A HYBRID ALGORITHM FOR APPROXIMATING A COMMON ELEMENT OF SOLUTIONS OF A VARIATIONAL INEQUALITY PROBLEM AND A CONVEX FEASIBILITY PROBLEM

A Thesis Presented to the Department of

Pure and Applied Mathematics

African University of Science and Technology

In Partial Fulfilment of the Requirements for the Degree of

Master of Science

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December, 2017.

CERTIFICATION

This is to certify that the thesis titled ”A HYBRID ALGORITHM FOR APPROXIMATING A COMMON ELEMENT OF SOLUTIONS OF A VARIATIONAL INEQUALITY PROBLEM AND A CONVEX FEASIBILTY PROBLEM” submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research work carried out by Okereke Lois Chinwendu in the department of Pure and Applied Mathematics.

APPROVAL

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ABSTRACT

In this thesis, a hybrid extragradient-like iteration algorithm for approximating a common element of the set of solutions of a variational inequality problem for a monotone, *k-Lipschitz* map and common fixed points of a countable family of relatively nonexpansive maps in a uniformly smooth and 2-uniformly convex real Banach space is introduced. A strong convergence theorem for the sequence generated by this algorithm is proved. The theorem obtained is a significant improvement of the results of Ceng *et al.* (J. Glob. Optim. **46**(2010), 635-646). Finally, some applications of the theorem are presented.

Keywords: Relatively nonexpansive maps, monotone maps, Lipschitz continuous maps, generalized projection, variational inequality problem, fixed point problem hybrid extragradient-like approximation method.

ACKNOWLEDGEMENTS

My heartfelt thanks go to my amiable and indefatigable head of department and thesis supervisor professor Charles Ejike Chidume, whose guidiance and wealth of experience has contributed immensely to the success of this work. In a fatherly manner, you have taught me the essence of successful research in Mathematics and have truly inspired me to push the limits of my mathematical reasoning and for this, I will be eternally grateful to you.

To Abubakar Adamu, my industrious and reliable research colleague, I cannot fail to acknowledge the productive ideas you supplied in the course of this work, the sleepless nights you spent on the chalkboard with me, and your meticulous effort in typesetting parts of this work. Special thanks to the African Capacity Building Foundation (ACBF) for their financial support which had guaranteed the comfort and stability to carry out this work.

Words cannot express the depth of my gratitude to my parents Rufus and Priscilla Okereke, my siblings Phoebe, Bethel and Iyk and my brother in-law Okey, for their prayers and encouragement. This same feeling goes out to my "household" Ukamaka Nnyaba and Onyekachi Osisiogu, both of whom had stood by me through thick and thin academically, emotionally, financially, morally, ... just name it. I cannot thank you both enough. Also to Mr. Ademuyiwa Olalekan my "Kano father", I am especially grateful to you for creating the necessary support that enabled me to enroll and succeed in this program. Without forgetting to mention my "in-law" Eze Hillary and Sr. Chidinma Onyegbule who has been like a mother to me.

Furthermore, I owe my gratitude also to Professors Gane Samb Lo, Khalli Ezzinbi and Ngalla Djittè for their doggedness, devotion and steadfastness in teaching their courses in details which contributed to prepare me for this research work, also to Professor Eric Ofoedu my mentor, I thank you so much for introducing and encouraging me in words and deeds to undertake this program. I sincerely acknowledge the support of all other AUST staff especially Amaka Udigwe (Mathematics department administrative assistant) who never starved me of necessary information, Mrs Nsima for her prompt response to welfare issues, Mr. Osoba and Nafisa for their generous assistance.

I am further grateful to Dr. Ma'aruf, Dr. Usman, my senior colleagues, Chinedu Ezea, Ogonna Romanus, Monday Nnakwe, Emmanuel Otubo and all other members of the nonlinear operator theory research group including Kingsley, Stephen, Ikechukwu and Ejike for their constuctive criticisms and suggestions. Thank you to all my classmates, Aisha, Tony, Dave, Leonard, Aminu, Igwe, Haruna amongst others for the togetherness. Thanks to Irewole Iyomo, for magnanimously typesetting this work, and to my sweet friend and backbone Emmanuel Onwukwe for your roles behind the scene. To my other friends including Abraham Ebunu, Hillary Emeli, congregation members and all others too numerous to mention,thank you all.

Above all, my greatest thanks go to the Almighty God Jehovah for the priviledge of life and the wisdom to carry out this work, but most especially for surprising me positively in ways I never imagined.

PUBLICATIONS

1. C.E Chidume, L.O Chinwendu and A. Adamu; *A hybrid algorithm for approximating a common element of solutions of a variational inequality problem and a convex feasibility problem;* ANVI 21 (1) (2018) 46-64.

2. C.E Chidume, A. Adamu and L.O Chinwendu; *A Krasnoselskii-type algorithm for approximating solutions of variational inequality problem and convex feasibility probelm*(to appear).

DEDICATION

To the loving memory of my late elder brother, Exodus Chidi Okereke, whose passionate wish was for me to be educated.

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CHAPTER 1

GENERAL INTRODUCTION AND LITERATURE REVIEW

1.1 Background of study

"There is no branch of mathematics, however abstract which may not someday be applied to phenomena of the real world"

— Lobachevsky

Attesting to the authenticity of Lobachevsky's claim, the vast applicability of mathematical models whose constraints can be expressed as fixed point and (or) variational inequality problems in solving real life problems, such as in signal processing, networking, resource allocation, image recovery and so on, makes the field of variational inequality and fixed point theory a worthwhile area of research [See for example Maainge [2008] and Maainge [2010b] and the references contained in them].

In this thesis, we concentrate on approximating a common element of solutions of a variational inequality problem and common fixed point of a countable family of relatively nonexpansive maps in real Banach spaces. Hence, the results of this thesis will form major contributions to nonlinear operator theory, which falls within the general area of nonlinear functional analysis and applications.

1.1.1 Variational Inequality

It was the year 1958, in a classroom, at the *Instituto Nazionale di Alta Mathematica* Italy, that Antonio Signorini posed the problem "what will be the equilibrium configuration of a spherically shaped elastic body resting on a rigid frictionless plane?" A natural question is: what is special about this problem? Its ambiguous boundary condition. In fact, Signorini himself called it "problem with ambiguous boundary condition". The statement of the problem involves inequalities and according to Antman (1983), the essential difficulty is that the point of contact between the body and the plane is not known *a-priori*, conceivably too, the contact set could be especially complicated. Nevertheless, Signorini warmly invited young analyst to study the problem (Signorini [1959]).

Gaetano Fichera, a student in that class, decided to investigate the problem using the virtual work principle and in January 1963, he produced a complete proof of the existence and uniqueness of a solution for the problem. In honour of his teacher, Fichera renamed the problem as "Signorini problem" (Fichera [1963]). Fichera's solution of the Signorini problem became the bedrock that has metamorphosed into the field known as variational inequality today. Indeed, just as Antman puts it, the solution of the Signorini problem coincides with the birth of the field of variational inequalities (Antman [1983]).

Let E be a real Banach space with dual space E^*. Let C be a nonempty, closed and convex subset of E and $A : C \to E^*$. Let $\langle \cdot, \cdot \rangle : E \times E^* \to \mathbb{R}$ be the duality pairing.

The problem of finding a point $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \ \forall \ v \in C, \tag{1.1.1}$$

is called a variational inequality problem.
Let C be a nonempty, closed and convex subset of a real Banach space E with dual space E^* and $A : C \to E^*$ be a map. Then, A is said to be:

- *k-Lipschitz continuous* if there exists a constant $k \geq 0$ such that

$$\|Ax - Ay\| \leq k\|x - y\|, \ \forall x, y \in C. \tag{1.1.2}$$

- *monotone* if the following inequality holds:

$$\big\langle x - y, Ax - Ay \big\rangle \geq 0, \ \forall x, y \in C. \tag{1.1.3}$$

- *δ-inverse strongly monotone* if there exists a $\delta \geq 0$, such that

$$\big\langle x - y, Ax - Ay \big\rangle \geq \delta\|Ax - Ay\|^2, \ \forall x, y \in C. \tag{1.1.4}$$

- *maximal monotone* if A is monotone and the graph of A is not properly contained in the graph of any other monotone map.

It is immediate that if A is *δ-inverse strongly monotone*, then A is monotone and Lipschitz continuous.

In this thesis, we shall assume that the subset C of E is nonempty, closed and convex and the map A is monotone and k-Lipschitz. We shall denote the set of solutions of the variational inequality problem by $VI(C, A)$.

Remark 1.1.1 *It is easy to see that if u is a solution of the variational inequality problem (1.1.1) then,*

$$\langle x - y, Ax \rangle \geq 0, \ \forall x \in C.$$

1.1.2 Fixed Point Theory

Fixed point theory is one of the most important and useful tools of modern mathematics. Its interconnectedness with other fields such as game theory, optimization theory, approximation theory and variational inequality points to the fact that fixed point theory is a show piece of mathematical unification (Vandana and Chetan [2017]). Fixed point theory is based on a very simple mathematical setting. A point is called a fixed point, if it remains invariant under any form of transformation. For a self map f, i.e., $f : E \to E$, a fixed point is a point $x_0 \in E$ such that $f(x_0) = x_0$. This point however, may or may not exist. This gives rise to the problem: "what condition(s) guarantees existence of a fixed point?" This problem has been of interest since the 19th century and no doubt has attracted huge research from many mathematicians ranging from the contributions of Cauchy Fredhlin, Liouville, Lipschitz, Peano and Picard in establishing existence and uniqueness of solutions, particularly to differential equations using successive approximations (Vandana and Chetan [2017]). Several theorems on existence and properties of fixed points have been proved, amongst them include Banach fixed point theorem and Brouwer fixed point theorem referred to as the two fundamental theorem of fixed points (Vandana and Chetan [2017]).

In recent years, books, monographs and scholarly articles on fixed point theory abound (see e.g., Chidume [2009], Chidume et al. [2016],Berinde [2007],Zeidler [1985]). This thesis work focuses on the set of fixed points of a relatively nonexpansive map S and this is denoted by $F(S)$.

1.1.3 Variational Inequality and Fixed Point Problem

Many models for solving real life problems have their associated constraints captured as fixed point and variational inequality problems. Consequently, the problem of approximating a solution of the variational inequality problem that is also a fixed point of some operator is of great significance. [See e.g., Maainge [2010a], Ceng et al. [2010] and the references contained in them].

