THEORETICAL ANALYSIS OF GRAVITY-CONTROLLED WATERFLOODS

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THEORETICAL ANALYSIS OF GRAVITY-CONTROLLED

WATERFLOODS

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ABSTRACT

The concept of dumpflooding is to flow water from a high pressure aquifer zone to a low pressure oil producing zone by natural force of gravity through a well connecting the two zones to enhance oil sweep into a producing well.

The research described is aimed to find solutions to one of the flaws in dumpflooding that is, the inability to quantify the rate of water injection from a high pressure aquifer zone to a low pressure oil producing zone.

Theoretical dumpflood equations were developed for evaluating the rate of water injection from an aquifer into various types of oil reservoirs. Two cases were considered in the development of the rate of injection equations, which were; water from a finite aquifer injecting into a finite oil reservoir and water from an infinite aquifer injecting into a finite oil reservoir. In each case, the theory is applied to undersaturated oil reservoir as well as oil reservoir with a gas cap.

A simple single-well model was built using Excel Spreadsheet to solve the rate of injection equations by Bisection iteration technique.

Graphical sensitivity analysis ascertained in both cases that, the rate of water injection increases as productivity index increases at constant time interval, however, there is a decline in rate of injection for constant productivity index as time increases.

In conclusion, the rate of water injection recorded in both cases higher values in the finite oil reservoir with gas cap compared to the finite undersaturated oil reservoir because, large amount of water goes into compressing the gas cap gas. The boundary pressure in the finite aquifer zone depletes much faster compared to the infinite aquifer zone.

To the LORD who has been a lamp unto my feet and a light unto my path

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"*I waited patiently for the Lord and He inclined unto me and heard my cry*"

Psalm 40:1 (NKJV)

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Above all, may the grace of our Lord Jesus Christ, the love of God and the fellowship of the Holy Spirit be with you now and forevermore, amen.

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CHAPTER 1
INTRODUCTION

1.1 Motivation

The technique of injecting water into an oil producing reservoir for either pressure maintenance or secondary recovery is a well-established process. In an effort to reduce and reverse the declining pressure and increase oil recovery, a method of replacing the voidage created by the volume of oil extracted from the reservoir is required.

The sources of water available for injection purposes are; produced water, sea water, aquifer water, and fresh water, however, the economics associated with the injection project usually dictates which source should be used. The source of water used is dependent on the quality and quantity of water available.

This work deals with what is probably cheapest method of injecting water into an oil reservoir, that is, gravity waterflooding (also known as dumpflooding). The term dumpflooding and gravity waterflooding will be used interchangeably.

The concept of dumpflooding is to flow water from a high pressure aquifer zone to a low pressure oil producing zone by natural force of gravity through a well connecting the two zones to enhance oil sweep into a producing well.

Dumpflooding is economically beneficial over conventional waterflooding due to the absence of injection surface facilities and injection fluid cost.
Despite these benefits, there are challenges with monitoring the dumpflood wells and controlling the reservoir pressure. These include; difficulties with flood front control, water breakthrough, conformance management, and the inability to quantify the crossflow rate in each well [2] .

This work tries to find a solution to one of the flaws in dumpflooding, that is, the inability to quantify the rate of fluid transfer from a high pressure aquifer zone to a low pressure oil producing zone.

1.2 Objectives

The objectives of this work are as follows;

1. Develop theoretical equations to evaluate the rate of water injection into various types of oil reservoirs from an aquifer by dumpflooding.
2. Develop a simple single-well model to solve the fluid transfer rate equations numerically.

The following cases are considered;

a) Finite aquifer injecting into a finite reservoir, and
b) Infinite aquifer injecting into a finite reservoir

In each case, the theory is applied to undersaturated oil reservoir as well as oil reservoir with a gas cap.

1.3 Methodology

Theoretical dumpflood equations were developed from first principles based on the flow mechanisms. Material balance equations related the rate equations to the various oil reservoirs applied in both cases.

Excel spreadsheet program is written to solve the rate of flow equations by use of bisection iterative technique.

1.4 Scope and Limitation of Work

This work presents a theory of equations describing the fluid transfer rate in gravity-controlled waterfloods. The cases considered include; water from a finite aquifer injecting into a finite reservoir and water from an infinite aquifer injecting into a finite reservoir, where in each case, the equations are applied to an oil reservoir initially above its bubble point pressure and also to an oil reservoir with a gas cap. However, time factor could not permit to verify theoretical results with results from Reservoir Simulation followed by actual field data to ascertain any degree of error.

1.5 Organization of Work

This work is organized in six chapters. Chapter 1 introduces the work in terms of problem definition, objectives, methodology and scope and limitation of work. Chapter 2 addresses an overview of gravity-controlled waterflooding (dumpflooding) and possible applications. These include definitions and reviews from other authors, advantages, and disadvantages of dumpflooding. In chapter 3, flow mechanisms of dumpflooding are discussed with development of rate of water injection equations. Other factors considered are frictional pressure drop, material balance equation, and solution gas drive reservoir for both oil reservoirs initially above the bubble point pressure and an oil reservoir with a gas cap. Chapter 4 focuses on the application of the theory of dumpflooding as well as the bisection iterative technique to the various cases. Summary of analysis of results, conclusion, and recommendation are presented in chapter 5.

CHAPTER 2
OVERVIEW OF GRAVITY WATERFLOODING (DUMPFLOODING)

2.1 Definition of Dumpflooding

The Babylon dictionary defines dumpflooding as an unusual secondary recovery technique that uses water from a shallow water bed above the producing pay to flood the oil producing interval. It further described that water from the aquifer enters the injection string by its own pressure and the weight of the hydrostatic column produces the necessary force for it to penetrate the oil formation, pushing the oil ahead to the producing wells in the field.

Davies [1] explained dumpflooding as the process of allowing a water-bearing reservoir (aquifer) of high pressure potential to feed into an oil reservoir of lower pressure potential by placing the two zones in communication through a casing string. The aquifer zone can be above or below the oil reservoir (see figure 2.1), as long as there is sufficient pressure potential to effect the fluid transfer. In the upward flow case, a downhole pump is required.

According to Rawding et al [2], the term dumpflooding is a method by which fluids from one formation are allowed to flow into another formation, and thereby provide reservoir pressure support. A well is drilled to penetrate both a prolific aquifer (or gas) zone and a producing oil reservoir and under the right conditions, with a higher pressure aquifer (or gas) zone, significant quantities of water (or gas) flow from the aquifer (or gas) to the oil reservoir.

Quttainah and Al-Hunaif [3], described dumpflooding operation as basically injecting water into the recipient reservoir from a water source reservoir by the natural force of gravity and the pressure differential between the two reservoirs using the same well.

2.2 Review of Dumpflooding

Davies [1] developed a theory and technique for dumpflooding which permit monitoring of the fluid injection rate as well as the determination of wellbore properties of both the source zone and the injected zone. He presented the derivations of the equations describing the fluid transfer rate and a computer program to solve the equations of flow. He applied the theory to undersaturated oil reservoir and considered four cases namely;

- Both source and injected zones of infinite size
- Both source and injected zones of finite size
- Infinite injected zone and finite source zone, and
- Infinite source zone and finite injected zone.

He prepared a dumpflood chart for monitoring dumpflood rates from the output data of the computer program comprising of lines of constant source zone's productivity and injected zone's injectivity.

Rawding et al.,[2] described the philosophy and design of an intelligent well installation in a water dumpflood well in West Kuwait. They reported on the application of intelligent well completion for controlled dumpflood where water from the Zubair aquifer formation flows to the Minagish Oolite oil formation. The authors referred to the paper by Quttainah and Al-Hunaif, (2001) who first tried well dumpflood pilot project in the Umm Gudair field and showed that dumpflood can be expanded as a full field water dumpflood injection to pressure support the falling reservoir pressure. In their conclusion, the use of intelligent completion technology and remotely controlled hydraulic Interval Control Valve proved reliable and cost effective solution for a controlled dumpflood.

Quttainah and Al-Hunaif,[3] in their paper studied and analyzed the applicability of long term effects of dumpflood operation to enhance sweep and maintain reservoir pressure. A dumpflood pilot project was initiated from Umm Gudair reservoir to prove the viability of and quantify sweep benefits of water injection from a source aquifer into a recipient oil reservoir (injected zone). The paper recorded that, the dumpflood pilot project proved to be an excellent way to study the impact of water injection on the recipient reservoir. The paper concluded that, the use of dumpflood can be expanded as a full field water dumpflood injection to pressure support the falling reservoir pressure.

Yao et al.,[4] presented a paper on the case study of a 5-injection well pilot flood in three carbonate reservoirs in the Onbysk oil field using a pump-aided reverse dumpflood technique. Feasibility study includes laboratory core flood tests, waterflood computer simulation using a 3-D reservoir simulator and a 5-injection well pilot flood in three domes. In 1-1/2 years, the reverse dumpflood pilot project indicated that waterflooding is economically feasible, where the system reduced setup time and capital investment by a factor of approximately five. Oil recovery factor of about 22% was expected.

Koot and Konopczynski,[5] designed a well production system which utilizes a single well to perform both hydrocarbon fluid production and water dumpflooding processes. The well has a primary wellbore that is communicated with first and second zones respectively containing hydrocarbon fluid and high pressured water. A second lateral wellbore extends from the primary wellbore and communicates with the first zone. By remote control system and valve production tubing structure extending through the primary wellbore, hydrocarbon fluid flow from the first zone to the surface via the production tubing and simultaneously dumpflood water flow from the second zone into the first zone. This increases the pressure within the first zone thereby increasing the hydrocarbon fluid production rate.

Let us now consider some pros and cons of dumpflooding.

2.2.1 Advantages of Dumpflooding

The advantages of dumpflooding are discussed as follows;

- The capital costs of installing a conventional waterflooding system exceed the capital costs of dumpflooding by all the costs upstream of the injection wellhead. These capital costs include items such as flow lines, transfer pumps, water gathering systems, and water treating facilities.
- The operating costs of maintaining a conventional waterflood exceed the dumpflood operating costs by all those costs associated with equipment maintenance.
- In remote areas where there is inadequate ground water, dumpflooding could provide the necessary injection and avoid extremely expensive drilling of

water supply wells and the installation of the necessary pumping equipment to provide the desired injection rate.

- In areas where dumpflooding is being practiced, the injection rate could be increased by converting a watered-out producing well to dumpflooding.
- Dumpflooding is a self-regulating process; as the oil reservoir pressure rises, the rate declines and vice versa. Hence, there should be no tendency to over-pressure one area or to lower the pressure in another.
- The casing corrosion problems are reduced since the fluid transfer occurs in a closed system preventing oxygen from accelerating any corrosion tendency.

2.2.2 Disadvantages of dumpflooding

The disadvantages of dumpflooding are discussed as follows;

- It is difficult to measure the quantity of water transfer from one zone of high pressure to a second zone of lower pressure. In applying the techniques offered in this work, this measuring problem will be resolved.
- The rate of fluid transfer cannot readily be controlled below the natural transfer rate without the introduction of downhole chokes.
- With two zones open within one well, the servicing of either zone becomes more complicated, hence more expensive.
- Circulating the hole clean would be a problem if sand or other particles were deposited from the upper zone to the lower zone.

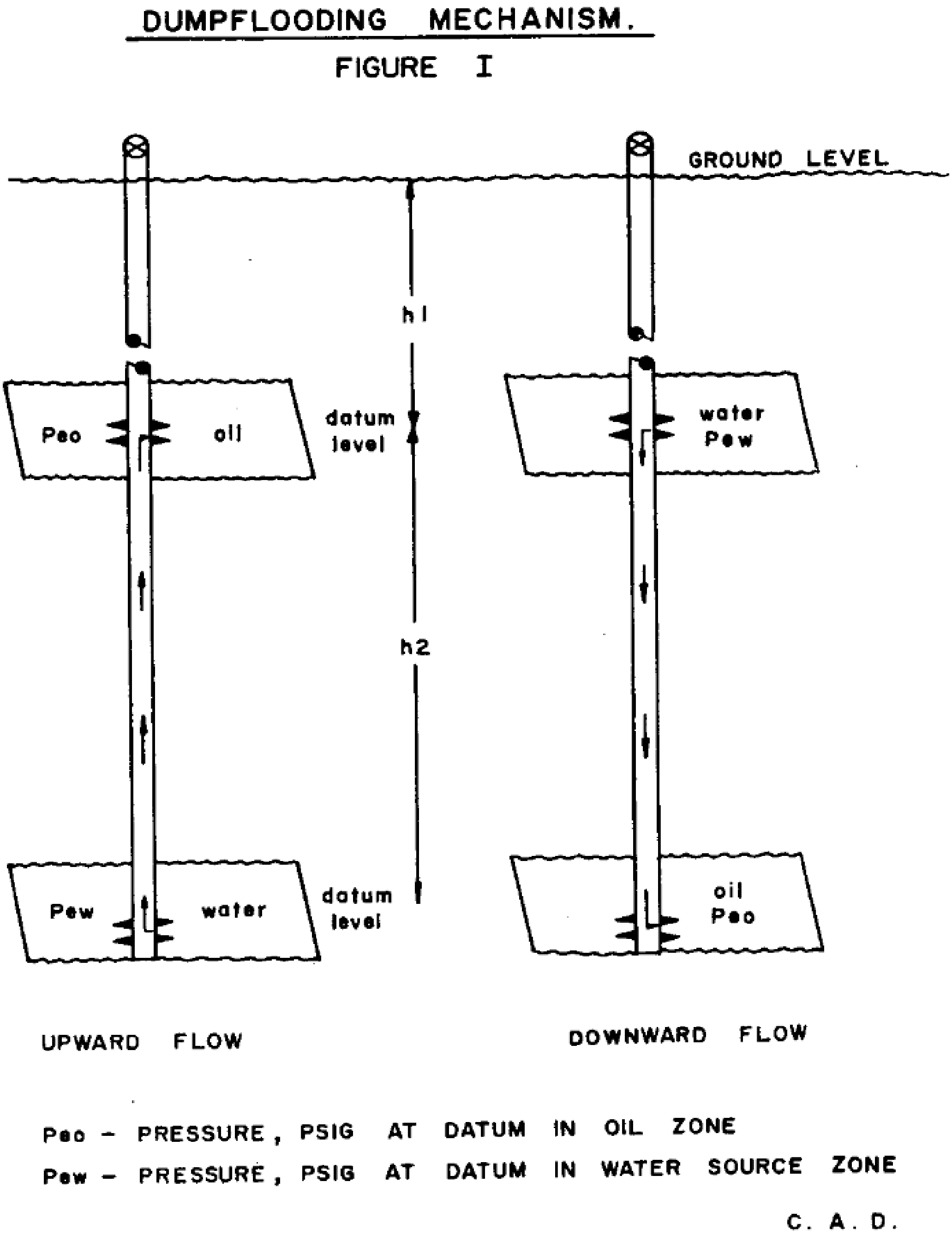


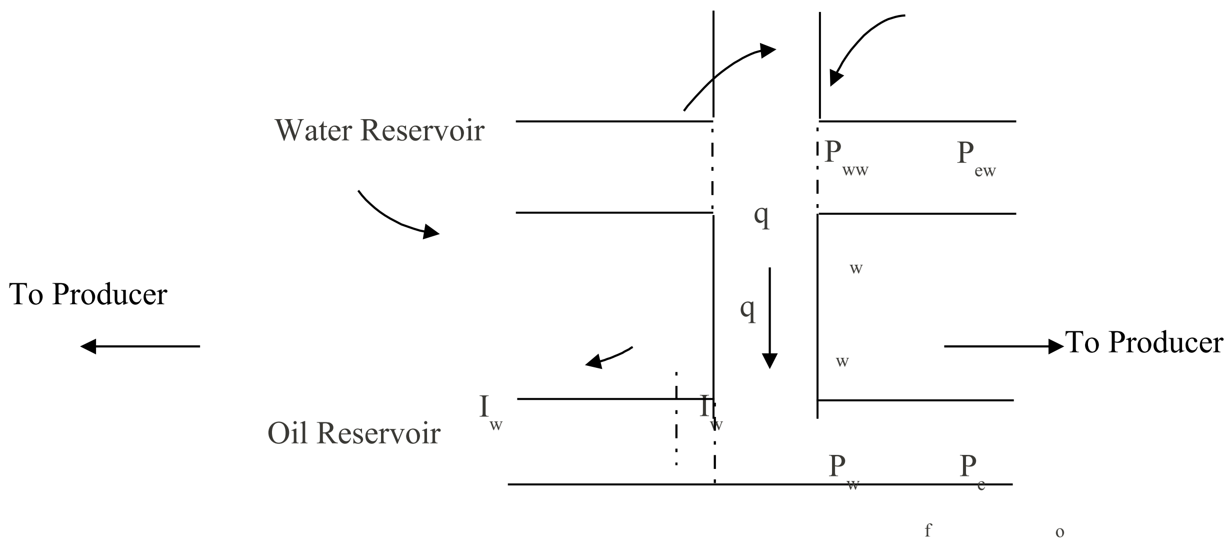
Figure 2.1 Dumpflooding Mechanism

CHAPTER 3
DUMPFLOODING FLOW MECHANISM

It is recalled in Chapter 2, Section 2.1 that, the term "Dumpflooding" as discussed by various authors had different word organization but same idea. The concept is to flow water from a high pressure aquifer zone to a low pressure oil producing zone by natural force of gravity through a well connecting the two zones to enhance oil sweep into a producing well. There are two mechanisms of flow that exist, that is, either the direction flow of water from the aquifer zone to oil producing zone is downward or upward (see figure 2.1).

3.1 Downward Flow Mechanism

Figure 3.1 Downward Flow Dumpflood Mechanism



The figure 3.1 describes the downward flow mechanism where the aquifer located above the oil reservoir flows high pressure water by gravity through a communicating well into the oil reservoir thus, displacing the oil ahead of it into the producing well. In the aquifer zone, the pressure differential ($P_{ew} - P_{ww}$) results in the movement of water which flows into the wellbore at a rate q_w. As the water flows down the communicating well between the two zones by gravity, there exists a pressure drop (i.e. $P_{ww} - P_{wf}$) due to friction and kinetic energy. The water finally injects into the oil reservoir at a rate I_w and together with the pressure differential ($P_{wf} - P_{eo}$) initiates the displacement of oil into the producing well.

The objective of the above description is to deduce an equation for the rate of water I_w injecting into the oil reservoir. The complete derivation is presented in Appendix A. The following assumptions are considered in the derivation.

- All pressures are datum corrected to the oil reservoir datum.
- Single phase (water) flows in the injection well (tubing).
- The fluid (water) is incompressible
- No loss of water in the wellbore, ($q_w = constant$).
- The injectivity and productivity indices are constants.

As derived in Appendix A, the rate of water injection, I_w into the oil reservoir is adapted from Davies[1] given in equation (A-4) as;

$$I_w\left[\frac{1}{I}+\frac{1}{J}\right]=(P_{wf}-P_{eo})+(P_{ew}-P_{ww}) \tag{3.10}$$

3.2 Upward Flow Mechanism

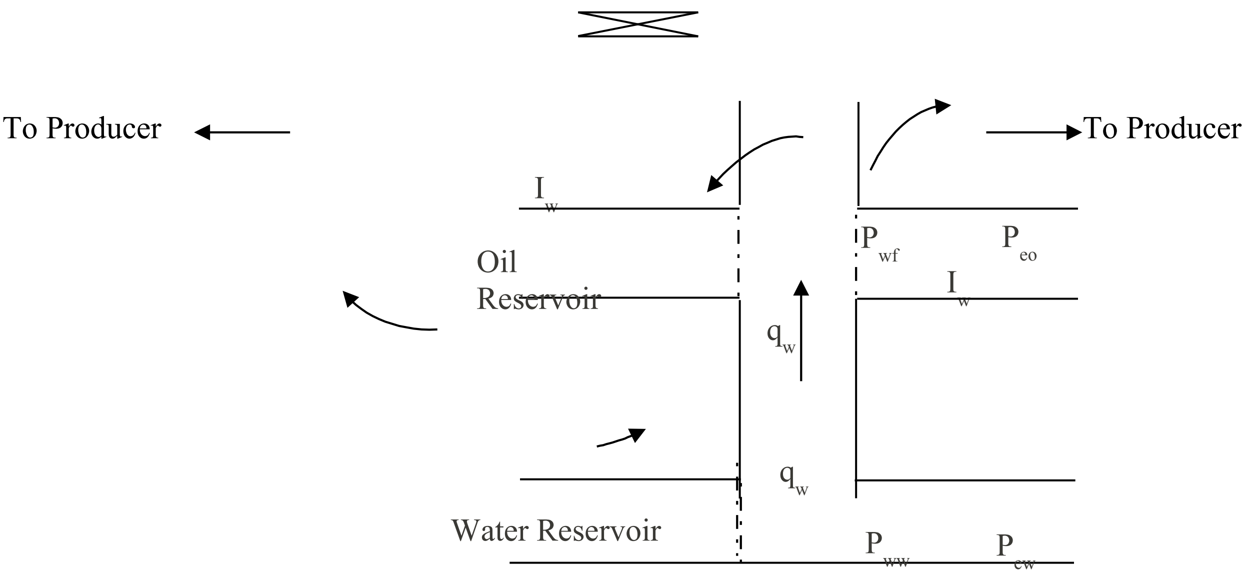


Figure 3.2 Upward Flow Dumpflood Mechanism

The upward flow mechanism described in figure 3.2 is the reverse of the downward flow mechanism. In this case, the aquifer is located below the oil reservoir and injects high pressure water with the aid of a downhole pump through a communicating well into the low pressure oil reservoir to displace oil into the producing well. Apart from the introduction of a downhole pump in this case, other sequence of operation is similar to that described in Section 3.1.

The objective and assumptions of this mechanism are the same as described in Section 3.1.

Similarly, as derived in Appendix A, the rate of water injection, I_w into the oil reservoir is given from equation (A-13) as[1];

$$I_w\left[\frac{1}{I}+\frac{1}{J}\right]=(P_{wf}-P_{eo})+(P_{ew}-P_{ww})$$

(3.11)

This is the same equation as in the downward case.

3.3 Pressure Drop in the Wellbore

In the discussion of above flow mechanisms, as water from the aquifer is injected either downward or upward into the oil reservoir through the wellbore (tubing string) connecting the two zones, there will exist a pressure drop due to friction and kinetic energy.

