SEMIGROUPS OF LINEAR OPERATORS AND APPLICATION TO DIFFERENTIAL EQUATIONS

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Epigraph

"if I were again begining my studies I would follow the advice of Plato and start with Mathematics."

Galileo Galilei

Preface

This work concerns one of the most important tools to solve well-posed problems in the theory of evolution equations (e.g diffusion equation, wave equations, ...) and in the theory of stochastic process, namely the semigroups of linear operators with application to differential equations.

A semigroup of linear operator on a Banach space X is a continuous operator valued function $T : [0, +\infty) \to \mathcal{B}(X, X)$ such that $T(t+s) = T(t)T(s)$ for every $t,\ s \geq 0$ and $T(0) = I$.

The fact that every non zero continuous complex function that satisfies $f(s+t) = f(s)f(t)$ for every $t,\ s \geq 0$ has the form $f(t) = exp(at)$, and that f is determined by the number $a = f'(0)$, motivates the association to $(T(t)_{t\geq 0})$ of an operator A defined by $Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t}$; $x \in \mathcal{D}(A)$ and called the infinitesimal generator of $(T(t))_{t\geq 0}$. Furthemore the study of the converse is of essential interest in the line of Hille-Yosida.

We divide this work into three chapters:

In the first chapter we present some preliminaries on the spectral theory, most of the materials follow from A. D. Andrew and W. L. Green[1]; C.E. Chidume [4], G. Barbatis; E.B. Davies and J.A. Erdos[3]; Erwin Kreyszig [5]; Khalil Ezzinbi [5].

In the second chapter we present the generation and representation of semigroups of linear operators and provide Hille-Yosida theorem which caracterizes the infinitesimal generator of a class of Continuous semigroup; essentially most of the materials follow from A. Pazy [7]; Khalil Ezzinbi [5].

Lastly we present the Abstract Cauchy problem as application, essentially most of the materials follow from Khalil Ezzinbi [5]; Alain Bensoussen, Guiseppe Da Prato, Michel C. Delphour, Sanjoy K. Mitter [2].

Acknowledgement

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Last but certainly not the least, I wish to express my heartfelt appreciation to my family in Democratique Republic of Congo for their genuine support during my stay away from home.

Dedication

This project is dedicated to my Lord and Savior, my God and King for the inspiration and wisdom.

Secondly I dedicate this project to my beloved parents- Padiani Petu Andre-Claude and Nduaya Kabombo Agnes for their parental care, support and love towards me during my course in African University of Science and Technology.

My brothers and sisters are not left out, their support helped me in achieving this goal, to them also I dedicate this work.

Contents

CHAPTER 1

Spectral Theory

1.1 Recall

Definition 1.1.1 *Let $T \in \mathcal{B}(X, X)$ be a bounded linear operator on a Banach space X. A complex number λ is said to be an eigenvalue of T if there exists a nonzero element $x \in X$ such that*

$$Tx = \lambda x. \tag{1.1.1}$$

The nonzero element x such that (1.1.1) holds is called an eigenvector of T corresponding to the eigenvalue λ.

$$\begin{aligned}
\lambda \text{ is an eigenvalue of } T &\iff \exists x \neq 0 \text{ such that } (\lambda I - T)x = 0. \\
&\iff x \in Ker(\lambda I - T). \\
&\iff \lambda I - T \text{ is not injective.} \\
&\iff Ker(\lambda I - T) \neq \{0\} \text{ i.e the null space of} \\
&\quad\;\; \lambda I - T \text{ is not trivial.}
\end{aligned}$$

Theorem 1.1.2 (Banach's Theorem) *Let X and Y be Banach spaces and assume $T \in \mathcal{B}(X, Y)$ is bijective. Then T^{-1} is a bounded linear operator from Y onto X, that is $T^{-1} \in \mathcal{B}(Y, X)$.*

Definition 1.1.3 *A complex valued function f is holomorphic(analytic) at z_0, if the limit of the function $z \mapsto \dfrac{f(z) - f(z_0)}{z - z_0}$ exists at $z = z_0$.*

Theorem 1.1.4 (Liouville Theorem) *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic(holomorphic) if f is bounded. Then f is constant.*

Definition 1.1.5 *Let* $ISO(X) := \{T \in \mathcal{B}(X,X) : T^{-1} \text{ exists } \}$

Lemma 1.1.6 *Let* X,Y *and* Z *be vector spaces.*

(i) If operators $B : X \to Y$ *and* $T : Y \to Z$ *are invertible, then* $TB : X \to Z$ *is invertible too and* $(TB)^{-1} = B^{-1}T^{-1}$.

(ii) If operators $B, T : X \to X$ *commute and* B *is invertible, then* B^{-1} *and* T *commute.*

Proof.(i) $B^{-1}T^{-1}TB = B^{-1}B = I_X$, $TBB^{-1}T^{-1} = TT^{-1} = I_Z$.

(ii) $B^{-1}T = B^{-1}TBB^{-1} = B^{-1}BTB^{-1} = TB^{-1}$. ■

Theorem 1.1.7 (Neumann Expansion) *Let* $T \in \mathcal{B}(X,X)$, *where* X *is a Banach space.*
If $\|T\| < 1$ *then* $I - T$ *is invertible, moreover*

$$(I - T)^{-1} = \sum_{k=0}^{+\infty} T^k.$$

Proof. Recall that $\mathcal{B}(X,X)$ is Banach space with

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|, \ \forall x \in X.$$

The series $\sum_{k=0}^{+\infty} T^k$ converge in $\mathcal{B}(X,X)$ since for $m < n$, we have
Setting $S_n = \sum_{k=0}^{n} T^k$

$$\|S_n - S_m\| = \|\sum_{k=0}^{n} T^k - \sum_{k=0}^{m} T^k\| = \left\| \sum_{k=m}^{n} T^k \right\|.$$

It follows

$$\begin{aligned} \left\| \sum_{k=m}^{n} T^k \right\| &\leq \sum_{k=m}^{n} \left\| T^k \right\| \\ &\leq \sum_{k=m}^{+\infty} \left\| T \right\|^k = \frac{\left\| T \right\|^m}{1 - \left\| T \right\|} \to 0 \text{ as } m \to +\infty. \end{aligned}$$

Therefore $(S_n)_{n \geq 0}$ is a cauchy sequence.
$(I - T)\big(\sum_{k=0}^{+\infty} T^k\big) = \sum_{k=0}^{+\infty} T^k - \sum_{k=0}^{+\infty} T^{k+1} = I + \sum_{k=1}^{+\infty} T^k - \sum_{k=1}^{+\infty} T^k = I,$
and
$\big(\sum_{k=0}^{+\infty} T^k\big)(I - T) = I.$
It now follows that $\big(I - T\big)^{-1} = \sum_{k=0}^{+\infty} T^k$.

Theorem 1.1.8 *$ISO(X)$ is open.*

Proof. We have to show that for all $T \in ISO(X)$, $\exists \varepsilon > 0 : B(T, \varepsilon) \subset ISO(X)$.
Let $T \in ISO(X)$ then T is invertible and we have $T.T^{-1} = I \neq O \Longrightarrow T^{-1} \neq O$.
Where O is the Zero operator. It follows that $\|T^{-1}\| \neq 0$ and then $\dfrac{1}{\|T^{-1}\|} > 0$.
Consider the open ball $B\big(T, \dfrac{1}{\|T^{-1}\|}\big)$.
Let $S \in \mathcal{B}(X, X)$ such that $S \in B\big(T, \dfrac{1}{\|T^{-1}\|}\big)$,one has $\|S - T\| < \dfrac{1}{\|T^{-1}\|}$;then

$$\|T^{-1}\|\|S - T\| < 1 \Longrightarrow \|T^{-1}S - I\| < 1$$

From the theorem(1.1.7) $T^{-1}S$ is invertible in X i.e $T^{-1}S \in ISO(X)$.
Since $T \in ISO(X)$ and $ISO(X)$ is a group with the composition of operator in $\mathcal{B}(X, X)$, it follows that $T(T^{-1}S) \in ISO(X)$ i.e $S \in ISO(X)$.
Take $\varepsilon = \dfrac{1}{\|T^{-1}\|}$,
we have shown that

$$\forall T \in ISO(X),\ \exists \varepsilon > 0 \text{ such that } B(T, \varepsilon) \subset ISO(X).$$

The proof is complete.

1.2 Spectral theory of bounded linear operators

Definition 1.2.1 *Let $T \in \mathcal{B}(X, X)$.*

1. *The resolvent set of T, denoted by $\rho(T)$, is the set of all complex numbers λ such that $\lambda I - T$ is bijective, i.e*

   $$\rho(T) = \big\{\lambda \in \mathbb{C} : (\lambda I - T)^{-1} \text{ exists in } \mathcal{B}(X, X)\big\}.$$

   If $\lambda \in \rho(T)$, then $\lambda I - T \in ISO(X)$.

2. *The spectrum of T, denoted by $\sigma(T)$, is defined by*

   $$\sigma(T) = \big\{\lambda \in \mathbb{C} : (\lambda I - T) \notin ISO(X)\big\}.$$

   The spectrum of T is the complement of $\rho(T)$ in $\mathbb{C}$.
   That is,

   $$\sigma(T) = \mathbb{C} \backslash \rho(T).$$

   The complex numbers λ in the spectrum of T may be classified according to the way in which $\lambda I - T$ fails to be invertible.

   (a) *The point spectrum of T, denoted by $\sigma(T)$, is the set of all eigenvalues of T. That is,*

   $$\sigma_p(T) = \Big\{\lambda \in \mathbb{C} : Ker(\lambda I - T) \neq \{0\}\Big\}$$

(b) The residual spectrum of T is the set

$$\sigma_r(T) = \Big\{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\} \text{ and } \overline{Im(\lambda I - T)} \text{ is a proper subset of } X\Big\},$$

that is ,

$$\sigma_r(T) = \Big\{\lambda \in \mathbb{C} : \lambda I - T \text{ is injective but not surjective,and } \overline{Im(\lambda I - T)} \neq X\Big\}.$$

(c) The continuous spectrum of T is the set

$$\sigma_c(T) = \Big\{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\} \text{ and } Im(\lambda I - T) \text{ is a proper dense subset of } X\Big\},$$

that is,

$$\sigma_c(T) = \Big\{\lambda \in \mathbb{C} : \lambda I - T \text{ is injective but not surjective, and } \overline{Im(\lambda I - T)} = X\Big\}.$$

Clearly, for each $T \in \mathcal{B}(X, X)$, the sets $\sigma_p(T)$, $\sigma_c(T)$, and $\sigma_r(T)$ are pairwise disjoint, $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$ and $\mathbb{C} = \rho(T) \cup \sigma(T)$. If $T \in \mathcal{B}(X, X)$ and $\lambda \in \rho(T)$, then $\lambda I - T$ is a bounded invertible linear operator of X onto itself. By Banach's Theorem (1.1.2), the inverse operator $(\lambda I - T)^{-1}$ is also a bounded linear operator of X onto itself. It therefore follows that $\rho(T)$ is precisely the set of all complex numbers λ such that $\lambda I - T$ has a bounded inverse defined on all of X.

Example 1.2.2 (Operator with a spectral value which is not an eigenvalue) *On the Hilbert sequence space $X = l_2$;*

$$l_2 = \Big\{(x_n)_{n \geq 1} : \sum_{n=1}^{+\infty} x_n^2 < +\infty\Big\}$$

we define a linear operator $T : l_2 \to l_2$ by $(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$. The operator T is called the right-shift operator. We show that $\sigma_p(T) = \emptyset$.
Suppose $\sigma_p(T) \neq \emptyset$ then $\exists \lambda \in \sigma_p(T)$, it follows that there exists $x = (x_1, x_2, \ldots) \in l_2 \backslash \{0\}$ such that $Tx = \lambda x$.
We have $(0, x_1, x_2, \ldots, x_n, \ldots) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n, \ldots)$.
If $\lambda = 0$ then $x = 0$ which is impossible since $x \neq 0$.
Consider now $\lambda \neq 0$ then

$$\begin{aligned} \lambda x_1 = 0 \quad &\Longrightarrow \quad x_1 = 0 \\ \lambda x_2 = x_1 \quad &\Longrightarrow \quad x_2 = 0 \\ &\vdots \\ \lambda x_n = x_{n-1} \quad &\Longrightarrow \quad x_n = 0 \end{aligned}$$

$\forall n \in \mathbb{N} : x_n = 0 \Longrightarrow x = 0$. *This is absurd since $x \neq 0$. But it is easy to see that T is not surjective.*
Therefore $\sigma_p(T) = \emptyset$.

Theorem 1.2.3 *Let X be a complex Banach space, $T \in \mathcal{B}(X,X)$ and $\lambda,\ \mu \in \rho(T)$. Define the resolvent $R(\lambda;T)$ of T by $R(\lambda;T) = (\lambda I - T)^{-1}$ instead of $R(\lambda,T)$ we also write R_λ if it is clear to what operator T we refer in a specific discussion. Then*

(a) satisfies the ***resolvent equation***

$$R(\lambda;T) - R(\mu;T) = (\mu - \lambda)R(\lambda;T)R(\mu;T) \text{ for } \lambda,\ \mu \in \rho(T).$$

(b) R_λ commutes with T.

(c) We have $R_\lambda R_\mu = R_\mu R_\lambda$.

Theorem 1.2.4 *Let $T \in \mathcal{B}(X,X)$ then the following are true.*

1. *$\rho(T)$ is open in $\mathbb{C}$.*
2. *Let $R(\lambda;T) = (\lambda I - T)^{-1}$,*
$$\lim_{\lambda \to +\infty} \|R(\lambda;T)\| = 0.$$
3. *Let $\phi : \rho(T) \longrightarrow \mathcal{B}(X,X)$, $\lambda \longmapsto R(\lambda;T)$ is holomorphic in $\rho(T)$.*
4. *$\sigma(T) \neq \emptyset$.*
5. *The spectrum $\sigma(T)$ is compact in $\mathbb{C}$, where $\sigma(T) \subseteq B(0, \|T\|)$.*
6. *$\|R(\lambda;T)\| \geq \dfrac{1}{d(\lambda, \sigma(T))}$, $\lambda \in \rho(T)$.*

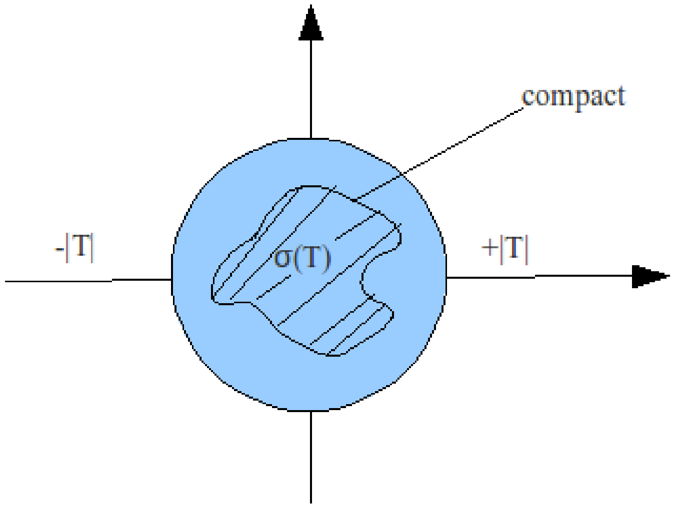

Figure 1.1: $\sigma(T) \subseteq \overline{B}(0, \|T\|)$.

Proof. 1. Let $\lambda \in \rho(T)$. We want to show that there exists $r > 0$: $B(\lambda, r) \subseteq \rho(T)$.
Claim: $B\left(\lambda, \dfrac{1}{\|R(\lambda;T)\|}\right) \subseteq \rho(T)$.
Let $\mu \in \mathbb{C}$ such that $\mu \in B\left(\lambda, \dfrac{1}{\|R(\lambda;T)\|}\right)$, that is

$$|\lambda - \mu| < \frac{1}{\|R(\lambda;T)\|}. \tag{1.2.1}$$

Let now show that $(\mu I - T)^{-1}$ exists.
One has

$$\begin{aligned} \mu I - T &= \mu I - \lambda I + \lambda I - T \\ &= (\lambda I - T)[R(\lambda, T)(\mu - \lambda) + I]. \end{aligned}$$

It follows from theorem (1.1.7) and relation (1.2.1) that $R(\lambda, T)(\mu - \lambda) \in ISO(X)$, and since we have $\lambda \in \rho(T)$ one has $\lambda I - T \in ISO(X)$.
Hence applying lemma(1.1.6(i)), we obtain $\mu I - T \in ISO(X)$ and then $\mu \in \rho(T)$.
It is natural to conclude that $B\left(\lambda, \dfrac{1}{\|R(\lambda;T)\|}\right) \subseteq \rho(T)$.