1.1.4 Convex Feasibility Problem

Let E be a real Banach space, and let $\{C_n\}_{n\geq 1}$ be a countable family of closed convex subsets of E. The problem of finding a point $x_0 \in \bigcap_{n=1}^{\infty} C_n$ is called the convex feasibility problem.

1.2 Statement of Problem

This thesis is concerned with the problem of approximating a common element of the set of solutions of the variational inequality problem for a monotone and k-Lipschitz map A and the set of fixed points of a relatively non-expansive map S, in a uniformly smooth and 2-uniformly convex real Banach space.

1.3 Objective of the Study

It is our aim in this thesis to:

- Study and analyse the work done in Hilbert space by [Ceng et al, 2010].
- Introduce an iterative algorithm for approximating an element of $VI(C, A) \cap F(S)$ in a uniformly smooth and 2-uniformly convex real Banach space.
- Establish the well-definedness of our algorithm.
- Prove a strong convergence theorem for the sequence generated by our algorithm.
- Give some applications of our theorem.

1.4 Literature Review

The evolution of variational inequality problems dates back to the late 1960's by Lions and Stampacchia Lions and Stampacchia [1967] and over the years, extensive study, analysis and generalisation of these problems have been done by numerous researchers in the field of nonlinear operator theory.The literature abounds with iterative algorithms for approximating solutions of variational inequality problems and fixed points of some operators (see, for example, Chidume [2009], Nilsrakoo and Saejung [2011], Buong [2010], Censor et al. [2012, 2011], Hieu et al. [2006], Dong et al. [2016], Gibali et al. [2015], Iiduka and Takahashi [2008], Chidume et al., Censor et al. [2010], Xu and Kim [2003], and the references contained in them).

Antipin [2000] studied methods for finding a solution of a variational inequality problem that satisfies some additional constraints in a finite dimensional space. Takahashi and Toyoda [2003], investigated the problem of finding a solution of a variational inequality problem which is also a fixed point of some map in a Hilbert space. By assuming A to be δ-inverse strongly monotone, S to be a nonexpansive map of C into C, and $VI(C, A) \cap F(S) \neq \emptyset$, they proposed an iterative algorithm and established *weak convergence* result of the sequence generated by their algorithm to an element of $VI(C, A) \cap F(S)$, where $F(S)$ is the set of fixed points of S. Later, Iiduka and Takahashi [2008], using a modified algorithm, while retaining the same assumptions on A and S proved strong convergence of the sequence generated by their algorithm to a point of $VI(C, A) \cap F(S)$. However, the assumption that A is δ-inverse strongly monotone excludes some important classes of maps (see, Nadezhkina and Takahashi [2006]).

In order to weaken the δ-inverse monotonicity condition on A, Ceng and Yao [2007] and Nadezhkina and Takahashi [2006], proved the following strong convergence theorems.

Theorem 1.4.1 (**Ceng and Yao [2007]**) *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to C$ be a contractive mapping with a contractive constant $L \in (0,1)$. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in C, \\ y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(y_n) + \beta_n S P_C(x_n - \lambda A y_n), \end{cases} \tag{1.4.1}$$

where $\{\lambda_n\} \subset (0,1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

1. $\alpha_n + \beta_n \leq 1 \ \forall \ n \geq 0$;
2. $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
3. $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $q = P_{F(S)\cap VI(C,A)} f(q)$ if and only if $\{Ax_n\}$ is bounded and $\liminf_{n\to\infty} \langle y - x_n, Ax_n \rangle \geq 0, \ \forall \ y \in C$.

Theorem 1.4.2 (**Nadezhkina and Takahashi [2006]**) *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to C$ be a montone k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.4.2}$$

for all $n \geq 0$ where$\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$ and $\{\alpha_n\} \subset [0,c]$ for some $c \in [0,1)$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to the same point $q = P_{F(S)\cap VI(C,A)} x_0$.

Motivated by these two results, Ceng et al. [2010] in 2010 introduced a hybrid extragradient-like approximation method and proved the following strong convergence theorem.

Theorem 1.4.3 (**Ceng et al. [2010]**) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C,A) \neq \emptyset$. Define inductively the sequence $(x_n), (y_n), (z_n)$ by*

$$\begin{cases} x_0 \in C, \\ y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ z_n = (1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n S P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + (3 - 3\gamma_n + \alpha_n) b^2 \|Ax_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.4.3}$$

for all $n \geq 0$ where $\{\lambda_n\}$ is a sequence in $[a,b]$ with $a > 0$ and $b < \frac{1}{2k}$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the conditions:

1. $\alpha_n + \beta_n \leq 1 \; \forall \; n \geq 0$;

2. $\lim_{n\to\infty} \alpha_n = 0$;

3. $\liminf_{n\to\infty} \beta_n > 0$;

4. $\lim_{n\to\infty} \gamma_n = 1$ *and* $\gamma_n > \frac{3}{4} \; \forall \; n \geq 0$.

Then, the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ *are well defined and converge strongly to the same point* $q = P_{F(S)\cap VI(C,A)} x_0$.

In this thesis, motivated by the result of Ceng et al. [2010], we introduce a hybrid extragradient-like algorithm in a *uniformly smooth and 2-uniformly convex* real Banach space and prove strong convergence of the sequence generated by our algorithm to a point $u \in F(S) \cap VI(C, A)$.

CHAPTER 2

PRELIMINARIES

In this chapter, we will give definition of some terms and results of interest used in the thesis.

2.1 Definition of terms

Through out this thesis, we will always let E be a real Banach space with dual space E^* and $\langle \cdot, \cdot \rangle$ denote the duality pairing of E and E^*. Whenever a sequence $\{x_n\}$ in E, converges strongly (weakly), respectively, we denote the convergence by $x_n \to x$ ($x_n \rightharpoonup x$).

Definition 2.1.1 *A normed space E is called smooth if for every $x \in E$, $\|x\| = 1$, there exists a unique $x^* \in E^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.*

Definition 2.1.2 *Let $q > 1$ and $r > 0$, be two fixed real numbers. Then E is uniformly smooth if and only if there exists a continuous, strictly increasing and convex function*

$$g : \mathbb{R}^+ \to \mathbb{R}^+, \ g(0) = 0$$

such that

$$\|\lambda x + (1-\lambda)y\|^q \geq \lambda\|x\|^q + (1-\lambda)\|y\|^q - W_q(\lambda)g(\|x-y\|)$$

for all $x, y \in B_r, 0 \leq \lambda \leq 1$, where $W_q(\lambda) = \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$.

Definition 2.1.3 *A normed space E is called uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$, such that for any $x, y \in E$, with $\|x\| = 1$, $\|y\| = 1$ and $\|x-y\| \geq \epsilon$ then $\|\frac{x+y}{2}\| \leq 1-\delta$.*

Remark 2.1.4 *We note immediately that the following definition is also used :*
A normed linear space E is uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Definition 2.1.5 *A normed space E is called strictly convex if for all $x, y \in E$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1-\lambda)y\| < 1$, $\forall \lambda \in (0, 1)$.*

Definition 2.1.6 *Let E be a normed space with $dimE \geq 2$. The modulus of convexity of E is the fuction $\delta_E : (0, 2] \to [0, 1]$ defined by*

$$\delta_E := \inf\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1; \epsilon = \|x - y\|\}.$$

Remark 2.1.7

1. *In the particular case of an inner product space H, we have* $\delta_H(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}$.

2. *Every uniformly convex space is reflexive .*

3. *E is uniformly convex if and only if* $\delta_E(\epsilon) > 0$ *for all* $\epsilon \in (0, 2)$

Definition 2.1.8 *Let* $p > 1$ *be a real number. Then, a normed space E is said to be p-uniformly convex if there is a constant* $c > 0$ *such that*

$$\delta_E(\epsilon) \geq c\epsilon^p$$

Example 2.1.9 *If* $E = L_p(orl_p),\ 1 < p < \infty$, *then*

(a) $\delta_E(\epsilon) \geq \frac{1}{2^{p+1}}\epsilon^2$ *if* $1 < p < 2$;

(b) $\delta_E(\epsilon) \geq \epsilon^p$, *if* $2 \leq \infty$.

Definition 2.1.10 *A map A of E into* E^* *is said to be hemicontinuous if for all* $x, y \in C$, *the map* $f : [0, 1] \to E^*$ *defined by* $f(t) = A(tx + (1 - t)y)$ *is continuous with respect to the weak** *topology of* E^*

Definition 2.1.11 *The problem of finding a point* $u \in C := \bigcap_{i=1}^{\infty} C_i$, *where* C_i *is a convex set for each i, is called a convex feasibility problem.*

Definition 2.1.12 *A continuous strictly increasing function* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $g(0) = 0$ *and* $\lim_{t\to\infty} g(t) = \infty$ *is called a guage function.*

Definition 2.1.13 *Given a guage function g, the map* $J_g : E \to 2^{E^*}$ *defined by*

$$J_g x := \{u^* \in E^* : \langle x, u^* \rangle = \|x\|\|u^*\|; \quad \|u^*\| = g(\|x\|)\}$$

is called the duality map with the guage function g where E is any normed space.

Remark 2.1.14

- *In the particular case* $f(t) = t$, *the duality map* $J = J_g$ *is called the normalized duality map.*
- *If E is a reflexive, strictly convex and smooth real Banach space, then, J is single-valued and bijective.*
- *In a Hilbert space H, the duality map J and it is inverse and* J^{-1} *are the identity maps on H.*
- *If E is uniformly smooth and uniformly convex, then, the dual space* E^* *is also uniformly smooth and uniformly convex and the normalized duality map J and its inverse,* J^{-1}, *are both uniformly continuous on bounded sets.*

Definition 2.1.15 *The Lyapunov fuctional* $\phi : E \times E \to [0, \infty)$ *is defined by*

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv \rangle + \|v\|^2 \ \forall\ u, v \in E, \tag{2.1.1}$$

where J is the normalized duality mapfrom E to E^*.

