WEAK AND STRONG CONVERGENCE THEOREMS FOR NONSPREADING TYPE MAPPING IN A HILBERT SPACES

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CERTIFICATION

This is to certify that the thesis titled "WEAK AND STRONG CONVERGENCE THEOREMS FOR NONSPREADING TYPE MAPPING IN A HILBERT SPACES" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of review research work carried out by Abdulnasir Bala Nuhu in the department of Pure and Applied Mathematics.

APPROVAL

WEAK AND STRONG CONVERGENCE THEOREMS FOR NONSPREADING TYPE MAPPING IN A HILBERT SPACES

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ABSTRACT

The work of Osilike and Isiogugu, Nonlinear Analysis, **74** (2011), 1814-1822 on weak and strong convergence theorems for a new class of *k-strictly pseudononspreading* mappings in real Hilbert spaces is reviewed. We studied in detail this new class of mappings which is more general than the class of *nonspreading* mappings studied by Kurokawa and Takahashi, Nonlinear Analysis **73** (2010) 1562-1568. Many incisive examples establishing the relationship of the class of *k-strictly pseudononspreading* mappings and several other important classes of operators are presented. Interesting properties of k-strictly pseudononspreading mappings and weak and strong convergence theorems for approximation of its fixed points which appeared in the cited work of Osilike and Isiogugu were studied and presented.

DEDICATION

I dedicate this thesis to my Father and my Mother, Alhaji Bala Nuhu and Hajiya Safiya Muhammad may Allah grant them long life and prosperity amin, and also to all Brothers, Sisters, and Friends for their love and support during my stay at African University of Science and Technology (AUST).

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 General Introduction

The content of this thesis falls within the general area of functional analysis, in particular, nonlinear operator theory; an area which has attracted the attention of prominent mathematicians due to its diverse applications in numerous fields of science. In this thesis, we concentrate on an important topic in this area – *Weak and strong convergence theorems of nonspreading type mappings in a Hilbert space.*

In this chapter, the background of our research work will be given; this will reveal how relevant our work is. Then in chapter two, we shall review the research work carried out in the area of research described in this thesis. Some basic definitions and fundamental tools we used in our work will be given in chapter three as preliminaries, while our main results will be presented in chapter four. In chapter five, conclusions will be given.

1.2 Background of study

Fixed point theory has been an important area of mathematics due to its applications in several areas of research such as in Optimization, Economics, and Evolution Equations, to mention but a few. For example, consider the problem of finding the equilibrium points of the system described by the following equation

$$\frac{du}{dt} + Au(t) = 0 \tag{1.1}$$

where $A : D(A) \subset H \longrightarrow H$ is a nonlinear map and H a real Hilbert space. At equilibrium, $\frac{du}{dt} = 0$. Thus, the original problem is reduced to the problem of finding solutions of the equation:

$$Au = 0 \tag{1.2}$$

i.e, finding the zeros of A. Several problems arising in Reservoir Engineering, Economics, Physics to mention but a few, can be modelled in the form of equation (1.2). Since generally A is nonlinear, there is no closed form solution of this equation. To solve equation (1.2), where A is a multi-valued monotone map, Felix Browder defined an operator $T : H \longrightarrow H$ by $T := I - A$, where I is the identity map on H. He called such T a pseudocontraction. It is easy to see that fixed points of T correspond to zeros of A which in turn correspond to equilibrium points of dynamical system described by equation(1.1). As a result of this, the study of fixed point theory of pseudocontractive maps and their types has attracted the interest of numerous scientists and researchers.

Kohsaka and Takahashi introduced an important class of mappings called the class of nonspreading mappings. They obtained a fixed point theorem for a single valued nonspreading mapping in Banach space. Furthermore, they obtained a common fixed point theorem for a commutative family of nonspreading mappings in Banach space.

Let C be a nonempty closed and convex subset of a real smooth, strictly convex and reflexive real Banach space, a map $T : C \longrightarrow C$ is nonspreading if $\forall \ x, y \in C$

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x). \tag{1.3}$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2, \ \forall \ x, y \in E$ and $J : E \longrightarrow 2^{E^*}$ defined by $Jx = \{j(x) \in E^* : \langle x, j(x)\rangle = \|x\|\|j(x)\|\}, \forall x \in E$, where $\langle ., .\rangle$ is the duality pairing between $x \in E$ and $j(x) \in E^*$, so that $\langle y, j(x)\rangle = (j(x))(y)$.

We observe that if E is a real Hilbert space, then j is the identity and

$$\phi(x, y) = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 = \|x - y\|^2.$$

Thus, if C is a nonempty closed and convex subset of a real Hilbert space, then

$T : C \longrightarrow C$ is nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \ \forall \ x, y \in C. \tag{1.4}$$

It is shown in Lemma 3.22 that inequality 1.4 is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \qquad \forall x, y \in C.$$

S. Lemoto and W. Takahashi obtained some fundamental properties for nonspreading mapping in a Hilbert space. Furthermore, they studied the approximation of common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space.

Y. Kurokawa and W. Takahashi obtained a weak convergence theorem of Bailon's type for nonspreading mapping in Hilbert space, using an idea of mean convergence they proved a strong convergence theorem for nonspreading mapping in a Hilbert space.

Osilike and Isiogugu [Osilike and Isiogugu, 2011] introduced a new class of nonspreading type mappings which is more general than the class studied by Kurokawa and Takahashi. Following the terminology of Browder and Petryshn they called a mapping

$T : C \longrightarrow C$ $k-$ *strictly pseudononspreading* if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle \qquad \forall x, y \in C \tag{1.5}$$

Our main focus in this thesis, is to review the work done by Osilike and Isiogugu in their paper titled "Weak and strong convergence theorem for nonspreading type mapping in Hilbert space" which appeared in Nonlinear Analysis, **Vol 74** (2011), 1814-1822.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, we review other works done in the area of research carried out in this thesis; so that the results obtained in this work will be well appreciated.

2.2 Review

The origin of nonspreading type mapping is traced as far as 2008, when Fumiaki Kohsaka and Wataru Takahashi considered the class of nonspreading mappings to study the resolvents of a maximal monotone operator in Banach spaces [Kohsaka et al., 2008]. This class of mappings contains the important class of firmly nonexpansive mappings, (i.e., a mapping $T : D(T) \subset H \longrightarrow H$ such that $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \quad \forall \ x, y \in D(T)$).

Firmly nonexpansive mappings have intimate connection with maximal monotone operators on Hilbert spaces where an operator $T : D(T) \subset H \longrightarrow 2^H$ with effective domain
$D(T) = \{x \in H : Tx \neq \emptyset\}$ is maximal monotone if
$\langle u - v, x - y\rangle \geq 0, \quad \forall \ x, y \in D(T), \ u \in Tx, \ v \in Ty$ and its graph
$G(T) = \{(X, u) : x \in D(T), \ u \in Tx\}$ is not properly contained in the graph of any other monotone operator.
It is proved in [Kohsaka et al., 2008] that if T is maximal monotone then the resolvent $J_\lambda = (I + \lambda T)^{-1}$ is singled valued and firmly nonexpansive, where $\lambda > 0$ and I is the identity in H.

Furthermore, $F(J_\lambda) = T^{-1}0 = \{x \in D(T) : 0 \in Tx\}$. Thus, the problem of finding zeros of maximal monotone operator in Hilbert space is reduced to fixed point problem for firmly nonexpansive mappings.

The class of firmly nonexpansive mappings is a proper subclass of nonexpansive mappings. For the class of nonexpansive mappings, apart from being an obvious generalization of the contraction mappings, they are important, as has been observed by Bruck [Bruck,1980], for the following two reasons:

- Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960s and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the geometric properties of the underlying Banach spaces instead of compactness properties.
- Nonexpansive mappings appear in applications as the transition operators for initial value problems of differential inclusions of the form $0 \in \dfrac{du}{dt} + A(t)u$, where the operator $\{A(t)\}$ are, in general, set-valued and are accretive or dissipative and minimally continuous.

Nonexpansive mappings have been studied extensively by numerous authors (see e.g. [Bruck, 1973], [Kirk, 1965],[Karlovitz, 1976], [Soardi, 1979]). Unlike as in the case of contraction mappings, trivial example shows that for a nonexpansive map T, mapping from a complete metric space into itself with $F(T) = \{x \in D(T) : Tx = x\} \neq \emptyset$, the Picard iterative sequence $x_{n+1} = Tx_n, \quad x_0 \in D(T), \; n \geq 0$, may fail to converge even when T has a unique fixed point.

It suffices, for example, to take T to be the anti-clockwise rotation of the closed unit ball in $\mathbb{R}^2$ around the origin of coordinates through a fixed acute angle. Clearly, T is nonexpansive, zero is the fixed point of T and the Picard iterative sequence $x_{n+1} = Tx_n, \quad x_0 \in D(T), \quad n \geq 0$, does not converge to zero [Chidume, 2009].

Krasnoselskii [Krasnoselskii et al., 1957], however, showed that in this example, if for any fixed element of the ball, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n), \quad n \geq 0, \tag{2.1}$$

then $\{x_n\}$ converges strongly to the unique fixed point of T.

Schaefer [Schaefer, 1957], gave a generalization of this scheme which has successfully been employed in approximating fixed points of nonexpansive maps mapping from a nonempty closed convex subset of a real normed space into itself. The recurrence relation of Schaefer is given by:

$x_0 \in C$,

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \quad n \geq 0 \tag{2.2}$$

where C is a nonempty closed convex subset of a real normed space and $\lambda \in (0,1)$ is fixed (see e.g., Schaefer [Schaefer, 1957]. The sequence $\{x_n\}_{n=1}^{\infty}$ generated by scheme (2.1) is called *Krasnoselskii sequence*.

Mann [Mann, 1953], however, introduced the following iterative scheme used in approximating fixed points of nonexpansive mappings: $x_0 \in C$,

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n, \quad n \geq 0, \tag{2.3}$$

where C is a nonempty closed convex subset of real normed space and $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ satisfying the following conditions:

$(i) \sum_{n=0}^{\infty} \lambda_n = \infty \quad (ii) \lim_{n\to\infty} \lambda_n = 0$. (see e.g. Edelstein and O'Brain [Edelstein et al.,1978], Chidume [Chidume, 1981], Ishikawa [Ishikawa, 1976]). The sequence $\{x_n\}_{n=1}^{\infty}$ given by (2.3) is now referred to as a *Mann sequence* in the light of Mann [Mann, 1953] while that given by (2.2) is consequently called (by many) the *Krasnoselskii-Mann*(K.M) sequence for finding fixed points of nonexpansive mappings. Assuming the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (2.3) is bounded, Ishikawa [Ishikawa, 1976] proved that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.4}$$

Any sequence satisfying 2.4 is called *an approximate fixed point sequence of T*. Studying the sequence given by (2.3) and introducing the notion of admissible sequence, Chidume proved that if C is bounded, then the convergence in (2.4) is uniform. Edelstein and O'Brain [Edelstein et al.,1978] proved that if $T : C \longrightarrow C$ is nonexpansive and C is nonempty closed convex and bounded and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (2.2), the limit in (2.4) is uniform.

It is worth noting that the Mann sequence generated by (2.3) converges weakly rather than strongly. However, if one have that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and impose some form of compactness conditions either on T or on its domain (see e.g, Chidume [Chidume, 2009]), then strong convergence of $\{x_n\}_{n=1}^{\infty}$ is guaranteed.

At this point, we would also like to emphasize the importance of iterative methods for approximating fixed points of nonexpansive mappings as observed by Byrne [Byrne, 2004].

> Many well-known algorithms in signal processing and image reconstruction are iterative in nature ... A wide variety of iterative procedures used in signal processing and image

reconstruction and elsewhere are special cases of the K.M iteration procedure, for particular choices of the operator ... ([Byrne, 2004]).

In 1967, Halpern [Halpern, 1967] was the first who introduced the following iteration scheme for a nonexpansive mapping T which was referred to as *Halpern iteration*. For any initialization $x_0 \in C$ and any anchor $u \in C$, $\alpha_n \in [0,1]$

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \qquad \forall \ \ n \geq 0 \tag{2.5}$$

He proved that the sequence (2.5) converges weakly to a fixed point of T, where $\alpha_n = \frac{1}{n^{\alpha}}, \quad \alpha \in (0,1)$.

