A MODIFIED SUBGRADIENT EXTRAGRADEINT METHOD FOR VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS IN REAL BANACH SPACES

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CERTIFICATION

This is to certify that the thesis titled **"MODIFIED SUBGRADIENT EXTRAGRADEINT METHOD FOR VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS IN REAL BANACH SPACES"** submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research carried out by Osisiogu, Onyekachi Oluseyi in the Department of Pure and Applied Mathematics.

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Abstract

Let E be a 2-uniformly convex and uniformly smooth real Banach space with dual space E^*. Let $A : C \to E^*$ be a monotone and Lipschitz continuous mapping and $U : C \to C$ be relatively nonexpansive. An algorithm for approximating the common elements of the set of fixed points of a relatively nonexpansive map U and the set of solutions of a variational inequality problem for the monotone and Lipschitz continuous map A in E is constructed and proved to converge strongly.

Keywords— Subgradient extragradient algorithm, monotone map, relatively nonexpansive map, Lipschitz map.

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Dedication

This thesis is dedicated to Almighty God.

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CHAPTER 1

Introduction

1.1 Background of study

The notion of monotone operators was introduced by Zarantonello [Zarantonello, 1960], Minty [Minty, 1962] and Kačurovskii [Kačurovskii, 1960]. Monotonicity conditions in the context of variational methods for nonlinear operator equations were also used by Vainberg and Kačurovskii [Vainberg *et al.*, 1959].

A map $A : D(A) \subset H \to H$ is monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0 \;\; \forall x, y \in H.$$

Consider the problem of finding the equilibrium states of the system described by

$$\frac{du}{dt} + Au = 0, \tag{1.1}$$

where A is a monotone-type mapping on a real Hilbert space. This equation describes the evolution of many physical phenomena which generate energy over time. It is known that many physically significant problems in different areas of research can be transformed into an equation of the form

$$Au = 0. \tag{1.2}$$

At equilibrium state, equation (1.1) reduces to equation (1.2) whose solutions, in this case, correspond to the equilibrium state of the system described by equation (1.1). Such equilibrium points are very desirable in many applications, for example, economics, ecology, physics and so on.

1.2 Variational inequality problem

Let C be a nonempty, closed and convex subset of a real normed space E with dual space E^*. Let $A : C \subset E \to E^*$ be a nonlinear operator. The classical variational inequality problem is the following: find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0 \; \forall \; y \in C. \tag{1.3}$$

The set of solutions of inequality (1.3) is denoted by $VI(C,\ A)$. The variational inequality problem is connected with convex minimization, fixed point problem, zero of nonlinear operator and so on.

Variational inequality has been shown to be an important mathematical model in the study of many real problems, in particular equilibrium problems. It provides us with a tool for formulating and qualitatively analyzing the equilibrium problems in terms of existence and uniqueness of solutions, stability, and sensitivity analysis, and provides us with algorithms for computational purposes.

For example, in optimization, we consider $f : [a,b] \to \mathbb{R}$ differentiable. It is well known that such f has a minimizer, say $x^* \in [a,b]$. We have the following cases:

1. $x^* = a \Rightarrow f'(x^*) \cdot (x - x^*) \geq 0 \;\; \forall \; x \in [a,b]$
2. $x^* = b \Rightarrow f'(x^*)(x - x^*) \geq 0 \;\; \forall \; x \in [a,b]$
3. $x^* \in (a,b) \Rightarrow f'(x^*)(x - x^*) = 0 \;\; \forall \; x \in [a,b]$

Thus, setting $C = [a,b]$, $A = f'$ we have

$$x^* \text{ is a minimizer} \Rightarrow \langle Ax^*, x - x^* \rangle \geq 0 \;\; \forall \; x \in C.$$

In general, in Euclidean n-dimensional $\mathbb{R}^n$, the variational inequality (1.3) becomes $(y - x^*)^\top Ax^* \geq 0 \; \forall \; y \in C$. This is equivalent to $y^\top Ax^* \geq x^{*\top} Ax^* \; \forall \; y \in C$. Thus, x^* is a solution to the minimization problem

$$\begin{cases} \min y^\top Ax^*; \\ y \in C, \end{cases}$$

i.e.

$$x^* \in VI(C,\ A) \Leftrightarrow x^* \text{ solves } \begin{cases} \min y^\top Ax^*; \\ y \in C. \end{cases}$$

1.3 Fixed Point Problem

In 1922, Banach [Banach, 1922] published his fixed point theorem known as Banach's Contraction Mapping Principle using the concept of Lipschitz mapping. A fixed point of an operator T is a solution of the equation $x = Tx$. The set of fixed points of T is denoted by $F(T)$. T is called a contraction if there exists a fixed $L < 1$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \text{ for all } x, y \in E. \tag{1.4}$$

A contraction mapping is also known as a Banach contraction. If inequality (1.4) holds for $L = 1$, then T is called nonexpansive and if inequality (1.4) holds for fixed $L < \infty$, then T is called Lipschitz continuous. Clearly, for the mapping T, the following obvious implications hold:

$$\boxed{\text{Contraction}} \implies \boxed{\text{Nonexapansive}} \implies \boxed{\text{Lipschitz continuous}}$$

The concept of fixed points makes sense only when the map T maps the space into itself, but this concept does not make sense when T maps the space into its dual.

Our main focus in this thesis is to construct an iterative algorithm that converges strongly to a solution of the set of fixed point problems of a relatively nonexpansive mapping and the set of variational inequality problems for monotone and Lipschitz continuous mapping on a 2-uniformly convex and uniformly smooth real Banach space.

CHAPTER 2

Literature Review

In this chapter, we deal with other work done in this area of research.

2.1 Review

The variational inequality theory has its origin in the works of Stampacchia (see [Stampacchia, 1964]) and Fichera (see [Ficher, 1963-1964]). This theory does not only provide powerful techniques for studying problems arising in various branches of mathematics, but also in mechanics, transportation, economics equilibrium or contact problems in elasticity. For instance, the moving boundary value problem, the traffic assignment problem, saddle point problem, the free boundary value problem can be characterized as variational inequality problems (see [Baiocchi *et al.*, 1984, Bertsekas *et al.*, 1982, Dafermos, 1990]).

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. The following variational inequality problem is studied: find $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0 \quad \forall v \in C, \tag{2.1}$$

where $A : H \to H$ is a single-valued map. Various iterative methods for solving problem (2.1) have been proposed and analyzed in Hilbert spaces or more general real Banach spaces when A is monotone and Lipschitz, strongly monotone and Lipschitz or inverse-strongly monotone.

In order to solve a saddle point problem, Korpelevič (1976) proposed the so-called *extragradient method* in a real Hilbert space H and is given as follows:

Algorithm 2.1.

$$\begin{cases} x_0 \in C \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = P_C(x_n - \lambda_n A y_n), \end{cases} \tag{2.2}$$

for all $n \geq 0$, where $\lambda_n \in (0, \frac{1}{k})$, C is closed convex subset of R^n and A is monotone and k-Lipschitz continuous map of C into R^n. He proved that if $VI(C, A)$ is nonempty, the sequence $\{x_n\}$ and $\{y_n\}$, generated by (2.2), converge to some point $z \in VI(C, A)$.

The extragradient method has received great attention by many authors who developed and improved it in various ways. In the case when C has a simple structure and the projections onto it can be evaluated readily, the extragradient method is very useful. Now, if C is any closed and convex set, one has to calculate in each iterate two projections onto C of H. Therefore, Censor *et al.* in 2011 modified the extragradient method and proposed the following iterative algorithm:

Algorithm 2.2.

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{w \in H : \langle x_n - \lambda A(x_n) - y_n, w - y_n \rangle \leq 0\} \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \end{cases} \tag{2.3}$$

for all $n \geq 0$.

We observe that Algorithm 2.2 replaces the second projection onto the closed and convex subset C in Algorithm 2.1 with the one onto the subgradient half-space T_n. The modified algorithm is called the *subgradient extragradient method* for variational inequality problem in real Hilbert space H. Censor *et al.* proved that Algorithm 2.2 converges weakly to a solution of variational inequality (2.1) in a real Hilbert space.

By modifying the extragradient method, Nadezhkina and Takahashi [Nadezhkina *et al.*, 2006] were able to prove a weak convergence result. More precisely, given a nonempty, closed and convex set $C \subset H$, a nonexpansive mapping $S : C \to C$ and a monotone and k-Lipschitz continuous mapping $A : C \to H$, they introduced the following iterative algorithm in order to find an element of $F(S) \cap VI(C, A)$.

Algorithm 2.3.

$$\begin{cases} x_0 = x \in C \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S y_n, \end{cases} \tag{2.4}$$

for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence on $(0, 1)$, $\{\lambda_n\}$ is a sequence in $(0, \frac{1}{k})$ and P_C is the metric projection of H onto C. It is shown that if $F(S) \cap VI(C, A) \neq \emptyset$, then the sequence generated by Algorithm (2.3) converges weakly to some $z \in F(S) \cap VI(C, A)$.