Remark 2.1.16

- *It is easy to see from the definition of ϕ that, in a real Hilbert space H, equation (2.1.1) reduces to $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in H$.*

- *Consider the map $V : E \times E^* \to \mathbb{R}$ defined by $V(u, u^*) = \|u\|^2 - 2\langle u, u^*\rangle + \|u^*\|^2$. It is easy to see that $V(u, u^*) = \phi(u, J^{-1}u^*)$ $\forall$ $u \in E$, $u^* \in E^*$.*

Furthermore, given $x, y, z \in E$, and $\tau \in (0, 1)$, we have the following properties (see, Nilsrakoo and Saejung [2011]):

P1 $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,

P2 $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle z - x, Jy - Jz\rangle$,

P3 $\phi(\tau x + (1 - \tau)y, z) \leq \tau\phi(x, z) + (1 - \tau)\phi(y, z)$

Definition 2.1.17 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. The map $P_C : H \to C$ defined by $\tilde{x} := P_C(x) \in C$ such that $\|x - \tilde{x}\| = \inf_{y \in C} \|x - y\|$ is called the metric projection of x onto C.*

Definition 2.1.18 *Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E . The map $\Pi_C : E \to C$ defined by $\tilde{x} := \Pi_C(x) \in C$ such that $\phi(\tilde{x}, x) = \inf_{y \in C} \phi(y, x)$ is called the generalized projection of x onto C.*

Remark 2.1.19 *Clearly, in a real Hilbert space, the generalized projection Π_C coincides with the metric projection P_C from E onto C.*

Definition 2.1.20 *Let $S : E \to E$ be a map. The set $\{x \in E : Sx = x\}$ is called the fixed point set of S. We denote the set by $F(S)$.*

Definition 2.1.21 *A map $S : E \to E$ is called quasi-nonexpansive if*

- $F(S) \neq \emptyset$;
- $\|Sx - p\| \leq \|x - p\|$ *for all* $x \in E$, $p \in F(S)$.

Definition 2.1.22 *Let $S : C \to E$ be a map. Then, S is called be relatively nonexpansive if the following conditions hold:*

(i) $F(S) := \{x \in C : Sx = x\} \neq \emptyset$;

(ii) $\phi(u, Sv) \leq \phi(u, v)$, $\forall$ $u \in F(S)$ *and* $v \in C$;

(iii) $(I - S)$ *is demi-closed at zero, i.e., whenever a sequence $\{v_n\}$ in C converges weakly to u and $\{v_n - Sv_n\}$ converges strongly 0, then $u \in F(S)$.*

Remark 2.1.23 *In a real Hilbert space, every nonexpansive map with nonempty fixed point set is relatively nonexpansive.*

2.2 Results of Interest

Lemma 2.2.1 Alber [1996]
Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive real Banach space E. Then,

1. *if $x \in E$ and $y \in C$, then $\tilde{x} = \Pi_C x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x}\rangle \geq 0$, for all $y \in C$,*
2. $\phi(y, \tilde{x}) + \phi(\tilde{x}, x) \leq \phi(y, x)$, *for all* $x \in E$, $y \in C$.

Lemma 2.2.2 *Xu [1991]*
Let E be a 2-uniformly convex and smooth real Banach space. Then, there exists a positive constant α such that

$$\alpha\|x-y\|^2 \le \phi(x,y),\ \forall\ x,y\in E. \tag{2.2.1}$$

Remark 2.2.3 *Without loss of generality, we may assume $\alpha \in (0,1)$.*

Lemma 2.2.4 *Xu [1991]*
Let E be a 2-uniformly convex real Banach space. Then, there exists a constant $c_2 > 0$ such that for every $x, y \in E$, $f_x \in J_2(x), f_y \in J_2(y)$, the following inequality holds:

$$\langle x-y, f_x - f_y\rangle \ge c_2\|x-y\|^2.$$

Lemma 2.2.5 *Kamimura and Takahashi [2002]*
Let E be a real smooth and uniformly convex Banach space, and let $\{u_n\}$ and $\{v_n\}$ be two sequences of E. If either $\{u_n\}$ or $\{v_n\}$ is bounded, then $\phi(u_n,v_n)\to 0 \Rightarrow \|u_n-v_n\|\to 0$.

Lemma 2.2.6 *Nilsrakoo and Saejung [2011]*
Let E be a uniformly smooth Banach space and $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, 1)$ such that $g(0) = 0$ and

$$\phi\left(u, J^{-1}[\beta Jx + (1-\beta)Jy]\right) \le \beta\phi(u,x) + (1-\beta)\phi(u,y) - \beta(1-\beta)g(\|Jx-Jy\|)$$

for all $\beta\in[0,1]$, $u\in E$ and $x,y\in B_r$

Lemma 2.2.7 *Rockafellar [1970]*
Let C be a nonempty closed and convex subset of a reflexive space E and A, a monotone, hemi-continuous map of C into E^*. Let $T\subset E\times E^*$ be an operator defined by:

$$Tu = \begin{cases} Au + N_C(u), & if\quad u\in\ C,\\ \emptyset, & if\quad u\notin\ C,\end{cases} \tag{2.2.2}$$

where $N_C(u)$ is defined as follows:

$$N_C(u) = \{w^*\in E^* : \langle u-z, w^*\rangle \ge 0\ \forall\ z\in C\}.$$

Then, T is maximal monotone and $T^{-1}0 = VI(C,A)$.

Lemma 2.2.8 *Kohsaka and Takahashi [2008]*
Let C be a closed convex subset of a uniformly smooth and uniformly convex Banach space E and let $(S_i)_{i=1}^\infty$ be a family of relatively nonexpansive maps such that $\bigcap_{i=1}^\infty F(S_i)\neq\emptyset$. Let $(\eta_i)_{i=1}^\infty\subset(0,1)$ and $(\mu_i)_{i=1}^\infty\subset(0,1)$ be sequences such that $\sum_{i=1}^\infty \eta_i = 1$. Consider the map $T : C\to E$ defined by

$$Tx = J^{-1}\left(\sum_{i=1}^{\infty}\eta_i(\mu_i Jx + (1-\mu_i)JS_i x\right)\quad for\ \ each\ x\in C. \tag{2.2.3}$$

Then, T is relatively nonexpansive and $F(T) = \bigcap_{i=1}^\infty F(S_i)$.

Lemma 2.2.9 *Let E be a 2-uniformly convex and uniformly smooth reflexive real Banach space . Then Lyapunov functional (2.1.1) is left weakly lower semi-continuous. That is, $\phi(.,x_0)$ is weakly lower semi-continuous for any fixed x_0 in E.*

Proof We show that $\phi(.,x_0)$ is lower semi-continuous and convex. Convexity follows from (P_3). Let $\{x_n\}\subset E$ such that $x_n\to x\in E$. We show that $\phi(x_n,x_0)\to\phi(x,x_0)$. By continuity of $\|.\|^2$ and Jx_0 we have,

$$\phi(x_n,x_0) = \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 \to \|x\|^2 - 2\langle x, Jx_0\rangle + \|x_0\|^2 = \phi(x,x_0).$$

Hence, $\phi(.,x_0)$ is left weakly lower semi-continuous. ■

The following result has recently been proved. But, for completeness, we reproduce the proof here.

Lemma 2.2.10 *Chidume and Otubo [2017]*
Let E be a 2-uniformly convex and smooth real Banach space and C be a nonempty, closed and convex subset of E. Let $x_1,\ x_2 \in E$ be arbitrary and $\Pi_C : E \to C$ be the generalized projection. Then, the following inequality holds:

$$\|\Pi_C x_1 - \Pi_C x_2\| \leq \frac{1}{c_2}\|Jx_1 - Jx_2\|, \tag{2.2.4}$$

where c_2 is the constant appearing in Lemma 2.2.4 and J is the normalized duality map on E.

Proof By Lemma 2.2.1 (1), for any $x_1, x_2 \in E$ we have

$$\langle \Pi_C x_2 - \Pi_C x_1, Jx_1 - J\Pi_C x_1\rangle \leq 0 \quad \text{and} \quad \langle \Pi_C x_1 - \Pi_C x_2, Jx_2 - J\Pi_C x_2\rangle \leq 0.$$

Adding thsee two inequalities, we obtain

$$\begin{aligned} & \langle \Pi_C x_1 - \Pi_C x_2, (Jx_2 - Jx_1) - (J\Pi_C x_2 - J\Pi_C x_1)\rangle \leq 0 \\ \Rightarrow\ & \langle \Pi_C x_1 - \Pi_C x_2, Jx_2 - Jx_1\rangle - \langle \Pi_C x_1 - \Pi_C x_2, J\Pi_C x_2 - J\Pi_C x_1\rangle \leq 0 \\ \Rightarrow\ & \langle \Pi_C x_1 - \Pi_C x_2, J\Pi_C x_1 - J\Pi_C x_2\rangle \leq \langle \Pi_C x_1 - \Pi_C x_2, Jx_1 - Jx_2\rangle. \end{aligned}$$

By Lemma 2.2.4, we obtain

$$\begin{aligned} c_2\|\Pi_C x_1 - \Pi_C x_2\|^2 &\leq \|\Pi_C x_1 - \Pi_C x_2\| \cdot \|Jx_1 - Jx_2\|, \\ \text{so that} \qquad & \\ \|\Pi_C x_1 - \Pi_C x_2\| &\leq \frac{1}{c_2}\|Jx_1 - Jx_2\|, \end{aligned} \tag{2.2.5}$$

completing the proof. ■

Remark 2.2.11 *Lemma 2.2.10 implies that the generalized projection Π_C is uniformly continuous whenever J is.*

Lemma 2.2.12 *(see, Ceng et al. [2010])*
Let C be a nonempty, closed and convex subset of a real Hilbert space H. Then,

1. *if $x \in H$ and $y \in C$, then $\tilde{x} = P_C x$ if and only if $\langle \tilde{x} - y, x - \tilde{x}\rangle \geq 0$, for all $y \in C$,*
2. *$\|p_C x - P_C y\| \leq \|x - y\|$, for all $x, y \in H$,*
3. *$\|x - P_C x\| + \|y - P_C y\| \leq \|x - y\|^2$, for all $x \in H$ and $y \in C$.*

Lemma 2.2.13 *(see, Kraikaew and Saejung [2014])*
Let $S : H \to H$ be a map. Then, the mapping $(I - S)$ is demeclosed at zero if and only if $x \in F(S)$ whenever $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$.

