APPROXIMATION METHOD FOR SOLVING VARIATIONAL INEQUALITY WITH MULTIPLE SET SPLIT FEASIBILITY PROBLEM IN BANACH SPACE

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Certification

This is to certify that the thesis titled " APPROXIMATION METHOD FOR SOLUTIONS OF VARIATIONAL INEQUALITY AND MULTIPLE SET SPLIT FEASIBILITY PROBLEM IN BANACH SPACE" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research work carried out by Aisha Aminu Adam in the department of Mathematical Sciences.

Approval

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Abstract

In this thesis, an iterative algorithm for approximating the solutions of a variational inequality problem for a strongly accretive, L-*Lipschitz* map and solutions of a multiple sets split feasibility problem is studied in a uniformly convex and 2-uniformly smooth real Banach space under the assumption that the duality map is weakly sequentially continuous. A strong convergence theorem is proved.

Dedication

This work is dedicated to my parents, Alhaji AMINU ADAM and Hajiya AISHA AHMAD. They made me believe in myself and always feel I can do it. Thanks for your love and support.

Contents

Chapter 1

General Introduction

In this chapter, we give a brief introduction of the subject matter and definitions of some basic terms which will be used in our subsequent discussions.

1.1 Introduction

The Multiple sets split feasibility problem is to find a point contained in the intersection of a family of closed convex sets in one space so that its image under a bonded linear transformation is contained in the intersection of a family of closed convex sets in the image space. It generalizes the convex feasibility problem and the two sets split feasibility problem. The problem is formulated as

$$find\ x \in \bigcap_{i=1}^{n} C_i \ such\ that\ A(x) \in \bigcap_{t=1}^{m} Q_t.$$

where $A : X \to Y$ is a bounded linear operator, $C_i \subset X, i = 1, 2, 3, \cdots, n$ and $Q_t \subset Y, t = 1, 2, 3, \cdots, m$ are nonempty closed convex sets.

When n = m = 1, the problem reduce to the Split feasibiliry problem (SFP) which is to find

$$x \in C\ such\ that\ A(x) \in Q.$$

where C and Q are two nonempty closed convex subsets of X and Y respectively.

In Banach space, the multiple sets split feasibility problem is formulated as finding an element $x \in X$ satisfying

$$x \in \bigcap_{i=1}^{n} C_i, \quad A(x) \in \bigcap_{t=1}^{m} Q_t.$$

where X and Y are two Banach spaces, m, n are two given integers, $A : X \to Y$ is a bounded linear operator, $C_i, i = 1, 2, 3, \cdots, n$ are closed convex sets in X, and $Q_t, t = 1, 2, 3, \cdots, m$ closed convex sets in Y.

The multiple sets split feasibility problem was first introduced by Censor and Elfving [9]. The problem arises in many practical fields such as signal processing, image reconstruction [11], Intensity modulated radiation therapy(IMRT)[10] and so on.

1.2 Preliminaries

Definition 1.2.1 *A vector space over some field say F is s nonempty set E together with two binary operations of addition(+) and scalar multiplication(.) satisfying the following conditions for any $v, w, z \in E, \delta, \beta \in F$.*

1. *$v + w = w + v$; the commutative law of addition,*
2. *$(v + w) + z = v + (w + z)$; the associative law for addition,*
3. *There exists $0 \in E$ satisfying $v + 0 = v$; the existence of an additive identity,*
4. *$\forall v \in E$ there exists $(-v) \in E$ such that v+(-v) = 0; the existence of an additive inverse,*
5. *$\delta \cdot (v + w) = \delta \cdot v + \delta \cdot w$;*
6. *$(\delta + \beta) \cdot v = \delta \cdot v + \beta \cdot v$;*
7. *$\delta \cdot (\beta \cdot v) = (\delta\beta) \cdot v$;*
8. *$1 \cdot v = v$.*

Here, the scalar multiplication $\delta \cdot v$ is often written as δv. The field of scalars will always be assumed to be either $\mathbb{R}$ or $\mathbb{C}$ and the vector space will be called real or complex depending on whether the field is $\mathbb{R}$ or $\mathbb{C}$. A vector space is also called a linear space.

Example 1.2.2 *Space $\mathbb{R}^n$. This is the Euclidean space, the underlying set being the set of all $n-$tuples of real numbers, written as $x = (x_1 ...., x_n)$, $y = (y_1 ...., y_n)$, etc., and we now see that this is a real vector space with the two algebraic operations defined in the usual fashion $x + y = (x_1 + y_1, ..., x_n + y_n)$ and $ax = (ax_1, ..., ax_n)$, $a \in \mathbb{R}$.*

Definition 1.2.3 *The vectors $\{x_1, x_2, x_3, \cdots\}$ are said to form a basis for E if they are linearly independent and $E = span\{x_1, x_2, x_3, \cdots\}$.*

Definition 1.2.4 *A vector space E is said to be finite dimensional if the number of vectors in a basis of E is finite.*

Note that if E is not finite dimensional, it is said to be infinite dimensional.

Example 1.2.5 *In analysis, infinite dimensional vector spaces are of greater interest than finite dimensional ones. For instance, $C[a, b]$ and l^2 are infinite dimensional, whereas $\mathbb{R}^n$ and $\mathbb{C}^k$ are finite dimensional for some $n, k \in \mathbb{N}$.*

Definition 1.2.6 *A normed space E is a vector space with a norm defined on it, here a norm on a (real or complex) vector space E is a real-valued function on E whose value at an $x \in E$ is denoted by $\|x\|$ and which satisfies the following properties, for $x, y \in E$ and $\alpha \in \mathbb{R}$*

1. $\|x\| \geq 0$;
2. $\|x\| = 0$ *iff* $x = 0$;
3. $\|\alpha x\| = |\alpha|\|x\|$;
4. $\|x + y\| \leq \|x\| + \|y\|$;

Definition 1.2.7 *A sequence $\{x_n\}$ in a normed linear space X is*
(i) convergent to $x \in X$ if given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|x_n - x\| < \epsilon \text{ whenever } n \geq N$$

(ii) said to be Cauchy if given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

Remark 1.2.8 *Every convergent sequence is Cauchy but the converse is not necessarily true.*

Definition 1.2.9 *A space X is said to be complete if every Cauchy sequence in X converges to an element of X.*

Definition 1.2.10 *A Banach space is a complete normed space (complete in the metric defined by the norm).*

Example 1.2.11 *The space l^p is a Banach space with norm given by*

$$\|x\| = (\sum_{j=1}^{\infty} |x|^p)^{\frac{1}{p}}$$

Definition 1.2.12 *An inner product space $(E, \langle\cdot,\cdot\rangle)$ is a vector space E together with an inner product $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{C}$ such that for all vectors x, y, z and scalar a we have*

1. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;

2. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$;

3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$;

4. $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0$ *iff* $x = 0$;

A norm on E can also be define as

1. $\|x\|^2 = \langle x, x\rangle$, $\forall x \in E$

2. x *and* y *are orthogonal if* $\langle x, y\rangle = 0$

Inner product space generalizes notion of dot product of finite dimensional spaces.

Definition 1.2.13 *A Hilbert space is a complete inner product space.*

In a Banach space E, beside the strong convergence defined by the norm, i.e., $\{x_n\} \subset E$ converges strongly to a if and only if $\lim_{n\to\infty} \|x_n - a\| = 0$, we shall consider the weak convergence, corresponding to the weak topology in E. We say that $\{x_n\} \subset E$ converges weakly to a if for any $f \in E^*$ $\langle x_n, f\rangle \to \langle a, f\rangle$ as $n \to \infty$.

Remark 1.2.14 *Any weakly convergent sequence* $\{x_n\}$ *in a Banach space is bounded.*

Definition 1.2.15 *Let E be a Banach space. Consider the following map* $J : E \to E^{**}$ *defined for each* $x \in E$, *by*

$$J(x) = \phi_x \in E^{**}$$

where

$$\phi_x : E^* \to \mathbb{R}$$

is given by

$$\phi_x(f) = \langle f, x\rangle, \text{ for each } f \in E^*.$$

Clearly J is linear, bounded and one-to-one. The mapping J defined above is called the canonical map(or canonical embedding) of E onto E^{**}.

Definition 1.2.16 *Let E be a normed linear space and J be the canonical embedding of E onto* E^{**}. *If J is onto, then E is called reflexive.*

Proposition 1.2.17 1. *In reflexive Banach space each bounded sequence has a weakly convergent subsequence.*

2. *The spaces* L_p *and* l_p, $p > 1$, *are reflexive.*

3. *The spaces L_1 and l_1 are non-reflexive.*

Definition 1.2.18 *A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in U$, where $U = \{z \in E : \|z\| = 1\}$ with $x \neq y$.*

Definition 1.2.19 *A Banach space E is said to be smooth, if for every $0 \neq x \in E$ there exists a unique $x^* \in E^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$ i.e., there exists a unique supporting hyperplane for the ball around the origin with radius $\|x\|$ at x.*

Definition 1.2.20 *The modulus of convexity of a normed space E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by*

$$\delta_E(\epsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\|; \|x\| = \|y\| = 1, \|x-y\| = \epsilon\}.$$

Definition 1.2.21 *The modulus of smoothness of a normed space E is the fuction $\rho_E : [0, \infty) \to [0, \infty)$ defined by*

$$\rho_E(r) = \frac{1}{2}\sup\{\|x+y\| + \|x-y\| - 2 : \|x\| = 1, \|y\| \leq r\}.$$

Definition 1.2.22 *A Banach space E is said to be uniformly convex, if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$, such that for any $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$ then $\|\frac{x+y}{2}\| \leq 1 - \delta$.*

Remark 1.2.23 1. *Every uniformly convex space is reflexive*

2. *E is uniformly convex iff $\delta_E(\epsilon) > 0 . \forall \epsilon \in (0, 2]$*

Definition 1.2.24 *A Banach space E is said to be uniformly smooth, if*

$$\lim_{r \to 0}(\frac{\rho_E(r)}{r}) = 0.$$

where $\rho_E(r)$ is the modulus of smoothness.