In 2006, to obtain strong convergence, Nadwzhkina and Takahashi [Nadezhkina *et al.*, 2006] introduced an iterative scheme by a hybrid method and proved strong convergence of the sequence generated by their algorithm to a point of $F(S) \cap VI(C, A)$ and it is as follows:

Algorithm 2.4.

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1 + \alpha_n) S P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{2.5}$$

for all $n \geq 0$, where $0 \leq \alpha_n \leq c < 1$ and $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$. Then the sequence $\{x_n\}$ converges strongly to the some point $z \in F(S) \cap VI(C, A)$.

In this thesis, we introduced a subgradient extragradient-like approximation method. The method produces sequences which are shown to converge strongly to a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions of a variational inequality problem for a monotone and Lipschitz continuous mapping.

2.1.1 Nonexpansive Mapping

The study of the existence of fixed points of nonexpansive mappings was initiated in 1965 by Browder [Browder, 1965], Gӧhde [Gӧhde *et al.*, 1965] and Kirk [Kirk, 1965] independently. Indeed, Browder and Gӧhde obtained an existence theorem for a nonexpansive mapping on a uniformly convex Banach space, while Kirk obtained the same result in a reflexive Banach space using the normal structure property. In this thesis we study the nonexpansivity of the so called *relatively nonexpansive mappings*.

In a paper of Eldred *et al.* (2005) two results about the existence of fixed points for relatively nonexpansive mapping were obtained. As the authors explain in the introduction, the significance of these two results lies in the fact that relatively nonexpansive assumption is much weaker than the assumption of nonexpansivity.

It is known that for a nonexpansive mapping T with $F(T) := \{x \in D(T) : Tx = x\} \neq \emptyset$, the classical Picard iterative sequence x_{n+1}, $x_0 \in D(T)$ does not always converge to a fixed point of T, assuming existence. To see this, consider the following example of the rotation of the unit ball around the origin of co-ordinates in $\mathbb{R}^2$ which is a nonexpansive map with the origin as its unique fixed point, the Picard iteration would not converge to the fixed point if for example, $x_0 = (1, 0)$. Krasnoselskii (1957) showed that in this example, the recursion formula:

$$x_0 \in E, x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}Tx_n, \quad n \geq 0$$

would converge to the fixed point. That is, taking the auxilliary nonexpansive mapping $\frac{1}{2}(I + T)$, where I denotes the identity transformation of the plane instead of by the usual Picard iterates, $x_{n+1} = Tx_n$, $x_0 \in K$, $n \geq 0$. Schacfer (1957) showed that the constant $\frac{1}{2}$ is not crucial. He proved that the recursion formula: $x_0 \in E$,

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n = 0, 1, 2, \ldots; \lambda \in (0, 1), \tag{2.6}$$

would converge to the fixed point. The recursion formula (2.6) is still being studied in connection with other nonlinear operators.

However, the most general iterative scheme now studied is the following:
$x_0 \in K$,

$$x_{n+1} = (1 - c_n)x_n + c_n Tx_n, \quad n \geq 0, \tag{2.7}$$

where $\{c_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:
$(i) \sum_{n=0}^{\infty} c_n = \infty$, $(ii) \lim_{n \longrightarrow \infty} c_n = 0$ (see for example [Chidume, 1981], [Edelstein *et al.*, 1973] and [Ishikawa, 1976]). The sequence $\{x_n\}$ generated by (2.7) is generally referred to as the *Mann sequence* in the light of Mann [Mann, 1953]. It is know that the sequence defined by (2.7) converges weakly to a fixed point of a nonexpansive map T. To obtain strong convergence which is desirable in several applications, a key step is to first establish that the sequence $\{x_n\}$ defined by (2.6) is an *approximate fixed point sequence*, i.e., that the sequence satisfies the following condition:

$$\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0. \tag{2.8}$$

That is, if the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded, Ishikawa [Ishikawa, 1976] proved that the sequence is an approximate fixed point sequence. The recursion formula (2.6) is consequently called the *Krasnoselskii-Mann formula* for finding fixed points of nonexpansive mappings. For several years, the study of Krasnoselskii-Mann iterative algorithm for approximating solutions of nonlinear equations became a flourishing area of research for many mathematicians. Edelstein and O'Brian [Edelstein *et al.*, 1973] considered the recursion formula (2.6) and proved that *if K is bounded*, then the convergence in (2.8) is uniform. Chidume (1981) considered the recursion formula (2.7), introduced the concept of admissible sequences and proved that *if K is bounded*, then the convergence in (2.8) is uniform for the sequence defined by (2.7).

In the recent years, the definition of relatively nonexpansive mapping has been presented and studied by many authors. It is known that if we are in a Hilbert space a relatively nonexpansive map reduces to a quasi-nonexpansive map.

CHAPTER 3

Theory and Methods

In this chapter, we give some definitions of most of the terms and concepts we shall use.

3.1 Definitions

Let H be a real Hilbert space. A nonlinear operator $A : D(A) \subset H \to 2^H$ is called monotone if

$$\langle u - v, x - y\rangle \geq 0 \ \ \forall \ u \in Ax, v \in Ay. \tag{3.1}$$

We do not require that Ax be nonempty. The domain of A is the set $D(A) = \{x \in E : Ax \neq \emptyset\}$.
If A is single-valued, it is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0 \ \forall \ x, y \in H \tag{3.2}$$

and it is called strongly monotone if there exists $\alpha \in (0,1)$ such that for all $x, y \in D(A)$, the following inequality holds:

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|x - y\|^2.$$

For example; let C be a closed and convex nonempty subset of a real Hilbert space H and let U be a nonexpansive map of C into itself: $\|U(x) - U(y)\| \leq \|x - y\|$ for all $x, y \in C$. Let I denote the identity map in H; then $A = I - U$ is monotone, with $D(A) = C$. Indeed, we have for all $x, y \in C$,

$$\begin{aligned}
\langle Ax - Ay, x - y\rangle &= \langle x - Ux - y + Uy, x - y\rangle \\
&= \langle x - y - (Ux - Uy), x - y\rangle \\
&= \langle x - y, x - y\rangle - \langle Ux - Uy, x - y\rangle \\
&= \|x - y\|^2 - \langle Ux - Uy, x - y\rangle \\
&\geq \|x - y\|^2 - \|Ux - Uy\| \cdot \|x - y\| \\
&\geq \|x - y\|^2 - \|x - y\|^2 = 0.
\end{aligned}$$

Hence, $A : H \to H$ defined by $A = I - U$ is monotone.

Also, consider the next example. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a convex and proper function. Then, the subdifferential of f at $x \in H$ is the map $\partial f : H \to 2^H$ defined by

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x) \ \forall \ y \in X\}. \tag{3.3}$$

Thus, for all $x, y \in X$, $u \in \partial f(x)$ and $v \in \partial f(y)$ implies

$$f(y) - f(x) \geq \langle u, y - x\rangle \text{ and } f(x) - f(y) \geq \langle v, x - y\rangle.$$

Adding the inequalities we get

$$0 \leq \langle u - v, x - y\rangle.$$

Hence, $\partial f : H \to 2^H$ is a monotone operator on H. Now, $0 \in \partial f(x) \Leftrightarrow f(x) \leq f(y) \quad \forall y \in H$, by definition This implies that $0 \in \partial f(x)$ if and only if x is a global minimizer of f. If $\partial f \equiv A$, it follows that solving $0 \in Au$ is solving for a minimizer of f. If the operator A is single-valued, then inclusion $0 \in Au$ reduces to equation (1.2).

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a closed convex nonempty subset of H.

Definition 3.1. A mapping $A : C \to H$ is called γ-inverse strongly monotone if there exists a real number $\gamma > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \gamma \|Ax - Ay\|^2 \ \ \forall \ x, y \in C. \tag{3.4}$$

Definition 3.2. Let $T : D(T) \subset E \to R(T) \subset E$ be a map, where $D(T)$ denotes the domain of T and $R(T)$ denotes the range of T. A point $x \in D(T)$ is called a fixed point of the map T if and only if $Tx = x$. The set of fixed points of a mapping T denoted by $F(T)$ is defined by $F(T) := \{x \in D(T) : Tx = x\}$.