Remark 2.2.14 *If $S : H \to H$ is nonexpansive, then $I - S$ is demiclosed at zero.*

Lemma 2.2.15 ? *Let H be a real Hilber space. Let $x, y \in H$, we have the following statements:*

1. *$|\langle x, y\rangle| \leq \|x\|\|y\|$;*
2. *$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ (the subdifferential inequalty)*

Lemma 2.2.16 *Chidume [2009] Let H be a real Hilber space. Let $x, y \in H$ and $\lambda \in (0, 1)$ then,*

$$\|x\lambda + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

CHAPTER 3

RESULTS OF CENG *ET AL.*

In this chapter we will give a detailed prove of the results of Ceng *et al.*

Theorem 3.0.1 *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C, A) \neq \emptyset$. We define inductively the sequences $\{x_n\}$ by*

$$\begin{cases} x_0 \in C, \\ y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ z_n = (1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n S P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + (3 - 3\gamma_n + \alpha_n)b^2\|Ax_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{3.0.1}$$

for all $n \geq 0$, where $\{\lambda_n\}$ is a sequence in $[a, b]$ with $a > 0$ and $b < \frac{1}{2k}$, and $\{\alpha_n\}, \{\beta_n\}\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leq 1$, *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$;

(iii) $\liminf_{n\to\infty} \beta_n > 0$;

(iv) $\lim_{n\to\infty} \gamma_n = 1$ *and* $\gamma_n > 3/4$ *for all* $n \geq 0$.

Then, the sequence $\{x_n\}$ is well-defined and converges to the point $q = P_{F(S)\cap VI(C,A)} x_0$.

Proof The authors Ceng et al. [2010], divided the proof of the theorem into six steps.
Step 1. Assuming that x_n is a well-defined element of C for some $n \in \mathbb{N}$, we show that $F(S) \cap VI(C, A) \subset C_n$.

Let $x^* \in F(S) \cap VI(C, A)$ be arbitrary. Set $t_n = P_C(x_n - \lambda_n A y_n)$ for all $n \geq 0$. Taking $x = x_n - \lambda_n A y_n$ and $y_n = x^*$ in inequality (), we obtain

$$\begin{aligned} \|t_n - x^*\|^2 &\leq \|x_n - \lambda_n A y_n - x^*\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\ &= \|x_n - x^*\|^2 + 2\lambda_n\langle Ay_n, x^* - y_n\rangle + 2\lambda\langle Ay_n, y_n - t_n\rangle - \|x_n - t_n\|^2. \end{aligned}$$

Using Remark 1, we deduce

$$\begin{aligned} \|t_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &= \|x_n - x^*\|^2 - \|(x_n - y_n) + (y_n - t_n)\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle. \end{aligned} \tag{3.0.2}$$

We estimate the last term, using $y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n)$:

$$\begin{aligned}
&\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle \\
&\quad = \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle + \lambda_n \langle A x_n - A y_n, t_n - y_n\rangle \\
&\quad \le \langle x_n - \lambda_n A x_n - (1-\gamma_n)x_n - \gamma_n P_C(x_n - \lambda_n A x_n), t_n - y_n\rangle \\
&\qquad + \lambda_n \|A x_n - A y_n\| \|t_n - y_n\| \\
&\quad \le \gamma_n \langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - y_n\rangle \\
&\qquad - (1-\gamma_n)\lambda_n \langle A x_n, t_n - y_n\rangle + \lambda_n k \|x_n - y_n\| \|t_n - y_n\| \qquad (3.0.3)
\end{aligned}$$

In addition, using properties of the projection $P_C(x_n - \lambda_n A x_n)$ we obtain

$$\begin{aligned}
&\langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - y_n\rangle \\
&\quad \le \langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - (1-\gamma_n)x_n - \gamma_n P_C(x_n - \lambda_n A x_n)\rangle \\
&\quad \le (1-\gamma_n)\langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - x_n\rangle \\
&\qquad + \gamma_n \langle x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n), t_n - P_C(x_n - \lambda_n A x_n)\rangle \\
&\quad \le (1-\gamma_n)\|x_n - \lambda_n A x_n - P_C(x_n - \lambda_n A x_n)\| \|t_n - x_n\| \\
&\quad \le (1-\gamma_n)\lambda_n \|A x_n\| (\|t_n - y_n\| + \|y_n - x_n\|). \qquad (3.0.4)
\end{aligned}$$

Gathering inequalities (3.0.2),(3.0.3),(3.0.4) and using $\gamma_n \le 1$ and $\lambda_n \le b$ we find

$$\begin{aligned}
\|t_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad 2\gamma_n(1-\gamma_n) b \|A x_n\| (\|t_n - y_n\| + \|y_n - x_n\|) \\
&\quad 2(1-\gamma_n) b \|A x_n\| \|t_n - y_n\| + 2bk \|x_n - y_n\| \|t_n - y_n\| \\
&\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad + (1-\gamma_n)\Big(2b^2 \|A x_n\|^2 + \|t_n - y_n\|^2 + \|y_n - x_n\|^2\Big) \\
&\quad (1-\gamma_n)\Big(b^2 \|A x_n\|^2 + \|t_n - y_n\|^2\Big) + bk\Big(\|x_n - y_n\|^2 + \|t_n - y_n\|^2\Big) \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 (\gamma_n - bk) \\
&\quad - \|y_n - t_n\|^2 (2\gamma_n - 1 - bk) + 3(1-\gamma_n) b^2 \|A x_n\|^2. \qquad (3.0.5)
\end{aligned}$$

Using our assumptions $b < \frac{1}{2k}$ abd $\gamma_n 3/4$, we obtain that for all $n \in \mathbb{N}$,

$$\|t_n - x^*\|^2 \le \|x_n - x^*\|^2 + 3(1-\gamma_n) b^2 \|A x_n\|^2. \qquad (3.0.6)$$

Also, using again relation () and properties of P_C, we obtain

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|(1-\gamma_n)(x_n - x^*) + \gamma_n\big(P_C(x_n - \lambda_n A x_n) - x^*\big)\|^2 \\
&\le (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n \|P_C(x_n - \lambda_n A x_n) - P_C x^*\|^2 \\
&(1-\gamma_n)\|x_n - x^*\| + \gamma_n \|x_n - x^* - \lambda_n A x_n\|^2 \\
&= (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n \Big[\|x_n - x^*\|^2 - 2\lambda_n \langle A x_n, x_n - x^*\rangle + \lambda_n^2 \|A x_n\|^2\Big] \\
&\le \|x_n - x^*\|^2 + b^2 \|A x_n\|^2. \qquad (3.0.7)
\end{aligned}$$

Since S is nonexpansive and $x^* \in F(S)$ we have $\|S t_n - x^*\| \le \|t_n - x^*\|$. Thus, relations (3.0.6) and (3.0.7) imply that

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|(1-\alpha_n-\beta_n)x_n + \alpha_n y_n \beta_n S t_n - x^*\|^2 \\
&\le (1-\alpha_n-\beta_n)\|x_n - x^*\|^2 + \alpha_n \|y_n - x^*\|^2 + \beta_n \|S t_n - x^*\|^2 \qquad (3.0.8) \\
&\le (1-\alpha_n-\beta_n)\|x_n - x^*\|^2 + \alpha_n \big[\|x_n - x^*\|^2 + b^2 \|A x_n\|^2\big] \\
&\quad + \beta_n \big[\|x_n - x^*\|^2 + 3(1-\gamma_n) b^2 \|A x_n\|^2\big] \\
&= \|x_n - x^*\|^2 + (3 - 3\gamma_n + \alpha_n) b^2 \|A x_n\|^2. \qquad (3.0.9)
\end{aligned}$$

Consequently, $x^* \in C_n$. Hence $F(S) \cap VI(C, A) \subset C_n$.

Step 2. We show that $\{x_n\}$ is well-defined and $F(S) \cap VI(C, A) \subset C_n \cap Q_n$ for all $n \geq 0$.

We show this assertion by mathematical induction. For $n = 0$, we have $Q_0 = C$. Hence by Step 1 we obtain $F(S) \cap VI(C, A) \subset C_0 \cap Q_0$. Assume that x_k is defined and $F(S) \cap VI(C, A) \subset C_k \cap Q_k$ for some $k \geq 0$. Note that C_k is a closed and convex subset of C since

$$C_k = \{z \in C : \|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - z\rangle \leq (3 - 3\gamma_k + \alpha_k)b^2\|Ax_k\|^2\}.$$

Also, it is obvious that Q_k is closed and convex. Thus, $C_k \cap Q_k$ is a closed convex subset, which is nonempty since by assumption it contains $F(S) \cap VI(C, A)$. Consequently, $x_{k+1} = P_{C_k \cap Q_k} x_0$ is well-defined.

The definition of x_{k+1} and Q_{k+1} imply that $C_k \cap Q_k \subset Q_{k+1}$. Hence, $F(S) \cap VI(C, A) \subset Q_{k+1}$. Using Step 1 we infer that $F(S) \cap VI(C, A) \subset C_{k+1} \cap Q_{k+1}$.

Step 3. We show that the following statements hold:

1. $\{x_\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists, and $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$;
2. $\lim_{n\to\infty}(z_n - x_n) = 0$.

Indeed, take any $x^* \in F(S) \cap VI(C, A)$. Using the fact that $x_{n+1} = P_{C_n \cap Q_n} x_0$ and $x^* \in F(S) \cap VI(C, A) \subset C_n \cap Q_n$, we obtain

$$\|x_{n+1} - x_0\| \leq \|x^* - x_0\|, \quad \forall\, n \geq 0. \tag{3.0.10}$$

Therefore, $\{x_n\}$ is bounded and so is $\{Ax_n\}$ due to the Lipschitz continuity of A. From the definition of Q_n it is clear that $x_n = P_{Q_n} x_0$. Since $x_{n+1} \in C_n \cap Q_n$, we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \quad \forall\, n \geq 0. \tag{3.0.11}$$

In particular, $\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$ exists. Then relation (3.0.11) implies that

$$\lim_{n\to\infty}(x_{n+1} - x_n) = 0. \tag{3.0.12}$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (3 - 3\gamma_n + \alpha_n)b^2\|Ax_n\|^2.$$

Since $\{Ax_n\}$ is bounded and $\lim_{n\to\infty} \gamma_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$, we deduce that $\lim_{n\to\infty}(z_n - x_{n+1}) = 0$. Combining with equation (3.0.12), we infer that $\lim_{n\to\infty}(z_n - x_n) = 0$..

Step 4. We show that the following statements hold:

1. $\lim_{n\to\infty}(x_n - y_n) = 0$;
2. $\lim_{n\to\infty}(Sx_n - x_n) = 0$.