In 1977, Lions [Lions et al., 1977] further proved that the sequence (2.5) converges strongly to a fixed point of T in a Hilbert spaces, where $\{\alpha_n\}_{n=1}^{\infty}$ satisfies the following conditions:

(C_1) $\lim\limits_{n\to\infty} \alpha_n = 0$

(C_2) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$

(C_3) $\lim\limits_{n\to\infty} \frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}^2} = 0.$

But, in [Halpern, 1967],[Lions et al., 1977], the real sequence $\{\alpha_n\}_{n=1}^{\infty}$ excluded the canonical choice $\alpha_n = \frac{1}{n+1}$. In 1992 ,Wittmann [Wittmann, 1992] proved still in Hilbert spaces, the strong convergence of the sequence (2.5) to a fixed point of T, where $\{\alpha_n\}_{n=1}^{\infty}$ satisfies the following conditions:

(C_1) $\lim\limits_{n\to\infty} \alpha_n = 0$

(C_2) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$

(C_4) $\sum\limits_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

The strong convergence of Halpern's iteration to a fixed point of T has also been proved in Banach spaces; see, (e.g. [Reich, 1980], [Reich, 1994], [Shioji et al., 1997] [Suzuki, 2007], [Takahashi et al., 1984], [Xu et al, 2002],[Xu et al.]). Reich ([Reich, 1980]-[Reich, 1994]) has showed the strong convergence of the sequence (2.5), where $\{\alpha_n\}_{n=1}^{\infty}$ satisfies the conditions (C_1), (C_2), (C_5), $\{\alpha_n\}_{n=1}^{\infty}$ is decreasing (noting that condition (C_5) is a special case of condition (C_4)). In 1997, Shioji and Takahashi [Shioji et al., 1997] extended Wittmann's result to Banach spaces. In 2002, Xu [Xu et al.] obtained a strong convergence theorem, where $\{\alpha_n\}_{n=1}^{\infty}$ satisfies the following conditions: $(C_1), (C_2)$ and (C_6) $\lim\limits_{n\to\infty} \frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}} = 0.$

In particular, the canonical choice of $\alpha_n = \frac{1}{n+1}$ satisfies the conditions $(C_1), (C_2)$ and (C_6). However, it remains an open question, (see [Halpern, 1967]) whether a real sequence $\{\alpha_n\}_{n+1}^{\infty}$ satisfying the conditions (C_1) and (C_2) is sufficient to guarantee the strong convergence of the Halpern's iteration (2.5) for nonexpansive mappings. Several mathematicians had addressed the open question in various research article. In [Song et al., 2009], Song proved that for a firmly nonexpansive mapping T, an important subclass of nonexpansive mappings,the answer to the Halpern open problem is affirmative.

Firmly nonexpansive mapping is a subclass of an important class of mapping called *nonspreading mappings*, introduced by Fumiaki Kohsaka and Wataru Takahashi [Kohsaka et al., 2008], [Kohsaka and Takahashi, 2008]. Let E be a smooth, strictly convex and reflexive real Banach space, let j be the normalized duality mapping of E, and let C be a nonempty closed and convex subset of E. Then, a mapping $T : C \longrightarrow C$ is said to be nonspreading if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) \quad \forall \ \ x, y \in C$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2, \ \ \forall \ \ x, y \in E.$

They considered such a mapping to study the resolvents of a maximal monotone operator in Banach space, they obtained a fixed point theorem for a single valued nonspreading mapping in Banach spaces. Furthermore, they obtained a common fixed point theorem for a commutative family of nonspreading mappings in Banach spaces. We observe that if E is a real Hilbert space, then j is the identity and $\phi(x, y) = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 = \|x - y\|^2 \quad \forall \ \ x, y \in H.$
Thus, if C is a nonempty closed and convex subset of a real Hilbert space, then
$T : C \longrightarrow C$ is nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \ \ \forall \ \ x, y \in C. \tag{2.6}$$

It is shown in Lemma 3.22 that inequality1.4 is equivalent to
$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \qquad \forall x, y \in C.$

S. Lemoto and W. Takahashi [Lemoto and Takahashi, 2009] obtained some fundamental properties for nonspreading mapping in a Hilbert space. They used Ishikawa [Ishikawa, 1974] two-step sequence to approximate a common fixed points of nonexpansive mapping and nonspreading mapping in a Hilbert space, this result is contained in the following theorem:

Theorem 2.1. *Let H be a real Hilbert space, let C be a nonempty closed and convex subset of H. Let S be a nonspreading mapping of C into itself, and let T be a nonexpansive mapping of C into* *itself such that* $F(S) \cap F(T) \neq \emptyset$. *Define a sequence* $\{x_n\}_{n=1}^{\infty}$ *as follows:*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n\{\beta_n S x_n + (1-\beta_n)T x_n\} \end{cases}$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \subset [0,1]$. *Then, the following hold:*

(*i*) *If* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\sum_{n=0}^{\infty}(1-\beta_n) < \infty$, *then* $\{x_n\}_{n=1}^{\infty}$ *converges weakly to* $v \in F(S)$.

(*ii*) *If* $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n) = \infty$ *and* $\sum_{n=0}^{\infty}\beta_n < \infty$, *then* $\{x_n\}_{n=1}^{\infty}$ *converges weakly to* $v \in F(T)$.

(*iii*) *If* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, *then* $\{x_n\}_{n=1}^{\infty}$ *converges weakly to* $v \in F(S) \cap F(T)$.

Yu Kurokawa and Wataru Takahashi [Kurokawa et al., 2010] obtained a weak convergence theorem of Bailon's type for nonspreading mapping in real Hilbert spaces; this result is contained in the following theorem:

Theorem 2.2. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H, *let* T *be a nonspreading mapping from* C *into itself. Define two sequences* $\{x_n\}_{n=1}^{\infty}$ *and* $\{z_n\}_{n=1}^{\infty}$ *in* C *as follows:*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T x_n, \\ z_n = \frac{1}{n}\sum_{k=1}^{n} x_k. \end{cases}$$

$\forall n \in \mathbb{N}$, *where* $0 \leq \alpha_n < 1$ *and* $\alpha_n \to 0$ *as* $n \to \infty$.

If $F(T) \neq \emptyset$,*then* $\{z_n\}_{n=1}^{\infty}$ *converges weakly to* $z \in F(T)$, *where* $z = \lim_{n\to\infty} P x_n$ *and* P *is the metric projection of* H *onto* $F(T)$.

In particular, for any $x \in C$, *define* $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$, *then* $\{S_n x\}_{n=1}^{\infty}$ *converges weakly to* $z \in F(T)$, *where* $z = \lim_{n\to\infty} P T^n x$.

Using an idea of mean convergence they proved a strong convergence theorem for nonspreading mapping in a Hilbert space, this result is contained in the following theorem:

Theorem 2.3. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H, *let* $T : C \longrightarrow C$ *be a nonspreading mapping, let* $u \in C$ *and define two sequences* $\{x_n\}_{n=1}^{\infty}$ *and* $\{z_n\}_{n=1}^{\infty}$ *in* C *as follows:*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k x_n. \end{cases}$$

$\forall n \in \mathbb{N}$, *where* $0 \leq \alpha_n < 1, \alpha_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *If* $F(T) \neq \emptyset$, *then* $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ *converge strongly to* Pu, *where* P *is the metric projection of* H *onto* $F(T)$.

In their final remark they posed an open problem whether a strong convergence theorem of Halpern's type for nonspreading mapping holds.
Osilike and Isiogugu [Osilike and Isiogugu, 2011] introduced a new class of nonspreading type mapping which is more general than the class studied by Kurokawa and Takahashi [Kurokawa et al., 2010]. Following the terminology of Browder and Petryshn they called a mapping
$T : C \longrightarrow C$ *kstrictly pseudononspreading* if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle \qquad \forall x, y \in C$$

They exhibited examples of this class of mappings and showed with examples that the new class of mapping is more general than the class of nonspreading mappings of Kurokawa and Takahashi [Kurokawa et al., 2010]. They further exhibited some important properties of the class of k-strictly pseudononspreading mappings; and extended the results of Kurokawa and Takahashi [Kurokawa et al., 2010] to the class of k- strictly pseudononspreading mappings. They used an auxilliary mapping to obtain a strong convergence theorem of Halpern's type [Halpern, 1967] for the class of k-strictly pseudononspreading mappings and for the case where T is averaged, this resolved in the affirmative an open problem posed by Kurokawa and Takahashi [Kurokawa et al., 2010] in their final remark.

Our main focus in this thesis, is to review the work done by Osilike and Isiogugu in their paper titled *"Weak and strong convergence theorem for nonspreading type mapping in Hilbert space"*.

CHAPTER 3

PRELIMINARIES

3.1 Definition of some terms

Let H be a real Hilbert space, $D(T)$ be domain of T, $R(T)$ be range of T, and $F(T)$ be fixed point set of T. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in H, we denote the weak convergence of $\{x_n\}_{n=1}^{\infty}$ to a point $x \in H$ by $x_n \rightharpoonup x$, and the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to a point $x \in H$ by $x_n \rightarrow x$.

Definition 3.1. Let C be a nonempty subset of a real Hilbert space H, and let $T : C \longrightarrow C$ be a mapping. A point x_0 is a *fixed point* of C if $Tx_0 = x_0$.

Definition 3.2. A mapping $T : D(T) \subseteq H \longrightarrow$ H is said to be *l-Lipschitzian* if there exists $l > 0$ such that

$$\|Tx - Ty\| \leq l\|x - y\|, \forall x, y \in D(T) \tag{3.1}$$

Definition 3.3. If $l < 1$ in inequality (3.1) then T is said to be *strictly contractive.*

Definition 3.4. If $l = 1$ in inequality (3.1) then T is *nonexpansive.*

Definition 3.5. T is *strongly nonexpansive* if T is nonexpansive and $x_n - y_n - (Tx_n - Ty_n) \rightarrow 0$ as $n \rightarrow \infty$, whenever $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty} \subset C$ such that $\{x_n - y_n\}_{n=1}^{\infty}$ is a bounded sequence and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Definition 3.6. T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and
$\|Tx - p\| \leq \|x - p\|, \quad \forall\, x \in\ D(T), \quad \forall\, p \in\ F(T).$

Definition 3.7. T is said to be *strongly quasi nonexpansive* if T has the following properties:
(1) $F(T) \neq \emptyset$
(2) $\|Tx - p\| \leq \|x - p\| \;\; \forall\, x \in\ C, \;\; p \ \in\ F(T).$

(3) $x_n - Tx_n \to 0$ as $n \to \infty$ whenever $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in C such that $\|x_n - p\| - \|Tx_n - p\| \to 0$ as $n \to \infty$ for some $p \in F(T)$.

Definition 3.8. T is said to be *firmly nonexpansive* if $\exists\ k > 0$ such that $\|Tx - Ty\|^2 \leq \|x - y\|^2 - k\|x - Tx - (y - Ty)\|^2, \quad \forall\ x, y\ \in\ C$ which is also equivalent to $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \forall\ x, y\ \in\ C.$
Every firmly nonexpansive map is strongly nonexpansive.

Definition 3.9. T is *quasi-firmly nonexpansive* if $F(T) \neq \emptyset, \exists\ k \in [0, 1)$ such that $\forall\ x\ \in\ C, p\ \in\ F(T)$ $\|Tx - p\|^2 \leq \|x - p\|^2 - k\|x - Tx\|^2.$

Definition 3.10. A mapping $T : C \longrightarrow C$ is *demi-contractive* if $\exists\ k \in [0, 1)$ such that $\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \quad \forall\ \ x\ \in C,\ \ \forall\ \ p\ \in F(T).$

Definition 3.11. An operator $T : D(T) \subseteq H \longrightarrow 2^H$ with effective domain $D(T) := \{x \in H : Tx \neq \emptyset\}$ is *maximal monotone* if $\langle u - v, x - y\rangle \geq 0, \quad \forall\ \ x, y\ \in D(T),\ u \in Tx,\ v \in Ty$ and its graph $G(T) = \{(x, u) : x \in D(T),\ u \in Tx\}$ is not properly contained in the graph of any other monotone operator.