Definition 3.3. A map T with domain $D(T)$ and range $R(T)$ in E is called L-Lipschitz if and only if there exists a constant $L > 0$ such that for all $x, y \in D(T)$,

$$||Tx - Ty|| \leq L||x - y||.$$

It is easy to see that a γ-inverse-strongly monotone mapping A is monotone and $\frac{1}{\gamma}$-Lipschitz continuous but converse is not true. In fact, for $x, y \in C$, from Definition 3.1 and $\gamma > 0$ we have

$$\begin{aligned} \langle Ax - Ay, x - y \rangle &\geq \gamma ||Ax - Ay||^2 \\ &\geq 0. \end{aligned}$$

Also, from Cauchy-Schwartz's like inequality and $\gamma > 0$ we have

$$\begin{aligned} \gamma ||Ax - Ax||^2 &\leq \langle Ax - Ay, x - y \rangle \\ &\leq ||Ax - Ay|| ||x - y||. \end{aligned}$$

Thus, $||Ax - Ay|| \leq \frac{1}{\gamma}||x - y||$. However, taking $Ax = \sin x$, $x \in C := [0, 2\pi]$, we see that

$$\begin{aligned} ||Ax - Ay|| &= ||\sin x - \sin y|| \\ &= |\cos a_{x,y}| \, ||x - y|| \qquad (\text{ for some } a_{x,y} \text{ between } x \text{ and } y, \text{ by Mean Value Theorem}) \\ &\leq ||x - y|| \;\; \forall \; x, y \in C. \end{aligned}$$

Thus, A is 1-Lipschitz, i.e., it is nonexpansive. However, A is not monotone (as the sine function is not monotone increasing). Hence, A is not γ-inverse strongly monotone.

Definition 3.4 (Convex function)**.** Let E be a real normed linear space. The function $f : C \to \mathbb{R} \cup \{+\infty\}$, C convex subset of E, is said to be convex if for all $x, y \in E$ and for every $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

3.2 Metric Projection Operator

We define some nonlinear functional and operators.

Metric projection operators in Hilbert and Banach spaces are widely used to solve many problems in different areas of mathematics such as fixed point theory, optimization theory, nonlinear programming, game theory and variational inequalities (see [Chidume *et al.*, 2005], [Singh, 1997], [Das *et al.*, 1981], [Kazmi, 1997]). In Hilbert spaces, these problems have been sufficiently studied and there are many interesting results (see [Deutsch, 2001], [Mhaskar *et al.*, 2000]). But it is difficult to transfer these results into Banach spaces using the metric projection operator because the metric projection operator in Banach spaces does not possess a number of properties which make them so effective in Hilbert spaces. For instance, in a Hilbert space, a metric projection operator is monotone (accretive) and nonexpansive which leads to a variety of applications of this operator in analysis. Now, metric projection operators in Banach space do not have the properties mentioned above although they were actively investigated and used in various applications. In 1994, Ya. I. Alber introduced other kinds of projections to replace the metric projection, which is a natural extension of the classical metric projection in Hilbert spaces ([Alber, 1996]).

Definition 3.5 (Metric Projection)**.** Let $C \subset$ H be a nonempty subset and $x \in H$. If there exists a point $y \in C$ such that

$$\|y-x\| \leq \|z-x\|$$

for any $z \in C$, then y is called a metric projection of x onto C and is denoted by $P_C x$ (see Figure 3.1). That is, the operator $P_C : H \to C \subseteq H$ is called metric projection operator if it yields the correspondence between an arbitrary point $x \in H$ and nearest point $y \in C$ according to minimization problem

$$P_C x = \{y : y \in C,\ \|y-x\| = \inf_{z \in C} \|z-x\|\}.$$

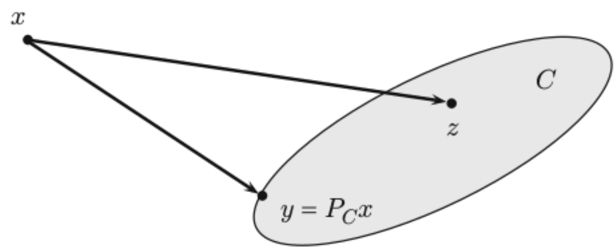

Figure 3.1: Metric Projection

In a Hilbert space H the metric projection operator satisfies the following inequality

$$\|P_C x - x\| \leq \|x-y\| \ \ \forall\ y \in C$$

by definition . Furthermore, we can obtain

$$\|P_C x - P_C y\| \leq \|x-y\| \ \ \forall x, y \in H.$$

It also satisfies a stronger property:

$$\|P_C x - x\|^2 \leq \|x-y\|^2 - \|P_C x - y\|^2 \ \ \forall y \in C.$$

It turns out that in Banach spaces these properties do not hold in general. Alber introduced a new operator which is call *generalized projection map*. First, we introduce the notion of projection defined by Ya. Alber which will be central to all the computation in this thesis. In what follows, we shall denote by $\langle f, x\rangle$ the duality paring of $x \in E$ and $f \in E^*$, i.e., $\langle f, x\rangle = f(x)$. We note that if E is an inner product space, the duality paring becomes the inner product.

Definition 3.6 (Duality map)**.** Let $\langle \cdot, \cdot \rangle$ denote the duality pairing of elements of E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* \mid \langle x^*, x\rangle = \|x\|^2,\ \|x\| = \|x^*\|\},\ \ x \in E.$$

Proposition 3.7. Let E be a real normed space. Then, the duality map $J : E \to 2^{E^*}$ is well defined. That is, for every $x \in E$, $Jx \neq \emptyset$.

Proof. Let $x \in E$. We consider two cases.
Case 1: Suppose $x = 0$. We take $x^* = 0$. Then the argument follows.
Case 2: Suppose $x \neq 0$, then $\|x\|x \neq 0$. As a consequence of the Hahn Banach theorem, there exists $y^* \in E^*$ such that $\|y^*\| = 1$ and $\langle y^*, \|x\|x\rangle = \big\|x\|x\|\big\| = \|x\|^2$. Now

$$\langle \|x\|y^*, x\rangle = \langle y^*, \|x\|x\rangle = \|x\|^2.$$

Take $x^* = \|x\|y^* \in E^*$. Then, $x^* \in Jx$. Hence, $Jx \neq \emptyset\ \ \forall\ x \in E$. □

Definition 3.8 (Reflexive)**.** Let E be a Banach space and let $G : E \to E^{**}$ be the canonical injection from E into E^{**}, that is $\langle Gx, f\rangle = \langle f, x\rangle,\ \ \forall\ x \in E,\ f \in E^*$. Then, E is said to be reflexive if G is subjective, i.e., $G(E) = E^{**}$.

Definition 3.9 (Smooth space)**.** A normed space E is called smooth if and only if for all $x \in E$ with $\|x\| = 1$, there exists a unique $x^* \in E^*$ such that $\|x^*\| = 1$ and $\langle x,\ x^*\rangle = \|x\|$.

Equivalently a normed space E is smooth if the

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3.5}$$

exists for all $x, y \in U$, where $U = \{x \in E : \|x\| = 1\}$.

Definition 3.10 (Uniformly smooth space)**.** A normed space E is said to be uniformly smooth if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x\| = 1$ and $\|y\| \leq \delta$, then

$$\|x + y\| + \|x - y\| < 2 + \epsilon\|y\|.$$

Equivalently a normed space E is uniformly smooth if (3.5) is attained uniformly in $x, y \in U$.

Definition 3.11 (Modulus of smoothness)**.** Let E be a normed linear space with $\dim(E) \geq 2$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\begin{aligned} \rho_E(\tau) \ &:= \ \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau\right\} \\ &= \ \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = 1; \|y\| = 1\right\}. \end{aligned}$$

Definition 3.12 (Strictly convexity)**.** A normed space E is said to be strictly convex if for any $x, y \in E$, $x \neq y$, $\|x\| = \|y\| = 1$ we have that $\|\lambda x + (1-\lambda)y\| < 1 \; \forall \; \lambda \in (0,1)$.

Definition 3.13 (Uniformly convexity)**.** A normed space E is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\left\|\frac{x+y}{2}\right\| < 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex.

Definition 3.14 (Modulus of convexity)**.** A function $\delta : [0,2] \to [0,1]$ called the modulus of convexity of E is defined as follows:

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in U, \|x - y\| \geq \epsilon\right\}.$$

Using this idea of modulus convexity, one can define uniform convexity of a normed linear space. In fact, a normed linear space E is uniformly convex if and only if $\delta(\epsilon) > 0$ for all $\epsilon \in (0,2]$.

Remark 3.15. Geometrically, a normed space E is uniformly convex if and only if the unit ball centred at the origin is "uniformly round". We list some examples of uniformly convex spaces.

1. Let E be the Cartesian plane, $\mathbb{R}^2$ with the norm defined for each $x = (x_1, x_2) \in \mathbb{R}^2$ by $\|x\|_2 = \left[|x_1|^2 + |x_2|^2\right]^{\frac{1}{2}}$. Then $\mathbb{R}^2$ endowed with this norm is uniformly convex. But the space $\mathbb{R}^2$ defined for each $x = (x_1, x_2) \in \mathbb{R}^2$ by $\|x\|_1 = |x_1| + |x_2|$ and $\|x\|_\infty = \max\{|x_1|, |x_2|\}$ are not uniformly convex.

2. Every real inner product space H is uniformly convex (see e.g., [Chidume, 2009]).

3. $L_p\,(or\; l_p)$ spaces, $1 < p < \infty$, are uniformly convex.

Some of the properties of modulus of convexity are:

1 The modulus of convexity δ_E is a non-decreasing function.

2 The modulus of convexity is continuous (see [Gurarri, 1967]).

Remark 3.16. Properties of the normalized duality map in different Banach spaces (see [Takahashi, 2000], [Vainberg, 1973] and [Chidume, 2009]).

