditions:
$(E1)$ B is α-inverse-strongly monotone;
$(E2)$ $||By|| \leq ||By - Bu||$ for all $y \in K$ and $u \in C(K, B)$;
$(E3)$ $C(K, B) \neq \emptyset$.

Theorem 8.3.2 *Let E be a 2-uniformly convex real Banach space which is also uniformly smooth and let K be a nonempty, closed and convex cone in E. Suppose $B : K \to E^*$ is an operator satisfying $(E1) - (E3)$ and let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of relatively-quasi nonexpansive mappings of K into itself such that $F := \cap_{n=1}^{\infty} F(T_n) \cap C(K, B) \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \neq$*

$\emptyset$. *Let* $\{x_n\}_{n=0}^{\infty}$ *be iteratively generated by* $x_0 \in C, \quad C_1 = C, \quad x_1 = \Pi_{C_1}x_0,$

$$\begin{cases} \upsilon_n = \Pi_K J^{-1}(Jx_n - \lambda_n Bx_n) \\ y_n = J^{-1}(\alpha_n J\upsilon_n + (1-\alpha_n)JT_n\upsilon_n) \\ z_n = J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n) \\ u_n = T_{r_{m,n}}^{G_m} T_{r_{m-1},n}^{G_{m-1}} ... T_{r_{2},n}^{G_2} T_{r_{1},n}^{G_1} z_n \\ C_{n+1} = \{w \in C_n : \phi(w,u_n) \leq \phi(w,x_n) + \beta_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad n \geq 1, \end{cases}$$

where J *is the duality mapping on* E. *Suppose* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *and* $\{\gamma_n\}_{n=1}^{\infty}$ *are sequences in* $(0,1)$ *such that* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, $\lim_{n\to\infty}\beta_n = 0$ *and* $\{\lambda_n\}_{n=1}^{\infty} \subset [a,b]$ *for some* a, b *with* $0 < a < b < \frac{c^2\alpha}{2}$, *where* $\frac{1}{c}$ *is the 2-uniformly convexity constant of* E *and* $\{r_{k,n}\}_{n=1}^{\infty} \subset (0,\infty)$, $(k = 1,2,...,m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1,2,...,m)$. *Suppose that for each bounded subset* D *of* K, *the ordered pair* $(\{T_n\}, D)$ *satisfies either condition AKTT or condition* **AKTT. Let* T *be the mapping from* C *into* E *defined by* $Tx := \lim_{n\to\infty} T_n x$ *for all* $x \in K$ *and suppose that* T *is closed and* $F(T) = \cap_{n=1}^{\infty} F(T_n)$. *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_F x_0$.

Proof Using Lemma 7.1.1 of [191], we have that $VI(K,B) = C(K,B)$. Hence, by Theorem 8.2.7 we obtain the desired conclusion. ■

Next we study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional and a common solution to a system of generalized mixed equilibrium problems in a Banach space.

Theorem 8.3.3 *Let* E *be a 2-uniformly convex real Banach space which is also uniformly smooth. Suppose* f *is a functional on* E *which satisfies the following conditions:*
(1) f *is a continuously Fréchet differentiable convex functional on* E *and* ∇f *is* $\frac{1}{\alpha}$*-Lipschitz continuous,*
(2) $(\nabla f)^{-1}0 = \{z \in E : f(z) = \min_{y\in E} f(y)\} \neq \emptyset$
and let $F := (\nabla f)^{-1}0 \cap \Big(\cap_{k=1}^{m} GMEP(F_k, \varphi_k)\Big) \neq \emptyset$. *Let* $\{x_n\}_{n=0}^{\infty}$ *be iteratively generated by* $x_0 \in E, \quad C_1 = E, \quad x_1 = x_0,$

$$\begin{cases} \upsilon_n = J^{-1}(Jx_n - \lambda_n \nabla f(x_n)) \\ y_n = J^{-1}(\alpha_n J\upsilon_n + (1-\alpha_n)J\upsilon_n) \\ z_n = J^{-1}(\beta_n Jx_0 + (1-\beta_n)Jy_n) \\ u_n = T_{r_{m,n}}^{G_m} T_{r_{m-1},n}^{G_{m-1}} ... T_{r_{2},n}^{G_2} T_{r_{1},n}^{G_1} z_n \\ C_{n+1} = \{w \in C_n : \phi(w,u_n) \leq \phi(w,x_n) + \beta_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad n \geq 1, \end{cases}$$

where J *is the duality mapping on* E. *Suppose* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *and* $\{\gamma_n\}_{n=1}^{\infty}$ *are sequences in* $(0,1)$ *such that* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, $\lim_{n\to\infty}\beta_n = 0$

and $\{\lambda_n\}_{n=1}^{\infty} \subset [a,b]$ *for some* a,b *with* $0 < a < b < \frac{c^2\alpha}{2}$, *where* $\frac{1}{c}$ *is the 2-uniformly convexity constant of* E *and* $\{r_{k,n}\}_{n=1}^{\infty} \subset (0,\infty)$, $(k = 1,2,...,m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1,2,...,m)$. *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_F x_0$.

Proof We know from condition (1) and Lemma 1.5.27 that ∇f is an α-inverse-strongly monotone operator from E into E^*. Using Theorem 8.2.7 we have the desired conclusion. ■

Finally, we apply our results to the problem of finding a zero of a maximal monotone operator which is also a common solution to the system of equilibrium problem and a solution to the variational inequality problems in a Banach space.

Theorem 8.3.4 *Let* E *be a 2-uniformly convex real Banach space which is also uniformly smooth. Let* $T \subset E \times E^*$ *be a maximal monotone operator. Let* $J_s = (J + sT)^{-1}J$ *for all* $s > 0$ *and let* C *be a nonempty, closed and convex subset of* E *such that* $D(T) \subset C \subset J^{-1}(\cap_{s>0} R(J + sT))$. *For each* $k = 1,2,...,m$, *let* F_k *be a bifunction from* $C \times C$ *satisfying* (A1)−(A4), $\varphi_k : C \to \mathbb{R}$ *be a lower semicontinuous and convex functional and* $A_k : C \to E^*$ *be a continuous and monotone mapping. Suppose that* $B : C \to E^*$ *an operator satisfying* $(B1)-(B3)$ *such that* $F := T^{-1}0 \cap VI(C,B) \cap \left(\cap_{k=1}^m EP(F_k)\right) \neq \emptyset$. *Let* $\{x_n\}_{n=0}^{\infty}$ *be iteratively generated by* $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1} x_0$,