Thus we have shown that

$$\forall \lambda \in \rho(T),\ \exists\, r = \frac{1}{\|R(\lambda;T)\|} > 0 \text{ such that } B\left(\lambda, \frac{1}{\|R(\lambda;T)\|}\right) \subseteq \rho(T).$$

2. Let $|\lambda| > \|T\|$. Then $\left\|\frac{T}{\lambda}\right\| < 1$, it follows from theorem (1.1.7) that $\left(I - \frac{T}{\lambda}\right)^{-1}$ exists. Moreover

$$\left(I - \frac{T}{\lambda}\right)^{-1} = \sum_{n\geq 0} \left(\frac{T}{\lambda}\right)^n,$$

then

$$\left\|\left(I - \frac{T}{\lambda}\right)^{-1}\right\| \leq \sum_{n\geq 0} \left\|\frac{T}{\lambda}\right\|^n = \frac{1}{1 - \left\|\frac{T}{\lambda}\right\|}.$$

Since

$$R(\lambda; T) = (\lambda I - T)^{-1} = \frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1},$$

therefore

$$\|R(\lambda; T)\| = \frac{1}{|\lambda|}\left\|\left(I - \frac{T}{\lambda}\right)^{-1}\right\| \leq \frac{1}{|\lambda| - \|T\|} \to 0 \text{ as } \lambda \to +\infty.$$

3. Let $\lambda_0 \in \rho(T)$ be such that $\lambda \to \lambda_0$. We have

$$\frac{R(\lambda; T) - R(\lambda_0; T)}{\lambda - \lambda_0} = -R(\lambda; T)R(\lambda_0; T) \to -R^2(\lambda_0; T) \in B(X, X).$$

4. Assume for the contrary that $\sigma(T) = \emptyset$, then $\rho(T) = \mathbb{C}$.
Take arbitrary $x \in X\backslash\{0\}$, $f \in X^*$.
We have from (3) above that $\lambda \mapsto R(\lambda; T)$ is holomorphic and bounded in $\rho(T) = \mathbb{C}$.
It follows that the function defined by

$$h(\lambda) := f(R(\lambda; T)x)$$

is holomorphic on $\mathbb{C}$, that is, h is an entire function.
Since holomorphy implies continuity, h is continuous and thus bounded on the compact disk $|\lambda| \leq \|T\|$.
But h is also bounded for $|\lambda| > \|T\|$ since by (2) above $\|R(\lambda; T)\| \leq \frac{1}{|\lambda| - \|T\|} < \frac{1}{\|T\|}$ and

$$\begin{aligned} |h(\lambda)| = |f(R(\lambda; T)x)| &\leq \|f\|\|R(\lambda; T)x\| \leq \|f\|\|R(\lambda; T)\|\|x\|\| \\ &\leq \|f\|\frac{\|x\|}{\|T\|}. \end{aligned}$$

Hence h is bounded on $\mathbb{C}$ and thus constant, by Liouville's theorem.
On the other hand $h(\lambda) \to 0$ as $|\lambda| \to +\infty$. We then conclude that $h \equiv 0$.
Let us take $f \in X^*$ such that $f(R(\lambda_0; T)x) = \|R(\lambda_0; T)x\|$ for the given $x \in X\backslash\{0\}$ and some $\lambda_0 \in \mathbb{C}$.

$$0 = h(\lambda_0) = f(R(\lambda_0; T)x) = \|R(\lambda_0; T)x\|$$

i.e

$$R(\lambda_0; T)x = 0$$

consequently

$$x = (\lambda_0 I - T)R(\lambda_0; T)x = 0 \text{ for } x \in X\backslash\{0\},$$

which is absurd.
Hence $\sigma(T) \neq \emptyset$.
5. **Claim:** If $T \in \mathcal{B}(X, X)$ and $\lambda \in \mathbb{C}$ such that $\|T\| < |\lambda|$, then $\lambda I - T$ is invertible.
That is

$$\big\{\lambda \in \mathbb{C} : \|T\| < |\lambda|\big\} \subset \rho(T), \text{ and consequently } \sigma(T) \subset \big\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\big\}.$$

Proof of the claim.
$\lambda I - T = \lambda(I - \lambda^{-1}T)$ and $\|\lambda^{-1}T\| = \dfrac{\|T\|}{|\lambda|} < 1$,
we have by theorem (1.1.7), that $(I - \lambda^{-1}T)$ is invertible, and so is $\lambda(I - \lambda^{-1}T) = \lambda I - T$.
Using that claim we have

$$\sigma(T) \subset \big\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\big\} \text{ that is } \sigma(T) \subset \overline{B}(0, \|T\|)$$

and so $\sigma(T)$ is bounded. It remains to show that $\sigma(T)$ is closed subset of $\mathbb{C}$. It sufficies to show that the resolvent $\rho(T)$ is an open subset of $\mathbb{C}$. But this has been established in 1.
Hence $\sigma(T) = \mathbb{C}\backslash\rho(T)$ is closed.
Thus $\sigma(T)$ is a closed and bounded subset of $\mathbb{C}$, and therefore compact.

6. Let $\lambda \in \rho(T)$. Then $d(\lambda, \sigma(T)) > 0$,
we know that $B(\lambda, \dfrac{1}{\|R(\lambda; T)\|}) \subseteq \rho(T)$, that is

$$\left\{\mu \in \mathbb{C} : |\lambda - \mu| < \frac{1}{\|R(\lambda; T)\|}\right\} \subset \rho(T).$$

It follows that

$$\sigma(T) \subset \left\{\mu \in \mathbb{C} : |\lambda - \mu| < \frac{1}{\|R(\lambda; T)\|}\right\}.$$

Therefore,

$$\begin{aligned} d(\lambda, \sigma(T)) &\geq d\bigg(\lambda, \left\{\mu \in \mathbb{C} : |\lambda - \mu| \geq \frac{1}{\|R(\lambda, T)\|}\right\}\bigg); \\ &\geq \frac{1}{\|R(\lambda; T)\|}. \end{aligned}$$

Example 1.2.5

Let $p \geq 1$.

$$l_p = \{(x_n)_{n\geq 1} \subset \mathbb{R} : \sum_{n=1}^{+\infty} |x_n|^p < +\infty\}.$$

$$\begin{array}{rcl} T : l_p & \rightarrow & l_p \\ (x_n)_{n\geq 1} & \rightarrow & (x_{n+1})_{n\geq 1}, \end{array}$$

that is $(x_n)_{n\geq 1} = (x_1, x_2, x_3, \ldots)$, $T((x_n)_{n\geq 1}) = (x_2, x_3, x_4, \ldots)$.

$$|T((x_n)_{n\geq 1})| = \left(\sum_{n=2}^{+\infty} |x_n|^p)^{\frac{1}{p}}\right) \leq \left(\sum_{n=1}^{+\infty} |x_n|^p)^{\frac{1}{p}}\right),$$

it follows that $\|T\| \leq 1$. Consequently $\sigma(T) \subset \overline{B}(0,1)$.
Claim: $\lambda \in \mathbb{C} : |\lambda| < 1 \Longrightarrow \lambda \in \sigma_p(T)$
Case 1: $\lambda \neq 1$. Let $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$. Set $x = (\lambda^n)_{n\geq 1} \in l_p$.

$$Tx = (\lambda^{n+1})_{n\geq 1} = \lambda(\lambda^n)_{n\geq 1} = \lambda x.$$

That is x is an eigen vector of T.
Hence $\lambda \in \sigma_p(T)$.
Case 2: $\lambda = 0$.
Let $x = (1, 0, 0, \ldots, 0)$, $Tx = (0, 0, \ldots, 0) = 0 \times x$, therefore $0 \in \sigma_p(T)$.

It follows that $B(0,1) \subseteq \sigma_p(T) \subseteq \sigma(T) \subseteq \overline{B}(0,1)$,
therefore $\overline{B}(0,1) \subseteq \overline{\sigma}(T) = \sigma(T) \subseteq \overline{B}(0,1)$.
Consequentely $\sigma(T) = \overline{B}(0,1)$.

Definition 1.2.6 *Let $T \in \mathcal{B}(X,X)$. The approximate spectrum of $\sigma_a(T)$ is defined by*

$$\sigma_a(T) = \left\{\lambda \in \mathbb{C} : \exists (x_n)_{n\geq 1} \subseteq X, \|x_n\| = 1 \text{ such that } \lambda x_n - Tx_n \rightarrow 0, n \rightarrow +\infty\right\}.$$

Lemma 1.2.7 *Let $T \in \mathcal{B}(X,X)$. Then $\sigma_a(T) \supseteq \sigma_p(T)$.*

Proof. Let $\lambda \in \sigma_p(T)$ then there exists $x \neq 0$ such that $Tx = \lambda x$.
We have $\lambda \dfrac{x}{\|x\|} - T\left(\dfrac{x}{\|x\|}\right) = 0 \Longrightarrow x_n = \dfrac{x}{\|x\|}, \forall n \in \mathbb{N}$.
Therefore $\lambda \in \sigma_a(T)$. This completes the proof.

Theorem 1.2.8 (Uniform boundedness principle) *Let X a Banach space and $(T_\alpha)_{\alpha\in I} \subset \mathcal{B}(X,X)$. Suppose that*

$$\sup_{\alpha\in I} \|T_\alpha(x)\| < +\infty \text{ for each } x \in X$$

Then,

$$\sup_{\alpha\in I} \|T_\alpha\| < +\infty.$$

Theorem 1.2.9 *Let $T \in \mathcal{B}(X, X)$. Then*

1. $\sigma(T) = \sigma_a(T) \cup \widetilde{\sigma}_r(T)$. *With* $\widetilde{\sigma}_r(T) = \{\lambda \in \mathbb{C} : Im(\lambda I - T)$ *is not dense in* $X\}$,
2. $\partial\sigma(T) \subseteq \sigma_a(T)$.

Lemma 1.2.10 *Let $T \in \mathcal{B}(X, X)$. Then*

$$\lambda \in \sigma_a(T) \iff \begin{cases} Ker(\lambda I - T) & \neq \{0\} \\ \quad or & \\ Im(\lambda I - T) & \text{is not closed.} \end{cases}$$

$\sigma_a(T) = \sigma_p(T) \cup \{\lambda Im(\lambda I - T)$ *is not closed* $\}$.

Proof. Let $\lambda \in \sigma_a(T)$, $\lambda \notin \sigma_p(T)$. Suppose that $Im(\lambda I - T)$ is closed.
$\lambda \notin \sigma_p(T) \Longrightarrow Ker(\lambda I - T) = \{0\}$ and $\lambda I - T$ is bijective from X to $Im(\lambda I - T)$.
Then $(\lambda I - T)^{-1}$ exists.
Since $Im(\lambda I - T)$ is closed, then by Banach theorem, we have $(\lambda I - T)^{-1} \in \mathcal{B}(Im(\lambda I - T), X)$ that is there exists $M > 0$ such that $\|(\lambda I - T)^{-1} y\| \leq M\|y\|$, $\forall y \in Im(\lambda I - T)$.
Set $x = (\lambda I - T)^{-1} y$, $\forall x \in X$.
$\|x\| \leq M\|\lambda x - Tx\|$, $\forall x \in X$. But $\lambda \in \sigma_a(T) \iff \exists (x_n)_{n \geq 1} \subset X$, $\|x_n\| = 1$ such that $\lambda x_n - Tx_n \to 0$ as $n \to +\infty$.
$1 = \|x_n\| \leq M\|\lambda x_n - Tx_n\| \to 0$. Contradiction.

Proof of theorem (1.2.9)

1. Let $\lambda \in \sigma(T) \backslash \sigma_a(T)$. We want to show that $\lambda \in \widetilde{\sigma}_r(T)$.
   But $\lambda \in \widetilde{\sigma}_r(T) \iff Im(\lambda I - T)$ is not dense in X.

$$\lambda \in \sigma_a(T) \iff \begin{cases} Ker(\lambda I - T) & \neq \{0\} \\ \quad \text{or} & \\ Im(\lambda I - T) & \text{is not closed.} \end{cases}$$

$$\lambda \in \sigma_a(T) \iff \begin{cases} Ker(\lambda I - T) & = \{0\} \\ \quad \text{and} & \\ Im(\lambda I - T) & \text{is closed.} \end{cases}$$

   It follows $(\lambda I - T)^{-1} \mathcal{B}(Im(\lambda I - T), X)$
   We have either $Im(\lambda I - T)$ equals X or is a proper subset of X.
   $Im(\lambda I - T) = X$ is impossible since $\lambda \in \sigma(T)$, then $Im(\lambda I - T) \neq X$, but $Im(\lambda I - T)$ is closed .
   Therefore $Im(\lambda I - T)$ is not dense in X. Consequently $\lambda \in \sigma_r(T)$

2. Let $\lambda \in \partial\sigma(T)$ then $\exists (\lambda_n)_{n \geq 1} \subset \rho(T)$ such that $\lambda_n \to \lambda$ as $n \to +\infty$.
   One has

$$\|R(\lambda_n; T)\| \geq \frac{1}{d(\lambda_n, \sigma(T))} \text{ and } \lim_{n \to +\infty} d(\lambda_n, \sigma(T)) = d(\lambda, \sigma(T)) = 0.$$

If follows that $\|R(\lambda_n;T)\| \to +\infty$.
By Banach-Steinhauss theorem there exists $x \in X$, $x \neq 0$ such that
$\sup_{n\in\mathbb{N}} \|R(\lambda_n;T)x\| \to +\infty$.
There exists $(\lambda_{n_k})_{k\geq 1}$ subsequence of $(\lambda_n)_{n\geq 1}$ such that $\|R(\lambda_{n_k};T)x\| \to +\infty$.
Set

$$\frac{R(\lambda_{n_k};T)x}{\|R(\lambda_{n_k};T)x\|},$$

we have,

$$\begin{aligned}(\lambda I - T)x_{n_k} &= \frac{(\lambda I - T)R(\lambda_{n_k};T)x}{\|R(\lambda_{n_k};T)x\|} \\ &= \frac{[(\lambda - \lambda_{n_k})I + (\lambda_{n_k}I - T)]R(\lambda_{n_k};T)x}{\|R(\lambda_{n_k};T)x\|} \\ &= \frac{[(\lambda - \lambda_{n_k})]R(\lambda_{n_k};T)x}{\|R(\lambda_{n_k};T)x\|} + \frac{[(\lambda_{n_k}I - T)]R(\lambda_{n_k};T)x}{\|R(\lambda_{n_k};T)x\|} \to 0 \text{ as } k \to +\infty.\end{aligned}$$

Our next result will be the important spectral mapping theorem for polynomials and we start with the following motivation.
Let $P \in \mathbb{R}[X] \equiv$ ring of polynomial ,$P(x) = a_0x^n + \ldots + a_n$.
Let $T \in \mathcal{B}(X,X)$, $P(T) = a_0T^n + \ldots + a_n.P(T) \in \mathcal{B}(X,X)$.
Question: $P(\sigma(T)) = \sigma(P(T))$?
Let $T \in \mathcal{B}(X,X)$, $f(z) = \sum\limits_{n\in\mathbb{N}} a_nz^n$ which converges in its disk of convergence $B(0,R)$,
R being the radius of convergence.
If $\|T\| < R$, define

$$f(T) = \sum_{n\in\mathbb{N}} a_nT^n.$$

Then $f(T) \in \mathcal{B}(X,X)$.

Theorem 1.2.11 (Spectral mapping theorem) *$P(\sigma(T)) = \sigma(P(T))$ for all $P \in \mathbb{R}[X]$ and for all $T \in \mathcal{B}(X,X)$.*

Proof. Let P be the polynomial given by

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \text{ with } a_0 \neq 0.$$

First let $\lambda \in \sigma(T)$. We need to show that $P(\lambda) \in \sigma(P(T))$, that is $P(\lambda) - P(T)$ is not invertible.

On the other hand λ is a root of $P(x) - P(\lambda)$. It follows that there exists $Q_\lambda \in \mathbb{R}[X]$ such that

$$P(\lambda) - P(x) = (\lambda - x)Q_\lambda(x).$$

Replacing x by T, one has $P(\lambda)I - P(T) = (\lambda I - T)Q_\lambda(T) = Q_\lambda(T)(\lambda I - T)$.
Assume for the contrary that $P(\lambda) \notin \sigma(P(T))$ then $(P(\lambda)I - (P(T))^{-1} \in \mathcal{B}(X,X)$.
We can write,

$I = (\lambda I - T)Q_\lambda(T)(P(T) - P(\lambda)I)^{-1} = (P(T) - P(\lambda)I)^{-1}(T - \lambda I)Q_\lambda(T)$.
It follows that $\lambda \in \rho(T)$ which is absurd.So we must have that $P(\lambda)I - (P(T)$ is not invertible, that is $P(\lambda) \in \sigma(P(T))$. Therefore we have shown that

$$P(\sigma(T)) \subseteq \sigma(P(T)).$$

Conversely, let $P \in \mathbb{R}[X]$ such that$d^\circ P \geq 1$.
Suppose that $\mu \in \sigma(P(T))$ and set $\mu - P(x) = c(\lambda_1 - x)\dots(\lambda_n - x)$, $c \neq 0$, $d^\circ P = n$.
We have $\lambda_1, \dots, \lambda_n$ are roots of $\mu - P(x)$ i.e $\mu = P(\lambda_j)$, $\forall j = 1, \dots, n$.
Substituting x by T, one gets $\mu I - P(T) = c(\lambda_1 I - T)\dots(\lambda_n I - T)$. Since $\mu I - P(T)$ is not invertible then there exists $j_0 \in \{1, \dots, n\}$ such that $(\lambda_{j_0} I - T)$ is not invertible i.e $\lambda_{j_0} \in \sigma(T)$.
Thus $\mu = P(\lambda_{j_0}) \in P(\sigma(T))$. Therefore,

$$\sigma(P(T)) \subseteq P(\sigma(T)).$$

Thus,

$$\sigma(P(T)) = P(\sigma(T)). \qquad \blacksquare$$

1.3 Spectral Theory of Compact Operators

Definition 1.3.1 *Let X, Y Banach space and $T \in \mathcal{B}(X, Y)$, T is said to be* ***compact*** *if for each sequence $(x_n)_{n\geq 1} \in X$ with $\|x_n\| = 1$ for each $n \in \mathbb{N}$, the sequence $(Tx_n)_{n\geq 1}$ has a subsequence which converges in Y. Equivalently, T is compact if for each bounded sequence $(x_n)_{n\geq 1} \in X$, the sequence $(Tx_n)_{n\geq 1}$ has a subsequence which converges in Y.*
That is T is compact if and only if $\overline{T(\overline{B}(0,1))}$ is compact if and only if $\forall B$ bounded set X, $\overline{T(B)}$ is sequentially compact on Y.