1. For any $x \in E$, $J(x)$ is nonempty, bounded, closed and convex.

2. J is a homogeneous operator in arbitrary Banach space E, that is , for any $x \in E$ and a real number α,

$$J(\alpha x) = \alpha J(x).$$

3. J is a monotone operator in arbitrary Banach space E, that is, for any $x, y \in E$, $k \in J(x)$ and $l \in J(y)$,

$$\langle k - l, x - y \rangle \geq 0.$$

4. If E is smooth, then J is a single-valued mapping.

5. If E is reflexive, then J is a map of E onto E^*.

6. If E is uniformly smooth, then J a is norm-to-norm uniformly continuous on each bounded subset of E.

7. If E is strictly convex, then J is one-to-one, that is,$x \neq y \Rightarrow J(x) \cap J(y) = \emptyset$.

8. J is the identity operator in Hilbert spaces.

9. If $E = L_p$ space $(2 \leq p < \infty)$, then $J : L_p \to L_p^*$ is Lipschitz.

10. If $E = L_p$ space $(1 < p < 2)$, then $J : L_p \to L_p^*$ is Hölder continuous.

11. If E is reflexive and strictly convex Banach with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$.

We have the following lattice below which shows the properties of J on different normed linear spaces.

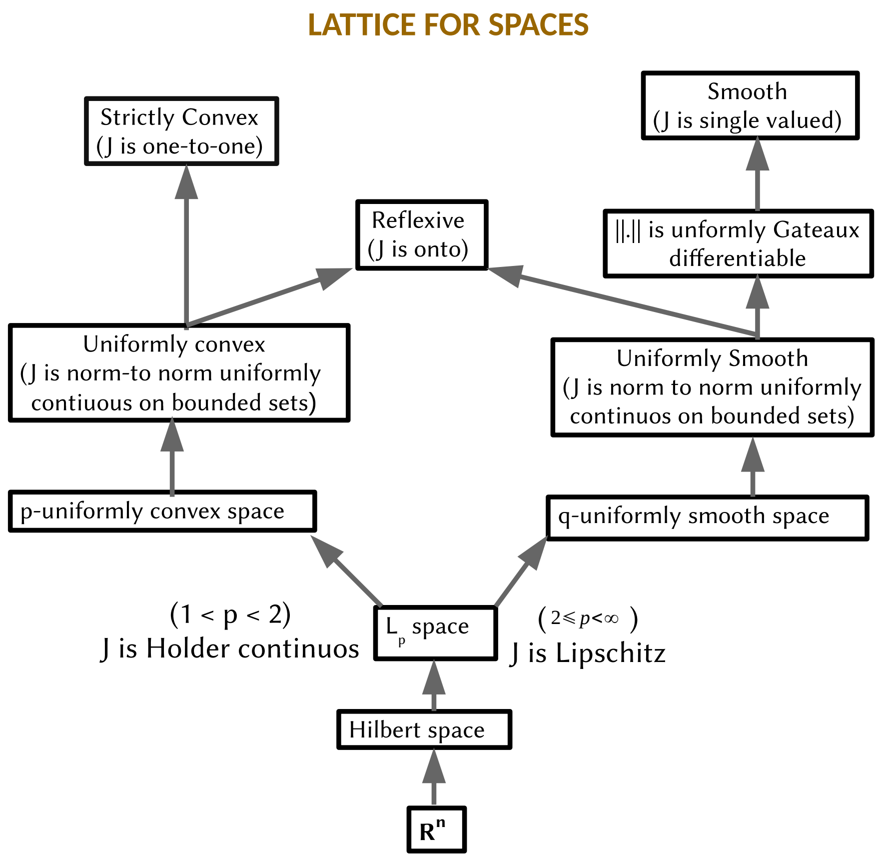

Figure 3.2: Lattice for Spaces

We are now ready to define the generalized duality map due to Alber [Alber, 1996].

Let E be a smooth real Banach space. We define the following Lyapunov functional by:

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \ \ \forall\ x,y \in E, \tag{3.6}$$

where J is the normalized duality mapping from E into E^*. This map has been studied by Alber and Guerre-Delabriere [Alber *et al.*, 2001].

Remark 3.17. From the definition of the Lyapunov function ϕ we have the following properties;

1. If $E = H$, a real Hilbert space, then equation (3.6) reduces to $\phi(x,y) = \|x-y\|^2$ for $x,y \in H$.

2. For all $x,\ y \in E$,
$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2. \tag{3.7}$$

3. For all $x,\ y,\ z \in E$
$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz - Jy\rangle. \tag{3.8}$$

Proof. Let $x,y \in E$. Using definition of ϕ and Cauchy-Schwartz's like inequality we have

$$\begin{aligned} \phi(x,y) &= \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Also, using $\langle x, Jy\rangle \leq \|x\|\|y\|$, we have

$$\begin{aligned} \phi(x,y) &= \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \\ &\geq \|x\|^2 - 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| - \|y\|)^2. \end{aligned}$$

Hence, $(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2$. Next, by expanding the RHS of equation (3.8), we have

$$\begin{aligned} \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz - Jy\rangle &= \|x\|^2 - 2\langle x, Jz\rangle + \|z\|^2 + \|z\|^2 - 2\langle z, Jy\rangle + \|y\|^2 \\ &\quad +2\langle x-z, Jz\rangle - 2\langle x-z, Jy\rangle \\ &= \|x\|^2 - 2\langle x, Jz\rangle + \|z\|^2 + \|z\|^2 - 2\langle z, Jy\rangle + \|y\|^2 \\ &\quad +2\langle x, Jz\rangle - 2\langle z, Jz\rangle - 2\langle x, Jy\rangle + 2\langle z, Jy\rangle \\ &= \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle = \phi(x,y). \end{aligned}$$

□

Lemma 3.18. Let E be a strictly convex and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$.

Proof. Let $x, y \in E$. We show first show that if $\phi(x,y) = 0$ then $x = y$. From equation (3.7), we have that $0 \leq (\|x\| - \|y\|)^2 \leq \phi(x,y) = 0 \Rightarrow \|x\| = \|y\|$. Then

$$\begin{aligned}\phi(x,y) &= \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 = 0 \\ &= 2\|x\|^2 - 2\langle x, Jy\rangle = 0\end{aligned}$$

This implies $\langle x, Jy\rangle = \|x\|^2 = \|y\|^2$. From the definition of J, we have $Jx = Jy$. Using the fact that J is strictly convex, we have that J is one-to-one, hence $x = y$. We can easily see that if $x = y$ then by definition of ϕ, we have that $\phi(x,y) = 0$. □

Lemma 3.19. Let E be a reflexive, strictly convex and smooth real Banach space and C be a nonempty, closed and convex subset of E. For each $x \in E$, there exists a unique element $z_x \in C$ such that

$$\phi(z_x, x) = \min_{y \in C} \phi(y, x).$$

To prove the Lemma, we use the following result:

Lemma 3.20 ([Chidume, 2009])**.** Let E be a reflexive real Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex proper lower semi-continuous function. Suppose $\lim\limits_{\|x\| \to \infty} f(x) = +\infty$. Then $\exists\ \bar{x} \in C$ such that $f(\bar{x}) \leq f(x)\ \forall\ x \in E$, i.e.,

$$f(\bar{x}) = \inf_{x \in E} f(x) = \min_{x \in E} f(x).$$

Proof. Let C be a closed, convex and nonempty subset of a reflexive real Banach space E and let $\phi_x : C \to \mathbb{R}$ defined by

$$\phi_x(y) = \phi(y, x) \ \ \forall\ y \in C.$$

To show existence, we first show that the function ϕ_x is convex and lower semi-continuous.
Let $y_1, y_2 \in C$ and $\lambda \in (0,1)$. We want to show that

$$\phi_x(\lambda y_1 + (1-\lambda) y_2) \leq \lambda \phi_x(y_1) + (1-\lambda)\phi_x(y_2).$$

Let $x \in E$. Since E is strictly convex, $\|\cdot\|^2$ is a strictly convex function. Therefore, we have

$$\begin{aligned}\phi_x(\lambda y_1 + (1-\lambda)y_2) &= \phi(\lambda y_1 + (1-\lambda y_2, x) \\ &= \|\lambda y_1 + (1-\lambda)y_2\|^2 - 2\langle \lambda y_1 + (1-\lambda)y_2, Jx\rangle + \|x\|^2 \\ &< \lambda\|y_1\|^2 + (1-\lambda)\|y_2\|^2 - 2\langle \lambda y_1, Jx\rangle - 2\langle (1-\lambda)y_2, Jx\rangle + \|x\|^2 \\ &= \lambda\|y_1\|^2 + (1-\lambda)\|y_2\|^2 - 2\langle \lambda y_1, Jx\rangle - 2\langle (1-\lambda)y_2, Jx\rangle + \lambda\|x\|^2 + (1-\lambda)\|x\|^2 \\ &= \lambda\phi(y_1, x) + (1-\lambda)\phi(y_2, x) \\ &= \lambda\phi_x(y_1) + (1-\lambda)\phi_x(y_2).\end{aligned}$$

