CONTROLLABILITY AND STABILIZABILITY OF LINEAR SYSTEMS IN HILBERT SPACES

A Project
Submitted to African University of Science and Technology,
Abuja-Nigeria
in partial fulfilment of the requirement for

MASTER DEGREE IN PURE AND APPLIED MATHEMATICS

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November 2010

Acknowledgement

Firstly, I would like to thank the Almighty Allah for being with me through out my studies. My sincere gratitude goes to my supervisor, Professor Khalil Ezzinbi for his guidance, support and patience from the begining to the end of this project. I also want acknowlegde the acting president of African University of Science and Technology, Abuja, Professor Charles Ejike Chidume, Professor Djitte Ngalla and Doctor Guy Degla for their advice, caring and encouragement through out my stay in AUST, same goes to entire community of AUST, which has been a great pleasure to live in.

Lastly but not the least, I would like to express my heart-felt appreciation and gratitude to my parents, brothers, sisters, relatives and friends for their continuous assistance, encouragement, guidance, support and prayer.

To my beloved parents:
Hashim Hassan
and
Jamila Lawal

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CHAPTER 1

Introduction

Questions about controllability and stability arise in almost every dynamical system problem. As a result, controllability and stability are one of the most extensively studied subjects in system theory. A departure point of control theory is the differential equation

$$\dot{x} = f(x, u), \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{1.0.1}$$

with the right hand side depending on a parameter u from a set $U \subset \mathbb{R}^m$. The set U is called *the set of control parameters.* Controls are of two types: *open* and *closed loops. An open loop control* can be basically an arbitrary function $u(\cdot) : [0, +\infty) \longrightarrow U$, for which the equation

$$\dot{x}(t) = f(x(t), u(t)), \qquad t \geq 0,\ x(0) = x_0 \in \mathbb{R}^n, \tag{1.0.2}$$

has a well defined solution.

A closed loop control can be identified with a mapping $k : \mathbb{R}^n \longrightarrow U$, which may depend on $t \geq 0$, such that the equation

$$\dot{x}(t) = f(x(t), k(x(t))), \qquad t \geq 0,\ x(0) = x_0 \in \mathbb{R}^n, \tag{1.0.3}$$

has a well defined solution. The mapping $k(\cdot)$ is called feedback. Controls are called also strategies or inputs, and the corresponding solutions of (1.0.2) or (1.0.3) are outputs of the system.

We do not consider all the system theory concepts here, we will concentrate mainly here on controllability and stability of linear system. To motivate our approach we present a brief survey of finite dimensional theory concepts and results which we will generalize later.

Linear System Theory A linear system is described by a (linear) differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \geq 0,\ x(0) = x_0, \tag{1.0.4}$$

where u is the control, x is the state. These functions take their values in linear spaces U and X, respectively. Furthermore, A and B are linear mappings between appropriate spaces. If the spaces U and X are finite dimensional, then the system is called finite dimensional. Otherwise, we have an infinite dimensional linear system.

1.1 Finite dimensional linear systems theory

Consider the linear finite dimensional control system:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \geq 0,\ x(0) = x_0, \tag{1.1.1}$$

where $x_0, x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, the linear transformations $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $B : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ will be identified with the representing matrices. System (1.1.1) is a differential equation on the state space $\mathbb{R}^n$ and the unique solution

$$x(t; x_0, u) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s. \tag{1.1.2}$$

The following concepts of controllability, stability and stablizability are now standard.

Controllability One says that a state $x_0 \in \mathbb{R}^n$ is reachable (or attainable) from x_1 in time T, if there exists an open loop control $u(\cdot)$ such that, for the output $x(\cdot)$, one has $x(0) = x_0, x(T) = x_1$. If an arbitrary state x_0 is reachable from an arbitrary state x_1 in a time T, then the system (1.1.1) is said to be controllable.

The proposition below gives a formula for a control transferring x_0 to x_1. In this formula the matrix Q_T, called the *controllability matrix* or *controllability Gramian*, appears:

$$Q_T = \int_0^t e^{A(T-s)}B(s)e^{A^*(T-s)}B(s)\,\mathrm{d}s, \quad T > 0,$$

where A^* is the transpose matrix of A. Then Q_T is symmetric and nonnegative definite.

Proposition 1.1.1. *Assume that for some $T > 0$ the matrix Q_T is nonsingular. Then for arbitrary $x_0, x_1 \in \mathbb{R}^n$ the control*

$$\hat{u}(s) = -B^*e^{A^*(T-s)}Q_T^{-1}(e^{A(T-s)}x_1 - x_0), \quad s \in [0, T],$$

transfers x_0 to x_1 at time T.

We now formulate algebraic conditions equivalent to controllability of (1.1.1).

Theorem 1.1.2. *The folowing conditions are equivalent:*

(i) An arbitrary state $x_1 \in \mathbb{R}^n$ is attainable from 0.

(ii) System (1.1.1) *is controllable.*

(iii) System (1.1.1) *is controllable at a given time $T > 0$.*

(iv) Matrix Q_T is nonsingular for some $T > 0$.

(v) Matrix Q_T s nonsingular for an arbitrary $T > 0$.

(vi) $\operatorname{rank}\left[B : BA : BA^2 : \ \cdots \ : BA^{n-1}\right] = n$.

Condition (vi) is called the *Kalman rank condition* or the *rank condition* for short.

Stability An important problem in the control of linear systems is the study of its sability. Consider the stability problem for the linear system

$$\dot{x}(t) = Ax(t), \quad t \geq 0,\ x(0) = x_0 \in \mathbb{R}^n \text{ and } A : \mathbb{R}^n \longrightarrow \mathbb{R}^n. \tag{1.1.3}$$

The system (1.1.3) is said to be *assymptotically stable* or *strongly stable* if for arbitrary $x_0 \in \mathbb{R}^n$

$$\lim_{t \to +\infty} \|x(t; x_0)\| = 0;$$

and *exponentially stable* or uniformly stable if there exist positive constants M and α such that

$$\|x(t; x_0)\| \leq Me^{-\alpha t}\|x_0\|, \quad t \geq 0.$$

Instead of saying that (1.1.3) we will often say that the matrix A is stable. We have the following well known result for finite dimensional linear system such as (1.1.3).

Theorem 1.1.3. *The following condition are equivalent:*

(i) *System* (1.1.3) *is assymptotically stable.*

(ii) *System* (1.1.3) *is exponentially stable.*

(iii) $\sup\{\mathrm{Re}\lambda; \lambda \in \sigma(A)\} < 0$.

(vi) $\int_0^{+\infty} \|x(t; x_0)\|^2 \,\mathrm{d}t < +\infty$.

(v) *The matrix equation*

$$A^*Q + AQ = -I \tag{1.1.4}$$

has a nonnegative symmetric solution Q.

Equation (1.1.4) is called *Lyapunov equation.*

Stabilizability The central theme within the system theory is to design a control u such that the corresponding state has desired behaviour. A typical example is stabilization: Design an control u such that $x(t) \to 0$ as $t \to +\infty$. A control designed by feedback achieves this goal. Feedback means that we do not calculate u only based on x_0, A, and B but also on the current state. This results in a control of the form $u(t) = Dx(t)$, and the question of stabilization turns into a question of finding D.

One says that the system (1.1.1) is stablizable if there exists an $m \times n$ marix D such that $A + BD$ is stable.

1.2 Motivation

From finite to infinite dimensional system To get an appreciation of the technical problems associated with the control of infinite dimensional systems, consider the distributed control of wave motion. Let $y(x, t)$ denote the transverse displacement at timt $t \geq 0$ of a vibrating medium in an $n-$dimensional bounded open region Ω with smooth boundary $\partial\Omega \subset \mathbb{R}^n$. Let us assume that

$$y(x,t) = 0, \quad \text{for } x \in \partial\Omega \text{ and } t \geq 0$$

and let the initial data at time $t = 0$ be

$$y(x,0) = y_0 \quad \text{and} \quad \frac{\partial}{\partial t} y(x,0) = y_1,$$

for some sufficiently smooth functions y_0 and y_1 defined on Ω. Consider the partial differential equation

$$\frac{\partial^2 y}{\partial t^2}(t) + Ay(t) = Bu(t) \tag{1.2.1}$$

or the equivalent first order system

$$\begin{cases} \dfrac{\partial y}{\partial t}(t) - z(t) = 0 \\ \\ \dfrac{\partial z}{\partial t}(t) + Ay(t) = Bu(t), \end{cases} \tag{1.2.2}$$

where the operator

$$A := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + a_0(x), \quad a_0(x) \geq \alpha > 0$$

is uniformly elliptic operator with $C^\infty-$coefficients in $\overline{\Omega}$, that is $a_{ij} \in C^\infty(\overline{\Omega})$. Let the controller $u(\cdot) \in L^2(0,\infty;\mathbb{R}^n)$ and let

$$B(x) := \big[B_1(x) \cdots B_m(x)\big]$$

be an $n \times m$ matrix with each column in $C^\infty(\bar{\Omega})$. We consiedr the state space

$$w := \begin{bmatrix} y \\ z \end{bmatrix} \in \mathcal{W} := H_0^1(\Omega) \otimes L^2(\Omega),$$

where $H_0^1(\Omega)$ denotes the usual Sobolev space of L^2-functions, with derivatives, in the distribution sense, belong to $L^2(\Omega)$ and vanishing at the boundary. Then (1.2.2) can be formally written as

$$\frac{\mathrm{d}w}{\mathrm{d}t}(t) + \tilde{A}w(t) = \tilde{B}u(t), \tag{1.2.3}$$

where

$$\tilde{A} = \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix} \quad \text{and} \quad \tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}.$$

Here $\tilde{B} \in \mathcal{L}(\mathbb{R}^m, \mathcal{W})$ and the operator $\tilde{A}$ is an unbounded opertor with a dense domain

$$\mathcal{D}(\tilde{A}) = [H_0^1(\Omega) \cap H^2(\Omega)] \otimes H_0^1(\Omega) \subset \mathcal{W}.$$

Because of the presence of the unbounded operator $\tilde{A}$, it is clear that the concept of a solution for (1.2.3) is not immediate. Intuitively, if we want (1.2.3) to have classical solution, then we would need

$$w_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \in \mathcal{D}(\tilde{A})$$

and $u(\cdot) \in C^1(\Omega)$. On the other hand, $-\tilde{A}$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on $\mathcal{W}$, i.e., $S(t)$ is a bounded linear opearator on $\mathcal{W}$ satisfying

(i) $S(0) = I$.

(ii) $S(t_1 + t_2) = S(t_1)S(t_2), \quad 0 < t_1, t_2 < +\infty$.

(iii) $t \longmapsto S(t)w : [0, +\infty) \longrightarrow \mathcal{W}$ is continuous for each $w \in \mathcal{W}$.

Then we say that $w(t)$ is a solution of (1.2.3) if it satisfies

$$w(t) = S(t)w_0 + \int_0^t S(t-s)\tilde{B}u(s)\,\mathrm{d}s, \tag{1.2.4}$$

where the integral is interpreted in the Bochner sense.

Now, given a strongly continuous semigroup S(t) on X (say a Banach space), it has a unique infinitesimal generator A, a closed unbounded operator that is defined on some dense domain D(A). The converse question, namely, when does A generate a strongly continuous semigroup, is a far more difficult issue and is the content of the Hille-Yosida theorem. Furthermore, there are different kinds of semigroups, and each plays an important role in the study of controllability and stability of infinite dimensional systems.

1.3 Organization of the work

The project is divided into two parts. Firstly, a general framework for the concepts of controllability and stability of linear systems in a Hilbert spaces is developed. Secondly, the concepts of controllability and stability of linear system in a Banach space is discussed.

Chapter 2

In chapter 2 we review some basic concepts of spectral theory and semigroup of linear operators, all the results discussed in this chapter are preliminary and can be found in any materials on the subject. The materials follows from Khalil Ezzinbi [3]; Klaus-Jochen Engel and Rainer Nagel [5]; R. F. Curtain and H. J. Zwart [6]; and Zheng-Hua Luo, Zhu Guo and Ömer Mörgul [8].

Chapter 3

Chapter 3 is dealed with the controllability and stabilizability of linear systems in Hilbert spaces. A general framework of concept of controllability of linear systems with bounded control is discussed and then the various concepts of controllability and sabilizability of linear systems and the the relationships between them is stated. almost all the results in this chapter are from A. J. Pritchard and T. Zabczyk [1]; Alain Bensoussen, Guiseppe Da Prato, Michel C. Delphour, Sanjoy K. Mitter [2]; Jerzy Zabczyk [4] and Ruth F. Curtain and Anthony J. Pritchard [7].

CHAPTER 2

Spectral Theory and Semigroup Properties

In this chapter we will review some basic concepts and results concerning spectral theory and semigroups of linear operators that will be needed in the whole project. The material is very standard and likely to have been found in any book on the subject, and so we state just the important results without proofs. A comprehensive discussions on the materials can be found in [3], [5], [6] and [8].

Notation. Throughout this chapter, we consier X to be a complex Banach space with the norm $\|\cdot\|$. If X and Y are Banach spaces, then $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y with the usual (induced) operator norm. We write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. sometimes we put a subscript near a norm such as in $\|x\|_X$, to indicate which norm we are using. H will denote the complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$.

2.1 Spectral Theory of Linear operators

Definition 2.1.1. *Let $A : \mathcal{D}(A) \subset X \to X$ be a closed linear operator. Then the* resolvent set *of A, denoted by $\rho(A)$, is defined by*

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A : \mathcal{D}(A) \to X \text{ is bijective}\}.$$

The spectrum *of A, denoted by $\sigma(A)$, is the complement of $\rho(A)$ in $\mathbb{C}$. For $\lambda \in \rho(A)$, the inverse*

$$R(\lambda, A) := (\lambda I - A)^{-1} \tag{2.1.1}$$

is, by closed graph theorem, a bounded linear operator on X and will be called the resolvent *(of A at the point λ).*

Remark 2.1.2.

(i) It follows immediately from the definition that the identity

$$AR(\lambda, A) = \lambda R(\lambda, A) - I \tag{2.1.2}$$

hold for every $\lambda \in \rho(A)$ and $R(\lambda, A)$ commutes with A.

(ii) For $\lambda, \mu \in \rho(\mathbb{C})$, one has

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \tag{2.1.3}$$

and

$$R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A). \tag{2.1.4}$$

(iii) If $\rho(A) \neq \emptyset$, then A is closed. Indeed, if $\lambda \in \rho(A)$, then the graph $\mathcal{G}(\lambda I - A)$ is the same as $\mathcal{G}((\lambda I - A)^{-1})$, except the coordinates are in reverse order.

Theorem 2.1.3. *Let $A : \mathcal{D}(A) \subset X \longrightarrow X$ be a closed linear operator, then the following properties hold.*

(i) The resolvent set $\rho(A)$ is open in $\mathbb{C}$, and for $\mu \in \rho(A)$ one has

$$R(\lambda, A) = \sum_{n=0}^{+\infty} (\mu - \lambda)^n R(\mu, A)^{n+1} \tag{2.1.5}$$

for $\lambda \in \mathbb{C}$ satisfying $|\mu - \lambda| < \frac{1}{\|R(\mu,A)\|}$.

(ii) The resolvent map *$\rho(A) \ni \lambda \longmapsto R(\lambda, A) \in \mathcal{L}(X)$ is locally analytic with*

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1} \quad \textit{for all } n \in \mathbb{N}. \tag{2.1.6}$$

(iii) For every $\lambda \in \rho(A)$ one has

$$\|R(\lambda, A)\| \geq \frac{1}{\mathrm{dist}(\lambda, \sigma(A))}. \tag{2.1.7}$$

(iv) Let $(\lambda_n)_{n\in\mathbb{N}} \subset \rho(A)$ with $\lambda_n \longrightarrow \lambda_0$ as $n \longrightarrow +\infty$. Then $\lambda_0 \in \sigma(A)$ if and only if

$$\lim_{n\to+\infty} \|R(\lambda_n, A)\| = +\infty.$$

As an immediate consequences, we have that the spectrum $\sigma(A)$ is a closed subset of $\mathbb{C}$. Nothing more can be said in general (see Example 2.1.16). However, if A is a bounded, it follows that

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\},$$

since

$$R(\lambda, A) = \frac{1}{\lambda}\left(1 - \frac{A}{\lambda}\right)^{-1} = \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}} \tag{2.1.8}$$

exists for all $|\lambda| > \|A\|$. In addition an applcation of *Liouvile's theorem* to the resolvent map implies $\sigma(A) \neq \emptyset$.

Corollary 2.1.4. *For a bounded operator A on X, the spectrum $\sigma(A)$ is alway compact and nonempty; hence its* spectral radius *defined by*

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}$$

is finite and satisfies $r(A) \leq \|A\|$.

The following proposition is known as *Gelfand formula.*

Proposition 2.1.5. *If $A \in \mathcal{L}(X)$, then $r(A) = \lim_{n\to+\infty} \|A^n\|^{\frac{1}{n}}$.*

Definition 2.1.6. *For a closed operator $A : \mathcal{D}(A) \subset X \to X$, the* point spectrum *of A, denoted by $\sigma_p(A)$, is defined by*

$$\sigma_p(A) := \{\lambda \in \mathbb{C} : \lambda I - A \textit{ is not injective}\}.$$

Moreover, each $\lambda \in \sigma_p(A)$ is called an eigenvalue, *and each $0 \neq x \in \mathcal{D}(A)$ satisfying $(\lambda I - A)x = 0$ is an* eigenvector *of A (corresponding to λ).*

In most casesd, the eigenvalues are simpler to determine than arbitrary spectral values. However, they do not, in general, exhaust the entire spectrum (see Example 2.1.16.)

Definition 2.1.7. *For a closed operator $A : \mathcal{D}(A) \subset X \to X$, the* approximate point spectrum *of A, denoted by $\sigma_a(A)$, is defined by*

$$\sigma_a(A) := \{\lambda \in \mathbb{C} : \exists (x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A) \text{ such that } \|x_n\| = 1, \lim_{n\to+\infty} \|Ax_n - \lambda x_n\| = 0\}.$$

Theorem 2.1.8. *For a closed operator $A : \mathcal{D}(A) \subset X \to X$, one has*

$$\sigma_a(A) := \{\lambda \in \mathbb{C} : \lambda I - A \textit{ is not injective or } \mathcal{R}(\lambda I - A) \textit{ is not closed in } X\}.$$

The inclusion $\sigma_p(A) \subset \sigma_a(A)$ is evident from the above theorem. The approximate point spectrum generalizes the point spectrum. However, as we see in the following proposition, it has the advantage that it is never empty unless $\sigma(A) = \emptyset$ or $\sigma(A) = \mathbb{C}$.