Definition 3.12. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. A map $T : C \longrightarrow C$ is *nonspreading* if
$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \qquad \forall x, y \in C$

Definition 3.13. A mapping $T : C \longrightarrow C$ is called *averaged* if it can be written as the averaged of the identity operator and a nonexpansive mapping, (i.e.,
$T = \beta I + (1 - \beta)S,$
where $\beta \in (0, 1)$ and $S : C \longrightarrow C$ is nonexpansive.

3.1.1 Basic facts in Hilbert Spaces

(1) Every strictly contractive mapping T with a contractive constant β is firmly type nonexpansive mapping with a constant $k = \frac{1-\beta}{1+\beta}$.

Proof. Indeed, $\|x - Tx - (y - Ty)\| \leq (1+\beta)\|x - y\|$, and hence

$$\begin{aligned}
\|Tx - Ty\|^2 \leq \beta^2\|x-y\|^2 &= \|x-y\|^2 - (1-\beta^2)\|x-y\|^2 \\
&= \|x-y\|^2 - \frac{1-\beta}{1+\beta}(1+\beta)^2\|x-y\|^2 \\
&\leq \|x-y\|^2 - \frac{1-\beta}{1+\beta}\|x - Tx - (y - Ty)\|^2
\end{aligned}$$

$\Rightarrow \|Tx - Ty\|^2 \leq \|x-y\|^2 - \frac{1-\beta}{1+\beta}\|x - Tx - (y-Ty)\|^2$

$\Rightarrow$ T is firmly type nonexpansive. □

The converse is not true in general.

Example 3.14. Consider $\mathbb{R}$ with usual norm, let $C = [0,1]$ and let $T : C \longrightarrow C$ be defined by $Tx = \frac{1}{2}x^2, \quad \forall x \in C.$

Claim 1: T is firmly type nonexpansive. In fact, since $\frac{x+y}{2} \leq 1 \quad \forall x, y \in C$, we have

$$\begin{aligned}
|Tx - Ty|^2 = (Tx - Ty)^2 &= (x-y)^2 - \Big((x-y)^2 - \frac{1}{4}(x^2-y^2)^2\Big) \\
&= (x-y)^2 - (x-y)^2\Big(1 - \frac{1}{4}(x+y)^2\Big) \\
&= (x-y)^2 - (x-y)^2\Big(1 - \frac{x+y}{2}\Big)\Big(1 + \frac{x+y}{2}\Big) \\
&\leq |x-y|^2 - (x-y)^2\Big(1 - \frac{x+y}{2}\Big)^2 \\
&= |x-y|^2 - \Big((x-y) - \frac{x^2-y^2}{2}\Big)^2 \\
&= |x-y|^2 - |x - Tx - (y - Ty)|^2.
\end{aligned}$$

Hence T is firmly type nonexpansive mapping.

Claim 2: T is not strictly contractive mapping

Suppose that there exists $\beta \in (0,1)$ such that $|Tx - Ty|^2 = \frac{|x^2-y^2|}{2} \leq \beta|x-y|, \qquad \forall x, y \in C$

Then $\frac{|x+y|}{2} \leq \beta$ whenever $x \neq y$.

Taking $x = 1, y = 1 - \frac{1}{n}, \quad n \in \mathbb{N}$, we have $\frac{1}{2}|1 + 1 - \frac{1}{n}| = \frac{1}{2}|2 - \frac{1}{n}| = 1 - \frac{1}{2n} \leq \beta < 1$

as $n \longrightarrow \infty \qquad 1 - \frac{1}{2n} \longrightarrow 1 \Rightarrow 1 \leq \beta < 1 \Rightarrow 1 < 1 \qquad$ (impossible)

Hence, T is firmly type nonexpansive, but not strictly contractive. So the class of strictly contractive mappings is a proper subclass of the class of firmly type nonexpansive mappings.

(2) Every nonspreading mapping with a nonempty fixed point set is quasi-nonexpansive. Also every nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive. The converse in

both cases is not true.

(3) Class of nonspreading type mappings and class of nonexpansive mappings are independent.

Example 3.15. Consider $\mathbb{R}$ with usual norm.

We define a map $T : [0,1] \longrightarrow [0,1]$ by $Tx = 1 - x \ \ \forall x \in [0,1]$.

Claim 1: T is nonexpansive.

Proof of Claim 1

Let $x, y \in [0,1] \Rightarrow Tx = 1 - x, \quad Ty = 1 - y$.

$\|Tx - Ty\| = \|1 - x - (1 - y)\| = \|y - x\| \leq \|x - y\|$.

Claim 2: T is not nonspreading mapping.

Proof of Claim 2:

Suppose T is nonspreading mapping, that is $\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \quad \forall x, y \in C$.

Take $x = 0$ and $y = 1 \Rightarrow T(0) = 1$ and $T(1) = 0 \Rightarrow \|T(0) - T(1)\|^2 = 1$.

$\|0 - 1\|^2 = 1, \ 2\langle 0 - 1, 1 - 0\rangle = 2\langle -1, 1\rangle = -2$.

Since T is nonspreading mapping we have

$\|T(0) - T(1)\|^2 = 1 \leq -1 = 1 + (-2) = \|0 - 1\|^2 + 2\langle 0 - 1, 1 - 0\rangle$, which is impossible.

Hence T is not a nonspreading mapping even though T is nonexpansive mapping.

Example 3.16. Consider $\mathbb{R}$ with usual norm.

We define a map $T : [0,3] \longrightarrow [0,3]$ by

$$Tx = \begin{cases} 0 & x \neq 3 \\ 2 & x = 3 \end{cases}$$

Claim 1: T is nonspreading mapping.

Case 1: $x \neq 3$ and $y = 3 \Rightarrow Tx = 0$ and $Ty = 2$.

$2\|Tx - Ty\|^2 = 2\|0 - 2\|^2 = 8 < 9 \leq \|Tx - y\|^2 + \|x - Ty\|^2$.

Case 2: $x, y \neq 3 \Rightarrow 2\|Tx - Ty\|^2 = 0 \leq \|Tx - y\|^2 + \|x - Ty\|^2$.

Case 3: $x = 3 = y \Rightarrow 2\|Tx - Ty\|^2 = 0 \leq \|Tx - y\|^2 + \|x - Ty\|^2$.

Claim 2: T is not nonexpansive mapping. This follows since if we take $x = 2, y = 3 \Rightarrow Tx = 0, Ty = 2$, then $\|Tx - Ty\| = 2 > 1 = \|x - y\|$.

Hence T is a nonspreading mapping which is not nonexpansive. So these two Examples of maps shows that the class of nonspreading type mappings and the class of nonexpansive mappings are independent.

(4) Every firmly nonexpansive mapping is nonexpansive, the converse is not true in general.

Example 3.17. Consider $\mathbb{R}^2$ with the usual norm defined $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$T(x,y) = (-y,x), \quad \forall \ (x,y) \in \mathbb{R}^2.$

Clearly, T is nonexpansive mapping, but not firmly nonexpansive.

In fact, take $x = (1,2)$ and $y = (4,6)$ $Tx = (-2,1), Ty = (-6,4)$

$\Rightarrow \|Tx - Ty\| = \|(-2,1) - (-6,4)\| = \|(4,-3)\| = \sqrt{16+9} = 5$ and

$\|x - Tx - (y - Ty)\| = \|(1,2) - (-6,4)\| = \|(4,-3)\| = 5\sqrt{2}$

Thus, we have $\|Tx - Ty\|^2 + \|x - Tx - (y - Ty)\|^2 = 25 + 50 > 25 = \|x - y\|^2$, so, T is not firmly nonexpansive.

This example shows that the class of firmly nonexpansive mappings is a proper subclass of the class of nonexpansive mappings.

(5) Every quasi-nonexpansive mapping is demicontractive, but the converse is not true in general.

Example 3.18. Consider $\mathbb{R}$ with usual norm and

let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$Tx = \begin{cases} x & x \in (-\infty, 0) \\ -3x & x \in [0, \infty) \end{cases}$$

We observe that $F(T) = (-\infty, 0]$

Case 1: Let $x \in (-\infty, 0)$ then $x \in F(T)$. For any $p \in F(T)$ we have

$|Tx - p|^2 = |x - p|^2$, and $|x - Tx|^2 = 0$,

thus,

$|Tx - p|^2 = |x - p|^2 + k|x - Tx|^2 \quad \forall \ k \in [0, 1).$

Case 2: Let $x \in [0, \infty)$ then for any $p \in F(T)$ we have

$|Tx - p|^2 = |-3x - p|^2 = 9x^2 + 6xp + p^2$ and $|x - Tx|^2 = |x + 3x|^2 = 16x^2$,

since $6xp \leq 0$ and $-2xp \geq 0$ we have

$$\begin{aligned} |Tx - p|^2 &= 9x^2 + 6xp + p^2 \\ &\leq x^2 - 2xp + p^2 + 8x^2 \\ &= |x - p|^2 + \tfrac{1}{2}|x - Tx|^2, \end{aligned}$$

hence T is $\frac{1}{2}-$ demicontractive.

We also observe that

$\forall \ x \in (0, \infty), |Tx - 0| = 3|x - 0| > |x - 0|,$

and so T is not a quasi-nonexpansive mapping.

In fact $\forall \ x, y \in (0, \infty)$ with $x \neq y$ we have

$|Tx - Ty| = 3|x - y| > |x - y|,$

and so T is not a nonexpansive mapping. If we take $x = 0, y = 3$ we get

$|Tx - Ty|^2 = 81, |Tx - y|^2 = 9, |Ty - x|^2 = 81$ and hence

$2|Tx - Ty|^2 = 162 > 9 + 81 = 90 = |Tx - y|^2 + |Ty - x|^2.$

Therefore, T is $\frac{1}{2}$-demicontractive but not quasi-nonexpansive, not nonexpansive and not nonspreading.

(6) If T is a maximal monotone then the resolvent $J_\lambda = (I + \lambda T)^{-1}$, where$\lambda > 0$ and I is the identity in H is single and firmly nonexpansive. Furthermore,

$F(J_\lambda) = T^{-1}0 = \{x \in D(T) : 0 \in Tx\}$.

Thus, the problem of finding zeros of maximal monotone operator in Hilbert space is reduced to fixed point problem for firmly nonexpansive mapping.

3.1.2 Demiclosedness Principle

Let E be a real Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at a point $p \in D(T)$ if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in $D(T)$ which converges weakly to a point $x \in D(T)$ and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to p, then $Tx = p$.

3.2 Preliminaries

Lemma 3.19. Let H be a real Hilbert space then the following are well-known results:

(i) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2 \;\; \forall \;\; x, y \in H$ and $\forall \;\; t \in [0, 1]$

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \;\; \forall \;\; x, y \in H$

(iii) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in H which converges weakly to $z \in H$ then

$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2 \quad \forall \;\; y \in H.$

Proof. (i) Let $x, y \in H$

Case 1: $t = 0$ or $t = 1$ the relation holds.

Case 2: $t \in (0, 1)$

$$\begin{aligned}
\|tx + (1-t)y\|^2 &= \langle tx + (1-t)y, tx + (1-t)y\rangle \\
&= t^2\|x\|^2 + (1-t)^2\|y\|^2 + 2t(1-t)\langle x, y\rangle - t(1-t)\|x-y\|^2 + t(1-t)\|x-y\|^2 \\
&= t^2\|x\|^2 + (1-t)^2\|y\|^2 + 2t(1-t)\langle x, y\rangle - t(1-t)\|x-y\|^2 + t(1-t)\langle x-y, x-y\rangle \\
&= t^2\|x\|^2 + \|y\|^2 - 2t\|y\|^2 + t^2\|y\|^2 + 2t\langle x, y\rangle - 2t^2\langle x, y\rangle - t(1-t)\|x-y\|^2 + t\|x\|^2 \\
&\quad - 2t\langle x, y\rangle + t\|y|^2 + t^2\|x\|^2 + 2t^2\langle x, y\rangle - t^2\|y\|^2 \\
&= \|y\|^2 - t\|y\|^2 - t(1-t)\|x-y\|^2 + t\|x\|^2 \\
&= t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2.
\end{aligned}$$

Hence, $\forall \;\; x, y \in H, \;\; \forall \;\; t \in [0, 1]$

$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2.$

(ii) Let $\forall \;\; x, y \in H$

Case 1: $x = y$, we have

$\|x + y\|^2 = 4\|x\|^2 \leq 5\|x\|^2 = \|x\|^2 + 2\langle y, x + y\rangle$

Case 2: $x \neq y$

$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$

$\leq \|x\|^2 + 2\|y\|^2 + 2\langle x, y\rangle$

$= \|x\|^2 + 2(\langle y, y\rangle + \langle y, x\rangle)$

$= \|x\|^2 + 2\langle y, x + y\rangle$

Hence, $\forall\ \ x, y \in H, \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$.

