SOBOLEV SPACES, EMBEDDING THEOREMS AND APPLICATIONS TO PDEs

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CERTIFICATION

This is to certify that the thesis titled "SOBOLEV SPACES, EMBEDDING THEOREMS AND APPLICATIONS TO PDEs" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research work carried out by Adams, Zekeri in the department of Pure and Applied Mathematics.

APPROVAL

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INTRODUCTION

The space of weak derivatives called the **Sobolev Spaces** which was discovered in the $1930's$ by the Russian Mathematician **Sergei Sobolev** have great importance in the theory of Partial differential equations and calculus of Variations.
Image denoising refers to the process of restoring a damaged image with missing information. Mathematical models often in the form of partial differential equations are often formulated to solve this problem. The boundary conditions on this models in the form of Diritchlet and Neumann conditions helps in solving the models formulated.
For example, consider the image denoising model which requires to solve the minimization problem given by

$$\min_{u\in X(\Omega)}\left\{\frac{1}{2}\int_{\Omega}(u(x)-f(x))^2dx+\int_{\Omega}J(\nabla u(x))dx\right\}, \tag{0.0.1}$$

where Ω is a bounded open domain of $\mathbb{R}^n$.
Here $f:\Omega\longrightarrow\mathbb{R}$ is a gray value image such that $f(x)$ is the light intensity of the noisy image at point x.
$u:\Omega\longrightarrow\mathbb{R}$ is the denoised image which is required to minimize the problem (0.0.1).
$J:\mathbb{R}^n\longrightarrow\mathbb{R}$ defined by

$$J(|\nabla u|)=|\nabla u|^p,\ p>1.$$

is required to be a convex, lower semi-continuous function in order to smoothen the observation u. $X(\Omega)$ is a space of funtions on Ω and is taken to be the space $W^{m,p}(\Omega)$, $1<p<\infty$.
Solving problem(0.0.1) is equivalent to solving the diffusion equation given by

$$\begin{cases} u-div(\beta(\nabla u))=f & \text{in} \quad \Omega\\ \beta(\nabla u).v=0 & \text{on} \quad \partial\Omega\end{cases} \tag{0.0.2}$$

Here $\beta:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is the subdifferential of J and v is the normal to the boundary of Ω, $\partial\Omega$.
Problem (0.0.3) describe the filtering process of the original corrupted image f.
In this work, we focus on seeking the existence of weak solutions to elliptic PDEs which are of the form of problem (0.0.3).
In chapter one of this work, we focussed on the study of the L^p Spaces.Also, we emphasized on relevant theroems in Linear functional analysis related to the L^p Spaces.
In chapter two of this work, we study the theory of weak derivatives and distributions which lead us to the study on Sobolev spaces. Also, relevant theorems and concepts in Sobolev spaces was discussed in the chapter.
In chapter three, we show the existence of a weak solution to some linear and nonlinear elliptic PDEs. Some of the elliptic partial differential equations to be considered are of Dirichlet and Neumann boundary conditions given below.

Example 0.0.1

$$\begin{cases} -\Delta u+u=f & in \quad \Omega\\ u=0 & on \quad \partial\Omega\end{cases} \tag{0.0.3}$$

This is a Dirichlet problem with homogeneous boundary condition.

Example 0.0.2

$$\begin{cases} -\Delta u + u = f & in \quad \Omega \\ \frac{\partial u}{\partial \boldsymbol{n}} = 0 & on \quad \partial\Omega \end{cases} \tag{0.0.4}$$

This a Neumann problem with homogeneous boundary condition.

Example 0.0.3

$$\begin{cases} -\Delta u + u = f & in\ \Omega,\ \ f \in L^2(\Omega) \\ \frac{\partial u}{\partial \boldsymbol{n}} = g & on\ \partial\Omega, \quad g \in L^2(\partial\Omega). \end{cases} \tag{0.0.5}$$

This a Neumann problem with non homogeneous boundary condition.

Example 0.0.4

$$\begin{cases} -\Delta u = f & in\ \Omega,\ \ f \in L^2(\Omega) \\ u = 0 & on\ \Gamma_0,\ \frac{\partial u}{\partial \boldsymbol{n}} = 0\ on\ \Gamma_1. \end{cases} \tag{0.0.6}$$

This is a mixed problem with Dirichlet and Neumann homogeneous boundary conditions.

DEDICATION

To God almighty and to my late Father.

CONTENTS

CHAPTER 1

L^P - SPACES

Introduction
The L^p- spaces plays an important role in mathematics, especially in functional analysis. It is a space of functions that was discovered by **Frigyes Riesz (1910)**. These spaces plays a vital role in measure and probability spaces. They also have great importance in the theoretical discussion of problems in physics, statistics, finance, engineering and other disciplines.

In statistics, problems on regression can be viewed as those of computing orthogonal projections on subspaces of L^2 spaces.

In physics,the space L^2 which is a Hilbert space is central to many applications especially in quantum mechanics in view of the fact that it helps in the interpretation of the wave functions which are been interpreted as a probability distribution functions.

The L^p- spaces are also vital in the theory of partial differential equations(PDEs) whose solutions are in the Sobolev spaces, $W^{m,p}$. These spaces consists of functions, u, in L^p whose derivatives of order less than or equal to m are also in L^p.

The objective of this chapter is to acquint us with the study on L^p- spaces and discuss some important properties of the L^p- spaces needed in this work. Also relevant Theorems on L^p- spaces in linear functional analysis needed in this work will be discussed.

In the following, Ω is an open subset of $\mathbb{R}^n$ with the lebesgue measure dx.

1.1 Definition and basic properties

Definition 1.1.1 *[6] Let $p \in \mathbb{R}$, $1 \leq p < \infty$. We define the L^p-space by*

$$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} \text{ measurable such that } \int_\Omega |f|^p dx < \infty\}$$

provided with the following norm

$$\|f\|_{L^p} = \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}, \text{ for all } f \in L_p(\Omega), \text{and } 1 \leq p < \infty.$$

For $p = \infty$, we define $L^\infty(\Omega)$ by

$$L^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} \text{ measurable such that there exists a constant } C > 0, \text{ satisfying } |f(x)| \leq C \text{ a.e in } \Omega\}$$

provided with the norm

$$\|f\|_{L^\infty} = ess\sup_{x\in\Omega} |f(x)| = inf\{C > 0 : |f(x)| < C \ a.e \ in \ \Omega\}.$$

Remark 1.1.2 *Let $1 \le p \le \infty$. The conjugate exponent q of p is defined by*

$$\tfrac{1}{p} + \tfrac{1}{q} = 1.$$

Lemma 1.1.3 *[6] Let $\lambda \in (0, 1)$. Then,*

$$x^\lambda \le (1 - \lambda) + \lambda x, \ for \ x > 0.$$

Proof . Set $f(x) = (1 - \lambda) + \lambda x - x^\lambda$. Then, we have that

$$f'(x) = \lambda - \lambda x^{\lambda-1}.$$

We obtain that $x = 1$ is a critical point(in fact, a minimum pont). Hence,

$$f(1) = 0 \le f(x).$$

Then,

$$0 \le (1 - \lambda) + \lambda x - x^\lambda.$$

Hence, we have that

$$x^\lambda \le (1 - \lambda) + \lambda x.$$

■

Lemma 1.1.4 *[6] Let $a, b \ge 0$ and $\lambda \in (0, 1)$, then the following inequality holds*

$$a^\lambda b^{1-\lambda} \le \lambda a + (1 - \lambda)b.$$

Proof . For $a = 0$ or $b = 0$, the inequality holds. Now, $a, b > 0$ and from Lemma 1.1.3, set $x = \frac{a}{b}$. Then, we obtain that

$$\left(\tfrac{a}{b}\right)^\lambda \le (1 - \lambda) + \lambda \tfrac{a}{b}.$$

Thus, we obtain that

$$a^\lambda b^{1-\lambda} \le \lambda a + (1 - \lambda)b.$$

■

Lemma 1.1.5 *[6] **(Cauchy Young's Inequality)***
Let $a, b \ge 0$. Then, for $\frac{1}{p} + \frac{1}{q} = 1$, we have that

$$ab \le \tfrac{a^p}{p} + \tfrac{b^q}{q}.$$

Proof . The result is trivial if either $a = 0$ or $b = 0$.
Let $a, b > 0$. Then,

$$\begin{aligned} ab &= exp(logab) \\ &= exp(loga + logb) \\ &= exp(\frac{1}{p}loga^p + \frac{1}{q}logb^q). \end{aligned}$$

By convexity of the exponential funtion, we get that

$$\begin{aligned} ab &\le \frac{1}{p}exp(loga^p) + \frac{1}{q}exp(logb^q) \\ ab &\le \frac{1}{p}a^p + \frac{1}{q}b^q. \end{aligned}$$

■

Theorem 1.1.6 *[6] (Hölder's inequality)*
Suppose that $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$. Moreover,

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Proof . From Lemma (1.1.4), we have that
Set $a = \frac{|f|^p}{\|f\|^p_{L^p}}$, $b = \frac{|g|^p}{\|g\|^q_{L^q}}$, $\lambda = \frac{1}{p}$, $f \neq 0$, $g \neq 0$, for all $x \in \Omega$. Then,

$$\frac{|f|}{\|f\|_{L^p}}.\frac{|g|}{\|g\|_{L^q}} \leq \frac{1}{p}\frac{|f|^p}{\|f\|^p_{L^p}} + \frac{1}{q}\frac{|g|^q}{\|g\|^q_{L^q}}.$$

Integrating both sides, we get that

$$\begin{aligned}\frac{1}{\|f\|_{L^p}\|g\|_{L^q}}\int_\Omega |f||g|dx &\leq \frac{1}{p\|f\|^p_{L^p}}\int_\Omega |f|^p dx + \frac{1}{q\|g\|^q_{L^q}}\int_\Omega |g|^q dx \\ &= \frac{\|f\|^p_{L^p}}{p\|f\|^p_{L^p}} + \frac{\|g\|^q_{L^q}}{q\|f\|^q_{L^q}} \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1.\end{aligned}$$

Hence, $fg \in L^1(\Omega)$. Furthermore,

$$\frac{1}{\|f\|_{L^p}\|g\|_{L^q}}\int_\Omega |fg|dx \leq 1.$$

Consequently, we have that

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

■

Remark 1.1.7 *[1](Generalized Hölder's inequality)*
Assume $f_1, f_2, ..., f_k$ are functions such that $f_i \in L^{p_i}$, for $1 \leq i \leq k$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + .. + \frac{1}{p_k} = 1$. Then, the product $f = f_1.f_2...f_k$ belongs to L^p and

$$\|f\|_{L^p} \leq \|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}...\|f_k\|_{L^{p_k}}.$$

Let $(n-1)$ non-negative functions $g_1, g_2, ..., g_n \in L^1(\Omega)$ be given. Then $g_i^{\frac{1}{n-1}} \in L^{n-1}(\Omega)$ for each i. By the generalized Holder's inequality, we obtain that,

$$\int_\Omega g_1^{\frac{1}{n-1}} g_2^{\frac{1}{n-1}} ... g_{n-1}^{\frac{1}{n-1}} dx \leq \prod_{i=1}^{n-1} \|g_i\|_{L^1}^{\frac{1}{n-1}}(\Omega).$$

Theorem 1.1.8 *[6] (Minkowski's inequality)*
Let $1 \leq p \leq \infty$ and $f, g \in L^p(\Omega)$. Then,

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Proof . If $f + g = 0$ a.e on Ω, the inequality holds. Assume $f + g \neq 0$ a.e on Ω. Then,

$$|f + g|^p = |f + g||f + g|^{p-1} \leq (|f| + |g|)\,|f + g|^{p-1}.$$

integrating, we have that

$$\int_\Omega |f+g|^p dx \le \int_\Omega (|f|+|g|)|f+g|^{p-1} dx.$$

By Hölder's inequality, we obtain that

$$\int_\Omega |f+g|^p dx \le (\|f\|_{L^p} + \|g\|_{L^p}) \|f+g\|_{L^p}^{\frac{p}{q}}$$
$$\|f+g\|_{L^p}^p \le (\|f\|_{L^p} + \|g\|_{L^p}) \|f+g\|_{L^p}^{\frac{p}{q}}$$
$$\|f+g\|_{L^p}^{p(1-\frac{1}{q})} \le \|f\|_{L^p} + \|g\|_{L^p}.$$

Consequently, we have that

$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}.$$

■

Lemma 1.1.9 *[6] (**L^p interpolation inequality**)*
Let $1 \le r \le s \le t \le \infty$ and suppose that $u \in L^r(\Omega) \cap L^t(\Omega)$. Then, for $\frac{1}{s} = \frac{a}{r} + \frac{1-a}{t}$, we have that,

$$\|u\|_{L^s} \le \|u\|_{L^r}^a \|u\|_{L^t}^{1-a}.$$

Proposition 1.1.10 *[3] If $\mu(\Omega) < \infty$, then for $1 \le p < q \le \infty$, $L^q(\Omega) \hookrightarrow L^p(\Omega)$.*

1.2 Main properties of L^P- spaces

Definition 1.2.1 *[6] Let $(f_n)_{n\ge0}$ be a sequence in $L^p(\Omega)$. Then, $(f_n)_{n\ge0}$ converges to $f \in L^p(\Omega)$ if*

$$\|f_n - f\|_{L^p} \longrightarrow 0 \quad as \quad n \to \infty.$$

Theorem 1.2.2 *[1] (**Monotone convergence Theorem (MCT)**)*
Let $(f_n)_{n\ge0}$ be a sequence of functions in $L^1(\Omega)$ that satisfies

$$f_1 \le f_2 \le ... \le f_n \le f_{n+1} \le ... \text{ a.e on } \Omega.$$

Moreover, if

$$\|f_n\|_{L^1} \le C < \infty, \text{ for all } n \in \mathbb{N}.$$

Then,

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ exists with } f \in L^1(\Omega) \text{ and } \|f_n - f\|_{L^1} \longrightarrow 0 \text{ as } n \to \infty.$$

Theorem 1.2.3 *[1] (**Dominated convergence Theorem(DCT)**)*
Let$(f_n)_{n\ge0}$ be a sequence of functions in $L^p(\Omega)$ such that $f_n(x) \longrightarrow f(x)$ a.e on Ω as $n \to \infty$. If there exist a function $g \in L^p(\Omega)$ such that $|f_n(x)| \le |g(x)|$ a.e on Ω for all $n \in \mathbb{N}$. Then $f_n \longrightarrow f$ in $L^p(\Omega)$, that is, $\|f_n - f\| \longrightarrow 0$ as $n \to \infty$.