Proposition 2.1.9. *For a closed operator $A : \mathcal{D}(A) \subset X \to X$, the topological boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$ is contained in the approximate point spectrum $\sigma_a(A)$.*

The remaining part of the spectrum is now taken care of by the following definition.

Definition 2.1.10. *For a closed operator $A : \mathcal{D}(A) \subset X \to X$, the* residual spectrum *of A, denoted by $\sigma_r(A)$, is defined by*

$$\sigma_r(A) := \{\lambda \in \mathbb{C} : \lambda I - A \textit{ is injective but } \mathcal{R}(\lambda I - A) \textit{ is not dense in } X\}.$$

Definition 2.1.11. *For a closed operator $A : \mathcal{D}(A) \subset X \to X$, the* continuous spectrum *of A, denoted by $\sigma_c(A)$, is defined by*

$$\sigma_c(A) := \{\lambda \in \mathbb{C} : \lambda I - A \textit{ is injective and } \mathcal{R}(\lambda I - A) \textit{ is dense in } X\}.$$

All the possibilities for $\lambda I - A$ not being bijective are now covered by Defintions above, and hence

$$\sigma(A) = \sigma_a(A) \cup \sigma_c(A).$$

However, there is no reason for the union to be disjoint. It is easy to find an examples by applying the following very useful dual characterization of $\sigma_a(A)$.

Proposition 2.1.12. *For a closed, densely defined operator A, the residual spectrum $\sigma_a(A)$ coincides with the point spectrum $\sigma_p(A^*)$ of A^*.*

See (give ref in the appendix) for more information about the adjoint operator A^*.

In the next theorem we show that for each $\lambda_0 \in \rho(A)$ there is a canonical relation, called the spectral mapping theorem, between the spectrum of the unbounded operator A and the spectrum of the bounded operator $R(\lambda_0, A)$. This will allow us to transfer results from the spectral theory of bounded operators to the unbounded case.

Theorem 2.1.13 (Spectral Mapping Theorem for the Resolvent).
Let $A : \mathcal{D}(A) \subset X \to X$ be a closed operaor with nonempty resolvent set $\rho(A)$.

(1)

$$\sigma(R(\lambda_0, A))\backslash\{0\} = (\lambda_0 - \sigma(A))^{-1} := \left\{\frac{1}{\lambda_0 - \mu} : \mu \in \sigma(A)\right\} \tag{2.1.9}$$

for each $\lambda_0 \in \rho(A)$.

(2) The analogous statements hold for the point, approximate point, and residual spectra of A and $R(\lambda_0, A)$.

When we are working on a Hilbert space, the following results are useful.

Theorem 2.1.14. *Let $A : \mathcal{D}(A) \subset H \to H$ be a self-adjoint linear operator which is densely defined. Then $\lambda \in \rho(A)$ if and only if there exists a $c > 0$ such that for every $x \in \mathcal{D}(A)$,*

$$\|(A - \lambda I)x\| \geq c\|x\|. \tag{2.1.10}$$

Theorem 2.1.15. *The spectrum $\sigma(A)$ of a densely defined self-adjoint linear operator $A : \mathcal{D}(A) \subset H \to H$ is real and closed.*

Example 2.1.16. *On $X := C[0, 1]$ take the differential operators*

$$A_i = f' \qquad \text{for } i = 1, 2$$

with domain

$$\mathcal{D}(A_1) := C^1[0, 1] \qquad \text{and}$$

$$\mathcal{D}(A_2) := \{f \in C^1[0, 1] : f(1) = 0\}.$$

Then one can see that

$$\sigma(A_1) = \mathbb{C},$$

since for each $\lambda \in \mathbb{C}$ one has $(\lambda I - A)f_\lambda = 0$ for $f_\lambda(x) = e^{\lambda x}$, $x \in [0,1]$. On the other hand,

$$\sigma(A_2) = \emptyset,$$

since

$$R_\lambda(x) := \int_x^1 e^{\lambda(x-t)} f(x)\mathrm{d}t, \qquad x \in [0,1],\ f \in X,$$

yields the inverse of $(\lambda I - A)$ for every $\lambda \in \mathbb{C}$.

2.2 Semigroups of Linear operators

Definition 2.2.1. *Let X be a Banach space and let $(S(t))_{t\geq 0}$ be a family of bounded linear operators from X into X. $(S(t))_{t\geq 0}$ is called a semigroup of bounded linear operators, or simply a semigroup on X if*

(i)

$$S(0) = I,$$

(ii)

$$S(t+s) = S(t)S(s), \quad \textit{for} \quad t, s \geq 0.$$

A semigroup $(S(t))_{t\geq 0}$ is called uniformly continuous if

$$\lim_{t\downarrow 0} \|S(t) - I\| = 0, \tag{2.2.1}$$

and is called strongly continuous (or C_0-semigroup for short) if

$$\lim_{t\downarrow 0} S(t)x = x, \quad \textit{for all } x \in X. \tag{2.2.2}$$

We note that conditions (2.2.1) and (2.2.2) are the continuity requirements for $T(t)$ at $t = 0$ in uniform and strong operator topologies, respectively. It is also easy to see that continuity at $t = 0$ implies continuity at t for each $t > 0$ in each case. Actually, the above defined concept of semigroups can be extended to a one-parameter groups as follows.

Definition 2.2.2. *A one-parameter family $(S(t))_{t\in\mathbb{R}}$ of bounded linear operators on a Banach space X is called a strongly continuous group of bounded linear operators, or a C_0-group for short, if it satisfies*

(i)

$$S(0) = I.$$

(ii)

$$S(t+s) = S(t)S(s), \quad \textit{for all } t, s \in \mathbb{R}.$$

(iii)

$$\lim_{t \downarrow 0} S(t)x = x, \quad for\ all\ x \in X.$$

Definition 2.2.3. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X. The operator A defined by*

$$Ax = \lim_{t \downarrow 0} \frac{S(t)x - x}{t}, \quad \forall x \in \mathcal{D}(A) \tag{2.2.3}$$

with domain

$$\mathcal{D}(A) = \Big\{ x \in X : \lim_{t \downarrow 0} \frac{S(t)x - x}{t} \quad \text{exists} \Big\} \tag{2.2.4}$$

is called the infinitesimal generator *of the semigroup* $(S(t))_{t\geq 0}$.

Similarly, we defined the infinitesimal generators of C_0-groups.

Definition 2.2.4. *Let $(S(t))_{t\in\mathbb{R}}$ be a C_0-group on X. The operator A defined by*

$$Ax = \lim_{t \downarrow 0} \frac{S(t)x - x}{t}, \quad \forall x \in \mathcal{D}(A) \tag{2.2.5}$$

with domain

$$\mathcal{D}(A) = \Big\{ x \in X : \lim_{t \downarrow 0} \frac{S(t)x - x}{t} \quad \text{exists} \Big\} \tag{2.2.6}$$

is called the infinitesimal generator *of the group* $(S(t))_{t\in\mathbb{R}}$.

It is obvious that if $(S(t))_{t\geq 0}$ is a C_0-group with the generator A, then both $(S(t))_{t\geq 0}$ and $(S(-t))_{t\geq 0}$ are C_0-semigroups with the generators A and $-A$, respectively. Conversely, if both A and $-A$ generate C_0-semigroups $(S_+(t))_{t\geq 0}$ and $(S_-(t))_{t\geq 0}$, respectively, then A is the infinitesimal generator of the C_0-group $S(t)$ given by

$$\begin{cases} S_+(t), & t \geq 0, \\ S_-(-t), & t < 0. \end{cases}$$

Theorem 2.2.5. *A C_0-semigroup on a Banach space X has the following properties:*

(i) $\|S(t)\|$ *is bounded on every subinterval of* $[0, +\infty)$*;*

(ii)

$$If \quad \omega_0 = \inf_{t>0} \Big(\frac{1}{t} \log \|S(t)\| \Big), \quad then \quad \omega_0 = \lim_{t \to +\infty} \Big(\frac{1}{t} \log \|S(t)\| \Big) < +\infty;$$

(iii) For all $\omega > \omega_0$, there exists a constant M_ω such that

$$\|S(t)\| \leq M_\omega e^{\omega t} \quad for\ all\ t \geq 0. \tag{2.2.7}$$

This constant ω_0 is called the growth bound *or the* type *of the semigroup.*

(iv) For every $x \in X, t \longmapsto S(t)x$ is a continuous function from $[0, +\infty)$ into X.

Theorem 2.2.6. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with the infinitesimal generator A. The the following results hold:*

(i) For all $x \in X$,
$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} S(s)x\,\mathrm{d}s = S(t)x;$$

(ii) For $x \in \mathcal{D}(A)$, $S(t)x \in \mathcal{D}(A)$, $\forall t \geq 0$;

(iii) $\forall x \in \mathcal{D}(A), S(\cdot)x \in C^1([0,+\infty);X) \cap C([0,+\infty);\mathcal{D}(A))$ and
$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax \ for \ x \in \mathcal{D}(A), \ t > 0;$$

(vi) $\dfrac{d^n}{dt^n}(S(t)x) = A^nS(t)x = S(t)A^nx$ for $x \in \mathcal{D}(A^n)$, $t > 0$;

(v) $\int_0^t S(s)xds \in \mathcal{D}(A)$ and $A\int_0^t S(s)xds = S(t)x - x$ for all $x \in X$;

(vi) $S(t)x - S(s)x = \int_s^t S(\tau)x\,\mathrm{d}\tau$ for $x \in \mathcal{D}(A)$;

(vii) $\mathcal{D}(A)$ is dense in X and A is a closed linear operator;

(viii) $\bigcap_{n=1}^{+\infty}\mathcal{D}(A^n)$ is dense in X.

The space $\mathcal{D}(A^n)$ is recursively defined as:
$$\mathcal{D}(A^n) = \{x \in \mathcal{D}(A) : Ax \in \mathcal{D}(A^{n-1})\}.$$

The following theorem proves that C_0-semigroups are uniquely determined by their generators.

Theorem 2.2.7. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be C_0-semigroups, and let A and B be their infinitesimal generators, respectively. If $A = B$, then $T(t) = S(t)$ for $t \geq 0$.*

The following proposition gives the spectral properties of the infinitesimal generator of C_0-semigroup.

Proposition 2.2.8. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X. Assume that there exist real numbers $M \geq 1$ and $\omega \in \mathbb{R}$ such that*
$$\|S(t)\| \leq Me^{\omega t}, \ \forall t \geq 0.$$

Then the infinitesimal generator A of $(S(t))_{t\geq 0}$ has the following properties:

(i) $C_\omega = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$,

(ii) for any $\lambda \in C_\omega$ the resolvent of A is given by
$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t}S(t)xdt, \ \forall x \in X. \tag{2.2.8}$$

Moreover, we have

$$\lim_{\mathrm{Re}\lambda\to\infty}\|R(\lambda,A)\|=0. \tag{2.2.9}$$

In fact, equation (2.2.8) states that the resolvent $R(\lambda, A)$ is the *Laplace transform* of the semigroup $S(t)$. The complete characterization of those linear operators which generate C_0-semigroup is given in the following theorem.

Theorem 2.2.9 (Hille-Yosida). *Let $A : \mathcal{D}(A) \subset X \to X$ be a closed linear operator. The following statements are equivalent:*

(i) $\mathcal{D}(A)$ is dense in X, there exist real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that $\rho(A) \supset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \omega\}$ and the following inequalities hold:

$$\|R(\lambda,A)^k\|\le\frac{M}{(\mathrm{Re}\lambda-\omega)^k},\ \forall k\in\mathbb{N},\forall\lambda \textit{ such that } \mathrm{Re}\lambda>\omega. \tag{2.2.10}$$

(ii) A is the infinitesimal generator of C_0-semigroup $(S(t))_{t\ge0}$ and there exist real numbers $\omega \in \mathbb{R}$ and $M > 0$ such that

$$\|S(t)\|\le Me^{\omega t},\ \forall t\ge 0. \tag{2.2.11}$$

Remark 2.2.10. *Let A satisfies condition (i) of Theorem 2.2.9. Then for any $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > \omega$, we set*

$$J_\lambda=\lambda R(\lambda,A),\qquad A_\lambda=AJ_\lambda \tag{2.2.12}$$

and recall that

$$A_\lambda=\lambda^2R(\lambda,A)-\lambda I.$$

The bounded linear operators A_λ are called the Yosida approximations *of A and satisfy*

$$\lim_{\mathrm{Re}\to\infty}J_\lambda x=x,\quad x\in X, \tag{2.2.13}$$

$$\lim_{\mathrm{Re}\to\infty}A_\lambda x=Ax,\quad x\in\mathcal{D}(A). \tag{2.2.14}$$

Theorem 2.2.11. *Let A be the infinitesimal generator of a C_0-semigroup on X. If A_λ is the Yosida approximation of A, i.e., $A_\lambda = \lambda AR(\lambda, A)$ then*

$$S(t)x=\lim_{\lambda\to\infty}e^{tA_\lambda}x,\qquad x\in X. \tag{2.2.15}$$

For the case where $X = H$, the following theorem gives a sufficient condition for a linear operator A to generate a C_0-semigroup and is useful since the conditions are easier to verify.

Theorem 2.2.12. *Let A be a closed densely defined linear operator on a Hilbert space X, and assume there exists a real number ω such that*

$$\mathrm{Re}\langle Ax,x\rangle\le\omega\|x\|,\quad x\in\mathcal{D}(A). \tag{2.2.16}$$

If

$$\mathrm{Re}\langle A^*x,x\rangle\le\omega\|x\|,\quad x\in\mathcal{D}(A^*), \tag{2.2.17}$$

or for some positive $\lambda > \omega$, $\lambda I - A$ is onto X, then A generates a C_0-semigroup $(S(t))_{t\ge0}$ such that

$$\|S(t)\|\le e^{\omega t}.$$

Given a semigroup of bounded linear operators $(S(t))_{t\geq 0}$, on X. For $t \geq 0$ let $S^*(t)$ be the adjoint operator of $S(t)$. From the definition of adjoint operators it is clear that the family $(S^*(t))_{t\geq 0}$, of bounded linear operators on X^*, satifies

$$S^*(0) = I,$$

$$\forall t \geq 0, \quad \forall s \geq 0, \quad S^*(t+s) = S^*(t)S^*(s),$$

$$\forall x^* \in X^*, \quad t \mapsto S^*(t)x^* : [0, +\infty) \to X^* \text{ is weak}^* \text{ continuous.}$$

The family $(S^*(t))_{t\geq 0}$, is called the *adjoint semigroup* associated with $(S(t))_{t\geq 0}$. The adjoint semigroup however, need not be a C_0-semigroup on X^* since the mapping $S(t) \longmapsto S^*(t)$ does not necessarily conserve the strong continuity of $S(t)$. Furtunately we have the following result in reflexive Banach spaces.

Proposition 2.2.13. *Let X be a reflexive Banach space and $(S(t))_{t\geq 0}$ be a C_0-semigroup on X with infinitesimal generator A. The adjoint semigroup $(S^*(t))_{t\geq 0}$ of $(S(t))_{t\geq 0}$ is a C_0-semigroup on X^* whose infinitesimal generator is A^* the adjoint of A.*

Definition 2.2.14. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X, then $(S(t))_{t\geq 0}$ is called a* semigroup of contractions *if*

$$\|S(t)\| \leq 1, \qquad \forall t \geq 0. \tag{2.2.18}$$

If $(S(t))_{t\geq 0}$ is a semigroup of contractions, then condition (2.2.10) reduces to

$$\|R(\lambda, A)\| \leq \frac{1}{\mathrm{Re}\lambda}, \ \forall \lambda, \ \mathrm{Re}\lambda > 0. \tag{2.2.19}$$

In this case we will give another characterization of linear operators that generate semigroups of contractions. For this purpose we need the definition of *dissipative operators*. We first recall the concept of *duality set* $F(x) \subseteq X^*$ is defined by

$$F(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. It should be noted that $F(x) \neq \emptyset$ for any $x \in X$ by the Hahn-Banach theorem.

Definition 2.2.15. *Let $x \in X$ and $F(x)$ be the duality set. A linear operator A in X is said to be* dissipative *if for every $x \in \mathcal{D}(A)$ there exists $x^* \in F(x)$ such that*

$$\mathrm{Re}\langle Ax, x^*\rangle \leq 0. \tag{2.2.20}$$

Theorem 2.2.16. *Let $x, y \in X$. Then*

$$\|x\| \leq \|x - hy\|, \quad \forall h > 0$$

if and if there exists an $x^* \in F(x)$ such that $\mathrm{Re}\langle y, x^*\rangle \leq 0$.

Corollary 2.2.17. *Let A be a linear operator in X, then A is dissipative if and only if $\|x\| \leq \|x - hAx\|$ for each $h > 0$ and all $x \in \mathcal{D}(A)$.*

Proposition 2.2.18. *Let A be a linear operator in X. Then A generates a C_0-semigroup of contractions on X if and only if*

(i) $\overline{\mathcal{D}(A)} = X$.

(ii) *A is dissipative and* $\mathrm{Im}(\lambda I - A) = X$, $\forall \lambda > 0$.

Definition 2.2.19. *A linear operator A in X is called* m-dissipative *if A is dissipative and* $\mathrm{Im}(\lambda_0 I - A) = X$ *for some* $\lambda_0 > 0$.

Theorem 2.2.20 (Lümer-Phillips). *Let A be a linear operator in X. Then A generates a C_0-semigroup of contractions if and only if*

(i) $\overline{\mathcal{D}(A)} = X$.

(ii) *A is m-dissipative.*

Theorem 2.2.21. *Let A be a linear operator in X. Then A is the generators of a C_0-semigroup of contractions on X if and only if*

(i) *A is closed linear operator with dense domain in X and*

(ii) *A and its adjoint operator A^* are dissipative.*

Theorem 2.2.22 (Stone). *Assume that $X = H$. Then A generates a C_0-group of unitary operators on H if and only if iA is self-adjoint, or A is skew-adjoint, i.e., $A^* = -A$.*

The next result relates the semigroup $T(t)$ and the *inverse Laplace transform* of $R(\lambda, A)$.