The relationship between P_{wf} (in the oil reservoir zone), P_{ww} (in the aquifer zone) and the pressure drop due to friction and kinetic energy presented in Appendix A from equation (A-5) is given as;

$$P_{ww}-P_{wf}=\Delta P_{KE}+\Delta P_{fr}$$

Assuming the tubing string cross-section does not change, the pressure drop due to kinetic energy (ΔP_{KE}) is negligible. Thus, the equation reduces to;

$$P_{ww}-P_{wf}=\Delta P_{fr}$$

The pressure drop due to friction is a function of flow rate and controls the dumpflood rate to a degree, therefore, there is a relationship between frictional pressure drop (ΔP_{fr}) and rate of water injection, I_w which is presented in Appendix A from equation (A-7) is given as;

$$I_w\left[\frac{1}{I}+\frac{1}{J}\right]=P_{ew}-P_{eo}-\Delta P_{fr}$$

Where the expression for the frictional pressure drop in equation (A-8) is given as[1];

$$\Delta P_{fr}=\left[\frac{518\rho^{0.79}\mu^{0.207}h}{d^{4.79}\times1000\times1440^{1.79}}\right]q_w^{\ 1.79}=Tq_w^{\ 1.79}$$

Since $q_w=I_w$, $\Delta P_{fr}=TI_w^{\ 1.79}$

The rate of water injection equation that is applicable to both downward and upward flow mechanisms due to frictional pressure drop derived in Appendix A, from equation (A-10) or (A-15) is[1];

$$I_w\left[\frac{1}{I}+\frac{1}{J}\right]+TI_w^{1.79}=P_{ew}-P_{eo} \tag{3.12}$$

Equation (3.12) is the desired equation for the dumpflood injection rate. However, the RHS of the equation contains two unknown parameters, P_{ew} and P_{eo}.

3.4 Material Balance Equation (MBE) for Reservoir Boundary Pressures

It is noticed from equation (3.12) that the equation of the rate of water injection I_w has two unknown pressures quantities in the RHS. Since these pressures P_{ew} and P_{eo} are boundary pressures in the aquifer zone and oil reservoir zone respectively, they can be deduced from material balance equation.

The general MBE derived in Appendix B from equation (B-5) is given as;

$$N_p(B_o+(R_p-R_s)B_g)-W_eB_w-(W_{inj}-W_p)B_w=NC_T\Delta p \tag{3.13}$$

Where

Δp = initial pressure – current pressure ($P_i - P$)

$$C_T=\frac{(B_o-B_{oi})+(R_{si}-R_s)B_g}{\Delta p}+\frac{mB_{oi}}{\Delta p}\left(\frac{B_g}{B_{gi}}-1\right)+\frac{B_{oi}(1+m)}{1-S_{wc}}(C_wS_{wc}+C_f)$$

The boundary pressure in the oil zone presented in Appendix B from equation (B-11) is given as;

$$P_{eo}=P_{io}+\left(\frac{B_w}{NC_T}\right)W_{inj}-\left\{\frac{B_o+(R_p-R_s)B_g}{NC_T}\right\}N_p \tag{3.14}$$

Similarly, the boundary pressure in the aquifer zone presented in Appendix B from equation (B-13) is given as;

$$P_{ew}=P_{iw}-\left(\frac{B_w}{N_wC_{Tw}}\right)W_{inj} \tag{3.15}$$

Finally, substituting equations (3.14) and (3.15) into equation (3.12) gives;

$$I_w\left[\frac{1}{I}+\frac{1}{J}\right]+TI_w^{1.79}=P_{iw}-\left(\frac{B_w}{N_wC_{Tw}}\right)W_{inj}-P_{io}-\left(\frac{B_w}{NC_T}\right)W_{inj}+\left\{\frac{B_o+(R_p-R_s)B_g}{NC_T}\right\}N_p \tag{3.16}$$

This is simplified to;

$$I_w\left[\frac{1}{I}+\frac{1}{J}\right]+TI_w^{1.79}=(P_{iw}-P_{io})-B_w\left[\frac{1}{N_wC_{Tw}}+\frac{1}{NC_T}\right]W_{inj}+\left\{\frac{B_o+(R_p-R_s)B_g}{NC_T}\right\}N_p$$

(3.17)

This represents the general equation for the rate of water I_w injecting from an aquifer into an oil reservoir by dumpflooding.

In the next section, the various types of oil reservoirs in consideration will be discussed. The MBE will be simplified in accordance with the prevailing drive mechanisms and their boundary pressures deduced accordingly.

3.5 Solution Gas Drive Reservoir

We can recall in Section 1.2 that only two types of reservoirs are considered for water injection in this study. These are; undersaturated oil reservoir and oil reservoir with gas cap.

According to Ahmed [8] , the reservoir with the most effective drive mechanism and also considered as best candidate for waterfloods is solution gas drive reservoir. A solution gas drive reservoir is one in which the principal drive mechanism is the expansion of the oil and its originally dissolved gas [6] . The increase in fluid volume during the process is equivalent to the production and generally it's considered the best candidate for waterfloods. Two mechanisms can be distinguished, which are; undersaturated oil reservoir and oil reservoir with initial gas cap.

The MBE presented in Appendix B further discusses the various conditions applicable to these two types of oil reservoirs in consideration.

3.5.1 Undersaturated Oil Reservoir

This is an oil reservoir with its initial pressure greater than the bubble-point pressure of the reservoir fluid, hence, only liquid phase exists in the reservoir. Therefore, during this period, R_p will equal R_s and will equal R_{si} and $m=0$, i.e., no initial gas cap.

The boundary pressure in the oil zone presented in Appendix B from equation (B-11) reduces to;

$$P_{eo} = P_{io} + \left(\frac{B_w}{NC_T} \right) W_{inj} - \left(\frac{B_o}{NC_T} \right) N_p \tag{3.18}$$

The total compressibility in the oil zone presented in Appendix B reduces to

$$C_T = \frac{(B_o - B_{oi})}{\Delta p} + \frac{B_{oi}}{1 - S_{wc}} (C_w S_{wc} + C_f) \tag{3.19}$$

Similarly the boundary pressure in the aquifer zone presented in Appendix B from equation (B-13) reduces to;

$$P_{ew} = P_{iw} - \left(\frac{B_w}{N_w C_{Tw}} \right) W_{inj} \tag{3.20}$$

3.5.2 Oil Reservoir with Gas Cap

When an oil reservoir is with an initial gas cap (i.e., the oil is initially saturated), there is negligible liquid expansion energy. It is assumed that with an initial gas cap, both water and pore compressibilities (C_w and C_f) are negligible and also water influx (W_e) is negligible.

The boundary pressure in the oil zone presented in Appendix B from equation (B-11) reduces to;

$$P_{eo} = P_{io} + \left(\frac{B_w}{NC_T} \right) W_{inj} - \left\{ \frac{B_o + (R_p - R_s) B_g}{NC_T} \right\} N_p \tag{3.21}$$

The total compressibility in the oil zone presented in Appendix B reduces to

$$C_T = \frac{(B_o - B_{oi}) + (R_{si} - R_s) B_g}{\Delta p} + \frac{m B_{oi}}{\Delta p} \left(\frac{B_g}{B_{gi}} - 1 \right) \tag{3.22}$$

Similarly the boundary pressure in the aquifer zone presented in Appendix B from equation (B-13) reduces to;

$$P_{ew} = P_{iw} - \left(\frac{B_w}{N_w C_{Tw}} \right) W_{inj} \tag{3.23}$$

CHAPTER 4
APPLICATION OF DUMPFLOOD THEORY

Previous discussions indicate how the general fluid transfer rate theory of dumpflooding is deduced. However, we need to apply this theory to practical cases and develop real life solutions to various reservoir mechanisms in the oil and gas industry.

In this section of study, the theory would be applied to the various types of reservoirs discussed in Section 3.5 under two cases.

As a reminder, the assumptions that will be applicable to the cases in consideration are restated.

1. The inflow performance can be represented by a straight line
2. There is no water influx
3. Frictional pressure loss is constant
4. There is no loss of water in the wellbore
5. All pressures are datum corrected to the oil reservoir datum
6. Single phase (water) flows in the injection tubing
7. The fluid (water) is incompressible
8. The tubing has no change in diameter

4.1 Case 1: Water from a Finite Aquifer Injecting into a Finite Reservoir

4.1.1 Undersaturated Oil Reservoir

The general equation for solving the rate of water injection from an aquifer zone into an undersaturated oil producing zone is fully developed in Appendix C.

The final rate equation calculated at any time, t from equation (C-18) is given as:

$$I_w(t) = I_{iw} e^{-\left(\frac{B}{A}\right)} + \frac{Cq_o}{B}\left\{1 - e^{-\left(\frac{B}{A}\right)}\right\} \tag{4.1}$$

Where initial rate of injection is in the form;

$$I_{iw} = \left\{\frac{\Delta P_i - TI_{iw}^{1.79}}{\left[\frac{1}{I} + \frac{1}{J}\right]}\right\}$$

These are nonlinear equations which can be solved numerically by an iterative technique. All variables are defined in Appendix C

4.1.2 Oil Reservoir with Gas Cap

Similarly, the final rate of water injection equation developed for oil reservoirs with gas cap as presented in Appendix C from equation (C-37) is given as;

$$I_w(t) = I_{iw} e^{-\left(\frac{E}{D}\right)} + \frac{Fq_o}{E}\left\{1 - e^{-\left(\frac{E}{D}\right)}\right\} \tag{4.2}$$

Where initial rate of injection is in the form;

$$I_{iw} = \left\{\frac{\Delta P_i - TI_{iw}^{1.79}}{\left[\frac{1}{I} + \frac{1}{J}\right]}\right\}$$

These are nonlinear equations which can be solved numerically by an iterative technique. All variables are defined in Appendix C

4.2 Case 2: Water from an Infinite Aquifer injecting into a Finite Reservoir

For an infinite aquifer, i.e. original water in place, $N_w = \infty$

4.2.1 Undersaturated Oil Reservoir

The general equation related to undersaturated oil reservoir for solving the rate of water injection from an aquifer zone into an oil producing zone is fully developed in Appendix D.

The final rate equation calculated at any time, t from equation (D-18) is given as:

$$I_w(t) = I_{iw} e^{-\left(\frac{V}{U}\right)} + \frac{Wq_o}{V}\left\{1 - e^{-\left(\frac{V}{U}\right)}\right\} \tag{4.3}$$

Where initial rate of injection is in the form;

$$I_{iw} = \left\{\frac{\Delta P_i - TI_{iw}^{1.79}}{\left[\frac{1}{I} + \frac{1}{J}\right]}\right\}$$

These are nonlinear equations which can be solved numerically by an iterative technique. All variables are defined in Appendix D

4.2.2 Oil Reservoir with Gas Cap

Similarly, the final rate of water injection equation developed for oil reservoirs with gas cap as presented in Appendix D from equation (D-37) is given as;

$$I_w(t) = I_{iw} e^{-\left(\frac{Y}{X}\right)} + \frac{Zq_o}{Y}\left\{1 - e^{-\left(\frac{Y}{X}\right)}\right\} \tag{4.4}$$

Where initial rate of injection is in the form;

$$I_{iw} = \left\{ \frac{\Delta P_i - TI_{iw}^{\ 1.79}}{\left[\frac{1}{I} + \frac{1}{J}\right]} \right\}$$

These are nonlinear equations which can be solved numerically by an iterative technique. All variables are defined in Appendix D

4.3 Application of Bisection Method for Solving Fluid Rate Transfer

There is the need to find numerical solutions to the nonlinear equations developed for the rate of water injection applied to the previous cases. In this regard, various iterative techniques have been analyzed in Appendix E to consider the most effective but less risky method for use.

The bisection method, amongst other iterative methods appeared much preferred in solving these nonlinear rate equations due to its guarantee for convergence and less risky, though it is slow to converge. This technique will be used to solve the equations of rate of water injection in the above cases.

The initial dumping rate can be written as a continuous function in the form;

$$f(I_{iw}) = AI_{iw} + TI_{iw}^{\ 1.79} - (P_{iw} - P_{io}) \tag{4.5}$$

The continuous function of the rate of water injection applied to case 1 for undersaturated oil reservoir is given by;

$$f(I_w) = I_w - I_{iw} e^{-\left(\frac{B}{A}\right)} - \frac{Cq_o}{B}\left\{1 - e^{-\left(\frac{B}{A}\right)}\right\} \tag{4.6}$$

Similarly, the continuous function of the rate of water injection applied to case 1 for an oil reservoir with gas cap is given by;

$$f(I_w) = I_w - I_{iw} e^{-\left(\frac{E}{D}\right)} - \frac{Fq_o}{E}\left\{1 - e^{-\left(\frac{E}{D}\right)}\right\} \tag{4.7}$$

For undersaturated oil reservoir in case 2, the continuous function of the rate of water injection is given by;

$$f(I_w) = I_w - I_{iw} e^{-\left(\frac{V}{U}\right)} - \frac{Wq_o}{V}\left\{1 - e^{-\left(\frac{V}{U}\right)}\right\} \tag{4.8}$$

Similarly, for oil reservoir with gas cap in case 2, the continuous function of the rate of water injection is given by;

$$f(I_w) = I_w - I_{iw} e^{-\left(\frac{Y}{X}\right)} - \frac{Zq_o}{Y}\left\{1 - e^{-\left(\frac{Y}{X}\right)}\right\} \tag{4.9}$$

CHAPTER 5

DISCUSSIONS AND ANALYSES OF RESULTS

Discussions and analyses conducted are based on the variations in rate of water injection I_w at different productivity index, *J* as well as injectivity index, *I* values within a time interval of one week (7 days) for a total period of four weeks (28 days). Other parameters kept constant include; q_o, C_{Tw}, C_T, B_o, B_g, B_w, B_{wi}, R_s, R_p, *N*, and N_w. Varying parameters include; *I*, *J* and *t.* Assumed values of *J* are 25, 50, 100, and 200 while *I* values are 25, 30, 35, 40, 45, and 50.

Sensitivity analyses of graphs and tables presented in this section are limited to only injectivity index *I* = 40 for all productivity index *J* values and the total period of time.

5.1 Sensitivity Analysis for Case 1: Finite Aquifer Injecting into Finite Reservoir

5.1.1 Undersaturated Oil Reservoir

Graphical results shown in Appendix F present the relationship between rate of water injection, I_w productivity index, *J* and injectivity index, *I* at time interval of one week (7 days) for a total period of four weeks (28 days) when a finite aquifer is dumping into a finite undersaturated oil reservoir.

Table 5.1 Case 1a-Undersaturated Oil Reservoir: Variations in I_w at *I* = 40

	I=40			
	t=7days	t=14days	t=21days	t=28days
	Iw	*Iw*	*Iw*	*Iw*
J=25	**6012.892**	**5834.740**	**5662.674**	**5496.487**
J=50	**8062.310**	**7706.579**	**7368.269**	**7046.527**
J=100	**9622.180**	**9070.526**	**8553.362**	**8068.531**
J=200	**10601.772**	**9890.517**	**9230.866**	**8619.075**

Table 5.1 presents the variations in I_w measured at 7 days interval for a total period of 28 days for all *J* values when *I* = 40. It is indicated that, for each constant *J* value, rate of injection I_w decreases considerably as time increases. Since the aquifer is finite, its boundary pressure depletes much faster with time, hence the decrease in I_w. Analyzing the variations in I_w on the 28th day, there is a rate increase of 1550.04 BWPD representing 22% of water injection as *J* increases from 25 to 50. Similarly, a

rate increase of 1022.004 BWPD representing 12.7% is recorded as J increases from 50 to 100. Finally, an increase of 550.544 BWPD representing 6.4% is dumped into the finite oil reservoir as J increases from 100 to 200. The reduction in rate percent is as a result of the depletion of high aquifer pressure as time increases.

5.1.2 Oil Reservoir with Gas Cap

Appendix F presents graphs showing the relationship between dumping rate, I_w productivity index, J and injectivity index, I at time interval of one week (7 days) for a total period of four weeks (28 days) when a finite aquifer is dumping into a finite oil reservoir with gas cap.

Table 5.2 Case 1b-Oil Reservoir with Gas Cap: Variations in I_w at $I = 40$

	I=40			
	t=7days	t=14days	t=21days	t=28days
	Iw	*Iw*	*Iw*	*Iw*
J=25	**6025.323**	**5859.177**	**5698.708**	**5543.721**
J=50	**8080.129**	**7741.344**	**7419.150**	**7112.735**
J=100	**9644.928**	**9114.600**	**8617.428**	**8151.339**
J=200	**10628.170**	**9941.399**	**9304.455**	**8713.722**

Results presented in table 5.2 indicate a decreasing trend in rate of water injection as time increases for all values of J. The decreasing trend in the rate of water injection is as a result of the depletion of boundary pressure in the finite aquifer. Considering the injection rate on the 28th day, there is a rate increase of 1569.014 BWPD representing 22.1% of water injection as J increases from 25 to 50. Similarly, a rate increase of 1038.604 BWPD representing 12.7% is recorded as J increases from 50 to 100. Finally, an increase of 562.383 BWPD representing 6.5% is dumped into the oil reservoir as J increases from 100 to 200. The reduction in rate percent is as a result of the depletion of high aquifer pressure as time increases.

5.1.3 Comparing Case 1a & 1b: Undersaturated Oil Reservoir vs. Oil Reservoir with Gas Cap

Appendix G presents graphical results showing the relationship between dumping rate, I_w productivity index, *J* and injectivity index, *I* at time interval of one week (7 days) for a total period of four weeks (28 days). The graphs compare the effects of injection rate between two reservoirs; as water is dumped from a finite aquifer into a finite undersaturated oil reservoir as well as a finite oil reservoir with gas cap.

There is insignificant difference in the rate of water injection comparing the two finite reservoirs when time is 7 days as evident in Appendix G, figure G-1. The graphs of the two finite oil reservoirs deviate as time increases for each value of *J*, as shown in figures G-2 to G-4 and tables 5.1 to 5.2.

Analysis on the 28th day from graphical results as well as tables 5.1 and 5.2 indicate that, for $J = 25$, a rate increase of 47.234 BWPD representing 0.85% is dumped into the finite oil reservoir with gas cap. For $J = 50$, a rate increase of 66.208 BWPD representing 0.93% is dumped into the finite oil reservoir with gas cap. Similarly, for $J = 100$, a recorded rate increase of 82.808 BWPD representing 1.0% is dumped into the finite oil reservoir with gas cap. Finally, a rate increase of 94.647 BWPD representing 1.1% is dumped into the finite oil reservoir with gas cap.

The rate of water injection recorded is much higher in the finite oil reservoir with gas cap compared to the finite undersaturated oil reservoir because; much water goes into compressing the gas in the gas cap.

5.2 Sensitivity Analysis for Case 2: Infinite Aquifer Injecting into Finite Reservoir

5.2.1 Undersaturated Oil Reservoir

Appendix H presents graphical results showing the relationship between dumping rate, I_w productivity index, *J* and injectivity index, *I* at time interval of one week (7 days) for a total period of four weeks (28 days) when an infinite aquifer is dumping into a finite undersaturated oil reservoir.

Table 5.3 Case 2a-Undersaturated Oil Reservoir: Variations in I_w at $I = 40$

	I=40			
	t=7days	t=14days	t=21days	t=28days
	Iw	Iw	Iw	Iw
J=25	6132.866	6069.342	6006.761	5945.109
J=50	8295.674	8157.994	8023.258	7891.401
J=100	9981.688	9759.026	9542.464	9331.834
J=200	11065.263	10771.535	10487.174	10211.880

The rates of water injection recorded in table 5.3 indicate a declining trend as time increases for all J values. Though infinite aquifer is assumed to have constant boundary pressure, yet in real life, there is minor drop in pressure with time as water is dumped from the aquifer zone. The table records an increase in rate of injection as J values increase at each interval of time. When t = 28 days, the rate records an increase of 1946.292 BWPD representing 24.7% as J increases from 25 to 50. With an increase in J value from 50 to 100, a rate increase of 1440.433 BWPD representing 15.4% is recorded. Finally, 880.046 BWPD representing 8.6% is recorded as an increase in dumping rate from an infinite aquifer into a finite undersaturated oil reservoir.

5.2.2 Oil Reservoir with Gas Cap

Graphical results shown in Appendix H present the relationship between dumping rate, I_w productivity index, J and injectivity index, I at time interval of one week (7 days) for a total period of four weeks (28 days) when an infinite aquifer is dumping into a finite oil reservoir with gas cap.

Table 5.4 Case 2b-Oil Reservoir with Gas Cap: Variations in I_w at $I = 40$

	I=40			
	t=7days	t=14days	t=21days	t=28days
	Iw	Iw	Iw	Iw
J=25	6145.420	6094.264	6043.868	5994.219
J=50	8313.748	8193.756	8076.329	7961.412
J=100	10004.855	9804.725	9610.077	9420.762
J=200	11092.229	10824.607	10565.519	10314.692

Table 5.4 records a declining rate of water injection as time increases for all *J* values. An insignificant drop in the boundary pressure with time contributes to the declining rate from the infinite aquifer into a finite oil reservoir with gas cap. Analyzing the rate behavior on the 28th day, there is a rate increase of 1967.193 BWPD representing 24.7% as *J* increases from 25 to 50. Similarly, as *J* increases from 50 to 100, a rate increase of 1459.35 BWPD representing 15.5% is recorded. Finally, 893.93 BWPD representing 8.7% is recorded as an increase in dumping rate from an infinite aquifer into a finite oil reservoir with gas cap.

5.2.3 Comparing Case 2a & 2b: Undersaturated Oil Reservoir vs. Oil Reservoir with Gas Cap

In appendix I, graphical results comparing the effects of injection rate between two reservoirs; where water is dumped from an infinite aquifer into a finite undersaturated oil reservoir as well as a finite oil reservoir with gas cap are presented.

There is insignificant difference in the rate of water injection comparing the two finite reservoirs when time is 7 days as evident in Appendix I, figure I-1. The graphs of the two finite oil reservoirs deviate as time increases for each value of *J*, as shown in figures I-2 to I-4 and tables 5.3 to 5.4.

Analysis on the 28th day from graphical results as well as tables 5.3 and 5.4 indicate that, for $J = 25$, a rate increase of 49.11 BWPD representing 0.82% is dumped into the finite oil reservoir with gas cap. For $J = 50$, a rate increase of 70.011 BWPD representing 0.88% is dumped into the finite oil reservoir with gas cap. Similarly, for $J = 100$, a recorded rate increase of 88.938 BWPD representing 0.94% is dumped into the finite oil reservoir with gas cap. Finally, a rate increase of 102.812 BWPD representing 1.0% is dumped into the finite oil reservoir with gas cap.

The rate of water injection recorded is much higher in the finite oil reservoir with gas cap compared to the finite undersaturated oil reservoir because; much water goes into compressing the gas in the gas cap.

5.3 Sensitivity Analysis of Cases 1 & 2

5.3.1 Case 1a & 2a: Comparing Undersaturated Oil Reservoirs

This analysis is based on graphical presentation comparing the rate of water injection between two aquifers dumping into finite undersaturated oil reservoirs as shown in Appendix J. For case 1a, the rate of injection is from a finite aquifer zone while injection rate in case 2a is from an infinite aquifer zone all dumping into finite undersaturated oil reservoirs. Results in figures J-1 to J-4 show a wide rate margin as time increases for each value of J.