Indeed, from inequalities (3.0.8) and (3.0.9) we infer

$$\begin{aligned} \|z_n - x^*\|^2 - \|x_n - x^*\|^2 &\leq (-\alpha_n - \beta_n)\|x_n - x^*\|^2 + \alpha_n\|y_n - x^*\|^2 + \beta_n\|St_n - x^*\|^2 \\ &\leq (3 - 3\gamma_n + \alpha_n)b^2\|Ax_n\|^2. \end{aligned} \tag{3.0.13}$$

Since $\alpha_n \to 0$, $\gamma_n \to 1$, and $\{x_n\}, \{Ax_n\}, \{y_n\}$ are bounded, we from (3.0.13) that

$$\lim_{n\to\infty} \beta_n(\|St_n - x^*\|^2 - \|x_n - x^*\|^2) = 0.$$

Using $\liminf_{n\to\infty} \beta_n > 0$, we get $\lim_(\|St_n - x^*\|^2 - \|x_n - x^*\|^2) = 0$. Then, equation (3.0.6) implies

$$\lim_{n\to\infty}(\|St_n - x^*\|^2 - \|x_n - x^*\|^2) \le \lim_{n\to\infty}(\|t_n - -x^*\|^2 - \|x_n - x^*\|^2)$$
$$\lim_{n\to\infty}(3 - 3\gamma_n + \alpha_n)b^2\|Ax_n\|^2 = 0$$

thus, $\lim_{n\to\infty}(\|t_n - -x^*\|^2 - \|x_n - x^*\|^2) = 0$. Now, we rewrite inequality (3.0.5) as

$$\|x_n - y_n\|^2(\gamma_n - bk) + \|y_n - t_n\|^2(2\gamma_n - 1 - bk) \le \|x_n - x^*\|^2 - \|t_n - x^* + 3(1-\gamma_n)b^2\|Ax_n\|^2.$$

We deduce that

$$\lim_{n\to\infty}\left[\|x_n - y_n\|^2(\gamma_n - bk) + \|y_n - t_n\|^2(2\gamma_n - 1 - bk)\right] = 0.$$

Our assumptions on λ_n and γ_n imply that $\gamma_n - bk > 1/4$ and $2\gamma_n - 1 - bk > \frac{1}{2} - bk > 0$. Consequently, $\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty}(y_n - t_n) = 0$. Hence, $\lim_{n\to\infty}(x_n - t_n) = 0$. Using the fact that S is nonexpansive, we get $\lim_{n\to\infty}(Sx_n - St_n) = 0$. We write the definition of z_n as

$$z_n - x_n = -\alpha_n x_n + \alpha_n y_n + \beta_n(St_n - x_n).$$

From $\lim_{n\to\infty}(z_n - x_n) = 0, \lim_{n\to\infty}\alpha_n = 0$, the boundedness of x_n, y_n and $\liminf_{n\to\infty}\beta_n > 0$, we infer that $\lim_{n\to\infty}(St_n - x_n) = 0$. Thus, $\lim_{n\to\infty}(Sx_n - x_n) = 0$.

Step 5. We claim that $\omega_\omega(x_n) \subset F(S) \cap VI(C, A)$, where $\omega_\omega(x_n)$ denotes the set of weak subsequential limts of $\{x_n\}$.

Indeed, since $\{x_n\}$ is bounded, it has a subsequence which converges weakly to some point in C and hence $\omega_\omega(x_n) \neq \emptyset$. Let $u \in \omega_\omega(x_n)$ be arbitrary. Then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ which converges weakly to u. Since we also have $\lim_{j\to\infty}(x_{n_j} - Sx_{n_j}) = 0$, from the demiclosedness principle it follows that $(I - S)u = 0$. Thus $u \in F(S)$. We now show that $u \in VI(C, A)$.
Since $t_n = P_C(x_n - \lambda_n Ay_n)$, for every $x \in C$, we have

$$\langle x_n - \lambda_n Ay_n - t_n, t_n - x\rangle \ge 0,$$

hence,

$$\langle x - t_n, Ay_n\rangle \ge \langle x - t_n, \frac{x_n - t_n}{\lambda_n}\rangle.$$

Combining with monotonicity of A, we obtain

$$\langle x - t_n, Ax\rangle \ge \langle x - t_n, At_n\rangle$$
$$\langle x - t_n, At_n - Ay_n\rangle + \langle x - t_n, Ay_n\rangle$$
$$\ge \langle x - t_n, At_n - Ay_n\rangle + \langle x - t_n, \frac{x_n - t_n}{\lambda_n}\rangle.$$

Since $\lim_{n\to\infty}(x_n - t_n) = \lim_{n\to\infty}(y_n - t_n) = 0$, A is Lipschitz continuous and $\lambda_n > a > 0$, we deduce that

$$\langle x - u, Ax\rangle = \lim_{j\to\infty}\langle x - t_{n_j}, Ax\rangle \ge 0, \quad \forall\, x \in C$$

Then, relation () entails that $u \in F(S) \cap VI(C, A)$.

Step 6. We show that $\{x_n\}$ converges strongly to the point $q = P_{F(S)\cap VI(C,A)}x_0$.
Assume that $\{x_n\}$ does not converge strongly to q. Then, there exists $\epsilon > 0$ and a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\|x_{n_j} - q\| > \epsilon$ for all j. Without loss of generality we may assume that $\{x_{n_j}\}$ converges weakly to some point u. By Step 5, $u \in F(S) \cap VI(C, A)$. Using $q = P_{F(S)\cap VI(C,A)}x_0$, the weak lower semicontinuity of $\|\cdot\|$, and relation (3.0.10) for $x^* = q$, we obtain

$$\|q - x_0\| \le \|u - x_0\| \le \liminf_{j\to\infty}\|x_{n_j} - x_0\| = \lim_{n\to\infty}\|x_n - x_0\| \le \|q - x_0\|. \qquad (3.0.14)$$

It follows that $\|q - x_0\| = \|u - x_0\|$, hence since $u = q$ is the unique element in $F(S) \cap VI(C, A)$ that minimizes the distance from x_0. Also, relation (3.0.14) implies $\lim_{j\to\infty} \|x_{n_j} - x_0\| = \|q - x_0\|$. Since $\{x_{n_j} - x_0\}$ converges weakly to $q - x_0$, this shows that $\{x_{n_j} - x_0\}$ converges strongly to $q - x_0$, and hence $\{x_{n_j}\}$ converges strongly to q, a contradiction. Thus, $\{x_n\}$ converges strongly to q.

■

CHAPTER 4

MAIN RESULTS

In this chapter we will present an extension of the results of Ceng et al. [2010], in a uniformly smooth and 2-uniformly convex real Banach space. Furthermore, we will extend the class of map from one relatively nonexpansive map to a countable family of relatively nonexpansive maps.

Lemma 4.0.1 *Let C be a nonempty closed convex subset of a uniformly smooth and 2-uniformly convex real Banach space E such that $J(C)$ is convex. Let $A : C \to E^*$ be a monotone and k-Lipschitz map and let $S : C \to C$ be a relatively nonexpansive map, such that $F(S) \cap V(C, A) \neq \emptyset$. We define inductively the algorithm*

$$\begin{cases} x_0 \in C_0 = C; \\ y_n = \Pi_C(J^{-1}[Jx_n - \gamma_n \lambda Ax_n]); \\ z_n = J^{-1}\left((1-\alpha_n-\beta_n)Jx_n + \alpha_n Jy_n + \beta_n JS\Pi_C(J^{-1}[Jx_n - \lambda Ay_n])\right); \\ C_{n+1} = \{z \in C_n : \phi(z, z_n) \leq \phi(z, x_n) + (3-3\gamma_n)b^2||Ax_n||^2 + b\alpha_n\tau_n\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{4.0.1}$$

for all $n \geq 0$ where $\lambda \in (0, b]$ with $b < \frac{\alpha}{2k}$, $\tau_n = max\{\|x_n\|, 1\}\|Ax_n\|\sigma(\|\gamma_n\lambda Ax_n\|)$ and $\{\alpha_n\}, \{\beta_n\}$,and $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following conditions

1. $\alpha_n + \beta_n \leq 1 \ \forall \ n \geq 0$;
2. $\lim_{n\to\infty} \alpha_n = 0$;
3. $\liminf_{n\to\infty} \beta_n > 0$;
4. $\lim_{n\to\infty} \gamma_n = 1$ *and* $\gamma_n > 1 - \frac{\alpha}{4} \ \forall n \geq 0$.

Then, the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are well-defined and converge strongly to some point $q = \Pi_{F(S)\cap VI(C,A)} x_0$.

Proof We divide the proof into several steps.
Step 1. Suppose x_n is a well-defined element of $C_n \quad \forall \ n \geq 0$. We show that $F(S) \cap V(C, A) \subset C_n$. Since x_n is well-defined, then, it is easy to see that y_n, z_n are well-defined elements of C. Let $u \in F(S) \cap V(C, A)$ be arbitrary.
Set $t_n = \Pi_C(J^{-1}[Jx_n - \lambda Ay_n])$ for all $n \geq 0$. Then, applying Lemma 2.2.1 (2), we have

$$\begin{aligned} \phi(u, t_n) &\leq \phi(u, J^{-1}[Jx_n - \lambda Ay_n]) - \phi(t_n, J^{-1}[Jx_n - \lambda Ay_n]) \\ &= \|u\|^2 - 2\langle u, Jx_n\rangle + 2\langle u, \lambda Ay_n\rangle - \|t_n\|^2 + 2\langle t_n, Jx_n\rangle - 2\langle t_n, \lambda Ay_n\rangle \\ &= \phi(u, x_n) - \phi(t_n, x_n) + 2\langle u - t_n, \lambda Ay_n\rangle \\ &= \phi(u, x_n) - \phi(t_n, x_n) + 2\lambda\langle u - y_n, Ay_n\rangle + 2\lambda\langle y_n - t_n, Ay_n\rangle. \end{aligned}$$

Using remark 1 and equation P_2, we have

$$\begin{aligned}\phi(u,t_n) &\leq \phi(u,x_n) - \phi(t_n,x_n) + 2\lambda\langle y_n - t_n, Ay_n\rangle \\ &= \phi(u,x_n) - \phi(y_n,x_n) - \phi(t_n,y_n) + 2\langle t_n - y_n, Jx_n - \lambda Ay_n - Jy_n\rangle .\end{aligned} \tag{4.0.2}$$

Now, we estimate $\langle t_n - y_n, Jx_n - \lambda Ay_n - Jy_n\rangle$. We have

$$\begin{aligned}\langle t_n - y_n, Jx_n - \lambda Ay_n - Jy_n\rangle &= \langle t_n - y_n, Jx_n - \lambda Ax_n - Jy_n\rangle + \lambda\langle t_n - y_n, Ax_n - Ay_n\rangle \\ &= \langle t_n - y_n, Jx_n - \gamma_n\lambda Ax_n - Jy_n\rangle - (1-\gamma_n)\langle t_n - y_n, \lambda Ax_n\rangle \\ &\quad + \lambda\langle t_n - y_n, Ax_n - Ay_n\rangle .\end{aligned}$$