We shall denote by $\mathcal{K}(X, Y)$ the space of all compact operators from X to Y. And if $Y = X$, we shall write $\mathcal{K}(X)$ the space of all compact operators on X We shall needs some preliminaries in other to prove the theorem 1.3.7, let us state some theorems and lemmas which will be useful.

Theorem 1.3.2

1. $\mathcal{K}(X, Y)$ is closed in $\mathcal{B}(X, Y)$.
2. $\mathcal{K}(X, X)$ is an ideal of $\mathcal{B}(X, X)$.
3. $T \in \mathcal{K}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$ then $S \circ T \in \mathcal{K}(X, Z)$

Lemma 1.3.3 (Riesz-Fredholm) *Let X Banach space and $Y \subsetneqq X$, be a closed subspace of X. Then for each $\varepsilon \in (0, 1)$, there exists $x \in X$ such that $\|x\| = 1$ and*

$$\|y - x\| > 1 - \varepsilon \text{ for all } \quad y \in Y.$$

Theorem 1.3.4 (Riesz theorem) *$dimX < +\infty \iff \overline{B}(0, 1)$ is compact.*

Let us state the following theorem which claims several properties of the operator $\lambda - T$ where $\lambda \neq 0$. Without loss of generality, it can be assumed that $\lambda = 1$. Therefore we consider $I - T$, I being the identity operator.

Theorem 1.3.5 (Fredholm Alternative) *Let $T \in \mathcal{K}(X)$, then*

1. *$dimKer(I-T)$ is finite.*
2. *$Im(I-T)$ is closed.*
3. *$Ker(I-T) = \{0\} \iff Im(I-T) = X$.*

Let us state also the following lemma which will be useful for the proof of the following theorem.

Lemma 1.3.6 *Let X be Banach space, $T \in \mathcal{K}(X)$ and let $\lambda_n \in \sigma(T) \backslash \{0\}$ such that $\lambda_n \to \lambda$ then $\lambda = 0$.*

Theorem 1.3.7 *Let X be Banach space, $T \in \mathcal{K}(X)$ with $dimX = +\infty$ then the following are true,*

1. *$0 \in \sigma(T)$.*
2. *Every nonzero $\lambda \in \sigma(T)$ is an eigenvalue of T. That is $\lambda \in \sigma(T)\backslash\{0\} \subseteq \sigma_p(T)$.*
3. *$\sigma(T)$ is either finite or infinite.*
   If $\sigma(T)$ is infinite then $\sigma(T) = \left\{(\lambda_n)_{n\geq 0} \in \mathbb{C} : \lambda_n \to 0\right\}$.

Proof. 1. Suppose for the contrary that $0 \notin \sigma(T)$,then $T^{-1} \in \mathcal{B}(X, X)$.
$\mathcal{K}(X)$ being an ideal, then $T \circ T^{-1} = I_X$ is compact. By Riesz theorem $I_X(\overline{B}(0,1))$ compact implies $dimX < +\infty$. Which is a contradiction. ■

2. Without loss of generality, assume $\lambda = 1$. $\lambda \in \sigma(T)$ not being an eigenvalue means $(I-T)$ is injective but not surjective. By theorem (1.3.5) part.2, $Y_1 = Im(I-T)$ is a closed proper subspace of X.
Since $(I-T)$ is injective, $Y_2 = (I-T)Y_1$ is again a closed proper subspace of Y_1.
Define $Y_n = Im(I-T)^n$.
Consider the decreasing sequence of subspaces $Y_1 \supset \cdots \supset Y_n \cdots \supset Y_m \cdots$
where all inclusions are proper. By lemma (1.3.3), we can choose unit vectors $y_n \in Y_n$ such that $d(y_n, Y_{n+1}) >$. Compactness of T means $\{Ty_n\}$ must contain a convergent subsequence. But for $n < m$

$$\|Ty_n - Ty_m\| = \|(T-I)y_n + y_n - (T-I)y_m - y_m\|$$

and it is easy to see that

$$(T-I)y_n - (T-I)y_m - y_m \in Y_{n+1},$$

which implies $\|Ty_n - Ty_m\| > \frac{1}{2}$. This is a contradiction, and so λ must be an eigenvalue.

3. a. Suppose to the contrary that $\sigma(T)$ is not finite.
Let $k \in \mathbb{N}$, $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq \frac{1}{k}\}$ is finite otherwise there exists (λ_{n_k}) converging to λ such that $|\lambda_n| \geq \frac{1}{k}$. Therefore $|\lambda| \geq \frac{1}{k}$ is absurd since $\lambda = 0$.

$$\sigma(T) = \underset{k\geq 1}{\cup}\left(\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq \frac{1}{k}\}\right).$$

It follows that $\sigma(T)$ is countable, $\sigma(T) = \{\lambda_n, n \geq 1\}$. There exists $\lambda \in \sigma(T)$ such that $\lambda_{n_k} \to \lambda$ as $k \to +\infty$. By lemma 1.3.6, $\lambda = 0$. Each sequence converge to 0, therefore $\lambda_n \to 0$. ■

b. Suppose eigenvalues of T do not accumulate at 0. We can therefore assume that there exist a sequence of distinct eigenvalues $(\lambda_n)_{n\geq 1}$, with corresponding eigenvectors $(x_n)_{n\geq 1}$, such that there exist $\varepsilon > 0$, $|\lambda_n| > \varepsilon$ for all $n \in \mathbb{N}$.
Define $Y_n = span\{x_1, .., x_n\}$. The sequence $(Y_n)_{n\geq 1}$ is a strictly increasing sequence. Choose unit vectors such that $y_n \in Y_n$ and $d(y_n, Y_{n-1}) = 1$.
Then for n < m,

$$\|Ty_n - Ty_m\| = \|(T - \lambda_n)y_n + \lambda_n y_n - (T - \lambda_m)y_m - \lambda_m y_m\|.$$

But

$$(T - \lambda_n)y_n + \lambda_n y_n - (T - \lambda_m)y_m - \lambda_m y_m \in Y_{m-1},$$

therefore $\|Ty_n - Ty_m\| > \varepsilon$, a contradiction.

1.4 Spectral theory of Unbounded linear operators

Definition 1.4.1 *Let X be a Banach space , the linear map $T : \mathcal{D}(T) \subseteq X \to X$ said to be unbounded if*

$$\sup_{\|x\|=1} \|Ax\| = +\infty.$$

Definition 1.4.2 *Let X and Y be linear spaces over a field $\mathbb{F}$ and $T : X \to Y$.
The graph of T, denoted by $\mathcal{G}(T)$, is the subset of $X \times Y$ given by*

$$\mathcal{G}(T) = \{(x, Tx) : x \in X\}.$$

Since T is linear, $\mathcal{G}(T)$ is a linear subspace of $X \times Y$. Let $\|\cdot\|_X$ and $\|\cdot\|_Y$ be norms on X and Y respectively. Then, for $x \in X$ and $y \in Y$, $\|(x, y)\| := \|x\|_X + \|y\|_Y$ defines a norm on $X \times Y$. If X and Y are Banach spaces, then so is $X \times Y$.

Definition 1.4.3 *Let X and Y be normed linear spaces over $\mathbb{F}$.
A linear operator $T : X \to Y$ is closed if its graph $\mathcal{G}(T)$ is a closed linear subspace of $X \times Y$.*

Lemma 1.4.4 *Let X be Banach space Let T, S: $\mathcal{D}(T) = \mathcal{D}(S) \to X$, T bounded and S closed linear operators then for all λ, $\mu \in \mathbb{C}($ or $\mathbb{R})$ $\lambda T + \mu S$ is closed.*

Theorem 1.4.5 (Closed Graph Theorem) *Let X and Y be Banach spaces and $T : X \to Y$ a closed linear operator. Then T is bounded.*

Theorem 1.4.6 *Let X be Banach space. Assume $T : \mathcal{D}(T) \subset X \to X$ is closed and T^{-1} exists then T^{-1} is closed.*

Proof. Let $T : \mathcal{D}(T) \subset X \to X$ is closed i.e for arbitrary $x_n \in \mathcal{D}(T)$, with $x_n \to x$, and $Tx_n \to y$ we have $x \in \mathcal{D}(T)$ and $Tx = y$.
Assume T^{-1} exists, and let $y_n \in \mathcal{D}(T^{-1})$ with $y_n \to y_1$ and $T^{-1}y_n \to y_2$ we have to show that $y_1 \in \mathcal{D}(T^{-1})$ and $T^{-1}y_1 = y_2$. Since T^{-1} exists one can put $x_n = T^{-1}y_n \Longrightarrow y_n = Tx_n$. It follows that

1. $y_n = Tx_n \to y$ and then $y_1 = y$.
2. $T^{-1}y_n = x_n \to x$ and then $y_2 = x$.

Since T is closed one has $Tx = y$ which implies $T^{-1}y = x$ i.e $T^{-1}y_1 = y_2$,
We have also since $x \in \mathcal{D}(T)$ that $T^{-1}y = x \Longrightarrow y \in \mathcal{D}(T^{-1})$, i.e $y_1 \in \mathcal{D}(T^{-1})$.
Therefore T^{-1} is closed.

Remark 1.4.7

Definition 1.4.8 *Let T be an unbounded linear operator,*

1. *The spectrum of T, denoted by $\sigma(T)$ is defined by*

$$\sigma(T) = \{\lambda \in \mathbb{C} : (\lambda I - T)^{-1} \text{ is not bijectif } \}.$$

2. *The resolvent set of T, denoted by $\rho(T)$ is defined by*

$$\rho(T) = \mathbb{C} \backslash \sigma(T).$$

That is $\lambda \in \rho(T) \iff (\lambda I - T)^{-1}$ exists.

Example 1.4.9 *Consider $X = C([0,1];\mathbb{R})$, T is a linear operator defined by:*

$$\begin{cases} \mathcal{D}(T) &= \{f \in C^1([0,1];\mathbb{R})\} \\ Tf &= f' \end{cases}$$

Then $\sigma(T) = \mathbb{C}$.
Proof. Let $\lambda \in \mathbb{C}$,$\lambda \in \sigma_p(T) \Longrightarrow T\phi_\lambda = \lambda\phi_\lambda$.
$\phi_\lambda : s \mapsto e^{\lambda s}$. Therefore $\sigma_p(T) = \mathbb{C}$.
Let us change the domain

$$\mathcal{D}(T) = \{f \in C^1([0,1];\mathbb{R}) : f(1) = 0\}$$

Then $\sigma(T) = \emptyset$.
Proof. $\sigma(T) = \emptyset \iff \rho(T) = \mathbb{C}$.
Let $\lambda \in \mathbb{C} : \lambda \in \rho(T) \iff \forall g \in X,\ \exists!\ f \in \mathcal{D}(T)$ such that $\lambda f - f' = g$

$$\begin{cases} f'(x) &= & -\lambda f(x) + g(x) \qquad x \in [0,1] \\ f(1) &= & 0 \end{cases}$$

$$f(x) = e^{\lambda(x-1)} f(1) + \int_1^x e^{x(x-s)} g(s) ds,\ x \in [0,1]$$

It follows that

$$f(x) = \int_1^x e^{x(x-s)} g(s) ds,\ x \in [0,1].$$

Therefore $\forall g \in X,\ \exists! f \in \mathcal{D}(T)$ such that $\lambda f - Tf = g$.
Hence $\lambda \in \rho(T)$.

Theorem 1.4.10 *Let X be a banach space, $T : \mathcal{D}(T) \subset X \to X$ be a closed linear operator then the following are true:*

1. *$\rho(T)$ is open .*
2. *$\phi_\lambda : \rho(T) \to \mathcal{B}(X,X)$ such that $\lambda \mapsto R(\lambda;T) = (\lambda I - T)^{-1}$ is holomorphic.*
3. *$\lambda \in \rho(T)$ then $|R(\lambda;T)| \geq \dfrac{1}{d(\lambda, \sigma(T))}$*

Proof. Similar as in theorem (1.2.4).

Theorem 1.4.11 *Let $T : \mathcal{D}(T) \to X$ be a closed linear operator with a non empty resolvent set and let $\lambda \in \rho(T)$.*
Then

$$\sigma(R(\lambda;T)) \backslash \{0\} = [\lambda - \sigma(T)]^{-1};$$

where

$$[\lambda - \sigma(T)]^{-1} = \left\{ \frac{1}{\lambda - \mu} : \mu \in \sigma(T) \right\}.$$

Proof. Let $\mu \in \mathbb{C}$; $\lambda \in \rho(T)$, $x \in X$

$$\begin{aligned}(\mu - R(\lambda;T))(x) &= \mu[(\lambda - \frac{1}{\mu}) - T]R(\lambda;T)(x), \;\; x \in X \\ &= \mu R(\lambda;T)[(\lambda - \frac{1}{\mu}) - T](x), \;\; x \in \mathcal{D}(T).\end{aligned}$$

Let $x \neq 0$ such that

$$\begin{aligned}x \in Ker(\mu I - R(\lambda;T)) &\iff (\mu - R(\lambda;T))(x) = 0 \\ &\iff [(\lambda - \frac{1}{\mu}) - T]R(\lambda;T)(x) = 0.\end{aligned}$$

We get

- $Ker[\mu - R(\lambda;T)] = Ker[(\lambda - \frac{1}{\mu}) - T]$.
- $Im[\mu - R(\lambda;T)] = Im(\lambda - \frac{1}{\mu} - T)$.

We also have $\sigma(T) = \sigma_p(T) \cup \{\xi \in \mathbb{C} : \overline{Im(\xi I - T)} \neq X\}$.
One has

$$\mu \in \sigma_p(R(\lambda;T)) \iff \lambda - \frac{1}{\mu} \in \sigma_p(T)) \tag{1.4.1}$$

also,

$$\overline{Im(\mu - R(\lambda;T))} \neq X \iff \overline{Im(\lambda - \frac{1}{\mu} - T)} \neq X. \tag{1.4.2}$$

(1.4.1) and (2.2.1) yield.
So,
$\sigma(R(\lambda, T))\backslash\{0\} = (\lambda - \sigma(T))^{-1}$. ■

CHAPTER 2

Generation and Representation of Semigroup of linear operator

2.1 Definitions and Properties

Definition 2.1.1 *Let X be Banach space. A family $(T(t))_{t\geq 0}$ of bounded linear operators from X to X is a strongly continuous semigoup of bounded linear operators if*

(i) $T(0) = id_X$;

(ii) $T(t+s) = T(t)T(s),\ \forall t,\ s \geq 0$;

(iii) $\forall x \in X,\ \mathbb{R} \ni t \longmapsto T(t)x \in X$ *is continuous at 0.*

Remark 2.1.2 *A semigroup of bounded linear operators $(T(t))_{t\geq 0}$, is uniformly continuous if*

$$\lim_{t\to 0} \|T(t) - I\| = 0.$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0 semigroup.

If only (i) and (ii) are sastified we say that the family $(T(t))_{t\geq 0}$ of bounded linear operators is a semigroup.

Definition 2.1.3 *The linear operator A defined by*

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\} \tag{2.1.1}$$

and

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t} = \left.\frac{d^+T(t)x}{dt}\right|_{t=0} \text{ for } x \in \mathcal{D}(A) \tag{2.1.2}$$

is the infinitesimal generator of the semigroup $(T(t))_{t\geq 0}$; $\mathcal{D}(A)$ is the domain of A.

From the definition it is clear that if $(T(t))_{t\geq 0}$ is a uniformly continuous semigroup of bounded linear operators then $\underset{s\to t}{lim}\|T(s)-T(t)\|=0$.

Theorem 2.1.4 *Let $(T(t))_{t\geq 0}$ be a C_0 semigroup then there exist constants $\omega\in\mathbb{R}$ and $M\geq 1$, such that*

$$\|T(t)\|\leq Me^{\omega t}\quad for\quad 0\leq t<+\infty. \tag{2.1.3}$$

Proof. We show first there is $\eta\in(0,1]$ such that

$$\underset{t\in[0,\eta]}{Sup}\|T(t)\|<+\infty.$$

Assume the contrary, i.e $\forall\eta=\frac{1}{n}\in(0,1]$ with $n\in\mathbb{N}:\ \underset{t\in[0,\eta]}{Sup}\|T(t)\|=+\infty$.

It follows that

$$\forall n\in\mathbb{N},\ \exists\ t_n\in\left[0,\frac{1}{n}\right]\text{ such that }\underset{n\geq 1}{Sup}\|T(t_n)\|=+\infty.$$

By uniform boundedness principle $\exists x\in X$: $\underset{n\geq 1}{Sup}\|T(t_n)x\|=+\infty$, that is $\|T(t_n)x\|$ is unbounded.

On the other hand $\forall x\in X$, $\mathbb{R}\ni t\longmapsto T(t)x\in X$ is continuous at 0; that is

$\forall\epsilon>0,\ \exists\delta>0:|t|<\delta\Rightarrow\|\ T(t)x-x\ \|<\epsilon$.

In particular, let $\epsilon=1$.