Let us base our analysis on time is 28 days and I = 40. The rates of water injection recorded in tables 5.1 and 5.3 reveal that, at J = 25, a rate increase of 448.622 BWPD representing 7.5% is dumped from the infinite aquifer zone. Similarly, 844.874 BWPD representing 10.7% is the increase rate of injection dumped from the infinite aquifer zone when J = 50. At J = 100, the recorded rate increase is 1263.303 BWPD representing 13.5% dumped from the infinite aquifer zone. Lastly, 1592.805 BWPD representing 15.6% is recorded as the increase rate of injection dumped from the infinite aquifer zone for J = 200.

These results indicate that the infinite aquifer zone experiences insignificant boundary pressure depletion compared to depletion of boundary pressure in the finite aquifer zone. However, the rate of water injection increases as productivity index J values increase.

5.3.2 Case 1b & 2b: Comparing Oil Reservoirs with Gas Cap

This analysis is based on graphical presentation comparing the rate of water injection between two aquifers dumping into finite oil reservoirs with gas cap as shown in Appendix K. For case 1b, the rate of injection is from a finite aquifer zone while injection rate in case 2b is from an infinite aquifer zone all dumping into finite oil reservoirs with gas cap. Results in figures K-1 to K-4 show a wide rate margin as time increases for each value of J.

Let us base our analysis on time is 28 days and I = 40. The rates of water injection recorded in tables 5.2 and 5.4 indicate that, at J = 25, a rate increase of 450.498 BWPD representing 7.5% is dumped from the infinite aquifer zone. Similarly, 848.677 BWPD representing 10.7% is the increase rate of injection dumped from the infinite aquifer zone when J = 50. At J = 100, the recorded rate increase is 1269.423

BWPD representing 13.5% dumped from the infinite aquifer zone. Lastly, 1600.97 BWPD representing 15.5% is recorded as the increase rate of injection dumped from the infinite aquifer zone for $J = 200$.

These results indicate that the infinite aquifer zone experiences insignificant boundary pressure depletion compared to depletion of boundary pressure in the finite aquifer zone, hence, much water from the infinite aquifer zone go into compressing the gas in the oil reservoir with gas gap than from the finite aquifer zone. However, the rate of water injection increases as productivity index J values increase

5.4 Conclusions and Recommendations

From all discussions in the body of this work, the following conclusions are made;

1. Theoretical dumpflood equations have been developed to evaluate the rate of water injection into various types of oil reservoir from an aquifer.
2. A simple Excel Spreadsheet model is built to solve the rate equations using Bisection iteration technique.
3. In both cases, the rate of water injection records higher in the finite oil reservoir with gas cap compared to the finite undersaturated oil reservoir because, large amount of water goes into compressing the gas in the gas cap.
4. The boundary pressure in the finite aquifer zone depletes much faster compared to the infinite aquifer zone.
5. In both cases, the rate of water injection increases as productivity index increases at constant time interval, however, there is a decline in rate of injection for constant productivity index as time increases.

It is recommended that the model be tested for accuracy by comparing results from simulation run as well as actual field data. Future research should be conducted for cases such as; finite and infinite aquifers injecting into oil reservoirs with water influx to predict the rate of water injection.

NOMENCLATURE

API symbols are used whenever possible

Symbol	Description	Units
I	Injectivity Index	BWPD/psi
J	Productivity Index	BWPD/psi
N	Original Oil in Place	MMSTB
N_w	Original Water in Place	MMBW
N_p	Oil Production, Cumulative	MMSTB
W_{inj}	Water Injected, Cumulative	MMBW
q_o	Oil Producing Rate	BOPD
q_w	Water Producing Rate	BWPD
q_{iw}	Initial Water Producing Rate	BWPD
$q_{iw}(L)$	Lower Boundary of the function	BWPD
$q_{iw}(u)$	Upper Boundary of the function	BWPD
$q_{iw}(m)$	Estimated Root of the function	BWPD
n	Number of Iterations	Dimensionless
P_e	External Boundary Pressure	psig
P_i	P_e at initial Conditions	psig
P_{ew}	Boundary Pressure in Water Zone	psig
P_{iw}	P_{ew} at initial Conditions	psig
P_{eo}	Boundary Pressure in Oil Zone	psig
P_{io}	P_{eo} at initial Conditions	psig
P_{wf}	Flowing Bottom Hole Pressure	psig
ΔP	Initial Pressure minus Current Pressure	psig
t	Time	days
C_T	Total Compressibility (Oil Zone)	1/psi
C_{Tw}	Total Compressibility (Water Zone)	1/psi
B_o	Current Oil Formation Volume Factor	RB/STB

B_{oi}	B_o at Initial Condition	RB/STB
B_g	Current Gas Formation Volume Factor	RB/SCF
B_{gi}	B_g at Initial Condition	RB/SCF
B_w	Current Oil Formation Volume Factor	RB/STB
B_{wi}	B_w at Initial Condition	RB/STB
R_s	Free Gas	SCF/STB
R_{si}	R_s at Initial Condition	SCF/STB
R_p	Cumulative GOR	SCF/STB
S_{wc}	Connate Water Saturation	Fraction
FL	Friction Loss	psi/BWPD
ρ	Density of Dumping Fluid	gms/cc
μ	Viscosity of Dumping Fluid	cp
d	Diameter or Equivalent* Diameter of Pipe, Internal	inches
d_c	Internal Diameter of Casing	inches
d_t	Internal Diameter of Tubing	inches
h	Distance between Mid-point of Source Zone Producing Interval To Mid-point of Injected Zone Producing Interval	feet

The equivalent* diameter of the annular space between the casing and tubing is computed using the formula;

$$d^{4.79} = (d_c - d_t)^{2.79} \times (d_c + d_t)^2$$

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APPENDIX A
DUMPFLOODING FLOW MECHANISM

Downward Flow Mechanism

Reference to figure 3.1, the rate of water injection, I_w into the oil reservoir can be calculated by the equation:

$$I_w = I(P_{wf} - P_{eo}) \text{ or } \frac{I_w}{I} = P_{wf} - P_{eo} \tag{A-1}$$

Assuming that the injectivity index I is a measured constant, equation (A-1) has three unknowns.

Considering the water reservoir, the rate of water flow into the wellbore is given by;

$$q_w = J(P_{ew} - P_{ww}) \text{ or } \frac{q_w}{J} = P_{ew} - P_{ww} \tag{A-2}$$

Assuming that q_w is constant down the wellbore and there is no loss of fluid,

$q_w = I_w$, and equation (A-2) can be written as;

$$\frac{I_w}{J} = P_{ew} - P_{ww} \tag{A-3}$$

By adding equations (A-1) and (A-3) gives;

$$I_w\left[\frac{1}{I} + \frac{1}{J}\right] = (P_{wf} - P_{eo}) + (P_{ew} - P_{ww}) \tag{A-4}$$

The pressure drop equation in the wellbore which connects P_{wf} and P_{ww} is given by;

$$P_{ww} - P_{wf} = \Delta P_{KE} + \Delta P_{fr} \tag{A-5}$$

Assuming that the tubing cross-section does not change, the pressure drop due to kinetic energy ΔP_{KE} is negligible. Thus;

$$P_{ww} - P_{wf} = \Delta P_{fr} \tag{A-6}$$

Substituting equation (A-6) into equation (A-4) gives

$$I_w\left[\frac{1}{I} + \frac{1}{J}\right] = P_{ew} - P_{eo} - \Delta P_{fr} \tag{A-7}$$

The expression for the frictional pressure drop is given by;

$$\Delta P_{fr} = \left[\frac{518\rho^{0.79}\mu^{0.207}h}{d^{4.79} \times 1000 \times 1440^{1.79}}\right] q_w^{\ 1.79} = T q_w^{\ 1.79} \tag{A-8}$$

Since $q_w = I_w$

$$\Delta P_{fr} = T I_w^{1.79} \qquad \text{(A-9)}$$

Substituting equation (A-9) into equation (A-7) and introducing (A-8) gives;

$$I_w\left[\frac{1}{I}+\frac{1}{J}\right]+T I_w^{1.79} = P_{ew} - P_{eo} \qquad \text{(A-10)}$$

The RHS of the equation has two unknowns which can be evaluated from material balance equation (MBE).

Upward Flow Mechanism

From figure 3.2, the rate at which water from the aquifer flows into the wellbore is given by;

$$q_w = J(P_{ew} - P_{ww}) \text{ or } \frac{q_w}{J} = P_{ew} - P_{ww} \qquad \text{(A-11)}$$

Similarly, the rate at which water is injected into the oil reservoir is given by;

$$I_w = I(P_{wf} - P_{eo}) \text{ or } \frac{I_w}{I} = P_{wf} - P_{eo} \qquad \text{(A-12)}$$

Assuming that q_w is constant down the wellbore and there is no loss of fluid,

$$q_w = I_w$$

By adding equations (A-11) and (A-12) gives;

$$I_w\left[\frac{1}{I}+\frac{1}{J}\right] = (P_{wf} - P_{eo}) + (P_{ew} - P_{ww}) \qquad \text{(A-13)}$$

Finally, in the wellbore the frictional pressure drop is given by

$$P_{ww} - P_{wf} = \Delta P_{fr} = \left[\frac{518\rho^{0.79}\mu^{0.207}h}{d^{4.79}\times 1000\times 1440^{1.79}}\right] q_w^{1.79} = T q_w^{1.79} \qquad \text{(A-14)}$$

By introducing equation (A-14) into equation (A-13) gives;

$$I_w\left[\frac{1}{I}+\frac{1}{J}\right]+T I_w^{1.79} = P_{ew} - P_{eo} \qquad \text{(A-15)}$$

This equation is same as in the downward case.

The RHS of the equation has two unknowns which can be evaluated from material balance equation (MBE).

APPENDIX B

MATERIAL BALANCE EQUATION (MBE) FOR RESERVOIR BOUNDARY PRESSURE

The general form of the material balance equation was first presented by Schilthuis and derived as a volume balance which equates the cumulative observed production, expressed as an underground withdrawal, to the expansion of the fluids in the reservoir resulting from a finite pressure drop.

The volume balance can be evaluated in reservoir barrels as:

Underground Withdrawal (rb)	=	Expansion of oil + originally dissolved gas (rb) Expansion of gas cap (rb) + Reduction in HCPV due to connate water expansion and decrease in the pore volume (rb)

Where

HCPV = hydrocarbon pore volume of the reservoir

Before evaluating the various components in the above equation it is first necessary to define the following parameters.

N is the initial oil in place in stock tank barrels

$$= \frac{V\phi(1-S_{wc})}{B_{oi}} \qquad \text{(stb)}$$

m is the ratio

$$= \frac{initial\ hydrocarbon\ volume\ of\ the\ gas\ cap}{initial\ hydrocarbon\ volume\ of\ the\ oil}$$

(being defined under initial conditions, is a constant)

N_p is the cumulative oil production in stock tank barrels, and

R_p is the cumulative gas oil ratio

$$= \frac{Cumulative\ gas\ production(scf)}{cumulative\ oil\ production(stb)}$$

The total underground withdrawal term is therefore

$$N_p(B_o + (R_p - R_s)B_g) \qquad \text{(rb)} \qquad \text{(B-1)}$$

Expansion of oil plus originally dissolved gas term is

$$N((B_o - B_{oi}) + (R_{si} - R_s)B_g) \quad \text{(rb)} \qquad \text{(B-2)}$$

Expansion of the gas cap term is

$$mNB_{oi}\left(\frac{B_g}{B_{gi}}-1\right) \qquad \text{(rb)} \qquad \text{(B-3)}$$

Total change in the HCPV due to the connate water expansion and pore volume reduction term is;

$$(1+m)NB_{oi}\left(\frac{C_w S_{wc}+C_f}{1-S_{wc}}\right)\Delta p \qquad \text{(B-4)}$$

Therefore equating the underground withdrawal to the sum of the volume changes in the reservoir, i.e., equations (B-1), (B-2), (B-3), (B-4), gives the general expression for the material balance as;

$$N_p\left(B_o+(R_p-R_s)B_g\right)=NB_{oi}\left[\frac{(B_o-B_{oi})+(R_{si}-R_s)B_g}{B_{oi}}+m\left(\frac{B_g}{B_{gi}}-1\right)+(1+m)\left(\frac{C_w S_{wc}+C_f}{1-S_{wc}}\right)\Delta p\right] \qquad \text{(B-5)}$$

$$+(W_e-W_p)B_w+W_{inj}B_w$$

Equation (B-5) can be expressed as:

$$N_p(B_o+(R_p-R_s)B_g)-W_eB_w-(W_{inj}-W_p)B_w = NC_T\Delta p \qquad \text{(B-6)}$$

Where

Δp = initial pressure – current pressure ($P_i - P$)

$$C_T=\frac{(B_o-B_{oi})+(R_{si}-R_s)B_g}{\Delta p}+\frac{mB_{oi}}{\Delta p}\left(\frac{B_g}{B_{gi}}-1\right)+\frac{B_{oi}(1+m)}{1-S_{wc}}(C_w S_{wc}+C_f)$$

(B-7)

Assume that

Water influx (W_e) = 0

$(W_{inj} - W_p)B_w$, is the net water injected, i.e. $W_{inj}B_w$, equation (B-6) simplifies to;

$$N_p(B_o + (R_p - R_s)B_g) - W_{inj}B_w = NC_T(P_i - P) \quad \text{(B-8)}$$

Rearranging gives;

$$W_{inj}B_w - N_p(B_o + (R_p - R_s)B_g) = NC_T(P - P_i) \quad \text{(B-9)}$$

Thus for the oil zone, MBE is given as;

$$W_{inj}B_w - N_p(B_o + (R_p - R_s)B_g) = NC_T(P_{eo} - P_{io}) \quad \text{(B-10)}$$

The boundary pressure in the oil zone can be rearranged and given as;

$$P_{eo} = P_{io} + \left(\frac{B_w}{NC_T}\right)W_{inj} - \left\{\frac{B_o + (R_p - R_s)B_g}{NC_T}\right\}N_p \quad \text{(B-11)}$$

Applying the MBE to the aquifer zone where there is no oil and so $N_p = 0$ gives;

$$W_{inj}B_w = N_w C_{Tw}(P_{iw} - P_{ew}) \quad \text{(B-12)}$$

The boundary pressure in the aquifer zone can be rearranged and given as;

$$P_{ew} = P_{iw} - \left(\frac{B_w}{N_w C_{Tw}}\right)W_{inj} \quad \text{(B-13)}$$

APPENDIX C

Case 1: Water from a Finite Aquifer injecting into a Finite Reservoir

a) Undersaturated Oil reservoir.

The assumptions in undersaturated oil reservoirs are such that, $R_p = R_{si} = R_s$ and $m = 0$, thus MBE in the oil zone at time, t, from equation (B-10) is reduced to;

$$W_{inj} B_w - N_p B_o = NC_T (P_{eo} - P_{io}) \tag{C-1}$$

The boundary pressure in the oil zone from equation (C-1) is deduced as;

$$P_{eo} = P_{io} + \left(\frac{B_w}{NC_T} \right) W_{inj} - \left(\frac{B_o}{NC_T} \right) N_p \tag{C-2}$$

The total compressibility in the oil zone from Appendix B is reduced to;

$$C_T = \frac{(B_o - B_{oi})}{\Delta p} + \frac{B_{oi}}{1 - S_{wc}} (C_w S_{wc} + C_f) \tag{C-3}$$

Likewise, MBE in the aquifer zone at time, t, from equation (B-12) would give

$$W_{inj} B_w = N_w C_{Tw} (P_{iw} - P_{ew}) \tag{C-4}$$

The boundary pressure in the aquifer zone from equation (C-4) is deduced as;

$$P_{ew} = P_{iw} - \left(\frac{B_w}{N_w C_{Tw}} \right) W_{inj} \tag{C-5}$$

From equation (A-10) or (A-15), the rate of water injection into the oil zone in either downward or upward flow mechanisms is;

$$I_w \left[\frac{1}{I} + \frac{1}{J} \right] + T I_w^{1.79} = P_{ew} - P_{eo} \tag{C-6}$$

Substituting equations (C-2) and (C-5) into equation (C-6) gives;

$$I_w \left[\frac{1}{I} + \frac{1}{J} \right] + T I_w^{1.79} = (P_{iw} - P_{io}) - \left(\frac{B_w}{N_w C_{Tw}} \right) W_{inj} - \left(\frac{B_w}{NC_T} \right) W_{inj} + \left(\frac{B_o}{NC_T} \right) N_p \tag{C-7}$$

This is simplified to;

$$I_w \left[\frac{1}{I} + \frac{1}{J} \right] + T I_w^{1.79} = (P_{iw} - P_{io}) - B_w \left(\frac{1}{N_w C_{Tw}} + \frac{1}{NC_T} \right) W_{inj} + B_o \left(\frac{1}{NC_T} \right) N_p$$

(C-8)

Let $A = \left[\frac{1}{I} + \frac{1}{J}\right]$

$$B = B_w\left(\frac{1}{N_w C_{Tw}} + \frac{1}{NC_T}\right)$$

$$C = B_o\left(\frac{1}{NC_T}\right)$$

$$\Delta P_i = (P_{iw} - P_{io})$$

Equation (C-8) becomes;

$$AI_w + TI_w^{1.79} = \Delta P_i - BW_{inj} + CN_p \tag{C-9}$$

Neglecting the frictional pressure drop, i.e. $TI_w^{1.79} = 0$ and differentiating equation (C-9) with respect to time, *t*;

$$A\frac{\partial I_w}{\partial t} + B\frac{\partial W_{inj}}{\partial t} - C\frac{\partial N_p}{\partial t} = 0 \tag{C-10}$$

But

$$\frac{\partial W_{inj}}{\partial t} = I_w \qquad \frac{\partial N_p}{\partial t} = q_o$$

Thus;

$$A\frac{\partial I_w}{\partial t} + BI_w - Cq_o = 0 \tag{C-11}$$

If the oil production rate q_o is a specified constant, then;

$$A\frac{dI_w}{dt} + BI_w = cons\tan t \tag{C-12}$$

$$\frac{dI_w}{dt} + \left(\frac{B}{A}\right)I_w = \frac{Cq_o}{A} \tag{C-13}$$

The solution to this differential equation with respect to time *t*, is given as;

$$I_w(t) = Ke^{-\left(\frac{B}{A}\right)t} + \frac{Cq_o}{B} \tag{C-14}$$

Where K, is a constant. This constant is found by applying the initial condition.

At initial condition;

$$t = 0 \qquad I_w = I_{iw} \qquad P_{ew} = P_{iw} \qquad P_{eo} = P_{io}$$

Therefore,

$$I_{iw}\left[\frac{1}{I}+\frac{1}{J}\right]=P_{iw}-P_{io}-\Delta P_{fr}$$

$$AI_{iw}=\Delta P_i-\Delta P_{fr}$$

Or $$I_{iw}=\frac{\Delta P_i-\Delta P_{fr}}{A}$$, is the initial rate of water injection (C-15)

Hence

$$\frac{\Delta P_i-\Delta P_{fr}}{A}=\frac{Cq_o}{B}+K$$

$$K=\frac{\Delta P_i-\Delta P_{fr}}{A}-\frac{Cq_o}{B} \tag{C-16}$$

Substituting equation (C-16) into equation (C-14) gives;

$$I_w(t)=\left[\frac{\Delta P_i-\Delta P_{fr}}{A}-\frac{Cq_o}{B}\right]e^{-\left(\frac{B}{A}\right)t}+\frac{Cq_o}{B}$$

Alternatively;

$$I_w(t)=\left\{\frac{\Delta P_i-\Delta P_{fr}}{A}\right\}e^{-\left(\frac{B}{A}\right)t}+\frac{Cq_o}{B}\left\{1-e^{-\left(\frac{B}{A}\right)t}\right\} \tag{C-17}$$

Or

$$I_w(t)=I_{iw}e^{-\left(\frac{B}{A}\right)t}+\frac{Cq_o}{B}\left\{1-e^{-\left(\frac{B}{A}\right)t}\right\} \tag{C-18}$$

Where initial rate of injection is in the form;

$$I_{iw}=\left\{\frac{\Delta P_i-TI_{iw}^{1.79}}{\left[\frac{1}{I}+\frac{1}{J}\right]}\right\} \tag{C-19}$$

Equations (C-18) and (C-19) represent the final and initial rate of water injection at any time *t*, for undersaturated oil reservoir. These nonlinear equations can be solved by an iterative technique.

b) Oil Reservoir with a Gas Cap

The assumptions in oil reservoir with gas cap gas are such that, $C_w=0$ and $C_f=0$, thus MBE in the oil zone at time, t, from equation (B-10) is reduced to;

$$W_{inj}B_w-N_p(B_o+(R_p-R_s)B_g)=NC_T(P_{eo}-P_{io}) \tag{C-20}$$

The boundary pressure in the zone from equation (C-20) is given as;

$$P_{eo} = P_{io} + \left(\frac{B_w}{NC_T}\right)W_{inj} - \left\{\frac{B_o + (R_p - R_s)B_g}{NC_T}\right\}N_p \qquad \text{(C-21)}$$

The total compressibility in the oil zone from Appendix B is reduced to;

$$C_T = \frac{(B_o - B_{oi}) + (R_{si} - R_s)B_g}{\Delta p} + \frac{mB_{oi}}{\Delta p}\left(\frac{B_g}{B_{gi}} - 1\right) \qquad \text{(C-22)}$$

Likewise, MBE in the aquifer zone at time, t, from equation (B-12) would give

$$W_{inj}B_w = N_w C_{Tw}(P_{iw} - P_{ew}) \qquad \text{(C-23)}$$

The boundary pressure in the aquifer zone from equation (C-23) is deduced as;

$$P_{ew} = P_{iw} - \left(\frac{B_w}{N_w C_{Tw}}\right)W_{inj} \qquad \text{(C-24)}$$

From equation (A-10) or (A-15), the rate of water injection into the oil zone in either downward or upward flow mechanisms is;

$$I_w\left[\frac{1}{I} + \frac{1}{J}\right] + TI_w^{\,1.79} = P_{ew} - P_{eo} \qquad \text{(C-25)}$$

Substituting equations (C-21) and (C-24) into equation (C-25) gives;

$$I_w\left[\frac{1}{I} + \frac{1}{J}\right] + TI_w^{\,1.79} = P_{iw} - \left(\frac{B_w}{N_w C_{Tw}}\right)W_{inj} - P_{io} - \left(\frac{B_w}{NC_T}\right)W_{inj} + \left\{\frac{B_o + (R_p - R_s)B_g}{NC_T}\right\}N_p$$

(C-26)

This is simplified to;

$$I_w\left[\frac{1}{I} + \frac{1}{J}\right] + TI_w^{\,1.79} = (P_{iw} - P_{io}) - B_w\left[\frac{1}{N_w C_{Tw}} + \frac{1}{NC_T}\right]W_{inj} + \left\{\frac{B_o + (R_p - R_s)B_g}{NC_T}\right\}N_p$$

(C-27)

Let $D = \left[\frac{1}{I} + \frac{1}{J}\right]$

$$E = B_w\left(\frac{1}{N_w C_{Tw}} + \frac{1}{NC_T}\right)$$

$$F = \left(\frac{B_o + \left(R_p - R_s\right)B_g}{NC_T}\right)$$

$$\Delta P_i = (P_{iw} - P_{io})$$

Equation (C-27) becomes;

$$DI_w + TI_w^{\,1.79} = \Delta P_i - EW_{inj} + FN_p \qquad \text{(C-28)}$$

Neglecting the frictional pressure drop, i.e. $TI_w^{1.79} = 0$ and differentiating equation (C-28) with respect to time, *t*;

$$D\frac{\partial I_w}{\partial t} + E\frac{\partial W_{inj}}{\partial t} - F\frac{\partial N_p}{\partial t} = 0 \qquad \text{(C-29)}$$

But

$$\frac{\partial W_{inj}}{\partial t} = I_w \qquad\qquad \frac{\partial N_p}{\partial t} = q_o$$

Thus;

$$D\frac{\partial I_w}{\partial t} + EI_w - Fq_o = 0 \qquad \text{(C-30)}$$

If the oil production rate q_o is a specified constant, then;

$$D\frac{dI_w}{dt} + EI_w = cons\tan t \qquad \text{(C-31)}$$

$$\frac{dI_w}{dt} + \left(\frac{E}{D}\right)I_w = \frac{Fq_o}{D} \qquad \text{(C-32)}$$

The solution to this differential equation with respect to time *t*, is given as;

$$I_w(t) = Ke^{-\left(\frac{E}{D}\right)t} + \frac{Fq_o}{E} \qquad \text{(C-33)}$$

Where K, is a constant. This constant is found by applying the initial condition.