Remark 1.2.25 1. *ρ_E is continuous, convex and nondecreasing with $\rho_E(0) = 0$ and $\rho_E(r) \leq r$*

2. *The function $r \mapsto \frac{\rho_E(r)}{r}$ is nondecreasing and fulfills $\frac{\rho_E(r)}{r} > 0$ for all $r > 0$.*

Definition 1.2.26 *Let $q > 1$ be a real number. A normed space E is said to be q-uniformly smooth if there is a constant $d > 0$ such that*

$$\rho_E(r) \leq d\epsilon^q.$$

When $1 < q \leq 2$, E is said to be 2-uniformly smooth.

Definition 1.2.27 *A mapping $A : E_1 \to E_2$ is said to be bounded and linear if there exists real numbers c, α and β such that for $x, y \in E_1$,*

$$\|Ax\| \leq c\|x\|$$

and

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay.$$

Definition 1.2.28 *Let E_1 and E_2 be two reflexive, strictly convex and smooth Banach spaces. The mapping $A : E_1 \longrightarrow E_2$ is called a bounded linear operator endowered with the operator norm $\|A\| = \sup_{\|x\|\leq 1} \|Ax\|$. The dual operator $A^* : E_2^* \longrightarrow E_1^*$ defined by $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle \forall x \in E_1, y^* \in E_2^*$ is called the adjoint operator of A. The adjoint operator A^* has the property. $\|A^*\| = \|A\|$*

Definition 1.2.29 *A continuous strictly incresing function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $g(0) = 0$ and $\lim_{t\to\infty} g(t) = \infty$ is called a guage function.*

Definition 1.2.30 *The ganeralized daulity map $J_\phi : E \longrightarrow 2^{E^*}$ with respect to the guage function ϕ is defined by*

$$J_\phi(x) = \{x^* \in E^*; \langle x, x^*\rangle = \|x\|\|x^*\|, \|x^*\| = \phi(\|x\|)\}.$$

For $p > 1$, if $\phi(t) = t^{p-1}$, then $J_p : E \longrightarrow 2^{E^*}$ defined by

$$J_p(x) = \{x^* \in E^*; \langle x, x^*\rangle = \|x\|\|x^*\|, \|x^*\| = \phi(\|x\|) = \|x\|^{p-1}\}.$$

is also called the generalized duality map.

In particular, if $p = 2$ then

$$J_2x := Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

is called the normalized duality mapping

Proposition 1.2.31 *The duality map of a Banach space E has the following properties;*

1. *It is homogeneous*
2. *It is additive iff E is a Hilbert space.*
3. *It is single-valued iff E is smooth.*
4. *It is surjective iff E is reflexive.*
5. *It is injective or strictly monotone iff E is strictly convex*

6. *It is norm to weak* uniformly continous on bounded subsets of E if E is smooth*

7. *If E is Hilbert, J and J^{-1} are identity.*

If E is reflexive, strictly convex and smooth, then J is bijective. In this case the inverse $J^{-1} : E^* \longrightarrow E$ is given by $J^{-1} = J^*$ with J^* being the duality mapping of E^*.

Definition 1.2.32 *The duality mapping J_E^p is said to be weakly sequentially continuous if for each $x_n \to x$ weakly, we have $J_E^p(x_n) \to J_E^p(x)$ weakly*.*

Definition 1.2.33 *A mapping $P_C : E \to C$ is said to be a projection of x onto C if for all $y \in C$ there exists a unique element $P_C(x) \in C$ such that $\|x - P_C(x)\| = \min_{y \in C} \|x - y\|$. Moreover, if J^p is the duality mapping of E, then $x_0 \in C$ is the projection of x onto C iff*

$$\langle J^p(x_0 - x), y - x_0 \rangle \geq 0 \; \forall y \in C.$$

Definition 1.2.34 *A mapping $T : C \to C$ is said to be nonexpansive if*

$$\|Tx - Ty\| \leq \|x - y\| \quad for \; any \; x, \; y \; \in \; C.$$

Definition 1.2.35 *A mapping $T : E \to E$ is said to be accretive if*

$$\langle Tx - Ty, j(x - y) \rangle \geq 0 \;\; \forall x, y \; \in \; E.$$

Definition 1.2.36 *A mapping $T : E \to E$ is said to be strongly accretive if there exists $\beta > 0$ such that*

$$\langle Tx - Ty, j(x - y) \rangle \geq \beta \|x - y\|^2 \;\; \forall x, y \; \in \; E.$$

Remark 1.2.37 *If $E := H$ a Hilbert space, then a strongly accretive map T is strongly monotone.*

Definition 1.2.38 *A mapping $T : E \to E$ is said to be L-Lipschitz continuous on E if there exists $L > 0$ such that*

$$\|Tx - Ty\| \leq L\|x - y\| \;\; \forall \; x, y \; \in \; E.$$

Definition 1.2.39 *Let C be a nonempty, closed and convex subset of E. The problem of finding $x^* \in C$ such that*

$$\langle j(x - x^*), Tx^* \rangle \geq 0$$

is called a variational inequality problem(VI), where $T : E \to E$ is strongly accretive and $L-$Lipschitz continuous.

Lemma 1.2.40 *[8] Let E be q- uniformly smooth Banach space. Then, there exists a constant $d_q > 0$, such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, jx\rangle + d_q\|y\|^q.$$

Lemma 1.2.41 *[25] Let E be 2- uniformly smooth Banach space with best smoothness constant $k > 0$. Then,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, jx\rangle + 2\|ky\|^2.$$

Lemma 1.2.42 *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then ,*

$$z = P_C x \ iff \ \langle y - z, j(x - z)\rangle \leq 0 \ \forall y \in C.$$

Lemma 1.2.43 *[4] Let E be a normed linear space. Then, the following inequality hold:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle \quad for \ x, y \ \in E, \ j(x + y) \in J(x + y).$$

Lemma 1.2.44 *([19]see also [3]) Let E be a uniformly convex Banach space with modulus of convexity $\delta(\epsilon)$ of order ϵ^q, $q \geq 2$, then there exists $\lambda > 0$ such that the following inequality hold:*

$$\|P_C x - x\|^q \leq \|x - y\|^q - \lambda\|P_C x - y\|^q \quad \forall \ y \in C.$$

Lemma 1.2.45 *Let E be a uniformly smooth Banach space with best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$. Suppose $T : E \to E$ is strongly accretive and L- Lipschitz continuous on E , $0 < \alpha < 1$, $0 \leq \eta \leq 1 - \alpha$ and $0 < \mu < \frac{2\beta}{L^2}$. Then,*

$$\|(1 - \eta)x - \alpha\mu Tx - [(1 - \eta)y - \alpha\mu Ty]\| \leq (1 - \eta - \alpha\tau)\|x - y\| \ \ \forall \ x, y \ \in \ E,$$

where

$$\tau = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \quad \in \ (0, 1].$$

Proof

By L-Lipschitz contuinity and strongly accretivity of F, we have

$$\begin{aligned}
\|(x - y) - \mu(Tx - Ty)\|^2 &= \|\mu(Tx - Ty) - (x - y)\|^2 \\
&\leq \mu^2\|Tx - Ty\|^2 - 2\mu\langle Tx - Ty, j(x - y)\rangle \\
&+ 2k^2\|x - y\|^2 \\
&\leq \mu^2\|Tx - Ty\|^2 - 2\mu\langle Tx - Ty, j(x - y)\rangle \\
&+ \|x - y\|^2 \\
&\leq \mu^2 L^2\|x - y\|^2 - 2\mu\beta\|x - y\|^2 + \|x - y\|^2 \\
&= (1 - 2\mu\beta + \mu^2 L^2)\|x - y\|^2
\end{aligned}$$

Thus

$$\|(x-y)-\mu(Tx-Ty)\| \leq \sqrt{1-2\mu\beta+\mu^2L^2}\|x-y\|.j \tag{1.2.1}$$

Now, using (1.2.1)

$$\begin{aligned}
&\|(1-\eta)x-\alpha\mu Tx-[(1-\eta)y-\alpha\mu Ty]\| \\
&= \|(1-\eta)(x-y)-\alpha\mu(Tx-Ty)\| \\
&= \|(1-\eta)x-y)-\alpha[(x-y)-\mu(Tx-Ty)]\| \\
&\leq (1-\eta-\alpha)\|x-y)\| + \alpha\|(x-y)-\mu(Tx-Ty)\| \\
&\leq (1-\eta-\alpha)\|x-y)\| + \alpha\sqrt{1-2\mu\beta+\mu^2L^2}\|x-y\| \\
&= (1-\eta-\alpha\tau)\|x-y)\|
\end{aligned}$$

where

$$\tau = 1-\sqrt{1-\mu(2\beta-\mu L^2)}.$$

This completes the proof.

Lemma 1.2.46 *[2] Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that for any integer m, there exists an integer p such that $p \geq m$ and $a_p \leq a_{p+1}$. Let n_0 be an integer such that $a_{n_0} \leq a_{n_0+1}$ and define for all integer $n \geq n_0$ by*

$$\tau(n) = \max\{k \in \mathbb{N} : n_0 \leq k \leq n, a_k \leq a_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence satisfying $\lim_{n\to\infty}\tau(n) = \infty$ and the following inequalities hold true:

$$a_{\tau(n)} \leq a_{\tau(n)+1} \quad a_n \leq a_{\tau(n)+1} \quad \forall n \geq n_0.$$

Lemma 1.2.47 *[24] Assume $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the condition*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\xi_n, \quad \forall\ n\ \geq\ n_0.$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\xi_n\}$ is a sequence in $\mathbb{R}$ such that
(4i) $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(ii) $\limsup_{n\to\infty}\xi_n \leq 0$.
Then $\lim_{n\to\infty}a_n = 0$.