$$\begin{cases} \upsilon_n = \Pi_C J^{-1}(Jx_n - \lambda_n B x_n) \\ y_n = J^{-1}(\alpha_n J\upsilon_n + (1-\alpha_n) J J_{s_n} \upsilon_n) \\ z_n = J^{-1}(\beta_n J x_0 + (1-\beta_n) J y_n) \\ u_n = T_{r_{m,n}}^{G_m} T_{r_{m-1},n}^{G_{m-1}} \dots T_{r_2,n}^{G_2} T_{r_1,n}^{G_1} z_n \\ C_{n+1} = \{w \in C_n : \phi(w,u_n) \leq \phi(w,x_n) + \beta_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0\rangle)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1, \end{cases}$$

where J *is the duality mapping on* E. *Suppose* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *and* $\{\gamma_n\}_{n=1}^{\infty}$ *are sequences in* $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\{\lambda_n\}_{n=1}^{\infty} \subset [a,b]$ *for some* a,b *with* $0 < a < b < \frac{c^2\alpha}{2}$, *where* $\frac{1}{c}$ *is the 2-uniformly convexity constant of* E *and* $\{r_{k,n}\}_{n=1}^{\infty}$ *and* $\{s_n\}_{n=1}^{\infty}$ *are sequences in* $(0,\infty)$, $(k = 1,2,...,m)$ *such that* $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1,2,...,m)$ *and* $\liminf_{n\to\infty} s_n > 0$. *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_F x_0$.

Proof Replacing T_n in Theorem 8.2.7 with J_{s_n}, we obtain

$$\lim_{n\to\infty} ||J\upsilon_n - JJ_{s_n}\upsilon_n|| = 0.$$

Since the $\liminf_{n\to\infty} s_n > 0$, we obtain

$$\lim_{n\to\infty} ||A_{s_n}\upsilon_n|| = \lim_{n\to\infty} \frac{1}{s_n} ||J\upsilon_n - JJ_{s_n}\upsilon_n|| = 0.$$

Similarly, following the first few lines of proof of Theorem 8.2.7, we can obtain

$$\begin{aligned}\phi(x^*, u_n) &\leq \alpha_n \phi(x^*, \upsilon_n) + (1-\alpha_n)\phi(x^*, J_{s_n}\upsilon_n) \\ &\leq \alpha_n \phi(x^*, x_n) + (1-\alpha_n)\phi(x^*, J_{s_n}\upsilon_n). \qquad (8.3.1)\end{aligned}$$

Furthermore, using Lemma 1.5.19, we obtain

$$\begin{aligned}\phi(J_{s_n}\upsilon_n, \upsilon_n) &\leq \phi(x^*, \upsilon_n) - \phi(x^*, J_{s_n}\upsilon_n) \\ &\leq \phi(x^*, x_n) - \phi(x^*, J_{s_n}\upsilon_n). \qquad (8.3.2)\end{aligned}$$

Using (8.3.1) in (8.3.2), we have

$$\phi(J_{s_n}\upsilon_n, \upsilon_n) \leq \frac{1}{1-\alpha_n}\Big[\phi(x^*, x_n) - \phi(x^*, u_n)\Big]. \qquad (8.3.3)$$

Hence $\lim_{n\to\infty} ||J_{s_n}\upsilon_n - \upsilon_n|| = 0$. From Theorem 8.2.7, we have $\lim_{n\to\infty} ||\upsilon_n - p|| = 0$. Therefore, $\lim_{n\to\infty} ||J_{s_n}\upsilon_n - p|| = 0$. Let $(u, u^*) \in T$. It then follows from the monotonicity of T that

$$\langle u - J_{s_n}\upsilon_n, u^* - A_{s_n}\upsilon_n\rangle \geq 0.$$

Letting $n \to \infty$, we obtain $\langle u - p, u^*\rangle \geq 0$. Then the maximality of T implies that $p \in T^{-1}0 := \cap_{n=1}^{\infty} F(J_{s_n})$. Hence, by Theorem 8.2.7, we have the desired result. ■

Remark 8.3.5 *Our Theorem 8.2.7 and Theorem 8.2.10 extend the result of Martinez-Yanes and Xu [131] from a nonexpansive mapping in a Hilbert space to infinite family of relatively quasi-nonexpansive mappings and weak relatively nonexpansive mappings in a Banach space respectively. Also, Our Theorem 8.2.7 and Theorem 8.2.10 extend the result of Qin and Su [162] and the result of Matsushita and Takahashi [134] from relatively nonexpansive mapping to infinite family of relatively quasi-nonexpansive mappings and weak relatively nonexpansive mappings.*

Remark 8.3.6 *All the results in this chapter are due to Shehu [177, 178, 179].*

CHAPTER 9

Conclusions and Future Work

9.1 Conclusions

In Part I of this research work, we develop a simpler explicit coupled iterative scheme for solving nonlinear integral equation of Hammerstein type $u + KFu = 0$ and explicit coupled iterative scheme for solving generalized equations of Hammerstein type, i.e., equations of the type

$$u + \sum_{i=1}^{m} K_i F_i u = 0 \tag{9.1.1}$$

in *Banach spaces*. Our results extend and improve several existing results in the literature on approximation of solutions of Hammerstein integral equations.

Furthermore, in Part II of the thesis, we prove strong convergence theorem for common fixed points of L_n-Lipschitzian mappings in *real Banach spaces much more general than uniformly convex real Banach spaces* considered in Aoyama *et al.* (Theorem 1.3.32 above). A corollary of our theorem extends Theorem 1.3.32 from uniformly convex real Banach spaces with uniformly Gâteaux differentiable norm to real Banach spaces with uniformly Gâteaux differentiable norm and possessing uniform normal structure. Furthermore, in this our more general setting, condition

$$\sum_{n=1}^{\infty} \sup\{||T_{n+1}x - T_n x|| : x \in B\} < \infty$$

for any bounded subset B of K, which is not easily satisfied in several situations and which is assumed in Nilsrakoo and Saejung [145] and in Aoyama *et al.*[5] is replaced with a simpler condition which *is satisfied for example, for a sequence of nonexpansive mappings*. Our results improve and extend the results of Nilsrakoo and Saejung [145], Aoyama *et al.* [5] and many other important recent results.

Finally, in Part III of the thesis, we introduce new iterative schemes and prove strong convergence results using the iterative schemes for finding common element of the set of fixed points of nonlinear mapping, set of solutions to equilibrium problems and set of solutions to variational inequality problems in Banach spaces. Our results extend and improve several results in this direction.