Since $y_n = \Pi_C(J^{-1}[Jx_n - \gamma_n\lambda Ax_n])$, applying Lemma 2.1.1 (1), we have

$$\begin{aligned}\langle t_n - y_n, Jx_n - \lambda Ay_n - Jy_n\rangle &\leq -(1-\gamma_n)\langle t_n - y_n, \lambda Ax_n\rangle + \lambda\|t_n - y_n\|\|Ax_n - Ay_n\| \\ &\leq -(1-\gamma_n)\langle t_n - y_n, \lambda Ax_n\rangle + \lambda k\|t_n - y_n\|\|x_n - y_n\|.\end{aligned}$$

Since $(1-\gamma_n)\lambda\|Ax_n\|\left[\|t_n - y_n\| + \|x_n - y_n\|\right] \geq 0$ and $-(1-\gamma_n)\langle t_n - y_n, \lambda Ax_n\rangle \leq (1-\gamma_n)\lambda\|Ax_n\|\|t_n - y_n\|$, we have

$$\begin{aligned}\langle t_n - y_n, Jx_n - \lambda Ay_n - Jy_n\rangle &\leq (1-\gamma_n)\lambda\|Ax_n\|\left(\|t_n - y_n\| + \|x_n - y_n\|\right) \\ + (1-\gamma_n)\lambda\|Ax_n\|\|t_n - y_n\| &+ \lambda k\|t_n - y_n\|\|x_n - y_n\|.\end{aligned} \tag{4.0.3}$$

Putting together equations (4.0.2) and (4.0.3), using $\gamma_n \leq 1$ and $\lambda \leq b$ we deduce, by Lemma 2.2.2, that

$$\begin{aligned}\phi(u,t_n) &\leq \phi(u,x_n) - \phi(y_n,x_n) - \phi(t_n,y_n) + (1-\gamma_n)\left(2b^2||Ax_n||^2 + \|t_n - y_n\|^2 + ||y_n - x_n\|^2\right) \\ &\qquad + (1-\gamma_n)\left(b^2\|Ax_n\|^2 + \|t_n - y_n\|^2\right) + bk\left(\|x_n - y_n\|^2 + \|t_n - y_n\|^2\right) \\ &\leq \phi(u,x_n) - \phi(y_n,x_n) - \phi(t_n,y_n) + (1-\gamma_n)\left(2b^2\|Ax_n\|^2 + \frac{1}{\alpha}\phi(t_n,y_n) + \frac{1}{\alpha}\phi(y_n,x_n)\right) \\ &\qquad + (1-\gamma_n)\left(b^2\|Ax_n\|^2 + \frac{1}{\alpha}\phi(t_n,y_n)\right) + bk\left(\frac{1}{\alpha}\phi(y_n,x_n) + \frac{1}{\alpha}\phi(t_n,y_n)\right) \\ &= \phi(u,x_n) - \left(1 - \frac{(1-\gamma_n)}{\alpha} - \frac{bk}{\alpha}\right)\phi(y_n,x_n) - \left(1 - 2\frac{(1-\gamma_n)}{\alpha} - \frac{bk}{\alpha}\right)\phi(t_n,y_n) \\ &\qquad + 3(1-\gamma_n)b^2\|Ax_n\|^2.\end{aligned} \tag{4.0.4}$$

Using the assumption that $b < \frac{\alpha}{2k}$ and $\gamma_n > 1 - \frac{\alpha}{4}$, we have

$$\phi(u,t_n) \leq \phi(u,x_n) + 3(1-\gamma_n)b^2||Ax_n||^2. \tag{4.0.5}$$

Also, using Lemma 2.1.1, Lemma 2.2.6, the fact that $\gamma_n \leq 1$ and $\lambda \leq b$ and remark 1, we have

$$\begin{aligned}\phi(u,y_n) &\leq \phi(u, J^{-1}(Jx_n - \gamma_n\lambda Ax_n)) \\ &= \|u\|^2 - 2\langle u, Jx_n - \gamma_n\lambda Ax_n\rangle + \|Jx_n - \gamma_n\lambda Ax_n\|^2 \\ = \|u\|^2 - 2\langle u, Jx_n - \gamma_n\lambda Ax_n\rangle &+ \|Jx_n\|^2 - 2\langle x_n, \gamma_n\lambda Ax_n\rangle + max\{\|x_n\|, 1\}\|\gamma_n\lambda Ax_n\|\sigma(\|\gamma_n\lambda Ax_n\|) \\ &\leq \|u\|^2 - 2\langle u, Jx_n\rangle + 2\langle u, \gamma_n\lambda Ax_n\rangle + \|Jx_n\|^2 - 2\langle x_n, \gamma_n\lambda Ax_n\rangle + b\tau_n \\ &= \|u\|^2 - 2\langle u, Jx_n\rangle + \|Jx_n\|^2 + b\tau_n + 2\gamma_n\lambda\langle u - x_n, Ax_n\rangle \\ &\leq \phi(u,x_n) + b\tau_n.\end{aligned} \tag{4.0.6}$$

Next, we estimate $\phi(u,z_n)$, using Lemma 2.2.6 and inequalities (4.0.5) and (4.0.6). We obtain,

$$\begin{aligned}\phi(u,z_n) &= \phi(u, J^{-1}\left((1-\alpha_n-\beta_n)Jx_n + \alpha_n Jy_n + \beta_n JSt_n\right)) \\ &\leq (1-\alpha_n-\beta_n)\phi(u,x_n) + \alpha_n\phi(u,y_n) + \beta_n\phi(u,St_n).\end{aligned} \tag{4.0.7}$$

Thus,

$$\begin{aligned}\phi(u,z_n) &\le (1-\alpha_n-\beta_n)\phi(u,x_n)+\alpha_n\phi(u,y_n)+\beta_n\phi(u,t_n)\\ &\le (1-\alpha_n-\beta_n)\phi(u,x_n)+\alpha_n\Big(\phi(u,x_n)+b\tau_n\Big)+\beta_n\Big(\phi(u,x_n)+3(1-\gamma_n)b^2\|\|Ax_n\|^2\Big)\\ &= \phi(u,x_n)+(3-3\gamma_n)b^2\|Ax_n\|^2+b\alpha_n\tau_n.\end{aligned} \tag{4.0.8}$$

Thus, $u\in C_{n+1}$. Hence, $F(S)\cap VI(C,A)\subset C_{n+1}$.

Step 2. We show that the sequence $\{x_n\}$ is a well-defined element of C for all $n\ge 0$. It suffices to show that C_n is closed and convex for all $n\ge 0$. But, it is easy to see that C_n is closed and convex since

$$C_n=\{z\in C_{n-1}:\|z_n\|^2-\|x_n\|^2+2\langle z,Jx_n-Jz_n\rangle\le(3-3\gamma_n)b^2\|Ax_n\|^2+b\alpha_n\tau_n\},\ n\ge 1.$$

Thus, C_n is a closed and convex subset of C, which is nonempty since by Step 1, it contains $F(S)\cap VI(C,A)$. Hence, $x_{n+1}=\Pi_{C_{n+1}}x_0$ is well-defined.

Step 3. We show the following

- (i) $\{x_n\}$ is bounded;
- (ii) $\lim_{n\to\infty}\phi(x_n,x_0)$ exists;
- (iii) $\lim_{n\to\infty}\phi(x_n,x_{n+1})=0$;
- (iv) $\lim_{n\to\infty}\|z_n-x_n\|=0$.

For (i), let $u\in F(S)\cap VI(C,A)$. Using the fact that $F(S)\cap VI(C,A)\subset C_{n+1}$, we have

$$\phi(x_{n+1},x_0)\le\phi(u,x_0),\quad \forall\ n\ge 0. \tag{4.0.9}$$

Hence, $\{x_n\}$ is bounded.
For (ii) and (iii), by definition, $x_n=\Pi_{C_n}x_0$. Since $x_{n+1}\in C_{n+1}\subset C_n$, using Lemma 2.1.1 (2), we have

$$\phi(x_{n+1},x_n)\le\phi(x_{n+1},x_0)-\phi(x_n,x_0). \tag{4.0.10}$$

Thus,

$$\phi(x_n,x_0)\le\phi(x_{n+1},x_0). \tag{4.0.11}$$

Hence, $\lim_{n\to\infty}\phi(x_n,x_0)$ exists and $\phi(x_n,x_{n+1})\to 0$. Since $\{x_n\}$ is bounded, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{4.0.12}$$

For (iv), since $x_{n+1}\in C_n$, we have

$$\phi(x_{n+1},z_n)\le\phi(x_{n+1},x_n)+(3-3\gamma_n)b^2\|Ax_n\|^2+b\alpha_n\tau_n.$$

Observe that $\{\tau_n\}$ is bounded, since $\{x_n\},\{Ax_n\},\{\gamma_n\},\{\alpha_n\}$ are bounded. Furthermore, since $\gamma_n\to 1$ and $\alpha_n\to 0$, we deduce that

$$\lim_{n\to\infty}\phi(x_{n+1},z_n)=0 \tag{4.0.13}$$

Using the uniform continuity of J and J^{-1} on bounded sets and Lemma 2.2.10, we have that $\{y_n\}$ and $\{z_n\}$ are bounded. Hence, using Lemma 2.2.5, equation (4.0.13) implies $\lim_{n\to\infty}\|x_{n+1}-z_n\|=0$. Combining with equation (4.0.12), we have that
$\lim_{n\to\infty}\|z_n-x_n\|=0$.

Step4: We show the following:

(i) $\lim_{n\to\infty}\|x_n - y_n\| = 0$,

(ii) $\lim_{n\to\infty}\|St_n - t_n\| = 0$.