At initial condition;

$t = 0 \qquad I_w = I_{iw} \qquad P_{ew} = P_{iw} \qquad P_{eo} = P_{io}$

Therefore,

$$I_{iw}\left[\frac{1}{I} + \frac{1}{J}\right] = P_{iw} - P_{io} - \Delta P_{fr}$$

$$DI_{iw} = \Delta P_i - \Delta P_{fr}$$

Or $\qquad I_{iw} = \dfrac{\Delta P_i - \Delta P_{fr}}{D}$, is the initial rate of water injection $\qquad$ (C-34)

Hence

$$\frac{\Delta P_i - \Delta P_{fr}}{D} = \frac{Fq_o}{E} + K$$

$$K = \frac{\Delta P_i - \Delta P_{fr}}{D} - \frac{Fq_o}{E} \qquad \text{(C-35)}$$

Substituting equation (C-35) into equation (C-33) gives;

$$I_w(t)=\left[\frac{\Delta P_i-\Delta P_{fr}}{D}-\frac{Fq_o}{E}\right]e^{-\left(\frac{E}{D}\right)t}+\frac{Fq_o}{E}$$

Alternatively;

$$I_w(t)=\left\{\frac{\Delta P_i-\Delta P_{fr}}{D}\right\}e^{-\left(\frac{E}{D}\right)t}+\frac{Fq_o}{E}\left\{1-e^{-\left(\frac{E}{D}\right)t}\right\} \tag{C-36}$$

Or

$$I_w(t)=I_{iw}e^{-\left(\frac{E}{D}\right)t}+\frac{Fq_o}{E}\left\{1-e^{-\left(\frac{E}{D}\right)t}\right\} \tag{C-37}$$

Where initial rate of injection is in the form;

$$I_{iw}=\left\{\frac{\Delta P_i-TI_{iw}^{1.79}}{\left[\frac{1}{I}+\frac{1}{J}\right]}\right\} \tag{C-38}$$

Equations (C-37) and (C-38) represent the final and initial rate of water injection at any time *t*, for undersaturated oil reservoir. These nonlinear equations can be solved by an iterative technique.

APPENDIX D

Case 2: Water from an Infinite Aquifer injecting into a Finite Reservoir

a) Undersaturated Oil reservoir

The assumptions in undersaturated oil reservoirs are such that, $R_p = R_{si} = R_s$ and $m = 0$, thus MBE in the oil zone at time, t, from equation (B-10) is reduced to;

$$W_{inj} B_w - N_p B_o = N C_T (P_{eo} - P_{io}) \quad \text{(D-1)}$$

The boundary pressure in the oil zone from equation (C-1) is deduced as;

$$P_{eo} = P_{io} + \left(\frac{B_w}{NC_T} \right) W_{inj} - \left(\frac{B_o}{NC_T} \right) N_p \quad \text{(D-2)}$$

The total compressibility in the oil zone from Appendix B is reduced to;

$$C_T = \frac{(B_o - B_{oi})}{\Delta p} + \frac{B_{oi}}{1 - S_{wc}} (C_w S_{wc} + C_f) \quad \text{(D-3)}$$

Likewise, MBE in the aquifer zone at time, t, from equation (B-12) would give

$$W_{inj} B_w = N_w C_{Tw} (P_{iw} - P_{ew}) \quad \text{(D-4)}$$

The boundary pressure in the aquifer zone from equation (C-4) is deduced as;

$$P_{ew} = P_{iw} - \left(\frac{B_w}{N_w C_{Tw}} \right) W_{inj} \quad \text{(D-5)}$$

But for an infinite aquifer zone, $N_w = \infty$ therefore equation (D-5) reduces to;

$$P_{ew} = P_{iw} \quad \text{(D-6)}$$

From equation (A-10) or (A-15), the rate of water injection into the oil zone in either downward or upward flow mechanisms is;

$$I_w \left[\frac{1}{I} + \frac{1}{J} \right] + T I_w^{1.79} = P_{ew} - P_{eo} \quad \text{(D-7)}$$

Substituting equations (D-2) and (D-6) into equation (D-7) gives;

$$I_w \left[\frac{1}{I} + \frac{1}{J} \right] + T I_w^{1.79} = (P_{iw} - P_{io}) - \left(\frac{B_w}{NC_T} \right) W_{inj} + \left(\frac{B_o}{NC_T} \right) N_p \quad \text{(D-8)}$$

Let $U = \left[\frac{1}{I} + \frac{1}{J}\right]$

$V = B_w\left(\frac{1}{NC_T}\right)$

$W = B_o\left(\frac{1}{NC_T}\right)$

$\Delta P_i = (P_{iw} - P_{io})$

Equation (C-8) becomes;

$$UI_w + TI_w^{1.79} = \Delta P_i - VW_{inj} + WN_p \qquad \text{(D-9)}$$

Neglecting the frictional pressure drop, i.e. $TI_w^{1.79} = 0$ and differentiating equation (D-9) with respect to time, *t*;

$$U\frac{\partial I_w}{\partial t} + V\frac{\partial W_{inj}}{\partial t} - W\frac{\partial N_p}{\partial t} = 0 \qquad \text{(D-10)}$$

But

$$\frac{\partial W_{inj}}{\partial t} = I_w \qquad \frac{\partial N_p}{\partial t} = q_o$$

Thus;

$$U\frac{\partial I_w}{\partial t} + VI_w - Wq_o = 0 \qquad \text{(D-11)}$$

If the oil production rate q_o is a specified constant, then;

$$U\frac{dI_w}{dt} + VI_w = cons\tan t \qquad \text{(D-12)}$$

$$\frac{dI_w}{dt} + \left(\frac{V}{U}\right)I_w = \frac{Wq_o}{U} \qquad \text{(D-13)}$$

The solution to this differential equation with respect to time *t*, is given as;

$$I_w(t) = Ke^{-\left(\frac{V}{U}\right)t} + \frac{Wq_o}{V} \qquad \text{(D-14)}$$

Where K, is a constant. This constant is found by applying the initial condition.

At initial condition;

$t = 0 \qquad I_w = I_{iw} \qquad P_{ew} = P_{iw} \qquad P_{eo} = P_{io}$

Therefore,

$$I_{iw}\left[\frac{1}{I} + \frac{1}{J}\right] = P_{iw} - P_{io} - \Delta P_{fr}$$

$$UI_{iw} = \Delta P_i - \Delta P_{fr}$$

Or $\quad I_{iw} = \dfrac{\Delta P_i - \Delta P_{fr}}{U}$, is the initial rate of water injection (D-15)

Hence

$$\frac{\Delta P_i - \Delta P_{fr}}{U} = \frac{Wq_o}{V} + K$$

$$K = \frac{\Delta P_i - \Delta P_{fr}}{U} - \frac{Wq_o}{V} \qquad \text{(D-16)}$$

Substituting equation (D-16) into equation (D-14) gives;

$$I_w(t) = \left[\frac{\Delta P_i - \Delta P_{fr}}{U} - \frac{Wq_o}{V}\right] e^{-\left(\frac{V}{U}\right)t} + \frac{Wq_o}{V}$$

Alternatively;

$$I_w(t) = \left\{\frac{\Delta P_i - \Delta P_{fr}}{U}\right\} e^{-\left(\frac{V}{U}\right)t} + \frac{Wq_o}{V}\left\{1 - e^{-\left(\frac{V}{U}\right)t}\right\} \qquad \text{(D-17)}$$

Or

$$I_w(t) = I_{iw} e^{-\left(\frac{V}{U}\right)t} + \frac{Wq_o}{V}\left\{1 - e^{-\left(\frac{V}{U}\right)t}\right\} \qquad \text{(D-18)}$$

Where initial rate of injection is in the form;

$$I_{iw} = \left\{\frac{\Delta P_i - T I_{iw}^{1.79}}{\left[\frac{1}{I} + \frac{1}{J}\right]}\right\} \qquad \text{(D-19)}$$

Equations (D-18) and (D-19) represent the final and initial rate of water injection at any time *t*, for undersaturated oil reservoir. These nonlinear equations can be solved by an iterative technique.

b) Oil reservoir with gas cap

The assumptions in oil reservoir with gas cap are such that, $C_w = 0$ and $C_f = 0$, thus MBE in the oil zone at time, *t*, from equation (B-10) is reduced to;

$$W_{inj}B_w - N_p(B_o + (R_p - R_s)B_g) = NC_T(P_{eo} - P_{io}) \qquad \text{(D-20)}$$

The boundary pressure in the zone from equation (D-20) is given as;

$$P_{eo} = P_{io} + \left(\frac{B_w}{NC_T}\right)W_{inj} - \left\{\frac{B_o + (R_p - R_s)B_g}{NC_T}\right\}N_p \qquad \text{(D-21)}$$

The total compressibility in the oil zone from Appendix B is reduced to;

$$C_T = \frac{(B_o - B_{oi}) + (R_{si} - R_s)B_g}{\Delta p} + \frac{mB_{oi}}{\Delta p}\left(\frac{B_g}{B_{gi}} - 1\right) \qquad \text{(D-22)}$$

Likewise, MBE in the aquifer zone at time, t, from equation (B-12) would give

$$W_{inj}B_w = N_w C_{Tw}(P_{iw} - P_{ew}) \qquad \text{(D-23)}$$

The boundary pressure in the aquifer zone from equation (D-23) is deduced as;

$$P_{ew} = P_{iw} - \left(\frac{B_w}{N_w C_{Tw}}\right)W_{inj} \qquad \text{(D-24)}$$

But for an infinite aquifer zone, $N_w = \infty$ therefore equation (D-24) reduces to;

$$P_{ew} = P_{iw} \qquad \text{(D-25)}$$

From equation (A-10) or (A-15), the rate of water injection into the oil zone in either downward or upward flow mechanisms is;

$$I_w\left[\frac{1}{I} + \frac{1}{J}\right] + TI_w^{1.79} = P_{ew} - P_{eo} \qquad \text{(D-26)}$$

Substituting equations (D-21) and (D-25) into equation (D-26) gives;

$$I_w\left[\frac{1}{I} + \frac{1}{J}\right] + TI_w^{1.79} = (P_{iw} - P_{io}) - \left(\frac{B_w}{NC_T}\right)W_{inj} + \left\{\frac{B_o + (R_p - R_s)B_g}{NC_T}\right\}N_p \qquad \text{(D-27)}$$

Let $X = \left[\frac{1}{I} + \frac{1}{J}\right]$

$$Y = B_w\left(\frac{1}{NC_T}\right)$$

$$Z = \left(\frac{B_o + \left(R_p - R_s\right)B_g}{NC_T}\right)$$

$$\Delta P_i = (P_{iw} - P_{io})$$

Equation (D-27) becomes;

$$XI_w + TI_w^{1.79} = \Delta P_i - YW_{inj} + ZN_p \quad \text{(D-28)}$$

Neglecting the frictional pressure drop, i.e. $TI_w^{1.79} = 0$ and differentiating equation (D-28) with respect to time, *t*;

$$X\frac{\partial I_w}{\partial t} + Y\frac{\partial W_{inj}}{\partial t} - Z\frac{\partial N_p}{\partial t} = 0 \quad \text{(D-29)}$$

But

$$\frac{\partial W_{inj}}{\partial t} = I_w \qquad \frac{\partial N_p}{\partial t} = q_o$$

Thus;

$$X\frac{\partial I_w}{\partial t} + YI_w - Zq_o = 0 \quad \text{(D-30)}$$

If the oil production rate q_o is a specified constant, then;

$$X\frac{dI_w}{dt} + YI_w = cons\tan t \quad \text{(D-31)}$$

$$\frac{dI_w}{dt} + \left(\frac{Y}{X}\right)I_w = \frac{Zq_o}{X} \quad \text{(D-32)}$$

The solution to this differential equation with respect to time *t*, is given as;

$$I_w(t) = Ke^{-\left(\frac{Y}{X}\right)t} + \frac{Zq_o}{Y} \quad \text{(D-33)}$$

Where K, is a constant. This constant is found by applying the initial condition.

At initial condition;

$$t = 0 \qquad I_w = I_{iw} \qquad P_{ew} = P_{iw} \qquad P_{eo} = P_{io}$$

Therefore,

$$I_{iw}\left[\frac{1}{I} + \frac{1}{J}\right] = P_{iw} - P_{io} - \Delta P_{fr}$$

$$XI_{iw} = \Delta P_i - \Delta P_{fr}$$

Or $$I_{iw}=\frac{\Delta P_i-\Delta P_{fr}}{X}$$, is the initial rate of water injection (D-34)

Hence

$$\frac{\Delta P_i-\Delta P_{fr}}{X}=\frac{Zq_o}{Y}+K$$

$$K=\frac{\Delta P_i-\Delta P_{fr}}{X}-\frac{Zq_o}{Y} \qquad \text{(D-35)}$$

Substituting equation (D-35) into equation (D-33) gives;

$$I_w(t)=\left[\frac{\Delta P_i-\Delta P_{fr}}{X}-\frac{Zq_o}{Y}\right]e^{-\left(\frac{Y}{X}\right)t}+\frac{Zq_o}{Y}$$

Alternatively;

$$I_w(t)=\left\{\frac{\Delta P_i-\Delta P_{fr}}{X}\right\}e^{-\left(\frac{Y}{X}\right)t}+\frac{Zq_o}{Y}\left\{1-e^{-\left(\frac{Y}{X}\right)t}\right\} \qquad \text{(D-36)}$$

Or

$$I_w(t)=I_{iw}e^{-\left(\frac{Y}{X}\right)t}+\frac{Zq_o}{Y}\left\{1-e^{-\left(\frac{Y}{X}\right)t}\right\} \qquad \text{(D-37)}$$

Where initial rate of injection is in the form;

$$I_{iw}=\left\{\frac{\Delta P_i-TI_{iw}^{\ 1.79}}{\left[\frac{1}{I}+\frac{1}{J}\right]}\right\} \qquad \text{(D-38)}$$

Equations (D-37) and (D-38) represent the final and initial rate of water injection at any time *t*, for undersaturated oil reservoir. These nonlinear equations can be solved by an iterative technique.

APPENDIX E

ITERATION METHODS

Newton's Method for Solving Equations $f(x)=0$

This method is also known as Newton-Raphson and commonly used because of its simplicity and great speed. The function *f* is assumed to have a continuous derivative *f'*. The method is based on the principle that if the initial guess of the root of $f(x)=0$ is x_i, then if one draws a tangent to the curve at $f(x_i)$, the point x_{i+1} where the tangent crosses the x-axis is an improved estimate of the root.

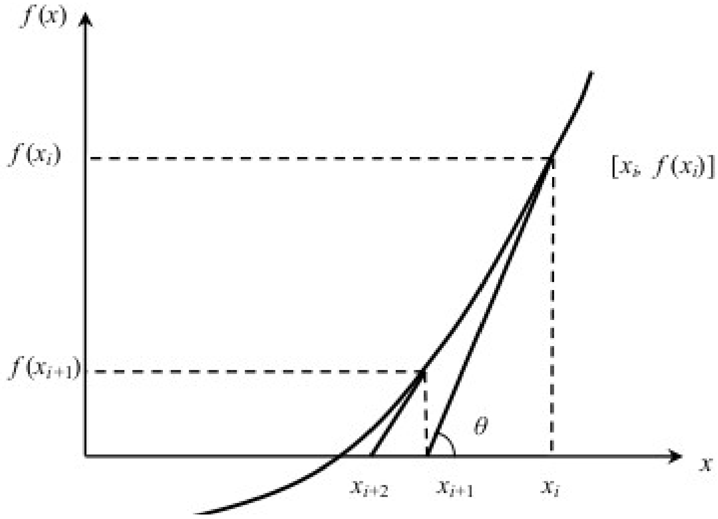

Figure E-1 Geometrical Illustration of the Newton-Raphson method

Using the definition of the slope of a function, at $x = x_i$

$$f'(x_1) = \tan\theta$$

$$= \frac{f(x_1)-0}{x_i - x_{i+1}}$$

This method of solving nonlinear equations is given by the iterative formula

$$x_{(i+1)} = x_i - \frac{f(x_i)}{f'(x_i)},\ i = 0, 1, 2, \ldots \qquad \text{(E-1)}$$

This equation requires two function evaluations per iteration, that of $f(x_i)$ and $f'(x_i)$ and with only one initial guess of root x_i, the other guess, x_{i+1} can be found and process repeated until the roots are within a desirable tolerance.

For the convergence of Newton-Raphson method,

$$g_x = x - \frac{f(x)}{f'(x)},$$ by differentiation,

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} \qquad \text{(E-2)}$$

$$= \frac{f(x)f''(x)}{f'(x)^2}$$

Since $f(s) = 0$, this shows that $g'(x) = 0$. Hence Newton-Raphson method is at least of second order.

Convergence is not guaranteed but if the method does converge, it does so much faster.

Algorithms

The steps of the Newton-Raphson method to find the root of an equation $f(x) = 0$ are

1. Evaluate $f'(x)$ symbolically
2. Use an initial guess of the root x_i, to estimate the new value of the root x_{i+1},

   as $x_{(i+1)} = x_i - \dfrac{f(x_i)}{f'(x_i)}$
3. Find the absolute relative approximate error $|\in_a|$ as

   $$|\in_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$
4. Compare the absolute relative approximate error with the pre-specified relative error tolerance $\in_s$. If $|\in_a| > \in_s$, then go to step 2, else stop the algorithm. Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

Drawbacks of Newton-Raphson Method

1. The Newton-Raphson method only works if you have a functional representation of $f'(x)$. Some functions may be difficult to impossible to differentiate.
2. There is divergence at inflection point if the selection of the initial guess or an iterated value of the root turns out to be close to the inflection point of the

function $f(x)$ in the equation $f(x)=0$. However, Newton-Raphson method is not guaranteed to find a root.

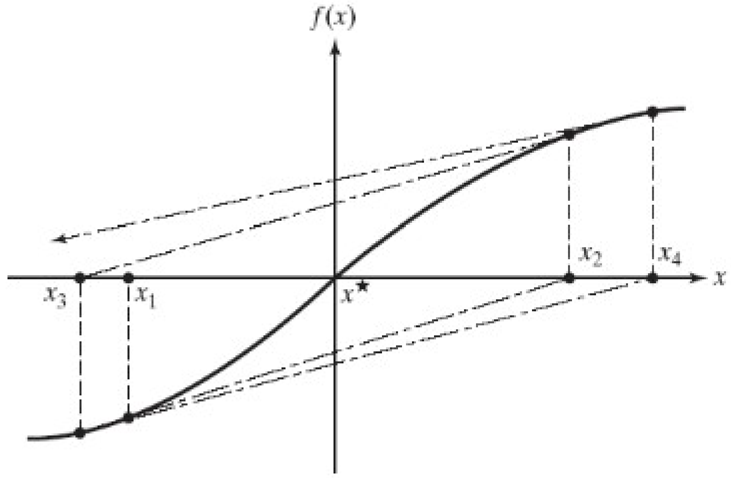

Figure E-2 Divergences at Inflection Point

3. If the derivative of a function at any tested point x_i is sufficiently close to zero, the next point x_{i+1} will be far away. There will be much delay in finding the roots.

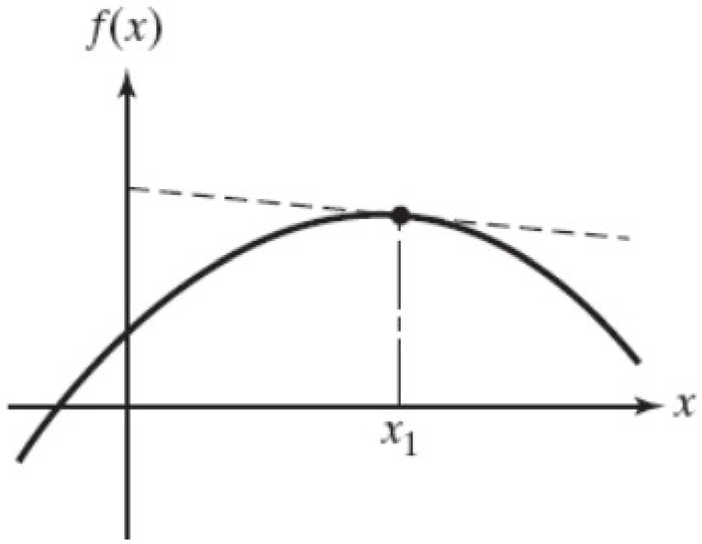

Figure E-3 Pitfall of Division by Zero or a Near Zero Number

4. If the derivative of the function changes sign near a tested point, the Newton-Raphson method may oscillate around a point nowhere near the nearest root but converge on the local maximum or minimum. Eventually, it may lead division by a number close to zero and may diverge.