Theorem 1.2.4 *[1] (**Fatou-Lemma**)*
Let $(f_n)_{n\ge0}$ be a sequence of functions in $L^1(\Omega)$ satisfying:
(i) for all n, $f_n \ge 0$, a.e.
(ii)$\|f_n\|_{L^1} < \infty$.
(iii) $\liminf_{n\to\infty} f_n(x) = f(x)$ a.e. on Ω. Then, $f \in L^1(\Omega)$ and

$$\int_{\Omega} f = \int_{\Omega} \liminf_{n\to\infty} f_n(x) \le \liminf_{n\to\infty} \int_{\Omega} f_n(x).$$

Theorem 1.2.5 *[6] $L^p(\Omega)$ is a Banach space for any $p \in [1, \infty]$.*

Proof . Let $\epsilon > 0$ be given.
Case I: for $p = \infty$, we show that $L^\infty(\Omega)$ is a Banach space.
Let $(u_n)_{n\ge 0}$ be a Cauchy sequence in $L^\infty(\Omega)$. Then, there exists $N_1 \in \mathbb{N}$ such that,

$$\|u_n - u_m\|_{L^\infty} = sup_{x\in\Omega}|u_n(x) - u_m(x)| < \epsilon \quad \text{for} \quad \text{all} \quad n, m \ge N_1$$

Hence, for each $x \in \Omega$,

$$|u_n(x) - u_m(x)| < \epsilon \quad \text{for} \quad \text{all} \quad n, m \ge N_1.$$

Hence, $(u_n(x))_{n\ge 0}$ is a Cauchy sequence in $\mathbb{R}$ a.e in Ω and converges to u a.e in Ω.
Fix n and letting $m \to \infty$, we get that,

$$|u_n(x) - u(x)| < \epsilon \quad \text{for all } n \ge N_1,$$

which implies that,

$$\|u_n - u\|_{L^\infty} = sup_{x\in\Omega}|u_n(x) - u(x)| < \epsilon \quad \text{for all } n \ge N_1.$$

Hence, $u_n \longrightarrow u$ in $L^\infty(\Omega)$. Thus, $L^\infty(\Omega)$ is Banach.
Case II: for $1 \le p < \infty$. Let $(u_n)_{n\ge 0}$ be a Cauchy sequence in $L^p(\Omega)$. Then, there exists $N \in \mathbb{N}$ such that,

$$\|u_n - u_m\|_{L^p} < \epsilon, \quad \text{for} \quad \text{all } n, m \ge N.$$

Then, by construction we get a subsequence $(u_{n_k})_{k\ge 0}$ such that,

$$\|u_{n_k} - u_{n_{k+1}}\|_{L^p} < \tfrac{1}{2^k} \text{ for } k \ge 0.$$

Define, $g_n = \sum_{k=1}^{n} |u_{n_k}(x) - u_{n_{k+1}}(x)|$.
Now, g_n is non-negative monotone increasing sequence in $L^p(\Omega)$. Thus, $g_n^p \in L^1(\Omega)$ for all $n \ge 0$.
Also, we have that

$$\begin{aligned}
\|g_n^p\|_{L^1} &= \|g_n\|_{L^p}^p \\
&\le \left(\sum_{k=1}^{n} \|u_{n_k} - u_{n_{k+1}}\| \right)^p \\
&< \left(\sum_{k=1}^{n} \frac{1}{2^k} \right)^p \\
&< \left(\sum_{k=1}^{\infty} \frac{1}{2^k} \right)^p \\
&= 1.
\end{aligned}$$

Hence, $\|g_n^p\|_{L^1} \le 1$. Then, by **MCT** 1.2.2, there exists $g^p \in L^1(\Omega)$ such that $g_n^p(x) \longrightarrow g^p(x)$ a.e in Ω. Hence, $g \in L^p(\Omega)$.
Now, for $r > m$, we have that

$$\begin{aligned}
|u_{n_m} - u_{n_r}| &\le |u_{n_m} - u_{n_{m+1}}| + |u_{n_{m+1}} - u_{n_{m+2}}| + ... + |u_{n_{r-1}} - u_{n_r}| \\
&= \sum_{k=1}^{r-1} |u_{n_k} - u_{nk+1}| - \sum_{k=1}^{m-1} |u_{n_k} - u_{nk+1}| \\
&= g_{r-1} - g_{m-1} \longrightarrow 0 \; as \; r, m \to \infty.
\end{aligned}$$

Hence, $(u_{n_k})_{k\ge 0}$ is Cauchy in $\mathbb{R}$ a.e in Ω. Thus,

$$u_{n_k}(x) \longrightarrow u(x) \text{ a.e in } \Omega.$$

Furthermore,

$$|u_{n_m} - u_{n_r}| \leq g_{r-1} - g_{m-1} \leq g_{r-1} \leq g.$$

Fix m and letting $r \to \infty$, we get that

$$|u_{n_m} - u| \leq g.$$

Consequently, we get that

$$|u_{n_m} - u|^p \leq g^p.$$

This implies that,

$$|u_{n_m} - u|^p \in L^1(\Omega).$$

Hence, $u_{n_m} - u \in L^p(\Omega)$. Thus, $u \in L^p(\Omega)$.
Since, $|u_{n_m} - u|^p \longrightarrow 0$ a.e in Ω and $|u_{n_m} - u|^p \leq g^p \in L^1(\Omega)$. Then, **DCT** 1.2.3 , we get that,

$$\|u_{n_k} - u\|_{L^p}^p = \|(u_{n_k} - u)^p\|_{L^1} \longrightarrow 0 \text{ as } k \to \infty.$$

That is, $u_{n_k} \longrightarrow u$ in $L^p(\Omega)$.
Since $(u_{n_k})_{k\geq 0}$ is a subsequence of the Cauchy sequence $(u_n)_{n\geq 0}$ and converges to $u \in L^p(\Omega)$. Hence, $u_n \longrightarrow u$ in $L^p(\Omega)$. ■

Theorem 1.2.6 *[3] **(Converse of DCT)***
Let $(f_n)_{n\geq 0}$ be a sequence of functions in $L^p(\Omega)$ such that $f_n \longrightarrow f$ in $L^p(\Omega)$ for $1 \leq p < \infty$. Then, there exists a subsequence $(f_{n_k})_{k\geq 0}$ of $(f_n)_{n\geq 0}$ and $g \in L^p(\Omega)$ such that $|f_{n_k}(x)| \leq |g(x)|$ a.e. on Ω and $f_{n_k}(x) \longrightarrow f(x)$ a.e. on Ω.

Theorem 1.2.7 *[6] For $1 \leq p < \infty$, suppose that $(f_n)_{n\geq 0} \subset L^p(\Omega)$ and $f_n(x) \longrightarrow f(x)$ a.e on Ω. If $\|f_n\|_{L^p} \longrightarrow \|f\|_{L^p}$ as $n \to \infty$, then $\|f_n - f\|_{L^p} \longrightarrow 0$ as $n \to \infty$, that is, $f_n \longrightarrow f$ in $L^p(\Omega)$.*

Proof . We know that for $a, b \geq 0$ and $1 \leq p < \infty$, that

$$\left(\tfrac{a+b}{2}\right)^p \leq \tfrac{1}{2}\left(a^p + b^p\right).$$

Hence,

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

Thus, we get that

$$|a-b|^p \leq |a+b|^p \leq 2^{p-1}(|a|^p + |b|^p).$$

Let $a = f_n$, $b = f$. Then,

$$|f_n - f|^p \leq 2^{p-1}(|f_n|^p + |f|^p),$$

and

$$0 \leq 2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p.$$

Since $f_n(x) \longrightarrow f(x)$ a.e. on Ω. Then,

$$\begin{aligned} 2^p \int_\Omega |f|^p &= \int_\Omega \lim_{n\to\infty} \left(2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p\right) \\ &= \int_\Omega \liminf_{n\to\infty} \left(2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p\right). \end{aligned}$$

By Fatou's Lemma 1.2.4, we get that

$$\begin{aligned} &\leq \liminf_{n\to\infty} \left(\int_\Omega 2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p \right) \\ \liminf_{n\to\infty} \int_\Omega |f_n - f|^p &\leq \liminf_{n\to\infty} 2^{p-1} \left(\int_\Omega |f_n|^p + |f|^p \right) - 2^p \int_\Omega |f|^p. \end{aligned}$$

Since $\|f_n\|_{L^p} \longrightarrow \|f\|_{L^p}$ as $n \to \infty$. Then, we have that

$$\liminf_{n\to\infty} \|f_n - f\|_{L^p}^p \leq 2^{p-1} \left(\liminf_{n\to\infty} \|f_n\|_{L^p}^p + \|f\|_{L^p}^p \right) - 2^p \|f\|_{L^p}^p.$$

Hence, we have that

$$\liminf_{n\to\infty} \|f_n - f\|_{L^p} = 0.$$

Furthermore, we get that

$$\begin{aligned} 2^p \int_\Omega |f|^p &\leq \liminf_{n\to\infty} \left(\int_\Omega 2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p \right). \\ &\leq \limsup_{n\to\infty} \left(\int_\Omega 2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p \right). \\ \limsup_{n\to\infty} \int_\Omega |f_n - f|^p &\leq 2^{p-1} \left(\limsup_{n\to\infty} \int_\Omega |f_n|^p + \int_\Omega |f|^p \right) - 2^p \int_\Omega |f|^p. \end{aligned}$$

Since $\|f_n\|_{L^p} \longrightarrow \|f\|_{L^p}$. Then, we have that

$$\limsup_{n\to\infty} \|f_n - f\|_{L^p}^p \leq 2^{p-1} \left(\limsup_{n\to\infty} \|f_n\|_{L^p}^p + \|f\|_{L^p}^p \right) - 2^p \|f\|_{L^p}^p.$$

Consequently, $\limsup_{n\to\infty} \|f_n - f\|_{L^p} = 0$.
Hence, we get

$$\limsup_{n\to\infty} \|f_n - f\|_{L^p} = \liminf_{n\to\infty} \|f_n - f\|_{L^p} = 0,$$

which implies that,

$$\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0.$$

■

Theorem 1.2.8 *[2] Let $1 \leq p < \infty$. Then, the dual of $L^p(\Omega)$ denoted $(L^p(\Omega))^* = L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof . Let $g \in L^q(\Omega)$ and for every $f \in L^p(\Omega)$. Consider the map

$$T_g : L^p(\Omega) \longrightarrow \mathbb{R}$$

defined as

$$T_g(f) = \int_\Omega fg dx \text{ for all } f \in L^p(\Omega).$$

Clearly, T_g is linear. Furthermore,

$$|T_g(f)| \leq \int_\Omega |fg| \leq \|f\|_{L^p} \|g\|_{L^q},$$

which shows that T_g is well-defined and bounded. Hence, $T_g \in (L^p(\Omega))^*$ and

$$\|T_g\|_{(L^P)^*} \leq \|g\|_{L^q}.$$

Now, let $h \in L^q(\Omega)$ be arbitrary. Define

$$\tilde{g} = \frac{sgn(h)|h|^{q-1}}{\|h\|_{L^q}^{q-1}}.$$

Then, we have that

$$\int_\Omega |\tilde{g}|^p \leq \int_\Omega \frac{|h|^{p(q-1)}}{\|h\|_{L^q}^{p(q-1)}} = \int_\Omega \frac{|h|^q}{\|h\|_{L^q}^q} = 1.$$

Hence, $\tilde{g} \in L^p(\Omega)$. Furthermore,

$$T_h(\tilde{g}) = \int_\Omega \frac{hsgn(h)|h|^{q-1}}{\|h\|_{L^q}^{q-1}} = \int_\Omega \frac{|h|^q}{\|h\|_{L^q}^{q-1}} = \|h\|_{L^q}.$$

Hence, we have that

$$T_h(\tilde{g}) = \|h\|_{L^q}.$$

Consequently, we have that

$$\|T_h\|_{(L^p)^*} \geq \|h\|_{L^q}.$$

Since $h \in L^q(\Omega)$ is arbitrary. Then, in particular for $h = g \in L^q(\Omega)$, we have that

$$\|T_g\|_{(L^p)^*} \geq \|g\|_{L^q}.$$

Consequently, we have that

$$\|T_g\|_{(L^p)^*} = \|g\|_{L^q},$$

which shows that $(L^p(\Omega))^*$ is isometrically isomorphic to $L^q(\Omega)$. This complete the proof. ■

Proposition 1.2.9 *[6] Let $(u_n)_{n\geq 0}$ be a sequence in $L^p(\Omega)$ such that $u_n \longrightarrow u$ in $L^p(\Omega)$. Then $u_n \rightharpoonup u$ in $L^p(\Omega)$.*

Proof . Given that $u_n \longrightarrow u$ in $L^p(\Omega)$. Then $\|u_n - u\|_{L^p} \longrightarrow 0$ as $n \to \infty$.
Let $v \in L^q(\Omega)$. Then,

$$\left|\int_\Omega u_n v - \int_\Omega uv\right| = \left|\int_\Omega (u_n - u)v\right|.$$

By Holder's inequality, we obtain that

$$\left|\int_\Omega u_n v - \int_\Omega uv\right| \leq \|u_n - u\|_{L^p}\|v\|_{L^q} \longrightarrow 0 \text{ as } n \to \infty$$

Hence,

$$\int_\Omega u_n v \longrightarrow \int_\Omega uv \text{ as } n \to \infty$$

which implies that, $u_n \rightharpoonup u$ in $L^p(\Omega)$. ■

Theorem 1.2.10 *[6] Suppose $1 \leq p < \infty$ and $u_n \rightharpoonup u$ in $L^p(\Omega)$. Then, $(u_n)_{n\geq 0}$ is bounded in $L^p(\Omega)$ and*

$$\|u\|_{L^p(\Omega)} \leq \liminf_{n\to\infty} \|u_n\|_{L^p}.$$

Theorem 1.2.11 *[6] Supppose $1 \leq p < \infty$ and the sequence $(u_n)_{n\geq 0}$ is bounded in $L^p(\Omega)$. Then, there is a subsequence still denoted by $(u_n)_{n\geq 0}$ and a function $u \in L^p(\Omega)$ such that*
$u_n \rightharpoonup u$ in $L^p(\Omega)$.

Definition 1.2.12 *[2] A Banach space E is called uniformly convex if for any $\epsilon \in (0,2]$ there exists a $\delta = \delta(\epsilon) \in (0,1)$ such that for all $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \epsilon$, then $\|\frac{1}{2}(x+y)\| \leq 1-\delta$.*

Theorem 1.2.13 *[2]* ***(Milman Pettis)***
Every uniformly convex real Banach space E is reflexive.

Theorem 1.2.14 *[2]* ***(Clarkson's inequality)***
Let $f, g \in L^p(\Omega)$. Then,
$(i)\|\frac{f+g}{2}\|^q_{L^p} + \|\frac{f-g}{2}\|^q_{L^p} \leq \left(\frac{1}{2}\|f\|^p_{L^p} + \frac{1}{2}\|g\|^p_{L^p}\right)^{\frac{1}{p-1}}$, $1 < p \leq 2$.
$(ii)\|\frac{f+g}{2}\|^p_{L^p} + \|\frac{f-g}{2}\|^p_{L^p} \leq \left(\frac{1}{2}\|f\|^p_{L^p} + \frac{1}{2}\|g\|^p_{L^p}\right)$, for $2 \leq p < \infty$.

Theorem 1.2.15 *[1] Let $1 < p < \infty$. Then, $L^p(\Omega)$ is uniformly convex. Hence, it is reflexive.*

Proof . Let $\epsilon \in (0,2]$, $f, g \in L^p(\Omega)$ with $\|f\|_{L^p} \leq 1$, $\|g\|_{L^p} \leq 1$ and $\|f-g\|_{L^p} \geq \epsilon$. Then, for $2 \leq p < \infty$, by Clarkson's inequality, we get that

$$\|\tfrac{f+g}{2}\|^p_{L^p} \leq 1 - (\tfrac{\epsilon}{2})^p$$
$$\|\tfrac{f+g}{2}\|_{L^p} \leq 1 - \left(1 - (\tfrac{\epsilon}{2})^p\right)^{\frac{1}{p}}$$

We choose $\delta = \left(1 - (\frac{\epsilon}{2})^p\right)^{\frac{1}{p}}$. Hence, for $\epsilon \in (0,2]$, with $\|f-g\| \geq \epsilon$, then

$$\|\tfrac{f+g}{2}\|_{L^p} \leq 1 - \delta.$$

Consequently, $L^p(\Omega)$ is uniformly convex for $2 \leq p < \infty$.
Furthermore, for $1 < p \leq 2$.
Let $\epsilon \in (0,2]$, $f, g \in L^p(\Omega)$ with $\|f\|_{L^p} \leq 1$, $\|g\|_{L^p} \leq 1$ and $\|f-g\|_{L^p} \geq \epsilon$. Then, by Clarkson's inequality

$$\|\tfrac{f+g}{2}\|^q_{L^p} \leq 1 - (\tfrac{\epsilon}{2})^q.$$
$$\|\tfrac{f+g}{2}\|_{L^p} \leq 1 - \left(1 - (\tfrac{\epsilon}{2})^q\right)^{\frac{1}{q}}.$$

We choose $\delta = \left(1 - (\frac{\epsilon}{2})^q\right)^{\frac{1}{q}}$. Hence, for $\epsilon \in (0,2]$, with $\|f-g\| \geq \epsilon$, then

$$\|\tfrac{f+g}{2}\|_{L^p} \leq 1 - \delta.$$

Consequently, $L^p(\Omega)$ is uniformly convex for $1 < p \leq 2$. Thus, by Milman Pettis 1.2.13, $L^p(\Omega)$ is reflexive for $1 < p < \infty$. ■

1.3 Convolution

Theorem 1.3.1 *[1]* ***(Tonelli)***
Let $F : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}$ be a measurable function such that
(i)

$$\int_{\Omega_2} |F(x,y)| d\mu_2 < \infty \text{ for a.e } x \in \Omega.$$

(ii)

$$\int_{\Omega_1} d\mu_1 \int_{\Omega_2} |F(x,y)| d\mu_2 < \infty.$$

Then, $F \in L^1(\Omega_1 \times \Omega_2)$.

Theorem 1.3.2 *[1] **(Fubini)***
Assume that $F \in L^1(\Omega_1 \times \Omega_2)$. Then, for a.e $x \in \Omega_1, F(x,y) \in L^1_y(\Omega_2)$ and $\int_{\Omega_2} F(x,y)d\mu_2 \in L^1_x(\Omega_1)$. Similarly, for a.e. $y \in \Omega_2$, $F(x,y) \in L^1_x(\Omega_1)$ and

$$\int_{\Omega_1} F(x,y)d\mu_1 \in L^1_y(\Omega_2).$$

Moreover, we have that

$$\int_{\Omega_1} d\mu_1 \int_{\Omega_2} F(x,y)d\mu_2 = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} F(x,y)d\mu_1 = \int_{\Omega_1}\int_{\Omega_2} F(x,y)d\mu_1 d\mu_2.$$

Definition 1.3.3 *[1] Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, with $1 \leq p \leq \infty$. we define the convolution of f and g denoted as $f * g$ as follows*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Theorem 1.3.4 *[1] Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$. Then, $f * g \in L^p(\mathbb{R}^n)$. Moreover,*

$$\|f * g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}.$$

Proof . Case I: for $p = \infty$.

$$\begin{aligned}(f * g)(x) &= \int_{\mathbb{R}^n} f(x-y)g(y)dy \\ &\leq sup_{y\in\mathbb{R}^n}|g(y)| \int_{\mathbb{R}^n} |f(x-y)|dy \\ \|f * g\|_{L^\infty} &\leq \|f\|_{L^1}\|g\|_{L^\infty}\end{aligned}$$

Case II: for $p = 1$.