Then,

$$\|\ T(t)x-x\ \|<1.$$

Hence we obtain the estimates:

$$\|\ T(t)x\ \|-\|\ x\ \|\leq|\|T(t)x\|-\|x\||\leq\|T(t)x-1\|<1.$$

This implies that

$$\|T(t)x\|\leq 1+\|x\|.$$

But one has $0\leq t_n\leq\frac{1}{n}$ and then $t_n\to 0$ as $n\to+\infty$ i.e

Take $\epsilon=\delta$,

$$\exists n_0\in\mathbb{N}:\left|t_n\right|<\delta;\ \forall n>n_0$$

then

$$\|T(t_n)x\|\leq 1+\|x\|,n>n_0;$$

it follows that

$$\underset{n\geq n_0}{Sup}\|T(t_n)x\|\leq 1+\|x\|,\ n>n_0. \tag{2.1.4}$$

Now let $n=1,2,\ldots,n_0-1$ there is only a finite number of $T(t_n)x$.

Let $M^*=max\|T(t_n)x\|,n=1,2,\ldots,n_0-1$. Then for these,

$$Sup\|T(t_n)x\| \leq M^* \quad \text{for} n = 1, 2, \ldots, n_0 - 1. \tag{2.1.5}$$

So from (2.1.4) and (2.1.5) we have $\underset{n\geq 1}{Sup}\|T(t_n)x\| \leq 1 + \|x\| + M^*$.

Hence we get the contradiction,
Thus,

$$\exists \eta \in (0, 1] : \underset{t\in[0,\eta]}{Sup}\|T(t)\| < +\infty.$$

Let $M := \underset{t\in[0,\eta]}{Sup}\|T(t)\|$, since $\|T(0)\| = 1$ then $M \geq 1$.
Let $\omega = \eta^{-1}logM$. Given $t \geq 0$ with $t > \eta$ we have $t = n(t)\eta + \delta$, where $0 \leq \delta < \eta$ and $n(t) \in \mathbb{N}$.
By semigroup property

$$\begin{aligned}
\|T(t)\| &= \|T(\eta)^{n(t)}T(\delta)\| \\
&\leq \|T(\eta)^{n(t)}\|\|T(\delta)\| \\
&= MM^{n(t)} = MM^{\frac{t-\delta}{\eta}} \\
&\leq MM^{\frac{t}{\eta}} = Me^{\omega\eta\frac{t}{\eta}} = Me^{\omega t}.
\end{aligned}$$

This completes the proof.

Theorem 2.1.5 *If $(T(t))_{t\geq 0}$ is a C_0 semigroup then $\forall x \in X$, $t \mapsto T(t)x$ is continuous from $\mathbb{R}^+$(the positive real line) into X.*

Proof. Let $t_0 > 0$, $x \in X$.
We want to show that $\lim_{t\to t_0} T(t)x = T(t_0)x$.
Case 1: $t > t_0$

$$\begin{aligned}
T(t)x - T(t_0)x &= T(t_0)[T(t - t_0)x - x] \\
\|T(t)x - T(t_0)x\| &\leq \|T(t_0)\|\|T(t - t_0)x - x\| \to 0 \qquad \text{as } t \to t_0.
\end{aligned}$$

Therefore, $\underset{t\to t_0^+}{lim}T(t)x = T(t_0)x$.

Case 2: $t < t_0$

$$\begin{aligned}
\|T(t)x - T(t_0)x\| &\leq \|T(t)[T(t_0 - t)x - x]\| \\
&\leq \|T(t)\|\|T(t_0 - t)x - x\| \\
&\leq Me^{\omega t}\|T(t_0 - t) - x\| \to 0 \text{ as } t \to t_0.
\end{aligned}$$

Therefore, $\underset{t\to t_0^-}{lim}T(t)x = T(t_0)x$.

Hence $\underset{t\to t_0}{lim}T(t)x = T(t_0)x$.

Theorem 2.1.6 *Let $(T(t))_{t\geq 0}$ be a C_0 semigroup and A be its infinitesimal generator. Then*

a) For $x \in X$,

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)xds = T(t)x; \tag{2.1.6}$$

b) For $x \in X$, $\int_0^t T(s)xds \in \mathcal{D}(A)$ and

$$A\Big(\int_0^t T(s)xds\Big) = T(t)x - x. \tag{2.1.7}$$

c) For $x \in \mathcal{D}(A)$, $T(t)x \in \mathcal{D}(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax; \tag{2.1.8}$$

d) For $x \in \mathcal{D}(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau. \tag{2.1.9}$$

Proof. a) Let $x \in X$ and $h > 0$; let's write the estimates

$$\begin{aligned} \left\|\frac{1}{h}\int_t^{t+h} T(s)xds - T(t)x\right\| &= \left\|\frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_t^{t+h} T(t)xds\right\| \\ &\leq \frac{1}{h}\int_t^{t+h} \Big\|T(s)x - T(t)x\Big\|ds. \end{aligned} \tag{2.1.10}$$

Changing the variable, set $u + t = s$, $du = ds$; if $s = t$ then $u = 0$ and if $s = t + h$ then $u = h$.

$$\frac{1}{h}\int_t^{t+h} \Big\|T(s)x - T(t)x\Big\| = \frac{1}{h}\int_0^h \Big\|T(u+t)x - T(t)x\Big\|du,$$

Since u is a dummy variable one can write

$$\frac{1}{h}\int_t^{t+h} \Big\|T(s)x - T(t)x\Big\| = \frac{1}{h}\int_0^h \Big\|T(s+t)x - T(t)x\Big\|ds.$$

Since $t \mapsto T(t)x$ is a continuous function from $\mathbb{R}^+$ to X i.e, given $\varepsilon > 0$, $\exists\delta > 0$ such that $|t - to| < \delta$ then $\|T(t)x - T(t_0)x\| \leq \varepsilon$.
Take $h = t_0 - t$, we can write the continuity of $t \mapsto T(t)x$ equivalentely as follows given $\varepsilon > 0$, $\exists\delta > 0$ such that $|h| < \delta$ then $\|T(t)x - T(t+h)x\| < \varepsilon$.

$$\frac{1}{h}\int_0^h \|T(t)x - T(t+s)x\|ds < \frac{1}{h}\int_0^h \varepsilon ds = \varepsilon.$$

It is then natural to write

$$\frac{1}{h}\int_t^{t+h}\Big\|T(s)x - T(t)x\Big\| = \frac{1}{h}\int_0^h \Big\|T(s+t)x - T(t)x\Big\|ds < \frac{1}{h}\int_0^h \varepsilon ds = \varepsilon.$$

Therefore

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)xds = T(t)x.$$

b) Let $x \in X$ and $h > 0$;

$$\begin{aligned}\frac{T(h)-I}{h}\int_0^t T(s)xds &= \frac{1}{h}\int_0^t T(s+h)xds - \frac{1}{h}\int_0^t T(s)xds \\ &= \frac{1}{h}\int_h^{h+t} T(s)xds - \frac{1}{h}\int_0^t T(s)xds.\end{aligned}$$

In the right hand side we have set for the first integral $u = s + h$; $du = ds$; if $s = 0$ then $u = h$ and if $s = t$ then $u = t + h$.

$$\begin{aligned}\frac{1}{h}\int_h^{h+t} T(s)xds - \frac{1}{h}\int_0^t T(s)xds &= \frac{1}{h}\int_h^t T(s)xds + \frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_0^t T(s)xds \\ &= \frac{1}{h}\int_0^h T(s)xds + \frac{1}{h}\int_0^t T(s)xds \\ &+ \frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_0^t T(s)xds \\ &= \frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_0^h T(s)xds.\end{aligned}$$

and letting $h \to 0$ the right-hand side tends to $T(t)x - x \in X$, which proves $b)$

c) Let $x \in \mathcal{D}(A)$ and $h > 0$; then

$$\begin{aligned}\frac{T(h)-I}{h}T(t)x &= \frac{T(h+t)-T(t)}{h}x \\ &= T(t)\frac{T(h)-I}{h}x \to T(t)Ax \text{ as } h \to 0. \end{aligned} \tag{2.1.11}$$

Thus, $T(t)x \in \mathcal{D}(A)$ and $AT(t)x = T(t)Ax$. (2.1.11) implies also that

$$\frac{d^+}{dt}T(t)x = AT(t)x = T(t)Ax,$$

i.e the right derivative of $T(t)x$ is $T(t)Ax$ to prove (2.1.8) we have to show that for $t > 0$ the left derivative of $T(t)x$ exists and equals $T(t)Ax$.

This follows from,

$$\lim_{h\to 0}\left[\frac{T(t)x-T(t-h)x}{h}-T(t)Ax\right]=\lim_{h\to 0}T(t-h)\left[\frac{T(h)x-x}{h}-Ax\right]+\lim_{h\to 0}\Big(T(t-h)Ax-T(t)Ax\Big)$$

and the fact that both terms on the right-hand side are zero, the first since $x\in\mathcal{D}(A)$ and $\|T(t-h)\|$ is bounded on $0\le h\le t$ and the second by continuity of $T(t)$.
This conclude the proof of c).

d) Integrating (2.1.8) from s to t we obtain d).

Corollary 2.1.7 *If A is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\ge 0}$ then $\mathcal{D}(A)$ the domain of A, is dense in X and A is closed linear operator.*

Proof. Let $x\in X$, set $x_t:=\frac{1}{t}\int_0^t T(s)xds$. By part c) of Theorem 2.1.6, $x_t\in\mathcal{D}(A)$ for $t>0$ and by part a) of the same theorem $x_t\to x$ as $t\to 0$. Thus $\overline{\mathcal{D}(A)}=X$.

Let $(x,y)\in\overline{\mathcal{G}(A)}$ then there exist $(x_n)_{n\ge 1}\subset\mathcal{D}(A)$ such that $(x_n,Ax_n)\to(x,y)$ i.e $x_n\to x$ and $Ax_n\to y$.
By part b) of Theorem 2.1.6, we have

$$T(t)x_n-x_n=\int_0^t T(s)Ax_nds \tag{2.1.12}$$

Claim: $\int_0^t T(s)Ax_nds\to\int_0^t T(s)yds$ uniformly on bounded interval.
Let $t\in[0,a]$ with $a>0$;

$$\begin{aligned}\|\int_0^t T(s)Ax_nds-\int_0^t T(s)yds\| &\le \int_0^t\|T(s)(Ax_n-y)\|ds\\ &\le \int_0^t\|T(s)\|\|Ax_n-y\|ds\\ &\le Me^{\omega t}t\|Ax_n-y\|.\end{aligned}$$

Since $Ax_n\to y$, it follows that

$$\lim_{n\to+\infty}\sup_{t\in[0,a]}\|\int_0^t T(s)Ax_nds-\int_0^t T(s)yds\|=0,$$

therefore our claim is true.
Using the previous claim and letting $n\to+\infty$ in (2.1.12) yields

$$T(t)x-x=\int_0^t T(s)yds. \tag{2.1.13}$$

Dividing (2.1.13) by $t>0$ and letting $t\to 0$, we see, using part a) of Theorem 2.1.6 that $x\in\mathcal{D}(A)$ and $Ax=y$.

Theorem 2.1.8 *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Proof. a) It is known that the series $\sum_{n=0}^{\infty} \dfrac{(tA)^n}{n!}$ in norm for every $t \geq 0$ and defines for each such t a bounded linear operator $T(t)$.
It is easy to see that

- $T(0) = I$
- $T(t+s) = T(t)T(s)$, for all $t,\ s \geq 0$
-

$$\begin{aligned} e^{tA} &= \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = I + \sum_{n=1}^{\infty} \frac{(tA)^n}{n!} \\ e^{tA} - I &= tA \sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{n(n-1)!} \end{aligned}$$

Taking the norm of both side,one has

$$\begin{aligned} \|e^{tA} - I\| &\leq \|tA\| \| \sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{n(n-1)!} \| \\ &\leq |t| \|A\| \| \sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{(n-1)!} \\ &\leq t\|A\| e^{t\|A\|}. \end{aligned}$$

That is $\|T(t) - I\| \leq t\|A\|e^{t\|A\|}$ which goes to 0 as t goes to 0.

Now, we claim that A is the infinitesimal generator of $T(t)$.
Let us prove our claim, let $t > 0$.
We have

$$\begin{aligned} e^{tA} - I &= tA \sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{n(n-1)!} \\ \frac{e^{tA} - I}{t} &= A \sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{n(n-1)!} \\ \frac{e^{tA} - I}{t} - A &= A \left[\sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{n(n-1)!} - I \right]. \end{aligned}$$

Taking the norm of both side, one has

$$\begin{aligned} \| \frac{e^{tA} - I}{t} - A \| &\leq \|A\| \| \sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{n(n-1)!} - I \|. \\ &\leq \|A\| \| \sum_{n=1}^{\infty} \frac{(tA)^{n-1}}{(n-1)!} - I \| = \|A\| \|e^{tA} - I\|. \end{aligned}$$

That is $\|\dfrac{T(t)-I}{t} - A\| \leq \|A\|\|T(t)-I\|$.
Which implies as $t \to 0^+$ that $\lim_{t\to 0^+} \dfrac{T(t)-I}{t} = A$
We have have established that $T(t)$ is a uniformly continuous semgroup of bounded linear operators on X and that A is its infinitesimal generator.

b) Let $T(t)$ be a C_0 semigroup of bounded linear operator on X.
Fix $\rho > 0$, small enough such that $\left\| I - \rho^{-1}\int_0^\rho T(s)ds \right\| \leq 1$ this implies that $\rho^{-1}\int_0^\rho T(s)ds$ is invertible and therefore $\int_0^\rho T(s)ds$ is invertible .

Now,Let $h > 0$,
$h^{-1}(T(h)-I)\int_0^\rho T(s)ds = h^{-1}(\int_0^\rho T(s+h)ds - \int_0^\rho T(s)ds = h^{-1}(\int_\rho^{\rho+h} T(s)ds - \int_0^\rho T(s)ds)$
and therefore
$h^{-1}(T(h)-I) = \left(h^{-1}\int_\rho^{\rho+h} T(s)ds - h^{-1}\int_0^h T(s)ds\right)\left(\int_0^\rho T(s)ds\right)^{-1}$ and letting $h \to 0$
it follows that $h^{-1}(T(h)-I)$ converges in norm to a bounded linear operator
$(T(\rho)-I)\left(\int_0^\rho T(s)ds\right)^{-1}$ which is the infinitesimal generator of $T(t)$. ■

Theorem 2.1.9 *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be two C_0 semigroup on X, generated respectively by A and B.*
If $A = B$ then $(T(t) = S(t)$, $t \geq 0$.

Proof. Assume $A = B$ and let $x \in \mathcal{D}(A) = \mathcal{D}(B)$.
Define $\alpha(s) := T(t-s)S(s)x$, $s \in [0,t]$.
From theorem(2.1.6)part (c) it follows that α is differentiable and that
$\alpha'(s) = \frac{d}{ds}T(t-s)S(s)x = -T(t-s)AS(s) + T(t-s)BS(s)x = 0$, since $A = B$.
It follows $\alpha(s) = constant$. In particular, its values at $s = 0$ and $s = t$ are the same that is $T(t)x = S(s)x \quad \forall x \in \mathcal{D}(A)$. By corollary(2.1.7) $\mathcal{D}(A)$ is dense in X, and $T(t)$, $S(s)$ are bounded ;
therefore $T(t)x = S(s)x$, $\forall x \in X$.

Definition 2.1.10 *$(T(t))_{t\geq 0}$ is a C_0 semigroup of contraction if and only if*

$$\|T(t)\| \leq 1, \ \forall t \geq 0.$$

2.2 Hille-Yosida Theorem

Theorem 2.2.1 (Hille-Yosida) *A linear (unbounded) operator A is the infinitesimal generator of a C_0 semigroup of contraction if and only if :*

(i) A is closed and $\overline{\mathcal{D}(A)} = X$.

(ii) The resolvent set of A, $\rho(A)$ contains $\mathbb{R}_*^+$ and for every $\lambda > 0$

$$\|R(\lambda; A)\| \leq \frac{1}{\lambda}.$$

Proof of the necessary condition. If A is the infinitesimal generator of C_0 semigroup then by corollary(2.1.7), A is closed and $\overline{\mathcal{D}(A)} = X$. Therefore (i) is proved.
Now, for $\lambda > 0$, define $R(\lambda) = \int_0^{+\infty} e^{-\lambda t} T(t) dt$.
The integral is well defined since $\| T(t) \| \leq 1$.

$$\begin{aligned} \| R(\lambda) \| = \| \int_0^{+\infty} e^{-\lambda t} T(t) dt \| &\leq \int_0^{+\infty} \| e^{-\lambda t} T(t) \| dt \\ &\leq \int_0^{+\infty} \| e^{-\lambda t} \| \| T(t) \| dt \\ &= \lim_{\eta \to +\infty} \int_0^{\eta} e^{-\lambda t} dt = \lim_{\eta \to +\infty} - \frac{e^{-\lambda t}}{\lambda} \Big|_0^{\eta} = \frac{1}{\lambda}. \end{aligned}$$

Then $\| R(\lambda) \| \leq \frac{1}{\lambda}$. It follows $R(\lambda) \in \mathcal{B}(X, X)$.