Theorem 2.2.23 (The exponential formula). *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X, and let A be its infinitesimal generator. Then the following holds.*

$$S(t)x = \lim_{n\to+\infty} \left(I - \frac{tA}{n}\right)^{-n} x = \lim_{n\to+\infty} \left[\frac{t}{n} R(\frac{n}{t}, A)\right]^{n} x, \quad x \in X, \tag{2.2.21}$$

where the limit is uniform in t on any bounded interval.

Definition 2.2.24. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X. $(S(t))_{t\geq 0}$ is said to be* differentiable *if for every $x \in X$, $t \mapsto S(t)x$ is differentiable for $t > 0$.*

Recall that by Theorem 2.2.6 (ii), if $(S(t))_{t\geq 0}$ is a C_0-semigroup generated by A and $x \in \mathcal{D}(A)$, then $S(t)x$ is differentiable $t > 0$, and $\frac{d}{dt}S(t)x = AS(t)x$. If $S(t)$ is differentiable, then this property holds for all $x \in X$ and $t > 0$. Note that if $S(t)x$ is differentiable for every $x \in X$ and $t \geq 0$ then $\mathcal{D}(A) = X$ and since A is closed it is necessarily bounded and $S(t)$ must be a uniformly continuous semigroup.

Theorem 2.2.25. *Let $(S(t))_{t\geq 0}$ be a differentiable C_0-semigroup on X and let A be its infinitesimal generator. Then the following hold*

(i) *For every $x \in X$ and $t > 0$,* $S(t)x \in \bigcap_{n=1}^{+\infty} \mathcal{D}(A^n)$.

(ii) For $t > 0$ and $n \in \mathbb{N}$, we have $S^{(n)}(t) = A^n S(t)$, $S^{(n)}(t)$ is a bounded linear operator and is continuous in the uniform operator topology and the following holds.

$$S^{(n)}(t) = \Big[AS(\frac{t}{n})\Big]^n = \Big[\frac{d}{dt}S(\frac{t}{n})\Big]^n. \tag{2.2.22}$$

Definition 2.2.26. *Let $A : \mathcal{D}(A) \subset X \to X$ be a closed operator. Then*

$$s(A) := \begin{cases} \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}, & \text{if } \sigma(A) \neq \emptyset, \\ -\infty, & \text{if } \sigma(A) = \emptyset, \end{cases} \tag{2.2.23}$$

is called the spectral bound *of A.*

Definition 2.2.27. *Let A be the infinitesimal generator of a C_0-semigroup of type ω_0. Then we say that A satisfies the* spectral determining growth condition *if $s(A) = \omega_0$.*

Proposition 2.2.28. *Let A be the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on X of type ω_0. Then*

$$\forall t \geq 0, \quad e^{\sigma(A)t} := \{e^{\lambda t} : \lambda \in \sigma(A)\} \subset \sigma(S(t)), \tag{2.2.24}$$

and

$$s(A) \leq \omega_0. \tag{2.2.25}$$

2.3 Examples

Example 2.3.1. *Let $X = L^2(0, +\infty$ be a Banach space which is specified by*

$$L^2(0,+\infty) = \Big\{f : [0,+\infty) \longrightarrow \mathbb{C} : \int_0^{+\infty} |f(x)|^2 \,\mathrm{d}x < +\infty\Big\}$$

with its usual norm, i.e., for $f \in X, \|f\|_2 = \Big(\int_0^{+\infty} |f(x)|^2 \,\mathrm{d}x\Big)^{1/2}$. Consider the following shift (translation) operator defined by

$$(T(t)f)(x) = f(x+t) \qquad for\ f \in X\ and\ x \in [0,+\infty).$$

Then

(i) the family $(T(t))_{t\geq 0}$ is a C_0-semigroup on X with $\|T(t)\| = 1 \qquad \forall t \geq 0$,

(ii) the infintesimal generator A of $T(t)$ is given by

$$A = \frac{\mathrm{d}}{\mathrm{dx}}, \qquad \mathcal{D}(A) = H^1(0,+\infty),$$

(iii) the spectrum of A is given by

$$\sigma(A) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq 0\},$$

(iv) *the point spectrum of A is given by*

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}.$$

Solution.

(i) We show that $T(t)$ satisfies the conditions (i), (ii) and equation (2.2.2) of definition 2.2.1.

- For each $x \in [0, +\infty)$, one has
$$(T(0)f)(x) = f(x + 0) = f(x), \tag{2.3.1}$$
which entails that $T(0) = I$.
- For each $x \in [0, +\infty)$,
$$(T(t + s)f)(x) = f(x + t + s) \qquad \forall\, t, s \geq 0, \tag{2.3.2}$$
on one hand.
On the other hand, for each $x \in [0, +\infty)$,
$$T(t)(T(s)f)(x) = T(t)(f(x + s)) = f(x + s + t), \qquad \forall t, s \geq 0. \tag{2.3.3}$$
Combining equation (2.3.2) and (2.3.3) one has
$$T(t + t) = T(t)T(s) \qquad \forall t, s \geq 0. \tag{2.3.4}$$
Equations (2.3.1) and 2.3.4 entails that $T(t)$ defines a semigroup on X.
- For strong continuity, let us first prove that for each $t \geq 0$, $\|T(t)\| = 1$. Let $t \geq 0$ be fixed, then one has
$$\begin{aligned}\|T(t)f\|_2^2 &= \int_0^{+\infty} |T(t)f(x)|^2 \,\mathrm{d}x = \int_0^{+\infty} |f(x + s)|^2 \,\mathrm{d}x = \int_t^{+\infty} |f(x)|^2 \,\mathrm{d}x \\ &\leq \int_0^{+\infty} |f(x)|^2 \,\mathrm{d}x = \|f\|_2^2,\end{aligned}$$
this implies that $\|T(t)f\|_2 \leq \|f\|_2$, which yields $\|T(t)\| \leq 1$ on one hand.
On the other hand, for fixed $t \geq 0$, consider $f_{0t} = \chi_{[t,t+1]}$, then $f_{0t} \in X$ since
$$\int_0^{+\infty} |f_{0t}(x)|^2 \,\mathrm{d}x = \int_0^{+\infty} |\chi_{[t,t+1]}(x)|^2 \,\mathrm{d}x = \int_t^{t+1} |\chi_{[t,t+1]}(x)|^2 \,\mathrm{d}x = 1 < +\infty,$$
therefore
$$\begin{aligned}\int_0^{+\infty} |T(t)f_{0t}(x)|^2 \,\mathrm{d}x &= \int_0^{+\infty} |f_{0t}(x + t)|^2 \,\mathrm{d}x = \int_0^{+\infty} |\chi_{[t,t+1]}(x + t)|^2 \,\mathrm{d}x \\ &= \int_t^{+\infty} |\chi_{[t,t+1]}(x)|^2 \,\mathrm{d}x = \int_t^{t+1} |\chi_{[t,t+1]}(x)|^2 \,\mathrm{d}x = 1,\end{aligned}$$

which implies that

$$1 = \|T(t)f_{0t}\|_2 \leq \sup_{f\in X:\|f\|=1} \|T(t)f\| = \|T(t)\|.$$

Hence

$$\|T(t)\| = 1 \qquad \forall t \geq 0. \tag{2.3.5}$$

Now let $\varphi \in C_0(0,+\infty)$, i.e., φ is a continuous function defined on $[0,+\infty)$ with compact support in $[0,+\infty)$. Then

$$\begin{aligned}
\lim_{t\downarrow 0}\|T(t)\varphi-\varphi\|_2^2 &= \lim_{t\downarrow 0}\int_0^{+\infty} |\varphi(x+t)-\varphi(x)|^2\,\mathrm{d}x \\
&= \lim_{t\downarrow 0}\int_0^{K} |\varphi(x+t)-\varphi(x)|^2\,\mathrm{d}x, \quad \text{for some } K\in\mathbb{R}.
\end{aligned}$$

Therefore using *Lebesgue Dominated Convergence Theorem*, one has

$$\lim_{t\downarrow 0}\|T(t)\varphi-\varphi\|_2 = 0. \tag{2.3.6}$$

Since $C_0(0,+\infty)$ is dense in X, it follows that for a function $f\in X$ and any given $\varepsilon>0$, there exists $\varphi\in C_0(0,+\infty)$ such that

$$\|f-\varphi\|_2 < \frac{\varepsilon}{4}. \tag{2.3.7}$$

Also from equation (2.3.6) it follows that there exists a $\delta>0$ such that for any t satisfying $0<t<\delta$ one has

$$\|T(t)\varphi-\varphi\|_2 < \frac{\varepsilon}{2}. \tag{2.3.8}$$

Therefore from (2.3.7) and (2.3.8) it follows that for any $f\in X$ and an $\varepsilon>0$, there exists a $\delta>0$ such that $0<t<\delta$ implies

$$\begin{aligned}
\|T(t)f-f\|_2 &= \|T(t)f-T(t)\varphi+T(t)\varphi-\varphi+\varphi-f\|_2 \\
&\leq \|T(t)f-T(t)\varphi\|_2+\|T(t)\varphi-\varphi\|_2+\|\varphi-f\|_2 \\
&\leq \|T(t)\|\|f-\varphi\|_2+\|T(t)\varphi-\varphi\|_2+\|f-\varphi\|_2 \\
&= 2\|f-\varphi\|_2+\|T(t)\varphi-\varphi\|_2 \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,
\end{aligned}$$

therefore one has

$$\lim_{t\downarrow 0}\|T(t)f-f\|_2 = 0.$$

Thus $T(t)$ defines a C_0-semigroup on X.

(ii) The infinitesimal generator of $T(t)$ is found to be

$$Af(x) = \lim_{t\downarrow 0} \frac{f(x+t)-f(x)}{t} = \frac{\mathrm{d}^+}{\mathrm{dx}} f(x), \quad \forall x \in [0,+\infty),$$

i.e.,

$$Af = \frac{\mathrm{d}^+ f}{\mathrm{dx}} \qquad \text{with the domain} \qquad \mathcal{D}(A) = \left\{ f \in X : \frac{\mathrm{d}^+ f}{\mathrm{dx}} \in X \right\}.$$

We claim that the infinitesimal generator of $T(t)$ is

$$A = \frac{\mathrm{d}}{\mathrm{dx}} \quad \text{and} \quad \mathcal{D}(A) = H^1(0,+\infty).$$

Since $T(t)$ is a C_0-semigroup and $\|T(t)\| = 1 \quad \forall t \geq 0$, it follows from *Hille-Yosida Theorem* 2.2.9 with $M = 1$ and $\omega = 0$ that $1 \in \rho(A)$ and for every $f \in X$ one has

$$[(I-A)^{-1}f](x) = \int_0^{+\infty} e^{-t} f(x+t)\mathrm{d}t = e^x \int_x^{+\infty} e^{-t} f(t)\mathrm{d}t,$$

holds for almost every $x \in [0,+\infty]$. Denoting $\varphi = (I-A)^{-1}f$, it follows that φ is continuous and the above formula holds for all $x \in [0,+\infty)$. Rearranging the formula one has

$$\begin{aligned} \varphi(x) &= e^x \int_x^{+\infty} e^{-t} f(t)\mathrm{d}t \\ &= e^x \int_0^{+\infty} e^{-t} f(t)\mathrm{d}t - e^x \int_0^x e^{-t} f(t)\mathrm{d}t \\ &= e^x \varphi(0) - \int_0^x e^{-t} f(t)\mathrm{d}t \qquad \forall x \geq 0. \end{aligned}$$

This shows that φ is locally absolutely continuous and

$$\begin{aligned} \varphi'(x) &= e^x \varphi(0) - f(x) - e^x \int_0^x e^{-t} f(t)\mathrm{d}t \\ &= e^x \varphi(0) - f(x) - e^x \int_0^{+\infty} e^{-t} f(t)\mathrm{d}t + e^x \int_x^{+\infty} e^{-t} f(t)\mathrm{d}t \\ &= e^x \varphi(0) - f(x) - e^x \varphi(0) + \varphi(x) = \varphi(x) - f(x) \end{aligned}$$

holds for almost every $x \geq 0$, which gives $\varphi' = \varphi - f$. Since both φ and f are in X, it follws that $\varphi' \in X$, hence $\varphi \in H^1(0,+\infty)$. Thus $\mathcal{D}(A) \subset H^1(0,+\infty)$. By definition of φ, we have $(I-A)\varphi = f$ which implies that $A\varphi = \varphi - f$. Comparing this with the formula $\varphi' = \varphi - f$, it follows that $A\varphi = \varphi'$. Since $\forall \varphi \in \mathcal{D}(A)$ there exists a unique $f \in X$ such that $\varphi = (I-A)^{-1}f$, we have

$$A\varphi = \varphi' \qquad \forall \varphi \in \mathcal{D}(A).$$

If the inclusion $\mathcal{D}(A) \subset H^1(0,+\infty)$ were strict, then there would exists $\psi \in H^1(0,+\infty)$ such that $\psi \notin \mathcal{D}(A)$. Denote $f = \psi - \psi'$ and put $\varphi = (I-A)^{-1}f$, then $\varphi - \varphi' = f$. Denoting $\eta = \psi - \varphi$ we obtain that $\eta \in H^1(0,+\infty)$ and

$$\eta' = \psi' - \varphi' = (\psi - f) - (\varphi - f) = \psi - \varphi = \eta,$$

hence $\eta = 0$, therefore $\psi = \varphi \in \mathcal{D}(A)$ which is a contradiction, so we have proved our claim.

(iii) Let $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda < 0$. Consider the function f_λ defined by $f_\lambda(x) = e^{\lambda x}$ for all $x \in [0,+\infty)$, then $f_\lambda \in X$ since

$$\int_0^{+\infty} |f_\lambda(x)|^2 \,\mathrm{d}x = \int_0^{+\infty} |e^{\lambda x}|^2 \,\mathrm{d}x = \int_0^{+\infty} e^{2\mathrm{Re}\lambda x} \,\mathrm{d}x = -\frac{1}{2\mathrm{Re}\lambda} < +\infty,$$

and

$$(\lambda I - A)f_\lambda = \lambda f_\lambda - Af_\lambda = \lambda f_\lambda - f_\lambda' = \lambda f_\lambda - \lambda f_\lambda = 0,$$

hence λ is an eigenvalue of A, and therefore

$$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\} \subset \sigma(A).$$

Since $\sigma(A)$ is closed, it follows that

$$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq 0\} \subset \sigma(A). \tag{2.3.9}$$

On the other hand, one has from *Hille-Yosida Theorem* (2.2.9) that

$$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\} \subset \rho(A),$$

which implies that

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq 0\}. \tag{2.3.10}$$

Thus (2.3.9) and (2.3.10)) together entails that

$$\sigma(A) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq 0\}. \tag{2.3.11}$$

(iv) From (iii) above it follows that $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\} \subset \sigma_p(A)$. So it is enough to show that the points on the imaginary axis are not eigeinvalues of A.
Let $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda = 0$, then λ is of the form $\lambda = i\mu$, $\mu \in \mathbb{R}$. Assume that

$$(\lambda I - A)f = 0, \text{ we show that } f = 0.$$

But $(\lambda I - A)f = 0$ implies

$$f' = \lambda f \Longrightarrow f(x) = Ce^{\lambda x}, \quad \forall x \in [0,+\infty) \text{ and some constant } C.$$

Therefore

$$\int_0^{+\infty} |f(x)|^2 \,\mathrm{d}x = \int_0^{+\infty} |C|^2 |e^{\lambda x}|^2 \,\mathrm{d}x = \int_0^{+\infty} |C|^2 |e^{i\mu x}| \,\mathrm{d}x = \begin{cases} 0 & \text{if } C = 0, \\ +\infty & \text{if } C \neq 0, \end{cases}$$

implies that $f = 0$ since $f \in X$, hence λ is not eigenvalues of A, so that

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}.$$

From equations (2.3.5) and (2.3.11) we have $\omega_0 = s(A)$, hence the semigroup satisfies the spectral determining growth condition. ■

Example 2.3.2. *Let $\{\lambda_n, n \geq 1\}$ be a sequence of complex numbers and $\{\phi_n, n \geq 1\}$ be an orthonormal basis in a separable Hilbert space H. Now define on H the operator A by*

$$Ax = \sum_{n=1}^{+\infty} \lambda_n \langle x, \phi_n \rangle \phi_n,$$

with the domain

$$\mathcal{D}(A) = \left\{ x \in X : \sum_{n=1}^{+\infty} |\lambda_n \langle x, \phi_n \rangle|^2 < +\infty \right\}.$$

Then A is a closed, densely defined linear operator on H, and $(\lambda I - A)$ is invertible if and only if $\inf_n |\lambda - \lambda_n| > 0$. Moreover, A generates a strongly continuous semigroup if $\sup_n \mathrm{Re}\lambda_n < +\infty$, given by

$$T(t)x = \sum_{n=1}^{+\infty} e^{\lambda_n t} \langle x, \phi_n \rangle \phi_n. \tag{2.3.12}$$

Solution.

- First we show that A is closed and densely defined operator.
  Let $(x_n)_{n \geq 1}$ be a sequence in $\mathcal{D}(A)$ such that $x_n \longrightarrow x$ and $Ax_n \longrightarrow y$ as $n \longrightarrow +\infty$. We show that $x \in \mathcal{D}(A)$ and $Ax = y$.
  For each $k \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{j=1}^{k} |\lambda_j \langle x, \phi_j \rangle|^2 &= \sum_{j=1}^{k} |\lambda_j \langle \lim_{n \to +\infty} x_n, \phi_j \rangle|^2 = \lim_{n \to +\infty} \sum_{j=1}^{k} |\lambda_j \langle x_n, \phi_j \rangle|^2 \\
&\leq \lim_{n \to +\infty} \sum_{j=1}^{+\infty} |\lambda_j \langle x_n, \phi_j \rangle|^2 = \lim_{n \to +\infty} \|Ax_n\|^2 \\
&= \|y\|^2 < +\infty,
\end{aligned}$$

  which implies that

$$\sum_{j=1}^{+\infty} |\lambda_j \langle x, \phi_j \rangle|^2 < +\infty,$$

  so that $x \in \mathcal{D}(A)$.