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Then,

$$|(f * g)(x)| \leq \int_{\mathbb{R}^n} |f(x-y)||g(y)|dy$$

Set $h(x,y) = f(x-y)g(y)$ for a.e $y \in \mathbb{R}^n$.
Thus,

$$\int_{\mathbb{R}^n} |h(x,y)|dx = |g(y)| \int_{\mathbb{R}^n} |f(x-y)|dx < \infty.$$

Also,

$$\int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} |h(x,y)|dx = \int_{\mathbb{R}^n} |g(y)|dy \int_{\mathbb{R}^n} |f(x-y)|dx = \|g\|_{L^1}\|f\|_{L^1} < \infty.$$

Hence, by Fubini's Theorem 1.3.2, $h \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. We apply Tonelli's Theorem 1.3.1 to obtain that,

$$\int_{\mathbb{R}^n} |(f * g)(x)|dx \leq \int_{\mathbb{R}^n} |g(y)|dy \int_{\mathbb{R}^n} |f(x-y)|dx.$$

That is,

$$\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}.$$

Case III: $1 < p < \infty$.

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

But,

$$|f(x-y)| = |f(x-y)|^{\frac{1}{p}} |f(x-y)|^{\frac{1}{q}}, \; where \frac{1}{p} + \frac{1}{q} = 1.$$
$$|f(x-y)||g(y)| = |f(x-y)|^{\frac{1}{p}} |f(x-y)|^{\frac{1}{q}} |g(y)|$$
$$\int_{\mathbb{R}^n} |f(x-y)||g(y)|dy = \int_{\mathbb{R}^n} |f(x-y)|^{\frac{1}{p}} |f(x-y)|^{\frac{1}{q}} |g(y)|dy.$$

Thus, by Holder's inequality we get that,

$$|(f * g)(x)| \leq \left(\int_{\mathbb{R}^n} |f(x-y)|dy \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dy \right)^{\frac{1}{p}}$$
$$= \|f\|_{L^1}^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dy \right)^{\frac{1}{p}}$$
$$|(f * g)(x)|^p \leq \|f\|_{L^1}^{\frac{p}{q}} \left(\int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dy \right)$$
$$\int_{\mathbb{R}^n} |(f * g)(x)|^p dx \leq \|f\|_{L^1}^{\frac{p}{q}} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dy dx \right).$$

By Tonelli's Theorem 1.3.1 we obtain that,

$$\|f * g\|_{L^p}^p \leq \|f\|_{L^1}^{\frac{p}{q}} \|f\|_{L^1} \|g\|_{L^p}^p$$

Then,

$$\|f * g\|_{L^p} \leq \|f\|_{L^1}^{\frac{1}{q}} \|f\|_{L^1}^{\frac{1}{p}} \|g\|_{L^p}$$
$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}.$$

■

CHAPTER 2

SOBOLEV SPACES AND EMBEDDING THEOREMS

Introduction
The Sobolev space, $W^{m,p}$ is a vector space consisting of functions, u in L^p with derivatives of order less than or equal to m in L^p. These spaces is equiped with a norm that is a combination of the norm of the function, u in L^p and that of the derivatives up to order m.

The space which was discovered in the 1930's by the Russian mathematician **Sergei Sobolev** was introduced mainly for the need of the theory of partial differential equations(PDEs). These spaces are special class of distribution which makes them closely related with the theory of distributions. Also, they spaces is well connected with the spaces of continuous and uniformly continuous functions.

The Sobolev spaces have great importance in the theory of partial differential equations and its applications in mathematical physics. Solutions to partial differential equations are well situated in the Sobolev spaces. Furthermore, the Sobolev spaces are vital tools in the theory of approximations, spectral theory and differential geometry.

This chapter is aimed at exposing us to the Sobolev spaces in details. Also, the theory of distributions will be discussed in this chapter. The sobolev inequalities mainly the Morrey's inequality and the Gagliardo-Nirenberg inequality which are vital in the theory of the Sobolev spaces will be discussed . Furthermore, the theory of embeddings in Sobolev spaces are of great importance. We shall discuss the general sobolev embeddings and compact embeddings of Sobolev spaces such as the Rellich-Kondrachov compact embedding theorem will be discussed.

In the following Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ is the boundary of Ω.

2.1 Theory of distributions

Definition 2.1.1 *[3] We define a multi-index, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ as an n-turple of non-negative integer numbers. Its length is defined as follows*

$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n.$$

Each multi index α determines a partial differential operator of order $|\alpha|$ namely

$$D^\alpha u = (\frac{\partial^{\alpha_1+\alpha_2+...+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}})u.$$

Given two multi-indices, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$, for $\beta \leq \alpha$ (means $\beta_i \leq \alpha_i$, $\forall i = 1, 2, ..., n$). Then,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \frac{\alpha_1!}{\beta_1!(\alpha_1-\beta_1)!}\cdots\frac{\alpha_n!}{\beta_n!(\alpha_n-\beta_n)!}.$$

Definition 2.1.2 *[3] We define the support of a continuous function, ϕ, denoted by, $supp(\phi)$ as the closure of the set $\{x \in \Omega : \phi(x) \neq 0\}$.*
In general, we define the support of a function, f, as the complement of union of all open sets, θ, in Ω such that $f = 0$ a.e. in θ.

Definition 2.1.3 *[3] We define $D(\Omega)$ as the space of C^∞ functions with compact support in Ω . It is usually called test functions.*

Definition 2.1.4 *[3] Let $\Omega \subset \mathbb{R}^n$ and K be a compact subset of Ω. We define*

$$D_K(\Omega) = \{\phi \in D(\Omega) : supp(\phi) \subseteq K\}.$$

Definition 2.1.5 *[3] We define $L^p_{loc}(\Omega)$, $1 \leq p < \infty$ as the set of all measurable functions u in Ω such that*

$$\int_K |u(x)|^p dx < \infty,$$

for any K compact set contained in Ω.

Definition 2.1.6 *[3] Let $T : D(\Omega) \longrightarrow \mathbb{R}$ be linear. Then, T is a distribution on Ω if for every compact set $K \subseteq \Omega$, there exists $C > 0$ and $n \in \mathbb{N}$ such that*

$$|T(\phi)| \leq C\|\phi\|_{C^n}, \text{ for all } \phi \in D_K(\Omega),$$

where $\|\phi\|_{C^n} = \sup_{x \in k, |\alpha| \leq n} |D^\alpha \phi(x)|$.

Definition 2.1.7 *[3] We define $D^{'}(\Omega)$ the space of distributions on Ω.*

Example 2.1.8 *[3] Let $f \in L^1_{loc}(\Omega)$. Then, $T_f : D(\Omega) \longrightarrow \mathbb{R}$ defined as follows*

$$T_f(\phi) = \int_\Omega f(x)\phi(x)dx, \text{ for all } \phi \in D(\Omega)$$

is a distribution on Ω.
Proof . *Indeed, let $\lambda \in \mathbb{R}$ and $\phi, \psi \in D(\Omega)$, then*

$$\begin{aligned} T_f(\lambda\phi + \psi) &= \int_\Omega f(x)(\lambda\phi(x) + \psi(x))dx \\ &= \lambda T_f(\phi) + T_f(\psi) \end{aligned}$$

So T_f is linear.
Furthermore, let $K \subseteq \Omega$ be compact such that $supp(\phi) \subseteq K$. Then,

$$|T_f(\phi)| \leq \left(\int_K |f(x)|dx\right) \|\phi\|_{c^0} \quad for \quad all \phi \in D_K(\Omega).$$

■

Example 2.1.9 *[3]**(Dirac distribution)** Let $\delta : D(\mathbb{R}) \longrightarrow \mathbb{R}$ defined by $\langle \delta, \psi \rangle = \phi(0)$ for all $\phi \in D(\mathbb{R})$. Then, δ is a distribution called the Dirac distribution.*
Clearly, δ is linear. Let $\phi \in D(R)$, and $K \subset \mathbb{R}$ such that $supp(\phi) \subseteq K$. Then,

$$|\langle \delta, \psi \rangle| = |\phi(0)| \leq \|\phi(x)\|_{C^0}, \quad for\ \phi \in D_k(\mathbb{R}).$$

Example 2.1.10 *[3] Let $T : D(\mathbb{R}) \longrightarrow \mathbb{R}$ defined by $\langle T, \psi \rangle = \sum_{n\geq 1} a_n \psi(\frac{1}{n})$ and we assume that $\sum_{n\geq 1} |a_n| < \infty$. Then, $T \in D^{'}(\Omega)$.*
Indeed, T is well-defined and linear. Let K be a compact subset of $\mathbb{R}$ such that $supp(\psi) \subseteq K$, then

$$\begin{aligned} |\langle T, \psi \rangle| &= \left| \sum_{n\geq 1} a_n \psi(\frac{1}{n}) \right| \\ &\leq \sup_{x\in K} |\psi(x)| \sum_{n\in \mathbb{N}\cap K} |a_n|. \end{aligned}$$

Set $C = \sum_{n\in\mathbb{N}\cap K} |a_n|$.
Hence, $T \in D^{'}(\mathbb{R})$.

Definition 2.1.11 *[3] Let $(T_n)_{n\geq 1}$ be a sequence of distributions in $D^{'}(\Omega)$ and $T \in D^{'}(\Omega)$. We say that $T_n \longrightarrow T$ in $D^{'}(\Omega)$ if and only if for all $\phi \in D(\Omega)$,*

$$\langle T_n, \phi \rangle \longrightarrow \langle T, \phi \rangle \ as\ n \to \infty.$$

Example 2.1.12 *[3] Let $T_n = cosnx \in D^{'}(\mathbb{R})$. Let $\phi \in D(\mathbb{R})$ and $a > 0$ such that $supp(\phi) \subseteq [-a, a]$. Then,*

$$\begin{aligned} \langle T_n, \phi \rangle &= \int_{\mathbb{R}} cos(nx)\phi(x)dx \\ &= \int_{-a}^{a} cos(nx)\phi(x)dx \\ &= \left[\frac{1}{n} sin(nx)\phi(x)\right]_{-a}^{a} - \frac{1}{n}\int_{-a}^{a} sin(nx)\phi^{'}(x)dx \end{aligned}$$

Now,

$$0 \leq \left| \frac{1}{n}\int_{-a}^{a} sin(nx)\phi^{'}(x)dx \right| \leq \frac{1}{n}\int_{-a}^{a} \left|\phi^{'}(x)\right| dx \longrightarrow 0 \ as\ n \to \infty$$

Also,

$$\left[\tfrac{1}{n} sin(nx)\phi(x)\right]_{-a}^{a} \longrightarrow 0 \ as\ n \to \infty$$

It follows that;

$$\langle T_n, \phi \rangle \longrightarrow 0 = \langle 0, \phi \rangle \ as\ n \to \infty.$$

Hence, $T_n \longrightarrow 0$ in $D^{'}(\mathbb{R})$ as $n \to \infty$.

2.2 Derivatives in distribution sense

Definition 2.2.1 *[3] Let $T \in D^{'}(\Omega)$ and $\alpha \in \mathbb{N}^n$. We define $D^\alpha T$ by;*

$$\langle D^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle \ for\ all\ \phi \in D(\Omega).$$

Theorem 2.2.2 *[3] Let $T \in D^{'}(\Omega)$. Then, for all multi index, α we have that $D^\alpha T \in D^{'}(\Omega)$.*

Theorem 2.2.3 *[7] Let $(T_n)_{n\geq 0} \subseteq D^{'}(\Omega)$ such that $T_n \longrightarrow T$ in $D^{'}(\Omega)$. Then, for all $\alpha \in \mathbb{N}^n$,*

$$D^\alpha T_n \longrightarrow D^\alpha T \ in\ D^{'}(\Omega) \ as\ n \to \infty.$$

Proof . Since $T_n \longrightarrow T$ in $D^{'}(\Omega)$. Then, for all $\psi \in D(\Omega)$,

$$\langle T_n, \psi \rangle \longrightarrow \langle T, \psi \rangle \ \ as\ n \to \infty$$

Let $\alpha \in \mathbb{N}^n$. Then,

$$\langle D^\alpha T_n, \psi \rangle = (-1)^{|\alpha|} \langle T_n, D^\alpha \psi \rangle$$

But $\psi \in D(\Omega)$ implies that $D^\alpha \psi \in D(\Omega)$.
Hence, we have that

$$(-1)^{|\alpha|} \langle T_n, D^\alpha \psi \rangle \longrightarrow (-1)^{|\alpha|} \langle T, D^\alpha \psi \rangle .$$

Consequently,

$$\langle D^\alpha T_n, \psi \rangle \longrightarrow (-1)^{|\alpha|} \langle T, D^\alpha \psi \rangle = \langle D^\alpha T, \psi \rangle .$$

Hence, we obtain that,

$$D^\alpha T_n \longrightarrow D^\alpha T \ in\ D^{'}(\Omega)\ as\ n \to \infty.$$

∎

Definition 2.2.4 *[7] Let $f \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ a multi-index. Then, the distribution T_f admits a distributional derivative defined as*

$$D^\alpha T_f(\phi) = (-1)^{|\alpha|} T_{D^\alpha f}(\phi) = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \phi(x) dx,\ \ for\ all\ \phi \in D(\Omega).$$

If there exists $g \in L^1_{loc}(\Omega)$ such that

$$T_g = D^\alpha T_f,$$

That is,

$$\int_\Omega g(x)\phi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \phi(x) dx$$

Then, g is called the α^{th} derivative of f written as $g = D^\alpha f$.

Example 2.2.5 *[3] Consider the function*

$$f(x) = \begin{cases} 0, & if\, x \leq 0 \\ x, & if\, x > 0 \end{cases} \tag{2.2.1}$$

It derivative is the Heaviside function defined by,

$$H(x) = \begin{cases} 0, & if\, x \leq 0 \\ 1, & if\, x > 0 \end{cases} \tag{2.2.2}$$

Lemma 2.2.6 *[7] Let $f \in L^1_{loc}(\Omega)$ and suppose $g, \tilde{g} \in L^1_{loc}(\Omega)$ are the weak α^{th} derivative of f such that*

$$\int_\Omega f D^\alpha(\phi) dx = (-1)^{|\alpha|} \int_\Omega g\phi dx = (-1)^{|\alpha|} \int_\Omega \tilde{g}\phi dx\ \ for\ all\ \phi \in D(\Omega)$$

Then, we have that

$$g = \tilde{g}\ \ a.e\ in\ \Omega.$$

Proof . We begin the proof by stating the following

Lemma 2.2.7 *[3] Let $v \in L^1_{loc}(\Omega)$ with Ω non-empty open set in $\mathbb{R}^n$. If,*

$$\int_\Omega v(x)\phi(x)dx = 0, \ for \ all \ \phi \in D(\Omega).$$

Then, $v = 0$ a.e in Ω.