Hence, the function ϕ_x is convex, in fact strictly convex. Next we show lower semi-continuity. It suffices to show that the function ϕ_x is continuous. Let $(y_n)_n \subseteq E$ such that $y_n \to y$. We want to show that $\phi_x(y_n) \to \phi_x(y)$ as $n \to \infty$. By definition we have $\phi_x(y_n) = \phi(y_n, x) = \|y_n\|^2 - 2\langle y_n, Jx\rangle + \|x\|^2$. Using the fact that $\|\cdot\|^2$ and duality paring are continuous, taking limit as $n \to \infty$, we have

$$\begin{aligned}
\|y_n\|^2 - 2\langle y_n, Jx\rangle + \|x\|^2 &\to \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \\
&= \phi(y,x) = \phi_x(y).
\end{aligned}$$

Hence, the function ϕ_x is continuous which implies that it is lower semi-continuous. Secondly, we show that the function ϕ_x is coercive. By inequality (3.7) i.e., $\phi_x(y) = \phi(y,x) \geq (\|y\| - \|x\|)^2 \ \forall\ x, y \in E$. As $\|y\| \to \infty$ we have that $\phi(y,x) \to \infty$. This implies that ϕ_x is coercive. Clearly, ϕ_x is proper (in fact it is real-valued). Therefore, by Lemma 3.20 we have that there exists $y^* \in C$ such that $\phi_x(y^*) \leq \phi_x(y) \quad \forall y \in C$.

For uniqueness: suppose there exists $y_1, y_2 \in C$ such that $y_1 \neq y_2$ and $\phi_x(y_1) = \phi_x(y_2) \leq \phi_x(y)$ $\forall\ y \in E$. Then, by strict convexity of ϕ_x, we have

$$\begin{aligned}
\phi_x(y_1) = \phi(y_1, x) &\leq \phi(\lambda y_1 + (1-\lambda) y_2, x) \\
&< \lambda\phi(y_1, x) + (1-\lambda)\phi(y_2, x) \\
&= \phi(y_1, x)
\end{aligned}$$

a contradiction. Hence $y_1 = y_2$. This implies that it is unique. □

Definition 3.21 (Generalized projection of Alber [Alber, 1996])**.** The map $\Pi_C : E \to C$, defined by $\Pi_C x = z_x$, is called the generalized projection map from E onto C .

Remark 3.22. In Hilbert space, $\Pi_C = P_C$.

Define a map $V : E \times E^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2.$$

If E is reflexive and strictly convex Banach with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*$. That is J^{-1} exists. Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}(x^*)) \ \ \forall\ x \in E,\ x^* \in E^*. \tag{3.9}$$

Proof. Let $x \in E$ and $x^* \in E^*$.
Using definition and the fact that J^{-1} is a duality map, i.e., $\|J^{-1}(x^*)\| = \|x^*\|$, we have

$$\begin{aligned}
V(x, x^*) &= \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \\
&= \|x\|^2 - 2\langle x, J(J^{-1}(x^*))\rangle + \|J^{-1}(x^*)\|^2 \\
&= \phi(x, J^{-1}(x^*)).
\end{aligned}$$

$\square$

Using the definition of V above, Alber proved the following lemma which we shall use in the sequel.

Lemma 3.23 (Alber, [Alber, 1996])**.** Let E be a reflexive, strictly convex and smooth real Banach space with E^* as its dual. Then,

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \le V(x, x^* + y^*) \tag{3.10}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Now, we describe the properties of the operator Π_C:

Lemma 3.24 ([Alber, 1996], [Kamimura *et al.*, 2002])**.** Let C be a nonempty closed and convex subset of a smooth real Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0 \rangle \ge 0, \ \forall y \in C$.

Lemma 3.25. Let E be a reflexive, strictly convex and smooth real Banach space and C be a nonempty closed and convex subset of E. Then for any $x \in E$,

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x), \ \ \forall \ y \in C. \tag{3.11}$$

Proof. By definition Lemma 3.24 and putting $x_0 = \Pi_C x$, we have

$$\begin{aligned}
\phi(y,x) - \phi(x_0,x) - \phi(y,x_0) &= ||y||^2 - 2\langle y, Jx\rangle + ||x||^2 - ||x_0||^2 + 2\langle x_0, Jx\rangle - ||x||^2 \\
&\quad -||y||^2 + 2\langle y, Jx_0\rangle - ||x_0||^2 \\
&= -2\langle y, Jx\rangle - ||x_0||^2 + 2\langle x_0, Jx\rangle + 2\langle y, Jx_0\rangle - ||x_0||^2 \\
&= -2\langle y - x_0, Jx\rangle + 2\langle y, Jx_0\rangle - 2||x_0||^2 \\
&= -2\langle y - x_0, Jx\rangle + 2\langle y, Jx_0\rangle - 2\langle x_0, Jx_0\rangle \\
&= -2\langle y - x_0, Jx\rangle + 2\langle y, Jx_0\rangle + 2\langle y - x_0, Jx_0\rangle - 2\langle y, Jx_0\rangle \\
&= \langle y - x_0, Jx_0 - Jx\rangle \ge 0 \ \ \forall y \in C.
\end{aligned}$$

Hence, $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x), \ \ \forall \ y \in C.$ $\square$

Remark 3.26. The operator Π_C is fixed in each point $y \in C$, i.e., $\Pi_C y = y$.

3.2.1 Calculating the projection onto a closed convex set in Hilbert spaces

Iterative algorithms involving projection onto closed, convex sets abound in the literature. While these algorithms can be shown to converge strongly to the desired points, implementation of these algorithms can be very difficult when the convex set is arbitrary. For this reason, much effort has been made to replace arbitrary convex sets with, for example, half-spaces. This is because projection onto half-spaces can be computed with ease.

In this section, we give formulas that can be used to calculate projections onto a half-space. In this thesis, convergence of the algorithm is established using a special choice of half-space.

Example 3.27. Suppose that u is a non-zero vector in H and $\eta \in \mathbb{R}$. We set $C = \{x \in H : \langle x, u\rangle = \eta\}$. Then C is convex, closed and nonempty. Indeed, for $\frac{\eta u}{\|u\|^2} \in C$ and continuity of inner product together with its linearity in the first component makes C closed and convex respectively. For this set C, we have (see, for example [Heinz et al., 2011])

$$P_C x = x + \frac{\eta - \langle x, u\rangle}{\|u\|^2} u.$$

In the next example, we provide a closed-form expression for the projection onto a half space.

Example 3.28. Let $u \in H$, $\eta \in \mathbb{R}$, $u \neq 0$ and set $C = \{x \in H : \langle x, u\rangle \leq \eta\}$. As in the example above, C is a closed, convex and nonempty subset of H. In this case, we have (see, for example [Heinz et al., 2011])

$$(\forall\ x \in H)\ \ P_C x = \begin{cases} x, & if \langle x, u\rangle \leq \eta; \\ x + \frac{\eta - \langle x, u\rangle}{\|u\|^2} u, & if \langle x, u\rangle > \eta. \end{cases}$$

Lemma 3.29. Let E be a 2-uniformly convex and smooth real Banach space. Then, for every $x, y \in E$, $\phi(x, y) \geq c_1\|x - y\|^2$, where $c_1 > 0$.

Lemma 3.30 ([Kamimura *et al.*, 2002])**.** Let E be a real smooth and uniformly convex Banach space, and let$\{y_n\}$ and $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.

Next we define a relatively nonexpansive mapping.

Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive real Banach space E and T be a map from C into itself. We recall that a point $x \in C$ is said to be a fixed point of T if $Tx = x$. We denote the set of fixed points of T by $F(T)$. A point $p \in C$ is said to be an asymptotic fixed point of T if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. Following Matsushita and Takahashi [Matsushita *et al.*, 2004], a map T of C into itself is said to be relatively nonexpansive if the following conditions are satisfied:

(i) $F(T)$ is nonempty;

(ii) $\phi(u, Tx) \leq \phi(u, x)\ \ \forall\ u \in F(T),\ x \in C$;

(iii) $\hat{F}(T) = F(T)$.

Lemma 3.31. Let E be a strictly convex and smooth real Banach space and C be a closed convex subset of E. Let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.