Since $\{\phi_n, n \geq 1\}$ is an orthonormal basis of H, it follws that

$$\begin{aligned}
y &= \sum_{j=1}^{+\infty} \langle y, \phi_j \rangle \phi_j = \sum_{j=1}^{+\infty} \langle \lim_{n\to+\infty} Ax_n, \phi_j \rangle \phi_j = \sum_{j=1}^{+\infty} \lim_{n\to+\infty} \langle Ax_n, \phi_j \rangle \phi_j \\
&= \sum_{j=1}^{+\infty} \lim_{n\to+\infty} \left\langle \sum_{k=1}^{+\infty} \lambda_k \langle x_n, \phi_k \rangle \phi_k, \phi_j \right\rangle \phi_j = \sum_{j=1}^{+\infty} \lim_{n\to+\infty} \lambda_j \langle x_n, \phi_j \rangle \phi_j \\
&= \sum_{j=1}^{+\infty} \lambda_j \langle x, \phi_j \rangle \phi_j = Ax.
\end{aligned}$$

Hence A is closed.
For the density of $\mathcal{D}(A)$ in H, we have

$$\sum_{j=1}^{+\infty} |\lambda_j \langle \phi_n, \phi_j \rangle|^2 = |\lambda_n|^2 < +\infty \quad \forall n \in \mathbb{N},$$

therefore $\phi_n \in \mathcal{D}(A) \quad \forall n \in \mathbb{N}$. Let $x \in H$, by setting $x_n = \sum_{j=1}^{n} \langle x, \phi_j \rangle \phi_j$ we have $x_n \in \mathcal{D}(A) \quad \forall n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \sum_{j=1}^{n} \langle x, \phi_j \rangle \phi_j = \sum_{j=1}^{+\infty} \langle x, \phi_j \rangle \phi_j = x,$$

which entails that $\mathcal{D}(A)$ is dense H.

- We shall now show that $(\lambda I - A)$ is invertible if and only if $\inf_n |\lambda - \lambda_n| > 0$. Assume that $\inf_n |\lambda - \lambda_n| > 0$, then define the linear operator A_λ on H by

$$A_\lambda x = \sum_{n=1}^{+\infty} \frac{1}{\lambda - \lambda_n} \langle x, \phi_n \rangle \phi_n \tag{2.3.13}$$

it follows that

$$\begin{aligned}
\|A_\lambda x\|^2 &\leq \sum_{n=1}^{+\infty} \frac{1}{|\lambda - \lambda_n|^2} |\langle x, \phi_n \rangle|^2 \\
&\leq \frac{1}{\inf_n |\lambda - \lambda_n|^2} \sum_{n=1}^{+\infty} |\langle x, \phi_n \rangle|^2 \\
&\leq \frac{1}{\inf_n |\lambda - \lambda_n|^2} \|x\|^2, \qquad \forall x \in H,
\end{aligned}$$

hence A_λ defines a bounded linear operator on H. Let $x \in H$, then $A_\lambda x = y$ for some $y \in H$. We claim that $y \in \mathcal{D}(A)$. By setting

$$y = \sum_{n=1}^{+\infty} \langle y, \phi_n \rangle \phi_n$$

one has

$$\sum_{n=1}^{+\infty} \frac{1}{\lambda-\lambda_n}\langle x,\phi_n\rangle\phi_n = \sum_{n=1}^{+\infty}\langle y,\phi_n\rangle\phi_n,$$

which yields

$$\lambda_n\langle y,\phi_n\rangle = \lambda\langle y,\phi_n\rangle - \langle x,\phi_n\rangle \qquad \forall n\in\mathbb{N},$$

which implies that

$$\sum_{n=1}^{+\infty}|\lambda_n\langle y,\phi_n\rangle|^2 < +\infty,$$

hence $y\in\mathcal{D}(A)$ which proved our claim, therefore $A_\lambda x\in\mathcal{D}(A) \quad \forall x\in H$. Also,

$$\begin{aligned}
(\lambda I-A)A_\lambda x &= \lambda\sum_{n=1}^{+\infty}\frac{1}{\lambda-\lambda_n}\langle x,\phi_n\rangle\phi_n - \sum_{n=1}^{+\infty}\lambda_n\langle A_\lambda x,\phi_n\rangle\phi_n\\
&= \sum_{n=1}^{+\infty}\frac{\lambda}{\lambda-\lambda_n}\langle x,\phi_n\rangle\phi_n - \sum_{n=1}^{+\infty}\lambda_n\left\langle\sum_{j=1}^{+\infty}\frac{1}{\lambda-\lambda_j}\langle x,\phi_j\rangle\phi_j,\phi_n\right\rangle\phi_n\\
&= \sum_{n=1}^{+\infty}\frac{\lambda}{\lambda-\lambda_n}\langle x,\phi_n\rangle\phi_n - \sum_{n=1}^{+\infty}\frac{\lambda_n}{\lambda-\lambda_n}\langle x,\phi_n\rangle\phi_n\\
&= \sum_{n=1}^{+\infty}\langle x,\phi_n\rangle\phi_n = x.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
A_\lambda(\lambda I-A)x &= \lambda\sum_{n=1}^{+\infty}\frac{1}{\lambda-\lambda_n}\langle x,\phi_n\rangle\phi_n - \sum_{n=1}^{+\infty}\frac{1}{\lambda-\lambda_n}\langle Ax,\phi_n\rangle\phi_n\\
&= \sum_{n=1}^{+\infty}\frac{\lambda}{\lambda-\lambda_n}\langle x,\phi_n\rangle\phi_n - \sum_{n=1}^{+\infty}\frac{1}{\lambda-\lambda_n}\left\langle\sum_{j=1}^{+\infty}\lambda_j\langle A_\lambda x,\phi_j\rangle\phi_j,\phi_n\right\rangle\phi_n\\
&= \sum_{n=1}^{+\infty}\frac{\lambda}{\lambda-\lambda_n}\langle x,\phi_n\rangle\phi_n - \sum_{n=1}^{+\infty}\frac{\lambda_n}{\lambda-\lambda_n}\langle x,\phi_n\rangle\phi_n\\
&= \sum_{n=1}^{+\infty}\langle x,\phi_n\rangle\phi_n = x.
\end{aligned}$$

Hence $(\lambda I-A)$ is invertible and the inverse is given by,

$$(\lambda I-A)^{-1} = A_\lambda.$$

Conversely, assume that $(\lambda I-A)^{-1}$ is invertible. Since $(\lambda_n I-A)\phi_n = 0$ one has $\lambda_n\in\sigma(A)$ for each $n\in\mathbb{N}$. Therefore,

$$\inf_{n\in\mathbb{N}}|\lambda-\lambda_n| \geq \inf_{\mu\in\sigma(A)}|\lambda-\mu| = \operatorname{dist}(\lambda,\mu) \geq \frac{1}{\|(\lambda I-A)^{-1}\|} > 0.$$

- Assume that $\sup_n \mathrm{Re}\lambda_n < +\infty$. From Hille-Yosida Theorem 2.2.9, we know that A is the infinitesimal generator of a C_0-semigroup if there exists some constants M and ω such that
$$\|R(\lambda, A)^k\| \leq \frac{M}{(\mathrm{Re}\lambda - \omega)^k} \qquad \forall k \in \mathbb{N}, \quad \forall \lambda : \mathrm{Re}\lambda > \omega.$$
Now
$$R(\lambda, A)^k x = \sum_{n=1}^{+\infty} \frac{1}{(\lambda - \lambda_n)^k} \langle x, \phi_n \rangle \phi_n$$
and so
$$\|R(\lambda, A)^k\| \leq \sup_{n \in \mathbb{N}} \frac{1}{|\lambda - \lambda_n|^k} = \left[\sup_{n \in \mathbb{N}} \frac{1}{|\lambda - \lambda_n|} \right]^k \qquad \forall k \in \mathbb{N}. \tag{2.3.14}$$
But since $\sup_n \mathrm{Re}\lambda_n < +\infty$, there exists some $\omega \in \mathbb{R}$ such that $\sup_n \mathrm{Re}\lambda_n \leq \omega$, if $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > \omega$ one has
$$0 < \mathrm{Re}\lambda - \omega \leq \mathrm{Re}\lambda - \mathrm{Re}\lambda_n \leq |\mathrm{Re}\lambda - \mathrm{Re}\lambda_n| = |\mathrm{Re}(\lambda - \lambda_n)| \leq |\lambda - \lambda_n| \quad \forall n \in \mathbb{N},$$
hence
$$\sup_{n \in \mathbb{N}} \frac{1}{|\lambda - \lambda_n|} \leq \frac{1}{\mathrm{Re}\lambda - \omega}. \tag{2.3.15}$$
So equations (2.3.1) and (2.3.15) together entails that
$$\|R(\lambda, A)^k\| \leq \frac{1}{(\mathrm{Re}\lambda - \omega)^k} \qquad \forall k \in \mathbb{N}.$$
Thus, A is the infinitesimal generator of a strongly continuous semigroup.
Let $T(t)$ be the semigroup generated by A, then from Theorem 2.2.6 (ii), one has
$$\frac{\mathrm{d}}{\mathrm{dt}} T(t)\phi_n = T(t)A\phi_n = \lambda_n T(t)\phi_n$$
and so $T(t)\phi_n = e^{\lambda_n t} f_n$ for some $f_n \in H$. since $T(0) = I$, one conclude that $T(t)\phi_n = e^{\lambda_n t}\phi_n$. Since $T(t)$ is linear and bounded and $\{\phi_n, n \in \mathbb{N}\}$ forms an orthonormal basis in H, it follows that
$$T(t)x = \sum_{n=1}^{+\infty} e^{\lambda_n t} \langle x, \phi_n \rangle \phi_n \qquad \forall x \in H.$$

■

CHAPTER 3

Controllability and Stablizability in Hilbert Spaces

Notation. Throughout this chapter we will consider H and U to two Hilbert spaces, we denote by $\langle \cdot, \cdot \rangle_H$ the scalar product in H, by $\langle \cdot, \cdot \rangle_U$ the scalar product in U, by $\|\cdot\|_H$ the norm in H and by $\|\cdot\|_U$. In this chapter we will consider the stability of a linear control system in Hilbert spaces. Throughout this chapter H will be a Hilbert space identified by its dual.

3.1 Introduction

We first deal with abstract linear dynamical system (also called abstract Cauchy problem):

$$\begin{cases} \dot{x}(t) = Ax(x) + f(t) & \text{a.e } t \geq 0, \\ x(0) = x_0, \quad x_0 \in H, \end{cases} \tag{3.1.1}$$

evolving in a Hilbert space H, where A is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ and f is an $H-$valued funtion.

Definition 3.1.1. *Any continuous function $x : [0, T] \mapsto H$ such that*

(i) $x(0) = x_0$ and $x(t) \in \mathcal{D}(A)$ for all $t \in [0, T]$,

(ii) x is continuously differentiable on $[0, T]$ and

$$\frac{\mathrm{d}x}{\mathrm{dt}}(t) = Ax(t) + f(t), \qquad \forall t \in [0, T]$$

is called a classical solution *of* (3.1.1) *on* $[0, T]$.

Theorem 3.1.2. *Assume that $x_0 \in \mathcal{D}(A)$, $f \in C(0, T; H)$ and x is a classical solution of* (3.1.1)*. Then*

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)\,\mathrm{d}s \qquad \forall t \in [0, T]. \tag{3.1.2}$$

Proof. Suppose that x is a solution on $[0,T]$, then for any $t>0$ fixed and $s\in(0,t)$ one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}[S(t-s)x(s)] &= -AS(t-s)x(s)+S(t-s)(Ax(s)+f(s))\\ &= -AS(t-s)x(s)+AS(t-s)x(s)+S(t-s)f(s))\\ &= S(t-s)f(s).\end{aligned}$$

Integrating the above identity from 0 to t one obtain

$$x(t)-S(t)x_0=\int_0^t S(t-s)f(s)\,\mathrm{d}s$$

from which the result follows. □

It may be thought that (3.1.2) is always a solution of (3.1.1) but this is not always true in general. Additional conditions either on f or on the semigroup $(S(t))_{t\geq 0}$ are needed. The following result gives the converse.

Theorem 3.1.3. *If A generates a C_0-semigroup on H, and*

(i) $f\in C^1(0,T;H)$,

(ii) $x_0\in\mathcal{D}(A)$, *then* (3.1.2) *is continuously differentiable on* $[0,T]$ *and is a classical solution of* (3.1.1).

Proof.
- Let $u(t)=S(t)x_0$, then u is continuous on $[0,T]$ and $\dot{u}(t)=AS(t)x_0=Au(t)$, it follows that u is a classical solution of (3.1.1) with $f=0$.
- Let $v(t)=\int_0^t S(t-s)f(s)\,\mathrm{d}s$, then

$$\begin{aligned}\frac{S(h)-I}{h}v(t) &= \frac{S(h)-I}{h}\int_0^t S(t-s)f(s)\,\mathrm{d}s\\ &= \frac{1}{h}\int_0^t S(t+h-s)f(s)\,\mathrm{d}s-\frac{1}{h}\int_0^t S(t-s)f(s)\,\mathrm{d}s\\ &= \frac{1}{h}\int_0^{t+h} S(t+h-s)f(s)\,\mathrm{d}s-\frac{1}{h}\int_0^{t+h} S(t-s)f(s)\,\mathrm{d}s\\ &\quad -\frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,\mathrm{d}s\\ &= \frac{v(t+h)-v(t)}{h}-\frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,\mathrm{d}s. \qquad (3.1.3)\end{aligned}$$

Since f is continuous the second term on the right-hand side of (3.1.3) converges to $f(t)$ as $h\longrightarrow 0$. On the other hand,

$$\begin{aligned}\frac{v(t+h)-v(t)}{h} &= \frac{1}{h}\int_0^{t+h} S(t+h-s)f(s)\,\mathrm{d}s-\frac{1}{h}\int_0^{t+h} S(t-s)f(s)\,\mathrm{d}s\\ &= \int_0^{t+h} S(s)\frac{f(t+h-s)-f(t-s)}{h}\,\mathrm{d}s,\end{aligned}$$

hence it follows that

$$\begin{aligned}\frac{\mathrm{d}^+v}{\mathrm{dt}}(t) &= \lim_{h\downarrow 0}\frac{v(t+h)-v(t)}{h}\\ &= \lim_{h\downarrow 0}\int_0^{t+h} S(s)\frac{f(t+h-s)-f(t-s)}{h}\,\mathrm{d}s\\ &= \int_0^{t+h} S(s)\lim_{h\downarrow 0}\frac{f(t+h-s)-f(t-s)}{h}\,\mathrm{d}s\\ &= \int_0^t S(s)f'(t-s)\,\mathrm{d}s = \int_0^t S(t-s)f'(s)\,\mathrm{d}s.\end{aligned}$$

Consequetly it follows that $v(t)\in\mathcal{D}(A)$ and

$$Av(t) = \frac{\mathrm{d}^+v}{\mathrm{dt}}(t) - f(t)$$

which implies that

$$\frac{\mathrm{d}^+v}{\mathrm{dt}}(t) = Av(t) + f(t).$$

So one has

$$\dot{v}(t) = Av(t) + f(t), \quad t>0$$

hence v is a classical solution of (3.1.1) with $x(0)=0$. Combining the two parts of the proof one has

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)\,\mathrm{d}s$$

is a classical solution of (3.1.1).

□

The conditions of Theorem 3.1.3 are too restrictive for control theory where in general f is assumed to be measurable, so we need to weaken the hypothesis on f.

Definition 3.1.4. *If $f\in L^1(0,T;H)$, we say that (3.1.2) is a* mild solution *of (3.1.1).*

Lemma 3.1.5. *The function x defined by (3.1.2) is strongly continuous on $[0,T]$.*

Proof. Without lost of generality we assume $x_0=0$. For $\delta>0$, consider

$$\begin{aligned}x(t+\delta)-x(t) &= \int_0^{t+\delta} S(t+\delta-s)f(s)\,\mathrm{d}s - \int_0^t S(t-s)f(s)\,\mathrm{d}s\\ &= \int_0^t [S(t+\delta-s)-S(t-s)]f(s)\,\mathrm{d}s - \int_t^{t+\delta-s} S(t+\delta-s)f(s)\,\mathrm{d}s\\ &= \int_0^t [S(\delta)S(t-s)-S(t-s)]f(s)\,\mathrm{d}s - \int_t^{t+\delta-s} S(t+\delta-s)f(s)\,\mathrm{d}s\\ &= [S(\delta)-I]\int_0^t S(t-s)f(s)\,\mathrm{d}s - \int_t^{t+\delta-s} S(t+\delta-s)f(s)\,\mathrm{d}s\\ &= [S(\delta)-I]x(t) - \int_t^{t+\delta-s} S(t+\delta-s)f(s)\,\mathrm{d}s.\end{aligned}$$

Then

$$\|x(t+\delta)-x(t)\| \;\leq\; \|(S(\delta)-I)x(t)\| + \left(\int_t^{t+\delta}\|S(t+\delta-s)\|^2\,\mathrm{d}s\right)^{1/2}\left(\int_t^{t+\delta}\|f(s)\|^2\,\mathrm{d}s\right)^{1/2}$$

which goes to zero as $\delta \longrightarrow 0^+$.
Now consider

$$\begin{aligned} x(t-\delta)-x(t) &= \int_0^{t-\delta} S(t-\delta-s)f(s)\,\mathrm{d}s - \int_0^t S(t-s)f(s)\,\mathrm{d}s \\ &= \int_0^{t-\delta}[S(t-\delta-s)-S(t-s)]f(s)\,\mathrm{d}s - \int_{t-\delta}^t S(t-s)f(s)\,\mathrm{d}s. \end{aligned}$$

Then

$$\|x(t-\delta)-x(t)\| \;\leq\; \int_0^{t-\delta}\|[S(t+\delta-s)-S(t-s)]f(s)\|\,\mathrm{d}s + \int_{t-\delta}^t\|S(t-s)f(s)\|\,\mathrm{d}s.$$

So the first term converges to zero as $\delta \to 0^+$ by *Lebesgue Dominated Convergence Theorem*, the second term also tends to zero as $\delta \to 0^+$ by Theorem 2.2.5 (iv). $\square$

We can now show that this concept of *mild solution* is the same as the concept of *weak solution* used in the study of Partial Differential Equations.