1.3 Statement of the Problem

The split feasiblity problem (SFP) in finite-dimensional Hilbert spaces was first introduced by Censor et al.[9] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Since then, various algorithms have been introduced and studied for solving SFP and MSSFP

Recently, Anh[1] proposed a parallel method for solving the variational inequality with the MSSFP and proved a strong convergence of the iterative process in frame work of a Hilbert space. Now the problem is to establish new convergence theorems that hold in more general Banach space than Hilbert space.

1.4 Significance of the Study

When a multiple set split feasibility problem exists in a setting of a Banach space more general than Hilbert, the result of Anh[1] cannot be used to get solutions. While our result in this thesis can be applied to such problem to some extent.

1.5 Aim and Objectives

The aim of this work is to present a new theorem for solution of some non-linear operator problem. The aim is achieved through the following objective

To establish some existance results of solutions and the convergence of an iterative process for solving variational inequalities with multiple sets split feasibility problem in a uniformly convex and 2-uniformly smooth Banach spaces

1.6 Scope and Limitations

The theorems considered in this research work hold in a uniformly convex and 2-uniformly smooth Banach space whose duality mapping is weakly sequentially continuous. It involves solution of a variational inequality problem.

The work is limited to Banach spaces with weakly sequentially continuous duality maps which excludes some Banach spaces such as L_p spaces and sobolev spaces.

1.7 Methodology

Various well known computational techniques, theorems and results in the literature related to approximations of solutions of the multiple sets split feasibility problems are used.

Chapter 2

Literature Review

2.1 Literature review

The study of multiple sets split feasibility problem attracts the interest of many authors due to its various applications. Many projection methods have been developed by several authors to solve the problem see for example [16, 20, 26, 27, 28, 30] .

The split feasibility problem in finite dimensional Hilbert space was first introduced in 1994 by Censor and Elfving [7]. Byrne(2002) in [5] proposed the following projection method called the CQ-algorithm for solving SFP, which generates the new iterate as follows:

$$x_{n+1} = P_C[x_n - \gamma A^T(I - P_Q)Ax_n],$$

where $\gamma \in (0, \frac{2}{L})$ and L denotes the largest eigenvalue of the matrix A^TA.

Yang [26] presented a relaxed CQ-algorithm for solving SFP, where at n-th iteration, the projections onto C and Q were replaced with the halfspaces C_n and Q_n, respectively. Qu and Xiu [20] proposed a modified relaxed CQ-algorithm

$$x_{n+1} = P_{C_n}[x_n - \alpha_n A^T(I - P_{Q_n})Ax_n].$$

Censor et al. [9] defined the proximity function p to measure the distance of a point to all sets:

$$p(x) := \frac{1}{2}\sum_{i=1}^{n}\alpha_i\|x - P_{c_i}x\|^2 + \frac{1}{2}\sum_{j=1}^{m}\lambda_j\|A(x) - P_{Q_j}A(x)\|^2$$

where $\alpha_i > 0, \lambda_j > 0 \ \ \forall\ i$ and j and $\sum_{i=1}^{n}\alpha_i + \sum_{j=1}^{m}\lambda_j = 1$.
By Censor et al. [9] we see that

$$\bigtriangledown p(x) := \sum_{i=1}^{n}\alpha_i(x - P_{c_i}x) + \sum_{j=1}^{m}\lambda_j A^T(I - P_{Q_j})Ax$$

and they proposed an iterative algorithm to solve the MSSFP with the iteration step as

$$x^{k+1} = P_{\Omega}(x^k - s \bigtriangledown p(x^k))$$

where s is a positive scalar and $\Omega \subseteq \mathbb{R}^{\mathbb{N}}$ is a nonempty closed convex set. They proved a strong convergengence under the assumption that $s \in (0, \frac{2}{L})$ where L is the Lipschitz constant of $\bigtriangledown p$. This iterative algorithm use a fixed step size restricted by the Lipschitz constant of $\bigtriangledown p$, which depend on the operator norm of the linear transformation.

Motivated by the result of Qu and Xu's idea, Zhao and Yang [29] introduced a self adaptive projection method by adapting Armijo-like searches to solve the multiple sets split feaasibility problem.

However, these iterative algorithms need an inner iteration numbers to obtain a suitatble step size. the work of Zhao and Yang [30] suggested a new self adaptive way to compute directly the step size in each iteration. It need not estimate the Lipschitz constant or choose the inner iteration number.

In [24], Xu proposed an iterative method which used a fixed step size but rely on Lipschitz constant. To overcome this shortcomings, Wen et al.[17] proposed a cyclic and simultaneous iteration method with self adaptive step size for solving the multiple sets split feasibility problem.

In 2007, Schöpfer [21] developed iterative methods for the solution of the SFP in Banach spaces and also analyse stability and regularization properties. These iterative methods are as follows:

$$x_{n+1} = J_{E_1}^*(J_{E_1}(x_n) - \mu_n A^* J_{E_2}(Ax_n - P_Q(Ax_n)).$$

In 2008, Schöpfer et al. [22] proposed in Banach space the following algorithm: for $x_0 \in X$ and $n \geq 0$, set

$$x_{n+1} = \Pi_C j^*[j(x_n) - tA^* j(Ax_n - P_Q(Ax_n))]$$

where Π_C denotes the Bregmann projection and j the duality mapping and established a weak convergence of the algorithm under the assumption that X is p-uniformly convex, uniformly smooth and the duality map is sequentially weak to weak continuous.

Wang [23] in order to modify this work, used the idea of Nakajo and Takahashi [18] in which, for finding a fixed point of nonexpansive mapping T in a Hilbert space used the following scheme

$$\begin{cases} y_n = Tx_n, \\ D_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ E_n = \{z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0\} \\ x_{n+1} = P_{D_n \cap E_n}(x_0). \end{cases} \tag{2.1.1}$$

Combining the CQ Algorithm and (2.1.1), Wang obtained the following scheme. For initial guess x_0, define (x_n) recursively by

$$\begin{cases} y_n = T_n x_n, \\ D_n = \{u \in E : \triangle_(y_n, u) \leq \triangle(x_n, u)\}, \\ E_n = \{u \in E : \langle x_n - u, j_p x_0 - j_p x_n \rangle \geq 0\} \\ x_{n+1} = \Pi_{D_n \cap E_n}(x_0). \end{cases}$$

and established a strong convergence.

Recently, Anh [1] proposed a parallel merhod for solving a strongly variational inequalty over the multiple sets split feasibility problem . He supposed that $F : H_1 \to H_2$ is β-strongly monotone and L-Lipschitz continuous on H_1, to ivestigate the following variational inequality:

$$find\ x^* \in \Omega \quad such\ that \quad \langle F(x^*), x - x^* \rangle \geq 0 \quad \forall\ x \in \Omega.$$

where H_1 and H_2 are two Hilbert spaces and Ω is the solution set of the multiple sets split feasibility problem. It can be seen clearly that the variational inequality problem reduces to a problem of finding the minimum norm solution when F is the identity mapping in Hilbert space. He establish strong convergence of the sequence $\{x^k\}$ generated by the following algorithm in the frame work of Hilbert space.

Algorithm

Step 0: Choose $0 < \mu < \frac{2\beta}{L^2}, 0 < \delta \leq \delta_k \leq \bar{\delta} < \frac{2}{\|A\|^2+1}$, $\{\alpha_k\}$, $\{\eta_k\} \subset (0,1)$ $such\ that\ lim_{k\to\infty}\alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $0 \leq \eta_k \leq 1 - \alpha_k\ \forall k \geq 0$. $lim_{k\to\infty}\eta_k = \eta < 1$.

Step 1: let $x^0 \in E_1$. Set $k := 0$

Step 2: Compute

$$u^k = A(x^k)\ and\ P_{Q_t}(u^k),\ t = 1, 2, 3, \cdots N$$

Step 3: find

$$t_k = argmax\{\|P_{Q_t}(u^k) - u^k\| : \quad t = 1, 2, 3, \cdots N\}, \quad v^k = P_{Q_{t_k}}(u^k).$$

Step 4: compute

$$y^k = x^k - \delta_k A^*(u^k - v^k)$$

Step 5: Compute

$$P_{C_i}(y^k),\ i = 1, 2, 3, \cdots M,$$

Step 6: Find

$$i_k = argmax\{\|P_{C_i}(y^k) - y^k\| : i = 1, 2, 3, \cdots M\},\ z^k = P_{C_{i_k}}(y^k),$$

Step 7: Compute

$$x^{k+1} = \eta_k x^k + (1 - \eta_k) z^k - \alpha_k \mu F(z^k).$$

Step 8: Set $k := k + 1$ and go to step 2

Inspired by these works, this thesis is aimed at modifying and improving the work of Anh[1]. We propose a new algorithm and prove its norm convergence under the assumption that E_1 is 2-uniformly smooth, uniformly convex and its duality map is weakly sequentially continous.

Chapter 3

Strong convergence theorem for solving variational inequality with multiple set split feasibility problem

3.1 introduction

In this chapter, we initiate and also establish an iterative algorithm for solving variational inequality over the multiple sets split feasibiliry problem in Banach spaces and proved a strong convergence of the iterative algorithm.

Throughout this chapter, we assume that E_1 is uniformly convex and 2-uniformly smooth Banach space with best smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$, E_2 is a reflexive, strictly convex Banach space and $J_1 : E_1 \to E_1^*$, $J_2 : E_2 \to 2^{E_2^*}$ are the normalized duality mappings on E_1 and E_2 respectively.