(iii) Suppose $\{x_n\}_{n=1}^{\infty} \subseteq H$, such that $x_n \rightharpoonup z$ as $n \to \infty$ for some $z \in$ H.

We want to show that $\limsup\limits_{n\to\infty}\|x_n - y\|^2 = \limsup\limits_{n\to\infty}\|x_n - z\|^2 + \|z - y\|^2, \quad \forall\ \ y \in H.$

Now,

$x_n \rightharpoonup z \Rightarrow \{\|x_n\|\}_{n=1}^{\infty}$ is bounded $\quad \Rightarrow \limsup\|x_n\|^2 \in \mathbb{R}$.

Let $y \in H$

$\|x_n - y\|^2 = \|x_n - z + z - y\|^2$

$\qquad = \|x_n - z\|^2 + 2\langle x_n - z, z - y\rangle + \|z - y\|^2.$

By taking lim sup we obtain

$\limsup\limits_{n\to\infty}\|x_n - y\|^2 = \limsup\limits_{n\to\infty}\|x_n - z\|^2 + 2\lim\limits_{n\to\infty}\langle x_n - z, z - y\rangle + \|z - y\|^2,$

since $x_n \rightharpoonup z \Rightarrow \langle x_n - z, z - y\rangle \to 0$ as $n \to \infty$

$\Rightarrow \limsup\limits_{n\to\infty}\|x_n - y\|^2 = \limsup\limits_{n\to\infty}\|x_n - z\|^2 + \|z - y\|^2.$ $\square$

Projection map

Let C be a nonempty closed and convex subset of a real Hilbert space H. The *nearest point projection* $P_C : H \longrightarrow C$ defined from H onto C is the function which assigns to each $x \in H$ its nearest point denoted by $P_C x$ in C. Thus, $P_C x$ is the unique point in C such that

$\|x - P_C x\| \leq \|x - y\| \quad \forall\ \ y \in C.$

It is known that for each $x \in H$

$$\langle x - P_C x, y - P_C x\rangle \leq 0 \qquad \forall\ \ y \in C, \tag{3.2}$$

and projection map is firmly type nonexpansive in a Hilbert space.

Therefore, in a Hilbert space, metric projection $\Rightarrow$ firmly type nonexpansive mappings $\Rightarrow$ non-spreading mappings. In some text, $P_C x$ is called the shadow of x in C.

Note that:

If $x \in C$ then $P_C x = x$.

Lemma 3.20. Let C be a nonempty closed and convex subset of a real Hilbert space H, let $P_C : H \longrightarrow C$ be the projection of H onto C. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C and let $\|x_{n+1} - u\| \leq \|x_n - u\| \quad \forall\ \ u \in C$. Then, $\{P_C x_n\}_{n=1}^{\infty}$converges strongly in C.

Proof. Take $u_n = P_C x_n$. Let $m > n$ we have

$$\begin{aligned}
\|u_n - u_m\|^2 &= \|u_n - x_m + x_m - u_m\|^2 \\
&= \|(u_n - x_m) + (x_m - u_m)\|^2 \\
&= 2\|x_m - u_n\|^2 + 2\|x_m - u_m\|^2 - \|(u_n - x_m) - (x_m - u_m)\|^2 \\
&= 2\|x_m - u_n\|^2 + 2\|x_m - u_m\|^2 - \|x_m - u_m - u_n + x_m\|^2 \\
&= 2\|x_m - u_n\|^2 + 2\|x_m - u_m\|^2 - \|2x_m - u_m + u_n\|^2 \\
&= 2\|x_m - u_n\|^2 + 2\|x_m - u_m\|^2 - 4\|x_m - \tfrac{u_m+u_n}{2}\|^2 \\
&= 2\|x_m - u_m\|^2 + 2\|x_m - u_n\|^2 - 4\|x_m - \tfrac{u_m+u_n}{2}\|^2 \\
&\leq 2\|x_m - u_m\|^2 + 2\|x_m - u_n\|^2 - 4\|x_m - u_m\|^2 \\
&= 2\|x_m - u_n\|^2 - 2\|x_m - u_m\|^2 \\
&\leq 2\|x_n - u_n\|^2 - 2\|x_m - u_m\|^2.
\end{aligned}$$

Thus,

$$\|u_n - u_m\|^2 \leq 2\|x_n - u_n\|^2 - 2\|x_m - u_m\|^2. \tag{3.3}$$

Since $\|x_m - u_m\|^2 \leq \|x_n - u_n\|^2, \quad \forall\ m, n \in \mathbb{N}, \Rightarrow \lim_{n\to\infty} \|x_n - u_n\|^2$ exists.
Let $\lim_{n\to\infty} \|x_n - u_n\|^2 = k \Rightarrow \lim_{m\to\infty} \|x_m - u_m\|^2 = k$,
$\|u_n - u_m\|^2 \leq 2\|x_n - u_n\|^2 - 2\|x_m - u_m\|^2 \to 2k - 2k = 0$ as $n, m \to \infty$
$\Rightarrow \{u_n\}_{n=1}^{\infty}$ is Cauchy in C. Since C is a closed subset of $H \Rightarrow \lim_{n\to\infty} u_n$ exists in C.
Hence, $\{P_C x_n\}_{n=1}^{\infty}$ converges strongly in C. □

Lemma 3.21. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the following condition

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \beta_n, \qquad n \geq 1, \tag{3.4}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences such that
(i) $\{\alpha_n\}_{n=1}^{\infty} \subseteq [0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$
Then, $\lim_{n\to\infty} a_n = 0$.

Proof. Claim: $\liminf_{n\to\infty} a_n = 0$.
Suppose for contradiction that $\liminf_{n\to\infty} a_n \neq 0$,then $\exists\ a \in [0, \infty)$ such that $\liminf_{n\to\infty} a_n = a$
$\Rightarrow \forall\ \ \epsilon > 0\ \ \exists N_\epsilon \in \mathbb{N}$ such that $a_n > a - \epsilon, \qquad \forall\ \ n \geq N_\epsilon$.
By taking $\epsilon = \frac{3a}{4}$, we have $a_n > \frac{a}{4}, \quad \forall n \geq N_\epsilon$,

$$\Rightarrow \alpha_n a_n \geq \frac{a\alpha_n}{4}, \qquad \forall\ \ n \geq N_\epsilon. \tag{3.5}$$

Also, $\limsup_{n\to\infty} \beta_n = \beta \leq 0, \Rightarrow \forall \epsilon > 0, \quad \exists N_0 \in \mathbb{N}$ such that $\beta_n < \beta + \epsilon, \forall n \geq N_0$.

By taking $\beta + \epsilon = \frac{a}{8}$, we have $\beta_n < \frac{a}{8}$, $\quad \forall \; n \geq N_0$,

$$\Rightarrow \alpha_n \beta_n < \frac{a\alpha_n}{8} \qquad \forall n \geq N_0 \tag{3.6}$$

Now from inequalities (3.4),(3.5),(3.6) we have,

$a_{n+1} \leq a_n - \alpha_n a_n + \alpha_n \beta_n$

$\leq a_n - \frac{a\alpha_n}{4} + \frac{a\alpha_n}{8}$

$= a_n - \frac{a\alpha_n}{8} \leq a_n,$

$\Rightarrow a_{n+1} \leq a_n, \quad \forall \; n \geq 1, \; \Rightarrow \{a_n\}_{n=1}^{\infty}$ is monotone non-increasing.

Since $\{a_n\}_{n=1}^{\infty}$ is bounded below by 0, $\Rightarrow \{a_n\}_{n=1}^{\infty}$ converges,. (i.e $\lim_{n\to\infty} a_n$ exists).

Now from (3.4) we also have,

$\alpha_n a_n \leq a_n - a_{n+1} + \alpha_n \beta_n$

$\leq a_n - a_{n+1} + \beta_n$

$\frac{a\alpha_n}{4} \leq \alpha_n a_n \leq a_n - a_{n+1} + \beta_n$

$\Rightarrow \frac{a}{4}\sum_{k=1}^{n} \alpha_n \leq \sum_{k=1}^{n}(a_n - a_{n+1}) + \sum_{k=1}^{n}\beta_n = a_1 - a_{n+1} + \sum_{k=1}^{n}\beta_n,$

$\Rightarrow \frac{a}{4}\sum_{k=1}^{N} \alpha_n \leq a_1 + \sum_{k=1}^{N}\beta_n$

$\Rightarrow \frac{a}{4}\sum_{k=1}^{\infty} \alpha_n \leq a_1 + \sum_{k=1}^{\infty}\beta_n \leq a_1 + M_0$

Since $\sum_{k=1}^{\infty}\alpha_n = \infty \Rightarrow \infty \leq a_1 + M_0$ (Impossible).

Hence, $\liminf_{n\to\infty} a_n = 0$ since $\lim_{n\to\infty} a_n$ exists $\Rightarrow \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = 0.$ ☐

Lemma 3.22. Let C be a nonempty closed convex subset of H. Then a mapping $T : C \longrightarrow C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \qquad \forall x, y \in C \tag{3.7}$$

Proof. We have $\forall x, y \in C$

$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2$

$\Leftrightarrow 2\|Tx - Ty\|^2 \leq \|Tx - x + x - y\|^2 + \|x - Tx + Tx - Ty\|^2$

$\Leftrightarrow 2\|Tx-Ty\|^2 \leq \|Tx-x\|^2+2\langle Tx-x, x-y\rangle+\|x-y\|^2+\|x-Tx\|^2+2\langle x-Tx, Tx-Ty\rangle+\|Tx-Ty\|^2$

$\Leftrightarrow 2\|Tx - Ty\|^2 \leq 2\|Tx - x\|^2 + 2\langle Tx - x, x - Tx - (y - Ty)\rangle + \|x - y\|^2 + \|Tx - Ty\|^2$

$\Leftrightarrow 2\|Tx - Ty\|^2 - \|Tx - Ty\|^2 \leq 2\|Tx - x\|^2 + \|x - y\|^2 + 2\langle Tx - x, x - Tx - (y - Ty)\rangle$

$\Leftrightarrow \|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle Tx - x, Tx - x + x - Tx - (y - Ty)\rangle$

$\Leftrightarrow \|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle.$

So, we get the conclusion. ☐

CHAPTER 4

MAIN RESULT

4.1 k-strictly Pseudononspreading Mapping

Let H be a real Hilbert space, following the terminology of Browder-Petryshyn, we say that a mapping $T : D(T) \subseteq H \longrightarrow H$ is k-strictly pseudononspreading if there exists $k \in [0, 1)$ such that $\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle \quad \forall \; x, y \in D(T)$.

Clearly, every nonspreading mapping is 0-strictly pseudononspreading. The following example shows that the class of k-strictly pseudononspreading is a generalization of the class of nonspreading mappings.

Example 4.1. Let $\mathbb{R}$ denote reals with the usual norm, let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$Tx = \begin{cases} x, & x \in (-\infty, 0) \\ -2x, & x \in [0, \infty) \end{cases}$$

Claim 1: T is k-strictly pseudononspreading.

Case 1: $x, y \in (-\infty, 0)$

$Tx = x, \qquad Ty = y$

$|Tx - Ty|^2 = |x - y|^2$ and $k|x - Tx - (y - Ty)| = 2\langle x - Tx, y - Ty\rangle = 0$

Hence,

$|Tx - Ty|^2 = |x - y|^2 = |x - y|^2 + \frac{1}{3}|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle$.

