A KRASNOSELSKII-TYPE ALGORITHM FOR APPROXIMATING SOLUTIONS OF VARIATIONAL INEQUALITY PROBLEMS AND CONVEX FEASIBILITY PROBLEMS

A Thesis Presented to the Department of

Pure and Applied Mathematics

African University of Science and Technology

In Partial Fulfilment of the Requirements for the Degree of

Master of Science

By

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40477

December, 2017

Acknowledgements

My first gratitude goes to the Almighty. Without His guidance and protection I would not have been where I am today. This thesis owes a great deal to the support and encouragement of many people: My supervisor, Prof. Charles Ejike Chidume, who has been a constant source of inspiration, ideas and advice, and has always been incredibly generous with his time. Thus, I was really encouraged by his support and guidance.

My warm-hearted thanks go to my family: My eldest brother Ibrahim Adamu Abdullahi, Haruna Adamu Abdullahi, Abdullahi Adamu Abdullahi, Isyaka Adamu Abdullahi among others for their endless support and encouragement. I Love You All! for your kind hospitality and I am indebted to all of you for your love and support throughout the period of my study.

I would love to thank you Miss Amaka Udigwe for your goodwill. I would also like to send my gratitude to all staff members of the Mathematics Institute, both academic and non-academic who have in one way or the other supported me, during my stay at AUST. May God reward you all abundantly.

Finally, I would like to thank my friends at AUST; Sulaiman Abdulfatah, Ammar Sani Ibrahim, Ibrahim Yahaya, Muktar Muhammad Lawal, Muhammad Maikudi Isah, Agbo Ejike, Ifanyi Stephen, Aisha Aminu Adam, Ossiogu Onyekhaci, Nnyaba Victoria, Okereke Lois, to mention but a few and also my friends at home for their caring and support. I will never forget your kindness towards me.

This is an opportunity for me to thank my best friend Nasiru Salihu for always being by my side.

Certification

This is to certify that the thesis titled "A KRASNOSELSKII-TYPE ALGORITHM FOR APPROXIMATING SOLUTIONS OF VARIATIONAL INEQUALITY PROBLEMS AND CONVEX FEASIBILITY PROBLEMS" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research work carried out by Adamu Abubakar in the department of Pure and Applied Mathematics.

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Dedication

This work is dedicated to my beloved parents, Alhaji ADAMU ABDULLAHI a father like no other and Hajiya HAFSAH ADAMU. They made me believe in myself and always feel I can do it. Thanks for your love and support.

Abstract

A Krasnoselskii-type algorithm for approximating a common element of the set of solutions of a variational inequality problem for a monotone, *k-Lipschitz* map and solutions of a convex feasibility problem involving a countable family of relatively nonexpansive maps is studied in a uniformly smooth and 2-uniformly convex real Banach space. A strong convergence theorem is proved. Some applications of the theorem are presented.

Keywords

Relatively nonepxansive maps, monotone maps and Lipschitz continuous maps, generalized projection, variational inequality problems, countable family, subgradient method.

Publications

1. C.E. Chidume, **A. Adamu** and L.O. Chinwendu, *A Krasnoselskii-type Algorithm for Approximating Solutions of Variational Inequality Problems and Convex Feasibility Problems*, Journal of Nonlinear Variational Analysis, (Accepted, to appear)

2. C.E. Chidume, L.O. Chinwendu and **A. Adamu**, *A Hybrid Algorithm for Approximating Solutions of a Variational Inequality Problem and a Convex Feasibility Problem*, Advances in Nonlinear Variational Inequalities Vol. 21 (2018), No. 1, 46 - 65 .

3. C.E. Chidume, S.I. Ikechukwu and **A. Adamu,** *Inertial Algorithm for Approximating a Common Fixed Point for a Countable Family of Relatively Nonexpansive Maps,* Fixed Point Theory and Applictions https://doi.org/10.1186/s13663-018-0634-3.

Contents

CHAPTER 1

General Introduction and Literature Review

In this chapter, we give a general introduction on variational inequality problem, fixed point problem and finally we give a brief review of existing results on variational inequality and fixed point problem.

1.1 Background of study

The contributions of this thesis fall within the general area of nonlinear functional analysis and applications, in particular, nonlinear operator theory. We are interested in finding or approximating solution(s) of a variational inequality problem for a monotone k-Lipschitz map and a convex feasibility problem for a countable family of relatively nonexpansive maps, in Banach spaces.

1.1.1 Variational Inequality Problems

Variational inequality problems were formulated in the late 1960s by Lions and Stampacchia, and since then, they have been studied extensively. Numerous researchers have proposed and analyzed various iterative schemes for approximating solutions of variational inequality problems. The literature on this is extensive (see, for example, [Chidume, 2009], [Nilsrakoo and Saejung, 2011], [Buong, 2010],[Hieu et al., 2006], [Iiduka and Takahashi, 2008],[Censor et al., 2012], [Censor et al., 2011], [Dong et al., 2016], [Gibali et al., 2015], [Chidume et al., 2017], [Censor et al., 2010], and the references contained in them).

Let C be a nonempty, closed and convex subset of a real Banach space E with dual space E^* and $A : C \to E^*$ be a map. Then, A is said to be:

- *k-Lipschitz* if there exists a constant $k \geq 0$, such that

$$\|Ax - Ay\| \leq k\|x - y\|, \ \forall x, y \in C. \tag{1.1.1}$$

Remark 1.1.1 *If $k \in (0, 1)$, A is called a contraction. If $k = 1$, A is called nonexpansive.*

- *monotone* if the following inequality holds:

$$\big\langle x-y, Ax-Ay\big\rangle \geq 0, \ \forall x,y \in C. \tag{1.1.2}$$

- *δ-inverse strongly monotone* if there exists a $\delta \geq 0$, such that

$$\big\langle x-y, Ax-Ay\big\rangle \geq \delta\|Ax-Ay\|^2, \ \forall x,y \in C. \tag{1.1.3}$$

- *maximal monotone* if A is monotone and the graph of A is not properly contained in the graph of any other monotone map.

It is immediate that if A is *δ-inverse strongly monotone*, then A is monotone and Lipschitz continuous.

The problem of finding a point $u \in C$, such that

$$\langle v-u, Au\rangle \geq 0, \ \forall\ v \in C, \tag{1.1.4}$$

is called a *variational inequality problem.* We denote the set of solutions to the variational inequality problem (1.1.4) by $VI(C,A)$.

Remark 1.1.2 *It is easy to see that if u is a solution of the variational inequality problem (1.1.4) then,*

$$\langle x-u, Ax\rangle \geq 0, \ \forall x \in C.$$

1.1.2 Fixed Point Problems

The theory of fixed point proves to be a useful tool in modern mathematics. This comes from the fact that most important nonlinear problems in applications can be transformed to a fixed point problem.

Theorems concerning the existence and properties of fixed points are known as fixed point theorems. Several theorems have been proved on the existence and uniqueness of fixed point(s) of self-maps. These theorems include the Banach contraction mapping principle, Brouwer fixed point theorem, Schauder fixed point theorems and a host of other authors (see for example [Asati et al., 2013], [Khamsi, 2002], [Lee, 2013], [Smith, 2015])

Example 1.1.3 *Let E be a real normed space and $A : E \to E$, be an accretive operator; most real life problems can be modelled into an equation of the form*

$$\frac{du}{dt} + Au = 0. \tag{1.1.5}$$

At equilibrium, $\frac{du}{dt} = 0$. Thus, (1.1.5) reduces to

$$Au = 0. \tag{1.1.6}$$

[Browder, 1967], introduced an operator $T : E \to E$, by $T = I - A$ and called the map T, pseudo-contractive . It is easy to see that zeros of A corresponds to fixed points of T (i.e., $Au = 0$ if and only if $Tu = u$).

Also, several existence theorems have been proved for the equation (1.1.6), where A is of the monotone-type (or accretive-type) (see for example, [Brezis, 1974], [Browder, 1967], [Deimling, 1974], [Pascali and Sburian, 1978], and the references contained in them).

1.1.3 Variational Inequality and Fixed Point Problems

In numerous models for solving real-life problems, such as in signal processing, networking, resource allocation, image recovery, and so on, the constraints can be expressed as variational inequality problems and (or) as fixed point problems. Consequently, the problem of finding common elements of the set of solutions of variational inequality problems and the set of fixed points of operators has become a flourishing area of contemporary research for numerous mathematicians working in nonlinear operator theory (see, for example, [Mainge, 2010a, Mainge, 2008, Ceng et al., 2010] and the references contained in them).

1.2 Statement of the Problem

Let $A : E \to E^*$ be a monotone and *k-Lipschitz* map and $S : E \to E$ be a nonexpansive map. In studying variational inequality problems and fixed point problems on real Banach spaces more general than Hilbert spaces, several algorithms have been constructed for approximating solutions of variational inequality problems and fixed point problems (see, e.g., the following monographs: [Alber, 1996], [Berinde, 2007], [Browder, 1967], [Chidume, 2009], [Goebel and Reich, 1994] and the references contained in them). Consequently, since most real life problems exist in spaces more general than Hilbert spaces, this induced mathematicians to ask if such results can be obtained for a monotone, *k-Lipschitz* map and a nonexpansive map in Banach spaces.

However, the pursuit of analogous results for variational inequality problems and fixed point problems in more general Banach space with nonexpansive maps seem not to be feasible. The main difficulty (or challenge) is that most properties of the *Lyapunov functional* and *generalized projection* are proved using relatively nonexpansive maps.

1.3 Motivation of Research and Objectives

Motivated by the results of [Kraikaew and Saejung, 2014], and [Nakajo, 2015], it is our purpose in this thesis to introduce a *Krasnoselskii-type algorithm* in a *uniformly smooth and 2-uniformly convex real Banach space* and prove *strong convergence* of the sequence generated by our algorithm to a point $q \in F(S) \cap VI(C, A)$. The objectives are:

- To use the normalized duality map and Lyapunov functional for estimations;
- To extend the class of maps from one nonexpansive to a countable family of relatively nonexpansive maps; and
- To propose an algorithms with less computational cost when compared with existing algorithms in the Banach space.

1.4 Literature Review

Numerous researchers in nonlinear operator theory have studied various iterative methods for approximating solutions of variational inequality problems, approximating fixed points of nonexpansive maps and their generalizations (see, e.g., the following monographs: [Alber, 1996], [Berinde, 2007], [Browder, 1967], [Chidume, 2009], [Goebel and Reich, 1994] and the references contained in them). In most of the early results on iterative methods for approximating these solutions, the map A was often assumed to be *inverse-strongly monotone* (see, e.g., [Buong, 2010], [Censor et al., 2012], [Chidume et al., 2016], and the references contained in them). To relax the inverse-strong monotonicity condition on A, [Korpelevic, 1967] introduced, in a finite dimensional Euclidean space $\mathbb{R}^n$, the following *extragradient method*

$$\begin{cases} x_1 = x \in C; \\ x_{n+1} = P_C(x_n - \lambda A[P_C(x_n - \lambda A x_n)]), \forall\ n \in \mathbb{N}, \end{cases} \tag{1.4.1}$$

where A was assumed to be monotone and Lipschitz. The extragradient method has since then been studied and improved on by many authors in various ways. However, we observe that in the extragradient method, *two* projections onto a closed and convex subset C of H need to be computed in each step of the iteration process. As mentioned by [Censor et al., 2011], this may affect the efficiency of the method if the set C is not simple enough. Therefore, to improve on the extragradient method, [Censor et al., 2011] modified the the extragradient method and proposed the following iterative algorithm:

$$\begin{cases} x_0 \in H; \\ y_n = P_C(x_n - \tau A x_n); \\ T_n = \{w \in H : \langle x_n - \tau A x_n - y_n, w - y_n \rangle \le 0\}; \\ x_{n+1} = P_{T_n}(x_n - \tau A y_n). \end{cases} \tag{1.4.2}$$

The method (1.4.2) replaces the second projection onto the closed and convex subset C in (1.4.1) with a projection on to the *half-space* T_n. Algorithm (1.4.2) is the so-called *subgradient extragradient method.* We note that, the set T_n is a half-space, and hence algorithm (1.4.2) is easier to execute than algorithm (1.4.1). Under some mild assumptions, [Censor et al., 2011] proved that algorithm (1.4.2) *converges weakly* to a solution of variational inequality (1.1.4) *in a real Hilbert space.*

In order to obtain the strong convergence, [Kraikaew and Saejung, 2014] combined the subgradient extragradient method (1.4.2) with the method introduced by [Halpern, 1967] and proposed the following iterative algorithm:

$$\begin{cases} x_0 \in H; \\ y_n = P_C(x_n - \tau A x_n); \\ T_n = \{w \in H : \langle x_n - \tau A x_n - y_n, w - y_n \rangle \le 0\}; \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) P_{T_n}(x_n - \tau A y_n), \end{cases} \tag{1.4.3}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. They proved that the sequence generated by algorithm (1.4.3) *converges strongly* to a solution of the variational inequality problem (1.1.4) *in a real Hilbert space.* We remark, however, that convergence theorems have also been proved in real Banach spaces more general than Hilbert space. For instance, [Iiduka and Takahashi, 2008], using the following scheme,

$$\begin{cases} x_1 \in C; \\ x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \end{cases} \tag{1.4.4}$$

obtained *weak convergence* of the sequence $\{x_n\}$ generated by equation (1.4.4) to a solution of the variational inequality problem (1.1.4) in a 2-*uniformly convex, uniformly smooth real Banach space* whose duality map J *is weakly sequentially continuous*, under the conditions that,

$(A1)$ A is α-inverse-strongly-monotone;

$(A2)$ $VI(C, A) \neq \emptyset$; and

$(A3)$ $\|Ay\| \leq \|Au - Ay\|, \ \forall \ y \in C$ and $u \in VI(C, A)$.