Proof. We first show that $F(T)$ is closed. Let $(x_n)_n \subseteq F(T)$ such that $x_n \to x^*$ as $n \to \infty$. We want to show that $x^* \in F(T)$. Using the fact the T is relatively nonexpansive, we have

$$\phi(x_n, Tx^*) \leq \phi(x_n, x^*) \ \forall x_n \in F(T) \ \forall \ n \in \mathbb{N}.$$

This implies that, using the fact that ϕ is continuous in the first component, we have

$$\begin{aligned} \phi(x^*, Tx^*) &= \lim_{n\to\infty} \phi(x_n, Tx^*) \\ &\leq \lim_{n\to\infty} \phi(x_n, x^*) \\ &= \phi(x^*, x^*) \\ &= 0. \end{aligned}$$

By Lemma 3.18, we get $x^* = Tx^*$. So we have $x^* \in F(T)$. Next, we show that $F(T)$ is convex. Let $x, y \in F(T)$ and $t \in (0,1)$, we put $k = tx + (1-t)y$. We show that $k \in F(T)$, i.e., $Tk = k$. Let $z \in T(k)$. Then, we have

$$\begin{aligned} \phi(k,z) &= ||k||^2 - 2\langle k, Jz\rangle + ||z||^2 \\ &= ||k||^2 - 2\langle tx + (1-t)y, Jz\rangle + ||z||^2 \\ &= ||k||^2 - 2t\langle x, Jz\rangle - 2(1-t)\langle y, Jz\rangle + ||z||^2 \\ &= ||k||^2 + t||x||^2 - t||x||^2 - 2t\langle x, Jz\rangle + (1-t)||y||^2 - (1-t)||y||^2 - 2(1-t)\langle y, Jz\rangle \\ &\quad +t||z||^2 + (1-t)||z||^2 \\ &= ||k||^2 + t\Big(||x||^2 - 2\langle x, Jz\rangle + ||z||^2\Big) + (1-t)\Big(||y||^2 - 2\langle y, Jz\rangle + ||z||^2\Big) - t||x||^2 - (1-t)||y||^2 \\ &= ||k||^2 + t\phi(x,z) + (1-t)\phi(y,z) - t||x||^2 - (1-t)||y||^2 \\ &\leq ||k||^2 + t\phi(x,k) + (1-t)\phi(y,k) - t||x||^2 - (1-t)||y||^2 \\ &= ||k||^2 + t\Big(||x||^2 - 2\langle x, Jk\rangle + ||k||^2\Big) + (1-t)\Big(||y||^2 - 2\langle y, Jk\rangle + ||k||^2\Big) - t||x||^2 - (1-t)||y||^2 \\ &= ||k||^2 - 2\langle tx, Jk\rangle - 2\langle (1-t)y, Jk\rangle + t||k||^2 + (1-t)||k||^2 \\ &= ||k||^2 - 2\langle tx + (1-t)y, Jk\rangle + ||k||^2 \\ &= ||k||^2 - 2\langle k, Jk\rangle + ||k||^2 \\ &= 0. \end{aligned}$$

By Lemma 3.18, we obtain $k = z$. Hence, $k = T(k)$. So, $k \in F(T)$. Therefore $F(T)$ is convex. $\square$

It is known that the generalized projection Π_C of E onto C is relatively nonexpansive if E is smooth, strictly convex, and reflexive.

We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \ \forall \ y \in C.$$

Lemma 3.32 ([Rockafellar, 1970])**.** Let C be a nonempty closed convex subset of a real Banach space E and A be a monotone and hemicontinuous map from C into E^* with C = D(A). Let T be a map defined by:

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{3.12}$$

Then, T is maximal monotone and $T^{-1}(0) = VI(C, \ A)$.

Lemma 3.33 ([Kohsaka *et al.*, 2008])**.** Let C be a closed convex subset of a uniformly smooth and 2-uniformly convex Banach space E and $(S_i)_{i=1}^{\infty}$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $(\eta_i)_{i=1}^{\infty} \subset (0,1)$ and $(\mu_i)_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Consider the map $T : C \to E$ defined by

$$Tx = J^{-1}\left(\sum_{i=1}^{\infty} \eta_i \Big(\mu_i Jx + (1 - \mu_i) JS_i x\Big)\right), \text{ for each } x \in C. \tag{3.13}$$

Then, T is relatively nonexpansive and $F(T) = \bigcap\limits_{i=1}^{\infty} F(S_i)$.

CHAPTER 4

Main Result

4.1 Introduction

In this chapter, we construct an iterative sequence which converges strongly to a point common to the set of fixed points of a relatively nonexpansive mapping U and the solution set of a variational inequality problems for a monotone and Lipschitz continuous mapping A.

In what follows, except if stated otherwise, E is a 2-uniformly convex and uniformly smooth real Banach space with dual space E^* and 2-uniform convexity constant c_1. Also C is a nonempty closed convex subset of E, $A : C \to E^*$ is monotone and Lipschitz continuous on C with Lipschitz constant $L > 0$, $U : C \to C$ is relatively nonexpansive and $F(U) \cap VI(C, A) \neq \emptyset$.

4.2 Convergence theorem

We shall study the following algorithm.

$$\begin{cases} x_0 \in C_0 = C, \\ y_n = \Pi_C J^{-1}(Jx_n - \mu A x_n), \\ T_n = \{x \in E : \langle Jx_n - \mu A x_n - J y_n, x - y_n \rangle \leq 0\}, \\ z_n = \Pi_{T_n} J^{-1}(Jx_n - \mu A y_n), \\ w_n = J^{-1}\Big((1-\alpha)Jx_n + \alpha J U z_n)\Big) \\ C_{n+1} = \Big\{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n) - \alpha c\,(\phi(z_n, y_n) + \phi(y_n, x_n))\Big\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{4.1}$$

where $\alpha \in (0,1)$, μ and c are positive constants.

Remark 4.1. We show that $\{x_n\}$ generated by the algorithm is well-defined. We observe that $C \subseteq T_n$. To see this, let $y \in C$, we show that $y \in T_n$. From Algorithm 4.1, $y_n = \Pi_C J^{-1}(Jx_n - \mu A x_n)$ implies that $\langle x - y_n, Jy_n - Jx_n + \mu A x_n\rangle \geq 0 \ \forall \ x \in C$. In particular for $y = x$ we get $\langle y - y_n, Jy_n - Jx_n + \mu A x_n\rangle \geq 0$, this implies that $y \in T_n$. Hence $C \subseteq T_n$. Thus, the half-space T_n is nonempty, closed and convex. Also we show that C_n is closed, convex and nonempty.

Claim: C_n is closed and convex for all $n \geq 0$.

Proof of Claim: To show that C_n is convex $\forall \ n \geq 0$. We proceed by induction. Clearly, for $n = 0$, $C_n = C$ is convex. Suppose C_n is convex for some $n \geq 0$. We show that C_{n+1} is convex.

From Algorithm 4.1, $C_{n+1} = \Big\{z \in C_n : \phi(z, w_n) \leq \phi(z, x_n) - \alpha c\,(\phi(z_n, y_n) + \phi(y_n, x_n))\Big\}$ which is equivalent to $\Big\{z \in C_n : 2\langle z, Jx_n - Jw_n\rangle \leq -\alpha c\,(\phi(z_n, y_n) + \phi(y_n, x_n)) + \|x_n\|^2 - \|w_n\|^2\Big\}$.

Let $x, y \in C_{n+1}$ and $\lambda \in [0,1]$. We show that $\lambda x + (1-\lambda)y \in C_{n+1}$.

$$\begin{aligned}
2\langle \lambda x + (1-\lambda)y, Jx_n - Jw_n\rangle &= 2\langle \lambda x, Jx_n - Jw_n\rangle + 2\langle (1-\lambda)y, Jx_n - Jw_n\rangle \\
&= 2\lambda\langle x, Jx_n - Jw_n\rangle + 2(1-\lambda)\langle y, Jx_n - Jw_n\rangle \\
&\leq \lambda\Big[-\alpha c\,(\phi(z_n, y_n) + \phi(y_n, x_n)) + \|x_n\|^2 - \|w_n\|^2\Big] \\
&\quad +(1-\lambda)\Big[-\alpha c\,(\phi(z_n, y_n) + \phi(y_n, x_n)) + \|x_n\|^2 - \|w_n\|^2\Big] \\
&= -\alpha c\,(\phi(z_n, y_n) + \phi(y_n, x_n)) + \|x_n\|^2 - \|w_n\|^2
\end{aligned}$$

Hence, we have that C_n is convex for all $n \geq 0$. Next we show that C_n is closed $\forall \ n \geq 0$. We proceed by induction. Clearly, for $n = 0$, $C_n = C$ is closed. Suppose C_n is closed for some $n \geq 0$, we show that C_{n+1} is closed. Let $(v_n)_n \subseteq C_{n+1}$ such that $v_n \to \bar{v}$ as $n \to \infty$. It suffices to show that $\bar{v} \in C_{n+1}$. Now $v_n \in C_{n+1}$ implies $\phi(v_n, w_n) \leq \phi(v_n, x_n) - \alpha c\,(\phi(z_n, y_n) + \phi(y_n, x_n))$. But ϕ is continuous in the first component, so taking limit as $n \to \infty$, we have $\phi(\bar{v}, w_n) \leq \phi(\bar{v}, x_n) - \alpha c\,(\phi(z_n, y_n) + \phi(y_n, x_n))$. Hence $\bar{v} \in C_{n+1}$. Therefore, C_n is closed $\forall \ n \geq 0$. Thus, C_n is convex and closed for all $n \geq 0$. Therefore $\{x_n\}$ is well-defined as C_n is closed, convex and nonempty $(\emptyset \neq \Omega \subset C_n)$.