Case 2: $x, y \in [0, \infty)$

$Tx = -2x, \qquad Ty = -2y, \quad |Tx - Ty|^2 = |-2x + 2y|^2 = 4|x - y|^2$

and

$\frac{1}{3}|x - Tx - (y - Ty)|^2 = \frac{1}{3}|x + 2x - (y + 2y)|^2 = \frac{1}{3}|3(x - y)|^2 = 9\frac{1}{3}|x - y|^2$

$2\langle x - Tx, y - Ty\rangle = 2\langle x + 2x, y + 2y\rangle = 2\langle 3x, 3y\rangle = 18xy \geq 0$,

$\Rightarrow 2\langle x - Tx, y - Ty\rangle \geq 0.$

Thus,

$$\begin{aligned}
|Tx - Ty|^2 &= 4|x-y|^2 \\
&= |x-y|^2 + \tfrac{9}{3}|x-y|^2 \\
&= |x-y|^2 + \tfrac{1}{3}(9|x-y|^2) \\
&= |x-y|^2 + \tfrac{1}{3}|x - Tx - (y - Ty)|^2 \\
&\leq |x-y|^2 + \tfrac{1}{3}|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle,
\end{aligned}$$

Hence,

$$|Tx - Ty|^2 \leq \|x-y\|^2 + \tfrac{1}{3}|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle$$

Case 3: $x \in (-\infty, 0), \quad y \in [0, \infty) \quad Tx = x, \qquad Ty = -2y$

$$\begin{aligned}
|Tx - Ty|^2 &= |x + 2y|^2 \\
&= x^2 + 4xy + 4y^2
\end{aligned}$$

$$2\langle x - Tx, y - Ty\rangle = 2\langle x - x, y + 2y\rangle = 0$$

$$\begin{aligned}
\tfrac{1}{3}|x - Tx - (y - Ty)|^2 &= \tfrac{1}{3}|x - x - (y + 2y)|^2 \\
&= \tfrac{1}{3}|3y|^2 \\
&= \tfrac{1}{3}(9y^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
|x-y|^2 + \tfrac{1}{3}|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle &= x^2 - 2xy + 4y^2 \\
&= x^2 + 4xy + 4y^2 - 6xy \\
&\geq x^2 + 4xy + 4y^2 \text{ (since } -6xy \geq 0) \\
&= (x + 2y)^2 \\
&= |x + 2y|^2 \\
&= |x - (-2y)|^2 \\
&= |Tx - Ty|^2.
\end{aligned}$$

Hence,

$\forall \ x, y \in \mathbb{R}, |Tx - Ty|^2 \leq |x-y|^2 + \tfrac{1}{3}|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle.$

Thus,

T is $\frac{1}{3}-$strictly pseudononspreading mapping.

Claim 2: T is not nonspreading

To see this we take $x = 0, y = 1$, we have

$|Tx - Ty|^2 = |0 + 2|^2 = 4$, and $|x - y|^2 = 1$

$|Tx - Ty|^2 = 4 > 1 = |x - y|^2 + 2\langle x - Tx, y - Ty\rangle$

Hence, T is not a nonspreading mapping.

The class of nonspreading mappings is properly contained in the class of k-strictly pseudononspreading mappings and, the class of k-strictly pseudononspreading mappings also contains the class of firmly nonexpansive mappings.

4.2 Properties of k-strictly Pseudononspreading Mappings

Proposition 4.2. If T is k-strictly pseudononspreading mapping and $F(T)$ is nonempty, then T is demicontractive.

Proof. Suppose T is k-strictly pseudononspreading mapping and $F(T) \neq \emptyset$.
Let $x \in D(T)$ and $p \in F(T)$ then,
$\|Tx - Ty\|^2 \leq \|x - p\|^2 + k\|x - Tx - (p - Tp)\|^2 + 2\langle x - Tx, p - Tp\rangle$
$\Rightarrow \|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \quad$ for some $k \in [0, 1)$,
$\Rightarrow T$ is demicontractive. □

Proposition 4.3. Let C be a nonempty closed and convex subset of a real Hilbert space H, let $T : C \to C$ be a k-strictly pseudononspreading mapping, if $F(T) \neq \emptyset$ then $F(T)$ is closed and convex.

Proof. Now, we assume $F(T) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty} \subseteq F(T)$ converges to $x \in C$.
Claim 1: $x \in F(T)$
We observe that,

$$\begin{aligned}\|Tx - x\| &= \|Tx - x_n + x_n - x\| \\ &\leq \|Tx - x_n\| + \|x_n - x\| \\ &= \|Tx - Tx_n\| + \|x_n - x\| \qquad (\text{since } x_n \in F(T) \qquad \forall n \in \mathbb{N})\end{aligned}$$

Since T is k-strictly pseudononspreading mapping we have,

$$\begin{aligned}\|Tx - Tx_n\|^2 &\leq \|x - x_n\|^2 + k\|x - Tx - (x_n - Tx_n)\|^2 + 2\langle x - Tx, x_n - Tx_n\rangle \\ &= \|x - x_n\|^2 + k\|x - Tx\|^2 \\ &\leq (\|x - x_n\| + \sqrt{k}\|x - Tx\|)^2\end{aligned}$$

$\Rightarrow \|Tx - Tx_n\| \leq \|x - x_n\| + \sqrt{k}\|x - Tx\|$
Therefore,
$\|Tx - x\| \leq 2\|x - x_n\| + \sqrt{k}\|x - Tx\|$
$\Rightarrow (1 - k^{\frac{1}{2}})\|Tx - x\| \leq 2\|x_n - x\|$
$\Rightarrow \|Tx - x\| \leq \frac{2}{1-\sqrt{k}}\|x_n - x\| \to 0$ as $n \to \infty$
$\Rightarrow \|Tx - x\| = 0$
$\Rightarrow Tx = x$
$\Rightarrow x \in F(T)$
Hence $F(T)$ is closed.

Claim 2: $F(T)$ is convex

To show $F(T)$ is convex we consider two cases:

Case 1 $F(T)$ is a singleton set then $F(T)$ is convex.

Case 2 $F(T)$ is not a singleton set.

Let $p_1, p_2 \in F(T)$ and $\lambda \in [0,1]$ we show that $\lambda p_1 + (1-\lambda)p_2 \in F(T)$.

Now, we let $z = \lambda p_1 + (1-\lambda)p_2$ then $p_1 - z = (1-\lambda)(p_1 - p_2)$ and $p_2 - z = \lambda(p_2 - p_1)$.

$$\|z - Tz\|^2 = \|\lambda p_1 + (1-\lambda)p_2 - Tz\|^2$$
$$= \|\lambda(p_1 - Tz) + (1-\lambda)(p_2 - Tz)\|^2$$
$$= \lambda\|p_1 - Tz\|^2 + (1-\lambda)\|p_2 - Tz\|^2 - \lambda(1-\lambda)\|p_1 - p_2\|^2$$
$$\leq \lambda(\|p_1 - z\|^2 + k\|z - Tz\|^2) + (1-\lambda)(\|p_2 - z\|^2 + k\|z - Tz\|^2) - \lambda(1-\lambda)\|p_1 - p_2\|^2$$
$$= \lambda(\|(1-\lambda)(p_1 - p_2)\|^2 + k\|z - Tz\|^2) + (1-\lambda)\|\lambda(p_1 - p_2)\|^2 + k\|z - Tz\|^2) - \lambda(1-\lambda)\|p_1 - p_2\|^2$$
$$= \lambda(1-\lambda)^2\|p_1 - p_2\|^2 + \lambda k\|z - Tz\|^2 + \lambda^2(1-\lambda)\|p_1 - p_2\|^2 + (1-\lambda)k\|z - Tz\|^2 - \lambda(1-\lambda)\|p_1 - p_2\|^2$$
$$= (\lambda - 2\lambda^2 + \lambda^3 - \lambda^2 - \lambda^3 - \lambda + \lambda^2)\|p_1 - p_2\|^2 + k\|z - Tz\|^2$$
$$= (0)\|p_1 - p_2\|^2 + k\|z - Tz\|^2$$
$$\Rightarrow \|z - Tz\|^2 \leq k\|z - Tz\|^2$$
$$\Rightarrow (1-k)\|z - Tz\|^2 \leq 0$$

since $1 - k > 0$ and $\|z - Tz\|^2 \geq 0$, we obtain

$$(1-k)\|z - Tz\|^2 = 0$$

But $1 - k > 0$

$$\Rightarrow \|z - Tz\|^2 = 0 \Rightarrow z \in F(T).$$

Hence, $F(T)$ is convex. □

4.2.1 Proposition 3

Let C be a nonempty closed and convex subset of a real Hilbert space H and let $T : C \to C$ be a k-strictly pseudononspreading mapping then $(I - T)$ is demiclosed at 0.

Proof

Let $\{x_n\}_{n=1}^{\infty} \subseteq C$ which converges weakly to some $p \in C$ and $\{x_n - Tx_n\}_{n=1}^{\infty}$ converges strongly to 0, we show that $p \in F(T)$.

Since $x_n \rightharpoonup p$ as $n \to \infty \Rightarrow \{\|x_n\|\}_{n=1}^{\infty}$ is bounded, so $\limsup\limits_{n\to\infty}\|x_n\| \in \mathbb{R}$

we define

$f : H \longrightarrow [0, \infty)$ by

$$f(x) = \limsup_{n\to\infty}\|x_n - x\|^2 \quad \forall x \in H$$

By Lemma 1.1(iii) we obtain

$$f(x) = \limsup_{n\to\infty}\|x_n - p\|^2 + \|p - x\|^2 \quad \forall x \in H$$

Thus,

$f(x) = f(p) + \|p - x\|^2 \quad \forall x \in H$ and $f(Tp) = f(p) + \|p - Tp\|^2$

We also observe that

$$\begin{aligned} f(Tp) &= \limsup_{n\to\infty} \|x_n - Tp\|^2 \\ &= \limsup_{n\to\infty} \|x_n - Tx_n + Tx_n - Tp\|^2 \\ &= \limsup_{n\to\infty} (\|x_n - Tx_n\|^2 + \|Tx_n - Tp\|^2 + 2\langle x_n - Tx_n, Tx_n - Tp\rangle) \end{aligned}$$

Since $x_n - Tx_n \to 0 \Rightarrow x_n - Tx_n \rightharpoonup 0$ as $n \to \infty$

$$\Rightarrow \langle x_n - Tx_n, Tx_n - Tp\rangle \to 0 \text{ as } n \to \infty$$

Also, since $x_n - Tx_n \to 0 \Rightarrow \|x_n - Tx_n\|^2 \to 0$ as $n \to \infty$

We have,

$$\begin{aligned} f(Tp) &= \limsup_{n\to\infty} \|Tx_n - Tp\|^2 \\ &\leq \limsup_{n\to\infty} (\|x_n - p\|^2 + k\|x_n - Tx_n - (p - Tp)\|^2 + 2\langle x_n - Tx_n, p - Tp\rangle) \\ &= \limsup_{n\to\infty} \|x_n - p\|^2 + k\|p - Tp\|^2 \\ &= f(p) + k\|p - Tp\|^2 \end{aligned}$$

$\Rightarrow f(Tp) \leq f(p) + k\|p - Tp\|^2$

$\Rightarrow f(p) + \|p - Tp\|^2 \leq f(p) + k\|p - Tp\|^2$

$\Rightarrow (1 - k)\|p - Tp\|^2 \leq 0 \Rightarrow (1 - k)\|p - Tp\|^2 = 0$

$\Rightarrow \|p - Tp\|^2 = 0$

$\Rightarrow p = Tp$

$\Rightarrow p \in F(T)$.

Remark

We used proposition 3 to show that the weak limit of a sequence in C is a fixed point of T.

4.2.2 Proposition 4

Let C be a nonempty subset of H, let $T : C \to C$ be a map with nonempty fixed point set, let $\beta \in \mathbb{R}$ and let $T_\beta : C \to C$ be a map defined by $T_\beta x = \beta x + (1 - \beta)Tx$, then $F(T_\beta) = F(T)$.