An example of such a real Banach space is $l_p, \ 1 < p \leq 2$. The space $L_p, \ 1 < p \leq 2$ is excluded since the duality map on it is not weakly sequentially continuous.

Motivated by the result of [Iiduka and Takahashi, 2008], in 2015, [Nakajo, 2015] proposed and studied the following CQ method in a 2-uniformly convex and uniformly smooth real Banach space.

$$\begin{cases} x_1 = x \in E; \\ y_n = \Pi_C J^{-1}[Jx_n - \lambda_n A(x_n)]; \\ z_n = Ty_n; \\ C_n = \{z \in C : \phi(z, z_n) \leq \phi(z, x_n) - \phi(y_n, x_n) - 2\lambda_n \langle y_n - z, Ax_n - Ay_n \rangle\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \quad n \geq 0. \end{cases} \tag{1.4.5}$$

He significantly improved the result of [Iiduka and Takahashi, 2008] in the following sense:

- The operator A is assumed to be monotone and Lipschitz.
- The sequence $\{x_n\}$ generated by his scheme *converges strongly* to an element of $VI(C, A)$.
- The requirement that J be weakly sequentially continuous is dispensed with; consequently, the result of Nakajo is applicable in L_p spaces, $1 < p \leq 2$.
- The condition $(A3)$ is also dispensed with.
- The sequences $\{x_n\}$ and $\{z_n\}$ generated by his algorithm, not only converge to a point in $VI(C, A)$ but also to a fixed point of a *relatively nonepxansive* self-map of C.

However, we note that the algorithm (1.4.5) of Nakajo, *at each step of the iteration process*, requires the computation of two convex subsets, C_n and Q_n, their intersection $C_n \cap Q_n$ and the projection of the initial vector onto this intersection. This is certainly not convenient in several possible applications.

CHAPTER 2

Preliminaries

In this chapter, we will give definition of some terms and results of interest used in the thesis.

2.1 Definition of terms

Throughout this thesis, we will always let E be a real Banach space with dual space E^* and $\langle \cdot, \cdot \rangle$ denoting the duality pairing of E and E^*. Whenever a sequence $\{x_n\}$ in E, converges strongly (weakly), respectively, we denote the convergence by $x_n \to x$ ($x_n \rightharpoonup x$).

Definition 2.1.1 *A normed space E is called smooth if for every $x \in E$, $\|x\| = 1$, there exists a unique $x^* \in E^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.*

Definition 2.1.2 *Let $q > 1$ and $r > 0$, be two fixed real numbers. Then E is uniformly smooth if and only if there exists a continuous, strictly increasing and convex function*

$$g : \mathbb{R}^+ \to \mathbb{R}^+, \; g(0) = 0$$

such that

$$\|\lambda x + (1-\lambda)y\|^q \geq \lambda\|x\|^q + (1-\lambda)\|y\|^q - W_q(\lambda)g(\|x-y\|)$$

for all $x, y \in B_r, 0 \leq \lambda \leq 1$, where $W_q(\lambda) = \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$.

Definition 2.1.3 *A normed space E is called uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$, such that for any $x, y \in E$, with $\|x\| = 1$, $\|y\| = 1$ and $\|x - y\| \geq \epsilon$ then $\|\frac{x+y}{2}\| \leq 1 - \delta$.*

Remark 2.1.4 *We note immediately that the following definition is also used:*
A normed linear space E is uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Definition 2.1.5 *A normed space E is called strictly convex if for all $x, y \in E$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1-\lambda)y\| < 1$, $\forall \lambda \in (0, 1)$.*

Definition 2.1.6 *Let E be a normed space with $dimE \geq 2$. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by*

$$\delta_E := \inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\Big\}.$$

Remark 2.1.7

1. *In the particular case of an inner product space H, we have* $\delta_H(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}$.
2. *Every uniformly convex space is reflexive .*
3. *E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2)$*

Definition 2.1.8 *Let $p > 1$ be a real number. Then, a normed space E is said to be p-uniformly convex if there is a constant $c > 0$ such that*

$$\delta_E(\epsilon) \geq c\epsilon^p$$

Example 2.1.9 *If $E = L_p($ or $l_p),\ 1 < p < \infty$, then*

(a) $\delta_E(\epsilon) \geq \frac{1}{2^{p+1}}\epsilon^2$ *if* $1 < p < 2;$ *and*

(b) $\delta_E(\epsilon) \geq \epsilon^p$, *if* $2 \leq \infty$.

Definition 2.1.10 *A map A of E into E^* is said to be hemicontinuous if for all $x, y \in C$, the map $f : [0,1] \to E^*$ defined by $f(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^**

Definition 2.1.11 *The problem of finding a point $u \in C := \bigcap_{i=1}^{\infty} C_i$, where C_i is a convex set for each i, is called a convex feasibility problem.*

Definition 2.1.12 *A continuous strictly increasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(0) = 0$ and $\lim_{t\to\infty} g(t) = \infty$ is called a gauge function.*

Definition 2.1.13 *Given a gauge function g, the map $J_g : E \to 2^{E^*}$ defined by*

$$J_g x := \{u^* \in E^* : \langle x, u^*\rangle = \|x\|\|u^*\|; \quad \|u^*\| = g(\|x\|)\}$$

is called the duality map with the gauge function g where E is any normed space.

Remark 2.1.14

- *In the particular case $f(t) = t$, the duality map $J = J_g$ is called the normalized duality map.*
- *If E is a reflexive, strictly convex and smooth real Banach space, then J is single-valued and bijective.*

– *In a Hilbert space H, the duality map J and it is inverse and J^{-1} are the identity maps on H.*

– *If E is uniformly smooth and uniformly convex, then the dual space E^* is also uniformly smooth and uniformly convex and the normalized duality map J and its inverse, J^{-1}, are both uniformly continuous on bounded sets.*

Definition 2.1.15 *Let E be a smooth real Banach space and $\phi : E \times E \to [0, \infty)$ be defined by*

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2 \ \forall \ u, v \in E, \tag{2.1.1}$$

where J is the normalized duality map from E to E^*.

Remark 2.1.16

- *It is easy to see from the definition of ϕ that in a real Hilbert space H, equation (2.1.1) reduces to $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in H$.*

- *Consider the map $V : E \times E^* \to \mathbb{R}$ defined by $V(u, u^*) = \|u\|^2 - 2\langle u, u^*\rangle + \|u^*\|^2$. It is easy to see that $V(u, u^*) = \phi(u, J^{-1}u^*) \ \forall \ u \in E, \ u^* \in E^*$.*

Furthermore, given $x, y, z \in E$, and $\tau \in (0, 1)$, we have the following properties (see, [Nilsrakoo and Saejung, 2011]):

P1 $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,

P2 $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle z - x, Jy - Jz\rangle$,

P3 $\phi(\tau x + (1 - \tau)y, z) \leq \tau\phi(x, z) + (1 - \tau)\phi(y, z)$

Definition 2.1.17 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. The map $P_C : H \to C$ defined by $\tilde{x} := P_C(x) \in C$ such that $\|x - \tilde{x}\| = \inf\limits_{y \in C} \|x - y\|$ is called the metric projection of x onto C.*

Definition 2.1.18 *Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E . The map $\Pi_C : E \to C$ defined by $\tilde{x} := \Pi_C(x) \in C$ such that $\phi(\tilde{x}, x) = \inf\limits_{y \in C} \phi(y, x)$ is called the generalized projection of x onto C.*

Remark 2.1.19 *Clearly, in a real Hilbert space, the generalized projection Π_C coincides with the metric projection P_C from E onto C.*

Definition 2.1.20 *Let $S : E \to E$ be a map. The set $\{x \in E : Sx = x\}$ is called the fixed point set of S. We denote the set by $F(S)$.*

Definition 2.1.21 *A map $S : E \to E$ is called quasi-nonexpansive if*

- *$F(S) \neq \emptyset$;*

- *$\|Sx - p\| \leq \|x - p\|$ for all $x \in E$, $p \in F(S)$.*

Definition 2.1.22 *Let $S : C \to E$ be a map. Then, S is called be relatively nonexpansive if the following conditions hold:*

(i) $F(S) := \{x \in C : Sx = x\} \neq \emptyset$;

(ii) $\phi(u, Sv) \leq \phi(u, v),\ \forall\ u \in F(S)$ *and* $v \in C$;

(iii) $I - S$ *is demi-closed at zero, i.e., whenever a sequence* $\{v_n\}$ *in* C *converges weakly to* u *and* $\{v_n - Sv_n\}$ *converges strongly 0, then* $u \in F(S)$.

Remark 2.1.23 *In a real Hilbert space, every nonexpansive map with nonempty fixed point set is relatively nonexpansive.*

Definition 2.1.24 *Let* C *be a nonempty subset of* E. *A map* $S : C \to E$ *is said to be pseudo-contractive if*

$$\langle Sx - Sy, J(x - y)\rangle \leq \|x - y\|^2 - \lambda\|x - y - (Sx - Sy\|^2$$

holds for $x, y \in C$ *and for some* $\lambda > 0$.

Remark 2.1.25 *It is easy to see that such maps are Lipschitz with Lipschitzian constant* $k = \frac{1+\lambda}{\lambda}$.

2.2 Important Results

Lemma 2.2.1 *[Alber, 1996]*
Let C *be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive real Banach space* E. *Then,*

1. *if* $x \in E$ *and* $y \in C$, *then* $\tilde{x} = \Pi_C x$ *if and only if* $\langle \tilde{x} - y, Jx - J\tilde{x}\rangle \geq 0$, *for all* $y \in C$,

2. $\phi(y, \tilde{x}) + \phi(\tilde{x}, x) \leq \phi(y, x)$, *for all* $x \in E$, $y \in C$.

Lemma 2.2.2 *[Xu, 1991]*
Let E *be a* 2*-uniformly convex and smooth real Banach space. Then, there exists a positive constant* α *such that*

$$\alpha\|x - y\|^2 \leq \phi(x, y),\ \forall\ x, y \in E. \tag{2.2.1}$$

Remark 2.2.3 *Without loss of generality, we may assume* $\alpha \in (0, 1)$.

Lemma 2.2.4 *[Xu, 1991]*
Let E *be a 2-uniformly convex real Banach space. Then, there exists a constant* $c_2 > 0$ *such that for every* $x, y \in E$, $f_x \in J_2(x), f_y \in J_2(y)$, *the following inequality holds:*

$$\langle x - y, f_x - f_y\rangle \geq c_2\|x - y\|^2.$$

Lemma 2.2.5 *[Kamimura and Takahashi, 2002]*
Let E *be a real smooth and uniformly convex Banach space, and let* $\{u_n\}$ *and* $\{v_n\}$ *be two sequences of* E. *If either* $\{u_n\}$ *or* $\{v_n\}$ *is bounded, then* $\phi(u_n, v_n) \to 0 \Rightarrow \|u_n - v_n\| \to 0$.

Lemma 2.2.6 *[Nilsrakoo and Saejung, 2011]*
Let E be a uniformly smooth Banach space and $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, 1)$ such that $g(0) = 0$ and

$$\phi\left(u, J^{-1}[\beta Jx + (1-\beta)Jy]\right) \leq \beta\phi(u, x) + (1-\beta)\phi(u, y) - \beta(1-\beta)g(\|Jx - Jy\|)$$

for all $\beta \in [0, 1]$, $u \in E$ and $x, y \in B_r$

Lemma 2.2.7 *[Rockafellar, 1970]*
Let C be a nonempty closed and convex subset of a reflexive space E and A, a monotone, hemicontinuous map of C into E^*. Let $T \subset E \times E^*$ be an operator defined by:

$$Tu = \begin{cases} Au + N_C(u), & if \quad u \in C, \\ \emptyset, & if \quad u \notin C, \end{cases} \tag{2.2.2}$$

where $N_C(u)$ is defined as follows:

$$N_C(u) = \{w^* \in E^* : \langle u - z, w^* \rangle \geq 0 \ \forall \ z \in C\}.$$

Then, T is maximal monotone and $T^{-1}0 = VI(C, A)$.

Lemma 2.2.8 *[Xu, 2002]*
Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the conditions

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n, \ n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

$$(i) \quad \{\alpha_n\} \subset [0, 1] \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty; \quad (ii) \ \limsup_{n\to\infty} \beta_n \leq 0. \text{ Then, } \lim_{n\to\infty} a_n = 0.$$

Lemma 2.2.9 *[Mainge, 2010b]*
Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{m_j}\}$ of $\{a_n\}$ such that $a_{m_j} < a_{m_{j+1}}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} n_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:

$$a_{n_k} \leq a_{n_{k+1}} \quad \text{and} \quad a_k \leq a_{n_{k+1}}.$$

In fact, n_k is the largest number n in the set $\{1, ..., k\}$ such that $a_n < a_{n+1}$ holds.