Claim : $R(\lambda) = R(\lambda; A) = (\lambda I - A)^{-1}$.
We have to show that $R(\lambda)(\lambda I - A) = Id_{\mathcal{D}(A)}$ and $(\lambda I - A)R(\lambda) = Id_X$.
Let us show that $R(\lambda)x \in \mathcal{D}(A)$.
Let $x \in \mathcal{D}(A)$ and $h > 0$,

$$\begin{aligned} \frac{T(h) - I}{h} R(\lambda)x &= \frac{T(h) - I}{h} \int_0^{+\infty} e^{-\lambda t} T(t) x dt \\ &= \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t + h) x dt - \frac{1}{h} \int_0^{+\infty} e^{-\lambda t} T(t) x dt \end{aligned}$$

Let $t + h = u$ then $dt = du$ and $t = u - h$;
it follows that

$$\begin{cases} u = h & \text{if } t = 0 \\ u = +\infty & \text{if } t = +\infty \end{cases}$$

$$\frac{1}{h} \int_0^{+\infty} e^{-\lambda(u-h)} T(u) x du = \frac{1}{h} \int_0^{+\infty} e^{-\lambda(t-h)} T(t) x dt.$$

Since u is a dummy variable.

$$\begin{aligned}\frac{1}{h}\int_0^{+\infty} e^{-\lambda t}T(t+h)xdt - \frac{1}{h}\int_0^{+\infty} e^{-\lambda t}T(t)xdt &= \frac{1}{h}\int_0^{+\infty} e^{-\lambda(t-h)}T(t)xdt \\ &- \frac{1}{h}\int_0^{+\infty} e^{-\lambda t}T(t)xdt \\ &= \frac{1}{h}\int_0^{+\infty}\Big(e^{-\lambda t}e^{\lambda h}T(t) - e^{-\lambda t}T(t)\Big)xdt \\ &- \frac{1}{h}\int_0^{h} e^{-\lambda t}e^{\lambda h}T(t)xdt \\ &= \frac{(e^{\lambda h}-1)}{h}\int_0^{+\infty} e^{-\lambda t}T(t)xdt \\ &- \frac{e^{\lambda h}}{h}\int_0^{h} e^{-\lambda t}T(t)xdt\end{aligned}$$

and as $h \to 0^+$ the right hand side tends to $\lambda R(\lambda)x - x$.
Then

$$\lim_{h\to 0^+}\frac{T(h)-I}{h}R(\lambda)x = AR(\lambda)x = \lambda R(\lambda)x - x.$$

One can write

$$AR(\lambda)x = R(\lambda)x - x \Longrightarrow (\lambda I - A)R(\lambda) = I_{\mathcal{D}(A)}. \tag{2.2.1}$$

Let $x \in \mathcal{D}(A)$,

$$\begin{aligned}R(\lambda)Ax &= \int_0^{+\infty} e^{-\lambda t}T(t)(Ax)dt = \int_0^{+\infty} e^{-\lambda t}AT(t)xdt \\ &= A\int_0^{+\infty} e^{-\lambda t}T(t)xdt = AR(\lambda)x\end{aligned}$$

Here we have used the theorem 2.1.6(c) and the closedness of A.
Using (2.2.1), we have

$$\begin{aligned}R(\lambda)Ax = \lambda R(\lambda)x - x &\Longrightarrow \lambda R(\lambda)x - R(\lambda)Ax = x \\ &\Longrightarrow R(\lambda)(\lambda I - A)x = x \\ &\Longrightarrow R(\lambda)(\lambda I - A) = Id_X \end{aligned}\tag{2.2.2}$$

then it follows from (2.2.1) and (2.2.2) that $R(\lambda) = R(\lambda; A)$.
We conclude that $\lambda > 0$, $R(\lambda) = R(\lambda; A)$ and $\|R(\lambda; A)\| \leq \frac{1}{\lambda}$.

In order to prove that the conditions (i) and (ii) of theorem 2.2.1 are sufficient for A to be the infinitesimal generator of a C_0 semigroup of contraction we will need some lemmas.

Lemma 2.2.2 *Let A satisfy the condition (i) and (ii) of theorem (2.2.1) and $R(\lambda; A) = (\lambda I - A)^{-1}$. Then*

$$\lim_{\lambda \longrightarrow +\infty} \lambda R(\lambda; A)x = x, \ \forall x \in X.$$

Proof. Suppose first that $x \in \mathcal{D}(A)$
One can write $R(\lambda; A)(\lambda I - A)x = x$ which implies $\lambda R(\lambda; A)x - x = R(\lambda; A)Ax$
It follows that

$$\begin{aligned} \|\lambda R(\lambda; A)x - x\| &= \|R(\lambda; A)Ax\| && (2.2.3) \\ &\leq \frac{1}{\lambda}\|Ax\| \longrightarrow 0 \text{ as } \lambda \longrightarrow +\infty. && (2.2.4) \end{aligned}$$

But $\mathcal{D}(A)$ is dense in X and $\|\lambda R(\lambda; A)\| \leq 1$. Therefore $\lambda R(\lambda; A)x \longrightarrow x$ as $\lambda \longrightarrow +\infty$ for every $x \in X$.
We now define, for every $\lambda > 0$, the *Yosida Approximation* of A by

$$A_\lambda = \lambda A R(\lambda; A) = \lambda^2 R(\lambda; A) - \lambda I. \tag{2.2.5}$$

Lemma 2.2.3 *Let A satisfy the condition (i) and (ii) of theorem (2.2.1). If A_λ is the Yosida Approximation of A, then*

$$\lim_{\lambda \longrightarrow +\infty} A_\lambda x = Ax, \quad \forall x \in \mathcal{D}(A).$$

Proof. Let $x \in \mathcal{D}(A)$;
$A_\lambda x = \lambda A R(\lambda; A)x = \lambda R(\lambda; A)Ax$, since the resolvent and the operator commutent.
As λ goes to $+\infty$ the right hand side converge to Ax by lemma 2.2.2.This completes the proof.

Lemma 2.2.4 *Let A satisfy the condition (i) and (ii) of theorem (2.2.1). If A_λ is the Yosida Approximation of A, then A_λ is the infinitesimal generator of a C_0 semigroup of contraction e^{tA_λ}. Futhermore,for every $x \in X$, λ, $\mu > 0$ we have*

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leq t\|A_\lambda x - A_\mu x\| \ ; \ t \geq 0$$

Proof of lemma 2.2.4. From (2.2.5) it is clear that A_λ is a bounded linear operator and thus it is the infinitesimal generator of a C_0 semigroup e^{tA_λ} of bounded linear operators (see theorem (2.1.8).
Claim:

$$\|e^{tA_\lambda}\| \leq 1 \ ; \quad \forall t \geq 0.$$

Proof.

$$\begin{aligned} e^{tA_\lambda} &= e^{(\lambda^2 t R(\lambda;A) - \lambda t Id_X)} \\ &= e^{t\lambda^2 R(\lambda;A)} e^{-\lambda t Id_X} \end{aligned}$$

Taking the norm, one has

$$\|e^{tA_\lambda}\| = e^{-\lambda t}\|e^{\lambda^2 tR(\lambda;A)}\| \leq e^{-\lambda t}e^{\lambda^2 t\|R(\lambda;A)\|} \leq 1$$

And therefore e^{tA_λ} is a C_0 semigroup of contraction. It is clear from the definition that e^{tA_λ}, e^{tA_μ}, A_λ and A_μ commute with each other.
By Mean value theorem, we have

$$e^{tA_\lambda}x - e^{tA_\mu}x = \int_0^1 \frac{d}{ds}\left(e^{tsA_\lambda}e^{t(1-s)A_\mu}x\right)ds,$$

on the other hand

$$\begin{aligned} \frac{d}{ds}e^{tsA_\lambda}e^{t(1-s)A_\mu}x &= tA_\lambda e^{tsA_\lambda}e^{t(1-s)A_\mu}x - tA_\mu e^{tsA_\lambda}e^{t(1-s)A_\mu}x \\ &= t(A_\lambda x - A_\mu x)e^{tsA_\lambda}e^{t(1-s)A_\mu}, \end{aligned}$$

it follows that

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leq t\|A_\lambda x - A_\mu x\|. \qquad \blacksquare$$

Proof of theorem 2.2.1 (Sufficient Condition).
Let $x \in \mathcal{D}(A)$. Then

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leq t\|A_\lambda x - A_\mu x\| \leq t\|A_\lambda x - Ax\| + t\|Ax - A_\mu x\|. \tag{2.2.6}$$

Claim: For $x \in X$,

$$\lim_{\lambda \longrightarrow +\infty} e^{tA_\lambda}x = T(t)x \ , \ \forall t \geq 0. \tag{2.2.7}$$

Proof. Let $x \in \mathcal{D}(A)$, by (2.2.6) and lemma 2.2.3 we get that $(e^{tA_\lambda}x)_{\lambda>0}$ is a cauchy sequence for all $x \in \mathcal{D}(A)$.
That is given $\epsilon > 0$ $\exists\delta > 0$ such that $\|e^{tA_\lambda}x - e^{tA_\mu}x\| < \frac{\epsilon}{3}$ $\forall\lambda,\ \mu > \delta$, and $\forall x \in \mathcal{D}(A)$.
Let's denote its limit by $T(t)x$ i.e $T(t)x := \lim_{\lambda \longrightarrow +\infty} e^{tA_\lambda}x$.
Since $\mathcal{D}(A)$ is dense in X we can write,
let $x \in X$ be arbitrary, given $\epsilon > 0$, $\exists x_0 \in \mathcal{D}(A)$ such that $\|x - x_0\| < \frac{\epsilon}{3}$.
We have

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leq \|e^{tA_\lambda}x - e^{tA_\lambda}x_0\| + \|e^{tA_\lambda}x_0 - e^{tA_\mu}x_0\| + \|e^{tA_\mu}x_0 - e^{tA_\mu}x\|;\ \lambda\ ,\ \mu > \delta$$

consequently

$$\begin{aligned} \|e^{tA_\lambda}x - e^{tA_\mu}x\| &\leq \|x - x_0\| + \frac{\epsilon}{3} + \|x - x_0\|;\ \lambda,\ \mu > \delta \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon;\ \lambda,\ \mu > \delta \end{aligned}$$

That is, let $x \in X$
$\forall\epsilon > 0$, $\exists\delta > 0$ such that $\|e^{tA_\lambda}x - e^{tA_\mu}x\| < \epsilon;\ \lambda,\ \mu > \delta$.

We get that $(e^{tA_\lambda}x)_{\lambda>0}$ is a cauchy sequence, $\forall x \in X$, on the other hand $\|e^{tA_\lambda}\| \leq 1;\ \forall t \geq 0$ (obviously $e^{tA_\lambda} \in \mathcal{B}(X,X)$) . Therefore,

$$\lim_{\lambda \longrightarrow +\infty} e^{tA_\lambda}x = T(t)x,\ \forall t \geq 0,\ \forall x \in X.$$

Then by Banach Steinhaus Theorem we get $T(t) \in \mathcal{B}(X,X)$.
Claim: Let $\lambda > 0$ and $x \in X$.
Then $\lim\limits_{\lambda \longrightarrow +\infty} e^{tA_\lambda}x = T(t)x$ is uniform with respect to t in each compact set of $\mathbb{R}^+$.
Proof. We have $(e^{tA_\lambda}x)_{\lambda>0}$ is a cauchy sequence, $\forall x \in X$ and $t \geq 0$.
Let $\lambda,\ \mu > 0$ and $x \in X$.
Let $\epsilon > 0$ be given, $\exists\ \delta > 0$ such that

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leq \epsilon;\ t \in [0;\alpha],\ \forall \alpha > 0;\ \lambda,\ \mu > \delta.$$

By uniform Cauchy criterion $e^{tA_\lambda}x$ converge uniformly in $[0;\alpha]$ for all $\alpha > 0$.
This completes the proof.

Let now check if the limit $T(t)$ is a C_0 semigroup of contraction .

(a) $T(0)x = \lim\limits_{\lambda \longrightarrow +\infty} e^0 x = x$ which implies $T(0) = I$ and since $\|T(t)x\| \leq \|T(t)\|\|x\|$, then $\|T(t)\| \leq 1$.

(b) $T(t+s)x = \lim\limits_{\lambda \longrightarrow +\infty} e^{(t+s)A_\lambda}x = \lim\limits_{\lambda \longrightarrow +\infty} e^{tA_\lambda}e^{sA_\lambda}x = T(t)T(s)$.

(c) $t \longmapsto T(t)x$ is continuous for $t \geq 0$ as a uniform limit of the continuous functions $t \longmapsto e^{tA_\lambda}x$.

Thus $T(t)$ is a C_0 semigroup of contraction on X.
To conclude the proof we will show that A is the infinitesimal generator of $T(t)$.
Let $x \in \mathcal{D}(A)$, then using (2.2.7) and theorem 2.1.6 part(d) we have

$$T(t)x - x = \lim_{\lambda \longrightarrow +\infty} (e^{tA_\lambda}x - x) = \lim_{\lambda \longrightarrow +\infty} \int_0^t e^{sA_\lambda}A_\lambda x ds = \int_0^t T(s)Axds \tag{2.2.8}$$

The last equality follows from the uniform convergence of $e^{tA_\lambda}A_\lambda x$ to $T(t)Ax$ on bounded intervals i.e

$$\begin{aligned} \|A_\lambda e^{sA_\lambda}x - T(s)Ax\| &\leq \|e^{sA_\lambda}A_\lambda x - e^{sA_\lambda}Ax\| + \|e^{sA_\lambda}Ax - T(s)Ax\| \\ &\leq e^{sA_\lambda}\|A_\lambda x - Ax\| + \|e^{sA_\lambda}Ax - T(s)Ax\|. \end{aligned}$$

As λ goes to zero, the right hand side goes to zero.
Therefore $\mathop{Sup}\limits_{s\in[0,t]} \|A_\lambda e^{sA_\lambda}x - T(s)Ax\| = 0$.

Let B be the infinitesimal generator of $(T(t))_{t\geq 0}$ and let $x \in \mathcal{D}(A)$.
Claim: $A = B$
Dividing 2.2.8 by $t > 0$ and letting $t \longrightarrow 0$ we see that $x \in \mathcal{D}(B)$ and that $Bx = Ax$. Thus $\mathcal{D}(A) \subseteq \mathcal{D}(B)$.
Since B is the infinitesimal generator of a C_0 semigroup of contractions $T(t)$ it follows from the necessary condition that $1 \in \rho(B)$.
On the other hand we assume assumption (ii) that $1 \in \rho(A)$.
Since $\mathcal{D}(B) \supseteq \mathcal{D}(A)$, $(I - B)\mathcal{D}(A) = (I - A)\mathcal{D}(A) = X$ (B and A are equivalente in $\mathcal{D}(A)$ and $I - A$ is bijective) then,
$(I - B)\mathcal{D}(A) = X$ which implies $(I - B)^{-1}X = \mathcal{D}(A)$ and therefore $A = B$, that is A is the infinitesimal generator of a C_0 semigroup of contraction.
This completes the proof.

2.3 The Lumer-phillips Theorem

We want to see an other caracterization of an infinitesimal generator of a C_0 semigroup.
In other to state and prove the result we need some preliminaries.
Let X be a Banach space and let X^* be its dual. We denote the value $x^* \in X$ at $x \in X$ by $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$. For every $x \in X$ we define the duality set $F(x) \subseteq X^*$ by

$$F(x) = \{x^* : x^* \in X^* \text{ and } \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}.$$

From Hahn-Banach theorem it follows that $F(x) \neq \emptyset$ for every x.

Definition 2.3.1 *A linear operator A is dissipative if for every $x \in \mathcal{D}(A)$ there is a $x^* \in F(x)$ such that $Re\langle Ax, x^*\rangle \leq 0$.*

A useful caracterisation of dissipative operators is given by the following theorem

Theorem 2.3.2 *A linear Operator $A : \mathcal{D}(A) \subseteq X \longrightarrow X$ is dissipative if and only if*

$$\|(\lambda I - A)x\| \geq \lambda\|x\| \text{ for all } x \in \mathcal{D}(A) \text{ and } \lambda > 0. \tag{2.3.1}$$

Proof. Let A be dissipative, $\lambda > 0$ and $x \in \mathcal{D}(A)$. If $x^* \in F(x)$ and $Re\langle Ax, x^*\rangle \leq 0$ then $\|\lambda x - Ax\|\|x\| \geq |\langle \lambda x - Ax, x^*\rangle| \geq \langle \lambda x - Ax, x^*\rangle \geq \lambda\|x\|^2$ and the conclusion follows at once.
Conversely, let $x \in \mathcal{D}(A)$ and assume that $\lambda\|x\| \leq \|\lambda x - Ax\|$ for all $\lambda > 0$. If $y_\lambda^* \in F(\lambda x - Ax)$ and $z_\lambda^* = \dfrac{y_\lambda^*}{\|y_\lambda^*\|}$ then $\|z_\lambda^*\| = 1$ and

$$\begin{aligned} \lambda\|x\| &\leq \|\lambda x - Ax\| = \langle \lambda x - Ax, z_\lambda^*\rangle \\ &= \lambda Re\langle x, z_\lambda^*\rangle - Re\langle Ax, z_\lambda^*\rangle \leq \lambda\|x\| - Re\langle Ax, z_\lambda^*\rangle \end{aligned}$$

for every $\lambda > 0$. Therefore

$$Re\langle Ax, z_\lambda^*\rangle \leq 0 \text{ and } Re\langle x, z_\lambda^*\rangle \geq \|x\| - \frac{1}{\lambda}\|Ax\|. \tag{2.3.2}$$

Since the unit ball of X^* is compact in the weak-star topology of X^* the net z^*_λ, $\lambda \to +\infty$, has a weak-star cluster point $z^* \in X^*$, $\|z^*\| \leq 1$. From (2.3.2) it follows that $Re\langle Ax, z^*\rangle \leq 0$ and $Re\langle x, z^*\rangle \geq \|x\|$. But $Re\langle x, z^*\rangle \leq |\langle x, z^*\rangle| \leq \|x\|$ and therefore $\langle x, z^*\rangle = \|x\|$. Taking $x^* = \|x\|z^*$ we have $x^* \in F(x)$ and $Re\langle Ax, x^*\rangle \leq 0$. Thus for every $x \in \mathcal{D}(A)$ there is an $x^* \in F(x)$ such that $Re\langle Ax, x^*\rangle \leq 0$ and A is dissipative. ■

Remark 2.3.3 *Let A be dissipative, $\lambda \in \rho(A)$ such that $\lambda > 0$ and $x \in \mathcal{D}(A)$.*
$\|x\| = \|(\lambda I - A)R(\lambda, A)x\| \geq \lambda\|R(\lambda, A)x\|$,
which implies

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}$$

$\rho(A) \neq \emptyset$ implies A closed then A is a generator of a C_0 semigroup of contractions.