Since $g, \tilde{g} \in L^1_{loc}(\Omega)$. Then $g - \tilde{g} \in L^1_{loc}(\Omega)$.
Let $\phi \in D(\Omega)$. Consequently, we have that

$$\int_\Omega (g - \tilde{g})\phi = 0.$$

Hence, $g = \tilde{g}$ a.e in Ω. ∎

Definition 2.2.8 *[6] We define $C^m(\Omega)$ the space of functions m times continuously differentiable on Ω.*

Theorem 2.2.9 *[7] Let $\Omega \subseteq \mathbb{R}^n$ be open connected set and assume $u \in D'(\Omega)$, such that $\frac{\partial u}{\partial x_i} = 0$, for $i = 1, 2, ..., n$. Then, u is a constant function.*

Definition 2.2.10 *[6] Let w be a function with $x \in \mathbb{R}^n$ such that*

$$w \in D(\Omega), w(x) \geq 0, w(x) = 0 \ if \ |x| \geq 1 \ and \ \int_{\mathbb{R}^n} w(x)dx = 1.$$

$$for \ \rho > 0, \ the \ function \ w_\rho(x) = \rho^{-n} w(\tfrac{x}{\rho}) \ , \ x \in \mathbb{R}^n$$

$$with \ w_\rho \in D(\mathbb{R}^n), \ w_\rho(x) \geq 0, w_\rho(x) = 0 \ if \ |x| \geq \rho \ and \ \int_{\mathbb{R}^n} w_\rho(x)dx = 1 \ is \ called \ a \ mollifier.$$

Definition 2.2.11 *[1] Let $u \in L^1_{loc}(\mathbb{R}^n)$. Then, the mollification (or regularization) of u denoted as u_ρ is the convolution of u with the standard mollifier, that is,*

$$u_\rho(x) = (w_\rho * u)(x) = \int_{\mathbb{R}^n} w_\rho(x - y)u(y)dy.$$

Theorem 2.2.12 *[3] Let $u \in C^k(\mathbb{R}^n)$. Then, for all multi index α, with length $|\alpha| \leq k$ we have that $u * w_\rho \in C^k(\mathbb{R}^n)$ and $D^\alpha(u * w_\rho) = D^\alpha u * w_\rho$.*

Definition 2.2.13 *(Absolute continuity)*
Let $u : [a, b] \longrightarrow \mathbb{R}$. Then, u is said to be absolutely continuous if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any finite set of disjoint intervals in $[a, b]$. If

$$(x_1, x_1'), ..., (x_n, x_n') \subseteq [a, b] \ with \ \textstyle\sum_{i=1}^n |x_i - x_i'| < \delta, \ then$$
$$\textstyle\sum_{i=1}^n |u(x_i) - u(x_i')| < \epsilon.$$

Remark 2.2.14 *The existence of deriavtive of a function is related to the absolute continuity of the function.*
Let $u : [a, b] \longrightarrow \mathbb{R}$. Then, u is absolutely continuous if and only if there exists a function $v \in L^1(a, b)$ such that

$$u(x) = u(a) + \int_a^x v(t)dt, x \in [a, b].$$

Corollary 2.2.15 *[7] Let $u \in L^1_{loc}(a,b)$ have a distributional derivative $u^{'} \in L^1(a,b)$. Then, there exists an absolutely continuous function $\tilde{u}$ such that*

$$\tilde{u}(x) = u(x) \text{ for a.e } x \in (a,b).$$
$$u^{'}(x) = \lim_{h\to 0} \frac{\tilde{u}(x+h)-\tilde{u}(x)}{h}.$$

Corollary 2.2.16 *[3](**Leibnitz formula**) Let $u, v \in C^\infty(\Omega)$ and $\alpha, \beta \in \mathbb{N}^n$ a multi-index sets. Then,*

$$D^\alpha(uv) = \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\beta u D^{\alpha-\beta} v.$$

Definition 2.2.17 *[3] Let $\alpha \in C^\infty(\Omega)$ and $T \in D^{'}(\Omega)$. Then,*

$$\langle \alpha T, \psi \rangle = \langle T, \alpha\psi \rangle, \text{ for all } \psi \in D(\Omega)$$

Theorem 2.2.18 *[3] Let $T \in D^{'}(\Omega)$. Then, $\alpha T \in D^{'}(\Omega)$ for all $\alpha \in C^\infty(\Omega)$.*

Proof Let $\psi \in D(\Omega)$ and $\alpha \in C^\infty(\Omega)$. Then, $\alpha\psi \in D(\Omega)$ (since $supp(\alpha\psi) \subseteq supp(\alpha) \cap supp(\psi)$). Let K be compact set such that $supp(\psi) \subseteq K$. Since $T \in D^{'}(\Omega)$, then by applying Leibnitz formula there exists $C > 0$ and $n \in \mathbb{N}$ such that

$$|\langle T, \alpha\psi \rangle| \leq C\|\psi\|_{c^n}.$$

■

2.3 Sobolev spaces

2.3.1 Basic definitions

Definition 2.3.1 *[1] The Sobolev space, $W^{m,p}(\Omega)$ is the space of all functions u belonging to $L^p(\Omega)$ such that for every multi-index α, with $|(\alpha)| \leq m$, the derivative $D^\alpha u$ exists and belong to $L^p(\Omega)$. That is; $W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L_p(\Omega), |\alpha| \leq m\}$.*
The standard norm in $W^{m,p}(\Omega)$ is given by

$$\|u\|_{m,p} = \left(\int_\Omega \sum_{|\alpha|\leq m} |D^\alpha u(x)|^p dx \right)^{\frac{1}{p}}, \ 1 \leq p < \infty.$$
$$\|u\|_{m,\infty} = \sum_{|\alpha|\leq m} esssup_{x\in\Omega} |D^\alpha u(x)|, \ for \ p = \infty.$$

Theorem 2.3.2 *[6] $W^{m,p}(\Omega)$, is a Banach space, for $p \in [1, \infty]$.*

Proof . Let $(u_n)_{n\geq 0}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. Then, for $n, k \in \mathbb{N}$,

$$\|u_n - u_k\|_{m,p} = \left(\int_\Omega \sum_{|\alpha|\leq m} |D^\alpha u_n - D^\alpha u_k|^p dx \right)^{\frac{1}{p}} \longrightarrow 0 \ as \ n, k \to \infty.$$

Hence, for each α with $|\alpha| \leq m$. Then,

$$\left(\int_\Omega |D^\alpha u_n - D^\alpha u_k|^p \right)^{\frac{1}{p}} \longrightarrow 0 \ as \ n, k \to \infty.$$

Hence, $(D^\alpha u_n)_{n\geq 0}$ is a Cauchy sequence in $L^p(\Omega)$ for each α, such that $|\alpha| \leq m$.
Since $L^p(\Omega)$ is Banach, there exist $u, v \in L^p(\Omega)$ such that

$$u_n \longrightarrow u \ in \ L^p(\Omega), \ D^\alpha u_n \longrightarrow v \ in \ L^p(\Omega).$$

But $L^p(\Omega) \subset L^1_{loc}(\Omega)$. Hence, $u_n \longrightarrow u \ in \ L^1_{loc}(\Omega)$ and $D^\alpha u_n \longrightarrow v \ in \ L^1_{loc}(\Omega)$.
Consequently, $v = D^\alpha u \in L^p(\Omega)$. Hence, $u \in W^{m,p}(\Omega)$.
For $p = \infty$. Let $\epsilon > 0$ be given.
Let $(u_n)_{n\geq 0}$ be a Cauchy sequence in $W^{m,\infty}(\Omega)$. Then, there exists $N_o \in \mathbb{N}$ such that,

$$\|u_n - u_k\|_{m,\infty} = \sum_{|\alpha|\leq m} sup_{x\in\Omega}|D^\alpha u_n - D^\alpha u_k| < \epsilon, \text{ for all } n, k \geq N_o.$$

Hence, for each α with $|\alpha| \leq m$, $(D^\alpha u_n)_{n\geq 0}$ is Cauchy in $L^\infty(\Omega)$. Consequently, there exist $v = D^\alpha u \in L^\infty(\Omega)$ such that $D^\alpha u_n \longrightarrow D^\alpha u$ as $n \to \infty$.
Hence, fix n and letting $k \to \infty$, we get that

$$\|u_n - u\|_{m,\infty} = \sum_{|\alpha|\leq m} sup_{x\in\Omega}|D^\alpha u_n - D^\alpha u| < \epsilon, \text{ for all } n \geq N_o$$

Thus, $u \in W^{m,\infty}(\Omega)$.
Hence, $W^{m,p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$.
■

Proposition 2.3.3 *[1] The norm $\sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p}$ is equivalent to the standard norm in $W^{m,p}(\Omega)$.*

Proof . Let $u \in W^{m,p}(\Omega)$, $1 \leq p < \infty$. Then, we show that the norm

$$\|u\|^*_{W^{m,p}} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p}$$

is equivalent to the standard norm given by

$$\|u\|_{W^{m,p}} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p}\right)^{\frac{1}{p}}.$$

Now, we have that

$$\begin{aligned}\|u\|_{W^{m,p}} &= \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p}\right)^{\frac{1}{p}} \\ &\leq \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p} \\ &= \|u\|^*_{W^{m,p}}\end{aligned}$$

It remain to show that there exists a constant $k > 0$ such that

$$\|u\|^*_{W^{m,p}} \leq k\|u\|_{W^{m,p}} \tag{2.3.1}$$

Suppose for contradiction that (2.3.1) does not hold. Then, for all $n \in \mathbb{N}$, there exists a sequence $(u_n)_{n\geq 1}$ in $W^{m,p}(\Omega)$ such that

$$\|u_n\|^*_{W^{m,p}} > n\|u_n\|_{W^{m,p}}.$$

Hence, we have that

$$\frac{\|u_n\|_{W^{m,p}}}{\|u_n\|^*_{W^{m,p}}} < \frac{1}{n}.$$

Set $v_n = \frac{u_n}{\|u_n\|^*_{W^{m,p}}}$. Then, $\|v_n\|^*_{W^{m,p}} = 1$ and $\|v_n\|_{W^{m,p}} < \frac{1}{n} \longrightarrow 0$ as $n \to \infty$.
Thus, for all multi index, α with length $|\alpha| \leq m$, we have that

$$\|D^\alpha v_n\|_{L^p} \longrightarrow 0 \quad as \quad n \to \infty.$$

Consequently, we have that

$$\|v_n\|^*_{W^{m,p}} \longrightarrow 0 \quad as \quad n \to \infty.$$

But, $\|v_n\|^*_{W^{m,p}} = 1$. Hence, we have that $1 = 0$ (condradiction). Thus, (2.3.1) holds. This complete the proof. ■

Definition 2.3.4 *[1] We consider a special case of the Sobolev space for $p = 2$ denoted by $H^m(\Omega) = W^{m,2}(\Omega)$. It is a Hilbert space endowed with the following inner product,*

$$\langle u, v \rangle_{H^m} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u . D^\alpha v dx.$$

Similarly, we define $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

Definition 2.3.5 *[6] We define the space $W_{loc}^{m,p}(\Omega)$ as the space of functions which are locally in $W^{m,p}(\Omega)$. That is,*

$$W_{loc}^{m,p}(\Omega) = \{u \in L_{loc}^p(\Omega) : D^\alpha u \in L_{loc}^p(\Omega), \ for\ all\ \alpha\ such\ that\ |\alpha| \leq m\}.$$

Proposition 2.3.6 *[7] Let (a, b) be an open interval and $u \in W^{1,p}(a, b)$. Then, there exists an absolutely continuous function $f : (a, b) \longrightarrow \mathbb{R}$ having derivative $f' \in L^p(a, b)$ which coincides with $u \in W^{1,p}(a, b)$.*

Proof . Let $u \in W^{1,p}(a, b)$, then $u \in L^p(a, b) \subseteq L_{loc}^1(a, b)$. Then, by Corollary 2.2.15, there is an absolutely continuous function $f = u$ a.e on Ω and $f' = u' \in L^p(a, b)$.
■

Proposition 2.3.7 *[1] Let $u \in W_0^{m,p}(\Omega)$, for $p \in [1, \infty]$ and let*

$$\tilde{u}(x) = \begin{cases} u(x) & if\, x \in \Omega \\ 0 & if\, x \in \mathbb{R}^n \backslash \Omega \end{cases} \tag{2.3.2}$$

Then, $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$.

Proof . Let $u \in W_0^{m,p}(\Omega)$. Then, there exists a sequence $(u_n)_{n \geq 0} \subseteq D(\Omega)$, such that $u_n \longrightarrow u$ in $W^{m,p}(\Omega)$ as $n \to \infty$.
Now,

$$\tilde{u_n}(x) = \begin{cases} u_n(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^n \backslash \Omega \end{cases} \tag{2.3.3}$$

Then, $\tilde{u_n} \in D(\mathbb{R}^n), \ for\ all\ n \geq 0$.
Furthermore,

$$\|\tilde{u}_n - \tilde{u}\|_{W^{m,p}(\mathbb{R}^n)} = \|u_n - u\|_{W^{m,p}(\Omega)} \longrightarrow 0\ as\ n \to \infty.$$

Hence, $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$.
■

2.3.2 The space $W_0^{m,p}(\Omega)$

Definition 2.3.8 *[1] The subspace $W_0^{m,p}(\Omega)$ is defined as the closure of $D(\Omega)$ with respect to the norm in $W^{m,p}(\Omega)$.*

Theorem 2.3.9 *[6] Let $u \in W^{m,p}(\Omega)$ and $v \in W_0^{m,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\int_\Omega (D^\alpha u)\, v dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha v dx \ where\ |\alpha| \ \leq m.$$

Proof . Since $D(\Omega)$ is dense in $W_0^{m,p}(\Omega)$. Then, for $v \in W_0^{m,p}(\Omega)$ there exists a sequence $(v_n)_{n \geq 0} \in D(\Omega)$ such that

$$\|v_n - v\|_{Wm,p(\Omega)} \longrightarrow 0 \ as \ n \ \to \infty.$$

Now, by definition of the derivative of $D^\alpha u$, we have that,

$$\int_\Omega (D^\alpha u)\, v_n = (-1)^{|\alpha|} \int_\Omega u D^\alpha v_n, \ for \ |\alpha| \leq m.$$

We need to show that,

$$\int_\Omega (D^\alpha u)\, v_n \longrightarrow \int_\Omega (D^\alpha u)\, v, \ as \ n \to \infty.$$

Also,

$$(-1)^{|\alpha|} \int_\Omega u D^\alpha v_n \longrightarrow (-1)^{|\alpha|} \int_\Omega u D^\alpha v, \ as \ n \to \infty.$$

Now,

$$\begin{aligned}
\left| \int_\Omega (D^\alpha u)\, v_n - \int_\Omega (D^\alpha u)\, v \right| &= \left| \int_\Omega D^\alpha u (v_n - v) \right| \\
&\leq \int_\Omega |D^\alpha u| |v_n - v| \\
&\leq \|D^\alpha u\|_{L_p} \|v_n - v\|_{L_q} \\
&\leq \|D^\alpha u\|_{Wm,p} \|v_n - v\|_{W^{m,q}} \longrightarrow 0, \ as \ n \to \infty.
\end{aligned}$$

Hence,

$$\int_\Omega (D^\alpha u)\, v_n \longrightarrow \int_\Omega (D^\alpha u)\, v, \ as \ n \to \infty$$

Furthermore,

$$\begin{aligned}
\left| (-1)^{|\alpha|} \left(\int_\Omega u D^\alpha v_n - \int_\Omega u D^\alpha v \right) \right| &= \left| \int_\Omega u D^\alpha (v_n - v) \right| \\
&\leq \int_\Omega |u| |D^\alpha (v_n - v)| \\
&\leq \|u\|_{L_p} \|D^\alpha (v_n - v)\|_{L_q} \\
&\leq \|u\|_{Wm,p} \|D^\alpha (v_n - v)\|_{W^{m,q}} \longrightarrow 0, \ as \ n \to \infty.
\end{aligned}$$

Which implies that,

$$(-1)^{|\alpha|} \int_\Omega u D^\alpha v_n \longrightarrow (-1)^{|\alpha|} \int_\Omega u D^\alpha v, \ as \ n \to \infty.$$

Hence,

$$\int_\Omega (D^\alpha u)\, v dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha v dx \ where \ |\alpha| \leq m.$$

■

Definition 2.3.10 *[1] We define the space $W_0^{1,p}(\Omega)$ for $p \in [1, \infty)$ as the closure of $D(\Omega)$ in $W^{1,p}(\Omega)$. It is associated with the norm in $W^{1,p}(\Omega)$.*
For $p = 2$, the space $W_0^{1,2}(\Omega) = H_0^1(\Omega)$ is a Hilbert space associated with the inner product in $H^1(\Omega) = W^{1,2}(\Omega)$.