Lemma 2.2.10 *[Alber and Ryazantseva, 2006]*
Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Then,

$$V(u, u^*) + 2\langle J^{-1}u^* - u, v^* \rangle \leq V(u, u^* + v^*) \tag{2.2.3}$$

for all $u \in E$ and $u^*, v^* \in E^*$.

Lemma 2.2.11 *[Kohsaka and Takahashi, 2008]*
Let C be a closed convex subset of a uniformly smooth and uniformly convex Banach space E and let $(S_i)_{i=1}^{\infty}$ be a family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $(\eta_i)_{i=1}^{\infty} \subset (0,1)$ and $(\mu_i)_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Consider the map $T : C \to E$ defined by

$$Tx = J^{-1}\left(\sum_{i=1}^{\infty} \eta_i(\mu_i Jx + (1-\mu_i)JS_i x)\right) \quad \textit{for each } x \in C. \tag{2.2.4}$$

Then, T is relatively nonexpansive and $F(T) = \bigcap_{i=1}^{\infty} F(S_i)$.

The following result has recently been proved. But, for completeness, we reproduce the proof here.

Lemma 2.2.12 *[Chidume and Otubo, 2017]*
Let E be a 2-uniformly convex and smooth real Banach space and C be a nonempty, closed and convex subset of E. Let $x_1, x_2 \in E$ be arbitrary and $\Pi_C : E \to C$ be the generalized projection. Then, the following inequality holds:

$$\|\Pi_C x_1 - \Pi_C x_2\| \leq \frac{1}{c_2}\|Jx_1 - Jx_2\|, \tag{2.2.5}$$

where c_2 is the constant appearing in Lemma 2.2.4 and J is the normalized duality map on E.

Proof By Lemma 2.2.1 (1), for any $x_1, x_2 \in E$ we have

$$\langle \Pi_C x_2 - \Pi_C x_1, Jx_1 - J\Pi_C x_1\rangle \leq 0 \quad \text{and} \quad \langle \Pi_C x_1 - \Pi_C x_2, Jx_2 - J\Pi_C x_2\rangle \leq 0.$$

Adding these two inequalities, we obtain

$$\begin{aligned}
&\langle \Pi_C x_1 - \Pi_C x_2, (Jx_2 - Jx_1) - (J\Pi_C x_2 - J\Pi_C x_1)\rangle \leq 0 \\
\Rightarrow\ &\langle \Pi_C x_1 - \Pi_C x_2, Jx_2 - Jx_1\rangle - \langle \Pi_C x_1 - \Pi_C x_2, J\Pi_C x_2 - J\Pi_C x_1\rangle \leq 0 \\
\Rightarrow\ &\langle \Pi_C x_1 - \Pi_C x_2, J\Pi_C x_1 - J\Pi_C x_2\rangle \leq \langle \Pi_C x_1 - \Pi_C x_2, Jx_1 - Jx_2\rangle.
\end{aligned}$$

By Lemma 2.2.4, we obtain

$$\begin{aligned}
c_2\|\Pi_C x_1 - \Pi_C x_2\|^2 &\leq \|\Pi_C x_1 - \Pi_C x_2\| \cdot \|Jx_1 - Jx_2\|, \\
\text{so that} \qquad\qquad & \\
\|\Pi_C x_1 - \Pi_C x_2\| &\leq \frac{1}{c_2}\|Jx_1 - Jx_2\|,
\end{aligned} \tag{2.2.6}$$

completing the proof. ■

Remark 2.2.13 *Lemma 2.2.12 implies that the generalized projection Π_C is uniformly continuous whenever J is.*

Lemma 2.2.14 *(see, [Ceng et al., 2010])*
Let C be a nonempty, closed and convex subset of a real Hilbert space H. Then,

1. *if $x \in H$ and $y \in C$, then $\tilde{x} = P_C x$ if and only if $\langle \tilde{x} - y, x - \tilde{x}\rangle \geq 0$, for all $y \in C$,*

2. $\|p_C x - P_C y\| \leq \|x - y\|$, *for all* $x, y \in H$,

3. $\|x - P_C x\| + \|y - P_C y\| \leq \|x - y\|^2$, *for all* $x \in H$ *and* $y \in C$.

Lemma 2.2.15 *(see, [Kraikaew and Saejung, 2014])*
Let $S : H \to H$ *be a map. Then, the mapping* $I - S$ *is demiclosed at zero if and only if* $x \in F(S)$ *whenever* $x_n \rightharpoonup x$ *and* $x_n - Tx_n \to 0$.

Remark 2.2.16 *If* $S : H \to H$ *is nonexpansive, then* $I - S$ *is demiclosed at zero.*

Lemma 2.2.17 *[Takahashi, 2009] Let* H *be a real Hilbert space. Let* $x, y \in H$, *we have the following statements:*

1. $|\langle x, y\rangle| \leq \|x\|\|y\|$;

2. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ *(the subdifferential inequality).*

Lemma 2.2.18 *[Chidume, 2009] Let* H *be a real Hilbert space. Let* $x, y \in H$ *and* $\lambda \in (0, 1)$ *then,*

$$\|x\lambda + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

CHAPTER 3

Results of Kraikaew and Saejung

3.1 Introduction

In this chapter we give a detailed proof of the results of [Kraikaew and Saejung, 2014] which they proved in a real Hilbert space.

3.2 The Subgradient Extragradient Algorithm

Inspired by the result of [Halpern, 1967], [Kraikaew and Saejung, 2014], introduce the subgradient extragradient algorithm which finds a solution of the variational inequality (1.1.4) and also proved a strong convergence theorem in a real Hilbert space. They divided the proof in to several Lemmas.

Lemma 3.2.1 *Let $f : H \to H$ be a monotone and L-Lipschitz mapping on C and τ be a positive number supposing that $VI(C, f)$ is nonempty. Let $x \in H$. Define*

$$\begin{aligned} U(x) &:= P_C\big(x - \tau f(x)\big); \\ T^x &:= \{w \in H : \langle x - \tau f(x) - U(x), w - U(x)\rangle \leq 0\}; \\ V(x) &:= P_{T^x}\Big(x - \tau f\big(U(x)\big)\Big). \end{aligned}$$

Then, for all $u \in VI(C, f)$, we have

$$\|V(x) - u\|^2 \leq \|x - u\|^2 - (1 - \tau L)\|x - U(x)\|^2 - (1 - \tau L)\|V(x) - U(x)\|^2. \quad (3.2.1)$$

In particular, if $\tau L \leq 1$, we have $\|V(x) - u\| \leq \|x - u\|$.

Proof Applying Lemma 2.2.14 (3), we have

$$\begin{aligned}
\|V(x)-u\|^2 &\le \|\big(x-\tau f(U(x))\big)-u\|^2-\|x-\tau f(U(x))-V(x)\|^2\\
&=\|x-u\|^2+2\tau\langle u-V(x),f(U(x))\rangle-\|x-V(x)\|^2\\
&=\|x-u\|^2+2\tau\langle u-U(x),f(U(x))-f(u)\rangle+2\tau\langle u-U(x),f(u)\rangle\\
&\quad+2\tau\langle U(x)-V(x),f(U(x))\rangle-\|x-V(x)\|^2\\
&\le \|x-u\|^2+2\tau\langle U(x)-V(x),f(U(x))\rangle-\|x-V(x)\|^2\\
&=\|x-u\|^2+2\tau\langle U(x)-V(x),f(U(x))\rangle-\|x-U(x)\|^2\\
&\quad-2\tau\langle x-U(x),U(x)-V(x)\rangle-\|U(x)-V(x)\|^2\\
&=\|x-u\|^2-\|x-U(x)\|^2-\|U(x)-V(x)\|^2\\
&\quad+2\langle x-\tau f(U(x))-U(x),V(x)-U(x)\rangle.
\end{aligned}$$

Now, we estimate

$$\begin{aligned}
\langle x-\tau f(U(x))-U(x),V(x)-U(x)\rangle &= \langle x-\tau f(x)-U(x),V(x)-U(x)\rangle\\
&\quad+\langle \tau f(x)-\tau f(U(x)),V(x)-U(x)\rangle\\
&\le \langle \tau f(x)-\tau f(U(x)),V(x)-U(x)\rangle\\
&\le \tau L\|x-U(x)\|\|V(x)-U(x)\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|V(x)-u\|^2 &\le \|x-u\|^2-\|x-U(x)\|^2-\|U(x)-V(x)\|^2\\
&\quad+2\tau L\|x-U(x)\|\|V(x)-U(x)\|\\
&=\|x-u\|^2-(1-\tau L)\|x-U(x)\|^2-(1-\tau L)\|U(x)-V(x)\|^2\\
&\quad-\tau L\big(\|x-U(x)\|-\|V(x)-U(x)\|\big)^2\\
&\le \|x-u\|^2-(1-\tau L)\|x-U(x)\|^2-(1-\tau L)\|U(x)-V(x)\|^2.
\end{aligned}$$

■

The next result is the demiclosedness-like property (Lemma 2.2.15) of the mapping $P_C(I-\tau f)$. Note that the authors [Kraikaew and Saejung, 2014] did not use the maximal monotonicity of $f+N_C$, where N_C is the normal cone of C, as it was the case in other papers (see, e.g., [Censor et al., 2010, Censor et al., 2011, Censor et al., 2012]).

Lemma 3.2.2 *Let $f: H\to H$ be a monotone and L-Lipschitz mapping on C. Let $U := P_C(I-\tau f)$, where $\tau>0$. If $\{x_n\}$ is a sequence in C satisfying $x_n \rightharpoonup \widehat{x}$ and $x_n-U(x_n)\to 0$, then $\widehat{x}\in VI(C,f)$.*

Proof Since f is monotone and hemicontinuous, it suffices to show that

$$\langle f(x),x-\widehat{x}\rangle \ge 0, \text{ for all } x\in C.$$

Let $x\in C$ and $\tau>0$. Observe that from Lemma 2.2.14 (1),

$$\langle x_n-\tau f(x_n)-U(x_n),U(x_n)-x\rangle \ge 0, \text{ for all } n\in\mathbb{N}.$$

Next, we consider

$$\begin{aligned}
\langle \tau f(x_n), x_n - x\rangle &= \langle \tau f(x_n), x_n - U(x_n)\rangle + \langle \tau f(x_n), U(x_n) - x\rangle \\
&= \langle \tau f(x_n), x_n - U(x_n)\rangle - \langle x_n - \tau f(x_n) - U(x_n), U(x_n) - x\rangle \\
&\quad + \langle x_n - U(x_n), U(x_n) - x\rangle \\
&\leq \langle \tau f(x_n), x_n - U(x_n)\rangle + \langle x_n - U(x_n), U(x_n) - x\rangle \\
&\leq \tau\|f(x_n)\|\|x_n - U(x_n)\| + \|x_n - U(x_n)\|\|U(x_n) - x\|.
\end{aligned}$$

Since $\{f(x_n)\}$, $\{U(x_n)\}$ are bounded, and $x_n - U(x_n) \to 0$, $\limsup_{n\to\infty}\langle \tau f(x_n), x_n - x\rangle \leq 0$. Using the monotonicity of f, we have

$$\begin{aligned}
\langle f(x), \widehat{x} - x\rangle &= \frac{1}{\tau}\limsup_{n\to\infty}\langle \tau f(x) - \tau f(x_n), x_n - x\rangle + \frac{1}{\tau}\limsup_{n\to\infty}\langle \tau f(x_n), x_n - x\rangle \\
&\leq \frac{1}{\tau}\limsup_{n\to\infty}\langle \tau f(x_n), x_n - x\rangle \leq 0.
\end{aligned}$$

This completes the proof. ■

Next, the authors [Kraikaew and Saejung, 2014] studied the following algorithm for approximating the solution of a variational inequality problem. Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $f : H \to H$ be a monotone map. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined by

$$\begin{cases} x_0 \in H, \\ y_n := P_C\big(x_n - \tau f(x_n)\big), \\ T_n := \{w \in H : \langle x_n - \tau f(x_n) - y_n, w - y_n\rangle 0\}, \\ x_{n+1} := \alpha_n x_0 + (1-\alpha_n)P_{T_n}\big(x_n - \tau f(y_n)\big), \end{cases} \tag{3.2.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Observe that that T_n in (3.2.2) is just T^{x_n} in Lemma 3.2.1.

In the sequel, we assume that $VI(C,f)$ is nonempty and we denote $\omega_\omega\{z_n\}$ the set of all cluster points of the sequence $\{z_n\}$.

Lemma 3.2.3 *Let $f : H \to H$ be a monotone and L-Lipschitz mapping on C and τ be a positive real number such that $\tau L \leq 1$. Then, the sequence $\{x_n\}$ generated by (3.2.2) satisfies the following inequality:*

$$\|x_{n+1} - z\| \leq \alpha_n\|x_0 - z\| + (1-\alpha_n)\|x_n - z\|,$$

for all $z \in VI(C,f)$. Furthermore, it follows inductively that $\{x_n\}$ is bounded.