The following Lemma will be used in what follows.

Lemma 4.2. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences generated by (4.1). Then,

$$\phi(p, z_n) \leq \phi(p, x_n) - \left(1 - \frac{\mu L}{c_1}\right)[\phi(z_n, y_n) + \phi(y_n, x_n)] \ \ \forall \ p \in VI(C,\ A). \tag{4.2}$$

Proof. Let $p \in F(U) \cap VI(C,\ A)$. Since $z_n = \Pi_{T_n} J^{-1}(Jx_n - \mu A y_n)$, using Lemma 3.25 and the definition of ϕ, we estimate as follows

$$\begin{aligned}
\phi(p,z_n) &= \phi(p,\Pi_{T_n}J^{-1}(Jx_n-\mu Ay_n)) \\
&\leq \phi(p,J^{-1}(Jx_n-\mu Ay_n))-\phi(z_n,J^{-1}(Jx_n-\mu Ay_n)) \\
&\leq \phi(p,x_n)-\phi(z_n,x_n)+2\mu\langle p-z_n,Ay_n\rangle \\
&\leq \phi(p,x_n)-\phi(z_n,x_n)+2\mu\langle y_n-z_n,Ay_n\rangle. \qquad (4.3)
\end{aligned}$$

Thus, from definition of ϕ, (3.8), Lemma 3.24, Lipschitz continuity of A and Lemma 3.29, we have

$$\begin{aligned}
\phi(z_n,x_n)-2\mu\langle y_n-z_n,Ayn\rangle &= \phi(z_n,y_n)+\phi(y_n,x_n)+2\langle y_n-z_n,Jx_n-Jy_n\rangle-2\mu\langle y_n-z_n,Ay_n\rangle \\
&= \phi(z_n,y_n)+\phi(y_n,x_n)-2\langle z_n-y_n,Jx_n-\mu Ay_n-Jy_n\rangle \\
&\geq \phi(z_n,y_n)+\phi(y_n,x_n)-2\mu\langle z_n-y_n,Ax_n-Ay_n\rangle \\
&\geq \phi(z_n,y_n)+\phi(y_n,x_n)-2\mu\|z_n-y_n\|\|Ax_n-Ay_n\| \\
&\geq \phi(z_n,y_n)+\phi(y_n,x_n)-2L\mu\|z_n-y_n\|\|x_n-y_n\| \\
&\geq \phi(z_n,y_n)+\phi(y_n,x_n)-L\mu(\|z_n-y_n\|^2+\|x_n-y_n\|^2) \\
&\geq \phi(z_n,y_n)+\phi(y_n,x_n)-\frac{\mu L}{c_1}(\phi(z_n,y_n)+\phi(y_n,x_n)) \\
&= c(\phi(z_n,y_n)+\phi(y_n,x_n)), \qquad (4.4)
\end{aligned}$$

where $c=1-\frac{\mu L}{c_1}$.
From inequalities (4.3) and (4.5), we have

$$\phi(p,z_n)\leq\phi(p,x_n)-c(\phi(z_n,y_n)+\phi(y_n,x_n)). \qquad (4.5)$$

□

We now prove the following theorem.

Theorem 4.3. *Let E be a 2-uniformly convex, uniformly smooth real Banach space and C be a nonempty closed convex subset of E. Let $U:C\to C$ be a relatively nonexpansive mapping and $A:C\to E^*$ be a monotone and L-Lipschitz mapping on C. Let μ be a real number satisfying $\mu<\frac{c_1}{L}$. Suppose that $F(U)\cap VI(C,\,A)$ is nonempty. Then the sequences $\{x_n\},\{y_n\},\{z_n\}$ generated by Algorithm 4.1 converge strongly to $\Pi_{F(U)\cap VI(C,\,A)}x_0$.*

Proof. We divide our proof into these steps.
Step 1: We show that $\Omega=F(U)\cap VI(C,\,A)\subset C_n$, for all $n\geq 0$. We proceed by induction. For $n=0$, we have that $\Omega\subset C_n$. Suppose $\Omega\subset C_n$ for some $n\geq 0$. We show that $\Omega\subset C_{n+1}$. Let $p\in\Omega$, then

using the fact that U is relatively nonexpansive and Lemma 4.2, we have that

$$\begin{aligned}
\phi(p, w_n) &= \phi(p, J^{-1}((1-\alpha)Jx_n + \alpha JUz_n)) \\
&= V(p, (1-\alpha)Jx_n + \alpha JUz_n) \\
&\leq (1-\alpha)\phi(p, x_n) + \alpha\phi(p, Uz_n) \\
&\leq (1-\alpha)\phi(p, x_n) + \alpha\,[\phi(p, x_n) - c(\phi(z_n, y_n) + \phi(y_n, x_n))] \\
&= \phi(p, x_n) - \alpha c(\phi(z_n, y_n) + \phi(y_n, x_n)).
\end{aligned}$$

This implies that $p \in C_{n+1}$. Hence $\Omega \subset C_n$, $\forall n \geq 0$.

Step 2: We show that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists.
Since $x_n = \Pi_{C_n} x_0$ and $\Omega \subset C_n$ $\forall$ $n \geq 0$, then using Lemma 3.25 , we have that for any $p \in \Omega$

$$\begin{aligned}
\phi(x_n, x_0) &\leq \phi(p, x_0) - \phi(p, x_n) \\
&\leq \phi(p, x_0). \qquad (4.6)
\end{aligned}$$

It follows that the sequence $\{\phi(x_n, x_0)\}$ is bounded and so by inequality (3.7) $\{x_n\}$ is bounded. Since $C_{n+1} \subset C_n$ $\forall$ $n \geq 0$ and $x_n = \Pi_{C_n} x_0$, we obtain that for $x_{n+1} \in C_{n+1}$

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0). \qquad (4.7)$$

Therefore, $\{\phi(x_n, x_0)\}$ is monotone nondecreasing and bounded above by $\phi(p, x_0)$. Hence, $\lim_{n\to\infty} \phi(x_n, x_0)$ exists.

Step 3: We show $\{x_n\}$ converges to $\Pi_{F(U)\cap VI(C,\,A)} x_0$.
Using the fact that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} \in C_n$, we have that for $m > n$,

$$\phi(x_m, x_n) \leq \phi(x_m, x_0) - \phi(x_n, x_0), \qquad (4.8)$$

this implies $\lim_{n\to\infty} \phi(x_m, x_n) = 0$ and by Lemma 3.30 , we have

$$\|x_n - x_m\| \to 0 \text{ as } n, m \to \infty. \qquad (4.9)$$

Hence $\{x_n\}$ is Cauchy which implies that there exists $x^* \in E$ such that $x_n \to x^*$ as $n \to \infty$. Since $\{x_n\}_{n\geq 1}$ is in C and C is closed, then $x^* \in C$.
So, from equation (4.9), we have for $m = n+1$. We get $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$.

Again, since $x_{n+1} \in C_{n+1}$, we have

$$\begin{aligned}
\phi(x_{n+1}, w_n) &\leq \phi(x_{n+1}, x_n) - \alpha c\Big(\phi(z_n, y_n) + \phi(y_n, x_n)\Big) \qquad (4.10) \\
&\leq \phi(x_{n+1}, x_n) \to 0 \text{ as } n \to \infty
\end{aligned}$$

and by Lemma 3.30, we have $\|x_{n+1} - w_n\| \to 0$ as $n \to \infty$.

Thus,

$$\|x_n - w_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \to 0.$$

We have $\|x_n - w_n\| \to 0$ as $n \to \infty$.

Similarly, using the fact that $x_{n+1} \in C_{n+1}$ and using inequality (4.10), we have that

1. $\phi(z_n, y_n) \to 0$ as $n \to \infty$ which implies by Lemma 3.30 that $\|z_n - y_n\| \to 0$ as $n \to \infty$.

2. $\phi(y_n, x_n) \to 0$ as $n \to \infty$ which implies by Lemma 3.30 that $\|y_n - x_n\| \to 0$ as $n \to \infty$.

Also,

$$\|x_n - z_n\| \leq \|x_n - y_n\| + \|y_n - z_n\| \to 0$$

i.e., $\|x_n - z_n\| \to 0$ as $n \to \infty$. Now, using the fact that J is norm-to-norm uniformly continuous on bounded sets and the fact that $\|x_n - w_n\| \to 0$, we have $\|Jx_n - Jw_n\| \to 0$, which implies that

$$\|Jx_n - JUz_n\| = \frac{1}{|\alpha|}\|Jw_n - Jx_n\| \to 0, \ \text{ as } n \to \infty.$$

By the $\|\cdot\|$ to $\|\cdot\|$ uniform continuity of J^{-1} on bounded sets, we have

$$\|x_n - Uz_n\| \to 0, \text{ as } n \to \infty.$$

Thus,

$$\|z_n - Uz_n\| \leq \|z_n - x_n\| + \|x_n - Uz_n\| \to 0, \ \text{ as } n \to \infty.$$

Therefore,

$$\|z_n - Uz_n\| \to 0, \text{ as } n \to \infty. \tag{4.11}$$

We now show that $x^* \in F(U) \cap VI(C, A)$. It suffices to show that $x^* \in F(U)$ and $x^* \in VI(C, A)$. But since $x_n \to x^*$ and $\|x_n - z_n\| \to 0$, we have that $z_n \to x^*$. Hence, using the fact that U is relatively nonexpansive and from $\|z_n - Uz_n\| \to 0$, $x^* \in F(U)$. Next, we show that $x^* \in VI(C, A)$.