Theorem 2.3.11 *[1] Let Ω be bounded, open subset of $\mathbb{R}^n$. Then,*

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u \mid_{\partial\Omega} = 0\}.$$

Proposition 2.3.12 *[1] Let $G \in C^1(\mathbb{R})$ be such that $G(0) = 0$ and $|G'(s)| \leq M$ for all $s \in \mathbb{R}$ and for some constant M. Let $u \in W^{1,p}(\Omega)$ with $1 \leq p \leq \infty$. Then, $Gou \in W^{1,p}(\Omega)$ and $\frac{\partial}{\partial x_i}(Gou) = G'(u(x))\frac{\partial u}{\partial x_i}$ for all $i = 1, 2, ..., n$.*

Lemma 2.3.13 *[1] Let $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$ and assume that supp(u) is a compact subset of Ω. Then, $u \in W_0^{1,p}(\Omega)$.*

Theorem 2.3.14 *[1] Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Let $u \in W_0^{1,p}(\Omega)$ for $1 \leq p < \infty$. Then, there exists a constant c such that*

$$\left| \int_\Omega u \frac{\partial \psi}{\partial x_i} \right| \leq c\|\psi\|_{L^q} \quad for \quad all \quad \psi \in D(\Omega), \quad i = 1, 2, ..., n.$$

Proof . Let $u \in W_0^{1,p}(\Omega)$. Then, there exists a sequence $(u_n)_{n\geq 0}$ in $D(\Omega)$ such that $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$. Consequently, $u_n \longrightarrow u$ in $L^p(\Omega)$ and $\frac{\partial u_n}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i}$ in $L^p(\Omega)$ *for all* $i = 1, 2, ..., n$.
Let $\psi \in D(\Omega)$. Then,

$$\int_\Omega u_n \frac{\partial \psi}{\partial x_i} = -\int_\Omega \frac{\partial u_n}{\partial x_i}\psi.$$

Hence, we have that

$$\left| \int_\Omega u_n \frac{\partial \psi}{\partial x_i} \right| \leq \int_\Omega \left| \frac{\partial u_n}{\partial x_i}\psi \right|.$$

Then, by Hölder's inequality we get that

$$\left| \int_\Omega u_n \frac{\partial \psi}{\partial x_i} \right| \leq \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^p} \|\psi\|_{L^q}.$$

Taking limits as $n \to \infty$, we have that

$$\left| \int_\Omega u \frac{\partial \psi}{\partial x_i} \right| \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p} \|\psi\|_{L^q}.$$

Set $\max\limits_{1\leq i \leq n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p} = c < \infty$. Consequently,

$$\left| \int_\Omega u \frac{\partial \psi}{\partial x_i} \right| \leq c\|\psi\|_{L^q} \quad for \quad all \quad \psi \in D(\Omega), \quad i = 1, 2, ..., n.$$

■

Definition 2.3.15 *[1] The dual space of $W_0^{1,p}(\Omega)$ for $1 \leq p < \infty$ is denoted by $W^{-1,q}(\Omega)$. We define the dual of $H_0^1(\Omega)$ by $H^{-1}(\Omega)$.*
If Ω is bounded, then

$$W_0^{1,p}(\Omega) \subset L_2(\Omega) \subset W^{-1,q}(\Omega), \ for \ 1 \leq p < \infty.$$

In general, the dual space of $W_0^{m,p}(\Omega)$, for $1 \leq p < \infty$ is denoted by $W^{-m,q}(\Omega)$.

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 2.3.16 *[7]* ***(Poincáre's inequality)***
Suppose $\Omega \subseteq \mathbb{R}^n$ is bounded. Then, there exists a constant $C > 0$ (depending on Ω) such that,

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L_p}, \ for \ all \ u \in W_0^{1,p}(\Omega).$$

Proof . Let $u \in D(\Omega)$, thus support of u is compact. Then, for each $i = 1, 2, ..., n$ and $x = (x_1, x_2, ..., x_n)$ with $supp(u) \subseteq [x_i, a]$, we have that

$$u(x) = \int_a^{x_i} \nabla u(x) ds_i, \; for \; all \; i = 1, 2, ..., n.$$

Hence, we obtain that

$$|u(x)|^p \leq \left(\int_a^{x_i} |\nabla u(x)| ds_i \right)^p.$$

Consequently, by Holder's inequality, there exists some constant $C > 0$ such that

$$\int_\Omega |u(x)|^p dx \leq C^p \int_\Omega |\nabla u(x)|^p dx, \; for \; all \; u \in D(\Omega).$$

Hence,

$$\|u\|_{L^P} \leq C^p \|\nabla u\|_{L^p}, \text{ for all } u \in D(\Omega).$$

Let $u \in W_0^{1,p}(\Omega)$. Since $\overline{D(\Omega)} = W_0^{1,p}(\Omega)$ in $W^{1,p}(\Omega)$ norm. Then, there exists a sequence $(u_n)_{n \geq 0} \subseteq D(\Omega)$ such that

$$\|u_n - u\|_{W^{1,p}(\Omega)} \longrightarrow 0 \; as \; n \to \infty.$$

which implies that,

$$\int_\Omega |u_n - u|^p dx + \int_\Omega |\nabla(u_n - u)|^p dx \longrightarrow 0 \; as \; n \to \infty.$$

Hence,

$$\int_\Omega |u_n|^p dx \longrightarrow \int_\Omega |u|^p dx \; and \; \int_\Omega |\nabla u_n|^p dx \longleftrightarrow \int_\Omega |\nabla u|^p dx \; as \; n \to \infty.$$

Thus, for $(u_n)_{n \geq 0} \subseteq D(\Omega)$, we have that

$$\|u_n\|_{L^p} \leq C \|\nabla u_n\|_{L^p}$$

Then, taking limits as $n \to \infty$, we get that

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}, \; for \; all \; u \in W_0^{1,p}(\Omega).$$

■

2.3.3 Extension operator

Theorem 2.3.17 *[7]*
Let $\Omega \subset\subset \tilde{\Omega} \subseteq \mathbb{R}^n$ be open set such that the closure of Ω is compact subset of $\tilde{\Omega}$ and assume that the boundary of Ω, $\partial\Omega$ is C^1.Then, there exists a bounded linear operator $E : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^n)$ and a constant C such that,
(i). $Eu(x) = u(x)$ for a.e $x \in \Omega$
(ii). $Eu(x) = 0$, for $x \notin \tilde{\Omega}$
(iii). $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$

2.3.4 Trace Theorem

The trace theorem plays an important role in the existence and uniqueness of solution to boundary value problem to Partial Differential equations. It helps in extending the restriction of a function to its boundary.

Theorem 2.3.18 *[7] Let $\Omega \subseteq \mathbb{R}^n, n \geq 2$ be a bounded open set with C^1 boundary and let $1 \leq p < \infty$. Then, there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \longrightarrow L_p(\partial\Omega)$$

and a constant $C > 0$ depending on p and on the set Ω such that,

$$\|Tu\|_{L_p(\partial\Omega)} = \|u\|_{L_p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

2.4 Embedding Theorems

Definition 2.4.1 *[1] Let X_1 and X_2 be Banach spaces. We say that X_1 is continuously embedded into X_2(denoted as $X_1 \hookrightarrow X_2$) if for any $u \in X_1$, we have $u \in X_2$ and there exists a constant $C > 0$ such that,*

$$\|u\|_{X_2} \leq C\|u\|_{X_1}.$$

We define the embedding operator(Linear bounded operator) as $J : X_1 \longrightarrow X_2$ which takes $u \in X_1$ into the same u considered as an element of X_2.
The embedding operator J is compact if any bounded set in X_1 is a compact set in X_2.
If $X_1 \subset X_2$ and the embedding operator $J : X_1 \longrightarrow X_2$ is compact, then we say that X_1 is compactly embedded in X_2 (denoted as $X_1 \subset\subset X_2$).
Furthermore, $X_1 \subset\subset X_2$ means that for any bounded sequence $(u_n)_{n\geq 0} \subseteq X_1$, there exists a subsequence $(u_{n_k})_{k\geq 0}$ which converges in X_2.

Theorem 2.4.2 *[1] The following embeddings hold:*
(a). $W^{m_1,p}(\Omega) \hookrightarrow W^{m_2,p}(\Omega),\ if\ m_1 > m_2$.
(b). $W^{m,p}(\Omega) \hookrightarrow L_p(\Omega),\ if\ m > 0$.

Proof . Consider the injection map

$$T : W^{m_1,p}(\Omega) \longrightarrow W^{m_2,p}(\Omega)$$
$$\text{defined as } Tu = u \text{ for all } u \in W^{m_1,p}(\Omega).$$

Now, T is linear. We show that the graph of T, $G(T)$, is closed.
Let $(u_n)_{n\geq 0} \subset W^{m_1,p}(\Omega)$ such that $u_n \longrightarrow u$ in $W^{m_1,p}(\Omega)$ and $Tu_n \longrightarrow v$ in $W^{m_2,p}(\Omega)$. Then, we show that $Tu = v$.
Then, $Tu_n = u_n \longrightarrow v$ in $W^{m_2,p}(\Omega)$, which implies that,

$$Tu = u = v$$

Hence, T is closed. Then, by closed graph theorem we have that T is continuous. Thus,

$$W^{m_1,p}(\Omega) \hookrightarrow W^{m_2,p}(\Omega)$$

Similarly, (b) holds. ■

Definition 2.4.3 ***(Hölder's space)*** *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Then, the function space defined as*

$$C^{k,\gamma}(\overline{\Omega}) = \left\{ u \in C^k(\overline{\Omega}) : \|u\|_{C^{k,\gamma}} < \infty \right\}.$$

is called the Hölder's space with exponent γ, where $0 \leq \gamma \leq 1$.
We define the Hölder norm as follows

$$\|u\|_{C^{k,\gamma}} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C^0} + \sum_{|\alpha|=k} [D^\alpha u]_\gamma.$$

where

$$[u]_\gamma := \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\gamma}.$$

Theorem 2.4.4 *[7]* ***(Morrey's inequality)***
Assume $n < p < \infty$ and set $\rho = 1 - \frac{n}{p} > 0$ and let $u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$. Then, u is Hölder continuous. Moreover, there exists a constant C, depending only on p and n such that

$$\|u\|_{C^{0,\rho}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Proof . We state the following Lemmas to help us in the prove of Morrey's inequality.

Lemma 2.4.5 *[6] For $B(x,r) \subset \mathbb{R}^n$, $y \in B(x,r)$ and $u \in C^1(\mathbb{R}^n)$. Then,*

$$\fint_{B(x,r)} |u(x) - u(y)| dy \leq C \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} dy.$$

Lemma 2.4.6 *[6] For $n < p \leq \infty$, let $B(x,r) \subset \mathbb{R}^n$ and let $y \in B(x,r)$. Then,*

$$|u(x) - u(y)| \leq C|x-y|^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)} \text{ for all } u \in C^1(\mathbb{R}^n).$$

We now proceed with the prove of Morrey's inequality. Let $u \in C^1(\mathbb{R}^n)$, then

$$\begin{aligned}|u(x)| &\leq |u(x) - u(y)| + |u(y)| \\ &\leq C_1 |x-y|^{\rho} \|Du\|_{L^p(\mathbb{R}^n)} + |u(y)|.\end{aligned}$$

Taking the average of the Right hand side over the ball centered at x with radius 1, we get that,

$$\begin{aligned}|u(x)| &\leq \fint_{B(x,1)} |u(y)| dy + C_1 \|Du\|_{L^p(\mathbb{R}^n)} \\ &\leq C_2 \|u\|_{L^p(\mathbb{R}^n)} + C_1 \|u\|_{W^{1,p}(\mathbb{R}^n)}.\end{aligned}$$

Hence,

$$\begin{aligned}\|u\|_{C^{0,\rho}(\mathbb{R}^n)} &= sup_{x\in\mathbb{R}^n} |u(x)| + sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{\rho}} \\ &\leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}, \ \textit{for all } u \in C^1(\mathbb{R}^n).\end{aligned}$$

For some constant C.
Since $D(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$. Then, there exists a sequence $(u_n)_{n\geq 1} \subset D(\mathbb{R}^n)$ such that

$$u_n \longrightarrow u \text{ in } W^{1,p}(\mathbb{R}^n).$$

Hence, for $n, k \in \mathbb{N}$

$$\|u_n - u_k\|_{C^{0,\rho}(\mathbb{R}^n)} \leq C \|u_n - u_k\|_{W^{1,p}(\mathbb{R}^n)}.$$

This implies that (u_n) is Cauchy in $C^{0,\rho}(\mathbb{R}^n)$ which is Banach. Hence, there exists $\tilde{u} \in C^{0,\rho}(\mathbb{R}^n)$ such that

$$u_n \longrightarrow \tilde{u} \text{ in } C^{0,\rho}(\mathbb{R}^n).$$

Thus, by uniqueness, $\tilde{u} = u$ a.e in $\mathbb{R}^n$. Consequently, we have that

$$\|u\|_{C^{0,\rho}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \text{ for all } u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n).$$

■

Corollary 2.4.7 *[7]* ***(Embedding)***
Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1 boundary. Assume $n < p < \infty$ and set $\rho = 1 - \frac{n}{p} > 0$. Then, every function $u \in W^{1,p}(\Omega)$ coincides a.e with a function $\tilde{u} \in C^{0,\rho}(\Omega)$. Moreover, there exists a constant C such that

$$\|\tilde{u}\|_{C^{0,\rho}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Proof . Let $\tilde{\Omega} = \{x \in \mathbb{R}^n : d(x,\Omega) < 1\}$. Then $\Omega \subset\subset \tilde{\Omega} \subseteq \mathbb{R}^n$. Hence, by the extension Theorem 2.3.17 for each $u \in W^{1,p}(\Omega)$, there exists an extension map

$$E : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^n)$$

such that $Eu = u$ a.e in Ω.
Since $C^1(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$. Then, there exists a sequence $(g_n)_{n\geq 0} \subseteq C^1(\mathbb{R}^n)$ such that

$$g_n \longrightarrow Eu \text{ in } W^{1,p}(\mathbb{R}^n).$$

By the Morrey's inequality 2.4.4. Then, we get that

$$\|g_n - g_m\|_{C^{0,\rho}(\mathbb{R}^n)} \leq C\|g_n - g_m\|_{W^{1,p}(\mathbb{R}^n)}$$

Hence, $(g_n)_{n\geq 0}$ is Cauchy in $C^{0,\rho}(\mathbb{R}^n)$ and so converges to $g \in C^{0,\rho}(\mathbb{R}^n)$ for a.e $x \in \mathbb{R}^n$.
Then, by uniqueness $g(x) = Eu(x)$ for a.e $x \in \mathbb{R}^n$. So that, $g(x) = u(x)$ a.e $x \in \Omega$.
By the Morrey's inequality 2.4.4 and the extension Theorem 2.3.17. Thus, there exist $C_1, C_2 > 0$ such that

$$\|g\|_{C^{0,\rho}(\Omega)} = \|g\|_{C^{0,\rho}(\mathbb{R}^n)} \leq C_1\|u\|_{W^{1,p}(\mathbb{R}^n)} \leq C_2\|u\|_{W^{1,p}(\Omega)}.$$

■

Definition 2.4.8 *[7] Let $1 \leq p < n$, we define the Sobolev conjugate of p by*

$$p^* = \frac{np}{n-p} > p, \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Theorem 2.4.9 *[7]* ***(Gagliardo-Nirenberg inequality)***
Assume $1 \leq p < n$. Then, there exists a constant C, depending only on p and n such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\| \nabla u\|_{L^p(\mathbb{R}^n)}, \text{ for all } u \in C_c^1(\mathbb{R}^n)$$

Proof . Let $p = 1$. Then, $p^* = \frac{n}{n-1}$.
Let $u \in C_c^1(\mathbb{R}^n)$, then u has a compact support. Then, for each $i \in 1, 2, ..., n$ and $x = (x_1, x_2, ..., x_n)$.We have that,

$$u(x) = \int_{-\infty}^{x_i} D_{x_i}u(x)ds_i$$

Then,

$$|u(x)| \leq \int_{-\infty}^{x_i} |D_{x_i}u(x)|ds_i, \ 1 \leq i \leq n.$$

Hence,

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty} |D_{x_i}u(x)|ds_i\right)^{\frac{1}{n-1}} \tag{2.4.1}$$

We integrate (2.4.1) with respect to x_1. Then,

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}dx_1 \leq \left(\int_{-\infty}^{\infty} |D_{x_1}u(x)ds_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty}\prod_{i=2}^{n}\left(\int_{-\infty}^{\infty} |D_{x_i}u(x)|ds_i\right)^{\frac{1}{n-1}} dx_1$$

Then, by the generalized Holder's inequality, we get that

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}dx_1 \leq \left(\int_{-\infty}^{\infty} |D_{x_1}u(x)ds_1\right)^{\frac{1}{n-1}} \prod_{i=2}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |D_{x_i}u(x)|ds_i dx_1\right)^{\frac{1}{n-1}}$$

Following the same argument for $x_2, x_3, ..., x_n$ and after n integrations we get that

$$\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}dx_1...dx_n \leq \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} |D_{x_i}u(x)|dx_1...dx_n\right)^{\frac{1}{n-1}}$$

Then,

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}}dx \leq \left(\int_{\mathbb{R}^n} |\nabla u(x)|dx\right)^{\frac{n}{n-1}}$$

Hence,

$$\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |\bigtriangledown u(x)| dx \tag{2.4.2}$$

Consequently,

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \|\bigtriangledown u\|_{L^1(\mathbb{R}^n)}$$

Case 2: for $1 < p < n$.
We set $u = |g|^\beta$, where $\beta = \frac{p(n-1)}{n-p}$. Then, equation (2.4.2) becomes

$$\left(\int_{\mathbb{R}^n} |g(x)|^{\frac{\beta n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} \beta |g(x)|^{\beta-1} |\bigtriangledown g| dx$$

By Holder's inequality, we get that

$$\left(\int_{\mathbb{R}^n} |g(x)|^{\frac{\beta n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq \beta \left(\int_{\mathbb{R}^n} |g|^{(\beta-1)q}\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |\bigtriangledown g|^p\right)^{\frac{1}{p}} \tag{2.4.3}$$