9.2 Suggestions For Future Work

Our future work will be concerned on developing coupled iterative schemes for solving nonlinear integral equation of Hammerstein type $u + KFu = 0$, when K and F are monotone operators in Banach spaces other than Hilbert spaces as considered in this research work.

Furthermore, the method of solution to equilibrium problem considered in this Ph.D research work, has a certain drawback, which should be at least be commented upon: the first step in the iteration procedure is highly implicit. This step consists of solving an equilibrium problem with a perturbed bifunction $\tilde{F}$, namely $\tilde{F}(z;y) = F(z;y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle$. We reckon that $\tilde{F}$ is better conditioned than F, but in principle the solution of the equilibrium problem for $\tilde{F}$ and C is a quite hard problem in itself. Thus, in fact all these methods do not solve the problem of finding a point in the intersection of the solution sets; it would be fairer to say that if one has an algorithm for solving the equilibrium problem, then all these methods do find a point in the intersection set, using such algorithm as an inner subroutine. Additionally, the subproblem the equilibrium problem for $\tilde{F}$ and C has as feasible set the original set C, which could be rather nasty. Since these problems must be solved at each iteration, it would seem better to use some approximation C_n of C at iteration n, e.g. a polyhedral outer approximation, like e.g. in [10, 40].
Our future work will also be concerned on the use of linear-search to solve fixed point problem, equilibrium and variational inequality problems in reflexive Banach space (see, [11, 107, 137]).

Bibliography

[1] Y. I. Alber; Metric and generalized projection operator in Banach spaces: properties and applications, *in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type* vol 178 of Lecture Notes in Pure and Applied Mathematics, pp 15-50, Dekker,New York, NY, USA, 1996.

[2] Y. I. Alber, C. E. Chidume, and H. Zegeye; Approximating fixed points of total asymptotically nonexpansive mappings, *Fixed Point Theory Appl.* 2006, article ID 10673.

[3] Y. I. Alber, S. Reich; An iterative method for solving a class of nonlinear operator equations in Banach spaces, *PanAmer. Math. J* 4 (1994), 39-54.

[4] A. Aleyner, Y. Censor; Best approximation to common fixed points of a semigroup of nonexpansive operators, *J. Nonlinear Convex Anal.*, **6**, no. 1, 2005, 137151.

[5] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda; Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, *Nonlinear Anal.* 67 (2006), 2350-2360.

[6] J. B. Baillon; On the'ore'me de type ergodique pour les contractions online'aires dans un espace de Hilbert, *C. R. Acad. Sci. Paris* **280** (1975), 1511-1514.

[7] J. B. Baillon, G. Haddad; Quelques proprietes des operateurs angle-bornes et n-cycliquement monotones, *Isreal J. Math.* **26** (1977), 137-150.

[8] S. Banach; Theorie des operations lineaires, Monografie Mat., PWN,Warszawa, 1932.

[9] H. H. Bauschke; The approximation of fixed points of compositions of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* **202** (1996), no. 1, 150-159.

[10] J. Y. Bello Cruz, A. N. Iusem; An explicit algorithm for monotone variational inequalities, *Optimization* (2011), doi: 10.1080/02331934.2010.536232.

[11] J. Y. Bello Cruz, A. N. Iusem; A strongly convergent direct method for monotone variational inequalities in Hilbert spaces, *Numer. Funct. Anal. Optim.* **30** (2009), 23-36.

[12] L. P. Belluce, W. A. Kirk; Fixed-point theorem for families of contraction mappings, *Pacific J. Math.*, **18** (1966), 213-217.

[13] L. P. Belluce, W. A. Kirk; Nonexpansive mappings and fixed-points in Banach spaces, *Illinois J. Math.*, **11**, 1967, 474479.

[14] T. D. Benavides, G. L. Acedo, and H.-K. Xu; Construction of sunny nonexpansive retractions in Banach spaces, *Bull. Aust. Math. Soc.*, **66**, no. 1, 2002, 916.

[15] V. Berinde; Iterative approximation of fixed points, *Springer Verlag Series:* Lecture Notes in Mathematics, Vol. 1912 (2007), ISBN 978-3-540-72233-5.

[16] V. Berinde; Iterative approximation of fixed points, *Editura Efemeride* Baia Mare, 2002.

[17] B. Beauzamy; Introduction to Banach spaces and their geometry, *Second Ed. North-Holland* 1985.

[18] E. Blum, W. Oettli; From optimization and variational inequalities to equilibrium problems, *Math. Student* **63** (1994), 123-145.

[19] H. Bréziz; Operateur maximaux monotones, in *mathematics studies*, vol. 5, North-Holland, Amsterdam, The Netherlands, (1973).

[20] H. Brežis, F. E. Browder; Some new results about Hammerstein equations, *Bull. Amer. Math. Soc.* **80** (1974), 567-572.

[21] H. Brežis, F. E. Browder; Existence theorems for nonlinear integral equations of Hammerstein type, *Bull. Amer. Math. Soc.* **81** (1975), 73-78.

[22] H. Brežis, F. E. Browder; Nonlinear integral equations and system of Hammerstein type, *Advances in Math.* **18** (1975), 115-147.

[23] F. E. Browder; Nonexpansive nonlinear operators in Banach spaces, *Proc. Nat. Acad. Sci., U.S.A.* **54** (1966), 1041-1044.

[24] F.E. Browder; Nonlinear operators and nonlinear equations of evolution in Banach spaces, in:*Proc. of Symposia in Pure Math., vol. XVIII, No. Part 2*,1976.

[25] F. E. Browder; The solvability of nonlinear functional equations, *Duke Math. J.* **30** (1963) 557-566.

[26] F. E. Browder; Nonlinear elliptic boundary value problems, *Bull. Amer. Math. Soc.* **69** (1963) 862-874.

[27] F. E. Browder; Nonlinear equations of evolution and nonlinear accretive operators in Banach spaces, *Bull. Amer. Math. Soc.* **73** (1967) 470-475.

[28] F. E. Browder; Nonlinear monotone and accretive operators in Banach space, *Proc. Natl. Acad. Sci. USA* **61** (1968) 388-393.

[29] F. E. Browder; Nonlinear mappings of nonexpansive and accretive type in Banach spaces, *Bull. Amer. Math. Soc.* **73** (1967), 875-882.

[30] F. E. Browder; Nonlinear functional analysis and nonlinear integral equations of Hammerstein and Urysohn type, *Contributions to Nonlinear Functional Analysis*, Academic Press, (1971), 425-500.

[31] F. E. Browder, D. G. Figueiredo, P. Gupta; Maximal monotone operators and a nonlinear integral equations of Hammerstein type, *Bull. Amer. Math. Soc.* **76** (1970), 700-705.