Proof Let $z \in VI(C,f)$. Set $w_n = P_{T_n}\big(I - \tau f P_C(I - \tau f)\big)x_n$. Hence, $x_{n+1} = \alpha_n x_0 + (1-\alpha_n)w_n$. It follows from Lemma 3.2.1 that $\|w_n - z\| \leq \|x_n - z\|$ and hence

$$\begin{aligned}
\|x_{n+1} - z\| &\leq \alpha_n\|x_0 - z\| + (1-\alpha_n)\|w_n - z\| \\
&\leq \alpha_n\|x_0 - z\| + (1-\alpha_n)\|x_n - z\|.
\end{aligned}$$

Thus, $\|x_{n+1} - z\| \leq max\{\|x_0 - z\|, \|x_n - z\|\}$. Hence, by induction we have $\|x_n - z\| \leq \|x_0 - z\|$ for all $n \in N$. Hence, the sequence $\{x_n\}$ is bounded. ■

Theorem 3.2.4 *Let $f : H \to H$ be a monotone and L-Lipschitz mapping on C and τ be a positive real number such that $\tau L < 1$. Let $\{x_n\} \subset H$ be a sequence generated by (3.2.2). Then, $x_n \to P_{VI(C,f)}x_0$.*

Proof We recall that $x_{n+1} = \alpha_n x_0 + (1-\alpha_n)w_n$. Set $z = P_{VI(C,f)}x_0$. Let us start from the following inequalities, which are consequences of (3.2.2) and the subdifferential inequality (Lemma 2.2.17)

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1-\alpha_n)^2\|w_n - z\|^2 + 2\alpha_n\langle x_0 - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)\|w_n - z\|^2 + 2\alpha_n\langle x_0 - z, x_{n+1} - z\rangle. \end{aligned} \tag{3.2.3}$$

To complete the proof, the authors [Kraikaew and Saejung, 2014] considered the following two cases.

Case 1. Assume there exists $n_0 \in \mathbb{N}$ such that $\|x_{n+1} - z\| \leq \|x_n - z\|$, for all $n \geq n_0$. Then $\lim_{n\to\infty}\|x_n - z\|$ exists. It follows from (3.2.3), using the fact that $\alpha_n \to 0$ and the boundedness of $\{x_n\}$ that $\|w_n - z\|^2 - \|x_n - z\|^2 \to 0$. By Lemma 3.2.1, we conclude that $x_n - P_C\big(x_n - \tau f(x_n)\big) \to 0$. Using Lemma 3.2.3, we have $\omega_\omega\{x_n\} \subset VI(C,f)$. Using a suitable subsequence $\{x_{p_i}\}$, we assume that

$$\limsup_{n\to\infty}\langle x_0 - z, x_{n+1} - z\rangle = \lim_{i\to\infty}\langle x_0 - z, x_{p_i} - z\rangle$$

and

$$x_{p_i} \rightharpoonup z' \quad \text{for some} \quad z' \in VI(C,f).$$

Consequently,

$$\limsup_{n\to\infty}\langle x_0 - z, x_{n+1} - z\rangle = \langle x_0 - P_{VI(C,f)}x_0, z' - P_{VI(C,f)}x_0\rangle \leq 0.$$

By Lemma 2.2.9, we have $\lim_{n\to\infty}\|x_n - z\|^2 = 0$. That is, $x_n \to z$.

Case 2. Otherwise, there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that

$$\|x_{m_j} - z\| < \|x_{m_{j+1}} - z\| \quad \text{for all } j \in \mathbb{N}.$$

From Lemma 2.2.10, there exists a nondecreasing $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} n_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{n_k} - z\| \leq \|x_{n_{k+1}} - z\| \quad \text{and} \quad \|x_k - z\| \leq \|x_{n_{k+1}} - z\|. \tag{3.2.4}$$

Observe that

$$\begin{aligned} \|x_{n_k} - z\| \leq \|x_{n_{k+1}} - z\| &\leq \alpha_{n_k}\|x_0 - z\| + (1-\alpha_{n_k})\|w_{n_k} - z\| \\ &\leq \alpha_{n_k}\|x_0 - z\| + (1-\alpha_{n_k})\|x_{n_k} - z\|. \end{aligned}$$

It follows from the fact that $\lim_{n\to\infty}\alpha_n = 0$ that

$$\|w_{n_k} - z\| - \|x_{n_k} - z\| \to 0.$$

By discarding the repeated terms of $\{n_k\}$, but still denoted by $\{n_k\}$, one can view $\{x_{n_k}\}$ as a subsequence of $\{x_n\}$. Hence, by Lemma 3.2.1 and Lemma 3.2.3, we have

$$x_{n_k} - P_C\big(x_{n_k} - \tau f(x_{n_k})\big) \to 0 \quad \text{and} \quad \omega_\omega\{x_{n_k}\} \subset VI(C,f).$$

Observe that $x_{n_k} - x_{n_k+1} \to 0$. In fact, it follows from Lemma 3.2.1 with the same notion U that $\|w_{n_k} - U(x_{n_k})\| \to 0$, $\|U(x_{n_k}) - x_{n_k}\| \to 0$ and

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &= \|\alpha_{n_k} x_0 + (1-\alpha_{n_k}) w_{n_k} - x_{n_k}\| \\ &\leq \alpha_{n_k}\|x_0 - x_{n_k}\| + (1-\alpha_{n_k})\|w_{n_k} - x_{n_k}\| \\ &\leq \alpha_{n_k}\|x_0 - x_{n_k}\| + (1-\alpha_{n_k})\big(\|w_{n_k} - U(x_{n_k})\| + \|U(x_{n_k}) - x_{n_k}\|\big) \\ &\to 0.\end{aligned}$$

As proved in the first case, we can conclude that

$$\limsup_{k\to\infty}\langle x_0 - z, x_{n_k+1} - z\rangle = \limsup_{k\to\infty}\langle x_0 - z, x_{n_k} - z\rangle \leq 0. \tag{3.2.5}$$

It follows from (3.2.3) and (3.2.4) that

$$\begin{aligned}\|x_{n_k+1} - z\|^2 &\leq (1-\alpha_{n_k})\|x_{n_k} - z\|^2 + 2\alpha_{n_k}\langle x_0 - z, x_{n_k+1} - z\rangle \\ &\leq (1-\alpha_{n_k})\|x_{n_k+1} - z\|^2 + 2\alpha_{n_k}\langle x_0 - z, x_{n_k+1} - z\rangle.\end{aligned}$$

Using (3.2.4) and the fact that $\alpha_{n_k} > 0$, we have $\|x_k - z\|^2 \leq \|x_{n_k+1} - z\|^2 \leq 2\langle x_0 - z, x_{n_k+1} - z\rangle$. Hence by (3.2.5), we have

$$\limsup_{k\to\infty}\|x_k - z\|^2 \leq 2\limsup_{k\to\infty}\langle x_0 - z, x_{n_k+1} - z\rangle.$$

Therefore, $x_k \to z$. This completes the proof. ■

3.3 The Modified Subgradient Extragradient Algorithm

Inspired by the second result of [Censor et al., 2011], the authors [Kraikaew and Saejung, 2014], introduce a modified subgradient extragradient algorithm for finding a solution of the variational inequality (1.1.4) which is also a fixed point of a given quasi-nonexpansive mapping. The algorithm is as follows:

Let $f, S : H \to H$ be maps and C be a nonempty, closed and convex subset of a real Hilbert space H. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences defined by

$$\begin{cases} x_0 \in H, \\ y_n := P_C\big(x_n - \tau f(x_n)\big), \\ T_n := \{w \in H : \langle x_n - \tau f(x_n) - y_n, w - y_n\rangle \leq 0\}, \\ z_n := \alpha_n x_0 + (1-\alpha_n) P_{T_n}\big(x_n - \tau f(y_n)\big), \\ x_{n+1} := \beta_n x_n + (1-\beta_n) S z_n, \end{cases} \tag{3.3.1}$$

where $\{\beta_n\} \subset (0,1)$ for some $a, b \in (0,1)$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Theorem 3.3.1 *Let $S : H \to H$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $f : H \to H$ be a monotone and L-Lipschitz mapping on C. Let τ be a positive real number such that $\tau L < 1$. Suppose that $VI(C, f) \cap F(S)$ is nonempty. Let $\{x_n\} \subset H$ be a sequence generated by (3.3.1). Then, $x_n \to P_{VI(C,f)\cap F(S)} x_0$.*

In a similar way, they split the proof into several lemmas.

Lemma 3.3.2 *The sequence $\{x_n\}$ is bounded.*

Proof Let $u \in VI(C, f) \cap F(S)$. Then, we have

$$\begin{aligned}
\|x_{n+1} - u\| &\le \beta_n \|x_n - u\| + (1 - \beta_n)\|S(z_n) - u\| \\
&\le \beta_n \|x_n - u\| + (1 - \beta_n)\|z_n - u\| \\
&= \beta_n \|x_n - u\| + (1 - \beta_n)\|\alpha_n x_0 + (1 - \alpha_n)w_n - u\| \\
&\le \beta_n \|x_n - u\| + (1 - \beta_n)\big(\alpha_n \|x_0 - u\| + (1 - \alpha_n)\|w_n - u\|\big) \\
&\le \beta_n \|x_n - u\| + (1 - \beta_n)\big(\alpha_n \|x_0 - u\| + (1 - \alpha_n)\|x_n - u\|\big) \\
&\le max\{\|x_0 - u\|, \|x_n - u\|\}.
\end{aligned}$$

By induction, the sequence $\{x_n\}$ is bounded. ■

Lemma 3.3.3 *The following inequality holds for all $u \in VI(C, f) \cap F(S)$ and $n \in \mathbb{N}$,*

$$\begin{aligned}
\|x_{n+1} - u\|^2 \le \big(1 - \alpha_n(1 - \beta_n)\big)\|x_n - u\|^2 &+ 2\alpha_n(1 - \beta_n)\langle x_0 - u, z_n - u\rangle \\
&- \beta_n(1 - \beta_n)\|x_n - S(z_n)\|^2.
\end{aligned} \tag{3.3.2}$$

Proof Let $u \in VI(C, f) \cap F(S)$. Set $w_n = P_{T_n}\big(I - \tau f(P_C(I - \tau f))\big)x_n$. It follows from Lemma 2.2.18, Lemma 3.2.1 with $\tau L < 1$ and Lemma 2.2.17 that

$$\begin{aligned}
\|x_{n+1} - u\|^2 &= \|\beta_n(x_n - u) + (1 - \beta_n)\big(S(z_n) - u\big)\|^2 \\
&= \beta_n \|x_n - u\|^2 + (1 - \beta_n)\|S(z_n) - u\|^2 - \beta_n(1 - \beta_n)\|x_n - S(z_n)\|^2 \\
&\le \beta_n \|x_n - u\|^2 + (1 - \beta_n)\|z_n - u\|^2 - \beta_n(1 - \beta_n)\|x_n - S(z_n)\|^2 \\
&= \beta_n \|x_n - u\|^2 + (1 - \beta_n)\|\alpha_n x_0 + (1 - \alpha_n)w_n - u\|^2 - \beta_n(1 - \beta_n)\|x_n - S(z_n)\|^2 \\
&\le \beta_n \|x_n - u\|^2 - \beta_n(1 - \beta_n)\|x_n - S(z_n)\|^2 \\
&\quad + (1 - \beta_n)\big((1 - \alpha_n)^2\|w_n - u\|^2 + 2\alpha_n\langle x_0 - u, z_n - u\rangle\big) \\
&\le \beta_n \|x_n - u\|^2 - \beta_n(1 - \beta_n)\|x_n - S(z_n)\|^2 \\
&\quad + (1 - \beta_n)\big((1 - \alpha_n)\|w_n - u\|^2 + 2\alpha_n\langle x_0 - u, z_n - u\rangle\big) \\
&= \big(1 - \alpha_n(1 - \beta_n)\big)\|x_n - u\|^2 + 2\alpha_n(1 - \beta_n)\langle x_0 - u, z_n - u\rangle \\
&\quad - \beta_n(1 - \beta_n)\|x_n - S(z_n)\|^2.
\end{aligned} \tag{3.3.3}$$

■

Lemma 3.3.4 *Let $u \in VI(C, f) \cap F(S)$. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1} - u\| - \|x_{n_k} - u\|) \ge 0$, then, $\omega_\omega\{x_{n_k}\} \subset VI(C, f) \cap F(S)$.*

Proof Observe that whenever $\liminf_{k\to\infty}(\|x_{n_k+1} - u\| - \|x_{n_k} - u\|) \ge 0$, we get

$$
\begin{aligned}
0 &\le \liminf_{k\to\infty} \big(\|x_{n_k+1}-u\| - \|x_{n_k}-u\|\big)\\
&\le \liminf_{k\to\infty} \big(\beta_{n_k}\|x_{n_k}-u\| + (1-\beta_{n_k})\|S\big(\alpha_{n_k}x_0 + (1-\alpha_{n_k})w_{n_k}\big) - u\| - \|x_{n_k}-u\|\\
&\le \liminf_{k\to\infty}(1-\beta_{n_k})\big(\alpha_{n_k}\|x_0-u\| + (1-\alpha_{n_k})\|w_{n_k}-u\| - \|x_{n_k}-u\|\big)\\
&\le \liminf_{k\to\infty}(1-\beta_{n_k})\big(\|w_{n_k}-u\| - \|x_{n_k}-u\|\big)\\
&\le (1-a)\liminf_{k\to\infty}\big(\|w_{n_k}-u\| - \|x_{n_k}-u\|\big)\\
&\le (1-a)\limsup_{k\to\infty}\big(\|w_{n_k}-u\| - \|x_{n_k}-u\|\big)\\
&\le 0.
\end{aligned}
$$