But, $(\beta - 1)q = (\beta - 1)\frac{p}{p-1} = \frac{np}{n-p} = p^*$.
Also, $\frac{\beta n}{n-1} = p^*$ and $\frac{n-1}{n} - \frac{1}{q} = \frac{1}{p^*}$. Hence, equation (2.4.3) becomes,

$$\left(\int_{\mathbb{R}^n} |g(x)|^{p^*} dx\right)^{\frac{n-1}{n}} \leq \beta \left(\int_{\mathbb{R}^n} |g|^{p^*}\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |\bigtriangledown g|^p\right)^{\frac{1}{p}}$$

Consequently, we have that

$$\|g\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\bigtriangledown g\|_{L^p(\mathbb{R}^n)}.$$

■

Corollary 2.4.10 *[7] (Embedding)*
Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open domain with C^1 boundary and assume $1 \leq p < n$. Then, for every $q \in [1, p^*]$, there exists a constant C such that

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \text{ for all } u \in W^{1,p}(\Omega).$$

Proof . Let $\tilde{\Omega} = \{x \in \mathbb{R}^n : d(x, \Omega) < 1\}$. Then $\Omega \subset\subset \tilde{\Omega} \subseteq \mathbb{R}^n$. Hence, by the extension Theorem 2.3.17 for each $u \in W^{1,p}(\Omega)$, there exists an extension map

$$E : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^n)$$

such that $Eu = u$ a.e in Ω.
Then, applying the Gagliardo-Nirenberg inequality 2.4.9 and the extension Theorem 2.3.17. Thus, there exists C_1,C_2,C_3 and C_4 such that for each $u \in W^{1,p}(\Omega)$.

$$\|u\|_{L^q(\Omega)} \leq C_1 \|u\|_{L^{p^*}(\Omega)} \leq C_2 \|\bigtriangledown u\|_{L^p(\mathbb{R}^n)} \leq C_3 \|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C_4 \|u\|_{W^{1,p}(\Omega)}.$$

■

Theorem 2.4.11 *[1] Let $\Omega \subseteq \mathbb{R}$ be bounded. Then, $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, for all $1 \leq p \leq \infty$.*

Proof . Let $\Omega = \mathbb{R}$ and $v \in D(\mathbb{R})$. Then, for $1 \leq p \leq \infty$. Set $G(v) = |v|^{p-1}v$ and $w = G(v)$. Thus, $w \in D(\mathbb{R})$ and $w' = G'(v).v' = p|v|^{p-1}v'$. Consequently, for $x \in \mathbb{R}$, we have that

$$G(v(x)) = w(x) = \int_{-\infty}^{x} w'(t) dt.$$

Consequently, we have that

$$|v(x)|^p \leq \int_{-\infty}^{x} p|v(t)|^{p-1}|v^{'}(t)|dt.$$

Thus, we get

$$\|v\|_{L^\infty}^p \leq p\|v\|_{L^\infty}^{p-1} \int_{\mathbb{R}} |v^{'}(t)|dt.$$

Then, by Hölder's inequality we get that

$$\|v\|_{L^\infty} \leq c\|v^{'}\|_{L^p}.$$

Consequently, we have that

$$\|v\|_{L^\infty} \leq c\|v^{'}\|_{W^{1,p}(\mathbb{R})}.$$

Now, let $u \in W^{1,p}(\mathbb{R})$. Since $D(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$. Then, there exists a sequence $(u_n)_{n\geq 0}$ in $D(\mathbb{R})$ such that $u_n \longrightarrow u$ in $W^{1,p}(\mathbb{R})$. Consequently, we have that

$$\|u_n\|_{L^\infty} \leq c\|u_n\|_{W^{1,p}(\mathbb{R})}.$$

Taking limits as $n \to \infty$, we have that

$$\|u\|_{L^\infty} \leq c\|u\|_{W^{1,p}(\mathbb{R})} \quad for \quad all \quad u \in W^{1,p}(\mathbb{R}).$$

Furthermore, let $u \in W^{1,p}(\Omega)$, for $1 \leq p \leq \infty$. Then, by applying the extension Theorem 2.3.17 and using the fact that $D(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$ we have that for some constants c_1 and c_2,

$$\|u\|_{L^\infty(\Omega)} \leq c_1\|Eu\|_{W^{1,p}(\mathbb{R})} \leq c_2\|u\|_{W^{1,p}(\Omega)}.$$

■

Remark 2.4.12 *Let $\Omega \subseteq \mathbb{R}^n$ be bounded. Then, for $n \geq 2$ we have the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for all $n < p < \infty$.*

Theorem 2.4.13 *[6](**Arzela-Ascoli's Theorem**) Let $\overline{\Omega}$ be compact and suppose that $(u_n)_{n\geq 0} \subseteq C^0(\overline{\Omega})$, $\|u_n\|_{C^0} \leq M < \infty$ and the family $\{u_n : n \geq 0\}$ is equicontinuous.Then, there exists a subsequence $(u_{n_k})_{k\geq 0}$ such that $u_{n_k} \longrightarrow u$ uniformly on $\overline{\Omega}$.*

Theorem 2.4.14 *[7] Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^1 boundary. Then, for $n < p < \infty$ and $q \in [p, \infty]$, $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$.*

Proof . Since for every $u \in W^{1,p}(\Omega)$, for $n < p < \infty$ is Hölder continuous. Let $(u_n)_{n\geq 1} \subseteq W^{1,p}(\Omega)$ be a bounded sequence. Then, each u_n is Hölder continuous and uniformly bounded. Consequently, each u_n is equicontinuous and uniformly bounded. Thus, by Ascela- Ascoli's compactness theorem 2.4.13, there exists a subsequence $(u_{n_k})_{k\geq 1}$ which converges uniformly to a continuous function u on Ω. Since Ω is bounded, then by (**DCT**) 1.2.3 it implies that $\|u_{n_k} - u\|_{L^q(\Omega)} \longrightarrow 0$ as $k \to \infty$, for all $q \in [p, \infty]$. This implies that $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ compactly, whenever $n < p < \infty$ and $q \in [p, \infty]$.
■

Theorem 2.4.15 *[7](Rellich's Theorem)*
Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open domain with C^1 boundary and assume $1 \leq p < n$. Then, for every $q \in [1, p^*]$ (where $p^* = \frac{np}{n-p} > p$ is called the Sobolev conjugate of p), then $W^{1,p}(\Omega) \subset\subset L_q(\Omega)$ for $q \in [1, p^*]$.

Proof . Let $(u_n)_{n\geq 0}$ be a bounded sequence in $W^{1,p}(\Omega)$ and $\tilde{\Omega}$ be a bounded, open domain such that $\Omega \subset\subset \tilde{\Omega} \subset \mathbb{R}^n$ and $supp(u_n) \subseteq \tilde{\Omega}$. Then, by the extension Theorem $(u_n)_{n\geq 0}$ is defined on $\mathbb{R}^n$ and vanishes outside it compact support.
Since $q < p^*$ and $\tilde{\Omega}$ is bounded, then by Gagliardo-Nirenberg inequality (2.4.9) we have that,

$$\|u_n\|_{L^q(\mathbb{R}^n)} = \|u_n\|_{L^q(\tilde{\Omega})} \leq C_1\|u_n\|_{L^{p^*}(\tilde{\Omega})} \leq C_2\|u_n\|_{W^{1,p}(\tilde{\Omega})}$$

For some constants C_1 and C_2. Hence, (u_n) is uniformly bounded in $L^q(\mathbb{R}^n)$.
Let w_γ be a standard mollifier. Consider the mollified function,

$$u_n^\gamma = w_\gamma * u_n$$

with $supp(u_n^\gamma) \subset \tilde{\Omega}$ for all $n \geq 1$.

We show that $\|u_n^\gamma - u_n\|_{L^q(\tilde{\Omega})} \longrightarrow 0$ as $\gamma \to 0$.
Now,

$$\begin{aligned} u_n^\gamma(x) &= \int_{\mathbb{R}^n} w_\gamma(x-y)u_n(y)dy \\ &= \int_{|y|<\gamma} \frac{1}{\gamma^n} w(\frac{y}{\gamma})u_n(x-y)dy \end{aligned}$$

Let $\tilde{y} = \frac{y}{\gamma}$, then $y = \gamma\tilde{y}$ and $dy = \gamma d\tilde{y}$. Hence,

$$u_n^\gamma(x) = \int_{|\tilde{y}|<1} w(\tilde{y})u_n(x-\gamma\tilde{y})d\tilde{y}$$

But,

$$u_n(x-\gamma\tilde{y}) - u_n = \int_0^1 \frac{d}{dt}\left(u_n(x-\gamma t\tilde{y})\right)dt = -\gamma\int_0^1 \triangledown u_n(x-\gamma t\tilde{y})\tilde{y}dt$$

Then,

$$\begin{aligned} u_n^\gamma(x) - u_n(x) &= \int_{|\tilde{y}<1} w(\tilde{y})\left(u_n(x-\gamma\tilde{y}) - u_n\right)d\tilde{y} \\ &= -\gamma\int_{|\tilde{y}|<1} w(\tilde{y})\left(\int_0^1 \triangledown u_n(x-\gamma t\tilde{y})\tilde{y}dt\right)d\tilde{y} \end{aligned}$$

Consequently,

$$\int_{\tilde{\Omega}} |u_n^\gamma(x) - u_n(x)|dx \leq \gamma\int_{\tilde{\Omega}}\int_0^1 |\triangledown u_n(x-\gamma t\tilde{y})|dtdx$$

Let $z = x - \gamma t\tilde{y}$. Then,

$$\int_{\tilde{\Omega}} |u_n^\gamma(x) - u_n(x)|dx \leq \gamma\int_{\tilde{\Omega}} |\triangledown u_n(z)|dz$$

Hence,

$$\|u_n^\gamma - u_n\|_{L^1(\tilde{\Omega})} \leq \gamma C\|u_n\|_{W^{1,p}(\tilde{\Omega})} \text{ for some constant } C.$$

Using the L^p interpolation Lemma 1.1.9 for $0 < a < 1$, we get that

$$\|\|u_n^\gamma - u_n\|_{L^q(\tilde{\Omega})} \leq \|u_n^\gamma - u_n\|^a_{L^1(\tilde{\Omega})}\|u_n^\gamma - u_n\|^{1-a}_{L^{p^*}(\tilde{\Omega})} \leq C_0\gamma^a$$

For some constant C_0. For fixed $\delta > 0$ choose $\gamma > 0$ small enough such that

$$\|\|u_n^\gamma - u_n\|_{L^q(\tilde{\Omega})} \leq \|u_n^\gamma - u_n\|^a_{L^1(\tilde{\Omega})}\|u_n^\gamma - u_n\|^{1-a}_{L^{p^*}(\tilde{\Omega})} \leq C_0\gamma^a \leq \frac{\delta}{2}$$

Hence, $\|u_n^\gamma - u_n\|_{L^q(\tilde{\Omega})} \longrightarrow 0$ as $\gamma \to 0$.
But, $u_n^\gamma = w_\gamma * u_n$. Then, for some constants C_1 and C_2 and for fixed $\gamma > 0$, we have that

$$\|u_n^\gamma\|_{L^\infty(\tilde{\Omega})} \leq \|w_\gamma\|_{L^\infty(\tilde{\Omega})}\|u_n\|_{L^1(\tilde{\Omega})} \leq C_1\gamma^{-n} < \infty. \tag{2.4.4}$$

Also,

$$\| \nabla u_n^{\gamma}\|_{L^{\infty}(\tilde{\Omega})} \leq \| \nabla w_{\gamma}\|_{L^{\infty}(\tilde{\Omega})}\|u_n\|_{L^1(\tilde{\Omega})} \leq C_2\gamma^{-n-1} < \infty. \tag{2.4.5}$$

Hence, from (2.4.4) and (2.4.5) we have that (u_n^{γ}) is uniformly bounded and equicontinuous. Then, by Arzela-Ascoli's compactness Theorem there exists a subsequence $(u_{n_k}^{\gamma})_{k\geq 0}$ which converges uniformly to u^{γ} on $\tilde{\Omega}$.
Thus, we have that

$$\begin{aligned}\limsup_{j,k\to\infty}\|u_{n_k}-u_{n_j}\|_{L^q} &\leq \limsup_{j,k\to\infty}\Big(\|u_{n_k}-u_{n_k}^{\gamma}\|_{L^q}+\|u_{n_k}-u^{\gamma}\|_{L^q}+\|u^{\gamma}-u_{n_j}^{\gamma}\|_{L^q}+\|u_{n_j}-u_{n_j}^{\gamma}\|_{L^q}\Big)\\ &\leq \delta\end{aligned}$$

Then, for $\delta=\frac{1}{2},\frac{1}{4},\frac{1}{8}...$, we construct a subsequence $(u_{n_k})_{k\geq 0}$ such that

$$\limsup_{j,k\to\infty}\|u_{n_k}-u_{n_j}\|_{L^q(\tilde{\Omega})} \leq \frac{1}{2^k}$$

Thus,

$$\limsup_{j,k\to\infty}\|u_{n_k}-u_{n_j}\|_{L^q(\tilde{\Omega})} = 0$$

Hence, $(u_{n_k})_{n\geq 0}$ is a Cauchy sequence. Thus, it converges to $u\in L^q(\Omega)$. The proof is complete. ■

Corollary 2.4.16 *[7] Let Ω be an open subset of $\mathbb{R}^n$ of class C^1 with compact boundary, $\partial\Omega$. If $n>2$, then*

$$H^1(\Omega)\hookrightarrow L_{p^*}(\Omega),\ \textit{for}\ \tfrac{1}{p^*}=\tfrac{1}{2}-\tfrac{1}{n}.$$

where p^* is the Sobolev conjugate of p.

Proof . We apply the Rellich Theorem for $p=2$. Then, we have that $H^1(\Omega)=W^{1,2}(\Omega)\hookrightarrow L^{p^*}(\Omega)$. ■

Theorem 2.4.17 *[7] Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with C^1 boundary, then the following embeddings holds for the Sobolev Space, $W^{m,p}(\Omega)$,*

1. *If $m-\frac{n}{p}<0$, then $W^{m,p}(\Omega)\hookrightarrow L_q(\Omega)$, where $\frac{1}{q}=\frac{1}{p}-\frac{m}{n}$.*
2. *If $m-\frac{n}{p}=0$, then $W^{m,p}(\Omega)\hookrightarrow L_q(\Omega)$, for every $1\leq q\leq\infty$.*
3. *If $k\geq 0$ and $\gamma>0$. Then, $W^{m,p}(\Omega)\hookrightarrow C^{k,\gamma}(\Omega)$.*

Proof (1) Let $m-\frac{n}{p}<0$. Then, we show that $W^{m,p}(\Omega)\hookrightarrow L^q(\Omega)$ with $\frac{1}{q}=\frac{1}{p}-\frac{m}{n}$. Let $u\in W^{m,p}(\Omega)$. Then, we show that $u\in L^q(\Omega)$ and there exists a constant $c>0$ such that

$$\|u\|_{L^q}\leq c\|u\|_{W^{m,p}}.$$

Since $u\in W^{m,p}(\Omega)$, then $D^{\alpha}u\in W^{1,p}(\Omega)$ for all $\alpha\in\mathbb{N}^n$ with $|\alpha|\leq m-1$. Then, by (**Gadliardo-Nirenberg**) inequality (2.4.9) we have that

$$\|D^{\alpha}u\|_{L^{p^*}}\leq c\|u\|_{W^{m,p}},\quad \textit{where}\ \frac{1}{p^*}=\frac{1}{p}-\frac{1}{n}.$$

Consequently, $u\in W^{m-1,p^*}(\Omega)$. Set $p_1=p^*$ and repeating the same argument, we have that $u\in W^{m-2,p_1^*}$ and

$$\|u\|_{W^{m-2,p_1^*}}\leq c\|u\|_{W^{m-1,p_1}},\quad \textit{where}\ \frac{1}{p_1^*}=\frac{1}{p}-\frac{1}{n}.$$

Following the same pattern and setting $p_2 = p_1^*, p_3 = p_2^*, ..., p_j = p_{j-1}^*$ we get that

$$W^{m,p}(\Omega) \hookrightarrow W^{m-1,p_1}(\Omega) \hookrightarrow W^{m-2,p_2}(\Omega) \hookrightarrow ... \hookrightarrow W^{m-j,p_j}(\Omega), \quad where \quad \frac{1}{p_j} = \frac{1}{p} - \frac{j}{n}.$$

Doing this after m steps and set $p_m = q$, we get that

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad where \quad \frac{1}{q} = \frac{1}{p} - \frac{m}{n}.$$

(2). In the case when $mp = n$. Then, from (1), repeating the argument $(m-1)$ steps, we get that

$$W^{m,p}(\Omega) \hookrightarrow W^{1,n}(\Omega) \hookrightarrow W^{1,n-\epsilon}(\Omega), \quad where \quad \frac{1}{p_{m-1}} = \frac{1}{p} - \frac{m-1}{n} \quad and \quad for \quad any \quad \epsilon > 0.$$

Consequently, we have that

$$W^{1,n-\epsilon}(\Omega) \hookrightarrow L^q(\Omega), \quad where \quad q = \frac{n(n-\epsilon)}{n-(n-\epsilon)}.$$

Since, $\epsilon > 0$ is arbitrary, we have that

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad for \quad 1 \le q < \infty.$$

(3). Furthermore, let $k \ge 0$ and $\gamma > 0$ and let $u \in W^{m,p}(\Omega)$. From (1), we choose j to be the smallest integer such that $p_j > n$. Then, we have that $u \in W^{m-j,p_j}(\Omega)$ and

$$\frac{1}{p} - \frac{j}{n} = \frac{1}{p_j} < \frac{1}{n} < \frac{1}{p} - \frac{j-1}{n}.$$

Hence, for every multi-index α with $|\alpha| \le m-j-1$. Then, by Morrey's inequality (2.4.4) we get that

$$D^\alpha u \in W^{1,p_j}(\Omega) \hookrightarrow C^{0,\gamma}(\Omega), \quad with \quad \gamma = 1 - \frac{n}{p_j} = 1 - \frac{n}{p} + j.$$

Consequently, for all multi-index α with $|\alpha| \le m-j-1$ we have $u \in C^{m-j-1,\gamma}(\Omega)$.
Set $k = m-j-1$. Hence, we have that

$$W^{m,p}(\Omega) \hookrightarrow C^{k,\gamma}(\Omega).$$

■

Example 2.4.18 *[7] Let Ω be open unit ball in $\mathbb{R}^5$ and assume $u \in W^{4,2}(\Omega)$. Then, by applying the Gadliardo Nirenberg inequality (2.4.9) and Morrey's inequality (2.4.4) twice, we get that*

$$W^{4,2}(\Omega) \hookrightarrow W^{3,\frac{10}{3}}(\Omega) \hookrightarrow W^{2,10}(\Omega) \hookrightarrow C^{1,\frac{1}{2}}(\Omega).$$

where the net smoothness of $u \in W^{4,2}(\Omega)$ given by $m - \frac{n}{p} = 4 - \frac{5}{2} = \frac{3}{2}$.