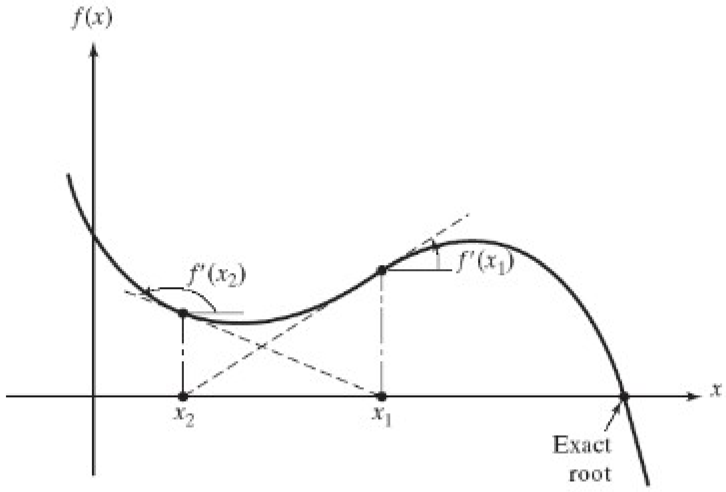

Figure E-4 Oscillations around Local Maxima

Secant Method for Solving Equations $f(x)=0$

This method is considered an approximation of Newton-Raphson method and suggests the idea of replacing the derivative with the difference quotient

$$f'(x_i) \approx \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} \qquad \text{(E-3))}$$

Hence the formula of the secant method is given by

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i)-f(x_{i-1})} \qquad \text{(E-4)}$$

Where i = 1, 2, 3… requires one function evaluation per iteration, following the initial step.

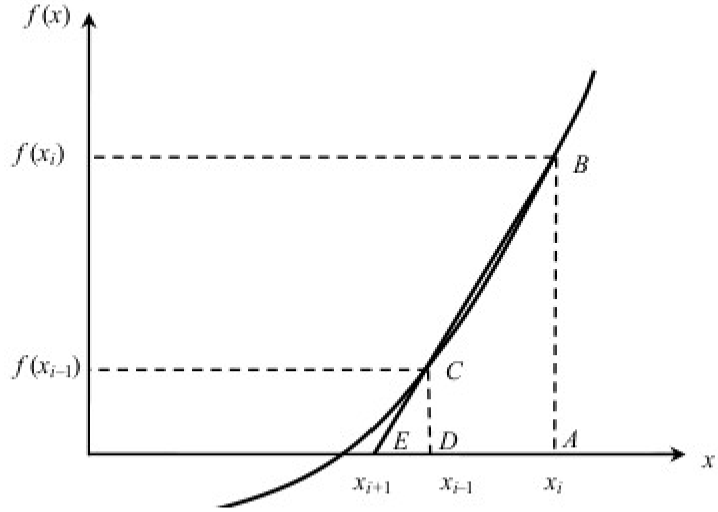

Figure E-5 Geometrical Representation of the Secant Method

The convergence is superlinear, that is, more rapid than linear and almost quadratic like Newton-Raphson Method.

It is not proper to combine the secant formula and write it in the form;

$$x_{i+1} = \frac{f(x_i)x_{i-1} - f(x_{i-1})x_i}{f(x_i) - f(x_{i-1})}$$, as this may lead to loss of significant digits if x_i and x_{i-1} are about equal.

Costs of Secant and Newton-Raphson Methods

1. Secant method requires one function evaluation per iteration, following an initial guess, as compared with Newton-Raphson method requires two function evaluations per iteration, that of $f(x_i)$ and $f'(x_i)$.
2. The secant method does not require the use of the derivative of the function, whereas, the Newton-Raphson method makes use of the derivative.
3. The secant method is often faster in time, even though more iterates are needed with it than Newton-Raphson method to attain a similar accuracy.

Bisection Method for Solving Equations $f(x)=0$

This is one of the first numerical methods developed to find the root of a nonlinear equation $f(x)=0$. The theorem $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_l and x_u

a. If $f(x_l)f(x_u)<0$

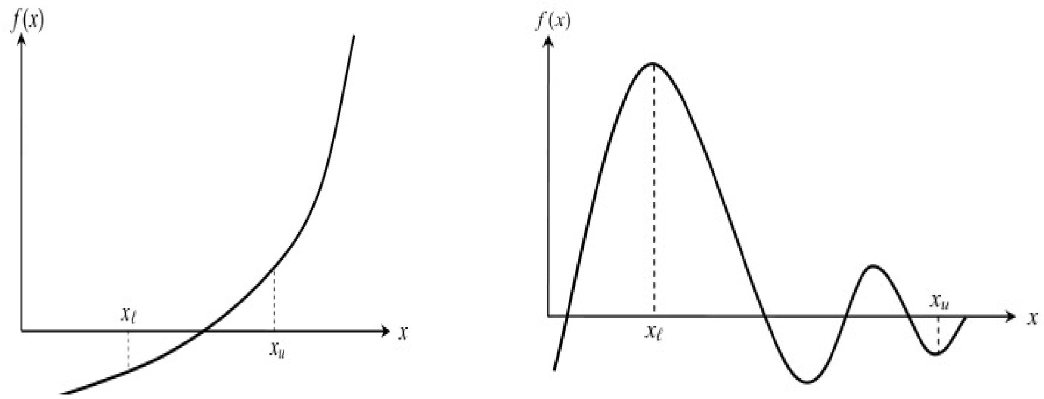

Figure E-6 At Least one Root exists between the Two Points if the function is real, Continuous, and changes sign.

b. If $f(x_l)f(x_u) > 0$, there may or may not be any root between x_l and x_u.

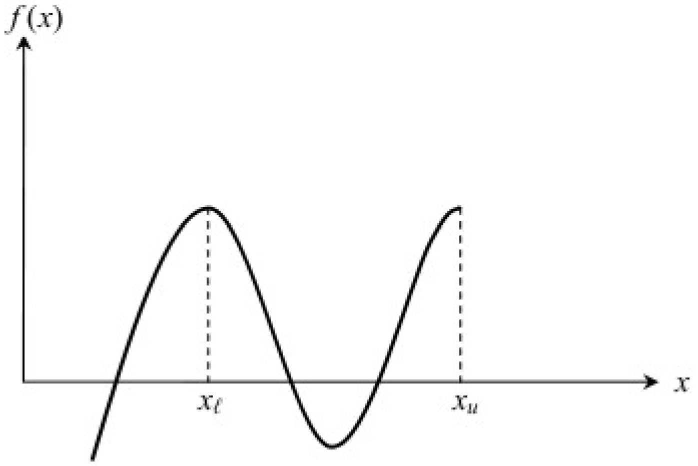

Figure E-7 Roots of the Equation $f(x) = 0$ may exist between the two points if function $f(x)$ does not change sign between two points

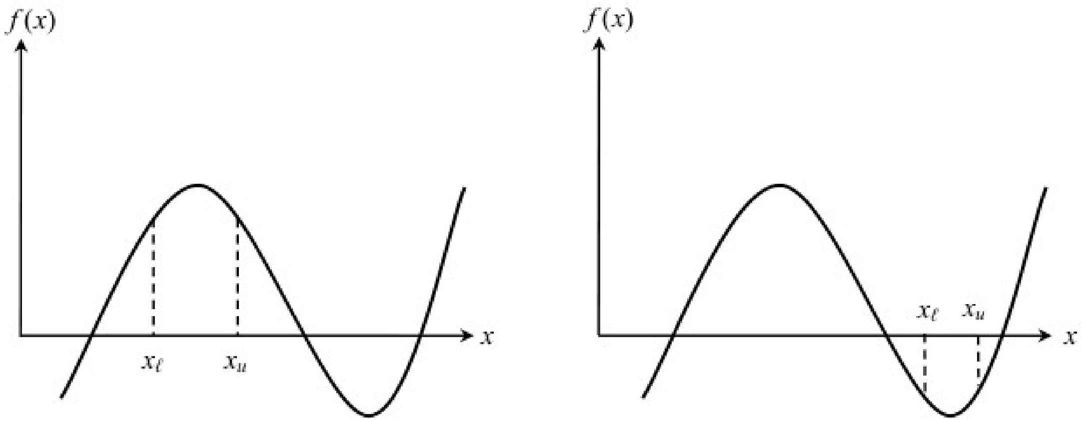

Figure E-8 Roots of the Equation $f(x) = 0$ may not exist between the two points if function $f(x)$ does not change sign between two points

a. Algorithm for the Bisection Method

The steps to apply the bisection method to find the roots of the equation $f(x) = 0$ are

1. Choose x_l and x_u as two guesses for the root such that $f(x_l)f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_l and x_u.

2. Estimate the root, x_m, of the equation $f(x)=0$ as the mid-point between x_l and x_u as; $x_m = \frac{x_l + x_u}{2}$
3. Check the following
   a. If $f(x_l)f(x_u)<0$, then the root lie between x_l and x_m; then x_l = x_l and $x_u = x_m$.
   b. If $f(x_l)f(x_u)>0$, then the root lie between x_m and x_u; then x_l = x_m and $x_u = x_u$.
   c. If $f(x_l)f(x_u)=0$, then the root is x_m. Stop the algorithm if this is true.
4. Find the new estimate of the root $x_m = \frac{x_l + x_u}{2}$ and also the absolute relative approximate error as $|\in_a| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$

   Where

   x_m^{new} = estimated root from present iteration

   x_m^{old} = estimated root from previous iteration
5. Compare the absolute relative approximate error $|\in_a|$ with the pre-specified relative error tolerance $\in_s$. If $|\in_a| > \in_s$, then go to step 3, else stop the algorithm.

b. Advantages of Bisection Method

a. The bisection method is always convergent. Since the method brackets the root, the method is guaranteed to converge.
b. As iterations are conducted, the interval gets halved; hence one can guarantee the error in the solution of the equation.

c. Drawbacks of Bisection Method

a. The convergence of the bisection method is slow as it is simply based on halving the interval.

b. If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root.
c. The bisection method only finds roots where the function crosses the x-axis but cannot find roots where the function is tangent to the x-axis.
d. The bisection method cannot find complex roots of polynomials.
e. For functions $f(x)$ where there is a singularity (a point where the function becomes infinite) and it reverses sign at the singularity, the bisection method may converge on the singularity. However, the function is not continuous and the theorem that a root exists is also not applicable.

In general,

The root of the function is estimated as;

$$I_w(m) = \frac{I_w(L) + I_w(u)}{2} \tag{E-5}$$

The absolute error is estimated as;

$$\left| I_w(m)^{new} - I_w(m)^{old} \right| \tag{E-6}$$

Similarly, the relative percentage error is given as;

$$\left| \frac{I_w(m)^{new} - I_w(m)^{old}}{I_w(m)^{new}} \right| \times 100 \tag{E-7}$$

After n steps of iterations, the approximate root is computed with an absolute error of at most;

$$\frac{\left[I_w(u) - I_w(L) \right]}{2^{n+1}} \tag{E-8}$$

The number of iterations, n, of the bisection method needed to determine the root within an error of at most 5×10^{-8} is given by;

$$\frac{I_w(u) - I_w(L)}{2^{n+1}} \leq 5 \times 10^{-8} \tag{E-9}$$

$$n \geq \left\{ \frac{\log\left[\frac{I_w(u) - I_w(L)}{5} \right] + 8}{\log(2)} \right\} - 1 \tag{E-10}$$

APPENDIX F

F.1 Case 1: Finite Aquifer Injecting into Finite Reservoir

a) Undersaturated Oil Reservoir

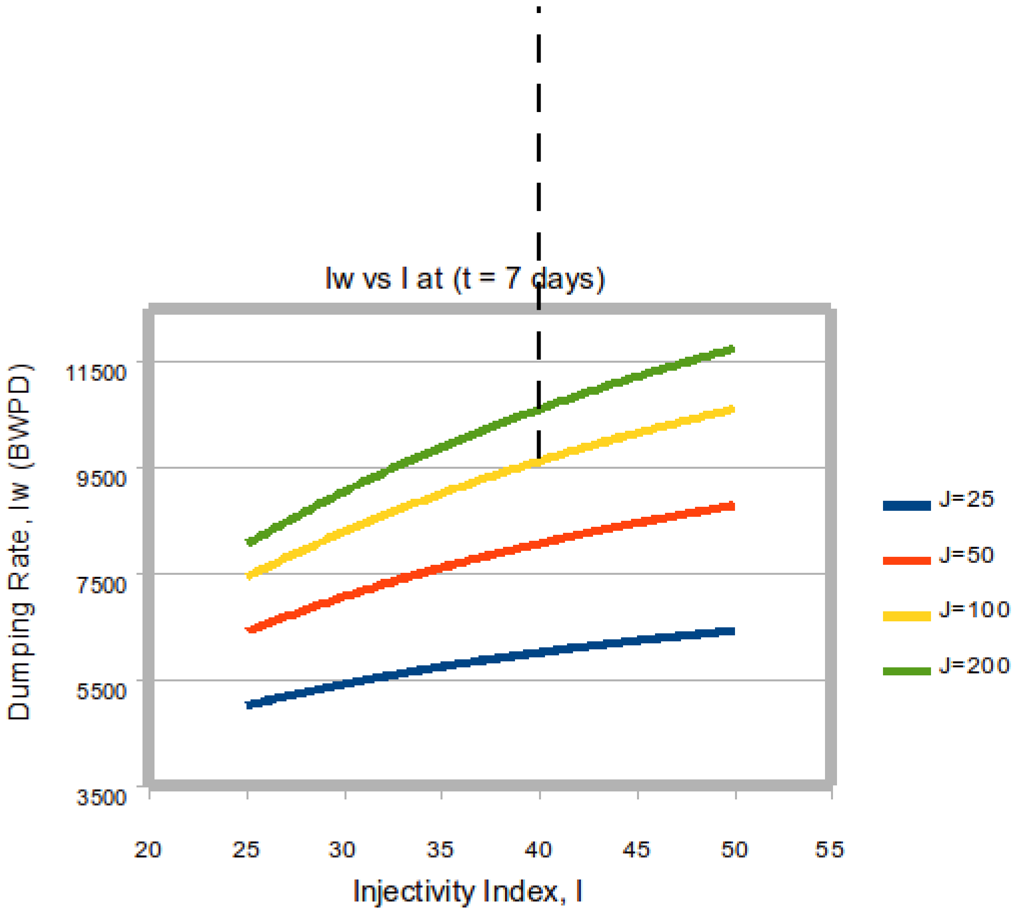

Figure F-1 Effects of J & I on I_w in Undersaturated Oil Reservoir (t = 7 days)

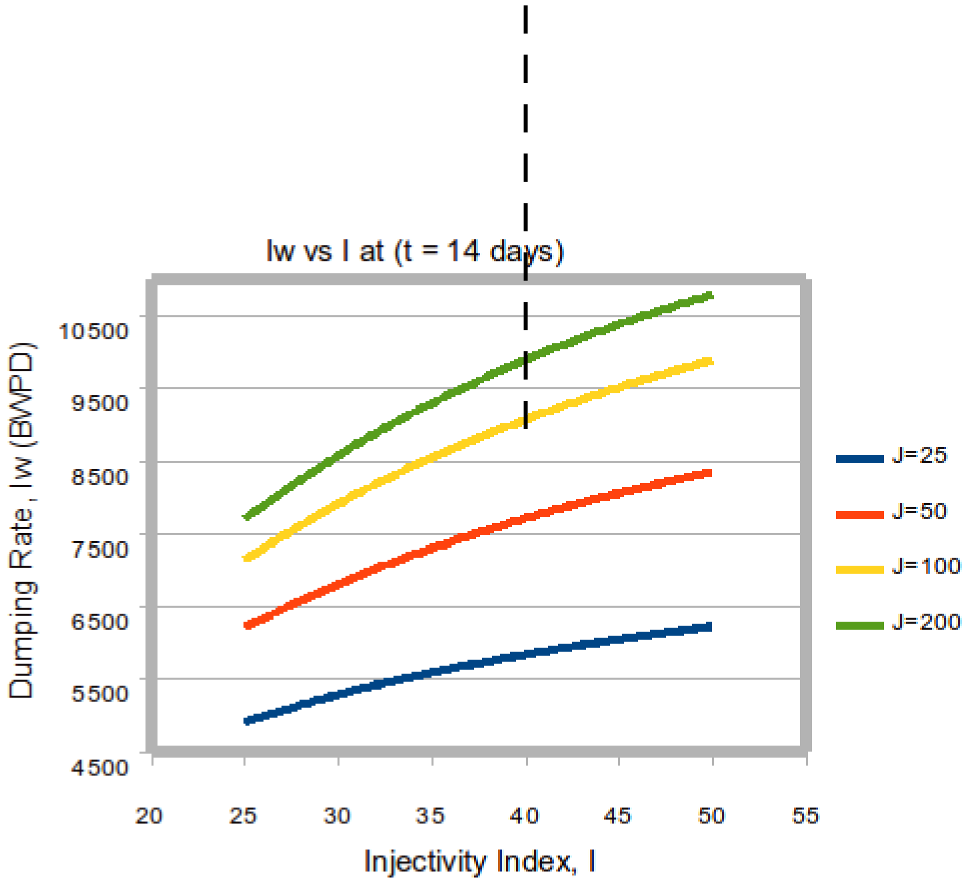

Figure F-2 Effects of *J* & *I* on I_w in Undersaturated Oil Reservoir (t = 14 days)

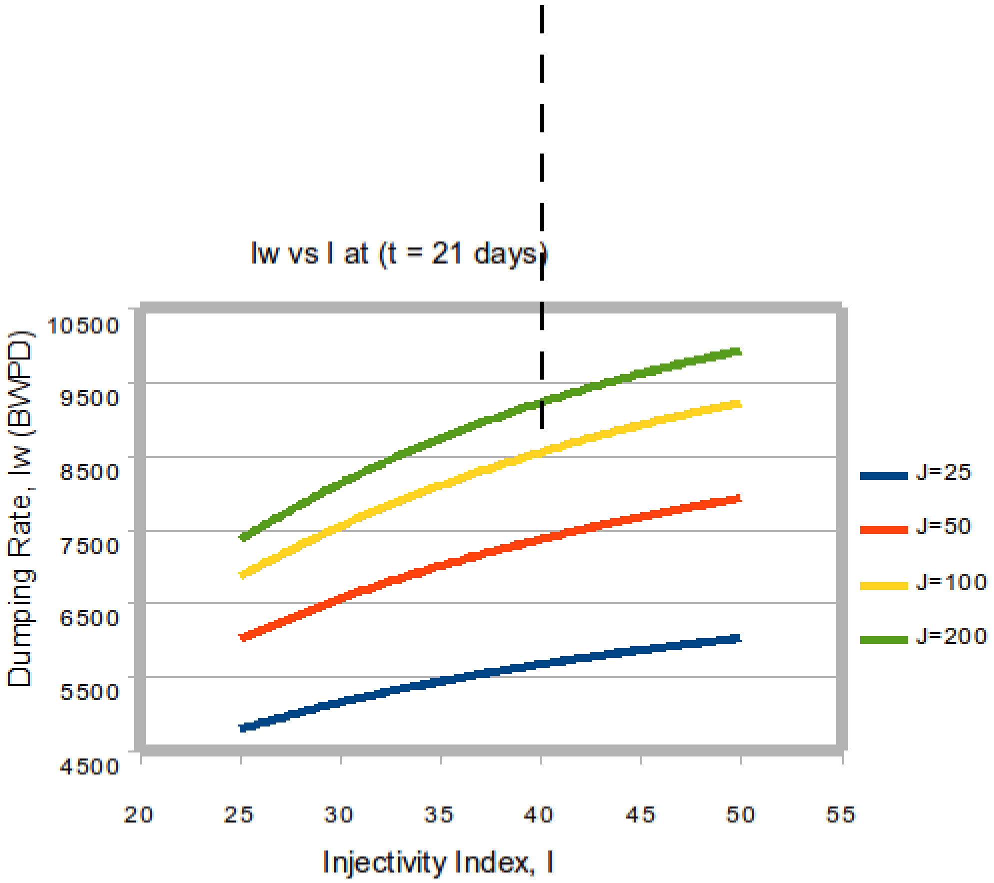

Figure F-3 Effects of *J* & *I* on I_w in Undersaturated Oil Reservoir (t = 21 days)

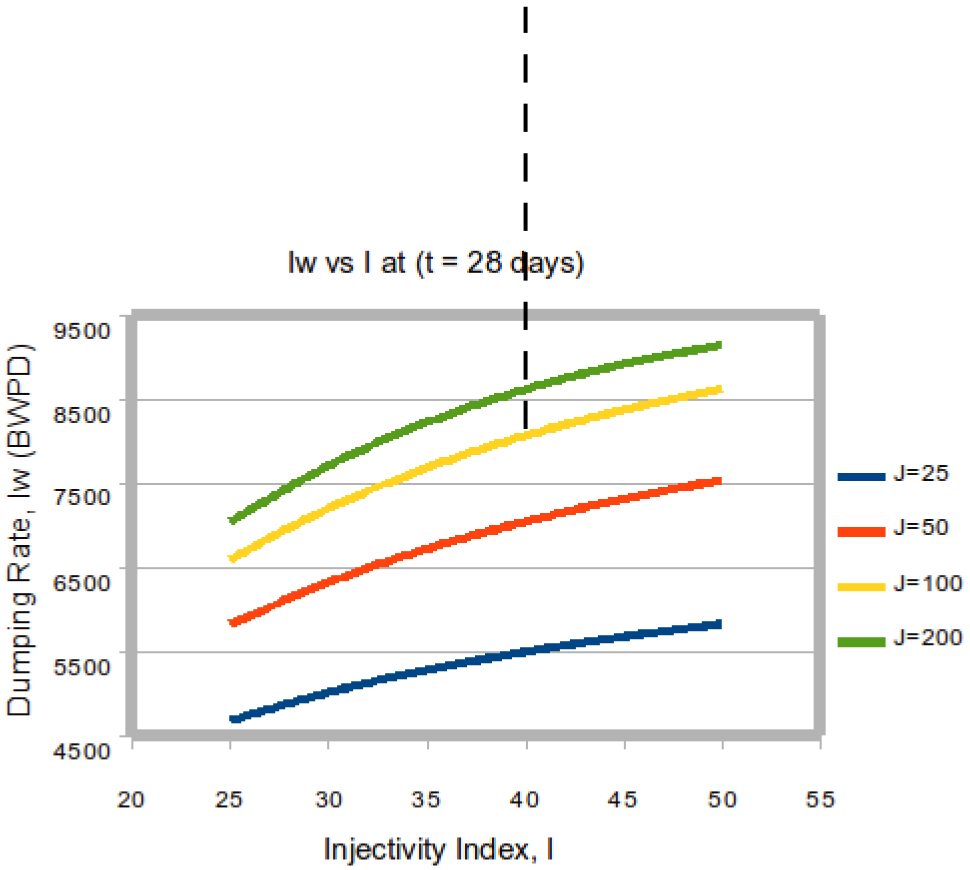

Figure F-4 Effects of *J* & *I* on I_w in Undersaturated Oil Reservoir (t = 28 days)