Hence, $\|w_{n_k}-u\| - \|x_{n_k}-u\| \to 0$. It follows from Lemma 3.2.1 and Lemma 3.2.3 that

$$x_{n_k} - w_{n_k} \to 0 \quad \text{and} \quad \omega_\omega\{x_{n_k}\} \subset VI(C,f). \tag{3.3.4}$$

Next, we show that $\omega_\omega\{x_{n_k}\} \subset F(S)$. Using inequality (3.3.3) and the fact that $\alpha_{n_k} \to 0$, we have

$$
\begin{aligned}
0 &\le \liminf_{k\to\infty}\big(\|x_{n_k+1}-u\|^2 - \|x_{n_k}-u\|^2\big)\\
&\le \liminf_{k\to\infty}\big(-\alpha_{n_k}(1-\beta_{n_k})\|x_{n_k}-u\|^2 + 2\alpha_{n_k}(1-\beta_{n_k})\langle x_0-u, z_{n_k}-u\rangle\\
&- \beta_{n_k}(1-\beta_{n_k})\|x_{n_k} - S(z_{n_k})\|^2\big)\\
&= -\limsup \beta_{n_k}(1-\beta_{n_k})\|x_{n_k} - S(z_{n_k})\|^2\\
&\le -a(1-b)\limsup \|x_{n_k} - S(z_{n_k})\|^2.
\end{aligned}
$$

Hence, $x_{n_k} - S(z_{n_k}) \to 0$. It follows from (3.3.4) that

$$z_{n_k} - x_{n_k} = \alpha_{n_k}(x_0 - x_{n_k}) + (1-\alpha_{n_k})(w_{n_k} - x_{n_k}) \to 0. \tag{3.3.5}$$

Therefore

$$\|z_{n_k} - S(z_{n_k})\| \le \|z_{n_k} - x_{n_k}\| + \|x_{n_k} - S(z_{n_k})\| \to 0.$$

By (3.3.5) and demiclosedness of the mapping $I - S$, we get

$$\omega_\omega\{z_{n_k}\} = \omega_\omega\{x_{n_k}\} \subset F(S).$$

Then,

$$\omega_\omega\{x_{n_k}\} \subset VI(C,f) \cap F(S).$$

■

Proof *of Theorem 3.3.1.* Let $z := P_{VI(C,f)\cap F(S)}x_0$. Since $\beta_n < 1$, for all $n \in \mathbb{N}$, it follows from inequality (3.3.3) that

$$\|x_{n+1} - z\|^2 \le \big(1 - \alpha_n(1-\beta_n)\big)\|x_n - u\|^2 + 2\alpha_n(1-\beta_n)\langle x_0 - u, z_n - u\rangle. \tag{3.3.6}$$

Case 1. There exists $n_0 \in \mathbb{N}$ such that $\|x_{n+1} - z\| \le \|x_n - z\|$ for all $n \ge n_0$. Then, $\lim_{n\to\infty}\|x_n - z\|$ exists. In particular, $\liminf_{n\to\infty}(\|x_{n+1} - z\| - \|x_n - z\|) = 0$. It follows from Lemma 3.3.4 that $\omega_\omega\{x_n\} \subset VI(C,f) \cap F(S)$ and $w_n - x_n \to 0$. Since

$z_n - x_n = \alpha_n(x_0 - x_n) + (1-\alpha_n)(w_n - x_n) \to 0$, we have $\omega_\omega\{z_n\} = \omega_\omega\{x_n\}$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \widehat{x}$ and

$$\lim_{k\to\infty} \langle x_0 - z, x_{n_k} - z\rangle = \limsup_{n\to\infty}\langle x_0 - z, x_n - z\rangle = \limsup_{n\to\infty}\langle x_0 - z, z_n - z\rangle.$$

Because $\omega_\omega\{x_n\} \subset VI(C,f)$, we have

$$\lim_{k\to\infty} \langle x_0 - z, x_{n_k} - z\rangle = \langle x_0 - z, \widehat{x} - z\rangle \leq 0.$$

Hence, $\limsup_{n\to\infty}\langle x_0 - z, z_n - z\rangle \leq 0$. By applying Lemma 2.2.8 to inequality (3.3.6), we have $\|x_n - z\| \to 0$, that is, $x_n \to z$.

Case 2. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\|x_{m_j} - z\| \leq \|x_{n_k} + 1 - z\| \quad \forall j \in \mathbb{N}.$$

From Lemma 2.2.9, there exists a nondecreasing sequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} n_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{n_k} - z\| \leq \|x_{n_k+1} - z\| \quad \text{and} \quad \|x_k - z\| \leq \|x_{n_k+1} - z\|. \tag{3.3.7}$$

By discarding the repeated terms $\{n_k\}$, but still denoted $\{n_k\}$, we can view $\{x_{n_k}\}$ as a subsequence of $\{x_n\}$. In this case, we have $\liminf_{k\to\infty}(\|x_{n_k+1} - z\| - \|x_{n_k} - z\|) \geq 0$. Hence, $\omega_\omega\{x_{n_k}\} \subset VI(C,f) \cap F(S)$ and, by using a similar argument as in the first case, $\omega_\omega\{z_{n_k}\} = \omega_\omega\{x_{n_k}\}$. It follows from the boundedness of $\{x_{n_k}\}$ that there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \rightharpoonup \widehat{x}$ and

$$\lim_{l\to\infty} \langle x_0 - z, x_{n_{k_l}} - z\rangle = \limsup_{k\to\infty}\langle x_0 - z, x_{n_k} - z\rangle = \limsup_{k\to\infty}\langle x_0 - z, \widehat{x} - z\rangle.$$

Because $\omega_\omega\{x_{n_k}\} \subset VI(C,f)$, we have

$$\limsup_{k\to\infty}\langle x_0 - z, z_{n_k} - z\rangle = \lim_{l\to\infty}\langle x_0 - z, x_{n_{k_l}} - z\rangle = \langle x_0 - z, \widehat{x} - z\rangle \leq 0.$$

It follows from (3.3.6) and (3.3.7) that

$$\begin{aligned}
\|x_{n_k+1} - z\|^2 &\leq \big(1 - \alpha_{n_k}(1 - \beta_{n_k})\big)\|x_{n_k} - u\|^2 + 2\alpha_{n_k}(1 - \beta_{n_k}\langle x_0 - u, z_{n_k} - u\rangle \\
&\leq \big(1 - \alpha_{n_k}(1 - \beta_{n_k})\big)\|x_{n_k} - u\|^2 + 2\alpha_{n_k}(1 - \beta_{n_k+1}\langle x_0 - u, z_{n_k} - u\rangle.
\end{aligned}$$

Since $\alpha_{n_k}(1 - \beta_{n_k}) > 0$, for all $k \in \mathbb{N}$,

$$\|x_k - z\|^2 \leq \|x_{n_k+1} - z\|^2 \leq 2\langle x_0 - z, x_{n_k+1} - z\rangle.$$

Consequently,

$$\limsup_{k\to\infty} \|x_{n_k} - z\|^2 \leq \limsup_{k\to\infty} 2\langle x_0 - z, x_{n_k+1} - z\rangle \leq 0.$$

Therefore $x_k \to z$. ∎

CHAPTER 4

Our Contributions

In this chapter we present an extension of the results of [Kraikaew and Saejung, 2014], to a uniformly smooth and 2-uniformly convex real Banach space. Furthermore, we extend the class of map from one nonexpansive map to a countable family of relatively nonexpansive maps.

4.1 Approximating a solution of a variational inequality problem

We prove the following theorem.

Theorem 4.1.1 *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Let $A : E \to E^*$ be a monotone map on C and k-Lipschitz on E. Assume $VI(C, A) \neq \emptyset$. Define the sequence $\{x_n\}$ by,*

$$\begin{cases} x_0 \in C; \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda Ax_n); \\ T_n = \{z \in E : \langle z - y_n, Jx_n - \lambda Ax_n - Jy_n\rangle \leq 0\}; \\ t_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda Ay_n); \\ x_{n+1} = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)Jt_n), \end{cases} \tag{4.1.1}$$

for all $n \geq 0$, where $\lambda \in (0, b]$, $b \in (0, 1)$ with $b < \frac{\alpha}{k}$, α being the constant in Lemma 2.2.2 and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly the point $q = \Pi_{VI(C,\ A)}x_0$.

Proof We divide the proof into two steps.

Step 1. We show that $\{x_n\}$ is bounded. Let $u \in VI(C, A)$. Then, applying

Lemma 2.2.1 (2), we have

$$\begin{aligned}\phi(u,t_n) &\leq \phi(u,J^{-1}[Jx_n-\lambda Ay_n])-\phi(t_n,J^{-1}[Jx_n-\lambda Ay_n])\\ &= \|u\|^2-2\langle u,Jx_n-\lambda Ay_n\rangle-\|t_n\|^2+2\langle t_n,Jx_n-\lambda Ay_n\rangle\\ &= \phi(u,x_n)-\phi(t_n,x_n)+2\langle u-t_n,\lambda Ay_n\rangle\\ &= \phi(u,x_n)-\phi(t_n,x_n)+2\lambda\langle u-y_n,Ay_n\rangle+2\lambda\langle y_n-t_n,Ay_n\rangle.\end{aligned} \tag{4.1.2}$$

Using Remark 1 and property P2, we have

$$\begin{aligned}\phi(u,t_n) &\leq \phi(u,x_n)-\phi(t_n,x_n)+2\lambda\langle y_n-t_n,Ay_n\rangle\\ &= \phi(u,x_n)-\phi(y_n,x_n)-\phi(t_n,y_n)+2\left\langle t_n-y_n,Jx_n-\lambda Ay_n-Jy_n\right\rangle.\end{aligned} \tag{4.1.3}$$

Now, we estimate $\langle t_n-y_n,Jx_n-\lambda Ay_n-Jy_n\rangle$, using the fact that $t_n\in T_n$, the Lipschitz continuity of A and Lemma 2.2.2. We obtain

$$\begin{aligned}\langle t_n-y_n,Jx_n-\lambda Ay_n-Jy_n\rangle &= \langle t_n-y_n,Jx_n-\lambda Ax_n-Jy_n\rangle+\lambda\langle t_n-y_n,Ax_n-Ay_n\rangle\\ &\leq \lambda\langle t_n-y_n,Ax_n-Ay_n\rangle\\ &\leq \lambda\|t_n-y_n\|\|Ax_n-Ay_n\|\\ &\leq \frac{k\lambda}{2}\Big(\|t_n-y_n\|^2+\|x_n-y_n\|^2\Big)\\ &\leq \frac{k\lambda}{2\alpha}\Big(\phi(t_n,y_n)+\phi(y_n,x_n)\Big).\end{aligned} \tag{4.1.4}$$

Thus,
$$\begin{aligned}\phi(u,t_n) &\leq \phi(u,x_n)-\phi(y_n,x_n)-\phi(t_n,y_n)+\frac{k\lambda}{\alpha}\Big(\phi(t_n,y_n)+\phi(y_n,x_n)\Big)\\ &= \phi(u,x_n)-\Big(1-\frac{k\lambda}{\alpha}\Big)\Big(\phi(y_n,x_n)+\phi(t_n,y_n)\Big) &\quad(4.1.5)\\ &\leq \phi(u,x_n). &\quad(4.1.6)\end{aligned}$$

Now,
$$\begin{aligned}\phi(u,x_{n+1}) &\leq \alpha_n\phi(u,x_0)+(1-\alpha_n)\phi(u,t_n)\\ &\leq \alpha_n\phi(u,x_0)+(1-\alpha_n)\phi(u,x_n)\\ &\leq max\{\phi(u,x_0),\phi(u,x_n)\}.\end{aligned} \tag{4.1.7}$$

By induction, we obtain that $\phi(u,x_{n+1})\leq\phi(u,x_0)$. Hence, the sequence $\{\phi(u,x_n)\}$ is bounded. By property P1, $\{x_n\}$ is bounded. Furthermore, $\phi(u,t_n)\leq\phi(u,x_n)$, $\forall\ n\ \geq 0$ implies that $\{t_n\}$ is also bounded. Using Lemma 2.2.10. We have

$$\begin{aligned}\phi(u,x_{n+1}) &= V(u,Jx_{n+1})\\ &\leq V(u,(1-\alpha_n)Jt_n+\alpha_nJu)+2\alpha_n\langle x_{n+1}-u,Jx_0-Ju\rangle\\ &\leq (1-\alpha_n)V(u,Jt_n)+2\alpha_n\langle x_{n+1}-u,Jx_0-Ju\rangle &\quad(4.1.8)\\ &= (1-\alpha_n)\phi(u,t_n)+2\alpha_n\langle x_{n+1}-u,Jx_0-Ju\rangle\\ &\leq (1-\alpha_n)\phi(u,x_n)+2\alpha_n\langle x_{n+1}-u,Jx_0-Ju\rangle. &\quad(4.1.9)\end{aligned}$$

Step 2. We show that the sequence $\{x_n\}$ converges strongly to the point $q=\Pi_{VI(C,\ A)}x_0$. To show this, we shall consider two cases. Let $u\in VI(C,A)$

Case 1. Assume there exists $n_0 \in \mathbb{N}$, such that

$$\phi(u, x_{n+1}) \leq \phi(u, x_n), \ \forall \ n \geq n_0.$$

Then, the sequence $\{\phi(u, x_n)\}$ is convergent.