Definition 3.1.6. *Let $f \in L^1(0,T;H)$, then we say that x is a* weak solution *of* (3.1.1) *if*

(i) the fucntion which maps $t \longmapsto x(t)$ is continuous on $[0,T]$,

(ii)

$$\int_0^T \langle g(t), x(t)\rangle\,\mathrm{d}t + \int_0^T\langle y(t), f(t)\rangle\,\mathrm{d}t + \langle y(0), x_0\rangle = 0 \tag{3.1.4}$$

for all $g \in C(0,T;H)$, where

$$y(t) = -\int_t^T S^*(s-t)g(s)\,\mathrm{d}s. \tag{3.1.5}$$

Proposition 3.1.7. *For every $x_0 \in H$, there exists a unique weak soluton of* (3.1.1) *and this solution coincides with the mild solution of* (3.1.1).

Proof. Substituting for (3.1.5) in (3.1.4) one has

$$\begin{aligned} \int_0^T\langle g(t), x(t)\rangle\,\mathrm{d}t &= \int_0^T\left\langle -\int_t^T S^*(s-t)g(s)\,\mathrm{d}s, f(t)\right\rangle\mathrm{d}t \\ &= -\int_0^T\left\langle \int_0^T \chi_{[t,T]}(s)S^*(s-t)g(s)\,\mathrm{d}s, f(t)\right\rangle\mathrm{d}t \\ &= -\int_0^T\int_0^T\langle \chi_{[t,T]}(s)S^*(s-t)g(s), f(t)\rangle\,\mathrm{d}s\,\mathrm{d}t \end{aligned} \tag{3.1.6}$$

$$
\begin{aligned}
\int_0^T \langle g(t), x(t)\rangle \,\mathrm{d}t &= -\int_0^T \int_0^T \langle \chi_{[t,T]}(s) S^*(s-t) g(s), f(t)\rangle \,\mathrm{d}t \,\mathrm{d}s \\
&= -\int_0^T \int_0^s \langle S^*(s-t) g(s), f(t)\rangle \,\mathrm{d}t \,\mathrm{d}s \\
&= -\int_0^T \int_0^s \langle g(s), S(s-t) f(t)\rangle \,\mathrm{d}t \,\mathrm{d}s \\
&= -\int_0^T \int_0^t \langle g(t), S(t-s) f(s)\rangle \,\mathrm{d}s \,\mathrm{d}t \\
&= -\int_0^T \langle g(t), \int_0^t S(t-s) f(s) \,\mathrm{d}s\rangle \,\mathrm{d}t. \qquad (3.1.7)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\langle y(0), x_0\rangle &= \left\langle -\int_0^T S^*(s) g(s) \,\mathrm{d}s, x_0 \right\rangle = -\int_0^T \langle S^*(s) g(s), x_0\rangle \,\mathrm{d}s \\
&= -\int_0^T \langle g(s), S(s) x_0\rangle \,\mathrm{d}s = -\int_0^T \langle g(t), S(t) x_0\rangle \,\mathrm{d}t \qquad (3.1.8)
\end{aligned}
$$

Combining (3.1.7) and (3.1.8) in (3.1.4) one has

$$
\int_0^T \left\langle g(t), x(t) - S(t)x_0 - \int_0^t S(t-s) f(s) \,\mathrm{d}s \right\rangle \mathrm{d}t = 0 \quad \forall g \in C(0,T;H),
$$

since $f \in C(0,T;H)$ and using the continuity property of $t \longmapsto x(t)$ on $[0,T]$ and the strong continuity of $S(t)$ it follows that

$$
x(t) - S(t)x_0 - \int_0^t S(t-s) f(s) = 0,
$$

thus

$$
x(t) = S(t)x_0 + \int_0^t S(t-s) f(s)
$$

which shows that the mild solution is weak solution and vice-versa. To prove uniqueness, let $\overline{x}(t)$ be another weak solution of (3.1.1), then

$$
\int_0^T \langle g(t), x(t) - \overline{x}(t)\rangle \,\mathrm{d}t = 0 \quad \forall g \in C(0,T;H)
$$

which implies that $x(t) = \overline{x}(t)$. $\square$

In application to control problems, the nonhomogeneous terms f in (3.1.1) is often determined by a control input.

3.2 Controllability in Hilbert Spaces

This section is devoted to the study of controllability of linear control systems.

3.2.1 A General Framework for Linear Control Systems

Let $S(t)$, $t \in [0, +\infty)$, be a C_0-semigroup on a Hilbert space H with generator $A : \mathcal{D}(A) \subset H \longrightarrow H$. As usual, we denote by $S^*(t)$ by the adjoint of $S(t)$, then $S^*(t)$, $t \in [0, +\infty)$ is a C_0-semigroup on H and the generator of this semigroup is the adjoit A^* of A. Let U be a Hilbert space and $B \in \mathcal{L}(U, H)$, we now consider the following linear control system

$$\dot{x}(t) = Ax(x) + Bu(t), \quad \text{a.e. } t \in [0, T] \tag{3.2.1}$$

where $T > 0$, and at time $t > 0$ the state of the system is $x(t) \in H$ and the control is $u(t) \in U$, H is called the *state space* and U is called the *control space*. We are interested in the Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(x) + Bu(t), & \text{a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{3.2.2}$$

where $x_0 \in H$, the control function u is assumed to belong to $L^2(0, T; U)$. It follows from Proposition (3.1.7) with $f = Bu$, that the solution of the system (3.2.2) satisfying $x(0) = x_0 \in H$ according to some control $u \in L^2(0, T; U)$ is unique and is given by

$$x(t; x_0, u) = S(t)x_0 + \int_0^t S(t-s)Bu(s)\,\mathrm{d}s, \quad \forall t \in [0, T]. \tag{3.2.3}$$

We will be particularly interested in the following two operators:

- $\mathcal{L}_T : L^2(0, T; U) \longrightarrow H$ defined by

$$\mathcal{L}_T u = \int_0^T S(T-s)Bu(s)\,\mathrm{d}s, \quad \text{for all } u \in L^2(0, T; U). \tag{3.2.4}$$

  Then for any $u \in L^2(0, T; U)$ one has

$$\begin{aligned} \|\mathcal{L}_T u\|_H &= \left\| \int_0^T S(T-s)Bu(s)\,\mathrm{d}s \right\|_H \leq \int_0^T \|S(T-s)Bu(s)\|_H\,\mathrm{d}s \\ &\leq \left(\int_0^T \|S(T-s)\|^2_{\mathcal{L}(H)}\,\mathrm{d}s \right)^{1/2} \left(\int_0^T \|Bu(s)\|^2_H\,\mathrm{d}s \right)^{1/2} \\ &\leq M\sqrt{T}\|B\|_{\mathcal{L}(U,H)}\|u\|_{L^2(0,T;U)} \end{aligned}$$

  for some constant $M > 0$, hence it follows that $\mathcal{L}_T \in \mathcal{L}(L^2(0, T; U), H)$. Also note from (3.2.3) and (3.2.4) that $x(T, u, x_0) = S(T)x_0 + \mathcal{L}_T u$. Computing the adjoint of

$\mathcal{L}_T$ one has:

$$\begin{aligned}\langle \mathcal{L}_T^* x, u\rangle_{L^2(0,T;U)} &= \langle x, \mathcal{L}_T u\rangle_H = \langle x, \int_0^T S(T-s)Bu(s)\,\mathrm{d}s\rangle_H \\ &= \int_0^T \langle x, S(T-s)Bu(s)\rangle_H\,\mathrm{d}s = \int_0^T \langle B^*S^*(T-s)x, u(s)\rangle_U\,\mathrm{d}s,\end{aligned}$$

and by the very definition of scalar product in $L^2(0,T;U)$

$$\mathcal{L}_T^* x(s) = B^*S^*(T-s)x, \quad \text{for a.e } s \in [0,T] \text{ i.e., } \mathcal{L}_T^* x(\cdot) = B^*S^*(T-\cdot)x. \tag{3.2.5}$$

- $Q_T : H \longrightarrow H$ defined by

$$Q_T x = \int_0^T S(s)BB^*S^*(s)x\,\mathrm{d}s, \quad \text{for all } x \in H. \tag{3.2.6}$$

Then for any $x \in H$ one has

$$\begin{aligned}\|Q_T x\|_H &= \left\|\int_0^T S(s)BB^*S^*(s)x\,\mathrm{d}s\right\|_H \leq \int_0^T \|S(s)BB^*S^*(s)x\|_H\,\mathrm{d}s \\ &\leq C\|x\|_H \quad \text{for constant } C > 0,\end{aligned}$$

hence it follows that $Q_T \in \mathcal{L}(H)$. It is also self-adjoint and positive definite:

$$\begin{aligned}\langle Q_T x, y\rangle_H &= \left\langle \int_0^T S(s)BB^*S^*(s)x\,\mathrm{d}s, y\right\rangle_H = \int_0^T \langle S(s)BB^*S^*(s)x, y\rangle_H\,\mathrm{d}s \\ &= \int_0^T \langle x, S(s)BB^*S^*(s)y\rangle_H\,\mathrm{d}s = \langle x, Q_T y\rangle_H \quad \forall x, y \in H,\end{aligned}$$

hence $Q_T^* = Q_T$., that is Q_T is self-adjoint.

$$\begin{aligned}\langle Q_T x, x\rangle_H &= \left\langle \int_0^T S(s)BB^*S^*(s)x\,\mathrm{d}s, x\right\rangle_H = \int_0^T \langle S(s)BB^*S^*(s)x, x\rangle_H\,\mathrm{d}s \\ &= \int_0^T \langle B^*S^*(s)x, B^*S^*(s)x\rangle_H\,\mathrm{d}s = \int_0^T \|B^*S^*(s)x\|_H^2\,\mathrm{d}s \\ &\geq 0 \quad \forall x \in H,\end{aligned}$$

hence Q_T is a positive definite.

Also let us remark that for every $x \in H$,

$$Q_T x = \int_0^T S(s)BB^*S(s)x\,\mathrm{d}s = \int_0^T S(T-s)BB^*S(T-s)x\,\mathrm{d}s = \mathcal{L}_T\mathcal{L}_T^* x.$$

The operator Q_T is called the *controllability operator*.

3.2.2 Various Concept of Controllability

We now introduce the three basic concepts of controllability.

Definition 3.2.1. *For any $x_0 \in H$ and $T > 0$, the system (3.2.1) is said to be* exactly controllable *from x_0 in time T if, for every $x_1 \in H$, there exists a control $u \in L^2(0,T;U)$ such that the solution x of the system (3.2.1), with $x(0) = x_0$, associated with the control u, satisfies $x(T;u,x_0) = x_1$.*

Remark 3.2.2. *It is clear that the system (3.2.1) is exactly controllable from x_0 in time T if and only if $\mathcal{L}_T$ is onto, i.e., $\mathrm{Im}\mathcal{L}_T = H$. In particular, if the system (3.2.1) is exactly controllable from x_0 in time T then, it is exactly controllable from any point $x_0' \in H$ in time T. One says that the system (3.2.1) is exactly controllable in time T.*

Definition 3.2.3. *The system (3.2.1) is said to be* approximately controllable *from $x_0 \in H$ in time T if, for every $x_1 \in H$ and any $\varepsilon > 0$, there exists a control $u \in L^2(0,T;U)$ such that the solution x of the system (3.2.1), with $x(0) = x_0$, associated with the control u, satisfies $\|x(T;u,x_0) - x_1\| < \varepsilon$.*

Remark 3.2.4. *As previously noted, this concept does not depend on the initial point, and the system (3.2.1) is approximately controlable if and if $\mathrm{Im}\mathcal{L}_T$ is dense in H, i.e., $\overline{\mathrm{Im}\mathcal{L}_T} = H$.*

Definition 3.2.5. *For $T > 0$, the system (3.2.1) is said to* exactly null controllable *or* exactly zero controllable *in time T if, for any $x_0 \in H$, there exists a control $u \in L^2(0,T;U)$ such that the solution x of (3.2.1) with $x(0) = x_0$, associated with the control u satisfies $x(T;u,x_0) = 0$. Or equivalently, for every $x_0 \in H$ and for every $x_1 \in H$, there exists an admissible control $u \in \mathscr{U}$ such that the solution x of the system (3.2.1) satisfies $x(T,u,x_0) = S(T)x_1$.*

Remark 3.2.6. *The concept of exactly null controllbility also does not depend on the initial point, and the system (3.2.1) is exactly null controllable if and only if $S(T)H \subset \mathrm{Im}\mathcal{L}_T$, i.e., $\mathrm{Im}S(T) \subset \mathrm{Im}\mathcal{L}_T$.*

Theorem 3.2.7. *Assume that for every $T > 0$, the control system (3.2.1) is exactly null controllable in time T. Then for every $T > 0$, the control system (3.2.1) is approximately controllable in time T.*

Proof. Let $x_0, x_1 \in H$, $T > 0$ and $\varepsilon > 0$. Since $S(t)$, $t \in [0,+\infty)$, is a C_0−semigroup it follows that there exists $\delta_1 > 0$ such that

$$\|S(t)x_1 - x_1\| < \varepsilon \text{ whenever } 0 < t < \delta_1,$$

choose $\delta = \min\{\delta_1, T\}$, then

$$\|S(t)x_1 - x_1\| < \varepsilon \text{ whenever } 0 < t < \delta,$$

in particular, there exists $\eta \in (0,\delta) \subset (0,T)$ such that

$$\|S(\eta)x_1 - x_1\| < \varepsilon. \tag{3.2.7}$$

Since the control system is null controllable in time T, for any $T > 0$, it is null controllable in time $T = \eta$, therefore from the null controllability assumption applied to the points $S(T-\eta)x_0, x_1 \in H$, it follows that there exists $\overline{u} \in L^2(0,\eta;U)$ such that the solution $\overline{x} \in C(0,\eta;H)$ of the Cauchy Problem

$$\begin{cases} \dot{\overline{x}}(t) = A\overline{x}(x) + B\overline{u}(t), & \text{a.e. } t \in [0,\eta], \\ \\ \overline{x}(0) = S(T-\eta)x_0, \end{cases} \tag{3.2.8}$$

satisfies

$$\overline{x}(\eta, \overline{u}, S(T-\eta)x_0) = S(\eta)x_1. \tag{3.2.9}$$

Let $u \in L^2(0,T;U)$ be defined by

$$u(t) = \begin{cases} 0, & t \in (0, T-\eta) \\ \overline{u}(t-T+\eta), & t \in (T-\eta, T). \end{cases}$$

Let $x \in C(0,T;U)$ be the solution of the Cauchy Problem

$$\begin{cases} \dot{x}(t) = Ax(x) + Bu(t), & \text{a.e. } t \in [0,T], \\ \\ x(0) = x_0. \end{cases}$$

Then

$$\begin{aligned} x(t;u,x_0) &= S(t)x_0 + \int_0^t S(t-s)Bu(s)\,\mathrm{d}s, \quad \forall t \in [0,T] \\ &= \begin{cases} S(t)x_0, & \forall t \in [0,T-\eta] \\ S(t)x_0 + \int_{T-\eta}^t S(t-s)B\overline{u}(s-T+\eta), & \forall t \in [T-\eta,T] \end{cases} \\ \\ &= \begin{cases} S(t)x_0, & \forall t \in [0,T-\eta] \\ S(t)x_0 + \int_0^{t-T+\eta} S(t-T+\eta-s)B\overline{u}(s), & \forall t \in [T-\eta,T] \end{cases} \\ \\ &= \begin{cases} S(t)x_0, & \forall t \in [0,T-\eta] \\ \overline{x}(t-T+\eta;\overline{u},S(T-\eta)x_0), & \forall t \in [T-\eta,T]. \end{cases} \end{aligned} \tag{3.2.10}$$

From (3.2.7), (3.2.9) and (3.2.10) one gets

$$\|x(T;u,x_0) - x_1\| = \|\overline{x}(\eta;\overline{u},S(T-\eta)x_0) - x_1\| = \|S(\eta)x_1 - x_1\| < \varepsilon.$$

Thus the system (3.2.1) is approximately controllable. □

It follows from the Definitions 3.2.1, 3.2.3, 3.2.5 and Theorem 3.2.7 that:

(exact controllability) $\Longrightarrow$ (approximate controllability) $\Longrightarrow$ (exactly null controllability).

The converses in general are not true. However, we have the following result if $S(t)$ is a C_0-group of linear operators.

Theorem 3.2.8. *Assume that $S(t)$, $t \in \mathbb{R}$, is a strongly continuous group of linear operators. Let $T > 0$ and assume that the control system (3.2.1) is null controllable in time T. Then the control system (3.2.1) is exactly controllable in time T.*

Proof. Let $x_0, x_1 \in H$. From the null controllability applied to initial data $x_0 - S(-T)x_1$, there exists $u \in \mathscr{U}_T$ such that the solution $\tilde{x}$ of the Cauchy problem

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x} + Bu(t), \\ \tilde{x}(0) = x_0 - S(-T)x_1, \end{cases}$$

is given by

$$\tilde{x}(t; u, x_0 - S(-T)x_1) = S(t)x_0 - S(t-T)x_1 + \int_0^t S(t-s)Bu(s)\,\mathrm{d}s$$

and satisfies

$$\tilde{x}(T; u, x_0 - S(-T)x_1) = 0. \tag{3.2.11}$$

Now consider the following system

$$\begin{cases} \dot{x}(t) = Ax + Bu(t), \\ x(0) = x_0, \end{cases}$$

then its solution is given by

$$\begin{aligned} x(t; u, x_0) &= S(t)x_0 + \int_0^t S(t-s)Bu(s)\,\mathrm{d}s \\ &= S(t-T)x_1 + S(t)x_0 - S(t-T)x_1 + \int_0^t S(t-s)Bu(s)\,\mathrm{d}s \\ &= S(t-T)x_1 + \tilde{x}(t, u, x_0 - S(-T)x_1), \quad \forall t \in [0, T]. \end{aligned} \tag{3.2.12}$$

Equations (3.2.11) and (3.2.12) entails that $x(T; u, x_0) = x_1$. Thus the system (3.2.1) is exactly controllable. □

We have the following characterizations for controllability concepts.