Theorem 2.4.19 *[7]* ***(Poincáire Wittinger inequality)***
Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, connected open set with C^1 boundary. Then, for $1 \le p < \infty$, there exists a constant C depending only on p and Ω, such that

$$\left\| u - \fint_\Omega u dx \right\|_{L^p} \le C \left\| \nabla u \right\|_{L^p} \quad for \ all \ u \in W^{1,p}(\Omega) \tag{2.4.6}$$

where

$$\fint_\Omega u dx = \frac{1}{mes(\Omega)} \int_\Omega u dx.$$

Proof . Suppose (2.4.6) is false. Then, for $n \geq 1$ there exists $(u_n)_{n\geq 1} \subset W^{1,p}(\Omega)$ such that

$$\|u_n - \fint_\Omega u_n dx\|_{L^p(\Omega)} > n\| \bigtriangledown u_n\|_{L^p(\Omega)}$$

Then,

$$\frac{\| \bigtriangledown u_n\|_{L^p}}{\|u_n - \fint_\Omega u_n dx\|_{L^p}} < \frac{1}{n}.$$

Let $v_n = \frac{u_n - \fint_\Omega u_n dx}{\|u_n - \fint_\Omega u_n dx\|_{L^p}}$.
Then, $\|v_n\|_{L^p} = 1$, $\| \bigtriangledown v_n\|_{L^p(\Omega)} < \frac{1}{n}$ and

$$\fint_\Omega v_n dx = 0$$

.
Hence, $\| \bigtriangledown v_n\|_{L^p} \longrightarrow 0$ as $n \to \infty$.
But $(v_n)_{n\geq 1}$ is bounded in $W^{1,p}(\Omega)$ for $1 \leq p < n$. Then, by Rellich Theorem (2.4.15), there exists a subsequence $(v_{n_k})_{k\geq 1} \subseteq (v_n)_{n\geq 1}$ such that

$$v_{n_k} \longrightarrow v \text{ in } L^p(\Omega)$$

Also, for $n < p < \infty$ by the embedding (2.4.7). Then (v_n) is uniformly bounded and Holder's continuous. Then, by Arzela- Ascoli's compactness theorem, there exists a subsequence (v_{n_k}) which converges to v in L^∞. Then,

$$\bigtriangledown v_{n_k} \longrightarrow \bigtriangledown v \text{ in } L^p(\Omega).$$

Furthermore,

$$\| \bigtriangledown v_{n_k}\|_{L^p} \longrightarrow \| \bigtriangledown v\|_{L^p}.$$

Hence, $\bigtriangledown v = 0$ and since Ω is connected we have that $v = k$ a.e in Ω.
Also,

$$v_{n_k} \longrightarrow v \text{ in } L^p(\Omega)$$

Then,

$$\fint_\Omega v_{n_k} dx \longrightarrow \fint_\Omega v dx$$

But,

$$\fint_\Omega v_{n_k} dx = 0.$$

Consequently, we have that

$$\fint_\Omega v dx = 0.$$

Hence,

$$k = v = 0$$

.
But $v = 0$, implies that $\|v\|_{L^p} = 0$. Also,

$$v_{n_k} \longrightarrow v$$

Then,

$$\|v_{n_k}\|_{L^p} \longrightarrow \|v\|_{L^p}$$

But, $\|v_{nk}\|_{L^p} = 1$. Then, $\|v\|_{L^p} = 1$, which is a contradiction.
Hence, inequality (2.4.6) holds. ∎

CHAPTER 3

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Introduction
Partial differential equations are used majorly to formulate models involving functions of several variables. They are used to point out differs happenings in sound, heat, diffusion, electrostatics, electrodynamics and fluid dynamics.

A partial differential equation is a differential equation involving partial derivatives of a dependent variable with more than one independent variable. It is often denoted as PDE.

The order of a partial differential equation is the order of the highest derivative that appear in the equation.

A partial differential equation is said to be linear if the dependent variable and all its partial derivatives are expressed as a linear combination in which the coefficients are independent of the dependent variable and it derivatives. A partial differential equation which is not linear is said to be non-linear.

The following examples gives an illustration of a linear PDE and a non-linear PDE.
Let $u(x, y)$ be a function of two independent variables x and y. Then, the equation

$$\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial u}{\partial y} = x^3$$

is a second order non-linear PDE.
The equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is a second order linear PDE.
Classification of partial differential equations.
Consider the general linear partial differential equation of second order of n variables given by

$$\sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i} + cu = d$$

The classification of the PDE depends on the signs of the eigenvalues of the $n \times n$ coefficient matrix, $A = [a_{i,j}]_{1 \leq i,j \leq n}$.
Let p denotes the number of positive eigenvalues and let z denotes the number of zero eigenvalues. Then, the partial differential equation is classified as follows
(a) It is elliptic if $z = 0$ and $p = n$ or $z = 0$ and $p = 0$.
(b) It is parabolic if $z > 0$.

(c) it is hyperbolic if $z = 0$ and $p = 1$ or $z = 0$ and $p = n - 1$.
Example of an elliptic PDE is the Laplace equation given by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0.$$

Example of a parabolic PDE is the heat equation given by

$$\frac{\partial u}{\partial t} = c^2 \Delta u, \; where \; c \; is \; constant.$$

Example of a hyperbolic PDE is the wave equation given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \; where \; c \; is \; constant.$$

In this chapter, we focus on solving some linear and non-linear elliptic partial differential equations with homogeneous and non-homogeneous boundary conditions. We show the existence of a unique weak solution to some Dirichlet and Neumann problems with homogeneous and non-homogeneous boundary conditions. Also, some mixed problems will be discussed.
Furthermore, we discuss the concept of monotone operators which aid us in solving the non-linear elliptic problems. We begin the chapter by discussing some important concepts needed to show the existence of the solution to the problems we shall solve.

3.1 Definitions and existence Theorem

Definition 3.1.1 *We define the Laplacian,*Δ*, of a function* u *by,*

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

We define $\frac{\partial u}{\partial \mathbf{n}} = \nabla u.\mathbf{n}$ as the outward normal vector of u, where $\mathbf{n}$ is the unit normal vector to $\partial\Omega$, pointing outward.

Theorem 3.1.2 *[3](**Green's formula**)*
Let $u \in H^2(\Omega)$ *and* $v \in H^1(\Omega)$*. Then,*

$$\int_{\Omega} (\Delta u) v = \int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{n}} v d\sigma - \int_{\Omega} \nabla u . \nabla v.$$

where $d\sigma$ *is the surface measure on* $\partial\Omega$*.*

Lemma 3.1.3 *[3] Let* $A \subseteq H$ *where* H *is a Hilbert space. Then,* $\overline{A} = H$ *if and only if* $A^{\perp} = \{0\}$*, where*

$$A^{\perp} = \{y \in H : \langle x, y \rangle = 0 \; for \; all \; x \in A\}.$$

Proof Suppose $\overline{A} = H$, then we show that $A^{\perp} = \{0\}$.
Let $y \in A^{\perp}$. Then, for all $x \in A$ we have that

$$\langle x, y \rangle = 0.$$

Consequently, $y \in H$ implies that there exists a sequence $(x_n)_{n \geq 0} \subseteq A$ such that $x_n \longrightarrow y$ as $n \to \infty$. Hence, we obtain that

$$\langle x_n, y \rangle = 0.$$

Taking limit as $n \to \infty$, we get that

$$\langle y, y \rangle = \|y\|^2 = 0.$$

Thus, $A^{\perp} = 0$ (since $y \in A^{\perp}$ was arbitrarily).

Conversely, suppose that $A^{\perp} = \{0\}$, then we show that $\overline{A} = H$.
For contradiction, suppose $\overline{A} \neq H$. Then, there exists $\tilde{y} \in H$, $\tilde{y} \neq 0$ such that $\langle \tilde{y}, x \rangle = 0$ for all $x \in A$.
But, $\langle x, y \rangle = 0$ for all $x \in A$, implies that $y = 0$. Hence, $\tilde{y} = 0$ which is a contradiction.
Thus, $\overline{A} = H$. ∎

Remark 3.1.4 *Let $u \in H^2(\Omega)$, then $\frac{\partial u}{\partial x_i} \in H^1(\Omega)$. Then, by Trace Theorem, we get that $\frac{\partial u}{\partial x_i} |_{\partial\Omega} \in L^2(\partial\Omega)$.*

Theorem 3.1.5 *[3](**Density of traces**)*
Let $\Omega \subseteq \mathbb{R}^n$ be smooth. Then,

$$\overline{\{u |_{\partial\Omega} : u \in H^1(\Omega)\}} = L^2(\partial\Omega).$$

Theorem 3.1.6 *[1](**Lax-Milgram Theorem**)*
Let H be a real Hilbert space and a, a bilinear continuous form on H. Suppose a is coercive, that is, there exists $\alpha > 0$ such that

$$a(u, u) \geq \alpha \|u\|^2, \;\; for \; all \; u \in H.$$

Then, for all $f \in H^*$, there exists a unique $v \in H$ such that

$$a(v, u) = f(u), \;\; for \; all \; u \in H.$$

3.2 Applications to elliptic partial differential equations

In this section, we solve some linear elliptic PDEs with homogeneous and non-homogeneous boundary conditions.
Consider the Dirichlet problem with homogeneous boundary condition given by

$$\begin{cases} -\Delta u + u = f & in \; \Omega, \;\; f \in L^2(\Omega) \\ u = 0 & on \; \partial\Omega. \end{cases} \tag{3.2.1}$$

Due to the Diritchlet boundary condition given, we look for the solution

$$u \in H_0^1(\Omega) = \left\{ u \in H^1(\Omega) : u |_{\partial\Omega} = 0 \right\},$$

which satisfies (3.2.1).
Let $u \in H_0^1(\Omega)$ be a solution of (3.2.1). Then, for all $v \in H_0^1(\Omega)$, we have that

$$\int_\Omega -\Delta u.v dx + \int_\Omega uv dx = \int_\Omega f v dx, \;\; for \; all \; v \in H_0^1(\Omega).$$

Then, by Green's formula, we get that

$$\int_\Omega \nabla u.\nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}.v d\sigma + \int_\Omega uv dx = \int_\Omega f v dx \tag{3.2.2}$$

Since $v \in H_0^1(\Omega)$, then (3.2.2) becomes

$$\int_\Omega \nabla u.\nabla v dx + \int_\Omega uv dx = \int_\Omega f v dx.$$

Set $H = H_0^1(\Omega)$. Let $T_1 : H \times H \longrightarrow \mathbb{R}$ defined by

$$T_1(u, v) = \int_\Omega \nabla u.\nabla v dx + \int_\Omega uv dx, \;\; for \; all \; u, v \in H.$$

and let $T_2 : H \longrightarrow \mathbb{R}$ defined by

$$T_2(v) = \int_\Omega fvdx, \;\; for \; all \; v \in H.$$

Now, T_1 is bilinear. Furthermore,

$$\begin{aligned} |T_1(u,v)| &\leq \int_\Omega |\nabla u||\nabla v|dx + \int_\Omega |u||v|dx \\ &\leq \|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + \|u\|_{L^2}\|v\|_{L^2} \end{aligned}$$

By Poincaire's inequality, for some $C > 0$, we get that

$$|T_1(u,v)| \leq C\|u\|_{H_0^1}.$$

Hence, T_1 is continuous. Also,

$$T_1(u,u) = \int_\Omega |\nabla u|^2 + \int_\Omega |u|^2 \geq \|u\|_{H_0^1}$$

Consequently, T_1 is coercive. Also, we have that $T_2 \in H^*$.
Then, by the Theorem 3.1.6, there exists a unique $u \in H_0^1(\Omega)$ which satisfies (3.2.1).
We show that $u \in H_0^1(\Omega)$ satisfies (3.2.1).
Now,

$$\int_\Omega \nabla u.\nabla v + \int_\Omega uv = \int_\Omega fv, \;\; for \; all \; v \in H_0^1(\Omega). \tag{3.2.3}$$

But,

$$\begin{aligned} \int_\Omega \nabla u.\nabla v &= \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}.\frac{\partial v}{\partial x_i} \\ &= \sum_{i=1}^n \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle \\ &= \sum_{i=1}^n - \left\langle \frac{\partial^2 u}{\partial x_i^2}, v \right\rangle \\ &= \langle -\Delta u, v \rangle . \end{aligned}$$

Hence, (3.2.3) becomes

$$\langle -\Delta u, v \rangle + \langle u, v \rangle = \langle f, v \rangle , \;\; for \; all \; v \in H_0^1(\Omega).$$

Consequently, we have that

$$-\Delta u + u = f \; in \; \Omega$$

Since $u \in H_0^1(\Omega)$, then $u = 0$ on $\partial\Omega$.