[32] F. E. Browder, P. Gupta; Monotone operators and nonlinear integral equations of Hammerstein type, *Bull. Amer. Math. Soc.* **75** (1969), 1347-1353.

[33] F. E. Browder, W. V. Petryshyn; Construction of fixed points of nonlinear mappings in Hilbert spaces, *J. Math. Anal. Appl.* **20** (1967), 197-228.

[34] R. E. Bruck; A strong convergent iterative method for the solution of $0 \in Ax$ for a maximal monotone operator A in a Hilbert space, *J. Math. Anal. Appl.* **48** (1974) 114-126.

[35] R. E. Bruck; A common fixed point theorem for a commuting family of nonexpansive mappings, *Pacific J. Math.*, **53** (1974), 59-71.

[36] R. E. Bruck;, On weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space, *J. Math. Anal. Appl.* **61** (1977), 159-164.

[37] R. E. Bruck; The iterative solution of the equation $y \in x + Tx$ for a monotone operator T in Hilbert space, *Bull. Amer. Math. Soc.* **79** (1973), 1258-1261.

[38] D. Butnariu, S. Reich, A. J. Zaslavski; Asymptotic behaviour of relatively nonexpansive operators in Banach spaces, *J. Appl. Anal.* 7 (2001), 151-174.

[39] D. Butnariu, S. Reich, A. J. Zaslavski; Weak convergence of orbits of nonlinear operator in reflexive Banach spaces, *Numer. Funct. Anal. Optim.* 24 (2003), 489-508.

[40] R. S. Burachik, J. O. Lopes, B. F. Svaiter; An outer approximation method for the variational inequality problem, *SIAM Journal Control on Optimization* **43** (2005), 2071-2088

[41] W. L. Bynum; Normal structure coefficients for Banach spaces, *Pacific J. Math.* **86** (1980) 427-436.

[42] J. Caristi; The fixed point theory for mappings satisfying inwardness conditions, Ph.D. Thesis, The University of Iowa, Iowa City, 1975.

[43] S. S. Chang, Y. J. Cho, H. Zhou; Iterative methods for nonlinear operator equations in Banach spaces, *Nova Science Publishers, Inc,.* Huntington, NY, 2002.

[44] S. S. Chang, H. W. J. Lee, J. K. Kim; A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization, *Nonlinear Anal.* **70** (2009), 3307-3319.

[45] S. Chang, J. K. Kim, X. R. Wang; Modified Block iterative algorithm for solving convex feasibility problems in Banach spaces, *J. Ineq. Appl.* (2010), Article ID 869684,14 pages, 2010.

[46] Y. Censor, S. Reich; Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, *Optimization* 37 (1996), 323-339.

[47] R. Sh. Chepanovich; Nonlinear Hammerstein equations and fixed points, *Publ. Inst. Math. (Beograd) N. S.* **35** (1984), 119-123.

[48] C. E. Chidume; Iterative approximation of fixed points of Lipschitz pseudo-contractive maps, *Proc. Amer. Math. Soc.* **129** (2001), 2245-2251. MR 2002e:47071

[49] C. E. Chidume; Geometric properties of Banach spaces and nonlinear iterations, *Springer Verlag Series: Lecture Notes in Mathematics* Vol. 1965 (2009), XVII, 326p, ISBN 978-1-84882-189-7.

[50] C. E. Chidume; Applicable Functional Analysis, *Abdus Salam International Centre for Theoretical Physics Publication* July (2009).

[51] C. E. Chidume, M. Abbas, B. Ali; Convergence of the Mann iteration algorithm for a class of pseudo-contractive mappings,*Appl. Math. Comput.***194** (2007) 1-6.

[52] C. E. Chidume, Bashir Ali; Convergence theorems for common fixed points for finite families of nonexpansive mappings in reflexive Banach spaces, *Nonlinear Anal., TMA* **68** (2008) 3410-3418.

[53] C. E. Chidume, Bashir Ali; Weak and strong convergence theorems for finite families of asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* **330** (2007), 377-387.

[54] C. E. Chidume, Bashir Ali; Approximation of common fixed points for finite families of non-self asymptotically nonexpansve mappings in Banach spaces, *J. Math. Anal. Appl.* **326** (2007), no. 2, 960-973.

[55] C. E. Chidume, C. O. Chidume; A convergence theorem for zeros of generalized Phi-quasi-accretive mappings, *Proc. Amer. Math. Soc.* **134** (2006), 243-251.

[56] C. E. Chidume, C.O. Chidume; Iterative approximation of fixed points of nonexpansive mappings, *J. Math. Anal. Appl.* **318** (2006) , no.1, 288-295.

[57] C. E. Chidume, C.O. Chidume; Iterative methods for common fixed points for a countable family of nonexpansive mappings in uniformly convex spaces, *Nonlinear Anal.* **71** (2009), 4346-4356.

[58] C. E. Chidume, C.O. Chidume and A. P. Nwogbaga; Approximation methods for common fixed points of a countable family of nonexpansive mappings, *Nonlinear Anal.* **71** (2009), 164-175.

[59] C. E. Chidume, C. O. Chidume, Y. Shehu, Strong convergence theorems for a Mann type iterative scheme for a countable family of Lipschitzian mappings, *J. Appl. Math. Comput.* **35** (2011), 251-261.

[60] C. E. Chidume, N. Djitte; Approximation of solutions of Hammerstein equations with bounded strongly accretive nonlinear operators, *Nonlinear Anal.* **70** (2009), 4071-4078.

[61] C. E. Chidume, N. Djitte; Iterative approximation of solutions of nonlinear equations of Hammerstein type, *Nonlinear Anal.* **70** (2009), 4086-4092.

[62] C. E. Chidume, N. Djitte; Strong convergence theorems for zeros of bounded maximal monotone nonlinear operators, *Abstract Appl. Anal.* volume 2012 (2012), Article ID 681348, 19 pages.

[63] C. E. Chidume, N. Djitte; Convergence Theorems for Solutions of Hammerstein Equations with Accretive-Type Nonlinear Operators, *PanAmer Math. J.*, **22** (2012), no.2, 19-29.

[64] C. E. Chidume, N. Djitte; Approximation of solutions of nonlinear integral equations of Hammerstein type, *ISRN Mathematical Anal.* volume 2012 (2012), Article ID 169751, 12 pages.

[65] C. E. Chidume, J. Li and A. Udomene; Convergence of paths and approximation of fixed points of asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* **133** (2005), no. 2, 473 - 480.