Proof. $F(T_\beta) = \{x \in C : T_\beta x = x\}$

$$\begin{aligned} &= \{x \in C : \beta x + (1 - \beta)Tx = x\} \\ &= \{x \in C : (1 - \beta)Tx = (1 - \beta)x\} \\ &= \{x \in C : Tx = x\} \\ &= F(T) \end{aligned}$$

Hence, $F(T_\beta) = F(T)$. □

4.3 Weak and strong convergence theorem for nonspreading-type mappings in a Hilbert space

Theorem 4.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $T : C \longrightarrow C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set $F(T)$. Let $\beta \in [k, 1)$ and let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)[\beta x_n + (1-\beta)Tx_n] & n \geq 1, \\ z_n = \frac{1}{n}\sum_{k=1}^{\infty} x_k & n \geq 1. \end{cases} \tag{4.1}$$

Then $\{z_n\}_{n=1}^{\infty}$ converges weakly to $z \in F(T)$ where $z = \lim_{n\to\infty} P_{F(T)}x_n$.
In particular, $\forall x \in C$, if we define $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T_\beta^k x, \quad n \geq 1$
Where,
$T_\beta = \beta I + (1-\beta)T$
Then
$\{S_n x\}_{n=1}^{\infty}$ converges weakly to $z \in F(T)$ where $z = \lim_{n\to\infty} P_{F(T)}T_\beta^n x$.

Proof. Let $T_\beta x = \beta x + (1-\beta)Tx$ then $\forall x, y \in C$, we have

$$\begin{aligned} \|T_\beta x - T_\beta y\|^2 &= \|\beta x + (1-\beta)Tx - (\beta y + (1-\beta)Ty)\|^2 \\ &= \|\beta(x-y) + (1-\beta)(Tx-Ty)\|^2 \\ &= \beta\|x-y\|^2 + (1-\beta)\|Tx-Ty\|^2 - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \\ &\leq \beta\|x-y\|^2 + (1-\beta)(\|x-y\|^2 + k\|x-Tx-(y-Ty)\|^2 + 2\langle x-Tx, y-Ty\rangle) \\ &\quad - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2 \end{aligned}$$

(since T is K-strictly pseudononspreading mapping)

$$\begin{aligned} &= \beta\|x-y\|^2 + (1-\beta)\|x-y\|^2 + k(1-\beta)\|x-Tx-(y-Ty)\|^2 + 2(1-\beta)\langle x-Tx, y \\ &\quad - Ty\rangle - \beta(1-\beta)\|x-Tx-(y-Ty)\|^2. \end{aligned}$$

Thus, $\|T_\beta x - T_\beta y\|^2 \leq \|x-y\|^2 - (1-\beta)(\beta-k)\|x-Tx-(y-Ty)\|^2 + 2(1-\beta)\langle x-Tx, y-Ty\rangle$

$$\|T_\beta x - T_\beta y\|^2 \leq \|x-y\|^2 + 2(1-\beta)\langle x-Tx, y-Ty\rangle \tag{4.2}$$

Claim: $(1-\beta)\langle x-Tx, y-Ty\rangle = \frac{1}{1-\beta}\langle x-T_\beta x, y-T_\beta y\rangle$

Proof of claim

$(1-\beta)\langle x-Tx, y-Ty\rangle = \frac{1}{1-\beta}\langle x-T_\beta x, y-T_\beta y\rangle \Leftrightarrow \langle x-T_\beta x, y-T_\beta y\rangle = (1-\beta)^2\langle x-Tx, y-Ty\rangle$

Now,

$$\begin{aligned}(1-\beta)^2\langle x-Tx, y-Ty\rangle &= \langle (1-\beta)x-Tx, (1-\beta)y-Ty\rangle \\ &= \langle x-Tx-\beta x+\beta Tx, y-Ty-\beta y+\beta Ty\rangle \\ &= \langle x-\beta x-Tx+\beta Tx, y-\beta y-\beta Ty\rangle \\ &= \langle x-\beta x-(1-\beta)Tx, y-\beta y-(1-\beta)Ty\rangle \\ &= \langle x-(\beta x+(1-\beta)Tx), y-(\beta y+(1-\beta)Ty\rangle \\ &= \langle x-T_\beta x, y-T_\beta y\rangle\end{aligned}$$

Hence,

$(1-\beta)\langle x-Tx, y-Ty\rangle = \frac{1}{1-\beta}\langle x-T_\beta x, y-T_\beta y\rangle.$

$$\Rightarrow 2(1-\beta)\langle x-Tx, y-Ty\rangle = \frac{2}{1-\beta}\langle x-T_\beta x, y-T_\beta y\rangle \tag{4.3}$$

By substituting (4.3) in (4.2) we obtain

$$\|T_\beta x-T_\beta y\|^2 \leq \|x-y\|^2 + \frac{2}{1-\beta}\langle x-T_\beta x, y-T_\beta y\rangle \tag{4.4}$$

Since $F(T) \neq \emptyset$ and by (4.4) we have that $\forall p \in F(T)$

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n x_n + (1-\alpha_n)T_\beta x_n - p\| \\ &= \|\alpha_n(x_n-p) + (1-\alpha_n)(T_\beta x_n - p)\| \\ &\leq \alpha_n\|x_n-p\| + (1-\alpha_n)\|T_\beta x_n - p\| \\ &= \alpha_n\|x_n-p\| + (1-\alpha_n)\|T_\beta x_n - T_\beta p + T_\beta p - p\|\end{aligned}$$

since $F(T_\beta) = F(T) \Rightarrow T_\beta p = p$

$$\Rightarrow \|x_{n+1}-p\| \leq \alpha_n\|x_n-p\| + (1-\alpha_n)\|T_\beta x_n - T_\beta p\| \tag{4.5}$$

Claim: $\|T_\beta x_n - T_\beta p\| \leq \|x_n - p\|$

Proof of claim

$$\begin{aligned}\|T_\beta x_n - T_\beta p\|^2 &\leq \|x_n-p\|^2 + \tfrac{2}{1-\beta}\langle x_n - T_\beta x_n, p - T_\beta p\rangle \\ &= \|x_n-p\|^2\end{aligned}$$

$$\|T_\beta x_n - T_\beta p\| \leq \|x_n - p\| \tag{4.6}$$

By substituting (4.6) in (4.5) we obtain

$\|x_{n+1}-p\| \leq \alpha_n\|x_n-p\| + (1-\alpha_n)\|x_n-p\| = \|x_n-p\|$

Hence,

$$\|x_{n+1}-p\| \leq \|x_n-p\| \qquad \forall n \geq 1 \tag{4.7}$$

We observe that

$\|x_{n+1}-p\| \leq \|x_n-p\| \quad \Rightarrow \|x_{n+1}-p\| \leq \|x_1-p\|$

Set $M_0 = \|x_1 - p\| + 1 \quad \Rightarrow M_0 > 0$

So we have $\|x_{n+1} - p\| \leq M_0 \quad \forall n \geq 1$

Claim: $\{x_n\}_{n=1}^{\infty}$ is bounded.

$\|x_n\| \leq \|x_{n-1} - p\| + \|p\|$

$\quad \leq \|x_1 - p\| + \|p\|$

$M_0 + \|p\| = M \quad \forall n \geq 1$

$\Rightarrow \{x_n\}_{n=1}^{\infty}$ is bounded.

Using lemma(1) and (3.4) we obtain $\quad \forall k \geq 1 \quad$ and $\forall y \in C$

$\|x_{k+1} - T_\beta y\|^2 = \|\alpha_k(x_k - T_\beta y) + (1-\alpha_k)(T_\beta x_k - T_\beta y)\|^2$

$\quad = \alpha_k\|x_k - T_\beta y\|^2 + (1-\alpha_k)\|T_\beta y - T_\beta x_k\|^2 - \alpha_k(1-\alpha_k)\|x_k - T_\beta x_k\|^2$

$$\|x_{k+1} - T_\beta y\|^2 \leq \alpha_k\|x_k - T_\beta y\|^2 + (1-\alpha_k)(\|x_k - y\|^2 + \frac{2}{1-\beta}\langle y - T_\beta y, x_k - T_\beta x_k\rangle) \tag{4.8}$$

Since,

$\|x_k - y\|^2 = \|x_k - T_\beta y + T_\beta y - y\|^2$

$$\Rightarrow \|x_k - y\|^2 = \|x_k - T_\beta y\|^2 + \|T - \beta y - y\|^2 + 2\langle x_k - T_\beta y, T_\beta y - y\rangle \tag{4.9}$$

By substituting (4.9) in (4.8) we have

$\|x_{k+1} - T_\beta y\|^2 \leq \alpha_k\|x_k - T_\beta y\|^2 + (1-\alpha_k)([\|x_k - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle x_k - T_\beta y, T_\beta y - y\rangle]$
$\quad + \frac{2}{1-\beta}\langle x_k - T_\beta x_k, y - T_\beta y\rangle)$

$\quad \leq \alpha_k\|x_k - T_\beta y\|^2 + (1-\alpha_k)\|x_k - T_\beta y\|^2 + (1-\alpha_k)\|T_\beta y - y\|^2 + 2(1-\alpha_k)\langle x_k -$
$T_\beta y, T_\beta y - y\rangle + \frac{2(1-\alpha_k)}{1-\beta)}\langle x_k - T_\beta x_k, y - T_\beta y\rangle$

$\quad = \|x_k - T_\beta y\|^2 + (1-\alpha_k)\|T_\beta y - y\|^2 + \frac{2}{1-\beta}\langle(1-\alpha_k)(x_k - T_\beta x_k), y - T_\beta y\rangle + (2 -$
$2\alpha_k)\langle x_k - T_\beta y, T_\beta y - y\rangle$

$\quad = \|x_k - T_\beta y\|^2 + (1-\alpha_k)\|T_\beta y - y\|^2 + \frac{2}{1-\beta}\langle(1-\alpha_k)(x_k - T_\beta x_k), y - T_\beta y\rangle + 2\langle x_k -$
$T_\beta y, T_\beta y - y\rangle - 2\alpha_k\langle x_k - T_\beta y, T_\beta y - y\rangle$

So,

$$\begin{aligned}\|x_{k+1} - T_\beta y\|^2 \leq\ & \|x_k - T_\beta y\|^2 + (1-\alpha_k)\|T_\beta y - y\|^2 + 2\langle x_k - T_\beta y, T_\beta y - y\rangle - 2\alpha_k\langle x_k - T_\beta y, T_\beta y - y\rangle \\ & + \frac{2}{1-\beta}\langle(1-\alpha_k)(x_k - T_\beta x_k), y - T_\beta y\rangle\end{aligned} \tag{4.10}$$

We observe that,

$$x_k - x_{k+1} = (1-\alpha_k)(x_k - T_\beta x_k) \tag{4.11}$$

Substituting (4.11) in (4.10) yields

$$\begin{aligned}\|x_{k+1} - T_\beta y\|^2 \leq\ & \|x_k - T_\beta y\|^2 + (1-\alpha_k)\|T_\beta y - y\|^2 + 2\langle x_k - T_\beta y, T_\beta y - y\rangle - 2\alpha_k\langle x_k - T_\beta y, T_\beta y\rangle \\ & + \frac{2}{1-\beta}\langle x_k - x_{k+1}, y - T_\beta y\rangle\end{aligned} \tag{4.12}$$

Summing (4.12) from $k=1$ to n and dividing through by n we obtain

$$\frac{1}{n}\sum_{k=1}^{n}\|x_{k+1}-T_\beta y\|^2 \le \frac{1}{n}\sum_{k=1}^{n}\|x_k-T_\beta y\|^2 + \frac{1}{n}\sum_{k=1}^{n}\|T_\beta y-y\|^2 + \frac{2}{n(1-\beta)}\sum_{k=1}^{n}\langle x_k-x_{k+1}, y-T_\beta y\rangle + \frac{2}{n}\sum_{k=1}^{n}\langle x_k-T_\beta y, T_\beta y-y\rangle - \frac{2\alpha_k}{n}\sum_{k=1}^{n}\langle x_k-T_\beta y, T_\beta y-y\rangle$$

$$\begin{aligned}\Rightarrow \frac{1}{n}\|x_{n+1}-T_\beta y\|^2 &\le \frac{1}{n}\|x_1-T_\beta y\|^2 + \|T_\beta y-y\|^2 + \frac{2}{1-\beta}\langle\frac{x_1}{n}-\frac{x_{n+1}}{n}y-T_\beta y\rangle \\ &+ 2\langle z_n-T_\beta y, T_\beta y-y\rangle - \frac{2}{n}\sum_{k=1}^{n}\alpha_k\langle x_k-T_\beta y, T_\beta y-y\rangle\end{aligned} \tag{4.13}$$