3.2 Main result

Algorithm 1:

Step 0: Choose $0 < \mu < \frac{2\beta}{L^2}, 0 < \delta \leq \delta_k \leq \bar{\delta} < \frac{2}{\|A\|^2+1}$, $\{\alpha_k\}$, $\{\eta_k\} \subset (0,1)$ *such that* $lim_{k\to\infty}\alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $0 \leq \eta_k \leq 1 - \alpha_k$ $\forall k \geq 0$. $lim_{k\to\infty}\eta_k = \eta < 1$.

Step 1: let $x^0 \in E_1$. Set $k := 0$

Step 2: Compute

$$u^k = A(x^k) \text{ and } P_{Q_t}(u^k),\ t = 1, 2, 3, \cdots N$$

Step 3: find

$$t_k = argmax\{\|P_{Q_t}(u^k) - u^k\| : \quad t = 1, 2, 3, \cdots N\}, \quad v^k = P_{Q_{t_k}}(u^k).$$

Step 4: compute

$$y^k = x^k - \delta_k J_1{}^{-1} A^* j_2(u^k - v^k), \quad j_2(u^k - v^k) \in J_2(u^k - v^k)$$

Step 5: Compute

$$P_{C_i}(y^k), \ i = 1, 2, 3, \cdots M,$$

Step 6: Find

$$i_k = argmax\{\|P_{C_i}(y^k) - y^k\| : i = 1, 2, 3, \cdots M\}, \ z^k = P_{C_{i_k}}(y^k),$$

Step 7: Compute

$$x^{k+1} = \eta_k x^k + (1 - \eta_k) z^k - \alpha_k \mu F(z^k).$$

Step 8: Set $k := k + 1$ and go to step 2
In the subsequent discussion, we assume $P_\Omega : \Omega \to E_1$ to be a nonexpansive projection of E_1 onto Ω. Where Ω is the solution set of the multiple set split feasibility problem.

Theorem 3.2.1 *Let E_1 be a uniformly convex and 2-uniformly smooth space with best smoothness constant k satisfying $o < k \leq \frac{1}{\sqrt{2}}$ whose duality map is weakly sequentially continuous and E_2 a reflexive and strictly convex Banach space. Let $C_1, C_2, \cdots C_M$ be M- nonempty closed convex subsets of E_1 and $Q_1,\ Q_2,\ Q_3, \cdots, Q_N$ be N-nonempty closed convex subsets of E_2. Let $F : E_1 \to E_1$ be strongly accretive and L- Lipschitz continous map and $A : E_1 \to E_2$ be a bounded linear operator with its adjoint A^*. Let $x_0 \in E_1$. Then, the sequence $\{x_k\}$ geenerated by Algoritm 1 converges strongly to a unique solution of the variational inequality problem which also solves the multiple set split feasibility problem.*

Proof

For all x, y in Ω, we have

$$\begin{aligned}
&\|P_\Omega(x - \mu F(x)) - P_\Omega(y - \mu F(y))\|^2 \\
&\leq \|(x - \mu F(x)) - (y - \mu F(y))\|^2 \\
&= \|\mu(F(x) - F(y)) - (x - y)\|^2 \\
&\leq \mu^2 \|F(x) - F(y)\|^2 - 2\mu\langle F(x) - F(y), J_1(x - y)\rangle \\
&+ 2k^2 \|x - y\|^2.
\end{aligned}$$

Since F is strongly accretive and L-Lipschitz continuous we get,

$$\begin{aligned}\|P_\Omega(x-\mu F(x))-P_\Omega(y-\mu F(y))\|^2 &\le \mu^2L^2\|x-y\|^2+\|x-y\|^2\\ &\quad -2\mu\beta\|x-y\|^2\\ &=(1+\mu^2L^2-2\mu\beta)\|x-y\|^2.\end{aligned}$$

Thus, $\|P_\Omega(x-\mu F(x))-P_\Omega(y-\mu F(y))\|\le\rho\|x-y\| \ \ \forall x,y\in\Omega,$
where

$$\rho=\sqrt{1-\mu(2\beta-\mu L^2)} \quad \in(0,1).$$

By Banach contraction mapping principle, there exists a unique point $x^*\in\Omega$ such that
$P_\Omega(x^*-\mu F(x^*))=x^*$.
By Lemma 1.2.42, we obtain $\langle(x^*-\mu F(x^*))-x^*,J_1(x-x^*)\rangle\le 0 \ \ \forall\ x\in\Omega.$
So,

$$\langle x^*-(x^*-\mu F(x^*)),J_1(x-x^*)\rangle\ge 0 \ \ \forall\ x\in\Omega.$$

This implies,

$$\langle F(x^*),J_1(x-x^*)\rangle\ge 0 \ \ \forall\ x\in\Omega. \qquad (3.2.1)$$

Since $x^*\in\Omega$.
We continue the proof in steps.
Step 1:

Using Lemma 1.2.45 and Lemma 1.2.43 , we have that for all $k \in \mathbb{N}$,

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|\eta_k x^k + (1-\eta_k)z^k - \alpha_k \mu F(z^k) - x^*\|^2 \\
&= \|\eta_k x^k + (1-\eta_k)z^k + \eta_k x^* - \eta_k x^* + \alpha_k \mu F(x^*) \\
&- \alpha_k \mu F(x^*) - \alpha_k \mu F(z^k) - x^*\|^2 \\
&= \|(1-\eta_k)z^k - \alpha_k \mu F(z^k) - [(1-\eta_k)x^* - \alpha_k \mu F(x^*)] \\
&+ \eta_k (x^k - x^*) - \alpha_k \mu F(x^*)\|^2 \\
&\leq \|(1-\eta_k)z^k - \alpha_k \mu F(z^k) - [(1-\eta_k)x^* - \alpha_k \mu F(x^*)] \\
&+ \eta_k (x^k - x^*)\|^2 - 2\alpha_k \mu \langle F(x^*), J_1(x^{k+1} - x^*)\rangle \\
&\leq [\|(1-\eta_k)z^k - \alpha_k \mu F(z^k) - [(1-\eta_k)x^* - \alpha_k \mu F(x^*)]\| \\
&+ \eta_k \|(x^k - x^*)\|]^2 + 2\alpha_k \mu \langle F(x^*), J_1(x^* - x^{k+1})\rangle \\
&\leq [(1-\eta_k - \alpha_k \tau)\|z^k - x^*\| + \eta_k \|x^k - x^*\|]^2 \\
&+ 2\alpha_k \mu \langle F(x^*), J_1(x^* - x^{k+1})\rangle \\
&= (1-\eta_k - \alpha_k \tau)^2 \|z^k - x^*\|^2 + 2(1-\eta_k - \alpha_k \tau)\eta_k \|z^k - x^*\| \\
&\ \|x^k - z^*\| + {\eta_k}^2 \|x^k - x^*\|^2 + 2\alpha_k \mu \langle F(x^*), J_1(x^* - x^{k+1})\rangle \\
&\leq (1-\eta_k - \alpha_k \tau)^2 \|z^k - x^*\|^2 + (1-\eta_k - \alpha_k \tau)\eta_k (\|z^k - x^*\|^2 \\
&+ \|x^k - x^*\|^2) + 2\alpha_k \mu \langle F(x^*), J_1(x^* - x^{k+1})\rangle + {\eta_k}^2 \|x^k - x^*\|^2 \\
&= (1-\eta_k - \alpha_k \tau)(1-\alpha_k \tau)\|z^k - x^*\|^2 + \eta_k (1-\alpha_k \tau)\|x^k - x^*\|^2 \\
&+ 2\alpha_k \mu \langle F(x^*), J_1(x^* - x^{k+1})\rangle \\
&\leq (1-\eta_k - \alpha_k \tau)\|z^k - x^*\|^2 + \eta_k \|x^k - x^*\|^2 \\
&+ 2\alpha_k \mu \langle F(x^*), J_1(x^* - x^{k+1})\rangle.
\end{aligned}$$

Hence, for all $k \in \mathbb{N}$

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 \quad &\leq \quad (1-\eta_k - \alpha_k \tau)\|z^k - x^*\|^2 + \eta_k \|x^k - x^*\|^2 \\
&+ \quad 2\alpha_k \mu \langle F(x^*), J_1(x^* - x^{k+1})\rangle. \qquad (3.2.2)
\end{aligned}$$

Step 2: The sequences $\{x^k\}$, $\{y^k\}, \{z^k\}$ *and* $\{F(x^k)\}$ are bounded.
Using property of projection we have,
$\langle v^k - A(x^*), j_2(u^k - v^k)\rangle = \langle P_{Q_{t_k}}(u^k) - A(x^*), j_2(u^k - P_{Q_{t_k}}(u^k))\rangle \geq 0.$

Therefore,

$$\begin{aligned}
\|y^k - x^*\|^2 &= \|x^k - \delta_k J_1^{-1} A^* j_2(u^k - v^k) - x^*\|^2 \\
&= \|\delta_k J_1^{-1} A^* j_2(u^k - v^k) - (x^k - x^*)\|^2 \\
&\leq {\delta_k}^2 \|A^*\|^2 \|j_2(u^k - v^k)\|^2 - 2\delta_k \langle x^k - x^*, J_1 J_1^{-1} A^* j_2(u^k - v^k)\rangle \\
&+ 2k^2 \|x^k - x^*\|^2 \\
&\leq {\delta_k}^2 \|A^*\|^2 \|j_2(u^k - v^k)\|^2 - 2\delta_k \langle x^k - x^*, J_1 J_1^{-1} A^* j_2(u^k - v^k)\rangle \\
&+ \|x^k - x^*\|^2 \\
&= {\delta_k}^2 \|A\|^2 \|u^k - v^k\|^2 - 2\delta_k \langle A(x^k - x^*), j_2(u^k - v^k)\rangle \\
&+ \|x^k - x^*\|^2 \\
&= {\delta_k}^2 \|A\|^2 \|u^k - v^k\|^2 - 2\delta_k \langle u^k - A(x^*), j_2(u^k - v^k)\rangle \\
&+ \|x^k - x^*\|^2 \\
&= {\delta_k}^2 \|A\|^2 \|u^k - v^k\|^2 - 2\delta_k \langle u^k - v^k, j_2(u^k - v^k)\rangle \\
&- 2\delta_k \langle v^k - A(x^*), j_2(u^k - v^k)\rangle + \|x^k - x^*\|^2 \\
&\leq {\delta_k}^2 \|A\|^2 \|u^k - v^k\|^2 - 2\delta_k \langle u^k - v^k, j_2(u^k - v^k)\rangle + \|x^k - x^*\|^2 \\
&= \|x^k - x^*\|^2 - \delta_k (2 - \delta_k \|A\|^2) \|u^k - v^k\|^2.
\end{aligned}$$