[66] C. E. Chidume, C. Moore; Fixed point iteration for pseudo-contractive maps, *Proc. Amer. Math. Soc.* **127** (4) (1999), 1163-1170. MR 99f:47068

[67] C. E. Chidume, S. Mutangadura; An example on the Mann iteration method for Lipschitz pseudo-contractions, *Proc. Amer. Math. Soc.* **129** (2001), (8), 2359-2363; MR 20022f: 47104.

[68] C. E. Chidume, E. U. Ofoedu; Solution of nonlinear integral equations of Hammerstein type, *Nonlinear Anal.* **74** (2011), 4293-4299.

[69] C. E. Chidume, E. U. Ofoedu; Approximation of common fixed points for finite families of total asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* **333** (2007) 128-141.

[70] C. E. Chidume, M.O. Osilike; Iterative solution for nonlinear integral equations of Hammerstein type, *J. Nig. Math. Soc.* **11** (1) (1993), 918.

[71] C. E. Chidume, E. U. Ofoedu and H. Zegeye; Strong and weak convergence theorems for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* **280** (2003), 364 -374.

[72] C. E. Chidume, Y. Shehu, Iterative approximation of solutions of equations of Hammerstein type in certain Banach spaces, *Appl. Math. Comput.* **219** (2013), 5657-5667.

[73] C. E. Chidume, Y. Shehu, Strong convergence theorem for approximation of solutions of equations of Hammerstein type, *Nonlinear Anal.* **75** (2012), 5664-5671.

[74] C. E. Chidume, Y. Shehu, Approximation of solutions of generalized equations of Harmmerstein type, *Comp. Math. Appl.* **63** (2012), 966-974.

[75] C. E. Chidume, H. Zegeye; Iterative approximation of solutions of nonlinear equations of Hammerstein type, *Abstract and Applied Analysis*, **6** (2003), 353- 367.

[76] C. E. Chidume, H. Zegeye; Approximate fixed point sequences and convergence theorems for Lipschitz pseudo-contractive maps, *Proc. Amer. Math. Soc.* **132** (2004) 831-840.

[77] C. E. Chidume, H. Zegeye; Approximation of solutions nonlinear equations of Hammerstein type in Hilbert space, *Proc. Amer. Math. Soc.* **133** (2005), 851-858.

[78] C. E. Chidume, H. Zegeye; Iterative approximation of solutions of nonlinear equations of Hammerstein type, *Abstr. Appl. Anal.* **6** (2003), 353-367.

[79] C. E. Chidume, H. Zegeye and N. Shahzad; Convergence theorems for a common fixed point of finite family of nonself nonexpansive mappings, *Fixed Point Theory and Appl.* (2005), no. 2, 233 - 241.

[80] A. Chinchuluun, P. Pardalos, A. Migdalas, L. Pitsoulis; (Eds), Pareto Optimality, Game Theory and Equilibria, *Springer* (2008), ISBN 978-0-387-77246-2.

[81] Y. J. Cho, X. Qin, J. I. Kang; Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems, *Nonlinear Anal.* **71** (2009), 4203-4214.

[82] P. Cholamjiak; A hybrid iterative scheme for equilibrium problems, variatonal inequality problems and fixed points problems in Banach spaces, *Fixed Point Theory Appl.* (2009), Article ID 719360, 18 pages.

[83] P. Cholamjiak, S. Suantai; Convergence analysis for a system of equilibrium problems and a countable family of relatively quasi-nonexpansive mappings in Banach spaces, *Abstract Appl. Anal.* vol 2010, Article ID 141376, 17 pages, 2010.

[84] W. Chuayjan, S. Thianwan; Strong convergence of a hybrid projection algorithm for equilibrium problems, variational inequality problems and fixed point problems in a Banach space *Fixed Points Theory Appl.* (2009), vol 2009, Article ID 613524,22 pages.

[85] F. Cianciaruso, G. Marino, L. Muglia; Iterative methods for equilibrium and fixed point problems for nonexpansive semigroups in Hilbert spaces, *J. Optim. Theory Appl.* **146** (2010), 491-509.

[86] I. Cioranescu; Geometry of Banach spaces, duality mappings and nonlinear problems, *Kluwer Academic* **Dordrecht,** 1990.

[87] P. L. Combettes, S. A. Hirstoaga;, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.* **6** (2005), 117-136.

[88] R. De Marr; Common fixed points for commuting contraction mappings, *Pacific J. Math.* **13** (1963), 1139-1141.

[89] K. Deimling; Nonlinear Functional Analysis, *Springer-Verlag* Berlin, Heidelberg, New York, Tokyo, 1985.

[90] V. Dolezale; Monotone operators and its applications in automation and network theory, *Studies in Automation and control (Elesevier Science Publ.* New York, 1979.

[91] D. G. De Figueiredo, C. P. Gupta; On the variational methods for the existence of solutions to nonlinear equations of Hammerstein type, *Bull. Amer. Math. Soc.* **40** (1973), 470-476.

[92] J. H. Fan, X. Liu, J. L. Li; Iterative schemes for approximating solutions of generalized variational inequalities in Banach spaces, *Nonlinear Anal.* 70 (2009), 3997-4007.

[93] S. D. Flam, A. S. Antipin;, Equilibrium programming using proximal-like algorithms *Math. Program* **78** (1997), 29-41.

[94] C. A. Floudas, A. Christodoulos, P. M. Pardalos; (Eds.), Encyclopedia of Optimization - 2ND EDITION, *Springer*, (2009), XXXIV, 4626 p., 613 illus., ISBN 978-0-387-74760-6.

[95] A. Genel, J. Lindenstrauss; An example concerning fixed points, *Isreal J. Math.* **22** (1975), 81-86.

[96] F. Giannessi, A. Maugeri, P. M. Pardalos; (Eds), Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models, *Springer* Vol. 58 (2002), ISBN 978-1-4020-0161-1.

[97] K. Goebel, W. A. Kirk; A fixed point theorem for asymptotically nonexpansive mappings, *Bull. Amer. Math. Soc.* **35** (1972), 171-174.

[98] K. Goebel, S. Reich; Uniform convexity, hyperbolic geometry and nonexpansive mappings, Dekker, New York and Basel, 1984.

[99] C. P. Gupta; Nonlinear equations of Urysohn's type in a Banach space, *Comm. Math. Univ. Carolinae* **16** (1975), 377-386.

[100] C. P. Gupta; On a class of nonlinear integral equations of Urysohn's type, *J. Math. Anal. Appl.* **58** (1977), 344-360.

[101] B. Halpern, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc. **3** (1967), 957-961.

[102] A. Hammerstein; Nichtlineare integralgleichungen nebst anwendungen, *ActaMath.* **54** (1930) 117-176.