For (i), from inequalities (4.0.7) and (4.0.8) we infer that,

$$\begin{aligned}\phi(u,z_n)-\phi(u,x_n) &\leq (-\alpha_n-\beta_n)\phi(u,x_n)+\alpha_n\phi(u,y_n)+\beta_n\phi(u,St_n)\\ &\leq (3-3\gamma_n)b^2\|Ax_n\|^2+b\alpha_n\tau_n.\end{aligned} \tag{4.0.14}$$

Using the fact that $\alpha_n\to 0$, $\gamma_n\to 1$ and the boundedness of $\{x_n\},\{Ax_n\},\{\tau_n\}$ and $\{y_n\}$, from inequality (4.0.14) and the fact that $\phi(u,z_n)-\phi(u,x_n)\to 0$ as $n\to\infty$, we obtain $\lim_{n\to\infty}\beta_n\Big(\phi(u,St_n)-\phi(u,x_n)\Big)=0$. Since $\liminf_{n\to\infty}\beta_n>0$, we have , $\lim_{n\to\infty}\Big(\phi(u,St_n)-\phi(u,x_n)\Big)=0$. Inequalities (4.0.14) and (4.0.5) imply that

$$\begin{aligned}\lim_{n\to\infty}\Big(\phi(u,St_n)-\phi(u,x_n)\Big) &\leq \lim_{n\to\infty}\Big(\phi(u,t_n)-\phi(u,x_n)\Big)\\ &\leq \lim_{n\to\infty}\Big((3-3\gamma_n)b^2\|Ax_n\|^2+b\alpha_n\tau_n\Big)=0.\end{aligned}$$

Thus, $\lim_{n\to\infty}(\phi(u,t_n)-\phi(u,x_n))=0$. Next, re-arranging inequality (4.0.4), we have

$$\begin{aligned}\left(1-\frac{(1-\gamma_n)}{\alpha}-\frac{bk}{\alpha}\right)\phi(y_n,x_n)+\left(1-2\frac{(1-\gamma_n)}{\alpha}-\frac{bk}{\alpha}\right)\phi(t_n,y_n) &\leq \phi(u,x_n)-\phi(u,t_n)\\ &\quad+3(1-\gamma_n)b^2\|Ax_n\|^2.\end{aligned}$$

Using the assumptions on γ_n and b, we deduce that

$$\left(1-\frac{(1-\gamma_n)}{\alpha}-\frac{bk}{\alpha}\right)>0 \quad\text{and}\quad \left(1-2\frac{(1-\gamma_n)}{\alpha}-\frac{bk}{\alpha}\right)>0.$$

Hence,

$$\begin{aligned}0 &\leq \left(1-\frac{(1-\gamma_n)}{\alpha}-\frac{bk}{\alpha}\right)\phi(y_n,x_n)+\left(1-2\frac{(1-\gamma_n)}{\alpha}-\frac{bk}{\alpha}\right)\phi(t_n,y_n)\\ &\leq \phi(u,x_n)-\phi(u,t_n)+3(1-\gamma_n)b^2\|Ax_n\|^2.\end{aligned}$$

Hence, $\lim_{n\to\infty}\left(\left(1-\frac{(1-\gamma_n)}{\alpha}-\frac{bk}{\alpha}\right)\phi(y_n,x_n)+\left(1-2\frac{(1-\gamma_n)}{\alpha}-\frac{bk}{\alpha}\right)\phi(t_n,y_n)\right)=0.$

This implies $\lim_{n\to\infty}\phi(y_n,x_n)=\lim_{n\to\infty}\phi(t_n,y_n)=0$. Thus, $\|y_n-x_n\|\to 0$ and $\|t_n-y_n\|\to 0$, so that $\lim_{n\to\infty}\|x_n-t_n\|=0$.

For (ii), we have that

$$\begin{aligned}Jz_n &= (1-\alpha_n-\beta_n)Jx_n+\alpha_nJy_n+\beta_nJStn\\ &= (1-\alpha_n-\beta_n)Jx_n+\alpha_nJy_n+\beta_nJStn-\beta_nJt_n+\beta_nJt_n.\end{aligned}$$

This gives that

$$Jz_n-Jx_n=\alpha_n(Jy_n-Jx_n)+\beta_n(Jt_n-Jx_n)+\beta_n(JSt_n-Jt_n).$$

Using the uniform continuity of J on bounded sets, the fact that β_n is bounded and Step 3, we obtain

$$\lim_{n\to\infty}\|JSt_n-Jt_n\|=0.$$

By the uniform continuity of J^{-1} on bounded sets, we deduce that

$$\lim_{n\to\infty}\|St_n-t_n\|=0.$$

Step 5. We claim that $\Omega_w(x_n) \subset F(S) \cap VI(C, A)$, where $\Omega_w(x_n)$ denotes the set of weak subsequential limit is of $\{x_n\}$. Since $\{x_n\}$ is bounded, $\Omega_w(x_n) \neq \emptyset$. Let $u \in \Omega_w(x_n)$ be arbitrary. Then, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup u$ as $k \to \infty$. This implies that $t_{n_k} \rightharpoonup u$ as $k \to \infty$. Since we have that $\lim_{n\to\infty} \|St_{n_k} - t_{n_k}\| = 0$, it follows that $u \in F(S)$. We show that $u \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C(v), & if\ v \in C, \\ \emptyset, & if\ v \notin C, \end{cases}$$

be as defined in Lemma 2.2.7. Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$. it is known that T is maximal monotone if and only if given $(x, x^*) \in E \times E^*$ if $\langle x - y, x^* - y^* \rangle \geq 0\ \forall\ (y, y^*) \in G(T)$, then $x^* \in Tx$.

Claim: $(u, 0) \in G(T)$.
Let $(v, u^*) \in G(T)$. To establish the claim, it suffices to show that $\langle v - u, u^* \rangle \geq 0$.
Now, $(v, u^*) \in G(T) \ \Rightarrow\ u^* \in Tv = Av + N_C(v) \ \Rightarrow\ u^* - Av \in N_C(v)$.
Therefore, $\langle v - y, u^* - Av \rangle \geq 0,\ \forall\ y \in C$. Since $t_n = \Pi_C J^{-1}(Jx_n - \lambda Ay_n)$ and $v \in C$, we have, by Lemma 2.1.1 (1) that $\langle t_n - v, Jx_n - \lambda Ay_n - Jt_n \rangle \geq 0$. Thus,

$$\left\langle v - t_n, \frac{Jt_n - Jx_n}{\lambda} + Ay_n \right\rangle \geq 0, \quad n \geq 0 \ .$$

Using the fact that $t_n \in C$ and $u^* - Av \in N_C(v)$, we have

$$\begin{aligned} \langle v - t_{n_k}, u^* \rangle &\geq \langle v - t_{n_k}, Av \rangle \\ &\geq \langle v - t_{n_k}, Av \rangle - \left\langle v - t_{n_k}, \frac{Jt_{n_k} - Jx_{n_k}}{\lambda} + Ay_{n_k} \right\rangle \\ &= \langle v - t_{n_k}, Av - At_{n_k} \rangle + \langle v - t_{n_k}, At_{n_k} - Ay_{n_k} \rangle - \left\langle v - t_{n_k}, \frac{Jt_{n_k} - Jx_{n_k}}{\lambda} \right\rangle \\ &\geq \langle v - t_{n_k}, At_{n_k} - Ay_{n_k} \rangle - \left\langle v - t_{n_k}, \frac{Jt_{n_k} - Jx_{n_k}}{\lambda} \right\rangle . \end{aligned}$$

Hence, as $k \to \infty$, we have

$$\langle v - u, u^* \rangle \geq 0.$$

Therefore, $\Omega_w(x_n) \subset F(S) \cap VI(C, A)$.

Step 6. We show that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to some point $q = \Pi_{F(S)\cap VI(C,A)} x_0$.

For contradiction, assume that $\{x_n\}$ does not converge strongly to q. Then, there exixt $\epsilon > 0$ amd a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\|x_{n_k} - q\| > \epsilon,\ \forall\ k$. Without loss of generality, we may assume that $\{x_{n_k}\}$ converges weakly to some point u. By Step 5, $u \in F(S) \cap VI(C, A)$. Using the fact that $q = \Pi_{F(S)\cap VI(C,A)} x_0$, the weak lower semi-continuity of $\phi(., x_0)$ and Step 3, we have

$$\phi(q, x_0) \leq \phi(u, x_0) \leq \liminf_{k\to\infty} \phi(x_{n_k}, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) \leq \phi(q, x_0), \tag{4.0.15}$$

the last inequality following from the fact that $x_n = \Pi_{C_n} x_0$.
This implies that $\phi(q, x_0) = \phi(u, x_0)$. But, since $u \in F(S) \cap VI(C, A)$, we have

$$\phi(u, q) \leq \phi(u, x_0) - \phi(q, x_0) = 0.$$

Hence, $\phi(u, q) = 0 \ \Rightarrow\ u = q \ \Rightarrow x_{n_k} \rightharpoonup q$.
Also, (4.0.15) gives $\lim_{k\to\infty} \phi(x_{n_k}, x_0) = \phi(q, x_0)$. Moreover, by (P_2), we have

$$\phi(x_{n_k}, q) = \phi(x_{n_k}, x_0) + \phi(x_0, q) + 2 \langle x_0 - x_{n_k}, Jq - Jx_0 \rangle .$$

Since $x_{n_k} \rightharpoonup q$ and $\phi(x_{n_k}, x_0) \to \phi(q, x_0)$, we deduce that

$$\lim_{k\to\infty} \phi(x_{n_k}, q) = \phi(q, x_0) + \phi(x_0, q) + 2\langle x_0 - q, Jq - Jx_0\rangle = 0.$$

Therefore, $x_{n_k} \to q$, a contradiction. Hence, $\{x_n\}$ converge strongly to q. It then follows immediately that $\{y_n\}$ and $\{z_n\}$ also converge strongly to q. ■

Theorem 4.0.2 *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E such that $J(C)$ is convex. Let $A : C \to E^*$ be a monotone, k-Lipschitz map and $\{S_i\}_{i=1}^{\infty}$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, where $S_i : C \to C$, $\forall$ i. Let $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\mu_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Assume $\Big(\bigcap_{i=1}^{\infty} F(S_i)\Big) \cap V(C,A) \neq \emptyset$. Define inductively the sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in C_0 = C; \\ y_n = \Pi_C(J^{-1}[Jx_n - \gamma_n\lambda Ax_n]); \\ z_n = J^{-1}\Big((1-\alpha_n-\beta_n)Jx_n + \alpha_n Jy_n + \beta_n JS\Pi_C(J^{-1}[Jx_n - \lambda Ay_n])\Big); \\ C_{n+1} = \{z \in C_n : \phi(z, z_n) \le \phi(z, x_n) + (3-3\gamma_n)b^2||Ax_n||^2 + b\alpha_n\tau_n\}; \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \end{cases} \tag{4.0.16}$$

for all $n \ge 0$ where $Sx = J^{-1}\Big(\sum_{i=1}^{\infty}\eta_i(\mu_i Jx + (1-\mu_i)JS_i x\Big)$ for each $x \in C$, $\lambda \in (0, b]$ with $b < \frac{\alpha}{2k}$, $\tau_n = max\{\|x_n\|, 1\}\|Ax_n\|\sigma(\|\gamma_n\lambda Ax_n\|)$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following conditions

1. $\alpha_n + \beta_n \le 1 \ \forall \ n \ge 0$;
2. $\lim_{n\to\infty} \alpha_n = 0$;
3. $\liminf_{n\to\infty} \beta_n > 0$;
4. $\lim_{n\to\infty} \gamma_n = 1$ *and* $\gamma_n > 1 - \frac{\alpha}{4}\ \forall n \ge 0$.