This implies

$$\|y^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \delta_k (2 - \delta_k \|A\|^2) \|u^k - v^k\|^2. \tag{3.2.3}$$

The last inequality together with $\{\delta_k\} \subset [\delta, \bar{\delta}]$ yields

$$\begin{aligned}
\|y^k - x^*\|^2 &\leq \|x^k - x^*\|^2 - \delta(2 - \bar{\delta}\|A\|^2)\|u^k - v^k\|^2 \\
&\leq \|x^k - x^*\|^2.
\end{aligned}$$

This implies,

$$\|yk - x^*\| \leq \|x^k - x^*\|. \tag{3.2.4}$$

Using (1.2.44), we get

$$\begin{aligned}
\|z^k - x^*\|^2 &= \|P_{C_{i_k}}(y^k) - x^*\|^2 \\
&\leq \|y^k - x^*\|^2 - \lambda \|z^k - y^k\|^2 \\
&\leq \|y^k - x^*\|^2.
\end{aligned}$$

Thus,

$$\|z^k - x^*\|^2 \leq \|y^k - x^*\|^2 - \lambda \|z^k - y^k\|^2 \tag{3.2.5}$$

and

$$\|z^k - x^*\|^2 \leq \|y^k - x^*\|^2. \tag{3.2.6}$$

From (3.2.4) and (3.2.6) we obtain,

$$\|z^k - x^*\| \leq \|x^k - x^*\|, \tag{3.2.7}$$

$$\begin{aligned}
\|x^{k+1} - x^*\| &= \|(1-\eta_k)z^k - \alpha_k \mu F(z^k) - [(1-\eta_k)x^* - \alpha_k \mu F(x^*)] \\
&\quad + \eta_k (x^k - x^*) - \alpha_k \mu F(x^*)\| \\
&\leq \|(1-\eta_k)z^k - \alpha_k \mu F(z^k) - [(1-\eta_k)x^* - \alpha_k \mu F(x^*)]\| \\
&\quad + \eta_k \|(x^k - x^*)\| + \alpha_k \mu \|F(x^*)\| \\
&\leq (1 - \eta_k - \alpha_k \tau)\|x^k - x^*\| + \eta_k \|x^k - x^*\| + \alpha_k \mu \|F(x^*)\| \\
&= (1 - \alpha_k \tau)\|x^k - x^*\| + \alpha_k \mu \|F(x^*)\| \\
&= (1 - \alpha_k \tau)\|x^k - x^*\| + \alpha_k \tau \frac{\mu \|F(x^*)\|}{\tau}.
\end{aligned}$$

Therefore,

$$\|x^{k+1} - x^*\| \leq (1 - \alpha_k \tau)\|x^k - x^*\| + \alpha_k \tau \frac{\mu \|F(x^*)\|}{\tau} \tag{3.2.8}$$

where

$$\tau = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \quad \in (0, 1).$$

From (3.2.8), we have

$$\begin{aligned}
\|x^{k+1} - x^*\| &\leq (1 - \alpha_k \tau)\|x^k - x^*\| + \alpha_k \tau \frac{\mu \|F(x^*)\|}{\tau} \\
&\leq (1 - \alpha_k \tau) \max\{\|x^k - x^*\|, \frac{\mu \|F(x^*)\|}{\tau}\} \\
&\quad + \alpha_k \tau \max\{\|x^k - x^*\|, \frac{\mu \|F(x^*)\|}{\tau}\} \\
&= \max\{\|x^k - x^*\|, \frac{\mu \|F(x^*)\|}{\tau}\}.
\end{aligned}$$

Claim: For $k \geq 0$, $\|x^k - x^*\| \leq \max\{\|x^0 - x^*\|, \frac{\mu \|F(x^*)\|}{\tau}\}$.
Proof of claim
We proceed by induction.
For $n = 0$, it is clear that
$\|x^0 - x^*\| \leq \max\{\|x^0 - x^*\|, \frac{\mu \|F(x^*)\|}{\tau}\}$.
Assume $\|x^n - x^*\| \leq \max\{\|x^0 - x^*\|, \frac{\mu \|F(x^*)\|}{\tau}\}$, for some $n \in \mathbb{N}$.
Now,

$$\begin{aligned}
\|x^{n+1} - x^*\| &\leq \max\{\|x^n - x^*\|, \frac{\mu \|F(x^*)\|}{\tau}\} \\
&\leq \max\{\max\{\|x^n - x^*\|, \frac{\mu \|F(x^*)\|}{\tau}\}, \frac{\mu \|F(x^*)\|}{\tau}\} \\
&= \max\{\|x^0 - x^*\|, \frac{\mu \|F(x^*)\|}{\tau}\}.
\end{aligned}$$

Hence, *for all* $k \geq 0$, $\|x^k - x^*\| \leq \max\{\|x^0 - x^*\|, \frac{\mu\|F(x^*)\|}{\tau}\}$.

$$\begin{aligned}\|x^k\| &\leq \|x^k - x^*\| + \|x^*\| \\ &\leq \max\{\|x^0 - x^*\|, \frac{\mu\|F(x^*)\|}{\tau}\} + \|x^*\| \\ &= M.\end{aligned}$$

Thus,
$\|x^k\| \leq M \ \forall \ k \geq 0$. Hence, $\{x^k\}$ is bounded.

$$\begin{aligned}\|y^k\| &\leq \|y^k - x^*\| + \|x^*\| \\ &\leq \|x^k - x^{\|} + \|x^*\| \\ &\leq \max\{\|x^0 - x^*\|, \frac{\mu\|F(x^*)\|}{\tau}\} + \|x^*\| \\ &= M.\end{aligned}$$

Hence, $\{y^k\}$ is bounded.

$$\begin{aligned}\|z^k\| &\leq \|z^k - x^*\| + \|x^*\| \\ &\leq \|x^k - x^{\|} + \|x^*\| \\ &\leq \max\{\|x^0 - x^*\|, \frac{\mu\|F(x^*)\|}{\tau}\} + \|x^*\| \\ &= M.\end{aligned}$$

Hence, $\{z^k\}$ is bounded.

$$\begin{aligned}\|F(x^k)\| &\leq \|F(x^k) - F(x^*)\| + \|F(x^*)\| \\ &\leq L\|x^k - x^*\| + \|F(x^*)\| \\ &\leq L\max\{\|x^0 - x^*\|, \frac{\mu\|F(x^*)\|}{\tau}\} + \|F(x^*)\| \\ &= M^*.\end{aligned}$$

Hence, $\{F(x^k)\}$ is bounded.
Step 3: Now, to show the convergence of the sequence $\{x^k\}$, let us consider two cases.
Case 1: There exists k_0 such that the sequence $\{\|x^k - x^*\|\}$ is decreasing for $k \geq k_0$. In this case, the limit of $\{\|x^k - x^*\|\}$ exists.
From (3.2.2) we have

$$\|x^{k+1} - x^*\|^2 \leq (1-\eta_k)\|z^k - x^*\|^2 + \eta_k\|x^k - x^*\|^2 + 2\alpha_k\mu\langle F(x^*), J_1(x^* - x^{k+1})\rangle, \tag{3.2.9}$$

using (3.2.9) we have

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\le (1-\eta_k)\|z^k - x^*\|^2 - \|x^k - x^*\|^2 + \|x^k - x^*\|^2 + \eta_k\|x^k - x^*\|^2 \\ &\quad + 2\alpha_k\mu\langle F(x^*), J_1(x^* - x^{k+1})\rangle \\ &= (1-\eta_k)\|z^k - x^*\|^2 - (1-\eta_k)\|x^k - x^*\|^2 + \|x^k - x^*\|^2 \\ &\quad + 2\alpha_k\mu\langle F(x^*), j_1(x^* - x^{k+1})\rangle.\end{aligned}$$

Thus, from (3.2.6) and (3.2.4) we obtain

$$\begin{aligned}\frac{1}{1-\eta_k}(\|x^{k+1} - x^*\|^2 - \|x^k - x^*\|^2) - \frac{2\alpha_k\mu}{1-\eta_k}\langle F(x^*), J_1(x^* - x^{k+1})\rangle \\ \le \|z^k - x^*\|^2 - \|x^k - x^*\|^2 \\ \le \|y^k - x^*\|^2 - \|x^k - x^*\|^2 \\ \le 0.\end{aligned}$$