Claim 1:

(*i*) $\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|t_n - y_n\| = 0.$

From inequality (4.1.8) and (4.1.9) we deduce that

$$\lim_{n\to\infty} \big(\phi(u, t_n) - \phi(u, x_n)\big) = 0.$$

Hence, from inequality (4.1.5), we obtain

$$\lim_{n\to\infty} \phi(y_n, x_n) = \lim_{n\to\infty} \phi(t_n, y_n) = 0.$$

By Lemma 2.2.5 $\lim_{n\to\infty} \|x_n - y_n\| = 0 = \lim_{n\to\infty} \|t_n - y_n\|$.

Next we show that $\Omega_w(x_n) \subset VI(C, A)$, where $\Omega_w(x_n)$ denotes the set of weak subsequential limits of $\{x_n\}$. Since $\{x_n\}$ is bounded, $\Omega_w(x_n) \neq \emptyset$. Let $u \in \Omega_w(x_n)$. Then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup u$. We show that $u \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C(v), & if \ v \in C, \\ \emptyset, & if \ v \notin C, \end{cases}$$

be as defined in Lemma 2.2.7. Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$. It is known that if T is maximal monotone, then given $(x, x^*) \in E \times E^*$ such that if $\langle x - y, x^* - y^* \rangle \geq 0$, $\forall \ (y, y^*) \in G(T)$, one has $x^* \in Tx$.

Claim 2: $(u, 0) \in G(T)$.
Let $(v, u^*) \in G(T)$. To establish the claim, it suffices to show that $\langle v - u, u^* \rangle \geq 0$.

Now, $(v, u^*) \in G(T) \ \Rightarrow \ u^* \in Tv = Av + N_C(v) \ \Rightarrow \ u^* - Av \in N_C(v)$.
Therefore, $\langle v - y, u^* - Av \rangle \geq 0, \ \forall \ y \in C$. Since $y_n = \Pi_C J^{-1}(Jx_n - \lambda Ax_n)$ and $v \in C$, we have by Lemma 2.2.1 (1) that $\langle y_n - v, Jx_n - \lambda Ax_n - Jy_n \rangle \geq 0$. Thus,

$$\left\langle v - y_n, \frac{Jy_n - Jx_n}{\lambda} + Ax_n \right\rangle \geq 0, \quad n \geq 0 \ .$$

Using the fact that $y_n \in C$ and $u^* - Av \in N_C(v)$, we have

$$\begin{aligned}
\langle v - y_{n_k}, u^* \rangle &\geq \langle v - y_{n_k}, Av \rangle \\
&\geq \langle v - y_{n_k}, Av \rangle - \left\langle v - y_{n_k}, \frac{Jy_{n_k} - Jx_{n_k}}{\lambda} + Ax_{n_k} \right\rangle \\
&= \langle v - y_{n_k}, Av - Ay_{n_k} \rangle + \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k} \rangle - \left\langle v - y_{n_k}, \frac{Jy_{n_k} - Jx_{n_k}}{\lambda} \right\rangle \\
&\geq \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k} \rangle - \left\langle v - y_{n_k}, \frac{Jy_{n_k} - Jx_{n_k}}{\lambda} \right\rangle.
\end{aligned}$$

Using the Lipschitz continuity of A, Claim 1 and uniform continuity of J on bounded sets, we have

$$\langle v-u, u^*\rangle \geq 0.$$

Therefore, $\Omega_w(x_n) \subset VI(C, A)$.

Finally, we show that $\{x_n\}$ converges strongly to the point $q = \Pi_{VI(C,\ A)}x_0$. Since $\Omega_w(x_n) \subset VI(C, A)$, from inequality (4.1.9), and using Lemma 2.2.8 and Lemma 2.2.5, it suffices to show that $\limsup_{n\to\infty}\langle x_{n+1}-q, Jx_0-Jq\rangle \leq 0$. Let $z \in \Omega_w(x_n)$. Then, there exists a suitable subsequence $\{x_{n_k}\}$ such that

$$\limsup_{n\to\infty}\langle x_{n+1}-q, Jx_0-Jq\rangle = \lim_{k\to\infty}\langle x_{n_k}-q, Jx_0-Jq\rangle = \langle z-q, Jx_0-Jq\rangle.$$

By Lemma 2.2.1 (1), we have

$$\limsup_{n\to\infty}\langle x_{n+1}-q, Jx_0-Jq\rangle = \langle z-q, Jx_0-Jq\rangle \leq 0.$$

Hence, $x_n \to q$.

Case 2. If Case 1 does not hold, then, there exists a subsequence $\{x_{m_j}\} \subset \{x_n\}$ such that $\phi(u, x_{m_j+1}) > \phi(u, x_{m_j})$, for all $j \in \mathbb{N}$, $u \in VI(C, A)$. From Lemma 2.2.9, there exists a nondecreasing sequence $\{n_k\} \subset \mathbb{N}$, such that $\lim_{k\to\infty} n_k = \infty$ and the following inequalities hold:

$$\phi(u, x_{n_k}) \leq \phi(u, x_{n_k+1}) \text{ and } \phi(u, x_k) \leq \phi(u, x_{n_k+1}), \text{ for each } k \in \mathbb{N}.$$

Observe that

$$\begin{aligned}\phi(u, x_{n_k}) \leq \phi(u, x_{n_k+1}) &\leq \alpha_{n_k}\phi(u, x_0) + (1-\alpha_{n_k})\phi(u, t_{n_k})\\ &\leq \alpha_{n_k}\phi(u, x_0) + (1-\alpha_{n_k})\phi(u, x_{n_k}).\end{aligned}$$

Since $\alpha_n \to 0$, it follows that

$$\lim_{k\to\infty}\big(\phi(u, t_{n_k}) - \phi(u, x_{n_k})\big) = 0.$$

Using inequality (4.1.5), we obtain in a similar way as in Claim 1 that

$$\lim_{k\to\infty}\|t_{n_k}-y_{n_k}\| = \lim_{k\to\infty}\|y_{n_k}-x_{n_k}\| = 0.$$

Also, using a similar argument as in Claim 2, we obtain that $\Omega_w(x_{n_k}) \subset VI(C, A)$.

Next, we show that $\{x_k\}$ converges strongly to the point $q = \Pi_{VI(C,\ A)}x_0$. From inequality (4.1.9), setting $u = q$ we have

$$\begin{aligned}\phi(q, x_{n_k+1}) &\leq (1-\alpha_{n_k})\phi(q, x_{n_k}) + 2\alpha_{n_k}\langle x_{n_k+1}-q, Jx_0-Jq\rangle\\ &\leq (1-\alpha_{n_k})\phi(q, x_{n_k+1}) + 2\alpha_{n_k}\langle x_{n_k+1}-q, Jx_0-Jq\rangle.\end{aligned}$$

Since $\alpha_{n_k} > 0$, we have $\phi(q, x_{n_k+1}) \leq 2\langle x_{n_k+1}-q, Jx_0-Jq\rangle$.

Thus, $\phi(q, x_k) \leq 2\langle x_{n_k+1}-q, Jx_0-Jq\rangle$. Hence, $\limsup_{k\to\infty}\phi(q, x_k) \leq 2\limsup_{k\to\infty}\langle x_{n_k+1}-q, Jx_0-Jq\rangle \leq 0$. So, By Lemma 2.2.5, we have $x_k \to q$, as $k \to \infty$. ■

4.2 Approximating a common element of solutions of a variational inequality problem and a fixed point of a relatively nonexpansive map

We present a modified subgradient extragradient algorithm for finding a solution of the variational inequality problem (1.1.4) which is also a fixed point of a given relatively nonexpansive map.

Theorem 4.2.1 *Let E be a uniformly smooth and 2-uniformly convex real Banach space and C be a nonempty, closed and convex subset of E. Let $A : E \to E^*$ be a monotone map on C and k-Lipschitz on E and let $S : E \to E$ b a relatively nonexpansive map. We define the sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in E; \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda Ax_n); \\ T_n = \{z \in E : \langle z - y_n, Jx_n - \lambda Ax_n - Jy_n\rangle \leq 0\}; \\ t_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda Ay_n); \\ z_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jt_n); \\ x_{n+1} = J^{-1}(\lambda Jx_n + (1-\lambda)JSz_n), \end{cases} \tag{4.2.1}$$

where $\lambda \in (0,1)$ such that $\lambda < \frac{\alpha}{k}$, α been the constant in Lemma 2.2.2 and $\{\alpha_n\} \subset [0,1]$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose $F(S) \cap VI(C,A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (4.2.1) converges strongly to the point $q = \Pi_{F(S)VI(C,A)} x_0$.

Proof We divide the proof into two steps.
Step 1. We show that $\{x_n\}$ is bounded. Let $u \in F(S) \cap VI(C,A)$. Then,

$$\begin{aligned} \phi(u, x_{n+1}) &\leq \lambda\phi(u,x_n) + (1-\lambda)\phi(u,z_n) \\ &\leq \lambda\phi(u,x_n) + (1-\lambda)\big(\alpha_n\phi(u,x_0) + (1-\alpha_n)\phi(u,t_n)\big) \\ &\leq \lambda\phi(u,x_n) + (1-\lambda)\alpha_n\phi(u,x_0) + (1-\lambda)(1-\alpha_n)\phi(u,x_n) \\ &= \big(1-(1-\lambda)\alpha_n\big)\phi(u,x_n) + (1-\lambda)\alpha_n\phi(u,x_0) \\ &\leq max\{\phi(u,x_n), \phi(u,x_0)\} \end{aligned} \tag{4.2.2}$$

By induction, we have $\phi(u,x_{n+1}) \leq \phi(u,x_0)$. Hence, the sequence $\{\phi(u,x_n)\}$ is bounded. By property P1, $\{x_n\}$ is bounded. Furthermore, by Lemma 2.2.12, we have that $\{y_n\}$ is also bounded.

Step 2. We show that $\{x_n\}$ converges strongly to some point $q = \Pi_{F(S)\cap VI(C,A)} x_0$. To show this, we first establish the following:

(*i*) $\lim\limits_{n\to\infty} \|x_n - y_n\| = \lim\limits_{n\to\infty} \|t_n - y_n\| = \lim\limits_{n\to\infty} \|z_n - Sz_n\| = 0$; *and*

(*ii*) $\Omega_w(x_n) \subset F(S) \cap VI(C,A)$.

Let $u \in F(S) \cap VI(C,A)$. We shall consider two cases.

Case 1. Suppose there exists an $n_0 \in \mathbb{N}$, such that

$$\phi(u, x_{n+1}) \leq \phi(u, x_n), \ \forall\ n \geq n_0.$$

Then, the sequence $\{\phi(u, x_n)\}$ is convergent.

Now, we estimate $\phi(u, x_{n+1})$ using inequality (4.1.5).

$$\begin{aligned}
\phi(u, x_{n+1}) &\leq \lambda\phi(u, x_n) + (1-\lambda)\phi(u, Sz_n) \\
&\leq \lambda\phi(u, x_n) + (1-\lambda)\phi(u, z_n) \\
&\leq \lambda\phi(u, x_n) + (1-\lambda)\big(\alpha_n\phi(u, x_0) + (1-\alpha_n)\phi(u, t_n)\big) \\
&\leq \lambda\phi(u, x_n) + (1-\lambda)\alpha_n\phi(u, x_0) + (1-\lambda)(1-\alpha_n)\Big(\phi(u, x_n) - (1 - \frac{\lambda k}{\alpha})\phi(y_n, x_n)\Big) \\
&= \phi(u, x_n) + (1-\lambda)\alpha_n\phi(u, x_0) + \alpha_n(\lambda - 1)\phi(u, x_n) - (1-\lambda)\left(1 - \frac{\lambda k}{\alpha}\right)\phi(y_n, x_n) \\
&\qquad + \alpha_n(1-\lambda)\left(1 - \frac{\lambda k}{\alpha}\right)\phi(y_n, x_n).
\end{aligned} \tag{4.2.3}$$

Thus,

$$\sigma\phi(y_n, x_n) \leq \phi(u, x_n) - \phi(u, x_{n+1}) + (1-\lambda)\alpha_n\phi(u, x_0) + \alpha_n(\lambda-1)\phi(u, x_n) + \alpha_n\sigma\phi(y_n, x_n),$$

where $\sigma = (1-\lambda)\big(1 - \frac{\lambda k}{\alpha}\big)$. Using the fact that $\alpha_n \to 0$, the boundedness of $\{x_n\}$ and $\{y_n\}$, we deduce that $\phi(y_n, x_n) \to 0$, as $n \to \infty$. Hence, by Lemma 2.2.5, we have $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Next, using inequality (4.1.5), we have

$$\begin{aligned}
\phi(u, x_{n+1}) &\leq \lambda\phi(u, x_n) + (1-\lambda)\alpha_n\phi(u, x_0) + (1-\lambda)(1-\alpha_n)\phi(u, t_n) \\
&= \lambda\phi(u, x_n) + (1-\lambda)\alpha_n\phi(u, x_0) + (1-\lambda)\phi(u, t_n) - (1-\lambda)\alpha_n\phi(u, t_n) \\
&\leq \lambda\phi(u, x_n) + (1-\lambda)\phi(u, x_n) + (1-\lambda)\alpha_n\phi(u, x_0) - (1-\lambda)\alpha_n\phi(u, t_n).
\end{aligned} \tag{4.2.4}$$

Hence, $\lim_{n\to\infty} \big(\phi(u, x_n) - \phi(u, t_n)\big) = 0$. Also, from inequality (4.1.5), we have

$$0 \leq \Big(1 - \frac{\lambda k}{\alpha}\Big)\phi(t_n, y_n) \leq \phi(u, x_n) - \phi(u, t_n) - \Big(1 - \frac{\lambda k}{\alpha}\Big)\phi(y_n, x_n).$$

Thus, $\phi(t_n, y_n) \to 0$, as $n \to \infty$. By Lemma 2.2.5, $\|t_n - y_n\| \to 0$, as $n \to \infty$. It follows that $\lim\limits_{n\to\infty} \|x_n - t_n\| = 0$.