From Lemma 3.32 we have that the map $T : E \to 2^{E^*}$ defined by

$$Tv = \begin{cases} Av + N_C(v), & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases}$$

where $N_C(v)$ is the normal cone of C at $v \in C$ is maximal monotone. For all $(v, u^*) \in G(T)$, we have the $u^* - A(v) \in N_C(v)$. By definition of $N_C(v)$, we find that

$$\langle v - y, u^* - Av \rangle \geq 0 \qquad \forall\ y \in C.$$

Since $y_n \in C$, we have

$$\langle v - y_n, u^* \rangle \geq \langle v - y_n, Av \rangle. \tag{4.12}$$

By the definition of y_n $(= \Pi_C J^{-1}(Jx_n - \mu A x_n))$ and Lemma 3.24 , we get

$$\langle v - y_n, Ax_n\rangle \geq \left\langle v - y_n, \frac{Jx_n - Jy_n}{\mu}\right\rangle. \tag{4.13}$$

Therefore, it follows from inequalities (4.12) and (4.13) and monotonicity of A that

$$\begin{aligned}\langle v - y_n, u^*\rangle &\geq \langle v - y_n, Av\rangle \\ &= \langle v - y_n, Av - Ay_n\rangle + \langle v - y_n, Ay_n - Ax_n\rangle + \langle v - y_n, Ax_n\rangle \\ &\geq \langle v - y_n, Ay_n - Ax_n\rangle + \left\langle v - y_n, \frac{Jx_n - Jy_n}{\mu}\right\rangle. \end{aligned} \tag{4.14}$$

Since $\|x_n - y_n\| \to 0$ as $n \to \infty$ and A is L-Lipschitz continuous, we have

$$\lim_{n\to\infty} \|Ay_n - Ax_n\| = 0. \tag{4.15}$$

Taking limit in inequality (4.14) and using (4.15) with $y_n \to x^*$, we have $\langle v - x^*, u^*\rangle \geq 0 \ \ \forall\ (v, u^*) \in G(T)$. Since T is maximal monotone, we have $x^* \in T^{-1}0 = VI(C,\ A)$. Hence, $x^* \in VI(C,\ A)$. Therefore, $x^* \in F(U) \cap VI(C,\ A)$.

Next, we show the $\lim\limits_{n\to\infty} x_n = x^* = \Pi_{F(U)\cap VI(C,\ A)} x_0$. Let $w = \Pi_{F(U)\cap VI(C,\ A)} x_0$. Using the fact that $x^* \in F(U) \cap VI(C,\ A)$, we have

$$\phi(w, x_0) \leq \phi(x^*, x_0). \tag{4.16}$$

Since $x_n = \Pi_{C_n} x_0$ and $w \in F(U) \cap VI(C,\ A) \subseteq C_n$, we have

$$\phi(x_n, x_0) \leq \phi(w, x_0).$$

But we have that $x_n \to x^*$ as $n \to \infty$. This implies that by continuity of $\phi(\,\cdot\,, x_o)$, we have

$$\phi(x^*, x_0) \leq \phi(w, x_0). \tag{4.17}$$

Hence, with inequalities (4.16) and (4.17), we have

$$\phi(x^*, x_0) = \phi(w, x_0). \tag{4.18}$$

We observe that, from Lemma 3.25 and Lemma 3.18 and equation (4.18), we have

$$\begin{aligned} 0 \leq \quad \phi(x^*, w) \quad &\leq \phi(x*, x_0) - \phi(w, x_0) = 0 \\ \Rightarrow \quad \phi(x^*, w) \quad &= 0. \end{aligned}$$

Thus, $x^* = w = \Pi_{F(U)\cap VI(C,\ A)} x_0$. $\square$

CHAPTER 5

Application

5.1 Strong Convergence Theorem for a Countable Family of Relatively Nonexpansive Mappings

In this section, we prove a strong convergence theorem for relatively nonexpansive mappings in 2-uniformly convex and uniformly smooth Banach spaces. To this end, we need the following lemma.

Lemma 5.1 ([Kohsaka *et al.*, 2008])**.** Let C be a closed convex subset of a uniformly smooth and 2-uniformly convex Banach space E and $(S_i)_{i=1}^{\infty}$ be a family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $(\eta_i)_{i=1}^{\infty} \subset (0,1)$ and $(\mu_i)_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$. Consider the map $T : C \to E$ defined by

$$Tx = J^{-1}\left(\sum_{i=1}^{\infty} \eta_i\Big(\mu_i Jx + (1-\mu_i)JS_i x\Big)\right) \text{ for each } x \in C. \tag{5.1}$$

Then, T is relatively nonexpansive and $F(T) = \displaystyle\bigcap_{i=1}^{\infty} F(S_i)$.

Theorem 5.2. *Let C be a nonempty, closed and convex of a 2-uniformly convex and uniformly smooth real Banach space E such that J(C) is convex. Let $A_i : E \to E^*, i = 1,2,\ldots,N$ be a countable family of monotone and L_i-Lipschitz continuous maps. Let $U_i : C \to C, i = 1,2,3,\ldots,$ be a countable family of relatively nonexpansive maps such that $\displaystyle\bigcap_{i=1}^{\infty} F(U_i) \neq \emptyset$. Suppose $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\beta_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$ and $U : C \to E$ defined by $Ux = J^{-1}\left(\displaystyle\sum_{i=1}^{\infty} \eta_i\Big(\beta_i Jx + (1-\beta_i)JU_i x\Big)\right)$ for each $x \in C$. Let $\{x_n\}$ be generated by the following algorithm:*

Algorithm 5.3.

$$\begin{cases} x_0 \in C, \\ y_n = \Pi_C J^{-1}(Jx_n - \mu A_i x_n), \\ T_n = \{x \in E : \langle Jx_n - \mu A_i x_n - Jy_n, x - y_n \rangle \le 0\}, \\ z_n = \Pi_{T_n} J^{-1}(Jx_n - \mu A_i y_n), \\ w_n = J^{-1}\Big((1-\alpha)Jx_n + \alpha JUz_n\Big) \\ C_{n+1} = \Big\{z \in C_n : \phi(z, w_n) \le \phi(z, x_n) - \alpha c\,(\phi(z_n, y_n) + \phi(y_n, x_n))\Big\} \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\alpha \in (0,1)$, μ and c are positive constants. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ generated by Algorithm 5.3 converge strongly to $\Pi_{F(U)\cap VI(C,\ A)} x_0$.

Proof. From Lemma 5.1, U is relatively nonexpansive and $F(U) = \bigcap_{i=1}^{\infty} F(U_i)$ and $VI(C,\ A) = \bigcap_{i=1}^{\infty} VI(C,\ A_i)$. The conclusion follows from Theorem 4.3. $\square$

CHAPTER 6

Conclusion

Construction of fixed points is an important subject in nonlinear operator theory and its applications; in particular in image recovery and signal processing. In addition, several physical problems can be reduced to variational inequality problems. Such problems can be found in the theories of lubrication, filtrations and flows, moving boundary problems, to mention but few.

In this thesis, a subgradient extragradient method for finding a common element of the set of fixed points of relatively nonexpansive mapping and the set of solutions of variational inequality problem for monotone and Lipschitz continuous mapping is proposed. As a consequence of the result, a strong convergence theorem for approximating a common fixed point for a countable family of relatively nonexpansive mappings and an element of the solution set of variational inequality problems is obtained.

Our result extends and improves many recent and important results. For example, firstly, our result is proved in more general real Banach space than real Hilbert space – in a uniformly smooth and 2-uniformly convex real Banach space. This is an improvement of the result of Nadwzhkina and Takahashi [Nadezhkina *et al.*, 2006] which was proved in a real Hilbert space. Secondly, our algorithm involves a parameter that is fixed. This reduces computational cost. This is an improvement of the result of Censor *et al.* [Censor *et al.*, 2011] which involves parameters that are computed for each iteration. Finally, strong convergence theorem for obtaining a common element of the set of fixed points of relative nonexpansive mapping and the set of solutions of variational inequality problem for monotone and Lipschitz continuous mapping is obtained in this work. This is an improvement on the result of Censor *et al.* [Censor *et al.*, 2011] where they proved a weak convergence theorem for obtaining a solution of a variational inequality problem and a fixed point problem.

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