Claim: $\{z_n\}_{n=1}^{\infty}$ is also bounded.

$$\begin{aligned}\|z_n\| = \|\tfrac{1}{n}\sum_{k=1}^{n}x_k\| &\le \tfrac{1}{n}\sum_{k=1}^{n}\|x_k\| \\ &\le \tfrac{1}{n}\sum_{k=1}^{n}M \qquad \text{(since } \{x_n\}_{n=1}^{\infty} \text{ is bounded)} \\ &= M\end{aligned}$$

Hence, $\|z_n\| \le M \qquad \forall n\ge 1$

$\Rightarrow \{z_n\}_{n=1}^{\infty}$ is also bounded since $\{z_n\}_{n=1}^{\infty}$ is in H there exists a subsequence $\{z_{n_j}\}_{j=1}^{\infty} \subseteq \{z_n\}_{n=1}^{\infty}$ such that $z_{n_j} \to w$ as $n\to\infty$

Replacing n by n_j in (4.13) yields

$$\begin{aligned}\Rightarrow \frac{1}{n_j}\|x_{n_j+1}-T_\beta y\|^2 &\le \frac{1}{n_j}\|x_1-T_\beta y\|^2 + \|T_\beta y-y\|^2 + \frac{2}{1-\beta}\langle\frac{x_1}{n_j}-\frac{x_{n_j+1}}{n_j}, y-T_\beta y\rangle \\ &+ 2\langle z_{n_j}-T_\beta y, T_\beta y-y\rangle - \frac{2}{n_j}\sum_{k=1}^{n_j}\alpha_k\langle x_k-T_\beta y, T_\beta y-y\rangle\end{aligned} \tag{4.14}$$

Claim: $\lim_{n\to\infty}\alpha_k\langle x_k-T_\beta y, T_\beta y-y\rangle = 0$

Proof of claim

$$\begin{aligned}|\alpha_k\langle x_k-T_\beta y, T_\beta y-y\rangle| &\le \alpha_k\|x_k-T_\beta y\|\|T_\beta y-y\| \\ &\le \alpha_k M \qquad \text{(since } \{x_n\}_{n=1}^{\infty} \text{ is bounded)}\end{aligned}$$

$\Rightarrow |\alpha_k\langle x_k-T_\beta y, T_\beta y-y\rangle| \le \alpha_k M \to 0$ as $k\to\infty$ (since $\alpha_k\to 0$ as $k\to\infty$).

Since $z_{n_j} \rightharpoonup w$ as $j\to\infty$ and $\lim_{j\to\infty}\frac{1}{n_j}\sum_{k=1}^{n_j}\alpha_k\langle x_k-T_\beta y, T_\beta y-y\rangle$

we obtain from (4.14) that

$$0 \le \|T_\beta y-y\|^2 + 2\langle w-T_\beta y, T_\beta y-y\rangle \tag{4.15}$$

Since $y\in H$ is arbitrary, setting $y=w$ in (4.15) we obtain,

$0 \le \|T_\beta w-w\|^2 + 2\langle w-T_\beta w, T_\beta w-w\rangle$

$\Rightarrow 0 \le \|T_\beta w-w\|^2 - 2\|T_\beta w-w\|^2 = -\|T_\beta w-w\|^2$

$\Rightarrow \|T_\beta w-w\|^2 = 0 \Rightarrow T_\beta w = w$

$\Rightarrow w\in F(T_\beta) = F(T)$.

Since T is k-strictly pseudononspreading and $F(T)\ne\emptyset$ then it follows from proposition 2 that

$F(T)$ is closed and convex.Thus we can define the projection $P_{F(T)}H :\longrightarrow F(T)$

Since $\|x_{n+1} - p\| \leq \|x_n - p\| \quad \forall p \in F(T)$

It follows from Lemma(3.18) that $\{P_{F(T)}x_n\}_{n=1}^{\infty}$ converges strongly in $F(T)$

Let $\lim_{n\to\infty} P_{F(T)}x_n = z \in F(T)$.

We show that $z = w$ since $w \in F(T)$ using (3.2) we obtain

$$\begin{aligned}\langle w - z, x_k - P_{F(T)}x_k\rangle &= \langle w - P_{F(T)}x_k, x_k - P_{F(T)}x_k\rangle + \langle P_{F(T)}x_k - z, x_k - P_{F(T)}x_k\rangle \\ &\leq \langle P_{F(T)}x_k - z, x_k - P_{F(T)}x_k\rangle \qquad (\text{since } \langle w - P_{F(T)}x_k, x_k - P_{F(T)}x_k\rangle \leq 0)\end{aligned}$$

Hence,

$\langle w - z, x_k - P_{F(T)}x_k\rangle \leq \langle P_{F(T)}x_k - z, x_k - P_{F(T)}x_k\rangle$

$$\langle w - z, x_k - P_{F(T)}x_k\rangle \leq \|x_k - P_{F(T)}x_k\|\|P_{F(T)}x_k - z\|. \tag{4.16}$$

Since$\{x_n\}_{k=1}^{\infty}$ $\{P_{F(T)}x_n\}_{k=1}^{\infty}$ are bounded we have

$$\begin{aligned}\|x_k - P_{F(T)}x_k\| &\leq \|x_k\| + \|P_{F(T)}x_k\| \\ &\leq M_1 + M_2 = M \qquad \forall k \geq 1 \quad \text{for some } M_1 > 0 \text{ and } M_2 > 0\end{aligned}$$

Hence

$$\|x_k - P_{F(T)}x_k\| \leq M \qquad \forall k \geq 1. \tag{4.17}$$

By substituting (4.17) in (4.16) we obtain

$$\langle w - z, x_k - P_{F(T)}x_k\rangle \leq \|P_{F(T)}x_k - z\|M \tag{4.18}$$

Summing (4.18) from $k = 1$ to n_j and dividing by n_j we obtain

$\langle w - z, z_{n_j} - \frac{1}{n_j}\sum_{k=1}^{n_j} P_{F(T)}x_k\rangle \leq \frac{M}{n_j}\sum_{k=1}^{n_j}\|P_{F(T)}x_k - z\|$

since $z_{n_j} \to w$ as $j \to \infty$ and $P_{F(T)}x_n \to z$ as $n \to \infty$

We have,

$$\begin{aligned}\langle w - z, w - z\rangle \leq 0 &\Rightarrow \|w - z\|^2 \leq 0 \\ &\Rightarrow w = z\end{aligned}$$

if we set $\alpha_n = 0 \quad \forall n \geq 1$ in (4.1) we obtain

$$\begin{cases} x_{n+1} = \beta x_n + (1-\beta)Tx_n & n \geq 1, \\ z_n = \frac{1}{n}\sum_{k=1}^{\infty} x_k = \frac{1}{n}\sum_{k=1}^{\infty} T_\beta^{k-1}x_1 & n \geq 1. \end{cases}$$

Hence for any $x = x_1 \in C$,if we define

Then $\{S_n\}_{n=1}^{\infty}$ converges weakly to $z \in F(T)$ where $z = \lim_{n\to\infty} P_{F(T)}T_\beta^n x$.

This completes the proof of theorem(4.1). □

Remark 1:

For nonspreading mapping T, $k = 0$ and we can take $\beta = 0$ in our theorem 3.1 to obtain theorem 3.1 of [5] as a corollary.

Theorem 4.5. *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $T : C \longrightarrow C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set $F(T)$, let $\beta \in [0,1)$ and let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{k=1}^{\infty} \alpha_n = \infty$. Let $u \in C$ and let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n & n \geq 1, \\ z_n = \frac{1}{n}\sum_{k=0}^{n-1} T_\beta^k x_n & n \geq 1. \end{cases}$$

Then $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $P_{F(T)}u$ where $P_{F(T)} : H \longrightarrow F(T)$ is the metric projection of H onto $F(T)$.

Proof. Let $p \in F(T)$ be arbitrary then

$$\begin{aligned} \|z_n - p\| &= \|\tfrac{1}{n}\sum_{k=0}^{n-1} T_\beta^k x_n - p\| \\ &\leq \tfrac{1}{n}\sum_{k=0}^{n-1} \|T_\beta^k - p\| \\ &\leq \tfrac{1}{n}\sum_{k=0}^{n-1} \|T_\beta^k x_n - T_\beta^k p\| \\ &\leq \tfrac{1}{n}\sum_{k=0}^{n-1} \|x_n - p\| \\ &= \|x_n - p\| \end{aligned}$$

Hence,

$$\|z_n - p\| \leq \|x_n - p\| \quad \forall n \geq 1. \tag{4.19}$$

Thus,
$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n u + (1-\alpha_n) z_n - p\| \\ &= \|\alpha_n (u-p) + (1-\alpha_n)(z_n - p)\| \\ &\leq \alpha_n \|u-p\| + (1-\alpha_n)\|z_n - p\| \\ &\leq \alpha_n \|u-p\| + (1-\alpha_n)\|x_n - p\| \qquad \text{(from 4.19)}. \end{aligned}$$

Since

$\|x_1 - p\| \leq \max\{\|u-p\|, \|x_1 - p\|\}$

and the assumption that

$\|x_n - p\| \leq \max\{\|u-p\|, \|x_1 - p\|\} \quad \forall n \geq 1$

Thus, $\{x_n\}_{n=1}^{\infty}$ is bounded. Also since

$\|z_n - p\| \leq \|x_n - p\| \leq \max\{\|u-p\|, \|x_1 - p\|\} \quad \forall n \geq 1$

$\Rightarrow \{z_n\}_{n=1}^{\infty}$ is also bounded.

Since, $\|T_\beta^n x_n - p\|^2 = \|T_\beta^n x_n - T_\beta^n p + T_\beta^n p\|^2$

$$= \|T_\beta^n x_n - T_\beta^n p\|^2$$
$$\leq \|x_n - p\|^2 \qquad \text{(from 4.4)}$$

Hence, $\|T_\beta^n x_n - p\| \leq \|x_n - p\| \quad \forall n \geq 1$, we have that $\{T_\beta^n x_n\}_{n=1}^\infty$ is also bounded. Thus for arbitrary $y \in C$ and $\forall k = 0, 1, 2, ..., n-1$ we have

$$\|T_\beta^{k+1} x_n - T_\beta y\|^2 = \|T_\beta(T_\beta^k x_n) - T_\beta y\|^2$$
$$\leq \|T_\beta^k x_n - y\|^2 + \tfrac{2}{(1-\beta)}\langle T_\beta^k x_n - T_\beta^{k+1} x_n, y - T_\beta y\rangle$$
$$= \|T_\beta^k x_n - T_\beta y + T_\beta y - y\|^2 + \tfrac{2}{(1-\beta)}\langle T_\beta^k x_n - T_\beta^{k+1} x_n, y - T_\beta y\rangle$$
$$= \|T_\beta^k x_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle T_\beta^k x_n - T_\beta y, y - y - T_\beta y\rangle + \tfrac{2}{1-\beta}\langle T_\beta^k x_k - T_\beta^{k+1} x_n, y - T_\beta y\rangle$$

Thus

$$\|T_\beta^{k+1} x_n - T_\beta y\|^2 \leq \|T_\beta^k x_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle T_\beta^k x_n - T_\beta y, y - y - T_\beta y\rangle$$
$$+ \frac{2}{1-\beta}\langle T_\beta^k x_k - T_\beta^{k+1} x_n, y - T_\beta y\rangle \tag{4.20}$$

Summing (4.20) from $k = 0$ to $n-1$ and dividing through by n we obtain:

$$\frac{1}{n}\|T_\beta^n x_n - T_\beta y\|^2 \leq \frac{1}{n}\|x_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_n - T_\beta y, T_\beta y - y\rangle \quad + \frac{2}{n(1-\beta)}\langle x_n - T_\beta^n, T_\beta y - y\rangle \tag{4.21}$$

Since $\{z_n\}_{n=1}^\infty$ is bounded then there exists $\{z_{n_j}\}_{j=1}^\infty$ of $\{z_n\}_{n=1}^\infty$ which converges weakly to $w \in C$, i.e $z_{n_j} \rightharpoonup w$ as $j \rightharpoonup \infty$

Replacing n by n_j in (4.21) we obtain

$$\frac{1}{n_j}\|T_\beta^{n_j} x_{n_j} - T_\beta y\|^2 \leq \frac{1}{n_j}\|x_{n_j} - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_{n_j} - T_\beta y, T_\beta y - y\rangle$$
$$+ \frac{2}{n_j(1-\beta)}\langle x_{n_j} - T_\beta^{n_j}, T_\beta y - y\rangle \tag{4.22}$$

Since $\{x_n\}_{n=1}^\infty$ and $\{T_\beta^n x_n\}_{n=1}^\infty$ are bounded, letting $j \to \infty$ in (4.22) yields

$$0 \leq \|T_\beta y - y\|^2 + 2\langle w - T_\beta y, T_\beta y - y\rangle \tag{4.23}$$

Since $y \in C$ is arbitrary if we set $y = w$ in (4.23) we obtain

$$0 \leq \|T_\beta w - w\|^2 + 2\langle w - T_\beta w, T_\beta w - w\rangle$$
$$= \|T_\beta w - w\|^2 - 2\langle w - T_\beta w - w, T_\beta w - w\rangle$$
$$= -\|T_\beta w - w\|^2$$
$$\Rightarrow 0 \leq -\|T_\beta w - w\|^2 \Rightarrow \|T_\beta w - w\|^2 \leq 0$$
$$\Rightarrow \|T_\beta w - w\|^2 = 0 \Rightarrow T_\beta w = w$$
$$\Rightarrow w \in F(T_\beta) = F(T).$$

Hence, $w \in F(T)$.