Next, observe that $\phi(x_n, z_n) \leq \alpha_n\phi(x_n, x_0) + (1-\alpha_n)\phi(x_n, t_n)$. Using the fact that $\alpha_n \to 0$, as $n \to \infty$, boundedness of $\{x_n\}, \{t_n\}$ and $\|x_n - t_n\| \to 0$, we have $\phi(x_n, z_n) \to 0$, as $n \to \infty$. Thus, $\|x_n - z_n\| \to 0$ as $n \to \infty$. Using Lemma 2.2.6, we have

$$
\begin{aligned}
\phi(u,x_{n+1}) &\le \lambda\phi(u,x_n) + (1-\lambda)\phi(u,Sz_n) - \lambda(1-\lambda)g(\|Jx_n - JSz_n\|)\\
&\le \lambda\phi(u,x_n) + (1-\lambda)\big(\alpha_n\phi(u,x_0) + (1-\alpha_n)\phi(u,t_n)\big) - \lambda(1-\lambda)g(\|Jx_n - JSz_n\|)\\
&\le \lambda\phi(u,x_n) + (1-\lambda)\alpha_n\phi(u,x_0) + (1-\lambda)(1-\alpha_n)\phi(u,x_n) - \lambda(1-\lambda)g(\|Jx_n - JSz_n\|)\\
&= \phi(u,x_n) + (1-\lambda)\alpha_n\big(\phi(u,x_0) - \phi(u,x_n)\big) - \lambda(1-\lambda)g(\|Jx_n - JSz_n\|)
\end{aligned} \tag{4.2.5}
$$

Thus,
$$
\begin{aligned}
0 \le \lambda(1-\lambda)g(\|Jx_n - JSz_n\|) &\le \phi(u,x_n) - \phi(u,x_{n+1})\\
&+ \alpha_n(1-\lambda)\big(\phi(u,x_0) - \phi(u,x_n)\big)
\end{aligned} \tag{4.2.6}
$$

Since $\alpha_n \to 0$, $\{x_n\}$ is bounded and $\lim_{n\to\infty}\phi(u,x_n)$ exists, we have that $g(\|Jx_n - JSz_n\|) \to 0$, as $n \to \infty$. This implies that $\|Jx_n - JSz_n\| \to 0$. By uniform continuity of J^{-1} on bounded sets, we have $\|x_n - Sz_n\| \to 0$. Hence, $\|z_n - Sz_n\| \to 0$ since $\|z_n - Sz_n\| \le \|z_n - x_n\| + \|x_n - Sz_n\|$.

Now, we show that $\Omega_w(x_n) \subset F(S)\cap VI(C,A)$. Since $\{x_n\}$ is bounded, $\Omega_w(x_n) \neq \emptyset$. Let $z \in \Omega_w(x_n)$. Then, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup z$. This implies that $z_{n_k} \rightharpoonup z$, as $k \to \infty$. Since $\lim_{k\to\infty}\|z_{n_k} - Sz_{n_k}\| = 0$, it follows that $z \in F(S)$. By similar argument as in the prove of Claim 2 in Theorem 4.1.1 above, $z \in VI(C,A)$. Therefore, $\Omega_w(x_n) \subset F(S)\cap VI(C,A)$.

Next, we show that $\{x_n\}$ converges strongly to the point $q = \Pi_{F(S)\cap VI(C,A)}x_0$. Since $\{x_n\}$ is bounded, then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \rightharpoonup z$ and

$$
\lim_{k\to\infty}\langle x_{n_k} - q, Jx_0 - Jq\rangle = \limsup_{n\to\infty}\langle x_n - q, Jx_0 - Jq\rangle = \limsup_{n\to\infty}\langle z_n - q, Jx_0 - Jq\rangle.
$$

Since $\Omega_w(x_n) \subset F(S)\cap VI(C,A)$, we have $\lim\langle x_{n_k} - q, Jx_0 - Jq\rangle = \langle z - q, Jx_0 - Jq\rangle \le 0$. Hence, we deduce that

$$
\limsup_{n\to\infty}\langle z_n - q, Jx_0 - Jq\rangle \le 0. \tag{4.2.7}
$$

But, from Lemma 2.2.10, we have

$$
\begin{aligned}
\phi(q,x_{n+1}) &= \phi\big(q, J^{-1}(\lambda Jx_n + (1-\lambda)JSz_n\big)\\
&\le \lambda\phi(q,x_n) + (1-\lambda)\phi(q,Sz_n)\\
&\le \lambda\phi(q,x_n) + (1-\lambda)\phi\Big(q, J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n)Jt_n\big)\Big)\\
&= \lambda\phi(q,x_n) + (1-\lambda)V\big(q, \alpha_n Jx_0 + (1-\alpha_n)Jt_n\big)\\
&\le \lambda\phi(q,x_n) + (1-\lambda)\Big(V\big(q, \alpha_n Jx_0 + (1-\alpha_n)Jt_n - \alpha_n(Jx_0 - Jq)\big)\\
&\qquad + 2\alpha_n\langle z_n - q, Jx_0 - Jq\rangle\Big)\\
&= \lambda\phi(q,x_n) + (1-\lambda)\Big(V\big(q, \alpha_n Jq + (1-\alpha_n)Jt_n\big) + 2\alpha_n\langle z_n - q, Jx_0 - Jq\rangle\Big)\\
&\le \lambda\phi(q,x_n) + (1-\lambda)(1-\alpha_n)V(q,Jt_n) + 2(1-\lambda)\alpha_n\langle z_n - q, Jx_0 - Jq\rangle\\
&\le \lambda\phi(q,x_n) + (1-\lambda)(1-\alpha_n)\phi(q,x_n) + 2(1-\lambda)\alpha_n\langle z_n - q, Jx_0 - Jq\rangle\\
&= \big(1 - (1-\lambda)\alpha_n\big)\phi(q,x_n) + 2(1-\lambda)\alpha_n\langle z_n - q, Jx_0 - Jq\rangle.
\end{aligned} \tag{4.2.8}
$$

Using (4.2.7) and Lemma 2.2.8, we have $\phi(q, x_n) \to 0$. Hence, by Lemma 2.2.5, we have $x_n \to q$.

Case 2. If Case 1 does not hold, then, there exists a subsequence $\{x_{m_j}\} \subset \{x_n\}$ such that $\phi(u, x_{m_j+1}) > \phi(u, x_{m_j})$, for all $j \in \mathbb{N}$. From Lemma 2.2.9, there exists a nondecreasing sequence $\{n_k\} \subset \mathbb{N}$, such that $\lim_{k\to\infty} n_k = \infty$ and the following inequalities hold

$$\phi(u, x_{n_k}) \leq \phi(u, x_{n_k+1}) \text{ and } \phi(u, x_k) \leq \phi(u, x_{n_k+1}), \text{ for each } k \in \mathbb{N}.$$

Now,

$$\begin{aligned}
\phi(u, x_{n_k}) \leq \phi(u, x_{n_k+1}) &\Rightarrow\ 0 \leq \phi(u, x_{n_k+1}) - \phi(u, x_{n_k}) \\
&\Rightarrow\ 0 \leq \liminf_{k\to\infty} \big(\phi(u, x_{n_k+1}) - \phi(u, x_{n_k})\big) \\
&\leq \liminf_{k\to\infty} \Big(\lambda\phi(u, x_{n_k}) + (1-\lambda)\big(\alpha_{n_k}\phi(u, x_0) \\
&\qquad + (1-\alpha_{n_k})\phi(u, t_{n_k})\big) - \phi(u, x_{n_k})\Big) \\
&= \liminf_{k\to\infty} (1-\lambda)\big(\phi(u, t_{n_k}) - \phi(u, x_{n_k})\big).
\end{aligned}$$

Since $\phi(u, t_{n_k}) \leq \phi(u, x_{n_k}),\ \forall k \geq 0$,

$$0 \leq \liminf_{k\to\infty} \big(\phi(u, t_{n_k}) - \phi(u, x_{n_k})\big) \leq \limsup_{k\to\infty} \big(\phi(u, t_{n_k}) - \phi(u, x_{n_k})\big) \leq 0.$$

Hence, $\lim_{k\to\infty} \big(\phi(u, t_{n_k}) - \phi(u, x_{n_k})\big) = 0$. Using similar argument as in Case 1 above, we obtain that

- $\|t_{n_k} - y_{n_k}\| \to 0,\ \|y_{n_k} - x_{n_k}\| \to 0$, as $k \to \infty$;
- $\|x_{n_k} - z_{n_k}\| \to 0,\ \|Sz_{n_k} - z_{n_k}\| \to 0$, as $k \to \infty$; and
- $\Omega_w(x_{n_k}) \subset F(S) \cap VI(C, A)$.

Next, we show that $\{x_k\}$ converges strongly to $q = \Pi_{F(S)\cap VI(C,\ A)} x_0$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup z$, as $j \to \infty$ and

$$\lim_{j\to\infty} \langle x_{n_{k_j}} - q, Jx_0 - Jq\rangle = \limsup_{k\to\infty} \langle x_{n_k} - q, Jx_0 - Jq\rangle = \limsup_{k\to\infty} \langle z_{n_k} - q, Jx_0 - Jq\rangle.$$

Since $\Omega_w(x_{n_k}) \subset F(S) \cap VI(C, A))$, we have $\limsup_{k\to\infty} \langle z_{n_k} - q, Jx_0 - Jq\rangle \leq 0$. From inequality (4.2.8), we have

$$\begin{aligned}
\phi(q, x_{n_k+1}) &\leq \big(1 - (1-\lambda)\alpha_{n_k}\big)\phi(q, x_{n_k}) + 2(1-\lambda)\alpha_{n_k}\langle z_{n_k} - q, Jx_0 - Jq\rangle \\
&\leq \big(1 - (1-\lambda)\alpha_{n_k}\big)\phi(q, x_{n_k+1}) + 2(1-\lambda)\alpha_{n_k}\langle z_{n_k} - q, Jx_0 - Jq\rangle.
\end{aligned}$$

Since $(1-\lambda)\alpha_{n_k} > 0$ for all $k \geq 0$, we have

$$\phi(q, x_k) \leq \phi(q, x_{n_k+1}) \leq 2\langle z_{n_k} - q, Jx_0 - Jq\rangle \Rightarrow \limsup_{k\to\infty} \phi(q, x_k) \leq \limsup_{k\to\infty} 2\langle z_{n_k} - q, Jx_0 - Jq\rangle.$$

Thus, $\limsup\limits_{k\to\infty} \phi(q, x_k) \leq 0$. Therefore, $x_k \to q$, as $k \to \infty$. ∎

4.3 Approximating a common element of variational inequality and convex feasibility problem.

We prove the following theorem.

Theorem 4.3.1 *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Let $A : E \to E^*$ be a monotone map on C and k-Lipschitz on E and $\{S_i\}_{i=1}^{\infty}$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, where $S_i : E \to E$, $\forall\ i$. Let $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\mu_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Define the sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in E; \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n); \\ T_n = \{z \in E : \langle z - y_n, Jx_n - \lambda Ax_n - Jy_n\rangle \leq 0\}; \\ t_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda A y_n); \\ z_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jt_n); \\ x_{n+1} = J^{-1}(\lambda J x_n + (1-\lambda) JSz_n), \end{cases} \tag{4.3.1}$$

where $Sx = J^{-1}\Big(\sum_{i=1}^{\infty} \eta_i(\mu_i Jx + (1-\mu_i)JS_i x\Big)$ for each $x \in E$, $\lambda \in (0, \frac{\alpha}{k})$ and $\{\alpha_n\} \subset [0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose $\Big(\bigcap_{i=1}^{\infty} F(S_i)\Big) \cap V(C,A) \neq \emptyset$, then, the sequence $\{x_n\}$ generated by (4.3.1) converges strongly to the point $q = \Pi_{F(S)\cap VI(C,A)} x_0$.

Proof By Lemma 2.2.11, S is relatively nonexpansive and $F(S) = \bigcap_{i=1}^{\infty} F(S_i)$. The conclusion follows from Theorem 4.2.1. ■

4.4 Applications

We give some applications of our main theorem.