Since $\lim_{k\to\infty}(\|x^{k+1} - x^*\|^2 - \|x^k - x^*\|^2) = 0$, $\lim_{k\to\infty}\alpha_k = 0$, $\lim_{k\to\infty}\eta_k = \eta < 1$ *and* $\{x^k\}$ is bounded, it follows from the above inequality that

$$\lim_{k\to\infty}(\|z^k - x^*\|^2 - \|x^k - x^*\|^2) = 0 \tag{3.2.10}$$

and

$$\lim_{k\to\infty}(\|y^k - x^*\|^2 - \|x^k - x^*\|^2) = 0. \tag{3.2.11}$$

Also from (3.2.10)) and (3.2.11) we get

$$\lim_{k\to\infty}(\|y^k - x^*\|^2 - \|z^k - x^*\|^2) = 0. \tag{3.2.12}$$

From (3.2.5) $0 \le \|z^k - y^k\|^2 \le \frac{1}{\lambda}(\|y^k - x^*\|^2 - \|z^k - x^*\|^2)$
and $\lim_{k\to\infty}(\|y^k - x^*\|^2 - \|z^k - x^*\|^2) = 0$ from (3.2.12), we have

$$\lim_{k\to\infty}\|z^k - y^k\|^2 = 0.$$

This implies,

$$\lim_{k\to\infty}\|z^k - y^k\| = 0. \tag{3.2.13}$$

From the relation $z^k = P_{C_{i_k}}(y^k)$, (3.2.13) and the definition of i_k we have

$$\lim_{k\to\infty}\|P_{C_{i_k}}(y^k) - y^k\| = 0, \qquad i = 1, 2, 3, \cdots, M. \tag{3.2.14}$$

From (3.2.3), $\|y^k - x^*\|^2 \le \|x^k - x^*\|^2 - \delta_k(2 - \delta_k\|A\|^2)\|u^k - v^k\|^2$
and $\{\delta_k\} \subset [\delta, \bar{\delta}] \subset (0, \frac{2}{\|A\|^2+1})$, we have

$$\delta(2 - \bar{\delta}\|A\|^2)\|u^k - v^k\|^2 \le \|x^k - x^*\|^2 - \|y^k - x^*\|^2. \tag{3.2.15}$$

From (3.2.11) $\lim_{k\to\infty}(\|y^k - x^*\|^2 - \|x^k - x^*\|^2) = 0$,
we have

$$\lim_{k\to\infty} \|u^k - v^k\|^2 = 0 \tag{3.2.16}$$

From (3.2.16), the relation $v^k = P_{Q_{t_k}}(u^k)$ and the definition of t_k, we obtain

$$\lim_{k\to\infty} \|P_{Q_{t_k}}(u^k) - v^k\| = 0, \qquad t = 1, 2, 3, \cdots, N. \tag{3.2.17}$$

It follows from $y^k = x^k - \delta_k J_1^{-1} A^* j_2(u^k - v^k)$ that

$$\begin{aligned} \|x^k - y^k\| &= \delta_k \|J_1^{-1} A^* j_2(u^k - v^k)\| \\ &\leq \delta_k \|A^*\| \|j_2(u^k - v^k)\| \\ &= \delta_k \|A\| \|u^k - v^k\| \\ &\leq \bar{\delta} \|A\| \|u^k - v^k\|. \end{aligned}$$

From (3.2.16) and the last inequality we have

$$\lim_{k\to\infty} \|x^k - y^k\| = 0. \tag{3.2.18}$$

Choose a subsequene $\{x^{k_v}\}$ of $\{x^k\}$ such that

$$\limsup_{k\to\infty} \langle F(x^*), J_1(x^* - x^{k+1})\rangle = \lim_{v\to\infty} \langle F(x^*), J_1(x^* - x^{k_v})\rangle.$$

Since $\{x^{k_v}\}$ is bounded, we may assume without loss of generality that $x^{k_v} \rightharpoonup \bar{x}$ as $v \to \infty$.
Also, since J_1 is weakly sequentially continuous, we have

$$\limsup_{k\to\infty} \langle F(x^*), J_1(x^* - x^{k+1})\rangle = \langle F(x^*), J_1(x^* - \bar{x})\rangle. \tag{3.2.19}$$

From (3.2.18) and the fact that $x^{k_v} \rightharpoonup \bar{x}$ as $v \to \infty$, we obtain

$$y^{k_v} \rightharpoonup \bar{x}.$$

Also, from (3.2.14) and the fact that $y^{k_v} \rightharpoonup \bar{x}$ as $v \to \infty$, we obtain

$$P_{C_i}(y^{k_v}) \rightharpoonup \bar{x},\ as\ v \to \infty,\ \forall\ i = 1, 2, 3, \cdots, M.$$

Since $\{P_{C_i}(y^{k_v})\} \subset C_i$ and C_i is weakly closed,
then,

$$\bar{x} \in C_i\ \forall\ i = 1, 2, 3, \cdots, M.$$

i.e, $\bar{x} \in \bigcap_{i=1}^{M} C_i$.
Since $x^{k_v} \rightharpoonup \bar{x}$ and A is a bounded linear operator, it follows that

$$u^{k_v} = A(x^{k_v}) \rightharpoonup A(\bar{x})\ as\ v \to \infty.$$

From (3.2.17) and the fact that $u^{k_v} \rightharpoonup A(\bar{x})$, we have

$$P_{Q_t}(u^{k_v}) \rightharpoonup A(\bar{x} \; as \; v \to \infty \; \forall \; t = 1, 2, 3, \cdots, N.$$

Since Q_t is weakly closed and $\{P_{Q_t}(u^{k_v})\} \subset Q_t$, then $A(\bar{x}) \in Q_t, \;\; t = 1, 2, 3, \cdots, N$. i.e, $A(\bar{x}) \in \bigcap_{t=1}^{N} Q_t$.
Thus, $\bar{x} \in \Omega$.

Therefore, from (3.2.1) we have

$$\langle F(x^*), J_1(\bar{x} - x^*)\rangle \;\geq\; 0. \tag{3.2.20}$$

From (3.2.2) and (3.2.7) we have

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq (1 - \eta_k - \alpha_k\tau)\|z^k - x^*\|^2 + \eta_k\|x^k - x^*\|^2 \\
&+ 2\alpha_k\mu\langle F(x^*), j_1(x^* - x^{k+1})\rangle \\
&\leq (1 - \eta_k - \alpha_k\tau)\|x^k - x^*\|^2 + \eta_k\|x^k - x^*\|^2 \\
&+ 2\alpha_k\mu\langle F(x^*), J_1(x^* - x^{k+1})\rangle \\
&= (1 - \alpha_k\tau)\|x^k - x^*\|^2 + \alpha_k\tau\xi_k
\end{aligned}$$

where

$$\xi_k = \frac{2\mu\langle F(x^*), J_1(x^* - x^{k+1})\rangle}{\tau}.$$

From (3.2.19) and (3.2.20) we have

$$\begin{aligned}
\limsup_{k\to\infty}\langle F(x^*), J_1(x^* - x^{k+1})\rangle &= \langle F(x^*), J_1(x^* - \bar{x})\rangle \\
&\leq 0
\end{aligned}$$

So,

$$\limsup_{k\to\infty}\langle F(x^*), J_1(x^* - x^{k+1})\rangle \leq 0.$$

Hence,

$$\limsup_{k\to\infty}\xi_k \leq 0.$$

By Lemma 1.2.47, we have

$$\lim_{k\to\infty}\|x^k - x^*\|^2 = 0.$$

This implies

$$\lim_{k\to\infty}\|x^k - x^*\| = 0.$$

Hence, $x^k \to x^*$ as $k \to \infty$.
Case 2: Suppose that for any integer m, there exists an integer k such that

$k \geq m$ and $\|x^k - x^*\| \leq \|x^{k+1} - x^*\|$.

According to Lemma 1.2.46, there exists a nondecreasing sequence $\{\tau(k)\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} \tau(k) = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$ sufficiently large.

$$\|x^{\tau(k)} - x^*\| \leq \|x^{\tau(k)+1} - x^*\|, \quad \|x^k - x^*\| \leq \|x^{\tau(k)+1} - x^*\|. \tag{3.2.21}$$

Using (3.2.8) and (3.2.21) we have

$$\begin{aligned}\|x^{\tau(k)} - x^*\| &\leq \|x^{\tau(k)+1} - x^*\| \\ &\leq (1 - \eta_{\tau(k)} - \alpha_{\tau(k)})\|z^{\tau(k)} - x^*\| + \eta_{\tau(k)}\|x^{\tau(k)} - x^*\| \\ &+ \alpha_{\tau(k)}\mu\|F(x^*)\| \\ &\leq (1 - \eta_{\tau(k)})\|z^{\tau(k)} - x^*\| + \eta_{\tau(k)}\|x^{\tau(k)} - x^*\| \\ &+ \alpha_{\tau(k)}\mu\|F(x^*)\|\end{aligned}$$

$$\|x^{\tau(k)} - x^*\| - \|z^{\tau(k)} - x^*\| \leq \frac{\alpha_{\tau(k)}\mu\|F(x^*)\|}{1 - \eta_{\tau(k)}}. \tag{3.2.22}$$

From inequalities (3.2.6),(3.2.4) and (3.2.22) we obtain

$$\begin{aligned}0 &\leq \|y^{\tau(k)} - x^*\| - \|z^{\tau(k)} - x^*\| \\ &\leq \|x^{\tau(k)} - x^*\| - \|z^{\tau(k)} - x^*\| \\ &\leq \frac{\alpha_{\tau(k)}\mu\|F(x^*)\|}{1 - \eta_{\tau(k)}},\end{aligned}$$

then, it follows from $\lim_{k\to\infty} \alpha_k = 0$, $\lim_{k\to\infty} \eta_k = \eta < 1$ that

$$\lim_{k\to\infty} (\|y^{\tau(k)} - x^*\| - \|z^{\tau(k)} - x^*\|) = 0 \tag{3.2.23}$$

and

$$\lim_{k\to\infty} (\|x^{\tau(k)} - x^*\| - \|z^{\tau(k)} - x^*\|) = 0.$$

Therefore,

$$\lim_{k\to\infty} \|x^{\tau(k)} - x^*\| - \|y^{\tau()} - x^*\|) = 0. \tag{3.2.24}$$