Observe that since $\{z_n\}_{n=1}^\infty$ is bounded and $\lim_{n\to\infty} \alpha_n = 0$ Then,

$$\|x_{n+1} - z_n\| = \|\alpha_n(u - z_n) + (1 - \alpha_n)(z_n - z_n)\|$$

$= \alpha_n\|u - z_n\| \longrightarrow 0$ as $n \longrightarrow \infty$

We may assume without loss of generality that there exists a subsequence $\{x_{n_j+1}\}_{n=1}^{\infty}$ of $\{x_{n+1}\}_{n=1}^{\infty}$ such that

$$\limsup_{n\to\infty}\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle = \lim_{j\to\infty}\langle u - P_{F(T)}u, x_{n_{j+1}} - P_{F(T)}u\rangle$$

and $x_{n_{j+1}} \rightharpoonup z$ as $j \to \infty$.

Since, $\|x_{n+1} - z_n\| \to 0$ as $n \to \infty$ then for arbitrary bounded linear functional f on H we have

$|f(z_{n_j}) - f(z)| \leq |f(z_{n_j}) - f(x_{n_{j+1}})| + |f(x_{n_{j+1}}) - f(z)|$

$= |f(z_{n_j} - x_{n_{j+1}})| + |f(x_{n_{j+1}}) - f(z)|$

$\leq \|f\|\|z_{n_j} - x_{n_{j+1}}\| + |f(x_{n_{j+1}}) - f(z)| \longrightarrow 0$ as $j \longrightarrow \infty$

$\Rightarrow z_{n_j} \rightharpoonup z$ as $j \to \infty$ and whence $z \in F(T)$.

(since $F(T)$ is closed and convex $\Rightarrow F(T)$ is weakly closed).

Since $P_{F(T)} : H \longrightarrow F(T)$ is the metric projection we have

$\lim_{j\to\infty}\langle u - P_{F(T)}u, x_{n_{j+1}} - P_{F(T)}u\rangle = \langle u - P_{F(T)}u, z - P_{F(T)}u\rangle \leq 0.$

It now follows that

$\limsup_{n\to\infty}\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \leq 0$

Using Lemma 1(ii) and (4.19) we obtain

$\|x_{n+1} - P_{F(T)}u\|^2 = \|\alpha_n(u - P_{F(T)}u) + (1 - \alpha_n)(z_n - P_{F(T)}u)\|^2$

$= \alpha_n^2\|u-P_{F(T)}u\|^2+2\alpha_n(1-\alpha_n)\langle u-P_{F(T)}u, z_n-P_{F(T)}u\rangle+(1-\alpha_n)^2\|z_n-P_{F(T)}u\|^2$

$\leq 2\alpha_n^2\|u-P_{F(T)}u\|^2+2\alpha_n(1-\alpha_n)\langle u-P_{F(T)}u, z_n-P_{F(T)}u\rangle+(1-\alpha_n)^2\|z_n-P_{F(T)}u\|^2$

$\leq (1 - \alpha_n)^2\|z_n - P_{F(T)}u\|^2 + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle$

$\|x_{n+1} - P_{F(T)}u\|^2 = (1 - \alpha_n)\|x_n - P_{F(T)}u\|^2 + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle$

Since $\lim_{n\to\infty}\alpha_n = 0, \sum_{k=1}^{\infty}\alpha_n = \infty.$ and $\limsup_{n\to\infty}\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \leq 0$

it follows from Lemma(3.19), that $\lim_{n\to\infty}\|x_n - P_{F(T)}u\| = 0$

Furthermore,

$0 \leq \|z_n - P_{F(T)}u\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - P_{F(T)}u\| \to 0$ as $n \to \infty$

Hence, $\lim_{n\to\infty}\|z_n - P_{F(T)}u\| = 0 \Rightarrow z_n \to P_{F(T)}u$ as $n \to \infty$. □

Remark 2:

For a nonspreading mapping T, we have $k = 0$ and we can choose $\beta = 0$ in Theorem 4.2 to obtain Theorem 4.1 of [Kurokawa et al., 2010].

Remark 3:

The introduction of the axillary mapping T_β also yields the following strong convergence theorem of Halpern's type for k-strictly pseudo-nonspreading mappings and hence resolves in the affirmative the open problem raised by Kurokawa and Takahashi in their final remark in [Kurokawa et al., 2010] for the case where the mapping T is averaged.

Theorem 4.6. *Let C be a nonempty closed and convex subset of a real Hilbert space H, and let $T : C \longrightarrow C$ be a k-strictly pseudo-nonspreading mapping with nonempty fixed point set, let $\beta \in [k, 1)$, let $T_\beta = \beta I + (1-\beta)T$, let $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0,1)$ satisfying the conditions:*

$C_1 : \lim_{n\to\infty} \alpha_n = 0$

$C_2 : \sum_{k=1}^{\infty} \alpha_n = \infty.$

Let u be a fixed anchor in C and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated from an arbitrary $x_1 \in C$ by

$x_{n+1} = \alpha_n u + (1-\alpha_n)T_\beta x_n, \quad n \geq 1$

Then $x_n \to p \in F(T)$.

Proof. $F(T_\beta) = F(T) \neq \emptyset$, as in the proof of 4.1 we have,

$$\begin{aligned}\forall x, y \in C\ \|T_\beta x - T_\beta y\|^2 &\leq \beta\|x-y\|^2 + (1-\beta)(\|x-y\|^2 + k\|x - Tx - (y - Ty)\|^2 + \\ &\quad 2\langle x - Tx, y - Ty\rangle) - \beta(1-\beta)\|x - Tx - (y - Ty)\|^2 \\ &= \|x-y\|^2 + 2(1-\beta)\langle x - Tx, y - Ty\rangle - (1-\beta)(\beta - k)\|x - Tx - (y-Ty)\|^2 \\ &= \|x-y\|^2 + \tfrac{2}{1-\beta}\langle x - T_\beta x, y - T_\beta y\rangle - \tfrac{\beta-k}{1-\beta}\|x - T_\beta x - (y - T_\beta y)\|^2\end{aligned}$$

$$\|T_\beta x - T_\beta y\|^2 \leq \|x-y\|^2 + \frac{2}{1-\beta}\langle x - T_\beta x, y - T_\beta y\rangle - (\beta - k)\|x - T_\beta x - (y - T_\beta y)\|^2 \tag{4.24}$$

Thus, $\forall x, y \in C$ and $\forall p \in F(T) = F(T_\beta)$ we have

$$\begin{aligned}\|T_\beta x - p\|^2 &= \|T_\beta x - T_\beta p + T_\beta p - p\|^2 \\ &= \|T_\beta x - T_\beta p\|^2 \\ &\leq \|x-p\|^2 - (\beta-k)\|x - T_\beta x - (p - T_\beta p)\|^2 \\ &= \|x-p\|^2 - (\beta-k)\|x - T_\beta x\|^2\end{aligned}$$

Hence, $\|T_\beta x - p\|^2 \leq \|x-p\|^2 - (\beta-k)\|x - T_\beta x\|^2$

Since $\beta - k < 1 \Rightarrow T_\beta$ is a quasi-firmly nonexpansive mapping.

We observe that $F(T_\beta) = F(T) \neq \emptyset$,

$\|T_\beta x - p\|^2 \leq \|x-p\|^2 - (\beta-k)\|x - T_\beta x\|^2 \leq \|x-p\|^2 \Rightarrow \|T_\beta x - p\| \leq \|x - p\|$,

Also, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C, then$\|x_{n+1} - p\| \leq \alpha_n\|u-p\| + (1-\alpha_n)\|x_n - p\|$

$\Rightarrow \|x_{n+1} - p\| \leq \max\{\|u-p\|, \|x_1 - p\|\} \Rightarrow \{x_n\}_{n=1}^{\infty}$ is bounded in $C \Rightarrow x_n - T_\beta x_n \to 0$ as $n \to \infty$

$\Rightarrow x_n - p - (T_\beta x_n - p) \to 0$ as $n \to \infty \Rightarrow \|x_n - p - (T_\beta x_n - p)\| \to 0$ as $n \to \infty$.

$\|x_n - p\| - \|T_\beta x_n - p\| \leq \|x_n - p - (T_\beta x_n - p)\| \to 0$ as $n \to \infty$

$\Rightarrow \|x_n - p\| - \|T_\beta x_n - p\| \to 0$ as $n \to \infty \Rightarrow T_\beta$ is strongly quasi-nonexpansive.

Hence, it follows from [[Song et al., 2009],[Saejung, 2010]](see theorem 3.1 and Remark 1 of [Song et al., 2009] or corollary 8 of [Saejung, 2010] that $x_n \to p \in F(T_\beta) = F(T)$. □

Corollary 1

Let C be a nonempty closed and convex subset of a real Hilbert space H, $T : C \longrightarrow C$ be a nonspreading mapping with $F(T) \neq \emptyset$, let $\beta \in (0,1)$ and $T_\beta = \beta I + (1-\beta)T$, let $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0,1)$ satisfy the conditions:

$C_1 : \lim\limits_{n\to\infty} \alpha_n = 0.$

$C_2 : \sum\limits_{k=1}^{\infty} \alpha_n = \infty.$

Let u be a fixed anchor in C, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated from an arbitrary $x_1 \in C$ by

$x_{n+1} = \alpha_n u + (1-\alpha_n)T_\beta x_n \ , \ \ n \geq 1.$

Then,

$x_n \to p \in F(T).$

CHAPTER 5

SUMMARY

In the first chapter, we discussed fixed point theory and its applications to several areas of research such as in Optimization, Economics, Evolution Equations e.t.c; we discussed the intimate relationship between fixed points of nonexpansive mappings and equilibrium points of certain dynamical systems. We also introduced the history of the class of nonspreading mappings and the class of strictly pseudononspreading mappings in Hilbert spaces.
In the second chapter, we gave a review of other works done in the area of research carried out in this thesis.

In the third chapter, we gave the definitions of some basic terms used in this thesis; we gave some basic facts concerning the class of k-strictly pseudononspreading mappings in Hilbert spaces with examples and at the end of the third chapter we presented some preliminary results used in this thesis.

In the fourth chapter, we presented with examples, the class of k-strictly pseudononspreading mappings in Hilbert spaces and its relationship with the class of nonspreading mappings in Hilbert spaces. We further presented the main results in the work of Osilike and Isiogugu, Nonlinear Analysis, **74** (2011), 1814-1822 (i.e weak and strong convergence theorems for nonspreading type mapping in a Hilbert space).

In the last chapter, we gave a summary of the thesis in general.

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