F.2 Case 1: Finite Aquifer Injecting into Finite Reservoir

b) Oil Reservoir with Gas Cap

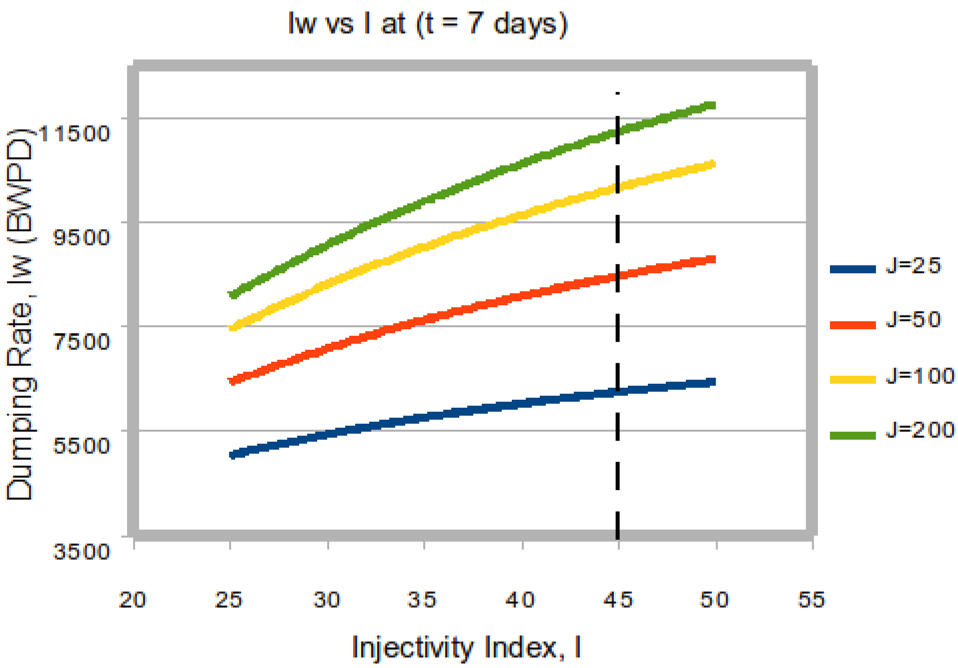

Figure F-5 Effects of J & I on I_w in Oil Reservoir with Gas Cap (t = 7 days)

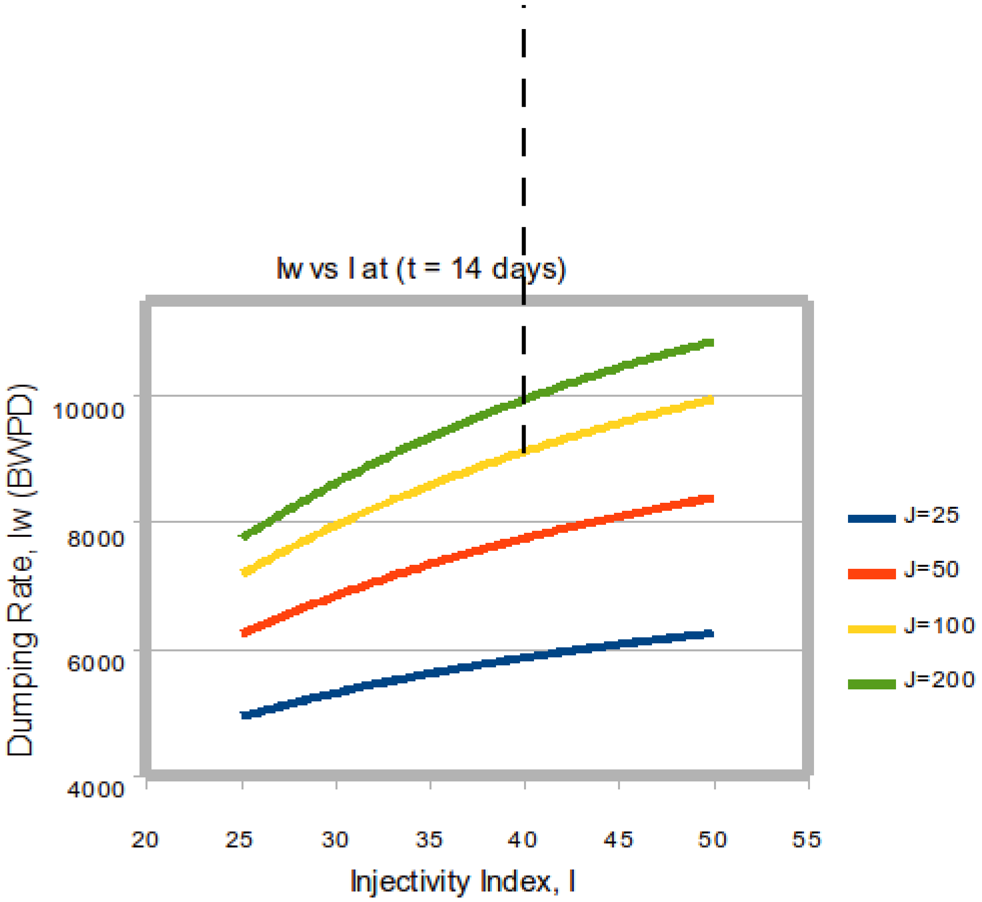

Figure F-6 Effects of *J* & *I* on I_w in Oil Reservoir with Gas Cap (t = 14 days)

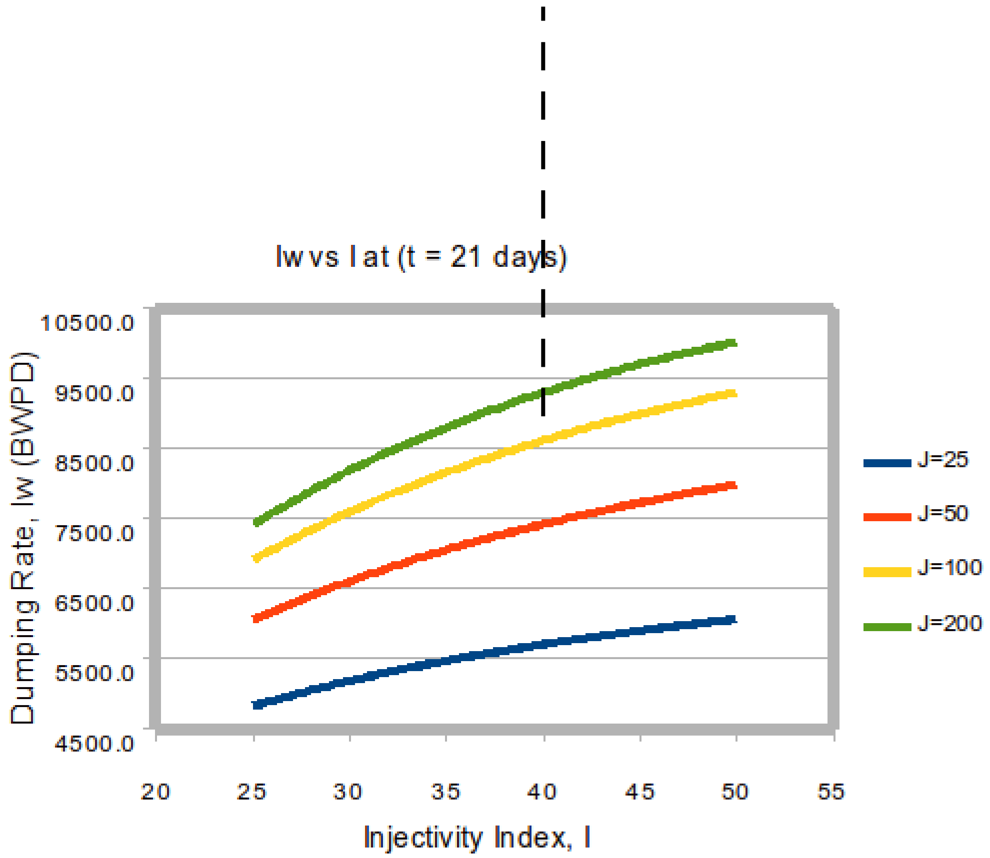

Figure F-7 Effects of J & I on I_w in Oil Reservoir with Gas Cap (t = 21 days)

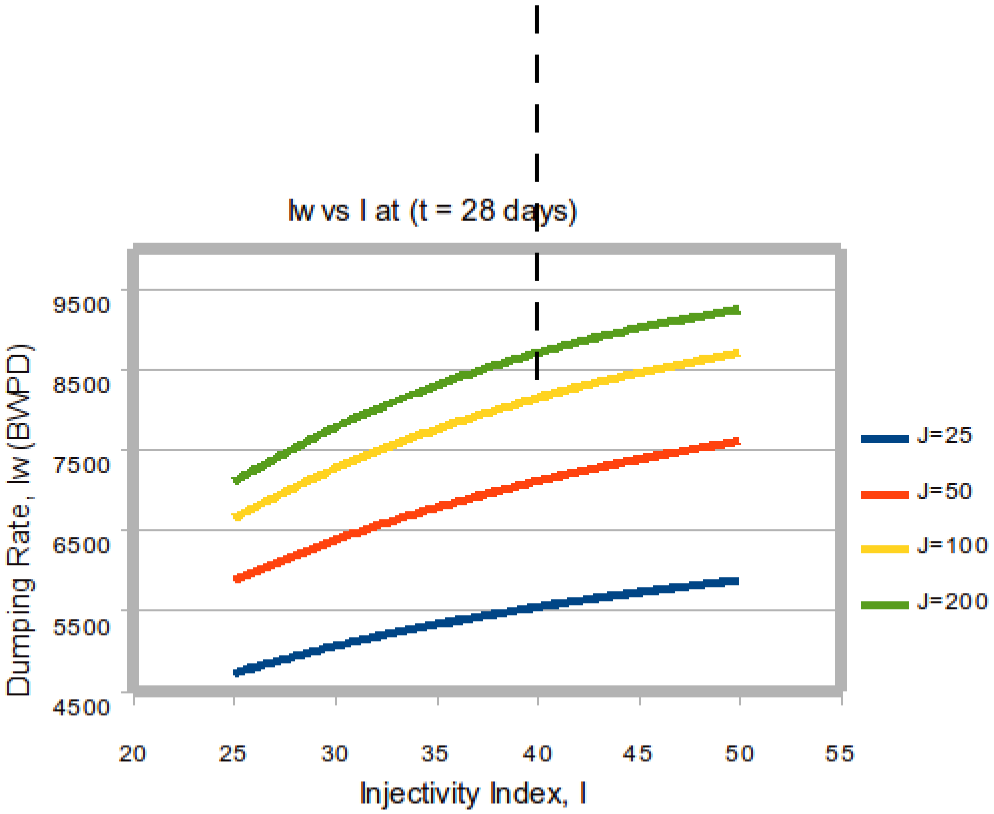

Figure F-8 Effects of J & I on I_w in Oil Reservoir with Gas Cap (t = 28 days)

APPENDIX G

Case 1: Comparing Undersaturated Oil Reservoir to Oil Reservoir with Gas Cap

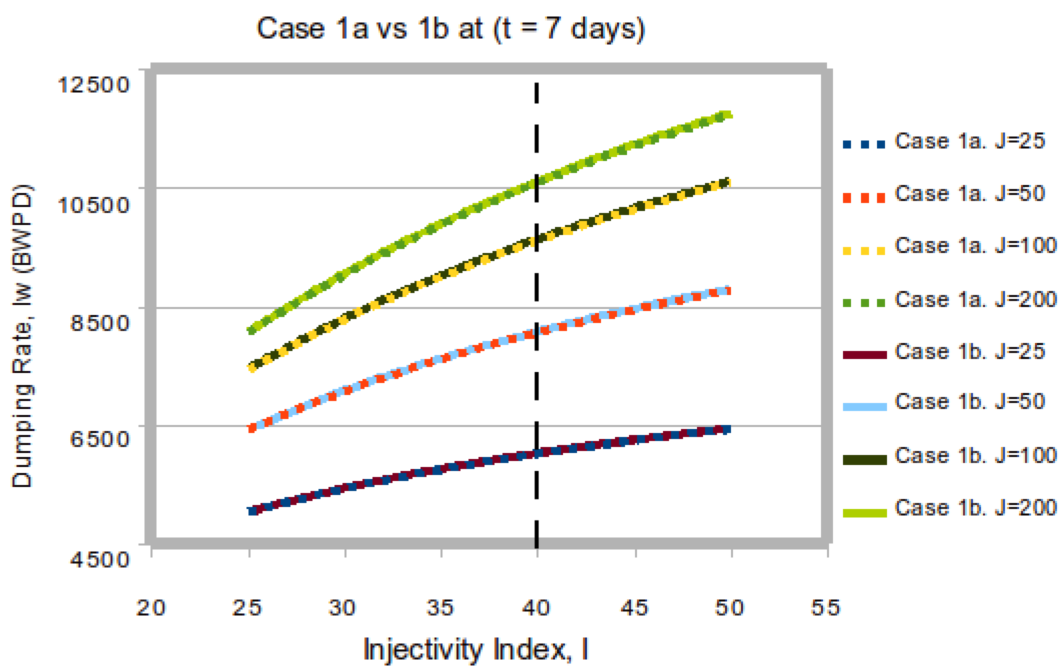

Figure G-1 Dumping Rate Comparison: Undersaturated Oil Reservoir and Oil Reservoir with Gas Cap (t = 7 days)

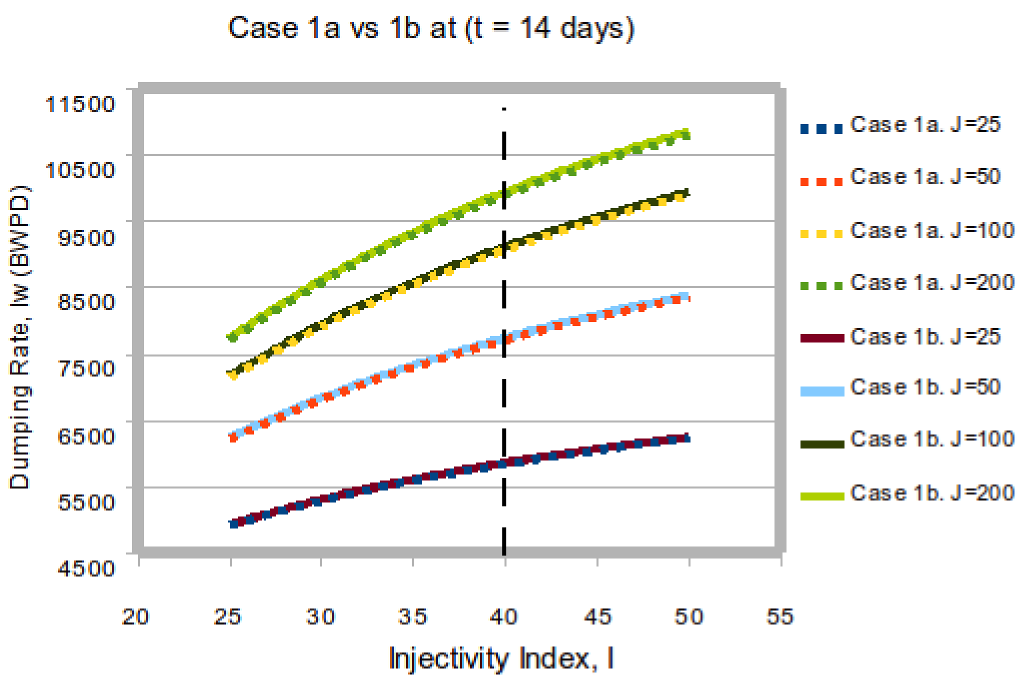

Figure G-2 Dumping Rate Comparison: Undersaturated Oil Reservoir and Oil Reservoir with Gas Cap (t = 14 days)

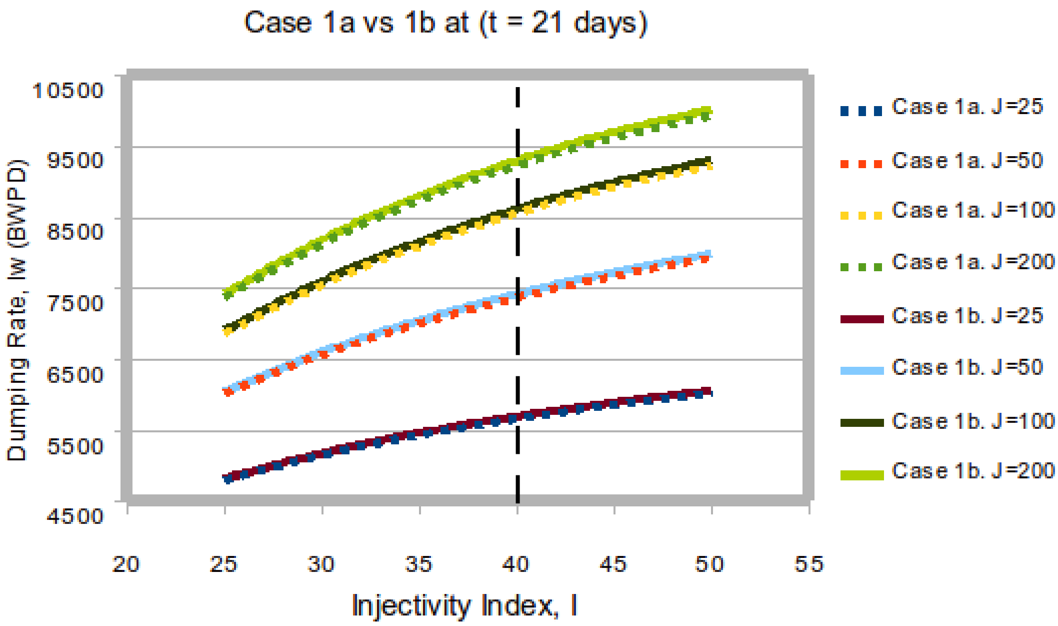

Figure G-3 Dumping Rate Comparison: Undersaturated Oil Reservoir and Oil Reservoir with Gas Cap (t = 21 days)

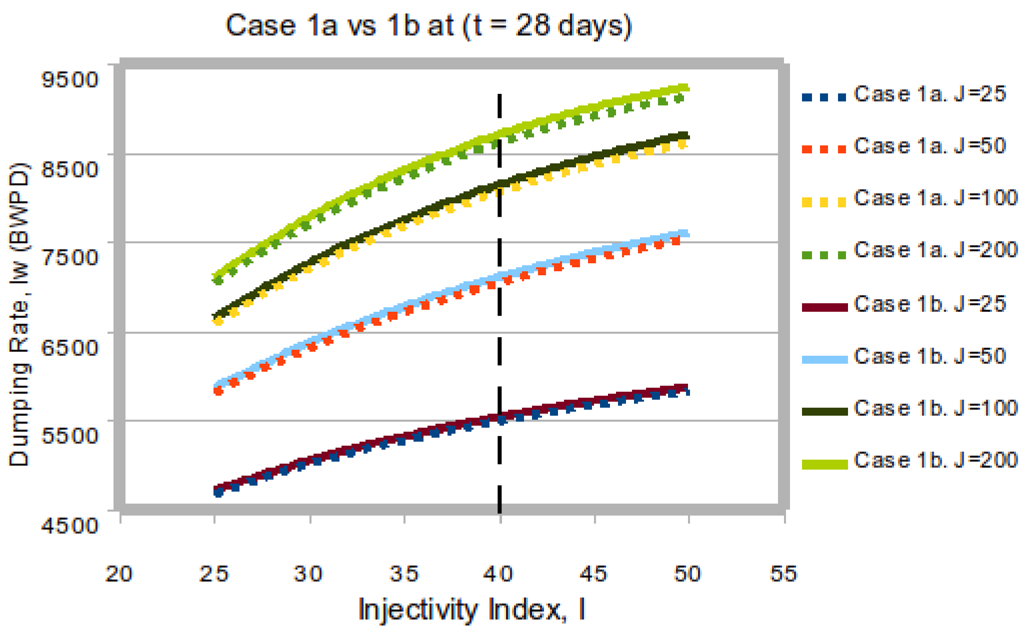

Figure G-4 Dumping Rate Comparison: Undersaturated Oil Reservoir and Oil Reservoir with Gas Cap (t = 28 days)

Appendix H

H.1 Case 2: Infinite Aquifer Injecting into Finite Reservoir

a) Undersaturated Oil Reservoir

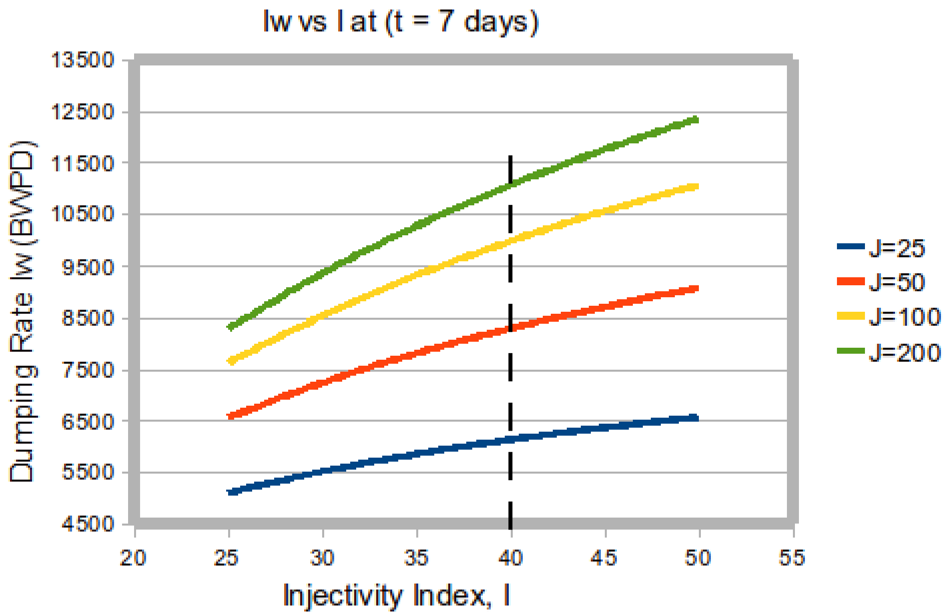

Figure H-1 Effects of J & I on I_w in Undersaturated Oil Reservoir (t = 7 days)