Then, the sequences $\{x_n\}, \{y_n\}$ and (z_n) are well-defined and converge strongly to some point $q = \Pi_{F(S)\cap\emptyset}x_0$.

Proof By Lemma 2.2.8, S is relatively nonexpansive and $F(S) = \bigcap_{i=1}^{\infty} F(S_i)$. The conclusion follows from Lemma 4.0.1. ■

4.1 Applications

We give some applications of our main Theorem.

Theorem 4.1.1 *Let C be a nonempty, closed and convex subset of L_p $\Big($or l_p or $W_p^m(\Omega)\Big)$, $1 < p \le 2$ such that $J(C)$ is convex. Let $A : C \to E^*$ be a monotone, k-Lipschitz map and $\{S_i\}_{i=1}^{\infty}$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, where $S_i : C \to C$, $\forall$ i. Let $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\mu_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Assume $\Big(\bigcap_{i=1}^{\infty} F(S_i)\Big) \cap V(C,A) \neq \emptyset$. Define inductively the sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in C_0 = C; \\ y_n = \Pi_C(J^{-1}[Jx_n - \gamma_n\lambda Ax_n]); \\ z_n = J^{-1}\Big((1-\alpha_n-\beta_n)Jx_n + \alpha_n Jy_n + \beta_n JS\Pi_C(J^{-1}[Jx_n - \lambda Ay_n])\Big); \\ C_{n+1} = \{z \in C_n : \phi(z, z_n) \le \phi(z, x_n) + (3-3\gamma_n)b^2\|Ax_n\|^2 + b\alpha_n\tau_n\}; \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \end{cases} \tag{4.1.1}$$

for all $n \geq 0$ *where* $Sx = J^{-1}\Big(\sum_{i=1}^{\infty}\eta_i(\mu_i Jx + (1-\mu_i)JS_i x\Big)$ *for each* $x \in C$, $\lambda \in (0, b]$ *with* $b < \frac{\alpha}{2k}$, $\tau_n = max\{\|x_n\|, 1\}\|Ax_n\|\sigma(\|\gamma_n\lambda Ax_n\|)$ *and* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0, 1]$ *satisfying the following conditions*

1. $\alpha_n + \beta_n \leq 1 \; \forall \; n \geq 0$;
2. $\lim_{n\to\infty} \alpha_n = 0$;
3. $\liminf_{n\to\infty} \beta_n > 0$;
4. $\lim_{n\to\infty} \gamma_n = 1$ *and* $\gamma_n > 1 - \frac{\alpha}{4} \; \forall n \geq 0$.

Then, the sequences $\{x_n\}, \{y_n\}$ *and* (z_n) *are well-defined and converge strongly to some point* $q = \Pi_{F(S)\cap\emptyset}x_0$.

Proof L_p $\Big($*or* l_p *or* $W_p^m(\Omega)\Big)$, $1 < p \leq 2$, are uniformly smooth and 2-uniformly convex. Hence, the conclusion follows from Theorem 4.0.2. ■

Corollary 4.1.2 *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be a monotone, k-Lipschitz map and* $\{S_i\}_{i=1}^{\infty}$ *be a countable family of nonexpansive maps such that* $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, *where* $S_i : C \to C$, $\forall$ *i. Let* $\{\eta_i\}_{i=1}^{\infty} \subset (0, 1)$ *and* $\{\mu_i\}_{i=1}^{\infty} \subset (0, 1)$ *be sequences such that* $\sum_{i=1}^{\infty}\eta_i = 1$. *Assume* $\Big(\bigcap_{i=1}^{\infty} F(S_i)\Big) \cap V(C, A) \neq \emptyset$. *Define inductively the sequence* $\{x_n\}$ *by*

$$\begin{cases} x_0 \in C_0 = C; \\ y_n = P_C(x_n - \gamma_n\lambda Ax_n); \\ z_n = (1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n SP_C(x_n - \lambda Ay_n); \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 \leq \|x_n - z\|^2 + (3 - 3\gamma_n + \alpha_n)b^2\|Ax_n\|^2\}; \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases} \tag{4.1.2}$$

for all $n \geq 0$ *where* $Sx = \Big(\sum_{i=1}^{\infty}\eta_i(\mu_i x + (1-\mu_i)S_i x\Big)$ *for each* $x \in C$, $\lambda \in (0, b]$ *with* $b < \frac{\alpha}{2k}$ *and* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0, 1]$ *satisfying the following conditions*

1. $\alpha_n + \beta_n \leq 1 \; \forall \; n \geq 0$;
2. $\lim_{n\to\infty} \alpha_n = 0$;
3. $\liminf_{n\to\infty} \beta_n > 0$;
4. $\lim_{n\to\infty} \gamma_n = 1$ *and* $\gamma_n > \frac{3}{4} \; \forall n \geq 0$.

Then, the sequences $\{x_n\}, \{y_n\}$ *and* (z_n) *are well-defined and converge strongly to some point* $q = \Pi_{F(S)\cap\emptyset}x_0$.

Observe that if we set $\alpha_n = 0$, $\beta_n = 1$ and $\gamma_n = 1$, for all $n \in \mathbb{N}$ in Theorem 4.0.2, we obtain the following Theorem:

Theorem 4.1.3 *Let* C *be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space* E *such that* $J(C)$ *is convex. Let* $A : C \to E^*$ *be a monotone and k-Lipschitz map and* $\{S_i\}_{i=1}^{\infty}$ *be a countable family of relatively nonexpansive maps such that* $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, *where* $S_i : C \to C \; \forall$ *i. Let* $\{\eta_i\}_{i=1}^{\infty} \subset (0, 1)$ *and* $\{\mu_i\}_{i=1}^{\infty} \subset (0, 1)$ *be sequences such that* $\sum_{i=1}^{\infty}\eta_i = 1$. *Assume* $(\bigcap_{i=1}^{\infty} F(S_i)) \cap V(C, A) \neq \emptyset$. *Define inductively the sequence* $\{x_n\}$

$$\begin{cases} x_0 \in C_0 = C; \\ y_n = \Pi_C(J^{-1}[Jx_n - \lambda Ax_n]); \\ z_n = S\Pi_C(J^{-1}[Jx_n - \lambda Ay_n]); \\ C_{n+1} = \{z \in C_n : \phi(z, z_n) \leq \phi(z, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}}, \end{cases} \tag{4.1.3}$$

for all $n \geq 0$ *where* $Sx = J^{-1}\Big(\sum_{i=1}^{\infty} \eta_i(\mu_i Jx + (1-\mu_i)JS_i x\Big)$ *for each* $x \in C$, $\lambda \in (0, b]$ *with* $b < \frac{\alpha}{2k}$. *Then, the sequences* $\{x_n\}, \{y_n\}$ *and* (z_n) *are well-defined and converge strongly to some point* $q = \Pi_{F(S)\cap\emptyset} x_0$.

Also, taking $S_i = I$, $\forall\ i$, $\alpha_n = 0$ and $\beta_n = 1$ in Theorem 4.0.2, we obtain the following theorem providing an iterative algorithm for finding the solution of a variational inequality problem.

Theorem 4.1.4 *Let* C *be a nonempty closed convex subset of a 2-uniformly convex and uniformly smooth reflexive space* E *such that* $J(C)$ *is convex. Let* $A : C \to E^*$ *be a monotone and k-Lipschitz map. We define inductively the scheme*

$$\begin{cases} x_0 \in C_0 = C; \\ y_n = \Pi_C(J^{-1}[Jx_n - \gamma_n \lambda A x_n]); \\ z_n = \Pi_C(J^{-1}[Jx_n - \lambda A y_n]); \\ C_{n+1} = \{z \in C_n : \phi(z, z_n) \leq \phi(z, x_n) + (3 - 3\gamma_n)b^2||Ax_n||^2\}; \\ x_{n+1} = \Pi_{C_{n+1}}, \end{cases} \tag{4.1.4}$$

for all $n \geq 0$ *where* $\lambda \in (0, b]$ *with* $b < \frac{\alpha}{2k}$, $\{\gamma_n\}$ *is a sequences in* $[0, 1]$ *satisfying the following conditions* $\lim_{n\to\infty} \gamma_n = 1$ *and* $\gamma_n > 1 - \frac{\alpha}{4}$ $\forall n \geq 0$. *Then, the sequences* $\{x_n\}, \{y_n\}$ *and* (z_n) *are well-defined and converge strongly to some point* $q = \Pi_{F(S)\cap\emptyset} x_0$.

Prototype

Prototypes for the parameters of our algorithm are: $\lambda \in (0, b), \alpha_n = \frac{1}{4n}, \beta_n = 1 - \frac{1}{2n}$ and $\gamma_n = 1 - \frac{\alpha}{8n}$.

CONCLUSION

Theorem 4.0.2 is a significant improvement of the result of Ceng et al. [2010] in the following sense:

1. The algorithm studied in Ceng et al. [2010] requires at *each step of the iteration process*, the computation of two subsets C_n and Q_n of C; their intersection $C_n \cap Q_n$, and the projection of the initial vector onto this intersection. In our iteration process, the subset Q_n has been dispensed with, thereby eleminating the need to compute $C_n \cap Q_n$ and the projection of an initial vector onto this intersection. Furthermore, the sequence $\{\lambda_n\}$ used in the algorithm of Ceng et al. [2010], which is also to be computed at each step of the iteration has been replaced by a fixed constant λ in our algorithm. This λ is to be computed once and used at each step of the iteration process. Consequently, our algorithm requires less computation time than the algorithm used in Ceng et al. [2010] thus reducing computational cost.

2. In Ceng et al. [2010], the authors proved a strong convergence theorem in a real Hilbert space for a monotone and *k-Lipschitz* map and *one* nonexpansive map, $S : C \to C$. In our Theorem, Theorem 4.0.2, strong convergence is proved for a monotone and *k-Lipschitz* map and a *countable family* of relatively nonexpansive maps, $S_i : C \to C$ and the much more general *uniformly smooth and 2-uniformly convex* real Banach spaces.

CHAPTER 5

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