[103] H. Iiduka, W. Takahashi; Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, *Nonlinear Anal.* **61** (2005), 341-350.

[104] G. Infante, P. Pietramala; Existence and multiplicity of non-negative solutions for systems of pertubed Hammerstein integral equations, *Nonlinear Anal.* **71** (2009), 1301-1310.

[105] G. Inoue, W. Takahashi, K. Zembayashi; Strong convergence theorems by hybrid methods for maximal monotone operators and relatively nonexpansive mappings in Banach spaces,*J. Convex Anal.* **16** (2009) 791-806.

[106] S. Ishikawa; Fixed points by a new iteration method, *Proc. Amer. Math. Soc.* **44** (1974), no.1; 147-150.

[107] A. N. Iusem, M. Nasri; Korpelevich's method for variational inequality problems in Banach spaces, *J. Global Optim.* (2011), doi: 10.1007/s10898-010-9613-x.

[108] M. Joshi; Existence theorem for a generalized Hammerstein type equation, *Comm. Math. Univ. Carolinae* **15** (1974) 283-291.

[109] J. S. Jung; Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* **302** (2005), no. 2, 509- 520.

[110] J. S. Jung, Y. J. Cho and R. P. Agarwal; Iterative shemes with some control conditions for family of finite nonexpansive mappings in Banach spaces, *Fixed Point Theory and Appl.* **2** (2005), 125-135.

[111] S. Kakutani; Two fixed point theorems concerning bicompact convex sets, *Proc. Imp. Acad. Tokyo* **14** (1938), 242-245.

[112] S. Kamimura, W. Takahashi; Strong convergence of a proximal-type algorithm in a Banach space, *SIAM J. Optim* **13** (2002) 938-945.

[113] J. Kang, Y. Su, X. Zhang; Hybrid algorithm for fixed points of weak relatively nonexpansive mappings and applications, *Nonlinear Analysis: Hybrid Systems* 4 (2010), 755-765.

[114] T. Kato; Nonlinear semigroups and evolution equations, *J. Math. Soc. Japan* **19** (1967) 508-520.

[115] D. Kinderlehrer, G. Stampacchia; An introduction to variational inequalities and their applications, *Academic Press, Inc.* 1980.

[116] X. Li, N. Huang, D. O'Regan; Viscosity approximation methods for pseudo-contractive semigroups in Banach spaces, *Nonlinear Anal.* **75** (2012) 3776-3786

[117] F. Kohsaka, W. Takahashi; Strong convergence of an iterative sequence for maximal monotone operators in a Banach space, *Abstr. Appl. Anal.* 2004 (2004), 239-249.

[118] D. Körnlein, U. Kohlenbach; Effective rates of convergence for Lipschitzian pseudocontractive mappings in general Banach spaces, *Nonlinear Anal.* **74** (2011) 5253-5267

[119] B. Lemaire; Which fixed point does the iteration method select?, in *Recent Advances in Optimization*, vol. 452, pp. 154-157, Springer, Berlin, Germany, (1997).

[120] S. Li, L. Li, Y. Su; General iterative methods for a one-parameter nonexpansive semigroup in Hilbert space, *Nonlinear Anal.* **70** (2009), 3065-3071.

[121] X. Li, N. Huang, D. O'Regan; Strong convergence theorems for relatively nonexpansive mappings in Banach spaces with applications, *Comp. Math. Appl.* **60** (2010), 1322-1331.

[122] T. C. Lim; A fixed point theorem for families of non expansive mappings, *Pacific J. Math.*, **53** (1974), 487-493.

[123] T. C. Lim; Characterization of normal structure, *Proc. Amer. Math. Soc.* **43** (1974) 313-319.

[124] T. C. Lim, H. -K. Xu; Fixed point theorems for asymptotically nonexpansive mappings, *Nonlinear Anal., TMA* **22** (1994) 1345-1355.

[125] L. Lin; System of Generalized Vector Quasi-Equilibrium Problems with Applications to Fixed Point Theorems for a Family of Nonexpansive Multivalued Mappings, *J. Global Optim.* **34**, no 1, (2006) 15-32.

[126] P. L. Lion; Approximation de points fixes de contractions, *C. R. Acad. Sci. Paris sér A-B* **284** (1977) A1357-A1359.

[127] Y. Liu; A general iterative method for equilibrium problems and strict pseudo-contractions in Hilbert spaces, *Nonlinear Anal.* **71** (2009), 4852-4861.

[128] P. E. Maingé; Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, *Set-Valued Anal.* **16** (2008), 899-912.

[129] P. E. Maingé; Approximation methods for common fixed points of nonexpansive mappings in Hilbert space, *J. Math. Anal. Appl.*, **325** (2007), 469-479.

[130] W. R. Mann; Mean value methods in iterations, *Bull. Amer. Math. Soc.* **4** (1953), 506-510.

[131] C. Martinez-Yanes, H. K. Xu; Strong convergence of the CQ method for fixed point iteration processes, *Nonlinear Anal.* **64** (2006), 2400-2411.

[132] R. H. Martin Jr.; A global existence theorem for authonomous differential equations in Banach spaces, *Proc. Amer. Math. Soc.* **26** (1970) 307-314.

[133] R. H. Martin Jr.; Nonlinear Operators and Differential Equations in Banach Spaces, Interscience, New York, 1976.

[134] S. Matsushita, W. Takahashi; A strong convergence theorem for relatively nonexpansive mappings in Banach spaces, *J. Approx. Theory* 134 (2005), 257-266.

[135] S. Matsushita, W. Takahashi; Weak and strong convergence theorems for relatively nonexpansive mappings in a Banach space, *Fixed Point Theory Appl.* 2004 (2004), 37-47.

[136] A. Markov; Quelques theoremes sur les ensembles Abeliens, *Doklady Akad. Nauk SSSR (N.S.)* **10** (1936), 311-314.

[137] J. Mashreghi, M. Nasri; Forcing strong convergence of Kopelevich's method in Banach spaces with its applications in game theory, *Nonlinear Anal.* **72** (2010), 2086-2099.

[138] C. H. Morales; Surjectivity theorems for multivalued mappings of accretive type, *Commentationes Mathematicae Universitatis Carolinae,* **26** (1985), 397-413.

[139] C. H. Morales; Zeros of accretive operators satisfying certain boundary conditions, *J. Math. Anal. Appl.* **105** (1985) 167-175.

[140] C. H. Morales; Variational inequalities for Φ-pseudo-contractive mappings, *Nonlinear Anal. TMA* **75** (2012) 477-484.