From (3.2.23) and (3.2.24)

$$\lim_{k\to\infty} \|y^{\tau(k)} - x^*\| = \lim_{k\to\infty} \|z^{\tau(k)} - x^*\|$$

$$\lim_{k\to\infty} \|x^{\tau(k)} - x^*\| = \lim_{k\to\infty} \|y^{\tau(k)} - x^*\|.$$

Since $\{x^k\}$, $\{y^k\}$ and $\{z^k\}$ are bounded, it follows that

$$\lim_{k\to\infty} \|y^{\tau(k)} - x^*\|^2 = \lim_{k\to\infty} \|z^{\tau(k)} - x^*\|^2$$

and

$$\lim_{k\to\infty} \|x^{\tau(k)} - x^*\|^2 = \lim_{k\to\infty} \|y^{\tau(k)} - x^*\|^2.$$

Thus,

$$\lim_{k\to\infty} (\|y^{\tau(k)} - x^*\|^2 - \|z^{\tau(k)} - x^*\|^2) = 0 \tag{3.2.25}$$

and

$$\lim_{k\to\infty} (\|x^{\tau(k)} - x^*\|^2 - \|y^{\tau(k)} - x^*\|^2) = 0. \tag{3.2.26}$$

From (3.2.5) we have

$$\|y^{\tau(k)} - z^{\tau(k)}\|^2 \le \frac{1}{\lambda}(\|y^{\tau_k} - x^*\|^2 - \|z^{\tau(k)} - x^*\|^2).$$

It follows from (3.2.25) that

$$\lim_{k\to\infty} \|y^{\tau(k)} - z^{\tau(k)}\| = 0. \tag{3.2.27}$$

Since $\{\delta_k\} \subset [\delta, \bar{\delta}] \subset (0, \frac{2}{\|A\|^2} + 1)$, we obtain from (3.2.3) that

$$\delta(2 - \bar{\delta}\|A\|^2)\|u^{\tau(k)} - v^{\tau(k)}\|^2 \le \|x^{\tau(k)} - x^*\|^2 - \|y^{\tau(k)} - x^*\|^2. \tag{3.2.28}$$

From (3.2.28) and (3.2.26) we have

$$\lim_{k\to\infty} \|u^{\tau(k)} - v^{\tau(k)}\| = 0. \tag{3.2.29}$$

On the other hand,

$$\begin{aligned}\|x^{\tau(k)} - y^{\tau(k)}\| &= \delta_{\tau(k)}\|J_1^{-1}A^*j_2(u^{\tau(k)} - v^{\tau(k)})\| \\ &\le \bar{\delta}\|A\|\|u^{\tau(k)} - v^{\tau(k)}\|.\end{aligned}$$

The above inequality and (3.2.29) implies that

$$\lim_{k\to\infty} \|x^{\tau(k)} - y^{\tau(k)}\| = 0. \tag{3.2.30}$$

Note that

$$\|x^{\tau(k)} - z^{\tau(k)}\| \le \|x^{\tau(k)} - y^{\tau(k)}\| + \|y^{\tau(k)} - z^{\tau(k)}\|.$$

This together with (3.2.27) and (3.2.30) implies

$$\lim_{k\to\infty} \|x^{\tau(k)} - z^{\tau(k)}\| = 0. \tag{3.2.31}$$

Following the method in Case1, we also obtain

$$\limsup_{k\to\infty} \langle F(x^*), J_1(x^* - x^{\tau(k)})\rangle \le 0. \tag{3.2.32}$$

Also

$$
\begin{aligned}
\|x^{\tau(k)+1} - x^{\tau(k)}\| &= \|(1-\eta_{\tau(k)})z^{\tau(k)} - \alpha_{\tau(k)}\mu F(z^{\tau(k)}) \\
&- [(1-\eta_{\tau(k)})x^{\tau(k)} - \alpha_{\tau(k)}\mu F(x^{\tau(k)})] - \alpha_{\tau(k)}\mu F(x^{\tau(k)})\| \\
&\le \|(1-\eta_{\tau(k)})z^{\tau(k)} - \alpha_{\tau(k)}\mu F(z^{\tau(k)}) \\
&- [(1-\eta_{\tau(k)})x^{\tau(k)} - \alpha_{\tau(k)}\mu F(x^{\tau(k)})]\| \\
&+ \alpha_{\tau(k)}\mu\|F(x^{\tau(k)})\| \\
&\le (1-\eta_{\tau(k)} - \alpha_{\tau(k)})\|z^{\tau(k)} - x^{\tau(k)}\| + \alpha_{\tau(k)}\mu\|F(x^{\tau(k)})\| \\
&\le \|z^{\tau(k)} - x^{\tau(k)}\| + \alpha_{\tau(k)}\mu\|F(x^{\tau(k)})\|.
\end{aligned}
$$

Hence

$$
\|x^{\tau(k)+1} - x^{\tau(k)}\| \le \|z^{\tau(k)} - x^{\tau(k)}\| + \alpha_{\tau(k)}\mu\|F(x^{\tau(k)})\|. \tag{3.2.33}
$$

Using (3.2.31), (3.2.33), the boundedness of $\{F(x^k)\}$ and the fact that $\lim_{k\to\infty}\alpha_k = 0$, we get

$$
\lim_{k\to\infty}\|x^{\tau(k)+1} - x^{\tau(k)}\| = 0. \tag{3.2.34}
$$

Since J_1 is norm to weak* uniformly continuous on bounded subsets of E_1, it follows from (3.2.32) and (3.2.34) that

$$
\begin{aligned}
\limsup_{k\to\infty}\langle F(x^*), J_1(x^* - x^{\tau(k)+1})\rangle &= \limsup_{k\to\infty}[\langle F(x^*), J_1(x^* - x^{\tau(k)+1}) \\
&- J_1(x^* - x^{\tau(k)})\rangle - \langle F(x^*), J_1(x^{\tau(k)} - x^*)\rangle] \\
&= \limsup_{k\to\infty}\langle F(x^*), J_1(x^* - x^{\tau(k)+1}) \\
&- J_1(x^* - x^{\tau(k)})\rangle \\
&- \limsup_{k\to\infty}\langle F(x^*), J_1(x^{\tau(k)} - x^*)\rangle \\
&\le \limsup_{k\to\infty}\langle F(x^*), J_1(x^* - x^{\tau(k)+1}) \\
&- J_1(x^* - x^{\tau(k)}\rangle \\
&\le 0.
\end{aligned}
$$

Thus,

$$
\limsup_{k\to\infty}\langle F(x^*), J_1(x^* - x^{\tau(k)+1})\rangle \le 0. \tag{3.2.35}
$$

From (3.2.2), (3.2.7) and (3.2.21) we get

$$\begin{aligned}\|x^{\tau(k)+1} - x^*\|^2 &\leq (1 - \eta_{\tau(k)} - \alpha_{\tau(k)}\tau)\|z^{\tau(k)} - x^*\|^2 + \eta_{\tau(k)}\|x^{\tau(k)} - x^*\|^2 \\ &+ 2\alpha_{\tau(k)}\mu\langle F(x^*), J_1(x^* - x^{\tau(k)+1})\rangle \\ &\leq (1 - \alpha_{\tau(k)}\tau)\|x^{\tau(k)} - x^*\|^2 \\ &+ 2\alpha_{\tau(k)}\mu\langle F(x^*), J_1(x^* - x^{\tau(k)+1})\rangle \\ &\leq (1 - \alpha_{\tau(k)}\tau)\|x^{\tau(k)+1} - x^*\|^2 \\ &+ 2\alpha_{\tau(k)}\mu\langle F(x^*), J_1(x^* - x^{\tau(k)+1})\rangle.\end{aligned}$$

Therefore, since $\alpha_{\tau(k)} > 0$

$$\|x^{\tau(k)+1} - x^*\|^2 \leq \frac{2\mu}{\tau}\langle F(x^*), J_1(x^* - x^{\tau(k)+1})\rangle. \quad (3.2.36)$$

From (3.2.21) and (3.2.36), we have

$$\|x^k - x^*\|^2 \leq \|x^{\tau(k)+1} - x^*\|^2 \leq \frac{2\mu}{\tau}\langle F(x^*), J_1(x^* - x^{\tau(k)+1})\rangle. \quad (3.2.37)$$

Taking limsup as $k \to \infty$ in (3.2.37) and using (3.2.35), we obtain that

$$\limsup_{k\to\infty} \|x^k - x^*\|^2 = 0.$$

This implies

$$\lim_{k\to\infty} \|x^k - x^*\| = 0.$$

Therefore, $x^k \to x^*$ *as* $k \to \infty$. This completes the proof.

Corollary 3.2.2 *[1] Let* $C_1,\ C_2,\ C_3, \cdots C_M$ *be M- nonempty closed convex subsets of* H_1 *and* $Q_1,\ Q_2,\ Q_3, \cdots, Q_N$ *be N-nonempty closed convex subsets of* H_2*, where* H_1 *and* H_2 *are two Hilbert spsces. Let* $F : H_1 \to H_1$ *be* β*-strongly monotone and L- Lipschitz continuous map and* $A : H_1 \to H_2$ *be a bounded linear operator with its adjoint* A^**. Let* $x_0 \in H_1$*. Then, the sequence* $\{x_k\}$ *geenerated by Algorithm 1 converges strongly to the unique solution of the problem (VI) provided the solution set* Ω *of the MSSFP is nonempty.*

Proof
Since the duality map is identity in Hilbert space, the result follows from Theorem 3.2.1.

Algorithm 2:

Step 0: Choose $0 < \delta \leq \delta_k \leq \bar{\delta} < \frac{2}{\|A\|^2+1}$,$\{\alpha_k\} \subset (0,1)$ $\{\eta_k\} \subset (0,1)$ such that $lim_{k\to\infty}\alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $0 \leq \eta_k \leq 1-\alpha_k \; \forall k \geq 0$. $lim_{k\to\infty}\eta_k = \eta < 1$.