Theorem 3.2.9. *The system (3.2.1) is exactly controllable in time $T > 0$ if and only if there exists some constant $M_T > 0$ such that*

$$\int_0^T \|B^* S^*(t)x\|_U^2\,\mathrm{d}t \geq M_T \|x\|_H^2 \quad \forall x \in H. \tag{3.2.13}$$

Proof. Necessity: Suppose that the system (3.2.1) is exactly controlable, then $\mathrm{Im}\mathcal{L}_T = H$. If $\mathcal{L}_T$ is injective, then $\mathcal{L}_T^{-1}$ exists on $\mathrm{Im}\mathcal{L}_T = H$. From the continuity of $\mathcal{L}_T$ it follows that $\mathcal{L}_T^{-1}$ is a closed linear operator. According to closed graph theorem, $\mathcal{L}_T^{-1}$ is a bounded linear on H, i.e., $\mathcal{L}_T^{-1} \in \mathcal{L}(H, L^2(0,T;U))$. Thus $(\mathcal{L}_T^{-1})^* \in \mathcal{L}(L^2(0,T;U), H)$, that is there exists $\delta_T > 0$ such that

$$\|(\mathcal{L}_T^{-1})^* u\|_H \leq \delta_T \|u\|_{L^2(0,T;U)}. \tag{3.2.14}$$

Now assume that $x_0 \in H$, then $u_0 = \mathcal{L}_T^* x_0 \in L^2(0,T;U)$. Thus, for every $x \in H$ one has

$$\begin{aligned}\langle x, (\mathcal{L}_T^{-1})^* u_0\rangle_H &= \langle x, (\mathcal{L}_T^{-1})^* \mathcal{L}_T^* x_0\rangle_H = \langle \mathcal{L}_T^{-1} x, \mathcal{L}_T^* x_0\rangle_{L^2(0,T;U)} \\ &= \langle \mathcal{L}_T \mathcal{L}_T^{-1} x, x_0\rangle_H = \langle x, x_0\rangle_H,\end{aligned}$$

which implies

$$\|x_0\|_H = \sup_{x\in H:\|x\|=1} \langle x, x_0\rangle_H = \sup_{x\in H:\|x\|=1} \langle x, (\mathcal{L}_T^{-1})^* u_0\rangle_H = \|(\mathcal{L}_T^{-1})^* u_0\|_H.$$

Using (3.2.14) one has

$$\|x_0\|_H = \|(\mathcal{L}_T^{-1})^* u_0\|_H \leq \delta_T \|u_0\|_{L^2(0,T;U)} = \delta_T \|\mathcal{L}_T^* x_0\|_{L^2(0,T;U)},$$

that is,

$$\begin{aligned}\int_0^T \|B^* S^*(t) x_0\|_U^2 \,\mathrm{d}t &= \int_0^T \|B^* S^*(T-t) x_0\|_U^2 \,\mathrm{d}t = \|\mathcal{L}_T^* x_0\|_{L^2(0,T;U)}^2 \\ &\geq \frac{1}{\delta_T^2} \|x_0\|_H^2 = M_T \|x_0\|_H^2, \text{ where } M_T = \frac{1}{\delta_T^2}.\end{aligned}$$

Hence

$$\int_0^T \|B^* S^*(t) x\|_U^2 \,\mathrm{d}t \geq M_T \|x\|_H^2, \quad \text{for all } x \in H.$$

If $\mathcal{L}_T$ is not injective, then $\ker\mathcal{L}_T = \{u : u \in L^2(0,T;U), \mathcal{L}_T u = 0\} \neq \{0\}$. We defined the Hilbert space

$$\overline{H} = L^2(0,T;U)/\ker\mathcal{L}_T = \{\overline{u} : \overline{u} = u + v : v \in \ker\mathcal{L}_T\},$$

which norm is

$$\|\overline{u}\|_{\overline{H}} = \inf_{v\in\ker\mathcal{L}_T} \|u + v\|_{L^2(0,T;U)}.$$

Let $\overline{\mathcal{L}}_T : \overline{H} \longrightarrow H$ be defined by $\overline{\mathcal{L}}_T \overline{u} = \mathcal{L}_T u$, then $\overline{\mathcal{L}}_T \in (\overline{H}, H)$ and $\overline{\mathcal{L}}_T$ is injective and $\mathrm{Im}\overline{\mathcal{L}}_T$=$\mathrm{Im}\mathcal{L}_T = H$. From the firs part of the proof it follows that

$$\|\overline{\mathcal{L}}_T^* x\|_H^2 \geq M_T \|x\|_H^2, \quad \text{for all } x \in H.$$

But from the definitions of $\overline{H}$ and $\overline{\mathcal{L}}_T$ one has

$$\begin{aligned}\|\overline{\mathcal{L}}_T^* x\|_{\overline{H}} &= \sup_{\overline{u}\in\overline{H}:\|\overline{u}\|=1} \langle \overline{\mathcal{L}}_T^* x, \overline{u}\rangle = \sup_{\overline{u}\in\overline{H}:\|\overline{u}\|=1} \langle x, \overline{\mathcal{L}}_T \overline{u}\rangle \\ &= \sup_{u:\|u\|=1} \langle x, \mathcal{L}_T u\rangle = \sup_{u:\|u\|=1} \langle \mathcal{L}_T^*, u\rangle \\ &= \|\mathcal{L}_T^* x\|_{L^2(0,T;U)}.\end{aligned}$$

Hence

$$\int_0^T \|B^*S^*(t)x\|_U^2 \,\mathrm{d}t \geq M_T\|x_0\|_H^2, \quad \text{for all } x \in H.$$

Sufficiency: Suppose (3.2.13) holds. It is enough to show that if $x \in H$ then $x \in \mathrm{Im}\mathcal{L}_T$. Since $\mathcal{L}_T \in \mathcal{L}(L^2(0,T;U),H)$, it follows that $\mathcal{L}_T^* \in \mathcal{L}(H,L^2(0,T;U))$. For $x \in H$, a linear functional f_x is defined on $\mathrm{Im}\mathcal{L}_T^*$ by

$$f_x(\mathcal{L}_T^*y) = \langle x, y\rangle, \quad y \in H. \tag{3.2.15}$$

If $\mathcal{L}_T^*y_n \longrightarrow 0$ as $n \longrightarrow +\infty$, from (3.2.13) it follows that $y_n \longrightarrow 0$ as $n \longrightarrow +\infty$ and $f_x(\mathcal{L}_T^*y_n) = \langle x, y_n\rangle \longrightarrow 0$ as $n \longrightarrow +\infty$, that is f is continuous. Thus f is a bounded linear functional on the subspace $\mathrm{Im}\mathcal{L}_T^*$ of $L^2(0,T;U)$. According to the Hahn-Banach theorem, f_x can be extended as a bounded linear functional on $L^2(0,T;U)$. Thus by Riesz Representation Theorem, there exists unique $u \in L^2(0,T;U)$ such that

$$f_x(\mathcal{L}_T^*y) = \langle u, \mathcal{L}_T^*y\rangle \quad y \in H. \tag{3.2.16}$$

Equations (3.2.15) and (3.2.16) together entails that

$$\langle x, y\rangle = \langle u, \mathcal{L}_T^*y\rangle = \langle \mathcal{L}_T u, y\rangle, \quad \text{for every } y \in H.$$

Therefore $x = \mathcal{L}_T u$, that is $x \in \mathrm{Im}\mathcal{L}_T$, hence $H \subset \mathrm{Im}\mathcal{L}_T$, which implies $\mathrm{Im}\mathcal{L}_T = H$, thus the system (3.2.1) is exactly controllable. □

Theorem 3.2.10. *The system (3.2.1) is exactly controllable in time $T > 0$ if and only if the controllability operator Q_T is invertible. Moreover, the control $u \in L^2(0,T;U)$ steering an initial state x_0 to a final state x_1 at time $T > 0$ is given by the formula*

$$u(t) = B^*S^*(T-t)Q_T^{-1}(x_1 - S(T)x_0). \tag{3.2.17}$$

Proof. Necessity: Suppose the system (3.2.1) is exactly controllable in time $T > 0$. Then from Theorem 3.2.9, there exists $M_T > 0$ such that

$$\begin{aligned} M_T\|x\|^2 &\leq \int_0^T \|B^*S^*(t)x\|^2 \,\mathrm{d}t = \int_0^T \langle B^*S^*(t)x, B^*S^*S(t)x\rangle^2 \,\mathrm{d}t \\ &= \int_0^T \langle S(t)BB^*S^*(t)x, x\rangle^2 \,\mathrm{d}t = \langle Q_T x, x\rangle, \quad \forall x \in H. \end{aligned}$$

Therefore

$$\langle Q_T x, x\rangle \geq M_T\|x\|^2, \quad \text{for all } x \in H. \tag{3.2.18}$$

This implies that Q_T is injective. Now we shall prove that Q_T is surjective, that is $\mathrm{Im}Q_T = H$. By contradiction, assume that $\mathrm{Im}Q_T$ is strictly contained in H. Using Cauchy-Schwartz's inequality and equation (3.2.18) one has

$$\|Q_T x\| \geq M_T\|x\|, \quad \text{for all } x \in H. \tag{3.2.19}$$

Now let $(Q_T x_n)_{n\geq 1}$ be a sequence in $\mathrm{Im}Q_T$ such that $Q_T x_n \longrightarrow y$ as $n \longrightarrow +\infty$, for some $y \in H$. Then inequality (3.2.19) implies that $x_n \longrightarrow x$ as $n \longrightarrow +\infty$, for some $x \in H$. Continuity of Q_T implies that $Q_T x_n$ converges to $Q_T x$, uniqueness of limit implies that $y = Q_T x$, hence $y \in \mathrm{Im}Q_T$. Therefore $\mathrm{Im}Q_T$ is closed. Then by Hahn-Banach Theorem there exists $x_0 \in H$ with $x_0 \neq 0$ such that

$$\langle Q_T x, x_0\rangle = 0, \quad \forall x \in H.$$

In particular, for $x = x_0$ one gets from (3.2.18) that

$$0 = \langle Q_T x_0, x_0\rangle \geq M_T \|x_0\|^2.$$

Then $x_0 = 0$, which contradicts the fact that $x_0 \neq 0$. Hence $\mathrm{Im}Q_T = H$, thus Q_T is bijective. Sufficiency: Now assume that Q_T is invertible. Then given arbitrary $x_0, x_1 \in H$ we shall prove the existence of a control $u \in L^2(0,T;U)$ such that $x_1 = S(T)x_0 + \mathcal{L}_T u$. The control u can be taking as follows

$$u(t) = B^* S^*(T-t) Q_T^{-1}(x_1 - S(T)x_0).$$

Then

$$\begin{aligned} \mathcal{L}_T u &= \int_0^T S(T-s)Bu(s)\,\mathrm{d}s = \int_0^T S(T-s)BB^*S^*(T-s)Q_T^{-1}(x_1 - S(T)x_0)\,\mathrm{d}s \\ &= Q_T Q_T^{-1}(x_1 - S(T)x_0) = x_1 - S(T)x_0, \end{aligned}$$

which implies that

$$x_1 = S(T)x_0 + \mathcal{L}_T u.$$

Hence the system is exactly controllable. □

Theorem 3.2.11. *Suppose system (3.2.1) is exactly controllable. Consider $x \in H$, the control*

$$u_0(t) = B^* S^*(T-t) Q_T^{-1} x$$

and the set

$$S_x = \{u \in L^2(0,T;U) : \mathcal{L}_T u = x\}.$$

Then

$$\|u_0\| = \inf\{\|u\| : u \in S_x\}.$$

Proof. Consider the following equalities

$$\begin{aligned} \langle u_0, u - u_0\rangle &= \int_0^T \langle B^*S^*(T-s)Q_T^{-1}x, u(s) - u_0(s)\rangle\,\mathrm{d}s \\ &= \int_0^T \langle Q_T^{-1}x, S(T-s)Bu(s) - S(T-s)Bu_0(s)\rangle\,\mathrm{d}s \\ &= \left\langle Q_T^{-1}x, \int_0^T S(T-s)Bu(s)\,\mathrm{d}s - \int_0^T S(T-s)Bu_0(s)\,\mathrm{d}s \right\rangle \\ &= \langle Q_T^{-1}x, \mathcal{L}_T u - \mathcal{L}_T u_0\rangle = \langle Q_T^{-1}x, x - x\rangle = 0. \end{aligned}$$

Hence,

$$\|u\|^2 - \|u_0\|^2 = \|u - u_0\|^2 \geq 0, \quad u \in S_x.$$

Therefore,

$$\|u_0\| \leq \|u\|, \quad \forall u \in S_x \quad \text{and} \quad \|u_0\| = \|u\| \quad \text{if and only if} \quad u_0 = u.$$

$\square$

Theorem 3.2.12. *The system (3.2.1) is approximately controllable in time $T > 0$ if and only if the following condition holds:*

$$\forall t \in [0,T] \quad B^* S^*(t)x = 0 \Longrightarrow x = 0.$$

This is equivalent to saying that $\mathcal{L}_T^*$ is injective.

Proof. Necessity: Suppose the system (3.2.1) is approximately controllable in time $T > 0$, that is $\overline{\mathrm{Im}\mathcal{L}_T} = H$. Assume that $B^* S^*(t)x = 0$ for all $t \in [0,T]$, and for some $x \in H$. Let $y \in \mathrm{Im}\mathcal{L}_T$, then $y = \mathcal{L}_T u$ for some $u \in L^2(0,T;U)$ and therefore

$$\langle x, y\rangle_H = \langle x, \mathcal{L}_T u\rangle_H = \langle \mathcal{L}_T^* x, u\rangle_{L^2(0,T;U)} = \int_0^T \langle B^* S^*(T-s)x, u(s)\rangle_U \,\mathrm{d}s = 0,$$

thus $\langle x, y\rangle_H = 0, \ \forall y \in \mathrm{Im}\mathcal{L}_T$ which implies that $\langle x, y\rangle_H = 0, \ \forall y \in \overline{\mathrm{Im}\mathcal{L}_T} = H$. Hence $x = 0$.
Sufficiency: Suppose that for all $t \in [0,T]$ $B^* S^*(t)x = 0$ implies $x = 0$. By contradiction, assume that $\overline{\mathrm{Im}\mathcal{L}_T} \subsetneq H$. Let $x \in H/\overline{\mathrm{Im}\mathcal{L}_T}$, then by Hahn-Banach Theorem there exists a bounded linear functional f defined on H such that $f(x) \neq 0$ and $f = 0$ on $\overline{\mathrm{Im}\mathcal{L}_T}$, that is

$$f(\mathcal{L}_T u) = 0, \quad \forall u \in L^2(0,T;U). \tag{3.2.20}$$

Using Riesz Representation Theorem there exists a unique $y \in H$ such that

$$f(z) = \langle y, z\rangle, \quad \forall z \in H. \tag{3.2.21}$$

Combining (3.2.20) and (3.2.21) one has

$$f(\mathcal{L}_T u) = \langle y, \mathcal{L}_T u\rangle = \langle \mathcal{L}_T^* y, u\rangle = 0, \quad \forall u \in L^2(0,T;U),$$

which implies that $\mathcal{L}_T^* y = 0$, thus $y = 0$ from the assumption, therefore $f(x) = \langle y, x\rangle = 0$ which contradicts the fact that $f(x) \neq 0$. Hence $\overline{\mathrm{Im}\mathcal{L}_T} = H$. $\square$

Theorem 3.2.13. *The system (3.2.1) is exactly null controllable in time $T > 0$ if and only if there exists some constant $M_T > 0$ such that*

$$\int_0^T \|B^* S^*(t)x\|_U^2 \,\mathrm{d}t \geq M_T \|S^*(T)x\|_H^2 \quad \forall x \in H. \tag{3.2.22}$$

Proof. Necessity: Suppose that the system (3.2.1) is exactly null controllable, then $\mathrm{Im}S(T) = \mathrm{Im}\mathcal{L}_T$. If $\mathcal{L}_T$ is injective, then $\mathcal{L}_T^{-1} : \mathrm{Im}\mathcal{L}_T \longrightarrow L^2(0,T;U)$ is well defined, and using the hypothesis $\mathrm{Im}S(T) \subset \mathrm{Im}\mathcal{L}_T$ it follows that $\mathcal{L}_T^{-1}S(T)$ is well defined. Moreover, $\mathcal{L}_T^{-1}S(T)$ is a closed linear operator from H into $L^2(0,T;U)$ and is therefore bounded by closed graph theorem. Thus the adjoint operator $(\mathcal{L}_T^{-1}S(T))^*$ defined from $L^2(0,T;T)$ into H is bounded, so there exists $\delta_T > 0$ such that

$$\|(\mathcal{L}_T^{-1}S(T))^*u\| \leq \delta_T\|u\|, \quad u \in L^2(0,T;U). \tag{3.2.23}$$

Now let $x_0 \in H$, we let $u_0 = \mathcal{L}_T^* x_0$, then for all $x \in H$ one has

$$\begin{aligned}\langle(\mathcal{L}_T^{-1}S(T))^*u_0, x\rangle &= \langle(\mathcal{L}_T^{-1}S(T))^*\mathcal{L}_T^*x_0, x\rangle = \langle\mathcal{L}_T^*x_0, (\mathcal{L}_T^{-1}S(T))x\rangle \\ &= \langle x_0, \mathcal{L}_T\mathcal{L}_T^{-1}S(T)x\rangle = \langle x_0, S(T)x\rangle = \langle S^*(T)x_0, x\rangle.\end{aligned}$$

Thus

$$\|S^*(T)x_0\| = \|\mathcal{L}_T^{-1}S(T)u\|,$$

which together with (3.2.23) entails that

$$\|S^*(T)x_0\| \leq \delta_T\|u_0\| = \delta_T\|\mathcal{L}_T^*x_0\| = \delta_T\left(\int_0^T \|B^*S^*(T-s)x_0\|\,\mathrm{d}s\right)^{1/2}.$$

Therefore

$$\int_0^T \|B^*S^*(T-s)x\|\,\mathrm{d}s \geq M_T\|S^*(T)x\|^2 \quad \forall x \in H,$$

where $M_T = \frac{1}{\delta_T^2}$. For the general case we define the Hilbert space $\overline{H} = L^2(0,T;U)/\ker\mathcal{L}_T$ with $\overline{u} \in \overline{H}$ being the equivalence class of $u + v$ for which $v \in \ker\mathcal{L}_T$, and

$$\|\overline{u}\|_{\overline{H}} = \inf_{v\in\ker\mathcal{L}_T}\|u+v\|_{L^2(0,T;U)}.$$