[141] A. Moudafi; Weak convergence theorems for nonexpansive mappings and equilibrium problems, *J. Nonlinear Convex Anal.* **9** (2008), 37-43.

[142] A. Moudafi; A partial complement method for approximating solutions of a primal dual fixed-point problem, *Optim. Lett.* **4** (3), (2010), 449-456.

[143] A. Moudafi, M. Thera; Proximal and dynamical approaches to equilibrium problems, in *Lecture notes in economics and mathematical systems* **477**, Springer, (1999), 187-201.

[144] N. Nadezhkina, W. Takahashi; Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* **128** (2006), 191-201.

[145] W. Nilsrakoo, S. Saejung; Weak and strong convergence theorems for countable Lipschitzian mappings and its applications, *Nonlinear Anal.*, **69** (2008), 26952708.

[146] W. Nilsrakoo, S. Saejung; Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings, *Fixed Points Theory Appl.* (2008), Article ID 312454,19 pages.

[147] W. Nilsrakoo, S. Saejung; Strong convergence theorems by Halpern-Mann iterations for relatively nonexpansive mappings in Banach spaces, *Appl. Math. Comput.* **217** (2011), 6577-6586.

[148] M. A. Noor; General variational inequalities and nonexpansive mappings, *J. Math. Anal. Appl.* **331** (2007), 810-822.

[149] J. G. O'Hara, P. Pillay, and H-K. Xu; Iterative approaches to finding nearest common fixed points of nonexpansive mappings in Hilbert spaces, *Nonlinear Anal.* **54** (2003), 1417-1426.

[150] M. O. Osilike; Iterative solution of nonlinear equations of the ϕ-strongly accretive type, *J. Math. Anal. Appl.* **200** (1996) 259-271.

[151] P. M. Pardalos, T. M. Rassias, A. A. Khan; (Eds), Nonlinear Analysis and Variational Problems, *Springer*(2010).

[152] D. Pascali, S. Sburlan; Nonlinear mappings of monotone type, *Editura Academiae*, Bucharest, Romania (1978).

[153] J. W. Peng, J. C. Yao; Strong convergence theorems of an iterative scheme based on extragradient method for mixed equilibrium problems and fixed points problems, *Math. Com. Model.* **49** (2009), 1816-1828.

[154] N. Petrot, K. Wattanawitoon, P. Kumam; A hybrid projection method for generalized mixed equilibrium problems and fixed point problems in Banach spaces, *Nonlinear Anal.: Hybrid Systems*, **4** (2010), 631-643.

[155] S. Plubtieng, P. Kumam; Weak convergence theorems for monotone mappings and countable family of nonexpansive mappings, *Com. Appl. Math.* **224** (2009), 614-621.

[156] S. Plubtieng, R. Punpaeng; A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings, *Appl. Math. Comput.* **197** (2008), 548-558.

[157] S. Plubtieng, K. Ungchittrakool; Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space, *J. Approx. Theory* **149** (2007), 103-115.

[158] L. Qihou; On Naimpally and Singh's open questions, *J. Math. Anal. Appl.* **124** (1987), 157- 164. MR 88j:47078

[159] L. Qihou; The convergence theorems of the sequence of Ishikawa iterates for hemicontractive mappings, *J. Math. Anal. Appl.* **148** (1990), 55-62. MR 92b:47094

[160] X. L. Qin, Y. J. Cho, S. M. Kang, H. Y. Zhou; Convergence of a hybrid projection algorithm in Banach spaces, *Acta Appl. Math.* **108** (2009), 299-313.

[161] X. Qin, M. Shang, Y. Su; Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, *Math. Com. Model.* **48** (2008), 1033-1046.

[162] X. Qin, Y. Su; Strong convergence theorems for relatively nonexpansive mappings in a Banach space, *Nonlinear Anal.* **67** (2007), 1958-1965.

[163] W. O. Ray; An elementary proof of surjective for a class of accretive operators, *Proc. Amer. Math. Soc.* **75** (1979) 255-258.

[164] S. Reich; Strong convergence theorems for resolvents of accretive mappings in Banach spaces, *J. Math. Anal. Appl.,* **75** (1980), 287-292.

[165] S. Reich; Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.* **183** (1994), 118-120.

[166] S. Reich; On the asymptotic behavior of nonlinear semigroups and the range of accretive operators, *J. Math. Anal. Appl.*, **79** (1981), 113-126.

[167] S. Reich; Almost convergence and nonlinear ergodic theorems, *J. Approx. Theory* **24** (1978), 269-272.

[168] S. Reich; Fixed points of contractive functions, *Boll. Un. Mat. Ital.* **5** (1972), no. 4, 26-42.

[169] B. E. Rhoades; Fixed point iterations for certain nonlinear mappings, *J. Math. Anal. Appl.* **183** (1994), 118-120.

[170] R. T. Rockafellar; Monotone operators and proximal point algorithm, *SIAM J. Control Optim.* **14** (1976), 877-898.

[171] J. Schu; Approximating fixed points of Lipschitzian pseudo-contractive mappings, *Houston J. Math.* **19** (1993), 107-115. MR 94f:47066

[172] J. Schu; Approximation of fixed points of asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* **112** (1991), no. 1, 143-151.

[173] Y. Shehu, An Iterative Method for Fixed Point Problems, Variational Inclusions and Generalized Equilibrium Problems, *Math. Comp. Model.* **54** (2011), 1394-1404.

[174] Y. Shehu, An Iterative Method for nonexpansive semigroups, Variational Inclusions and Generalized Equilibrium Problems, *Math. Comp. Model.* **55** (2012), 1301-1314.

[175] Y. Shehu, Fixed Point Solutions of Generalized Equilibrium Problems for Nonexpansive Mappings, *J. Comput. Appl. Math.* **234** (2010), 892-898.

[176] Y. Shehu, A new iterative scheme for a countable family of relatively nonexpansive mappings and an equilibrium problem in Banach spaces, *J. Glob Optim.* **54** (2012), 519-535.

[177] Y. Shehu, Strong convergence theorems for infinite family of relatively quasi nonexpansive mappings and systems of equilibrium problems, *Appl. Math. Comput.* **218** (2012), 5146-5156.

[178] Y. Shehu, Strong Convergence Theorems for infinite family of quasi-nonexpansive mappings and Generalized Equilibrium Problems and variational inequality problems, *Comp. Math. Appl.* **61** (2011), 357-366.

[179] Y. Shehu, Strong convergence theorems for nonlinear mappings, variational inequality problems and system of generalized mixed equilibrium problems, *Math. Comp. Model.* **54** (2011), 2259-2276.