Theorem 2.3.4 (Lumer-Phillips) *Let A be densily define on some Banach space X.*

(a) If A is dissipative and $\exists\lambda_0 > 0$ such that $Im(\lambda_0 I - A) = X$, then A is the generator of contractions on X.

(b) If A is the infinitesimal generator of a C_0 semigroup of contractions on X then $Im(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x \in \mathcal{D}(A)$ and $x^* \in F(x)$,

Proof. Let λ the dissipativeness of A implies Theorem 2.3.2 that

$$\|(\lambda I - A)x\| \geq \lambda\|x\| \text{ for every } \lambda > 0 \textit{ and } x \in \mathcal{D}(A). \tag{2.3.3}$$

Since $Im(\lambda_0 I - A) = X$, it follows from (2.3.3) with $\lambda = \lambda_0$ that $(\lambda_0 I - A)^{-1}$ is bounded linear operator and thus closed.
But then $\lambda_0 I - A$ is closed and therefore also A is closed. If $R(\lambda I - A) = X$ for every $\lambda > 0$ then $\rho(A) \supseteq]0, +\infty]$ and $\|R(\lambda, A)\| \leq \dfrac{1}{\lambda}$ by (2.3.3). It then follows from the Hille-Yosida theorem that A is the infinitesimal generator of a C_0 semigroup of contractions on X.
To complete the proof of (a) it remains to show that $Im(\lambda I - A) = X$ for all $\lambda > 0$. Consider the set
$\Lambda = \{\lambda : \lambda > 0 \ + \text{and } Im(\lambda I - A) = X\}$
We have to show that

1. Λ is open .
2. Λ is closed.
3. $\lambda_0 \in \Lambda \neq \emptyset$

Let $\lambda \in \Lambda$.By (2.3.3), $\lambda \in \rho(A)$.Since $\rho(A)$ is open, $\exists O$ a neighborhood of λ such that $O \subseteq \rho(A)$.
It follows that $O\cap]0, +\infty[\subseteq \Lambda$.Therefore Λ is open.

On the other hand, let $\lambda_n \in \Lambda$ such that $\lambda_n \to \lambda > 0$ we have to show that $\lambda \in \Lambda$ $\lambda_n \in \Lambda$ implies $R(\lambda_n, A)$ exists. Let $y \in X$ there exists an $x_n \in \mathcal{D}(A)$ such that

$$\lambda_n x_n - Ax_n = y. \tag{2.3.4}$$

From(2.3.1), it follows that$\|x_n\| \leq \frac{1}{\lambda}\|y\| \leq C$ for some $C > 0$. Now,
Claim: $(x_n)_{n\geq 0}$ is a cauchy sequence.
Let's prove our claim.
$\lambda_m\|x_n - x_m\| \leq \|\lambda_m(x_n - x_m) - A(x_n - x_m)\|$, since A is dissipative.
One has $\lambda_n x_n - Ax_n = y$ and $\lambda_m x_m - Ax_m = y$, it follows that $\lambda_n x_n - Ax_n = \lambda_m x_m - Ax_m$ and also $A(x_n - x_m) = \lambda_n x_n - \lambda_m x_m$.
Then ,

$$\begin{aligned}
\lambda_m\|x_n - x_m\| &\leq \|\lambda_m(x_n - x_m) - (\lambda_n x_n - \lambda_m x_m)\| \\
&= \|\lambda_m x_n - \lambda_n x_n\| \\
&= |\lambda_n - \lambda_m|\|x_n\| \\
&\leq C|\lambda_n - \lambda_m|.
\end{aligned}$$

Therefore $(x_n)_{n\geq 0}$ is cauchy.

Let $x_n \to x$. Then by (2.3.4) $Ax_n \to \ \lambda x - y$.
Since A is closed, $x \in \mathcal{D}(A)$ and $\lambda x - Ax = y$. Therefore $Im(\lambda I - A) = X$ and $\lambda \in \Lambda$. Thus Λ is also closed in $]0, +\infty[$ and since $\lambda_0 \in \Lambda$ by assumption then $\Lambda \neq \emptyset$ and therefore $\Lambda =]0, +\infty]$. This complete the proof of (a).

If A is the infinitesimal generator of C_0 semigroup of contractions, $T(t)$ on X, then by the Hille-Yosida theorem $\rho(A) \supseteq]0, +\infty]$ and therefore $Im(\lambda I - A) = X$ for all $\lambda > 0$. Futhermore, if $x \in \mathcal{D}(A)$, $x^* \in F(x)$ then

$$|\langle T(t)x, x^*\rangle| \leq \|T(t)x\|\|x^*\| \leq \|x\|^2.$$

and therefore,

$$Re\langle T(t)x - x, x^*\rangle = Re\langle T(t)x, x^*\rangle - \|x\|^2 \leq 0 \tag{2.3.5}$$

Dividing (2.3.5) by $t > 0$ and letting $t \to 0$ yields

$$Re\langle Ax, x^*\rangle \leq 0.$$

This holds for every $x^* \in F(x)$ and the proof is complete.

2.4 The Characterization of the Infinitesimal Generators of general C_0 semigroup

We want to turn now to the characterization of the infinitesimal generators of general C_0 semigroup of bounded operators. From Theorem (2.1.4) it follows that for such semigroups there exist real constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$. We will show that in other to to characterize the infinitesimal generator in general case it suffices to characterize the infinitesimal generators of uniformly bounded C_0 semigroups. This will be done renorming the Banach space X so that the uniformly bounded C_0 semigroup becomes, in the new norm, a C_0 semigroup of contractions and then using the previously proved characterizations of the infinitesimal generators of semigroups of contractions.

We start by stating the renorming lemma.

Lemma 2.4.1 *Let A be any linear operator for which $\rho(A) \supseteq]0, +\infty[$.*
If

$$\|R(\lambda; A)^n\| \leq \frac{M}{\lambda^n}, \ \forall \lambda > 0, \ \forall n \geq 1 \tag{2.4.1}$$

then there exists a norm $|\cdot|$ on X which is equivalent to the original norm $\|\cdot\|$ on X and satisfies:

$$\|x\| \leq |x| \leq M\|x\| \text{ for } x \in X \tag{2.4.2}$$

and

$$|R(\lambda; A)x| \leq \frac{|x|}{\lambda}; \ x \in X, \ \lambda > 0. \tag{2.4.3}$$

Theorem 2.4.2 *A linear operator A is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geq 0}$, satisfying $\|T(t)\| \leq M$ $(M \geq 1)$, if and only if*

(i) A is closed and $\mathcal{D}(A)$ is dense in X.

(ii) The resolvent set $\rho(A)$ contains $\mathbb{R}^+$ and $\|R(\lambda; A)^n\| \leq \frac{M}{\lambda^n}$, $\forall \lambda > 0$, $\forall n \geq 1$.

Remark 2.4.3 *Let $(T(t))_{t\geq 0}$ be a C_0 semigroup on a Banach space X and let A be its infinitesimal generator. If the norm in X is changed to an equivalent norm, $(T(t))_{t\geq 0}$ stays a C_0 semigroup on X with the new norm. The infinitesimal generator A does not change nor does the fact that A is closed and densely defined change when we pass to an equivalent norm on X. All these are topological properties which are independent of the particular equivalent norm with which X is endowed.*

Proof of theorem(2.4.2). Suppose (i) and (ii) are true,using lemma(2.4.1) there exists an equivalent norm $|\cdot|$ satisfying (2.4.2) of lemma (2.4.1) such that $|R(\lambda; A)| \leq \frac{1}{\lambda}$, $\forall \lambda > 0$. Applying Hille-Yoshida Theorem, A is the infinitesimal generator of contractions on X endowed with the norm $|\cdot|$.

Returning to the original norm, A is again the infinitesimal generator of $(T(t))_{t\geq 0}$ and, let $x \in X$, $\|T(t)x\| \leq |T(t)x| \leq |T(t)||x| \leq M\|x\|$ therefore $\|T(t)\| \leq M$ as required.
The condition (i) and (ii) are therefore sufficient.
Suppose that A is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geq 0}$ such that $\|T(t)\| \leq M$.
Define the following ,

$$|x| = \sup_{t\geq 0} \|T(t)x\|.$$

Then

$$\|x\| \leq |x| \leq M\|x\| \tag{2.4.4}$$

and therefore $|\cdot|$ is a norm on X which is equivalent to the original norm $\|\cdot\|$ on X.
Claim: $|T(x)| \leq 1$, $\forall t \geq 0$.
Proof.

$$\begin{aligned} |T(t)x| = \sup_{s\geq 0} \|T(s)T(t)x\| &= \sup_{s\geq 0} \|T(s+t)x\| \\ &= \sup_{s\geq t} \|T(s)x\| \\ &\leq \sup_{s\geq 0} \|T(s)x\| = |x| \end{aligned} \tag{2.4.5}$$

One has $|T(t)x| \leq |x|$. Therefore $|T(t)| \leq 1$ as we have claimed.
And $(T(t))_{t\geq 0}$ is a C_0 semigroup of contractions with the norm $|\cdot|$. It follows from the Hille-Yoshida theorem and the remark(2.4.3) that,

- A is closed and densily defined .
- $\rho(A) \supseteq]0, +\infty[$ and $|R(\lambda; A)| \leq \dfrac{1}{\lambda}$, $\forall \lambda > 0$.

Let $x \in X$, using (2.4.4)and(2.4.5) we have
$\|R(\lambda; A)^n x\| \leq |R(\lambda; A)^n x| \leq |R(\lambda; A)^n||x| < \dfrac{1}{\lambda^n}|x| \leq \dfrac{M}{\lambda^n}\|x\|$, $\forall \lambda > 0$ and $n \geq 1$.
Therefore

$$\|R(\lambda; A)^n x\| \leq \frac{M}{\lambda^n}\|x\|, \ \forall \lambda > 0, \ n \geq 1, \ \forall x \in X;$$

the condition (i) and (ii) are therefore also necessary.

Theorem 2.4.4 *A linear operator A is the infinitesimal generator of a C_0 semigroup satisfying $\|T(t)\| \leq e^{\omega t}$ (for some $\omega \geq 0$) if and only if*

(i) A is closed and $\overline{\mathcal{D}(A)} = X$.

(ii) $\rho(A) \supseteq]\omega, +\infty[$ and $\|R(\lambda, A)\| \leq \dfrac{1}{\lambda - \omega}$ for $\lambda > \omega$.

Proof. Consider the C_0 semigroup $S(t) := e^{-\omega t}T(t)$ then $S(t)$ is obviously a C_0 semigroup of contractions and A is the infinitesimal generator of $T(t)$ if and only if $A - \omega I$ is the infinitesimal of $S(t)$.
That is Let $x \in \mathcal{D}(A)$,

$$\begin{aligned} \frac{d}{dt}S(t)x &= -\omega e^{\omega t}T(t)x + e^{-\omega t}AT(t)x \\ \frac{d}{dt}S(t)x\Big|_{t=0} &= (A - \omega I)x. \end{aligned}$$

It follows that $A - \omega I$ is the generator of $(S(t))_{t\geq 0}$.
By Hille-Yosida theorem

(i) $\mathcal{D}(A - \omega I) = \mathcal{D}(A)$ implies $\overline{\mathcal{D}(A - \omega I)} = \overline{\mathcal{D}(A)} = X$; therefore $\overline{\mathcal{D}(A - \omega I)} = X$. On the other hand A and ωI are closed then $A - \omega I$ is closed.

(ii) $\rho(A - \omega I) \supseteq]0, +\infty[$ and $\|R(\lambda; A - \omega I)\| \leq \frac{1}{\lambda}$ for $\lambda > 0$.

That is

$$\rho(A - \omega I) \supseteq]0, +\infty[\Longleftrightarrow \rho(A) \supseteq]\omega, +\infty[\text{ for } \lambda > \omega$$

and

$$\begin{aligned} \|R(\lambda; A - \omega I)\| \leq \frac{1}{\lambda} \text{ for } \lambda > 0 &\Longleftrightarrow \|((\lambda + \omega)I - A)^{-1}\| \leq \frac{1}{\lambda} \text{ for } \lambda > 0 \\ &\Longleftrightarrow \|\gamma I - A)^{-1}\| \leq \frac{1}{\gamma - \omega},\ \forall \gamma > 0 \qquad (2.4.6) \end{aligned}$$

In relation 2.4.6 since $\lambda + \omega > \omega$ for all $\lambda > 0$ we have put $\gamma = \lambda + \omega$.

This completes the proof.

Theorem 2.4.5 (Hille-Yosida Theorem General case) *A linear operator A is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geq 0}$ such that $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$,$n \geq 1$,$\omega \in \mathbb{R}$ if and only if :*

(i) A is closed and $\mathcal{D}(A)$ is dense in X.

(ii) $\rho(A) \supseteq]\omega, +\infty[$ and $\|R(\lambda; A)^n\| \leq \dfrac{M}{(\lambda-\omega)^n}$, $\forall\lambda > \omega$, $\forall n > 1$.

Proof. Suppose that A is the infinitesimal generator of $(T(t))_{t\geq 0}$ such that $\|T(t)\| \leq Me^{\omega t}$, then $\|e^{-\omega t}T(t)\| \leq M$.
Set $S(t) := e^{-\omega t}T(t)$ and we know that $(S(t))_{t\geq 0}$ is generated by $A - \omega I$ then $\rho(A - \omega I) \supseteq$ $]0, +\infty[$
and $\|R(\lambda; A - \omega I)^n\| \leq \dfrac{M}{\lambda^n}$, $\lambda > 0$, $n \geq 1$.
By theorem 2.4.4, it follows that $\rho(A) \supseteq]0, +\infty[$ and $\|R(\lambda; A)^n| \leq \dfrac{M}{(\lambda-\omega)^n}$, $\forall\lambda > \omega$.

Assume (i) and (ii) are satisfied then we have by theorem 2.4.4 that A is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geq 0}$ such that $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$, $n \geq 1$, $\omega \in \mathbb{R}$.

CHAPTER 3

Abstract Cauchy Problem

3.1 Homogeneous Initial Value Problem

Let X be a Banach space and let A be a linear operator from $\mathcal{D}(A) \subset X$ into X. Given $x \in X$ the abstract Cauchy problem for A with initial data x consists of finding a solution $x(t)$ to the initial value problem

$$\begin{cases} x'(t) = Ax(t) \qquad t \geq 0 \\ \\ x(0) = x \end{cases} \tag{3.1.1}$$

where by solution we mean a X value function $x(t)$ such that $x(t)$ is continuous for $t \geq 0$, continuously differentiable and $x(t) \in \mathcal{D}(A)$ for $t > 0$ and (3.1.1) is satisfied. Note that since $x(t) \in \mathcal{D}(A)$ for $t > 0$ and u is continuous at $t = 0$ (3.1.1) can not have a solution for $x \notin \mathcal{D}(A)$.

Theorem 3.1.1 *Let $(T(t))_{t\geq 0}$ be a C_0 semigroup on X with infinitesimal generator A. Then for each x in $\mathcal{D}(A)$, the system (3.1.1) has a unique solution $x(t)$ in $C^1([0,\infty); X) \cap C([0,\infty); \mathcal{D}(A))$ and*

$$x(t) = T(t)x, \quad \forall t \geq 0.$$

Proof. The existence follows from theorem 2.1.6 part c).
To prove the uniqueness let v be another solution in $C^1([0,\infty); X) \cap C([0,\infty); \mathcal{D}(A))$. Fix $t > 0$ and set

$$z(s) = T(t-s)v(s), \ \ s \in [0,t].$$

From theorem 2.1.6 part c) and the properties of v and $z \in C^1([0,\infty);X) \cap C([0,\infty);\mathcal{D}(A))$, we have for all s in $[0,t]$,

$$\begin{aligned}\frac{dz}{ds}(s) &= -AT(t-s)v(s) + T(t-s)v'(s)\\ &= T(t-s)Ax(s) - AT(t-s)x(s)\\ &= 0.\end{aligned}$$

As result $z \in C^1([0,t];X)$ and $z(s) = z(0)$ for all $s \in [0,t]$; this implies
$v(t) = z(t) = T(t)v(0) = T(t)x$ for all $t \geq 0$.

Definition 3.1.2 *If A is the infinitesimal generator of a C_0 semigroup which is not differentiable then in general if $x_0 \notin \mathcal{D}(A)$, then the initial value problem (3.1.1) does not have a solution. The function $t \to T(t)x_0$ is then called mild solution of the initial value problem (3.1.1) .*

Definition 3.1.3 *Let T(t) be a C_0 semigroup on a Banach space X. The semigroup $T(t)$ is called differentiable for $t > t_0$ if for every $x \in X$; $t \to T(t)x$ is differentiable for $t > t_0$; $T(t)$ is called differentiable if it is differentiable for $t > 0$.*

Note that we have seen in theorem 2.1.6 part(c) that if $T(t)$ is a C_0 semigroup with infinitesimal generator A and $x \in \mathcal{D}(A)$ then $t \to T(t)x$ is differentiable for $t \geq 0$. If $T(t)$ is moreover differentiable then for every $x \in X$, $t \to T(t)x$ is differentiable for $t > 0$.
Note that $t \to T(t)x$ is differentiable for every $x \in X$ and $t > 0$ then $\mathcal{D}(A) = X$ and since A is closed it is necessary bounded.