We then defined $\overline{\mathcal{L}}_T\overline{u} = \mathcal{L}_Tu$, and so $\overline{\mathcal{L}}_T$ is injective on $\overline{H}$ with $\mathrm{Im}\mathcal{L}_T = \mathrm{Im}\overline{\mathcal{L}}_T$. From the first part of the proof it follows that

$$\|S^*(T)x_0\|_H \leq \delta_T\|\overline{\mathcal{L}}_T^*x\|_{\overline{H}} \quad \forall x \in H. \tag{3.2.24}$$

But from the definition of $\overline{H}$ one has

$$\|\overline{\mathcal{L}}_T^*x\|_{\overline{H}} = \|\mathcal{L}_T^*x\|_H,$$

which together with (3.2.24) yield

$$\int_0^T \|B^*S^*(T-s)x\|\,\mathrm{d}s \geq M_T\|S^*(T)x\|^2 \quad \forall x \in H,$$

where $M_T = \frac{1}{\delta_T^2}$.
Sufficiency: Suppose that (3.2.22) holds. It is enough to prove that $S(T)x \in \mathrm{Im}\mathcal{L}_T$ for all

$x \in H$, that is $S(T)x = \mathcal{L}_T u$ for some $u \in L^2(0,T;U)$. Since $\mathcal{L}_T \in \mathcal{L}(L^2(0,T;U),H)$ it follows that $\mathcal{L}_T^* \in (H, L^2(0,T;U))$, so for $x \in H$ a linear functional f_x is defined on $\mathrm{Im}\mathcal{L}_T^*$ by

$$f_x(\mathcal{L}_T^* y) = \langle x, S^*(T)y\rangle, \quad \forall y \in H. \tag{3.2.25}$$

If $\mathcal{L}_T^* y_n \longrightarrow 0$ as $n \longrightarrow +\infty$, from (3.2.22) it follows that $S^*(T)y_n \longrightarrow 0$ as $n \longrightarrow +\infty$ and $f_x(\mathcal{L}_T^* y_n) = \langle x, S^*(T)y_n\rangle \longrightarrow 0$ as $n \longrightarrow +\infty$, that is f is continuous. Thus f is a bounded linear functional on the subspace $\mathrm{Im}\mathcal{L}_T^*$ of $L^2(0,T;U)$. According to Hahn-Banach Theorem, f can be extended as a bounded linear functional on $L^2(0,T;U)$. Thus by Riesz Representation Theorem, there exists unique $u \in L^2(0,T;U)$ such that

$$f(\mathcal{L}_T^* y) = \langle u, \mathcal{L}_T^* y\rangle, \quad \forall y \in H. \tag{3.2.26}$$

Combining equation (3.2.25) and (3.2.26) one gets

$$\langle S(T)x, y\rangle = \langle x, S^*(T)y\rangle = \langle u, \mathcal{L}_T^* y\rangle = \langle \mathcal{L}_T u, y\rangle, \quad \forall y \in H.$$

Therefore $S(T)x = \mathcal{L}_T u$, that is $S(T)x \in \mathrm{Im}\mathcal{L}_T$, hence $\mathrm{Im}S(T) \subset \mathrm{Im}\mathcal{L}_T$, thus the system (3.2.1) is exactly null controllable. $\square$

3.3 Stability and Stablizability in Hilbert Spaces

We will first consider the following homogeneous linear dynamical system:

$$\begin{cases} \dot{x}(t) = Ax(x), & \forall t \geq 0, \\ \\ x(0) = x_0, & x_0 \in H, \end{cases} \tag{3.3.1}$$

where A is the infitesimal gnerator of a C_0-semigroup $S(t)$ on a Hilbert space H. We now introduce the three basic concepts of stability.

Definition 3.3.1. *A C_0-semigroup, $S(t)$, on a Hilbert space H is exponentially stable if there exist positive constants M and α such that*

$$\|S(t)\| \leq Me^{-\alpha t} \quad \forall t \geq 0. \tag{3.3.2}$$

If $S(t)$ is exponentially stable, then for any initial condition $x_0 \in H$, the corresponding mild solution of the system (3.3.1) $x(t)$ tends to zero exponentially as $t \longrightarrow +\infty$. We also say that the linear system (3.3.1) is exponentially stable if the corresponding semigroup $S(t)$ is exponentially stable.

Definition 3.3.2. *A C_0-semigroup, $S(t)$, on a Hilbert space H is strongly (or asymptotically) stable if for every $x \in H$,*

$$\lim_{t \to +\infty} \|S(t)x\| = 0. \tag{3.3.3}$$

If $S(t)$ is strongly stable, then for any initial condition $x_0 \in H$, the corresponding mild solution of the system (3.3.1) $x(t)$ tends to zero as $t \longrightarrow +\infty$. We also say that the linear system (3.3.1) is strongly stable if the corresponding semigroup $S(t)$ is strongly stable.

Definition 3.3.3. *A C_0-semigroup, $S(t)$, on a Hilbert space H is weakly stable if for every pair $x_1, x_2 \in H$,*

$$\lim_{t \to +\infty} \langle S(t)x_1, x_2 \rangle = 0. \tag{3.3.4}$$

If $S(t)$ is weakly stable, then for any initial condition $x_0 \in H$, the corresponding mild solution of the system (3.3.1) $x(t)$ tends to zero weakly as $t \longrightarrow +\infty$. We also say that the linear system (3.3.1) is weakly stable if the corresponding semigroup $S(t)$ is weakly stable.

Theorem 3.3.4. *Let $S(t)$ be a C_0-semigroup in a Hilbert space H. Then the following are equivalent:*

(i) $S(t)$ is exponentially stable, that is, there exists $\alpha > 0$ and $M \geq 1$ such that

$$\forall x \in H, \quad \forall t \geq 0, \quad \|S(t)x\| \leq Me^{-\alpha t}\|x\|; \tag{3.3.5}$$

(ii) the property

$$\forall x \in H, \quad \int_0^{+\infty} \|S(t)x\|^2 \,\mathrm{d}t < +\infty; \tag{3.3.6}$$

(iii) there exists a constant $C > 0$ such that

$$\forall x \in H, \quad \int_0^{+\infty} \|S(t)x\|^2 \,\mathrm{d}t \leq C^2\|x\|^2; \tag{3.3.7}$$

(iv) the type ω_0 of $S(t)$ verifies the condition

$$\omega_0 < 0; \tag{3.3.8}$$

(v) $S(t)$ verifies

$$\lim_{t \to +\infty} \|S(t)\| = 0 \tag{3.3.9}$$

in $\mathcal{L}(H)$.

Proof. (i) $\Longrightarrow$ (ii). If $\|S(t)x\| \leq Me^{-\alpha t}\|x\|, \quad \forall x \in H, \quad \forall t \geq 0$, then

$$\int_0^{+\infty} \|S(t)x\|^2 \,\mathrm{d}t \leq M^2\|x\|^2 \int_0^{+\infty} e^{-2\alpha t} \,\mathrm{d}t < +\infty.$$

(ii) $\Longrightarrow$ (iii). Define for each $k \in \mathbb{N}$, the set

$$U_k = \left\{ x \in H : \int_0^{+\infty} \|S(t)x\|^2 \,\mathrm{d}t \leq k^2 \right\}.$$

Claim: For each $k \in \mathbb{N}$, U_k is closed. Let $(x_n)_{n\geq 1}$ be a sequance in U_k such that $x_n \longrightarrow x$ in H. Then $(S(\cdot)x_n)_{n\geq 1}$ is a sequence in $L^2(0, +\infty; H)$ and since $\|S(\cdot)x_n\|_{L^2(0,+\infty;H)} \leq k \; \forall n \in \mathbb{N}$, it follows that $(S(\cdot)x_n)_{n\geq 1}$ is bounded in $L^2(0,+\infty;H)$. Hence up to a subsequence one has $S(\cdot)x_n \rightharpoonup h$ in $L^2(0,+\infty;H)$. So by Mazur Theorem, it follows that the sequence defined by

$$y_n := \sum_{j=n}^{N_n} \lambda_{k,n} S(\cdot)x_j \longrightarrow h \quad \text{in } L^2(0,+\infty;H).$$

Also up to a subsequence,

$$y_n(t) \longrightarrow h(t) \quad \text{for almost every } t \geq 0. \tag{3.3.10}$$

Using the continuity of $S(t)$ for each $t \geq 0$ one has

$$S(t)x_n \longrightarrow S(t)x, \quad \forall t \geq 0. \tag{3.3.11}$$

Let $\varepsilon > 0$ be fixed, then there exists some $m_0 \in \mathbb{N}$ such that for all $n \geq m_0$ one has $\|S(t)x_n - S(t)x\| \leq \varepsilon$. Now, take any $m \geq m_0$, then

$$\begin{aligned}
\|y_m(t) - S(t)x\| &= \left\| \sum_{j=m}^{N_m} \lambda_{k,m}(S(t)x_m - S(t)x) \right\| \leq \sum_{j=m}^{N_m} \lambda_{k,m} \|S(t)x_m - S(t)x\| \\
&\leq \sum_{j=m}^{N_m} \lambda_{k,m}\varepsilon = \varepsilon,
\end{aligned}$$

therefore $y_n(t) \longrightarrow S(t)$ for almost every $t \geq 0$, so from (3.3.10) and the uniqueness of limit it follows that $h(t) = S(t)x$ for almost every $t \geq 0$. Since $S(\cdot)x_n \rightharpoonup h$ in $L^2(0,+\infty;H)$ it follows that

$$\|h\|_{L^2(0,+\infty;H)} \leq \liminf_{n\to+\infty} \|S(\cdot)x_n\|_{L^2(0,+\infty;H)} \leq k.$$

Thus,

$$\int_0^{+\infty} \|S(t)x\|^2 \,\mathrm{d}t = \int_0^{+\infty} \|h(t)\|^2 \,\mathrm{d}t = \|h\|^2_{L^2(0,+\infty;H)} \leq k^2$$

and hence $x \in U_k$, which the proof of our claim.
By the hypothesis $H = \bigcup_{k=1}^{+\infty} U_k$ and because H is complete metric space, using Baire's Category Theorem, there exists $k_0 \in \mathbb{N}$ such that $\mathrm{int} U_{k_0} \neq \emptyset$, i.e., $\exists x_0 \in U_{k_0}$, $\exists r_0 > 0$ such that

$$\mathrm{B}(x_0, r_0) \subset U_{k_0}.$$

Hence

$$\int_0^{+\infty} \|S(t)(x_0 + y)\|^2 \,\mathrm{d}t \leq k_0^2, \quad \forall y \in \mathrm{B}(0, r_0),$$

and for all $y \in \mathrm{B}(0, r_0)$ one has

$$\begin{aligned}
\|T(\cdot)y\|_{L^2(0,+\infty;X)} &\leq \|T(\cdot)(x_0 + y)\|_{L^2(0,+\infty;X)} + \|T(\cdot)x_0\|_{L^2(0,+\infty;X)} \\
&= \left(\int_0^{+\infty} \|S(t)(x_0 + y)\|^2 \right)^{1/2} + \left(\int_0^{+\infty} \|S(t)x_0\|^2 \right)^{1/2} \\
&\leq 2k_0.
\end{aligned}$$

So for all $x \in H,\ x \neq 0$, one can apply the above inequality to $y_0 = r_0 x/\|x\|$ and obtains

$$\|T(\cdot)x\|_{L^2(0,+\infty;X)} \leq \frac{2k_0}{r_0}\|x\|,$$

hence,

$$\forall x \in H, \quad \int_0^{+\infty} \|S(t)x\|^2 \leq C^2\|x\|^2, \quad \text{where } C = \frac{2k_0}{r_0}.$$

(iii) $\Longrightarrow$ (iv). We know that for all $\omega > \omega_0$ there exists $M \geq 1$ such that

$$\forall t \geq 0, \quad \forall x \in H, \quad \|S(t)x\| \leq Me^{\omega t}\|x\|.$$

Choose $\omega = \max\{1, \omega_0\}$, then

$$\begin{aligned}
\frac{1-e^{-2\omega t}}{2\omega}\|S(t)x\|^2 &= \int_0^t e^{-2\omega s}\|S(t)x\|^2 \,\mathrm{d}s = \int_0^t e^{-2\omega s}\|S(s)S(t-s)x\|^2 \,\mathrm{d}s \\
&\leq \int_0^t e^{-2\omega s}\|S(s)\|^2\|S(t-s)x\|^2 \,\mathrm{d}s \leq M^2 \int_0^t \|S(t-s)x\|^2 \,\mathrm{d}s \\
&\leq M^2 \int_0^t \|S(s)x\|^2 \,\mathrm{d}s \leq M^2 \int_0^{+\infty} \|S(s)x\|^2 \,\mathrm{d}s \leq M^2C^2\|x\|^2.
\end{aligned}$$

As a result

$$\forall t > 0, \quad \|S(t)x\|^2 \leq \frac{2\omega}{1-e^{-2\omega t}} M^2C^2\|x\|^2.$$

Fix $T > 0$. As $\|T(t)\|$ is bounded on any compact interval $[0, T],\ T > 0$ and $\omega > 0$, then

$$\text{there exists } K > 0 \text{ such that} \quad \forall t \geq 0 \quad \|S(t)x\| \leq K\|x\|,$$

where

$$K = \max\left\{ \max\{\|S(t)\| : 0 \leq t \leq T\}, MC\left[\frac{2\omega}{1-e^{-2\omega T}}\right]^{1/2} \right\}.$$

As K is independent of x, one has $\|S(t)\| \leq K$ for all $t \geq 0$.

We now use a second estimate for $t > 0$

$$\begin{aligned}
t\|S(t)x\|^2 &= \int_0^t \|S(t)x\|^2 \,\mathrm{d}s \leq \int_0^t \|S(s)\|^2\|S(t-s)x\|^2 \,\mathrm{d}s \\
&\leq K^2 \int_0^t \|S(t-s)x\|^2 \,\mathrm{d}s = K^2 \int_0^t \|S(s)x\|^2 \,\mathrm{d}s \\
&\leq \int_0^{+\infty} \|S(s)x\|^2 \,\mathrm{d}s \leq K^2C^2\|x\|^2,
\end{aligned}$$

which implies

$$\|S(t)\| \leq \frac{KC}{\sqrt{t}} \quad \text{for all} \quad t \geq 0.$$

By choosing $t_0 = (KC+1)^2$ one has

$$\|S(t_0)\| \leq \frac{KC}{KC+1} < 1.$$

Consequently, $\log \|S(t_0)\| < 0$ which yields

$$\omega_0 = \inf_{t>0} \frac{1}{t} \log \|S(t)\| \leq \frac{1}{t_0} \log \|S(t_0)\| < 0.$$

(iv) $\Longrightarrow$ (i). It is sufficient to choose $\alpha = 1$ when $\omega_0 = -\infty$ and $\alpha = -\frac{1}{2}\omega_0$ when ω_0 is finite.

(i) $\Longrightarrow$ (v). If $\|S(t)x\| \leq Me^{-\alpha t}\|x\|, \quad \forall x \in H, \quad \forall t \geq 0$, then $\|S(t)\| \leq Me^{-\alpha t}, \quad \forall t \geq 0$ which yields

$$\lim_{t \to +\infty} \|S(t)\| = 0.$$

To complet the proof, we prove that (v) $\Longrightarrow$ (i). By definition of ω_0

$$\forall t \geq 0, \quad \|S(t)\| \geq e^{\omega_0 t}.$$

But by the hypothesis $S(t) \longrightarrow 0$ as t goes to $+\infty$, one has

$$0 = \lim_{t \to +\infty} \|S(t)\| \geq \lim_{t \to +\infty} e^{\omega_0 t},$$

which yields

$$\omega_0 < 0.$$

□

Theorem 3.3.5. *Let $S(t)$ be a C_0-semigroup on H with infinitesimal generator A. Then each statement in* Theorem 3.3.4 *is equivalent to:*

(vi) there exists a positive symmetric operator $Q \in \mathcal{L}(H)$ such that

$$\forall x, y \in \mathcal{D}(A), \quad \langle QAx, y\rangle + \langle Qx, Ay\rangle = -\langle x, y\rangle. \tag{3.3.12}$$

Equation (3.3.12) *is called* Lypunov equation.

Proof. It is sufficient to show that (ii) $\Longrightarrow$ (vi) and (vi) $\Longrightarrow$ (iii).

(ii) $\Longrightarrow$ (vi). For all $t \geq 0$ and $x, y \in H$ define

$$\langle Q(t)x, y\rangle = \int_0^t \langle S(s)x, S(s)y\rangle \,\mathrm{d}s, \qquad \langle Qx, y\rangle = \int_0^{+\infty} \langle S(s)x, S(s)y\rangle \,\mathrm{d}s,$$

that is ,

$$Q(t)x = \int_0^t S^*(s)S(s)x \,\mathrm{d}s \quad \text{and} \quad Qx = \int_0^{+\infty} S^*(s)S(s)x \,\mathrm{d}s$$

By hypothesis Q and $Q(t)$ are well defined elements of $\mathcal{L}(H)$. In addition,

$$\langle Q(t)x, x\rangle = \int_0^t \langle S(s)x, S(s)x\rangle \,\mathrm{d}s \geq 0, \quad \forall x \in H$$

and

$$\langle Q(t)x, x\rangle = \int_0^t \langle S(s)x, S(s)x\rangle \,\mathrm{d}s = \langle x, Q(t)x\rangle, \quad \forall x \in H,$$

hence, they are positive and symmetric. Moreover since (ii) $\Longrightarrow$ (iii) it follows that

$$\begin{aligned}
|\langle (Q - Q(t))x, y\rangle| &= \left| \int_t^{+\infty} \langle S(s)x, S(s)y\rangle \,\mathrm{d}s \right| \leq \int_t^{+\infty} |\langle S(s)x, S(s)y\rangle| \,\mathrm{d}s \\
&\leq \int_t^{+\infty} \|S(s)x\| \|S(s)y\| \,\mathrm{d}s \leq \|S(\cdot)x\|_{L^2(t,0;+\infty)} \|S(\cdot)y\|_{L^2(t,0;+\infty)} \\
&\leq C\|y\| \|S(\cdot)x\|_{L^2(t,0;+\infty)}
\end{aligned}$$

and, as t goes to $+\infty$, $Q(t)x \longrightarrow Qx$ in H for each $x \in H$.