Step 1: Set $k := 0$.
Step 2: Compute

$$u^k = A(x^k) \; and \; P_{Q_j}(u^k), \quad j = 1, 2, 3, \cdots N$$

Step 3: FInd

$$j_k = argmax\{\|P_{Q_j}(u^k) - u^k\| : \quad j = 1, 2, 3, \cdots N\}, \quad v^k = P_{Q_{j_k}}(u^k).$$

Step 4: Compute

$$y^k = x^k - \delta_k j_1^{-1} A^* j_2(u^k - v^k),$$

Step 5: Compute

$$P_{C_i}(y^k), \; i = 1, 2, 3, \cdots M\},$$

Step 6: Find

$$i_k = argmax\{\|P_{C_i}(y^k) - y^k\| : i = 1, 2, 3, \cdots M\}, \; z^k = P_{C_{i_k}}(y^k),$$

Step 7: Compute

$$x^{k+1} = \eta_k x^k + (1 - \eta_k - \alpha_k) z^k.$$

Step 8: Set $k := k + 1$ and go to step 2.

Corollary 3.2.3 *[1] Let $C_1, \; C_2, \; C_3, \cdots C_M$ be M- nonempty closed convex subsets of H_1 and $Q_1, \; Q_2, Q_3, \cdots, Q_N$ be N-nonempty closed convex subsets of H_2. Let $F : H_1 \to H_1$ be the identity map and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $x_0 \in H_1$. Then, the sequence $\{x_k\}$ geenerated by Algorithm 2, converges strongly to the mininmum norm solution provided MSSFP is nonempty.*

Proof
When F is the identity mapping, i.e, $F(x) = x$. The problem (VI) contains the problem of finding the minimum norm solution of the MSSFP as a special case. Therefore, the proof follows from Theorem 3.2.1.

When M = N = 1, the MSSFP becomes the split feasibility problem (SFP)

$$Find \; x^* \; \in \; C \; \; such \; that \; \; A(x^*) \in Q$$

Corollary 3.2.4 *Let E_1 be a uniformly convex and 2-uniformly smooth space with best smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$ whose duality map is weakly sequentially continuous and E_2 a reflexive and strictly convex Banach space. Let C and Q be nonempty closed convex subsets of*

E_1 and E_2 respectively. Let $F : E_1 \to E_1$ be strongly accretive and L- Lipschitz continuous map and $A : E_1 \to E_2$ be a bounded linear operator with its adjoint A^*. Let $x_0 \in E_1$. Then, the sequence $\{x_k\}$ geenerated by;

$$\begin{cases} y^k = x^k - \delta_k {j_1}^{-1} A^* j_2(A(x^k - P_Q(A(x^k))), \\ x^{k+1} = \eta_k x^k + (1-\eta_k) y^k - \alpha_k \mu F(y^k) \ \forall k \geq 0 \end{cases}$$

where $0 < \mu < \frac{2\beta}{L^2}, 0 < \delta \leq \delta_k \leq \bar{\delta} < \frac{2}{\|A\|^2+1}$, $\{\alpha_k\} \subset (0,1)$, $\{\eta_k\} \subset (0,1)$such that $lim_{k\to\infty}\alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$, $0 \leq \eta_k \leq 1 - \alpha_k \ \forall k \geq 0$, $lim_{k\to\infty}\eta_k = \eta < 1$, converges strongly to the unique solution of the variational inequality problem which also solves the split feasibility problem.

Proof
The proof follows from Theorem 3.2.1 and Algorithm 2.

Chapter 4

Summary and Conclusion

4.1 Introduction

This is the last chapter of this Thesis where we present the summary, conclusion and recommendation of our work.

4.2 Summary

In the main work, we studied the multiple-sets split feasibility problem in Hilbert spaces snd also studied it in the settings of Banach spaces much more general than Hilbert space. Next, we studied the multiple-sets split feasibility problem with variational inequalities in Banach spaces. In our main results, we initiated a new scheme and then establish strong convergence theorems for solving a strongly variational inequality with multiple sets split feasibility problem in uniformly convex and 2-uniformly smooth Banach spaces. We then obtained an algorithm for finding the minimum norm solution of the multiple-sets split feasibility prolem.

4.3 Conclusion

In our main work, we presented an iterative method for solving strongly variational inequality with multiple sets split feasibility problem and proved its norm convergence. The result in chapter three, generalized the result of Anh[1]

4.4 Recomendation

We strongly recomend other reseachers in this area to see if they can extend/or improve this work either by establishing norm convergence without the weak sequential continuity of the duality map or by extending the space such that it can include some important Banach spaces.

Bibliography

[1] T.V. Anh, *A parallel method for variational inequalities with the multiple-sets split feasibility problem constraints,* J.Fixed point theory Appl.sDoi:10.1007/s11784-017-0452-y(2017)

[2] T.V. Anh, *strongly convergent subgradient extragradient-Halpern method for solving a class of bilevel pseudomonotone variational inequalities.* Vietman J.Math.Doi:10.1007/s10013-01(2016)

[3] Ya.I. Alber, *Generalized projection operators in Banach spaces:prooerties and applications*, in: Theory and Applications of Nonlinear Operators of Accretive and monotone Types, Vol.178, lecture in Pure and Applied Mathematics, Dekker, New York, NY, USA,15-20(1996)

[4] B. Ali, *Common fixed points approximation for assymptotically nonexpansive semigroup in Banach spaces.* doi:10.5402/2011/68415(2011)

[5] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem.Inverse problems.* 18: 1441-53(2002)

[6] H.H. Cui, F.H. Wang, *Iterative methods for the split common fixed point problem in Hilbert space.* Fixed point theory Appl.78:1687-1812(2014)

[7] Y. Censor, T. Elfving, *A multiprojection algorithm using Bregman projection in a product space.* Numer. Algorithm, 8,221-239(1994)

[8] C.E. Chidume, *Geometric properties of Banach spaces and Nonlinear iterations.* Lecture notes in mathematics, Springer, London, UK,1965(2009)

[9] Y. Censor, T. Elfving, N. kopf, T. Bortfield, *The multiple-sets split feasibility problem and its application for inverse problems.* inverseprob.21,2071-2084(2005)

[10] Y. Censor, A. Ben-Israel, Y. Xiao, J.M. Galvin, *On linear infeasibility arising in intensity modulated radiation therapy inverse planning.* linear Algebra.Appl. 423,1404-1420(2008)

[11] P.L.Combettes, *The convex feasibility problem in image recovery.* Adv. Imaging Electron Phys.,95, 155-270(1996)

[12] P.L. Combettes, S.A. Hirstoaga, *Equilibrium programming in Hilbert space.* J.Nonlinear convex Anal. 6,117-136(2005)

[13] Y. Censor, T. Bortfield,B. Martin, A. Trofimov, *A unified approach for inversion problems in intensity radiation therapy.* Phys.Med.Biol. 51,2353-2365(2006)

[14] I.V. Konnov, *Combined Relaxation Methods for variational inequalities.* Springer, Berlin (2000)

[15] B. Liu, B. Qu, N. Zhang, *A successive projection method for solving the multiple sets split feasiblity problem.* Numer.Funct.Anal.Optim. 35,1459-1466(2014)

[16] G. Lopez, V. Martin and H.k. Xu, *Iterative algorithm for the multiple set split feasibility problem.* Medical physics publishing, Madison,243-279(2009)

[17] Meng Wen, Jigen Peng, Yuchao Tang, *A cyclic and simultaneous iterative method for solving the multiple sets split feasibility problem.* J. Optim.Theory Appl.166, 844-860(2005)

[18] K.Nakajo, W.Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups.* J.Math.Anal.Appl.279,372-379(2003)

[19] B. Prus, and R.Smarzewski, *Strongly unique best approximations and centres in uniformly convex spaces.* J.Math.Anal.Appl.,121,10-21(1987)

[20] B. Qu, N. Xiu, *Note on the CQ-algorithm for the split feasibility problem.* Inverse probl.21,1655-1665(2005)

[21] F. Schöpfer, *Iterative methods for the solution of the split feasibility problem in Banach space.* phD thesis, Saarbrücken(2007)

[22] F. Schöpfer, T. Schuster and A.K. louis,*An iterative regularization method for the solution of the split feasibility problem in Banach space.* inverse problem 24:055008(2008)

[23] F.Wang, *A new algorithm for solving the multiple set split feasibility problem in Banach space*. Numer.funct.Anal. and optim.35,99-110(2014)

[24] H.K. Xu, *Iterative algorithm for nonlinear operators.* J.Lond.Math.soc. 66,240-256(2002)

[25] Xuejin Tian, Lin Wang, Zhaoli Ma, *On the split equality common fxed point problem for quasi-nonexpansive multi-valued mappings in Banach spaces.* J. Nonlinear Sci. Appl. 9, 5536 5543(2016)

[26] Q. Yang, *The relaxed CQ-algorithm for solving the split feasibility problem.*Inverse probl.20,1261-1266(2004)

[27] Li Yang, S.S. Chang, Y.J. Cho and J.K. Kim, *Multiple sets split feasibility problem for total asymptotically strict pseudo contractive mappings.* Fixed Point Theory Appl.77(2011)

[28] J.L. Zhao, Y.J. Zhang, Q.Z. Yang, *Modified projection methods for the split feasibility problem and multiple sets split feasibility problem.* Appl.Math.Comput. 219,1640-1653(2012)

[29] J.L. Zhao, Q.Z. Yang, *Self adaptive projection method for solving the multiple sets split feasibility problem.* Inverse probl. 035009.13 pp.(2011)

[30] J.L. Zhao, Q.Z. Yang, *A simple projecion method for solving the multiple sets split feasibility problem.* Inverse probl. Sci.Eng. 21,537540(2013)