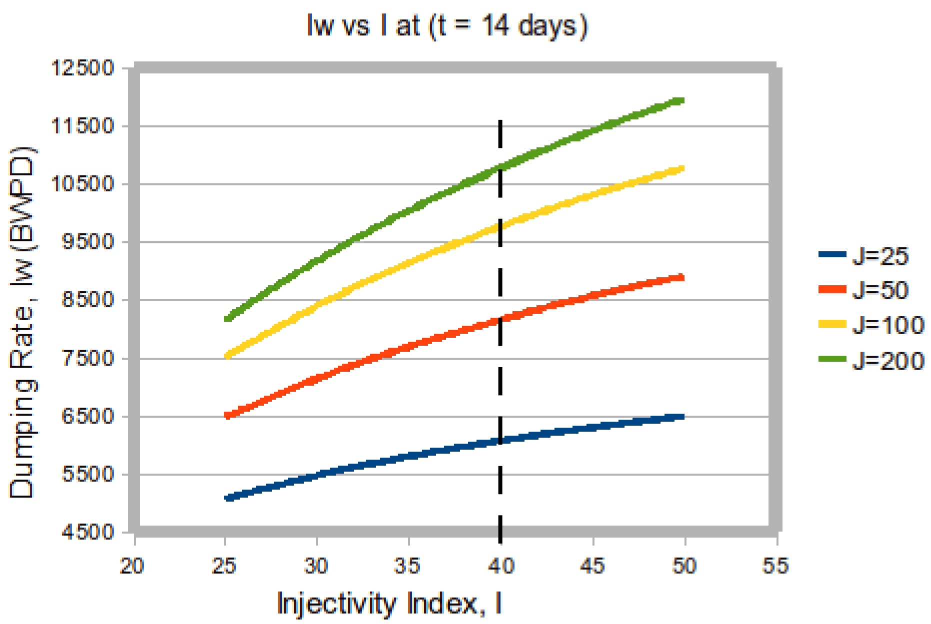

Figure H-2 Effects of J & I on I_w in Undersaturated Oil Reservoir (t = 14 days)

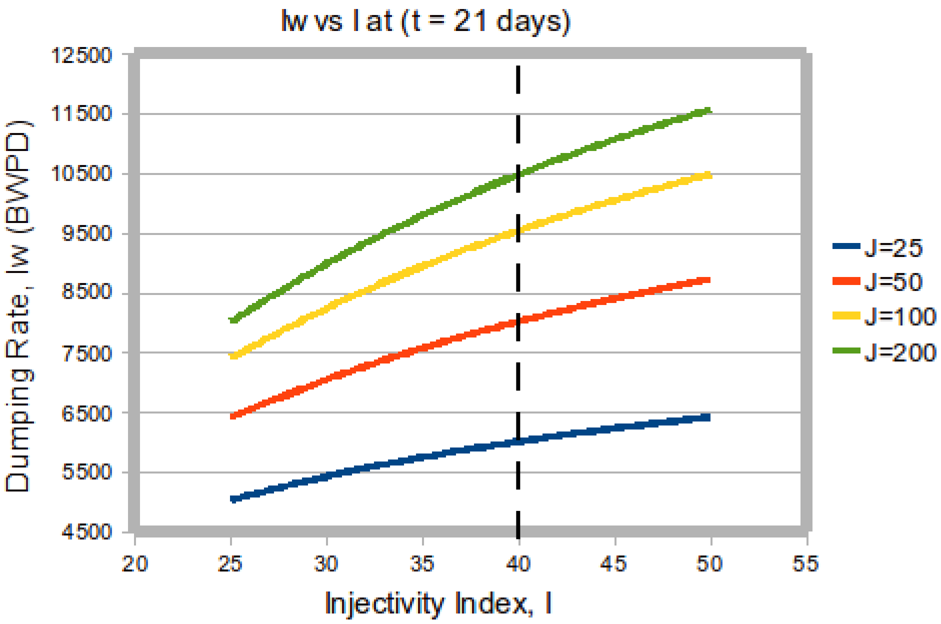

Figure H-3 Effects of J & I on I_w in Undersaturated Oil Reservoir (t = 21 days)

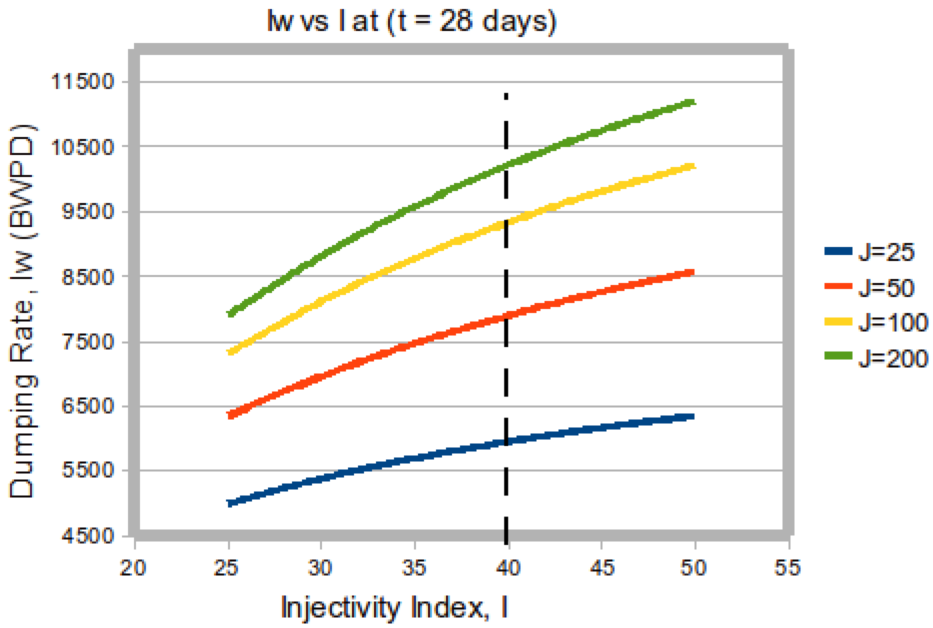

Figure H-4 Effects of J & I on I_w in Undersaturated Oil Reservoir (t = 28 days)

H.2 Case 2: Infinite Aquifer Injecting into Finite Reservoir

b) Oil Reservoir with Gas Cap

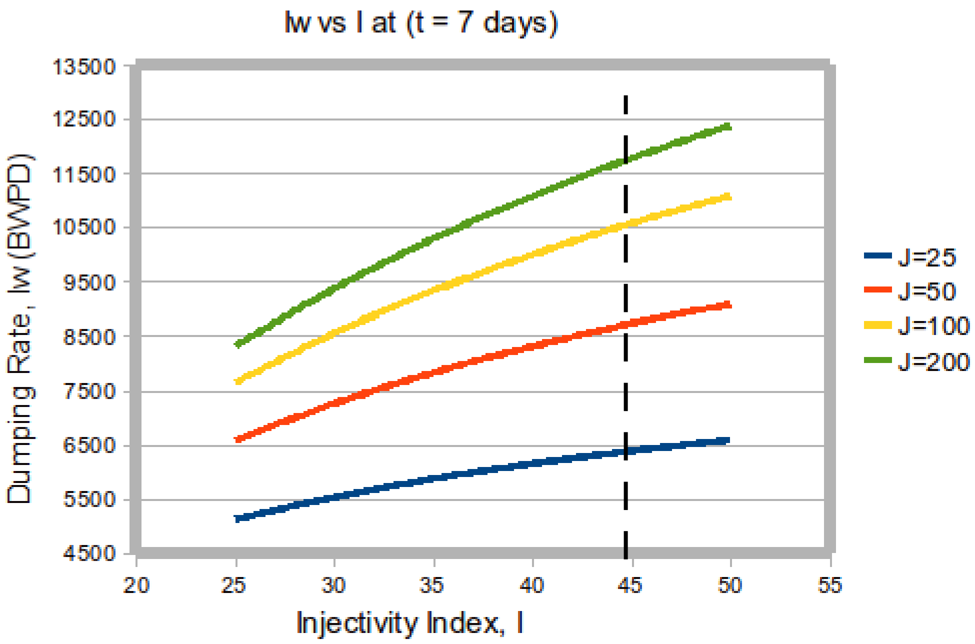

Figure H-5 Effects of J & I on I_w in Oil Reservoir with Gas Cap (t = 7 days)

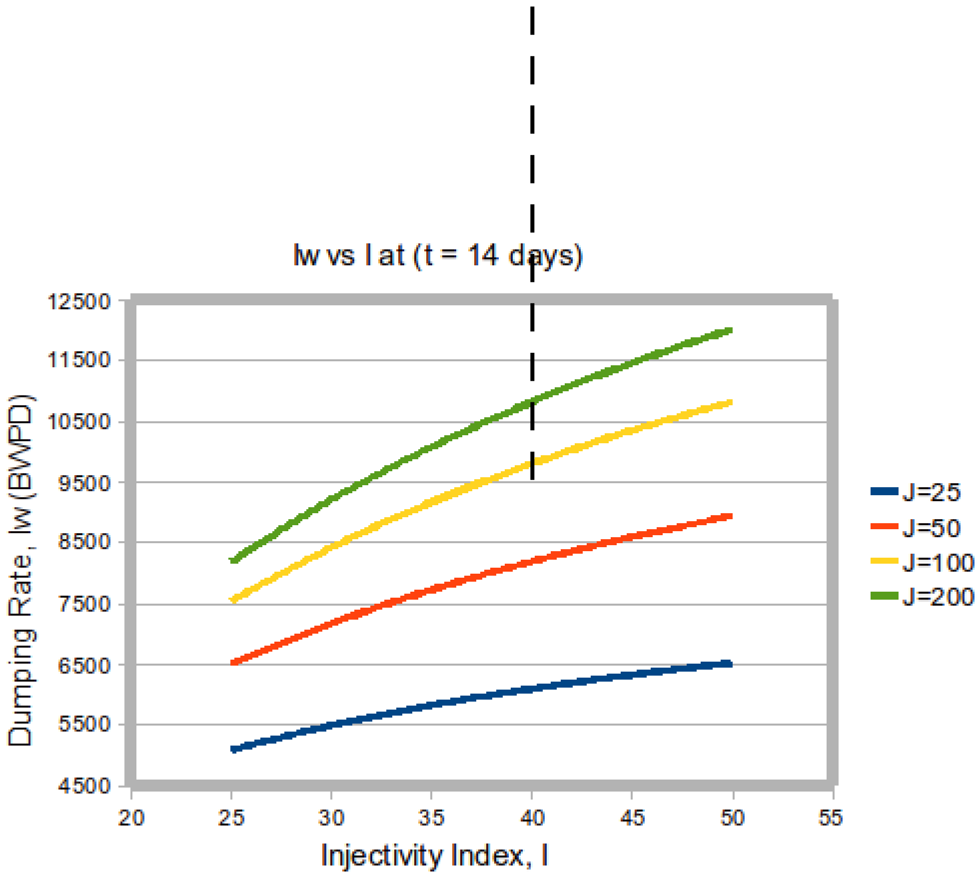

Figure H-6 Effects of J & I on I_w in Oil Reservoir with Gas Cap (t = 14 days)

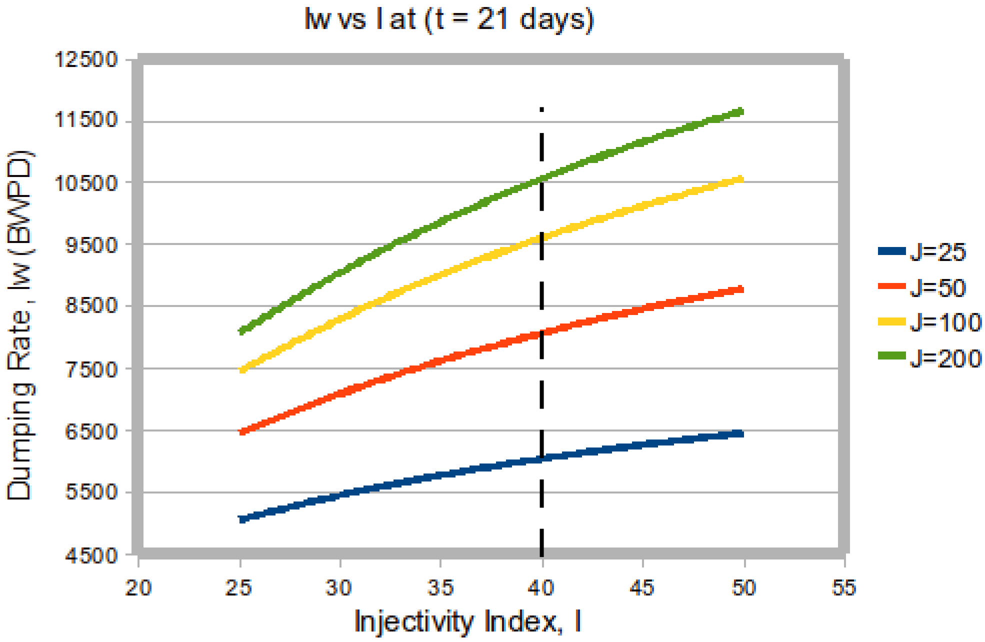

Figure H-7 Effects of J & I on I_w in Oil Reservoir with Gas Cap (t = 21 days)

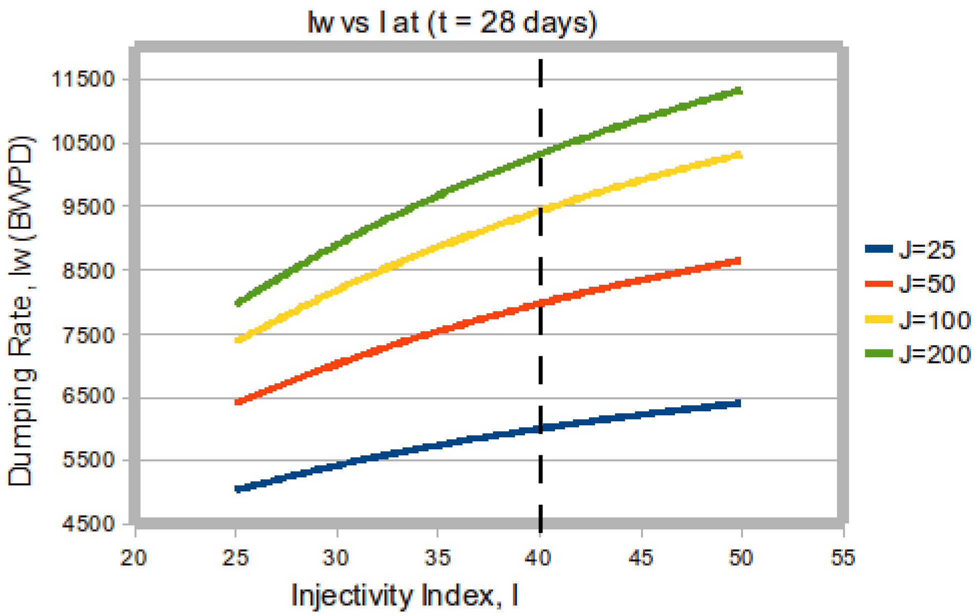

Figure H-8 Effects of J & I on I_w in Oil Reservoir with Gas Cap (t = 28 days)

APPENDIX I

Case 2: Comparing Undersaturated Oil Reservoir to Oil Reservoir with Gas Cap

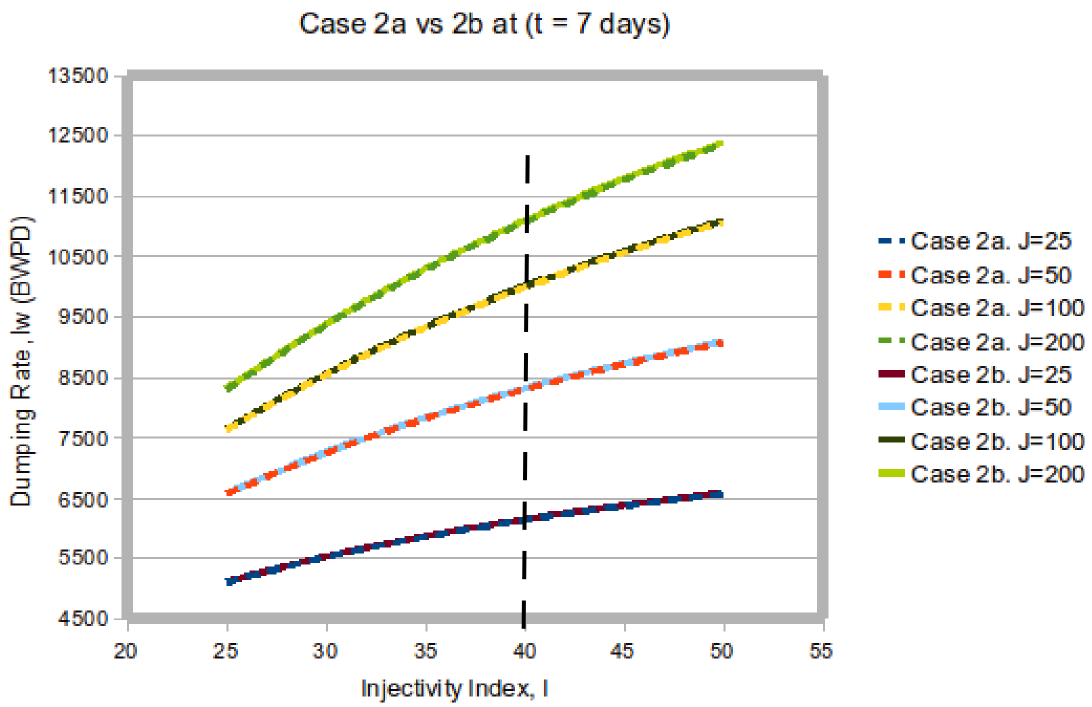

Figure I-1 Dumping Rate Comparison: Undersaturated Oil Reservoir and Oil Reservoir with Gas Cap (t = 7 days)

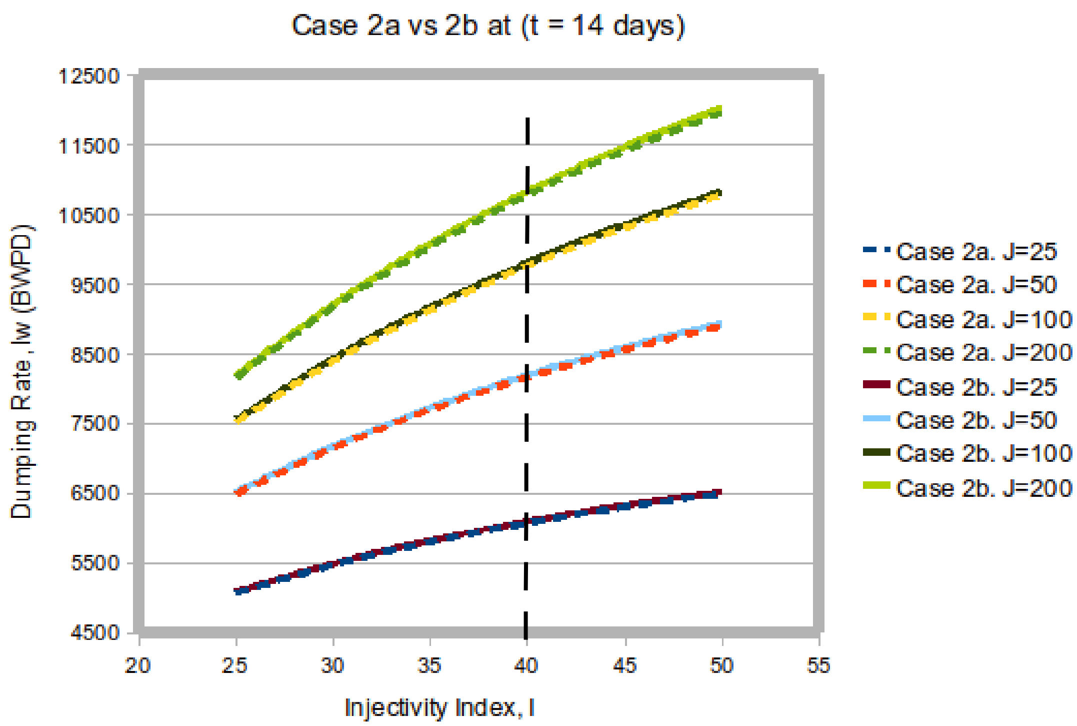

Figure I-2 Dumping Rate Comparison: Undersaturated Oil Reservoir and Oil Reservoir with Gas Cap (t = 14 days)

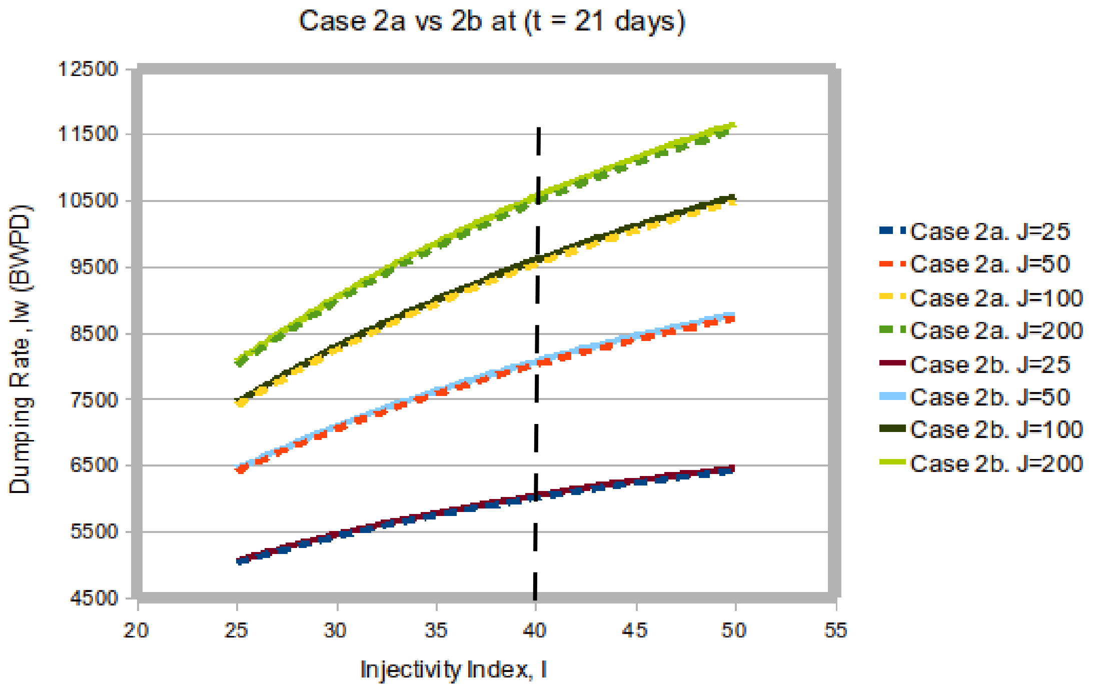

Figure I-3 Dumping Rate Comparison: Undersaturated Oil Reservoir and Oil Reservoir with Gas Cap (t = 21 days)