For all $x \in \mathcal{D}(A)$ and $y \in \mathcal{D}(A)$

$$\begin{aligned}
\langle Q(t)Ax, y\rangle + \langle Q(t)x, Ay\rangle &= \int_0^t (\langle S(s)Ax, S(s)y\rangle + \langle S(s)x, S(s)Ay\rangle) \,\mathrm{d}s \\
&= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \langle S(s)x, S(s)y\rangle \,\mathrm{d}s \\
&= \langle S(t)x, S(t)y\rangle - \langle x, y\rangle,
\end{aligned}$$

which implies

$$\langle S(t)x, S(t)y\rangle = \langle Q(t)Ax, y\rangle + \langle Q(t)x, Ay\rangle + \langle x, y\rangle. \tag{3.3.13}$$

Therefore for $y = x$

$$\lim_{t \to +\infty} \|S(t)x\|^2 = \langle QAx, x\rangle + \langle Qx, Ax\rangle + \|x\|^2.$$

But

$$\int_0^{+\infty} \|S(t)x\|^2 \,\mathrm{d}t < +\infty \Longrightarrow \liminf_{t \to +\infty} \|S(t)x\|^2 = 0,$$

otherwise there exists some positive number a such that

$$\liminf_{t\to+\infty}\|S(t)x\|^2 = \lim_{t\to+\infty}(\inf_{s\geq t}\|S(t)x\|^2) > a > 0,$$

which implies that there exists $t_0 \geq 0$ such that

$$\inf_{s\geq t_0}\|S(t)x\|^2 > a$$

which entails that

$$\begin{aligned}\int_0^{+\infty}\|S(t)x\|^2\,\mathrm{d}t &\geq \int_{t_0}^{+\infty}\|S(t)x\|^2\,\mathrm{d}t \geq \int_{t_0}^{+\infty}\inf_{s\geq t_0}\|S(t)x\|^2\,\mathrm{d}t\\ &\geq \int_{t_0}^{+\infty} a\,\mathrm{d}t = +\infty,\end{aligned}$$

which contradicts the fact that

$$\int_0^{+\infty}\|S(t)x\|^2\,\mathrm{d}t < +\infty.$$

Hence

$$\liminf_{t\to+\infty}\|S(t)x\|^2 = 0$$

and because the limit exists it coincides with its lim inf and

$$\forall x \in \mathcal{D}(A), \quad \lim_{t\to+\infty}\|S(t)x\|^2 = 0,$$

using this and letting t goes to $+\infty$ in equation (3.3.13) one has

$$\langle QAx, y\rangle + \langle Qx, Ay\rangle = -\langle x, y\rangle, \quad \forall x, y \in \mathcal{D}(A).$$

(vi) $\Longrightarrow$ (iii). For each $x \in \mathcal{D}(A)$, $S(t)x \in \mathcal{D}(A)$, $\forall t \geq 0$, one has

$$\langle QAS(t)x, S(t)x\rangle + \langle QS(t)x, AS(t)x\rangle + \langle S(t)x, S(t)x\rangle = 0$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle QS(t)x, S(t)x\rangle + \|S(t)x\|^2 = 0. \tag{3.3.14}$$

By integrating from 0 to T one gets

$$\langle QS(T)x, S(T)x\rangle - \langle Qx, x\rangle + \int_0^T \|S(t)x\|^2\,\mathrm{d}t = 0.$$

Since P is positive it follows that for all $x \in \mathcal{D}(A)$

$$\int_0^T \|S(t)x\|^2\,\mathrm{d}t = \langle Qx, x\rangle - \langle QS(T)x, S(T)x\rangle \leq \langle Qx, x\rangle,$$

which implies

$$\int_0^{+\infty}\|S(t)x\|^2\,\mathrm{d}t \leq \langle Qx, x\rangle \leq C^2\|x\|^2, \quad \text{where } C^2 = \|Q\|.$$

By density of $\mathcal{D}(A)$ in H, the result also holds for all x in H. □

Corollary 3.3.6. *If $\overline{Q}$ is any positive and symmetric solution of the Lyapunov's equation in $\mathcal{L}(H)$, then*

$$\forall x \in H, \quad \langle Qx, x\rangle \le \langle \overline{Q}x, x\rangle,$$

where $Q \in \mathcal{L}(H)$ is defined as

$$\forall x, y \in H, \quad \langle Qx, y\rangle = \int_0^{+\infty} \langle S(t)x, S(t)y\rangle \,\mathrm{d}t.$$

Proof. For all $x \in \mathcal{D}(A)$ we proceed as in the proof of Theorem 3.3.5, from equation (3.3.14) one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \overline{Q}S(t)x, S(t)x\rangle + \|S(t)x\|^2 = 0.$$

Hence

$$\int_0^T \|S(t)x\|^2 \,\mathrm{d}t = \langle \overline{Q}x, x\rangle - \langle \overline{Q}S(T)x, S(T)x\rangle \le \langle \overline{Q}x, x\rangle,$$

which implies

$$\langle Qx, x\rangle = \int_0^{+\infty} \|S(t)x\|^2 \,\mathrm{d}t \le \langle \overline{Q}x, x\rangle, \quad \forall x \in \mathcal{D}(A).$$

So by the density of $\mathcal{D}(A)$ in H the result holds for all $x \in H$. $\square$

We now proceed to stablizability problem, and we consider the case of linear control system (3.2.2).

Definition 3.3.7. *The system (3.2.2) is said to be* exponentially stablizable *if there exists a bounded linear operator $D \in \mathcal{L}(H, U)$ such that the operator A_D,*

$$A_D = A + BD \quad \text{with the domain} \quad \mathcal{D}(A_D) = \mathcal{D}(A),$$

generates an exponentially stable semigroup $S_D(t)$, $t \ge 0$, that is, the following linear system

$$\dot{x}(t) = (A + BD)x(t), \quad x(0) = x_0,$$

is exponentially stable.

Definition 3.3.8. *The system (3.2.2) is said to be* strongly stablizable *if there exixsts a bounded linear operator $D \in \mathcal{L}(H, U)$ such that the operator A_D,*

$$A_D = A + BD \quad \text{with the domain} \quad \mathcal{D}(A_D) = \mathcal{D}(A),$$

generates a strongly stable semigroup $S_D(t)$, $t \ge 0$, that is, the following linear system

$$\dot{x}(t) = (A + BD)x(t), \quad x(0) = x_0,$$

is strongly stable.

Definition 3.3.9. *The system* (3.2.2) *is said to be* weakly stablizable *if there exixsts a bounded linear operator $D \in \mathcal{L}(H, U)$ such that the operator A_D,*

$$A_D = A + BD \quad \text{with the domain} \quad \mathcal{D}(A_D) = \mathcal{D}(A),$$

generates a weakly stable semigroup $S_D(t)$, $t \geq 0$, that is, the following linear system

$$\dot{x}(t) = (A + BD)x(t), \quad x(0) = x_0,$$

is weakly stable.

Theorem 3.3.10. *The following conditions are equivalent:*

(i) *System* (3.2.2) *is exponentially stablizable.*

(ii) *For every initial condition $x_0 \in H$ there exists a control $u(\cdot)$ such that the corrresponding mild solution of* (3.2.2) *satisfies*

$$\int_0^{+\infty} (\|x(t)\|^2 + \|u(t)\|^2)\, \mathrm{d}t < +\infty. \tag{3.3.15}$$

(iii) *There exists a nonnegative operator P satisfying the following Ricatti equation:*

$$2\langle PAx, x\rangle + \langle x, x\rangle - \langle P^2x, x\rangle = 0, \quad x \in \mathcal{D}(A). \tag{3.3.16}$$

(iv) *For every initial condition $x_0 \in H$ there exists a control $u(\cdot)$ such that the control $u(t)$ corresponding to to the mild solution $x(t)$ of* (3.2.2) *tend to zero as $t \to +\infty$.*

Theorem 3.3.11. *(i)* *If a linear control system is exactly null controllable, then it is also exponentially stablizable.*

(ii) *If, in addition, the operator A generates a group, then exactly null controllability is a necessary and sufficient condition for exponential stablzability with an arbitrary prefixed exponential decay rate.*

3.4 Examples

Example 3.4.1 (Controllability of linear wave equation)**.** *Consider the system*

$$\ddot{z} + \alpha\dot{z} + Az = 0, \quad z(0) = z_0,\ \dot{z}(0) = z_1,\ \alpha \geq 0,$$

where A is a positive self adjoint operator on a real Hilbert space H with domain $\mathcal{D}(A)$ such that

$$\langle Az, z\rangle \geq K\|z\|^2 \quad \forall z \in \mathcal{D}(A),\ K > 0.$$

Let $\dot{z} = y$, then $\dot{y} = -\alpha y - Az$. We may write this formally as a first order system

$$\dot{w} = \mathscr{A}w, \quad \text{where } w = \begin{bmatrix} z \\ y \end{bmatrix}, \text{ and } \mathscr{A} = \begin{bmatrix} 0 & I \\ -A & -\alpha I \end{bmatrix}.$$

Introducing a Hilbert space $\mathscr{H} = \mathcal{D}(A^{1/2}) \times H$ *with inner product*

$$\langle w, \bar{w}\rangle_{\mathscr{H}} = \langle A^{1/2}z, A^{1/2}\bar{z}\rangle_H + \langle y, \bar{y}\rangle_H$$

where $w = \begin{bmatrix} z \\ y \end{bmatrix}$, $\bar{w} = \begin{bmatrix} \bar{z} \\ \bar{y} \end{bmatrix}$, *we have*

$$\begin{aligned} \langle w, \mathscr{A}w\rangle_{\mathcal{H}} &= \left\langle \begin{bmatrix} z \\ y \end{bmatrix}, \begin{bmatrix} y \\ -Az - \alpha y \end{bmatrix} \right\rangle_{\mathcal{H}} = \langle A^{1/2}z, A^{1/2}y\rangle_H + \langle y, -Az - \alpha y\rangle_H \\ &= \langle Az, y\rangle_H - \langle Az, y\rangle_H - \alpha\|y\|^2 = -\alpha\|y\|^2, \end{aligned}$$

for all $w \in \mathcal{D}(A) \times \mathcal{D}(A^{1/2})$. *The adjoint of* $\mathscr{A}$ *is computed to be*

$$\mathscr{A}^* = \begin{bmatrix} 0 & -I \\ A & -\alpha I \end{bmatrix}, \quad \mathcal{D}(\mathscr{A}^*) = \mathcal{D}(\mathcal{A}).$$

Thus

$$\begin{aligned} \langle w, \mathscr{A}^*w\rangle_{\mathscr{H}} &= \left\langle \begin{bmatrix} z \\ -y \end{bmatrix}, \begin{bmatrix} y \\ Az - \alpha y \end{bmatrix} \right\rangle_{\mathcal{H}} = \langle A^{1/2}z, A^{1/2}y\rangle_H - \langle y, Az - \alpha y\rangle_H \\ &= \langle Az, y\rangle_H - \langle Az, y\rangle_H - \alpha\|y\|^2 = -\alpha\|y\|^2, \end{aligned}$$

and so we apply Theorem (2.2.12) *to conclude that* A *generates a* C_0-*semigroup on* $\mathcal{H}$.

Now Consider the following linear wave equation

$$\begin{cases} \ddot{z} - z_{xx} = u(t,x), & 0 < x < 1 \\ z(t,0) = z(t,1) = 0, & t \in \mathbb{R}, \end{cases} \tag{3.4.1}$$

where $u \in L^2(0,T; L^2(0,1))$.

In the space $H = L^2(0,1)$ *this system can be written as an abstact first order ordinary differential equation. To this end, we consider the linear unbounded operator* $A : \mathcal{D}(A) \subset H \longrightarrow H$ *defined by* $A\phi = -\phi_{xx}$, *where* $\mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1)$. *Then*

$$\langle A\phi, \phi\rangle = -\int_0^1 \phi_{xx}(x)\phi(x)\,\mathrm{d}x = \int_0^1 |\phi_x(x)|^2\,\mathrm{d}x = \langle \phi, A\phi\rangle$$

and since $\phi(x) = \int_0^x \phi_x(s)\,\mathrm{d}s$ *which implies* $\int_0^1 |\phi(x)|^2 \leq \int_0^1 |\phi_x(x)|\,\mathrm{d}x$ *it follows that*

$$A^* = A \quad \text{and} \quad \langle A\phi, \phi\rangle \geq \|\phi\|_H^2.$$

So applying the preceeding result with $\alpha = 0$ *to conclude that the operator*

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \quad \mathcal{D}(A) \times \mathcal{D}(A^{1/2})$$

on $\mathscr{H} = \mathcal{D}(A) \times H$ *generates a* C_0-*semigroup* $S(t)_{t\geq 0}$. *Here the inner product on* $\mathscr{H}$ *is for* $w, \bar{w} \in H_0^1(0,1) \times L^2(0,1)$ *equivalent to*

$$\langle w, \bar{w}\rangle = \int_0^1 w_{1x}(x)\bar{w}_{1x}(x)\,\mathrm{d}x + \int_0^1 w_{2x}(x)\bar{w}_{2x}(x)\,\mathrm{d}x$$

and

$$S(t)\begin{bmatrix} w_1 \\ \\ w_2 \end{bmatrix} = \begin{bmatrix} \sum_1^{+\infty} 2(\langle w_1, \phi_n\rangle \cos n\pi t + \frac{1}{n\pi}\langle w_2, \phi_n\rangle \sin n\pi t)\phi_n \\ \\ \sum_1^{+\infty} 2(-n\pi\langle w_1, \phi_n\rangle \sin n\pi t + \langle w_2, \phi_n\rangle \cos n\pi t)\phi_n \end{bmatrix}, \tag{3.4.2}$$

where $\phi_n = \sin n\pi x$.

Also the adjoint operators $S^*(t)$ *and* B^* *are computed to be* $S^*(t) = S(-t)$, *and* $B^* = \begin{bmatrix} 0 & I \end{bmatrix}$, *so that if* $u \in L^2(0, T; H)$, *the system will be exactly controllable if and only if there exists* $M_T > 0$, *such that*

$$\int_0^T \|B^* S^*(t) w\|_H^2 \, \mathrm{d}t \geq M_T \|w\|_{\mathcal{H}}^2. \tag{3.4.3}$$

Lemma 3.4.2. *The constat* M_T *can be found in* (3.4.3) *if* $T > 0$.

So the system (3.4.1) *is exactly controllable on* $[0, T]$ *for any* $T>0$.

Example 3.4.3. *Let* $H = \ell^2(\mathbb{R})$, *and define the semigroup on* H *by*

$$S(t)x = (e^{-\lambda_n t}\xi_n), \quad x = (\xi_n), \quad t \geq 0$$

where (λ_n) *is a fixed sequence of positive numbers decreasing to zero.*

Let $\varepsilon > 0$ be given, then since

$$\sum_{n=1}^{+\infty} \xi_n^2 < +\infty,$$

there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=N_\varepsilon}^{+\infty} \xi_n^2 < \frac{\varepsilon}{2},$$

so it follows that

$$\begin{aligned} \|S(t)x\|^2 &= \sum_{n=1}^{+\infty} e^{-2\lambda_n t}\xi_n^2 = \sum_{n=1}^{N_\varepsilon - 1} e^{-2\lambda_n t}\xi_n^2 + \sum_{n=N_\varepsilon}^{+\infty} e^{-2\lambda_n t}\xi_n^2 \\ &\leq \sum_{n=1}^{N_\varepsilon - 1} e^{-2\lambda_n t}\xi_n^2 + \sum_{n=N_\varepsilon}^{+\infty} \xi_n^2 \leq \sum_{n=1}^{N_\varepsilon - 1} e^{-2\lambda_n t}\xi_n^2 + \frac{\varepsilon}{2}. \end{aligned}$$

Therefore letting t goes to $+\infty$ in the above yields

$$\lim_{t \to +\infty} \|S(t)x\| = 0,$$

hence $S(t)x \longrightarrow 0$ in $\ell^2(\mathbb{R})$ as t goes to $+\infty$. Thus $(S(t))_{t \geq 0}$ is assymptotically stable.

But

$$\|S(t)x\|^2 = \sum_{n=1}^{+\infty} e^{-2\lambda_n t}\xi_n^2 \leq \sum_{n=1}^{+\infty} \xi_n^2 = \|x\|^2,$$

which entails that $\|S(t)\| \le 1$ for all $t \ge 0$. On the other hand, for each $t \ge 0$ consider $x_{0t} = (e^{-\lambda_1 t}, 0, 0, \dots)$, then $x_{0t} \in \ell^2(\mathbb{R})$ for each $t \ge 0$ and $S(t)x_{0t} = (1, 0, 0, \dots)$. Therefore

$$1 = \|S(t)x_{0t}\| \le \sup_{x \in \ell^2(\mathbb{R}) : \|x\|=1} \|S(t)x\| = \|S(t)\|, \quad \forall t \ge 0.$$

Hence $\|S(t)\| = 1$ for all $t \ge 0$. Thus assymptotically stable semigroups are not in general exponentially satble.

Example 3.4.4. *Let $H = L^2(0, +\infty)$, and define a semigroup on H by*

$$S(t)x := \begin{cases} z(x-t) & \text{if } x \ge t, \\ 0 & \text{if } x < t. \end{cases} \tag{3.4.4}$$

Then

$$\|S(t)z\|^2 = \int_t^{+\infty} |z(x-t)|^2 \, \mathrm{d}x = \|z\|^2,$$

hence $\|S(t)\| = 1$, for all $t \ge 0$. But

$$|\langle S(t)z, \bar{z}\rangle| = \left| \int_t^{+\infty} z(x-t)\bar{z}(x) \, \mathrm{d}x \right| \le \|z\| \left(\int_t^{+\infty} |\bar{z}(x)|^2 \, \mathrm{d}x \right)^{1/2} \to 0 \text{ as } t \to +\infty.$$

So the semigroup $(S(t))_{t\ge 0}$ is weakly stable. Thus weakly stable semigroups are not in general assymptotically stable.

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