Theorem 4.4.1 *Let C be a nonempty, closed and convex subset of $E = L_p$ $\Big($or l_p or $W_p^m(\Omega)\Big)$, $1 < p \leq 2$. Let $A : E \to E^*$ be a monotone map on C, k-Lipschitz on E and $\{S_i\}_{i=1}^{\infty}$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, where $S_i : E \to E$, $\forall\ i$. Let $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\mu_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Define inductively the sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in E; \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A x_n); \\ T_n = \{z \in E : \langle z - y_n, Jx_n - \lambda Ax_n - Jy_n\rangle \leq 0\}; \\ t_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda A y_n); \\ z_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jt_n); \\ x_{n+1} = J^{-1}(\lambda J x_n + (1-\lambda) JSz_n), \end{cases} \tag{4.4.1}$$

where $Sx = J^{-1}\Big(\sum_{i=1}^{\infty}\eta_i\big(\mu_i Jx + (1-\mu_i)JS_i x\big)\Big)$ *for each* $x \in E$, $\lambda \in (0, \frac{\alpha}{k})$ *and* $\{\alpha_n\} \subset [0,1]$ *such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$. *Suppose* $\Big(\bigcap_{i=1}^{\infty}F(S_i)\Big) \cap V(C,A) \neq \emptyset$, *then, the sequences* $\{x_n\}$ *generated by* (4.4.1) *converges strongly to the point* $q = \Pi_{F(S)\cap VI(C,A)}x_0$.

Proof L_p $\Big($*or* l_p *or* $W_p^m(\Omega)\Big)$, $1 < p \leq 2$, are uniformly smooth and 2-uniformly convex. Hence, the conclusion follows from Theorem 4.3.1. ■

Corollary 4.4.2 *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* $A : H \to H$ *be a monotone map on* C *and* k*-Lipschitz on* E *and* $\{S_i\}_{i=1}^{\infty}$ *be a countable family of relatively nonexpansive maps such that* $\bigcap_{i=1}^{\infty}F(S_i) \neq \emptyset$, *where* $S_i : H \to H$, $\forall\ i$. *Let* $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ *and* $\{\mu_i\}_{i=1}^{\infty} \subset (0,1)$ *be sequences such that* $\sum_{i=1}^{\infty}\eta_i = 1$. *Define inductively the sequence* $\{x_n\}$ *by*

$$\begin{cases} x_0 \in E; \\ y_n = P_C J(x_n - \lambda A x_n); \\ T_n = \{z \in E : \langle z - y_n, x_n - \lambda A x_n - y_n\rangle \leq 0\}; \\ t_n = P_{T_n}(x_n - \lambda A y_n); \\ z_n = \alpha_n x_0 + (1-\alpha_n)t_n); \\ x_{n+1} = \lambda x_n + (1-\lambda)S z_n, \end{cases} \tag{4.4.2}$$

where $Sx = \Big(\sum_{i=1}^{\infty}\eta_i\big(\mu_i x + (1-\mu_i)S_i x\big)\Big)$ *for each* $x \in E$, $\lambda \in (0, \frac{\alpha}{k})$ *and* $\{\alpha_n\} \subset [0,1]$ *such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$. *Suppose* $\Big(\bigcap_{i=1}^{\infty}F(S_i)\Big) \cap V(C,A) \neq \emptyset$, *then, the sequences* $\{x_n\}$ *generated by* (4.4.2) *converges strongly to the point* $q = \Pi_{F(S)\cap VI(C,A)}x_0$.

Remark 4.4.3 *Our Theorems are improvements of the results of [Kraikaew and Saejung, 2014] and [Nakajo, 2015], in the following sense:*

1. *The algorithm (1.4.5) studied in [Nakajo, 2015] requires, at each step of the iteration process, the computation of two subsets* C_n *and* Q_n *of* C, *"their intersection* $C_n \cap Q_n$, *and the projection of the initial vector onto this intersection". In our algorithm (4.3.1), these subsets have been dispensed with. Furthermore, [Nakajo, 2015] proved a strong convergence theorem for a monotone and* k*-Lipschitz map and one relatively nonexpansive map,* $S : E \to E$. *In our theorem, strong convergence is proved for a monotone and* k*-Lipschitz map and a countable family of relatively nonexpansive maps,* $S_i : E \to E$.

2. *In the result of [Kraikaew and Saejung, 2014], the iteration parameter* β_n *used in their algorithm (1.4.3), which is to be computed at each step of the iteration has been replaced by a fixed constant* λ *in our algorithm (4.3.1). This* λ *is to be computed once and used at each step of the iteration process. Consequently, our algorithm reduces computational cost and possible computational complexity and errors. Furthermore, the theorem of [Kraikaew and Saejung, 2014] is proved in a real Hilbert space, while our theorem is proved in the much more general uniformly smooth and 2-uniformly convex real Banach spaces.*

3. *Finally, we remark that in some algorithms, the use of general sequences as iteration parameters instead of fixed constants may provide more general iteration algorithms. For example, the well-known Mann iteration process:* $x_0 \in K, \quad x_{n+1} = (1 - c_n)x_n + c_n T x_n, \quad n \geq 0$, *where* (i) $\lim_{n\to\infty} c_n = 0$ *and* (ii) $\sum_{n=0}^{\infty} c_n = \infty$ *provides a more general iteration scheme than the Krasnoselskii scheme:* $x_0 \in K, \quad x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \quad n \geq 0$, *where* $\lambda \in (0, 1)$. *While in this case, it is known that whenever the Krasnoselskii converges, it is preferred to the Mann scheme because it involves less computation than the Mann scheme and converges as a geometric progression, slightly faster than the convergence obtainable from any Mann sequence. However, there are problems where the Krasnoselskii scheme is not applicable but the Mann scheme is. Furthermore, whenever a general sequence* β_n *is introduced as an iteration parameter in any algorithm, it does not, in general, translate to more general algorithm than an algorithm with a fixed constant* β. *If the general sequence* β_n *introduced is bounded away from 0 and 1, it is easy to show that whenever the algorithm with* β_n *converges, the same algorithm with* β_n *replaced by* $\beta \in (0, 1)$ *converges. Thus, the use of* β_n *in such algorithm only increases computational cost and possible computational complexity and errors, and is therefore totally undesirable. The use of a constant iteration parameter* $\beta \in (0, 1)$ *is certainly preferred in such a case.*

4.5 Numerical Illustration

In this section, we give a numerical example to compare the computational cost of our algorithm (4.2.1) with the algorithm (1.4.5) studied in Nakajo [Nakajo, 2015].

Example. Let $E = \mathbb{R}$, $C = [\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$. Clearly, for $x \in \mathbb{R}$,

$$P_C x = \begin{cases} \alpha, & if \quad x < \alpha, \\ x, & if \quad x \in C, \\ \beta, & if \quad x > \beta. \end{cases}$$

Now, in algorithms (1.4.5) and (4.2.1), set $Ax = \frac{x}{3}$, $Sx = \sin x$, $C = [-1, 1]$. Then, it is easy to see that A is monotone and $\frac{1}{3}$-Lipschitz and S is relatively nonexpansive. It is also easy to see that $F(S) \cap VI(C, A) = \{0\}$. Furthermore, we take $x_1 = 5$, $\lambda_n = \frac{n}{2n+1}$ in (1.4.5) and $x_0 = 5$, $\lambda = \frac{1}{2}$, $\alpha_n = \frac{1}{2^n}$ in (4.2.1), $n = 0, 1, 2, \cdots$, as our parameters. Using a tolerance of 10^{-8}, the numerical results are sketched in Figure 4.1 below, where the y-axis represents the value of $|x_n - 0|$ while the x-axis represents the number of iteration (n).

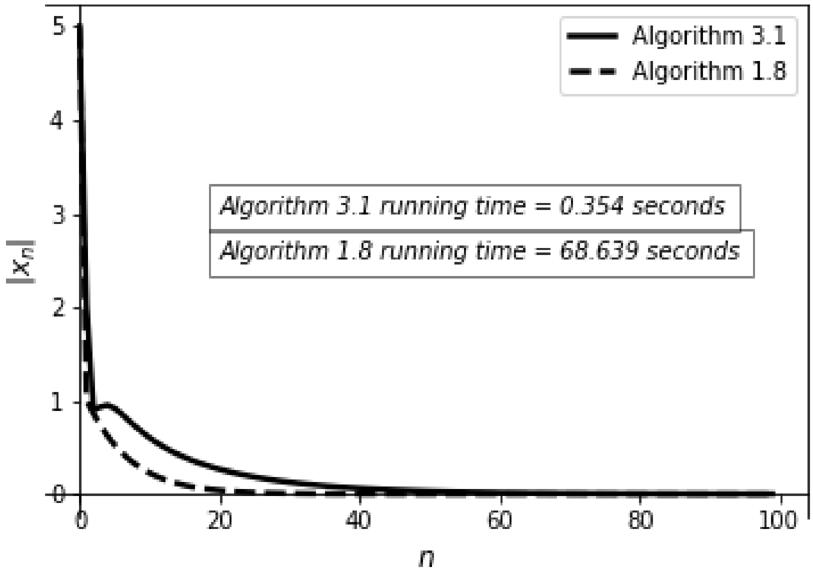


Figure 4.1

Remark 4.5.1 *Figure 4.1 compares the computational cost of our algorithm (4.2.1) with the algorithm (1.4.5) studied in Nakajo [Nakajo, 2015]. All computations and graphs were implemented in python 3.6 using some abstractions developed at AUST and other open source python library such as numpy and matplotlib on Zinox with intel core i7 4Gb RAM.*

Conlusion: It was observed that the number of iterations using algorithm (4.2.1) is greater than the number of iterations using algorithm (1.4.5). However, it took 0.354 seconds to obtain convergence for (4.2.1) while it took 68.639 seconds to obtain convergence for (1.4.5) using the same tolerance error. Consequently, looking at the time difference, we deduce that algorithm (1.4.5) requires much more computation time than algorithm (4.2.1).

CHAPTER 5

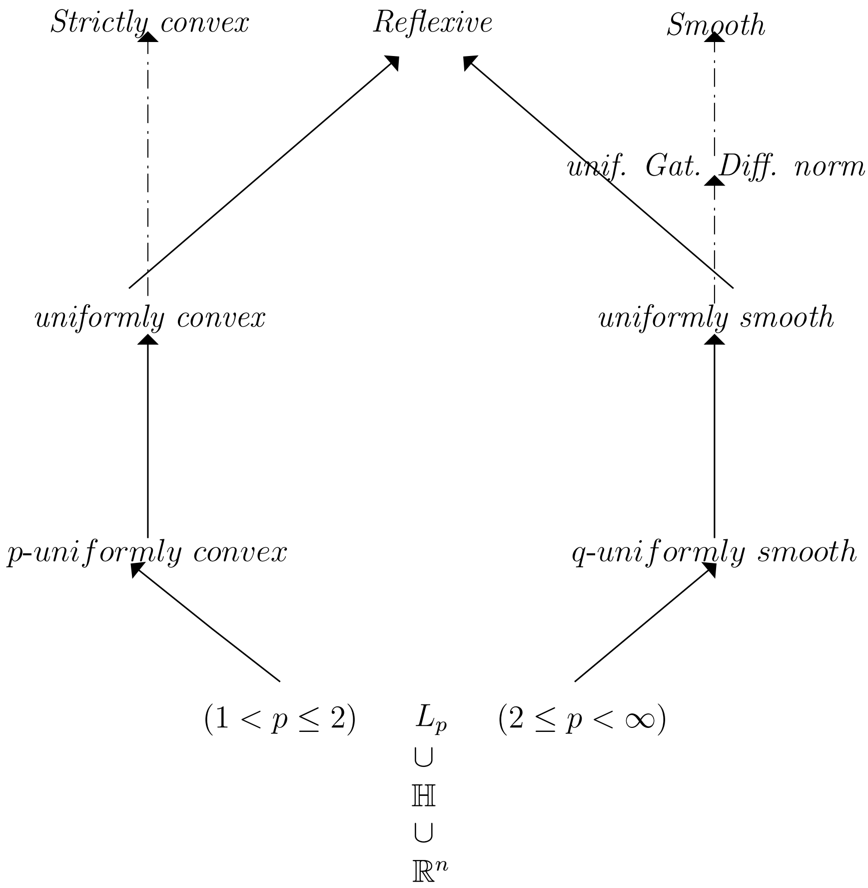
Appendix

5.1 Analytical representations of duality maps in L_p, l_p, and W^p_m, spaces, $1 < p < \infty$

The analytical representations of duality maps are known in L_p, l_p, and W^p_m, $1 < p < \infty$. Precisely, in the spaces l_p, $L_p(G)$ and $W^p_m(G)$, $p \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, respectively,

$$
\begin{aligned}
Jz &= \|z\|^{2-p}_{l_p} y \in l_q, \quad y = \{|z_1|^{p-2} z_1, |z_2|^{p-2} z_2, ...\}, \quad z = \{z_1, z_2, ...\}, \\
J^{-1}z &= \|z\|^{2-q}_{l_q} y \in l_p, \quad y = \{|z_1|^{q-2} z_1, |z_2|^{q-2} z_2, ...\}, \quad z = \{z_1, z_2, ...\}, \\
Jz &= \|z\|^{2-p}_{L_p} |z(s)|^{p-2} z(s) \in L_q(G), \quad s \in G, \\
J^{-1}z &= \|z\|^{2-q}_{L_q} |z(s)|^{q-2} z(s) \in L_p(G), \quad s \in G, \text{ and} \\
Jz &= \|z\|^{2-p}_{W^p_m} \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha (|D^\alpha z(s)|^{p-2} D^\alpha z(s)) \in W^q_{-m}(G), m > 0, s \in G
\end{aligned}
$$

(see for example [Alber and Ryazantseva, 2006]; p. 36).



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