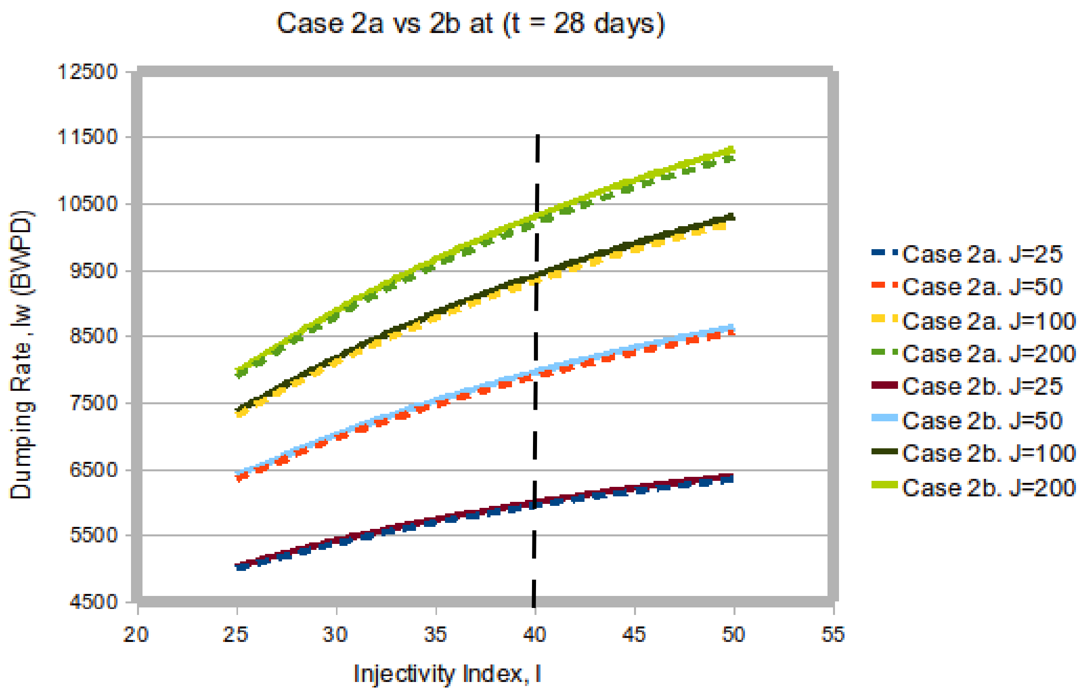

Figure I-4 Dumping Rate Comparison: Undersaturated Oil Reservoir and Oil Reservoir with Gas Cap (t = 28 days)

APPENDIX J

Case 1a & 2a: Comparing Undersaturated Oil Reservoirs

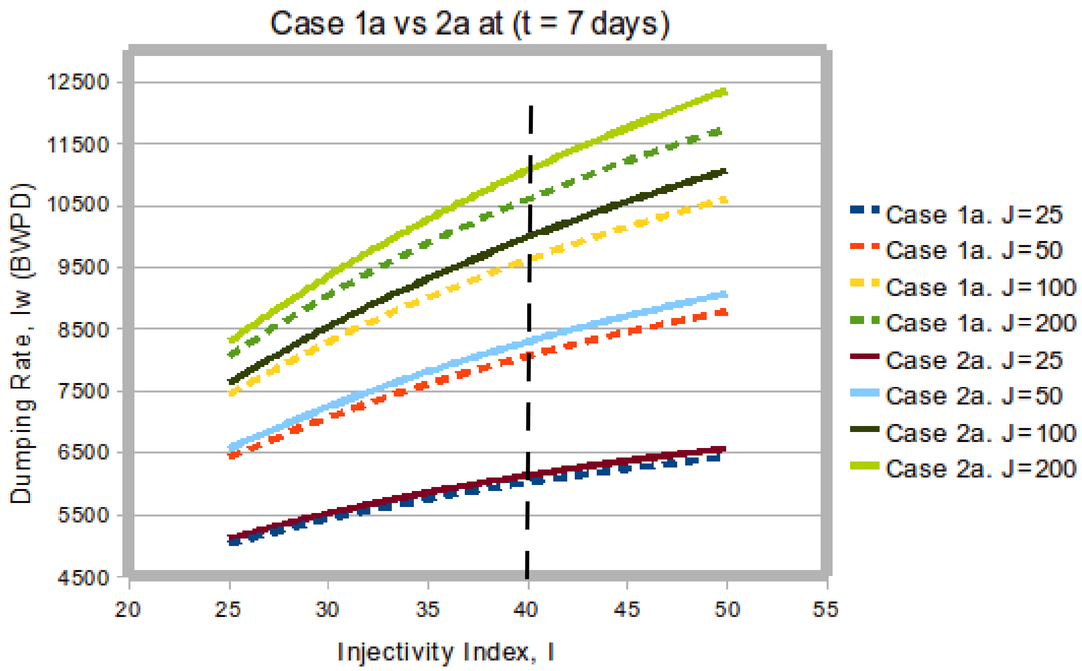

Figure J-1 Dumping Rate Comparison: Undersaturated Oil Reservoirs for Case 1a & 2a (t = 7 days)

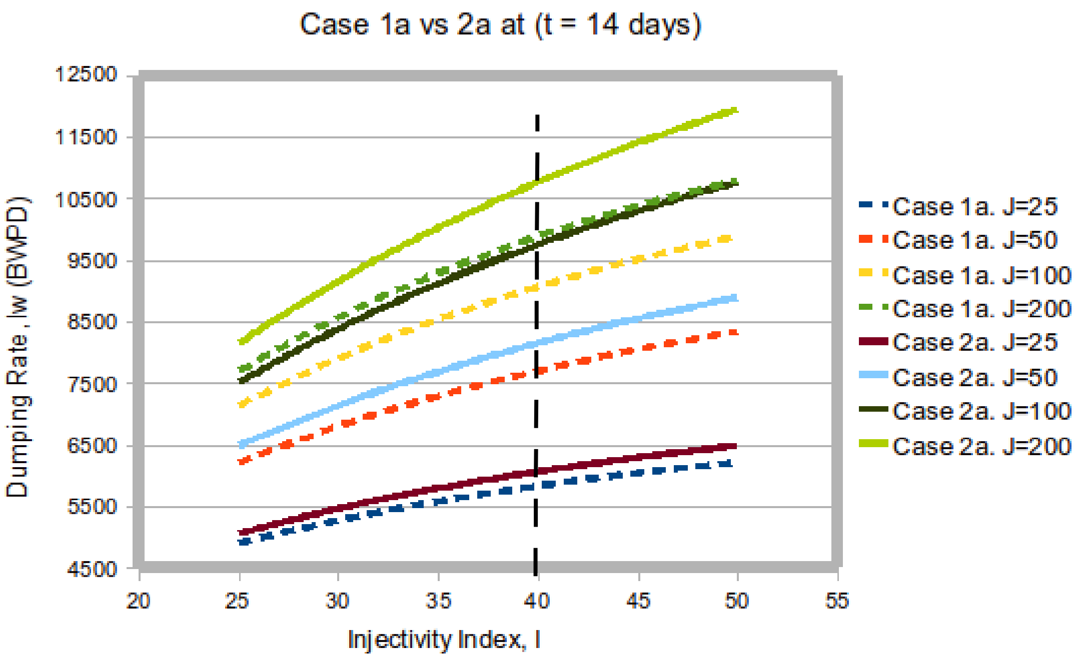

Figure J-2 Dumping Rate Comparison: Undersaturated Oil Reservoirs for Case 1a & 2a (t = 14 days)

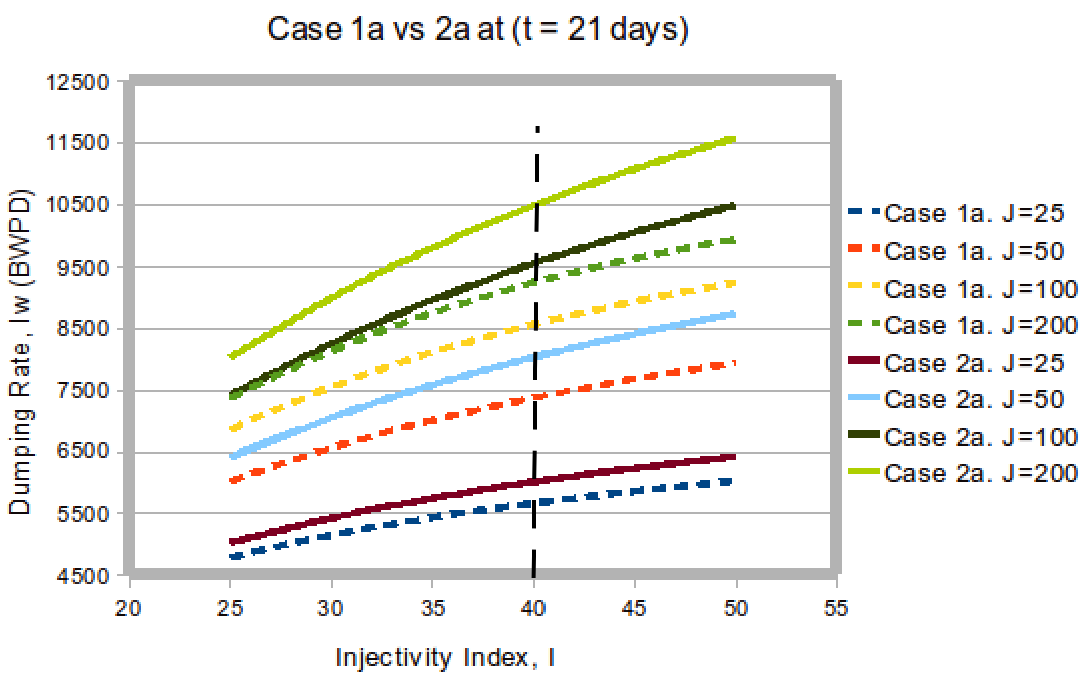

Figure J-3 Dumping Rate Comparison: Undersaturated Oil Reservoirs for Case 1a & 2a (t = 21 days)

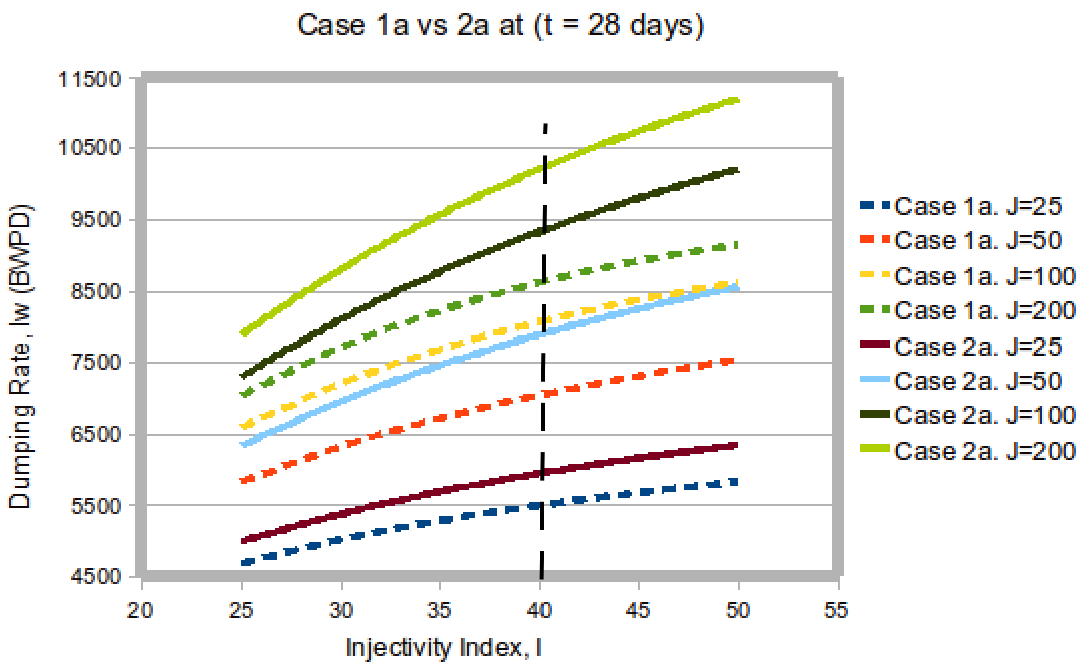

Figure J-4 Dumping Rate Comparison: Undersaturated Oil Reservoirs for Case 1a & 2a (t = 28 days)

APPENDIX K

Case 1b & 2b: Comparing Oil Reservoirs with Gas Cap

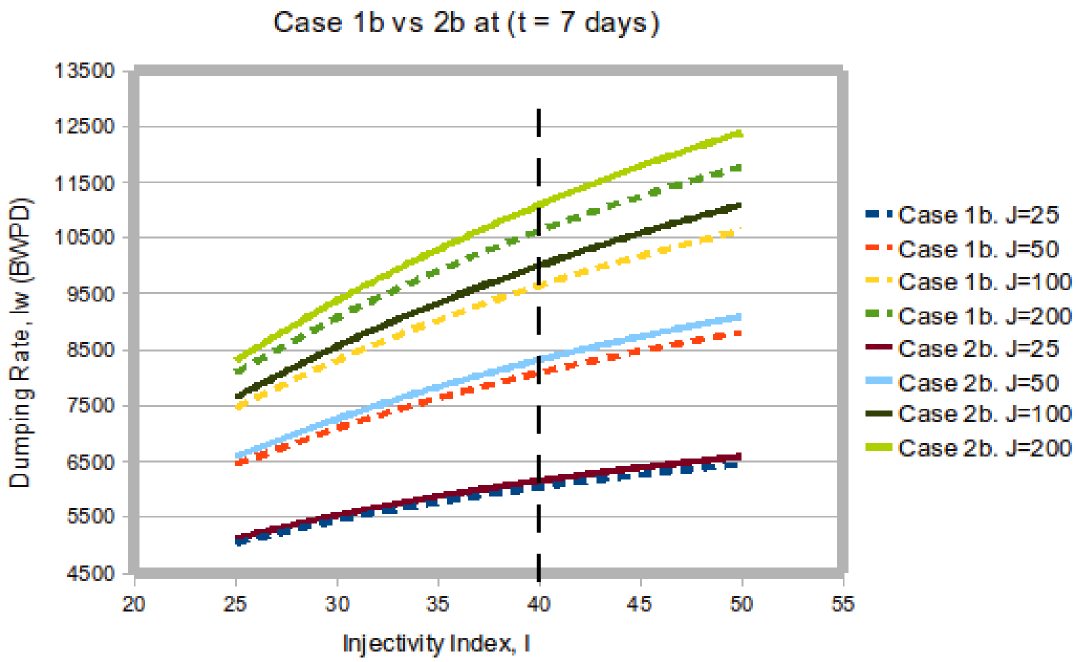

Figure K-1 Dumping Rate Comparison: Oil Reservoirs with Gas Cap for Case 1b & 2b (t = 7 days)

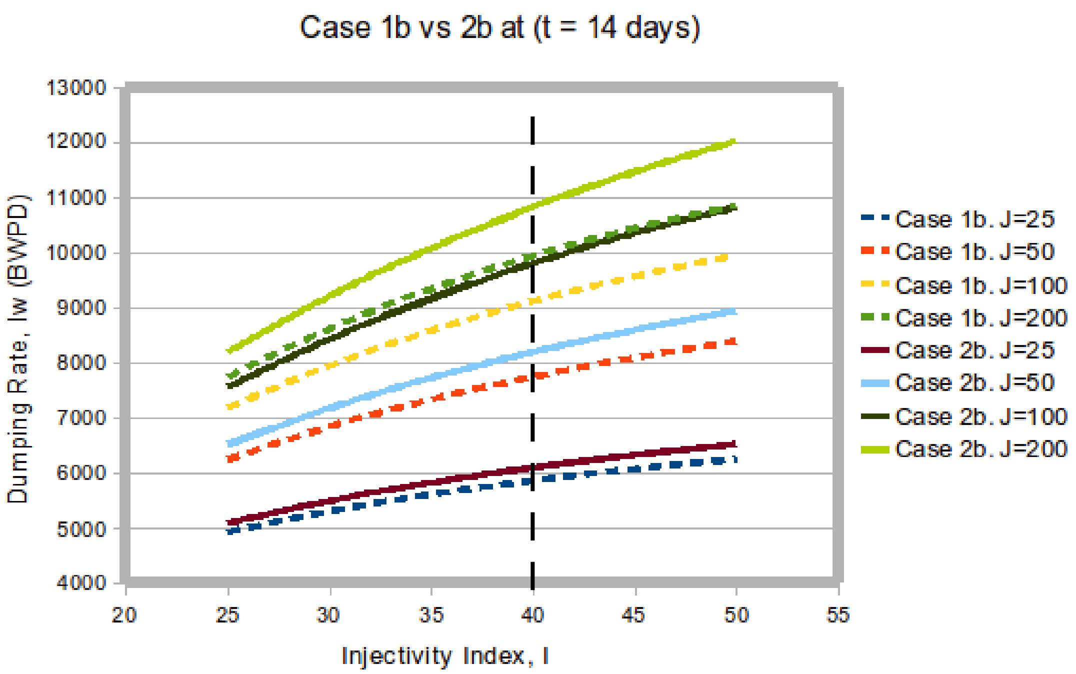

Figure K-2 Dumping Rate Comparison: Oil Reservoirs with Gas Cap for Case 1b & 2b (t = 14 days)

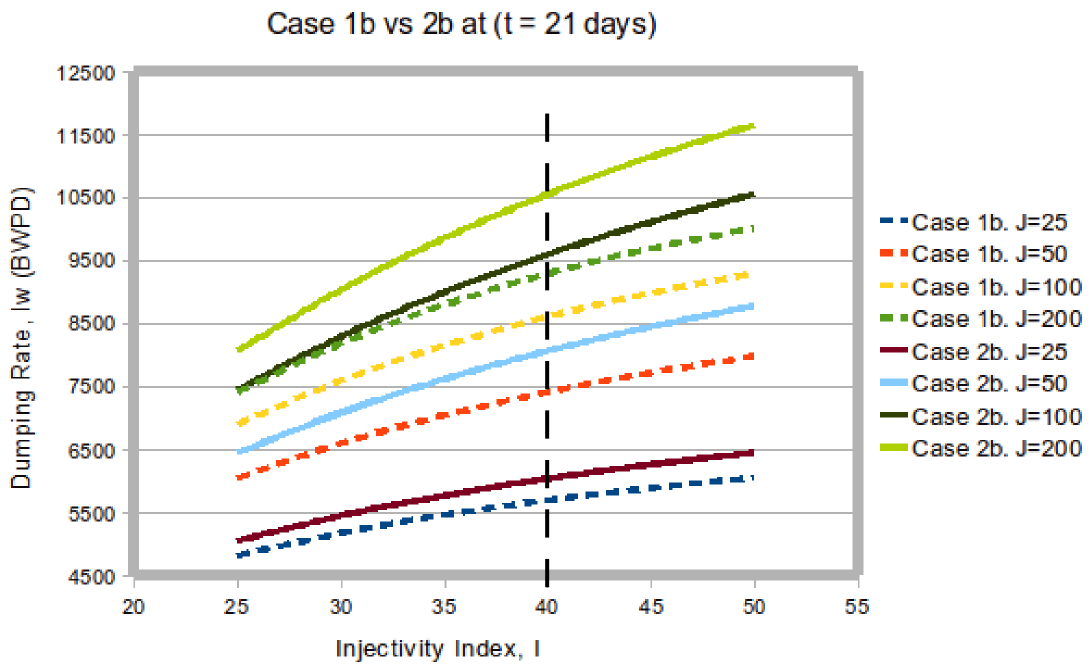

Figure K-3 Dumping Rate Comparison: Oil Reservoirs with Gas Cap for Case 1b & 2b (t = 21 days)

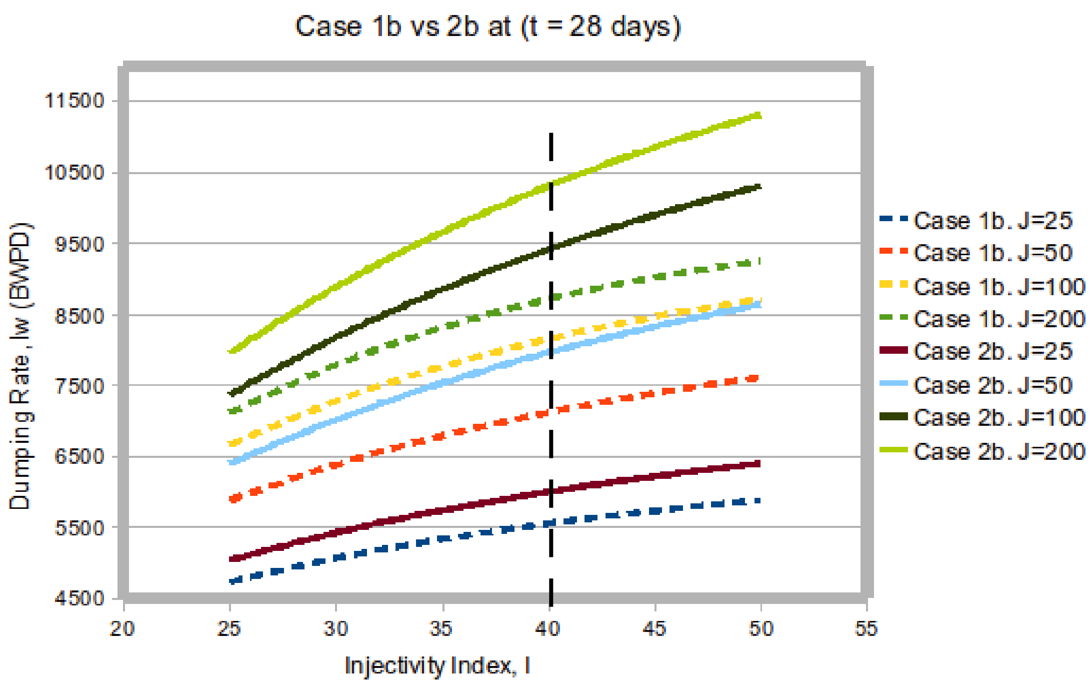

Figure K-4 Dumping Rate Comparison: Oil Reservoirs with Gas Cap for Case 1b & 2b (t = 28 days)