3.2 Nonhomogenous Linear Cauchy Problem

$$\begin{cases} x'(t) = Ax(t) + f(t) & t \geq 0 \\ \\ x(0) = x_0 \end{cases} \tag{3.2.1}$$

where $f : \mathbb{R}^+ \to X$ is continuous. We will assume throughout this section that A is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geq 0}$ so that the corresponding homogeneous equation, i.e with $f \equiv 0$, has a unique solution for every initial value $x_0 \in \mathcal{D}(A)$.

Definition 3.2.1 *A function $x : [o, a[\to X$ is a (classical) solution of (3.2.1) on $[0, a[$ if and only if*

i) x is continuous in $[0, a[$;

ii) $x(t)$ is continuously differentiable on $]0, a[$;

iii) $x(t) \in \mathcal{D}(A)$, for $t \geq 0$ and (3.2.1) is satisfied on $[0, a[$.

Let $T(t)$ be the C_0 semigroup generated by A and let x be a solution of (3.2.1). Then the X valued function $g(s) = T(t-s)x(s)$ is differentiable for $0 < s < t$ and

$$\begin{aligned}\frac{dg}{ds}(s) &= -AT(t-s)x(s) + T(t-s)x'(s) \\ &= T(t-s)(Ax(s) + f(s)) - AT(t-s)x(s) \\ &= -AT(t-s)x(s) + T(t-s)Ax(s) + T(t-s)f(s) \\ &= T(t-s)f(s). \end{aligned} \tag{3.2.2}$$

If $f \in L^1(0, a : X)$ then $T(t-s)f(s)$ is integrable and integrating (3.2.2) from 0 to t yields

$$g(t) = g(0) + \int_0^t T(t-s)f(s)ds$$

therefore,

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds. \tag{3.2.3}$$

In particular if $A \in \mathcal{B}(X, X)$, then $\forall x_0 \in X$, the equation (3.2.1)has a unique solution x on $\mathbb{R}^+$
which is $x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}f(s)ds$.

Corollary 3.2.2 *If $f \in L^1([0, a]; X)$ then for every $x \in X$ the initial value problem (3.2.1) has at most one solution. If it has a solution, this solution is given by (3.2.3).*

For every $f \in L^1([0, a]; X)$ the right-hand side of (3.2.3) is a continuous function on $[0, a]$. It is natural to consider it as a generalized solution of (3.2.1) even if it is not differentiable and does not strictly satisfy the equation in the sense of definition (3.2.1). We therefore define,

Definition 3.2.3 *Let A be the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geq 0}$.
Let $x \in X$ and $f \in L^1([0, a]; X)$. The function $x \in C([0, a]; X)$ given by*

$$x(t) = T(t)x + \int_0^t T(t-s)f(s)ds \ , \ 0 \leq t \leq a,$$

is the mild solution of the initial value problem (3.2.1) on $[0, a]$.

Remark 3.2.4 *The continuity of f, in general, is not sufficient to ensure the existence of solutions of (3.2.1) for $x_0 \in \mathcal{D}(A)$.*

Example 3.2.5 *Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ and let $y \in X$ be such that $T(t)y \notin \mathcal{D}(A)$ for any $t \geq 0$. Let $f(s) = T(s)y$.
Then f is continuous for $s \geq 0$. Consider the initial value problem*

$$\begin{cases} x'(t) = Ax(t) + T(t)y & t \geq 0 \\ x(0) = 0 \end{cases} \tag{3.2.4}$$

Claim: *(3.2.4) has no solution even though* $x(0) = 0 \in \mathcal{D}(A)$.
The mild solution of (3.2.4) is

$$x(t) = T(t)(0) + \int_0^t T(t-s)T(s)yds = \int_0^t T(t)yds = tT(t)y,$$

but $tT(t)y$ *is not differentiable for* $t > 0$ *since* $y \notin \mathcal{D}(A)$ *and therefore can not be the solution of (3.2.4) .*

Theorem 3.2.6 *Let* A *be the infinitesimal generator of a* C_0 *semigroup* $T(t)$*, let* $x_0 \in D(A)$ *and* $f : \mathbb{R}^+ \to X$ *is* C^1 *function then the mild solution becomes a classical solution of equation (3.2.1).*

Proof. The mild solution is $x(t) = T(t)(x_0) + \int_0^t T(t-s)T(s)f(s)ds = T(t)x_0 + v(t)$.
Claim: $v \in C^1(\mathbb{R}^+, X)$; $v(t) = Ax(t) + T(t)y,\ \forall t \geq 0$ and $v(0) = 0$.
Proof of Claim. Let $h > 0$,

$$\begin{aligned}
\frac{T(h)-I}{h}v(t) &= \frac{T(h)-I}{h}\int_0^t T(t-s)f(s)ds \\
&= \frac{1}{h}\int_0^t T(t+h-s)f(s)ds - \frac{1}{h}\int_0^t T(t-s)f(s)ds \\
&= \frac{1}{h}\int_0^{t+h} T(t+h-s)f(s)ds - \frac{1}{h}\int_0^t T(t-s)f(s)ds - \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds \\
&= \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds. \qquad (3.2.5)
\end{aligned}$$

From the continuity of f it is clear that the second term on the right-hand side of (3.2.5) has the limit $f(t)$ as $h \to 0$.
But we have also $v(t) = \int_0^t T(t-s)f(s)ds = \int_0^t T(s)f(t-s)ds$; then

$$\begin{aligned}
D^+v(t) = \frac{v(t+h)-v(t)}{h} &= \int_0^t T(s)\frac{f(t+h-s)-f(t-s)}{h} \text{as} h \to 0^+ \\
&= \int_0^t T(s)f'(t-s) \\
&= \int_0^t T(t-s)f'(s).
\end{aligned}$$

Since f' is continuous then $t \to \int_0^t T(t-s)f'(s)$ is continuous on $\mathbb{R}^+$
Finally : $Av(t) = D^+v(t) - f(t),\ t \geq 0$.

Lemma 3.2.7 *Let* $u : [a,b] \to X$.
Suppose $D^+u(t)$ *on* $[a,b]$ *and* $t \to D^+u(t)$ *is continuous on* $[a,b]$ *then* u *is of class* C^1 *on* $[a,b]$.

Using this lemma then v is of class C^1 and

$$\begin{cases} v'(t) = Av(t) + f(t), & t \geq 0. \\ v(0) = 0. \end{cases}$$

The mild solution of equation (3.2.1) is $x(t) = v(t) + T(t)x_0$.
We have to show that $x \in C^1$.
Let $x_0 \in D(A)$, then $\frac{d}{dt}T(t)x_0 = AT(t)x_0$ it follows that

$$\begin{aligned} x'(t) &= v'(t) + AT(t)x_0 \\ &= Av(t) + f(t) + AT(t)x_0 \\ &= A[v(t) + T(t)x_0] + f(t) \\ &= Ax(t) + f(t),\ t \geq 0\ ; \end{aligned} \tag{3.2.6}$$

the right-hand side of (3.2.6) is continuous since x and f are continuous.
Therefore $x \in C^1$.

3.3 Nonlinear Evolution Equation

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)) & t \geq 0 \\ \\ x(0) = x_0 \end{cases} \tag{3.3.1}$$

Where A is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geq 0}$ on a Banach space X and $f : \mathbb{R}^+ \times X \to X$ is continuous.

Definition 3.3.1 *We say that x is a solution (classical solution) of the equation (3.3.1) if and only if*

i) $x \in C(\mathbb{R}^+, X)$, $x(t) \in \mathcal{D}(A)$,$t \geq 0$;

ii) x *satisfies the equation (3.3.1).*

Theorem 3.3.2 *If x is a solution of the equation (3.3.1), then*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds,\ t \geq 0 \tag{3.3.2}$$

Proof. If x is a solution for the equation (3.3.1) then x is solution of the following equation

$$\begin{cases} y'(t) = Ay(t) + g(t) & t \geq 0 \\ \\ y(0) = y_0 \end{cases}$$

but $g(t) = f(t, x(t))$
$y(t) = x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds,\ t \geq 0$

Remark 3.3.3 *If x satisfies (3.3.2) then x is not necessary a solution of (3.3.1)*

Problem 3.3.4 *Does a mild solution of equation (3.3.1) exist?*

Example 3.3.5 $A \equiv 0$ *the equation (3.3.1) becomes*

$$\begin{cases} x'(t) = f(t, x(t)) & t \geq 0 \\ x(0) = x_0 \end{cases}$$

Continuity of f is not enough to get the existence of mild solution.

Theorem 3.3.6 *Suppose that $f : \mathbb{R}^+ \times X \to X$ is Lipschitz with respect to the second argument then for any $x_0 \in X$, the equation (3.3.1) has a unique mild solution on $\mathbb{R}^+$.*

Proof.
Let $x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds = (Hx)(t),\ t \geq 0$
We have to show that x is mild solution of equation (3.3.1) if and only if $Hx = x$
Define $C_a := C([0, a], X)$ such that $H : C_a \to C_a$.
Let $x_1\ ,\ x_2 \in C_a$: $(Hx_1)(t) - (Hx_2)(t) = \int_0^t T(t-s)(f(s, x_1(s)) - f(s, x_2(s))ds$.
Let $M_a = \sup_{t\in[0,a]} \|T(t)\| \leq Me^{\omega a} < +\infty, \omega > 0$.
$\|(Hx_1)(t) - (Hx_2)(t)\| \leq M_a K t \|x_1 - x_2\|$; K being the lipschitzian constant for f.
One has $H^2 = H \circ H$, then

$$\begin{aligned}
\|H^2x_1(t) - H^2x_2(t)\| &\leq M_a K \int_0^t \|H(x_1(s)) - H(x_2(s))\|ds \\
&\leq M_a K \int_0^t (M_a K s)\|x_1 - x_2\|ds,\ \forall t \in [0, a] \\
&= \frac{(M_a K)^2}{2!} t^2 \|x_1 - x_2\|,\ \forall t \in [0, a] \\
\vdots \qquad\qquad & \ \vdots \\
\|H^n x_1(t) - H^n x_2(t)\| &\leq \frac{(M_a K)^n}{n!} a^n \|x_1 - x_2\|\ ,\ \forall t \in [0, a];
\end{aligned}$$

it follows that $\|H^n x_1(t) - H^n x_2(t)\| \leq \dfrac{(M_a K)^n}{n!} a^n \|x_1 - x_2\|\ ,\ \forall x_1\ ,\ x_2 \in C_a$.
Since $\dfrac{(M_a K)^n}{n!} a^n \to 0$ as $n \to +\infty$ then $\exists p \in \mathbb{N}$ such that $\dfrac{(M_a K)^p}{p!} a^p < 1$
It follows that $H^p = H \circ \cdots \circ H$ is a contraction and then $\exists! x \in C_a$ such that $H^p x = x$.
Claim: $Hx = x$.
$H^p(x) = x$ implies that $H^{p+1}(x) = Hx$, one can write $H^p(H(x)) = H(x)$ it follows that $H(x)$is a fixed point of H^p and since the fixed point is unique then we get $H(x) = x$.
We conclude that $Hx = x$ is a mild solution of equation (3.3.1) on $[0, a]$, this is true for all $a > 0$.
Hence the equation (3.3.1)has a unique mild solution on $\mathbb{R}^+$.

3.3.1 Continuous dependence on the initial data

$$(\mathcal{P}_n)\begin{cases} x'(t) = Ax_n(t) + f(t, x_n(t)), & t \geq 0 \\ x(0) = x_{0,n} \end{cases}$$

$$(\mathcal{P}) \qquad \begin{cases} x'(t) = f(t, x(t)) & t \geq 0 \\ x(0) = x_0 \end{cases}$$

if $x_{0,n} \to x_0$ as $n \to +\infty$ then $x_n \to x$ uniformly in each compact set of $\mathbb{R}^+$.

Theorem 3.3.7 *Let $x(t, x_0)$ denotes the mild solution of (3.3.1) starting from x_0; $\forall a > 0 \quad \exists \alpha(a) > 0$; $\beta(a) > 0$*

$$|x(t, x_0) - x(t, y_0)| \leq \alpha(a)e^{\beta(a)t}|x_0 - y_0|, \ \forall t \in [0, a].$$

Proof. Let $t \in [0, a]$ and $x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds$

$$x(t) = x(t, x_0) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds$$

$$y(t) = x(t, y_0) = T(t)y_0 + \int_0^t T(t-s)f(s, x(s))ds$$

$$\begin{aligned} \|x(t) - y(t)\| &\leq \|T(t)(x_0 - y_0\| + \|\int_0^t T(t-s)(f(s, x(s)) - f(s, y(s))ds\| \\ &\leq \|T(t)\|\|x_0 - y_0\| + \int_0^t \|T(t-s)\|\|f(s, x(s)) - f(s, y(s))\|ds \\ &\leq Me^{\omega t}\|x_0 - y_0\| + \int_0^t Me^{\omega(t-s)}K\|x(s) - y(s)\|ds \\ &\leq Me^{\omega a}\|x_0 - y_0\| + KMe^{\omega a}\int_0^t \|x(s) - y(s)\|. \end{aligned} \tag{3.3.3}$$

Lemma 3.3.8 (Gronwall lemma) *Let $g : [a, b] \to \mathbb{R}^+$ continuous such that*

$$g(t) \leq \alpha + \int_a^t \beta(s)g(s)ds$$

where $\alpha > 0$ and $\beta : [a, b] \to \mathbb{R}^+$ is continuous. Then

$$g(t) \leq \alpha \times exp\big(\int_a^t \beta(s)ds\big).$$

Applying Gronwall lemma to (3.3.3) it follows$\|x(t)-y(t)\| \leq Me^{\omega a}\|x_0-y_0\|exp(\int_0^t KMe^{\omega a}ds)$ which implies
$\|x(t)-y(t)\| \leq Me^{\omega a}exp\big(KMe^{\omega a}\big)$
Take $Me^{\omega a} = \alpha(a)$ and $KMe^{\omega a} = \beta(a)$, therefore

$$\|x(t,x_0) - x(t,x_0)\| \leq \alpha(a)e^{\beta(a)t}, \ \forall t \in [0,a].$$

Lemma 3.3.9 *Let x be a mild solution of equation (3.3.1) on $[0,a]$ if we suppose*

i) $x_0 \in \mathcal{D}(A)$*;*
ii) $t \to f(t,x(t))$ *is C^1 function.*
Then x is a classical solution of equation (3.3.1).

Proof.
x can be seen as a mild solution of the following equation

$$\begin{cases} y'(t) = Ay(t) + g(t) \qquad t \geq 0 \\ \\ y(0) = y_0 \ , \ g(t) = f(t,x(t)) \end{cases} \tag{3.3.4}$$

If $f \in C^1([0,a],X)$ and $y_0 \in D(A)$ then $x = y \in C^1([0,a];X)$
Moreover

$$\begin{aligned} x'(t) = y'(t) &= Ay(t) + g(t) \\ &= Ax(t) + f(t,x(t)) \end{aligned}$$

Problem 3.3.10 *Find a sufficient condition ensuring (ii)*

Theorem 3.3.11 *Let $f : \mathbb{R}^+ \times X \to X$ is continuous and Lipschitzian with respect to the second argument.*
Moreover if $f \in C^1(\mathbb{R}^+ \times X, X)$ and the partial derivatives $D_t f(t,x)$ and $D_x f(t,x)$ are Lipschitz with respect to the second argument.
Let $x_0 \in D(A)$, then equation (3.3.1) has a classical solution on $\mathbb{R}^+$.

Proof. Let x be a mild solution of equation (3.3.1) on $\mathbb{R}^+$.