[180] T. Shimizu, W. Takahashi; Strong convergence of common fixed points of families of nonexpansive mappings, *J. Math. Anal. Appl.* **211** (1997), 71-83.

[181] N. Shioji, W. Takahashi; Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Amer. Math. Soc.* **125** (1997), 3641-3645.

[182] Y. Song, R. Chen; Convergence theorems of iterative algorithms for continuous pseudocontractive mappings, *Nonlinear Anal.*, **67** (2007) 3058-3063.

[183] G. Stampacchia; Formes bilineaires coercivities sur les ensembles convexes, *C. R. Acad. Sci. Paris* **258** (1964), 44134416.

[184] Y. Su, M. Shang, X. Qin; An iterative method of solution for equilibrium and optimization problems, *Nonlinear Anal.* **69** (2008), 2709-2719.

[185] Y. Su, H. K. Xu, X. Zhang,; Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications, *Nonlinear Anal.,* 73 (2010), 3890-3906.

[186] T. Suzuki; Strong convergence of Krasnoselkii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.* **305** (2005), 227-239.

[187] T. Suzuki; Common fixed points of one-parameter nonexpansive semigroups, *Bull. London Math. Soc.*, **38**, no. 6, 2006, 10091018.

[188] T. Suzuki; Fixed point property for nonexpansive mappings versus that for nonexpansive semigroups, *Nonlinear Anal. TMA*, **70**, no. 9, 2009, 33583361.

[189] T. Suzuki; On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces, *Proc. American Math. Soc.*, **131**, no. 7, (2003), 21332136.

[190] T. Suzuki; Browders type convergence theorems for one-parameter semigroups of nonexpansive mappings in Banach spaces, *Israel J. Math.*, **157**, no. 1, (2007), 239257.

[191] W. Takahashi; Nonlinear Functional Analysis-Fixed Point Theory and Applications, *Yokohama Publishers Inc., Yokohama,* (2000) (in Japanese).

[192] W. Takahashi; Nonlinear functional analysis, *Yokohama Publishers, Yokohama*, 2000

[193] S. Takahashi, W. Takahashi; Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, *Nonlinear Anal.* **69** (2008), 1025-1033.

[194] S. Takahashi, W. Takahashi; Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.* **331** (2007), 506-518.

[195] W. Takahashi, T. Tamura; Convergence theorems for pair of nonexpan- sive mappings, *J. Conv. Anal.* **5** (1998), 45-56.

[196] W. Takahashi, T. Tamura and M. Toyoda; Approximation of common fixed points of a family of finite nonexpansive mappings in Banach spaces, *Sci. Math. Jpn.* **56** (2002), no.3, 475-480.

[197] W. Takahashi, M. Toyoda; Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* **118** (2003) 417-428.

[198] W. Takahashi, K. Zembayashi; Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, *Fixed Point Theory Appl.* (2008), Article ID 528476, 11 pages.

[199] W. Takahashi, K. Zembayashi; Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, *Nonlinear Anal.* **70** (2009) 45-57.

[200] K. K. Tan, H. K. Xu; Approximating fixed point of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.* 178 (1993), 301-308.

[201] B. Turret; A dual view of a theorem of Baillon, in: S. P. Singh, J. H. Burry (Eds.), Nonlinear Analysis and Applications, Marcel Dekker, New York, 1982, pp.279-286.

[202] L. Wang; Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* **323** (2006), 550-557.

[203] S. Wang, C. Zhou; New iterative scheme for finite families of equilibrium, variational inequality and fixed point problems in Banach spaces *Fixed Point Theory Appl.* vol 2011, Article ID 372975, 18 pages, 2011.

[204] R. Wangkeeree; An extragradient approximation method for equilibrium problems and fixed point problems of a countable family of nonexpansive mappings, *Fixed Point Theory Appl.* vol 2008, Article ID 134148, 17 pages, 2008.

[205] R. Wangkeeree, R. Wangkeeree; The shrinking projection method for solving variational inequality problems and fixed point problems in Banach spaces, *Abstract Appl. Anal.* **volume 2009** (2009), Article ID 624798, 26 pages.

[206] R. Wittmann; Approximation of fixed points of nonexpasive mappings, *Arch. Math. Soc.* **59** (1992), 486-491.

[207] K. Q. Wu, N. J. Huang; The generalized f-projection operator with application, *Bull. Aust. Math. Soc.* 73 (2006),307-317.

[208] F. Q. Xia, N. J. Huang; Variational inclusions with a general H-monotone operator in Banach spaces, *Comput. Math. Appl.* **54** (2007), 24-30.

[209] H. K. Xu; Iterative algorithm for nonlinear operators, *J. London Math. Soc.* **66** (2) (2002), 1-17.

[210] H. K. Xu; Inequality in Banach spaces with Applications, *Nonlinear Anal.* **16** (1991), 1127-1138.

[211] Z. B. Xu, G. F. Roach; Characteristic inequalities of uniformly smooth Banach spaces, *J. Math. Anal. Appl.* **157** (1991) 189-210.

[212] Z. Yang, D. O'Regan; Positive solvability of systems of nonlinear Hammerstein integral equations, *J. Math. Anal. Appl.* **311** (2005), 600-614.

[213] Y. Yao, J. C. Yao; On modified iterative method for nonexpansive mappings and monotone mappings, *Appl. Math. Comput.* **186** (2) (2007), 1551-1558.

[214] Y. Yu; An Iterative Algorithm on Approximating Fixed Points of Pseudocontractive Mappings, Journal of Applied Mathematics Volume 2012, Article ID 341953, 11 pages doi:10.1155/2012/341953

[215] H. Zegeye, N. Shahzad; Strong convergence theorems for monotone mappings and weak relatively nonexpansive mappings, *Nonlinear Anal.* 70 (2009), 2707-2716.

[216] H. Zegeye, N. Shahzad; Strong convergence theorems for a common zero of a countably infinite family of α-inverse strongly accretive mappings, *Nonlinear Anal. TMA* **71** (2009) 531-538.

[217] H. Zegeye, N. Shahzad, T. Mekonen; Viscosity approximation methods for pseudo-contractive mappings in Banach spaces, *Appl. Math. Comp.* **185** (2007) 538-546.

[218] J. Zhao, S. He; A new iterative method for equilibrium problems and fixed points problems for infinite nonexpansive mappings and monotone mappings, *Appl. Math. Com.* **215** (2) (2009), 670-680.

[219] H. Zhou, L. Wei and Y. J. Cho; Strong convergence theorems on an iterative method for family of finite nonexpansive mappings in reflexive Banach spaces, *Appl. Math. Comput.* **173** (2006), 196-212.