Claim 1: $t \to f(t,x(t))$ is C^1 function on $\mathbb{R}^+$.
Let $a > 0$,$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s,x(s))ds$, $t \in [0,a]$.
Consider the following abstract cauchy problem,

$$\begin{cases} y'(t) = Ay(t) + D_t f(t,x(t)) + D_x f(t,x(t))(y(t)) \qquad t \in [0,a] \\ \\ y(0) = Ax_0 + f(0,x_0) \end{cases} \tag{3.3.5}$$

Equivalentely we have,

$$\begin{cases} y'(t) = Ay(t) + g(t, y(t)) & t \in [0, a] \\ y(0) = Ax_0 + f(0, x_0) \end{cases}$$

With $g(t, y(t)) = D_t f(t, x(t)) + D_x f(t, x(t))(y(t))$, g is continuous since f is C^1 function. On the other hand one has

$$\begin{aligned} |g(t, y_1) - g(t, y_2)| &= |D_x f(t, x(t))(y_1 - y_2)| \\ &\leq |D_x f(t, x(t))||y_1 - y_2| \\ &\leq \sup_{t\in[0,a]} |D_x f(t, x(t))||y_1 - y_2| \\ &\leq K|y_1 - y_2| \text{ where } K = \sup_{t\in[0,a]} |D_x f(t, x(t))| \end{aligned}$$

g is Lipschitzian with respect to the second argument then (3.3.5) has a unique mild solution y on $[0, a]$.
Let $z = y(0) + \int_0^t y(s)ds$, $t \in [0, a]$ it follows that $z \in C^1([0, a], X)$ since y is continuous

Claim 2: $x = z$.
$y(t) = T(t)[Ax_0 + f(0, x_0)] + \int_0^t T(t - s)\big[D_s f(s, x(s)) + D_x f(s, x(s))(y(s))\big]ds$
and

$$\begin{aligned} z(t) &= x(0) + \int_0^t T(s)Ax_0 ds + \int_0^t T(s)f(0, x_0)ds \\ &+ \int_0^t \int_0^s T(s - \tau)[D_\tau f(\tau, x(\tau)) + D_x f(\tau, x(\tau))y(\tau)]d\tau ds \\ &= x_0 + T(t)x_0 - x_0 + \int_0^t T(s)f(0, x_0)ds \\ &+ \int_0^t \int_0^s T(s - \tau)[D_\tau f(\tau, x(\tau)) + D_x f(\tau, x(\tau))y(\tau)]d\tau ds \\ &= T(t)x_0 + \int_0^t T(s)f(0, x_0)ds \\ &+ \int_0^t \int_0^s T(s - \tau)[D_\tau f(\tau, x(\tau)) + D_x f(\tau, x(\tau))y(\tau)]d\tau ds. \end{aligned} \tag{3.3.6}$$

$z \in C^1$ then $s \to f(s, x(s))$ is C^1 function on $[0, a]$.

$$\begin{aligned}
\frac{d}{ds}f(s,z(s)) &= D_t f(s,z(s)) + D_x f(s,z(s))(z'(s)) \\
&= D_t f(s,z(s)) + D_x f(s,z(s))(y(s))
\end{aligned}$$

$$\begin{aligned}
\frac{d}{ds}\int_0^t T(t-s)f(s,z(s))ds &= \frac{d}{ds}\int_0^t T(s)f(t-s,z(t-s))ds \\
&= T(t)f(0,x_0) + \int_0^t T(s)[D_t f(t-s,z(t-s)) \\
&+ D_x f\big(t-s,z(t-s)\big)y(t-s)]ds \\
&= T(t)f(0,x_0) + \int_0^t T(t-s)[D_t f(s,z(s)) + D_x f(s,z(s))(y(s))]ds
\end{aligned}$$

$$\begin{aligned}
\text{then,} \quad \int_0^t T(t-s)f(s,z(s))ds &= \int_0^t T(t-s)f(0,x_0)ds + \int_0^t\int_0^s T(s-\tau)[D_\tau f(\tau,z(\tau)) \\
&+ D_x f(\tau,z(\tau))(y(\tau))]dsd\tau.
\end{aligned}$$

One has also

$$\int_0^t T(t-s)f(0,x_0)ds = \int_0^t T(t-s)f(s,z(s))ds - \int_0^t\int_0^s T(s-\tau)[D_\tau f(\tau,z(\tau)) + D_x f(\tau,z(\tau))(y(\tau))]dsd\tau. \tag{3.3.7}$$

Finally combining (3.3.6) and (3.3.7), we get

$$\begin{aligned}
z(t) = T(t)x_0 + \int_0^t T(t-s)f(s,z(s))ds &+ \int_0^t\int_0^s T(s-\tau)[D_\tau f(\tau,x(\tau)) - D_\tau f(\tau,z(\tau))]d\tau ds \\
&+ \int_0^t\int_0^s T(s-\tau)[D_x f(\tau,x(\tau)) - D_x f(\tau,z(\tau))(y(\tau))]dsd\tau.
\end{aligned}$$

Now let's compare $z(t)$ and $x(t)$.
$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s,x(s))ds$ a mild solution of (3.3.5).

$$\begin{aligned}
z(t) - x(t) &= \int_0^t T(t-s)\big(f(s,z(s)) - f(s,x(s))\big)ds \\
&+ \int_0^t\int_0^s T(s-\tau)[D_\tau f(\tau,x(\tau)) - D_\tau f(\tau,z(\tau))]d\tau ds \\
&+ \int_0^t\int_0^s T(s-\tau)[D_x f(\tau,x(\tau)) - D_x f(\tau,z(\tau))(y(\tau))]dsd\tau.
\end{aligned}$$

Applying Gronwall lemma, since f, and D_t, D_x Lipschitzian then there exists $K(a) > 0$ such that

$|z(t) - x(t)| \leq K(a) \int_0^t |z(s) - x(s)| ds$, $t \in [0, a]$; it follows that $z = x$ therefore $x \in C^1([0, a], X)$ and $s \to f(s, x(s))$ is C^1 function.
Using lemma (3.3.9) we get that x is a classical solution on $[0, a]$. The proof is complete.

3.4 Application

Non linear heat equation

$$
(H.E) \begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial^2 x}(t,x) + g(u(t,x)) \qquad t \geq 0 \ , \ x \in [0,1] \\ u(t,0) = u(t,1) = 0 \qquad \text{(Dirichlet condition)} \\ \dfrac{\partial u}{\partial x}(t,0) = \dfrac{\partial u}{\partial x}(t,1) = 0 \qquad \text{(Neumann condition)} \\ u(t,0) = u_0(x), \ x \in [0,1] \end{cases}
$$

$u_0 \in C([0, 1], \mathbb{R})$, $g : \mathbb{R} \to \mathbb{R}$ is continuous.

Boundary conditions.
$u_0 \in C_0([0, 1], \mathbb{R}) = \{v \in C([0, 1], \mathbb{R}) : v(0) = v(1) = 0\}$
$\mathcal{D}(\triangle) = \{u \in C_0([0, 1], \mathbb{R}) \cap C^2([0, 1], \mathbb{R}) \ , \ u \ , \ u' \in C_0([0, 1], \mathbb{R})\}$

Lemma 3.4.1 *A is the infinitesimal generator of a C_0 semigroup on $C_0([0, 1], \mathbb{R})$.
Assume that u is a solution of $(H.E)$.
Let $v : \mathbb{R}^+ \to C_0([0, 1], \mathbb{R})$ such that $v(t)x = u(t, x)$, $t \geq 0$ and $x \in [0, 1]$.
Let $f : C_0([0, 1], \mathbb{R}) \to C_0([0, 1], \mathbb{R})$ such that $u \mapsto f(u)$ and $f(u)x = g(u(x))$
then*

$$
\begin{cases} v'(t) = Av(t) + f(v(t)) \qquad t \geq 0 \\ v(0) = u_0 \end{cases} \tag{3.4.1}
$$

$(H.E) \iff$ equation (3.4.1)

Theorem 3.4.2 *Let $u_0 \in \mathcal{D}(\triangle)$ if g is lipschitzian, g is C^1 function, and g' is lipschitzian then the equation (3.3.1) has a classical solution v and the function $u : \mathbb{R}^+ \times [0, 1] \to \mathbb{R}$ defined by $u(t, x) = v(t)x$, $t \geq 0$, $x \in [0, 1]$ is the solution of $(H.E)$.*

3.5 Stability and asymptotic behavior of solutions

3.5.1 Linear Equation

We consider the following linear equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) = Au(t) \qquad t \geq 0 \\ \\ u(0) = x_0 \end{cases}$$

We assume that A is the infinitesimal generator of a C_0 semigroup $(T(t))_{t\geq 0}$ on a Banach space X.
$u(t, x_0) = T(t)x_0$ is a mild solution since $x_0 \in X$.
We wish to study,

$$\lim_{t\to+\infty} T(t)x_0.$$

We know that there exist $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$.
If $\omega < 0$ then $\|T(t)\| \to 0$, as $t \to +\infty$.

Definition 3.5.1 *The type of $(T(t))_{t\geq 0}$ is defined by*

$$\omega_0(T) = \inf\{\omega \in \mathbb{R} : \sup_{t\geq 0}\{e^{-\omega t}\|T(t)\|\} < \infty\}.$$

Remark 3.5.2 *If $\omega_0(T) < 0$ then $T(t) \to 0$ as $t \to +\infty$ exponentially.*

Proposition 3.5.3 *The type $\omega_0 = \omega_0(T)$ of the semigroup T is computed by the following formula $\omega_0(T) = \lim\limits_{t\to+\infty} \dfrac{log\|T(t)\|}{t} = \inf\limits_{t>0}\dfrac{log\|T(t)\|}{t}$.*

Corollary 3.5.4 *If T is a C_0 semigroup of type ω_0, then for any $\omega > \omega_0$, there exists $M_\omega \geq 1$ such that*

$$\|T(t)\| \leq M_\omega e^{\omega t}, \quad t \geq 0.$$

Proof.
Choose t_ω as in the proof of Theorem 2.1.4 and set $t = n(t)t_\omega + r(t),\ n(t) \in \mathbb{N},\ 0 \leq r(t) < t_\omega$.
If $\omega_0 \geq 0$,then $\omega > \omega_0 \geq 0$ and we have

$$\|T(t)\| \leq \|T(t_\omega)\|^{n(t)}\|T(r(t))\| \leq M_\omega exp\Big(t_\omega n(t)\omega\Big) \leq M_\omega e^{\omega t},$$

where $M_\omega = sup\{\|T(s)\| : s \in [0, t_\omega]\}$.
If $\omega_0 < 0$, then we consider the semigroup $S(t) = e^{-\omega_0 t}T(t)$ for which the type is 0.
So for each $\omega > \omega_0$, $\omega - \omega_0 > 0$ and there exists $M_\omega > 1$ such that

$$e^{-\omega_0 t}\|T(t)\| = \|e^{-\omega_0 t}T(t)\| \leq M_\omega e^{(\omega-\omega_0)t},\ t \geq 0.$$

This completes the proof.

Definition 3.5.5 *We say that* $(T(t))_{t\geq 0}$ *is eventually compact iff there exists* $t_0 > 0$ *such that* $T(t_0)$ *is compact.*

Definition 3.5.6 *We say that* $(T(t))_{t\geq 0}$ *is compact if* $T(t)$ *is compact for each* $t > 0$.

Remark 3.5.7 *The semigroup law implies* $T(t)$ *is compact for* $t \geq t_0$ *if* $T(t)$ *is compact for* $t \geq t_0$.

Definition 3.5.8 *The spectral bound* $s(A)$ *is defined by*

$$s(A) = sup\{Re\lambda : \ \lambda \in \sigma(A)\}.$$

Theorem 3.5.9 *If* $(T(t))_{t\geq 0}$ *is eventually compact then*

$$\omega_0(T) = s(A).$$

Remark 3.5.10 *If* $s(A) < 0$ *then* $\|T(t)\| \to 0$, *as* $t \to +\infty$.
In general $s(A) \leq \omega_0(T)$.

Theorem 3.5.11 *If* $(T(t))_{t\geq 0}$ *is eventually compact then* $\sigma(A) = \sigma_p(A)$.

Example 3.5.12 $D(\triangle) = \{f \in C^2([0,1]) \cap C_0([0,1]), \ \triangle f = f''; \ f', \ f'' \in C_0([0,1])\}$

Lemma 3.5.13 $\triangle$ *is the infinitesimal generator of a compact semigroup on* $C_0([0,1])$.

3.5.2 Nonlinear case

Consider the following evolution equation

$$\begin{cases} x'(t) = Ax(t) + f(x(t)) \qquad t \geq 0 \\ \\ x(0) = x_0 \end{cases} \tag{3.5.1}$$

We suppose $f : X \to X$ is lipschitzian. Then the equation (3.5.1) has a unique mild solution $x(t, x_0)$ on $\mathbb{R}^+$.

Definition 3.5.14 *Let* $\overline{x} \in X$, $\overline{x}$ *is an equilibrum(stationary solution) of equation (3.5.1) iff*
$A\overline{x} + f(\overline{x}) = 0$ *that is* $x(t) = \overline{x}$ *is a constant solution (i.e* $\overline{x} = x(t, \overline{x})$, $\forall \geq 0$).

Definition 3.5.15 (Lyapunov) *Let* $\overline{x}$ *be an equilibrum of equation (3.5.1).*
We say that $\overline{x}$ *is stable if:* $\forall \epsilon > 0$, $\exists \delta > 0$ *such that* $\|x_0 - \overline{x}\| < \delta$ *then* $\|x(t, x_0) - \overline{x}\| < \epsilon$, $\forall t \geq 0$.

Definition 3.5.16 *We say that* $\overline{x}$ *is asymptotically stable if*

- $\overline{x}$ *is stable;*

- $\exists \delta_0 > 0$, $\forall x_0 \in B(\overline{x}, \delta_0) : x(t, x_0) \to \overline{x}$.

Definition 3.5.17 *We say that $\overline{x}$ is locally exponentially stable if*

- *$\overline{x}$ is asymptotically stable;*
- *The speed of the convergence is exponential i.e there exist $\delta_0 > 0$, $\alpha > 0$, $\beta > 0$ such that for all $x_0 \in B(\overline{x}, \delta_0)$ we have $\|x(t, x_0) - x(t, \overline{x})\| \leq \beta e^{-\alpha t}\|x_0 - \overline{x}\|$, $\forall t \geq 0$.*

Definition 3.5.18 *We say that $\overline{x}$ is globally asymptotically stable if*

- *$\overline{x}$ is stable;*
- *$\forall x_0 \in X : x(t, x_0) \to \overline{x}$ as $t \to +\infty$.*

3.5.3 Linearized stability

Suppose $\overline{x}$ is an equilibrum of equation (3.5.1) and f is differentiable at $\overline{x}$.
Let $B = f'(\overline{x}) \in \mathcal{B}(X, X)$.
Let consider the following linear system

$$\begin{cases} y'(t) = Ay(t) + By(t)) & t \geq 0 \\ \\ y(0) = y_0 \end{cases} \tag{3.5.2}$$

The equation (3.5.2) is well posed, $\forall y \in X$; that is the solution to the equation exists, it is unique and depends continuously to the initial data.

The equation (3.5.2) has a unique solution $y(t, y_0)$ given by

$$y(t, y_0) = T(t)y_0 + \int_0^t T(t-s)By(s, y_0)ds.$$

Lemma 3.5.19 *$A + B$ is the infinitesimal generator of a C_0 semigroup $(S(t))_{t\geq 0}$ defined by*

$$S(t)y_0 = y(t, y_0). \tag{3.5.3}$$

Proof. Obviously $(S(t))_{t\geq 0}$ given by relation (3.5.3) is a C_0 semigroup.
Let C its infinitesimal generator;
Claim: $C = A + B$.
Let $y_0 \in \mathcal{D}(C)$ then $\lim_{t\to 0^+} \dfrac{S(t)y_0 - y_0}{t}$ exists.
Or $\dfrac{S(t)y_0 - y_0}{t} = \dfrac{T(t)y_0 - y_0}{t} + \dfrac{1}{t}\int_0^t T(t-s)By(s, y_0)$.
As $t \to 0^+$, we get

$$Cy_0 = Ay_0 + By_0.$$

Therefore

$$C = A + B.$$

Theorem 3.5.20 *Suppose that $(T(t))_{t\geq 0}$ is compact.*
If $s(A+B)<0$ then $\overline{x}$ is locally exponentially stable.

Proof.
Claim: $s(A+B)=\omega_0(S)$.
It is enough to prove that $S(t)$ is compact for $t>0$.
But $S(t)y_0 = T(t)y_0 + \int_0^t T(t-s)BS(s)y_0 ds$.
Let $v(t)y_0 = \int_0^t T(t-s)BS(s)y_0 ds$, we have to show that $v(t)$ is compact.

$$\begin{aligned} v(t) &= \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} T(t-s)BS(s)y_0 ds \in \mathcal{B}(X,X) \\ &= \lim_{\varepsilon\to 0} K_\varepsilon \end{aligned}$$

Where $K_\varepsilon = \int_0^{t-\varepsilon} T(t-s)BS(s)ds$, we have to show that K_ε is compact.

$$\begin{aligned} &= \int_0^{t-\varepsilon} T(t-s)BS(s)ds \\ &= \int_0^{t-\varepsilon} T(t-\varepsilon+\varepsilon-s)BS(s)ds \\ &= T(\epsilon)\int_0^{t-\varepsilon-s} T(t-\varepsilon-s)BS(s)ds \qquad \text{is compact .} \end{aligned}$$

Having $T(\varepsilon)$ is compact and $\int_0^{t-\varepsilon} T(t-s)BS(s)y_0 ds \in \mathcal{B}(X,X)$, since composition of compact operator with linear operator is compact then K_ε is compact. Therefore $v(t)$ is compact.

Theorem 3.5.21 (Linearized stability principle) *Assume that the type of S, $\omega_0(S)<0$, then $\overline{x}$ is locally exponentially stable for the equation (3.5.1).*

3.5.4 Special case

Suppose, $f'(\overline{x}) = -\alpha Id$, $\alpha > 0$

Proposition 3.5.22 *There exist $\omega_0 > 0$ such that $\alpha > \omega_0$ then $\overline{x}$ is locally exponentially stable.*

Proof.(Linearized stability principle)
$\omega_0(S) = A - \alpha Id$ infinitesimal generator of $e^{-\alpha t}T(t)$ it follows that $S(t) = e^{-\alpha t}T(t)$.
$\|S(t)\| = \|e^{-\alpha t}T(t)\|$ but there exists $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega_0}t$, $\omega_0 > 0$, $\forall t \geq 0$.
One has $\|S(t)\| \leq Me^{(-\alpha+\omega_0)t}$
if $-\alpha + \omega_0 < 0 \iff \omega_0 < \alpha$. It is then natural to write $\omega_0(S) \leq -\alpha + \omega < 0$.
Consequently $\overline{x}$ is locally exponentially stable for our system (3.5.1).

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