Consider the Neumann problem with a homogeneous boundary condition given by

$$\begin{cases} -\Delta u + u = f & in \; \Omega, \;\; f \in L^2(\Omega) \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & on \; \partial\Omega. \end{cases} \tag{3.2.4}$$

We seek a unique weak solution $u \in H^1(\Omega)$ which satisfies (3.2.4).
Let $u \in H^1(\Omega)$ be a solution to (3.2.4). Then, for all $v \in H^1(\Omega)$, we get that

$$\int_\Omega -\Delta u.vdx + \int_\Omega uvdx = \int_\Omega fvdx.$$

Then, by Green's formula we get that

$$\int_{\Omega} \nabla u.\nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} v d\sigma + \int_{\Omega} uv dx = \int_{\Omega} fv dx.$$

But, $\frac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial\Omega$. Thus, we have that

$$\int_{\Omega} \nabla u.\nabla v dx + \int_{\Omega} uv dx = \int_{\Omega} fv dx.$$

Set $H = H^1(\Omega)$. Let $T_1 : H \times H \longrightarrow \mathbb{R}$ defined by

$$T_1(u, v) = \int_{\Omega} \nabla u.\nabla v + \int_{\Omega} uv, \text{ } for \text{ } all \text{ } u, v \in H.$$

Now, T_1 is bilinear, continuous and coercive. furthermore, let $T_2 : H \longrightarrow \mathbb{R}$ defined by

$$T_2(v) = \int_{\Omega} fv, \text{ } for \text{ } all \text{ } v \in H.$$

Then, $T_2 \in H^*$. Consequently, by Theorem (3.1.6), there exists a unique weak solution $u \in H^1(\Omega)$ which satisfies (3.2.4).
We show that $u \in H^1(\Omega)$ satisfies (3.2.4).
Now,

$$\int_{\Omega} \nabla u.\nabla v + \int_{\Omega} uv = \int_{\Omega} fv. \tag{3.2.5}$$

But,

$$\int_{\Omega} \nabla u.\nabla v = \langle -\Delta u, v \rangle, \text{ } for \text{ } all \text{ } v \in H^1(\Omega)$$

Hence, (3.2.5) becomes

$$\langle -\Delta u, v \rangle + \langle u, v \rangle = \langle f, v \rangle, \text{ } for \text{ } all \text{ } v \in H^1(\Omega).$$

Thus, we get that

$$-\Delta u + u = f \text{ } in \text{ } \Omega.$$

It remain to show that $\frac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial\Omega$.
Since $u \in H^1(\Omega)$ satisfies $-\Delta u + u = f$. Then, for all $v \in H^1(\Omega)$ we get that

$$-\int_{\Omega} \Delta u.v + \int_{\Omega} uv = \int_{\Omega} fv.$$

Hence,

$$\int_{\Omega} \nabla u.\nabla v - \int_{\Omega} \frac{\partial u}{\partial \mathbf{n}} v d\sigma + \int_{\Omega} uv = \int_{\Omega} fv. \tag{3.2.6}$$

from (3.2.5) and (3.2.6), we get that

$$\int_{\Omega} \frac{\partial u}{\partial \mathbf{n}} v d\sigma = 0, \text{ } for \text{ } all \text{ } v \in H^1(\Omega).$$

Consequently, $\frac{\partial u}{\partial \mathbf{n}} \in \left\{ u \mid_{\partial\Omega} : u \in H^1(\Omega) \right\}^{\perp}$. Then by the lemma(3.1.3), we have that

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ } on \text{ } \partial\Omega.$$

We consider a mixed problem of Diritchlet and Neumann homogeneous boundary conditions given by

$$\begin{cases} -\Delta u = f & in \text{ } \Omega, \text{ } f \in L^2(\Omega) \\ u = 0 & on \text{ } \Gamma_0, \text{ } \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ } on \text{ } \Gamma_1. \end{cases} \tag{3.2.7}$$

where $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $mes(\Gamma_0) > 0$, $mes(\Gamma_1) > 0$ and Ω is connected and smooth in $\mathbb{R}^n$.
Let $H = \left\{u \in H^1(\Omega) : u = 0 \ on \ \Gamma_0\right\}$. Then, we show that H is a closed subset of $H^1(\Omega)$.
Let $(u_n)_{n\geq 0} \subseteq H$ such that $u_n \longrightarrow u$ in $H^1(\Omega)$. Then, by Trace Theorem (2.3.18)

$$u_n \longrightarrow u, \ in \ L^2(\partial\Omega).$$

By converse of **DCT** (1.2.6), there exists a subsequence $(u_{n_k})_{k\geq 0} \subseteq (u_n)_{n\geq 0}$ such that

$$u_{n_k}(x) \longrightarrow u(x) \ a.e \ on \ \Gamma_0.$$

But, $u_{n_k} = 0$ on Γ_0. Hence, $u = 0$ on Γ_0.
Consequently, H is closed . Thus, H is a Hilbert space.
We seek a unique solution $u \in H$ which satisfies (3.2.7).
Let $u \in H$ be a solution of (3.2.7). Then, for all $v \in H$, we have that

$$-\int_\Omega \Delta u.v = \int_\Omega fv.$$

Then, by Green's formular, we get that

$$\int_\Omega \nabla u.\nabla v - \int_\Omega \frac{\partial u}{\partial \mathbf{n}}.v = \int_\Omega fv$$

Hence,

$$\int_\Omega \nabla u.\nabla v - \int_{\Gamma_0} \frac{\partial u}{\partial \mathbf{n}}.v - \int_{\Gamma_1} \frac{\partial u}{\partial \mathbf{n}}.v = \int_\Omega fv$$

But, $v = 0$ on Γ_0 and $\frac{\partial u}{\partial \mathbf{n}} = 0$ on Γ_1. Hence, we have that

$$\int_\Omega \nabla u.\nabla v = \int_\Omega fv.$$

Let $T_1 : H \times H \longrightarrow \mathbb{R}$ defined by

$$T_1(u,v) = \int_\Omega \nabla u.\nabla v, \ for \ all \ u,v \in H.$$

Also, let $T_2 : H \longrightarrow \mathbb{R}$ defined by

$$T_2(v) = \int_\Omega fv, \ for \ all \ v \in H.$$

Now, T_2 is a bounded linear map. Thus, $T_2 \in H^*$.
Furthermore, T_1 is bilinear and continuous. We show that T_1 is coercive. That is, there exists $\alpha > 0$ such that

$$T_1(u,u) \geq \alpha\|u\|^2_{H^1}.$$

For contradiction, suppose T_1 is not coercive. Then, for all $n \geq 1$, there exists $(u_n)_{n\geq 1} \subset H$ such that

$$T_1(u_n,u_n) < \frac{1}{n}\|u_n\|^2_{H^1}.$$

Then,

$$\int_\Omega |\nabla u_n|^2 < \frac{1}{n}\|u_n\|^2_{H^1}$$

Set $v_n = \frac{u_n}{\|u_n\|_{H^1}}$. Then,

$$\int_\Omega |\nabla v_n|^2 < \frac{1}{n}.$$

Hence,

$$\nabla v_n \longrightarrow 0 \ in \ L^2(\Omega)$$

But, $\|v_n\|_{H^1} = 1$. Since $(v_n)_{n\geq 1}$ is bounded in $H^1(\Omega)$. Then, by Rellich's Theorem 2.4.15 there exists a subsequence $(v_{n_k})_{k\geq 1} \subseteq (v_n)_{n\geq 1}$ which converges to v in $L^2(\Omega)$.
Consequently,

$$\nabla v_{n_k} \longrightarrow \nabla v \ in \ L^2(\Omega).$$

Then, by uniqueness of limits, we have that

$$\nabla v = 0 \ a.e \ on \ \Omega.$$

Since Ω is connected, then $v = c$, where c is constant.
Thus, $v_{n_k} \longrightarrow k$ in $H^1(\Omega)$. Then, by Trace Theorem 2.3.18 we have that

$$v_{n_k} \longrightarrow c \ in \ L^2(\partial\Omega).$$

Consequently, we get that

$$v_{n_k} \longrightarrow c \ in \ L^2(\Gamma_0).$$

That is,

$$\int_{\Gamma_0} |v_{n_k}|^2 d\sigma \longrightarrow c^2 mes(\Gamma_0).$$

But, $v_{n_k} = 0$ on Γ_0. Hence, $c^2 mes(\Gamma_0) = 0$
Which implies that, $c = 0$. Thus, we have that

$$v_{n_k} \longrightarrow 0 \ in \ H^1(\Omega).$$

But, $\|v_{n_k}\|_{H^1} = 1$ which implies that $1 = 0$ (Impossible). Consequently, T_1 is coercive. Then, by Theorem 3.1.6 there exist a unique weak solution $u \in H$ which satiefies (3.2.7). We show that $u \in H$ satisfies (3.2.7). Now,

$$\int_\Omega \nabla u.\nabla v = \int_\Omega fv, \ for \ all \ v \in H.$$

But,

$$\int_\Omega \nabla u.\nabla v = \langle -\Delta u, v\rangle.$$

Hence, we get that

$$\langle -\Delta u, v\rangle = \langle f, v\rangle \ \ for \ all \ v \in H,$$

which implies that

$$-\Delta u = f \ in \ \Omega.$$

Also, $u \in H$ implies that $u = 0$ on Γ_0.
It remain to show that $\frac{\partial u}{\partial \mathbf{n}} = 0$ on Γ_1.
Since $u \in H$ satiesfies $-\Delta u = f$. Then, for all $v \in H$, we have that

$$-\int_\Omega \Delta u.v = \int_\Omega fv.$$

Thus,

$$\int_\Omega \nabla u.\nabla v - \int_\Omega \frac{\partial u}{\partial \mathbf{n}} v d\sigma = \int_\Omega fv.$$

Consequently,

$$\int_{\Gamma_1} \frac{\partial u}{\partial \mathbf{n}} v d\sigma = 0 \ for \ all \ v \in H.$$

Hence, $\frac{\partial u}{\partial \mathbf{n}} \in \{u\,|_{\Gamma_1} \colon u \in H\}^{\perp}$.
By the Theorem 3.1.5 $\{u\,|_{\Gamma_1} \colon u \in H\}$ is dense in $L^2(\Gamma_1)$. Consequently, by lemma 3.1.3

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \ on\ \Gamma_1.$$

We consider the non-homogeneous Neumann problem given by

$$\begin{cases} -\Delta u + u = f & in\ \Omega,\ f \in L^2(\Omega) \\ \frac{\partial u}{\partial \mathbf{n}} = g & on\ \partial\Omega,\ \ g \in L^2(\partial\Omega). \end{cases} \tag{3.2.8}$$

We seek a unique solution $u \in H^1(\Omega)$ that satisfies (3.2.8).
Let $u \in H^1(\Omega)$ be a solution of (3.2.8). Then, for all $v \in H^1(\Omega)$ we have that

$$\int_\Omega -\Delta u.v dx + \int_\Omega u.v dx = \int_\Omega f.v dx.$$

Then, by Green's formula we get that

$$\int_\Omega \nabla u.\nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}.v d\sigma + \int_\Omega u.v dx = \int_\Omega f v dx.$$

But, $\frac{\partial u}{\partial \mathbf{n}} = g$ on $\partial\Omega$. Hence, we have that

$$\int_\Omega \nabla u.\nabla v + \int_\Omega u.v = \int_\Omega f v + \int_{\partial\Omega} g.v d\sigma.$$

Set $H = H^1(\Omega)$ and let $a : H \times H \longrightarrow \mathbb{R}$ defined by

$$a(u, v) = \int_\Omega \nabla u.\nabla v + \int_\Omega u.v \ for\ all\ u, v \in H.$$

Now, a is bilinear, continuous and coercive.
Let $L : H \longrightarrow \mathbb{R}$ defined by

$$L(v) = \int_\Omega f v + \int_{\partial\Omega} g.v d\sigma,\ for\ all\ v \in H.$$

Now, L is linear. Also,

$$\begin{aligned} |L(v)| &\leq \int_\Omega |f.v| + \int_{\partial\Omega} |g.v| d\sigma \\ &\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \\ &\leq \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right) \|v\|_{H^1} \end{aligned}$$

Hence, L is bounded. Consequently, we have that $L \in H^*$.
Then, by the Theorem 3.1.6, there exists a unique $u \in H^1(\Omega)$ that satisfies (3.2.8).
We show that $u \in H^1(\Omega)$ satisfies (3.2.8).
Now,

$$\int_\Omega \nabla u.\nabla v + \int_\Omega u.v = \int_\Omega f v + \int_{\partial\Omega} g.v d\sigma \tag{3.2.9}$$

For $v \sim \psi \in D(\Omega)$. Then, we have that

$$\int_{\partial\Omega} g.v = 0.$$

Hence, we have

$$\int_\Omega \nabla u.\nabla v + \int_\Omega u.v = \int_\Omega f v.$$

Which implies that

$$\langle -\Delta u, v\rangle + \langle u, v\rangle = \langle f, v\rangle\,, \ for\ all\ v \in H^1(\Omega).$$

Consequently, $-\Delta u + u = f$ in Ω, $f \in L^2(\Omega)$.
Since u satisfies $-\Delta u + u = f$. Then, for all $v \in H^1(\Omega)$ we get that

$$-\int_\Omega \Delta u.v + \int_\Omega u.v = \int_\Omega fv.$$

Hence,

$$\int_\Omega \nabla u.\nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}.vd\sigma + \int_\Omega u.v = \int_\Omega f.v \tag{3.2.10}$$

From equation (3.2.9) and equation (3.2.10), we get that

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial \mathbf{n}} - g\right).v = 0,\ for\ all\ v \in H^1(\Omega).$$

Consequently, $\frac{\partial u}{\partial \mathbf{n}} - g \in \left\{u\mid_{\partial\Omega}: u \in H^1(\Omega)\right\}^\perp$.
Thus, we apply Theorem 3.1.5 and lemma 3.1.3 to obtain that

$$\frac{\partial u}{\partial \mathbf{n}} - g = 0,\ on\ \partial\Omega.$$

Hence, we have that

$$\frac{\partial u}{\partial \mathbf{n}} = g\ on\ \partial\Omega.$$

Let Ω be smooth and connected in $\mathbb{R}^n$. We consider the mixed problem with a non-homogeneous boundary condition given by

$$\begin{cases} -\Delta u = f & in\ \Omega,\ f \in L^2(\Omega) \\ u = 0 & on\ \Gamma_0,\ \frac{\partial u}{\partial \mathbf{n}} = g\ on\ \Gamma_1. \end{cases} \tag{3.2.11}$$

where $mes(\Gamma_0) > 0$, $mes(\Gamma_1) > 0$ and $g \in L^2(\Gamma_1)$.
Let $H = \left\{u \in H^1(\Omega) : u = 0\ on\ \Gamma_0\right\}$ which is a closed subspace of the Hilbert space $H^1(\Omega)$. Thus, we seek a unique weak solution $u \in H$ which satisfies (3.2.11).
Let $u \in H$ be a solution of (3.2.11). Then, for all $v \in H$

$$\int_\Omega -\Delta u.vdx = \int_\Omega f.vdx.$$

Then, by Green's formula we get that

$$\int_\Omega \nabla u.\nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} vd\sigma = \int_\Omega fvdx.$$

Consequently, we get that

$$\int_\Omega \nabla u.\nabla v - \int_{\Gamma_1} g.vd\sigma = \int_\Omega fvdx.$$

Then,

$$\int_\Omega \nabla u.\nabla v = \int_\Omega fvdx + \int_{\Gamma_1} g.vd\sigma.$$

Now, let $T_1 : H \times H \longrightarrow \mathbb{R}$ defined by

$$T_1(u, v) = \int_\Omega \nabla u.\nabla v,\ for\ all\ u, v \in H.$$

T_1 is bilinear, continuous and coercive.
Also, let $T_2 : H \longrightarrow \mathbb{R}$ defined by

$$T_2(v) = \int_\Omega fvdx + \int_{\Gamma_1} g.vd\sigma,\ for\ all\ v \in H.$$

Indeed, $T_2 \in H^*$ since T_2 is linear and

$$\begin{aligned}|T_2(v)| &\leq \int_\Omega |f.v|dx + \int_{\Gamma_1} |g.v|d\sigma \\ &\leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_1)}\|v\|_{L^2(\Gamma_1)} \\ &\leq \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_1)}\right)\|v\|_{H^1}\end{aligned}$$

Which implies that T_2 is bounded. Consequently, $T_2 \in H^*$.
Then, by the Theorem 3.1.6 there exists a unique $u \in H$ which satisfies (3.2.11). We show that $u \in H$ satisfies (3.2.11).
Now, for all $v \in H$ we get that

$$\int_\Omega \nabla u.\nabla v = \int_\Omega fvdx + \int_{\Gamma_1} g.vd\sigma. \tag{3.2.12}$$

For $v \sim \psi \in D(\Gamma_1)$. Then, we get that

$$\int_{\Gamma_1} g.vd\sigma = 0.$$

Hence,

$$\int_\Omega \nabla u.\nabla v = \int_\Omega fvdx,$$

which implies that

$$\langle -\Delta u, v\rangle = \langle f, v\rangle\,,\ for\ all\ v \in H.$$

Thus,

$$-\Delta u = f\ in\ \Omega,\ \ f \in L^2(\Omega)$$

Also, $u \in H$ implies that $u = 0$ on Γ_0.
Since $u \in H$ satisfies

$$-\Delta u = f$$

Then, we get that

$$\int_\Omega \nabla u.\nabla vdx - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}.vd\sigma = \int_\Omega fvdx$$

Hence, we get that

$$\int_\Omega \nabla u.\nabla vdx - \int_{\Gamma_1} \frac{\partial u}{\partial \mathbf{n}}.vd\sigma = \int_\Omega fvdx,\ for\ all\ v \in H \tag{3.2.13}$$

Consequently, from (3.2.12) and (3.2.13) we get that

$$\int_{\Gamma_1} \left(\frac{\partial u}{\partial \mathbf{n}} - g\right) d\sigma = 0.$$

Hence, $\frac{\partial u}{\partial \mathbf{n}} - g \in \{u\,|_{\Gamma_1}: u \in H\}^\perp$. Then, we apply the lemma (3.1.3) and we obtain that

$$\frac{\partial u}{\partial \mathbf{n}} = g\ on\ \Gamma_1.$$

3.3 Monotone operators

Notations.
Let X and Y be real Banach spaces and $T : X \longrightarrow Y$ be a multi-valued operator. We define the graph of T, $G(T)$ as follows;

$$G(T) = \{(x, y) \in X \times Y : y \in Tx\}\,.$$

We define the effective domain of T, $D(T)$ as follows;

$$D(T) = \{x \in X : Tx \neq \emptyset\}\,.$$

Let X be a Banach space. We denote it dual by X^*.

Definition 3.3.1 *[9] A subset $A \subset X \times X^*$ is said to be a monotone set provided that*

$$\langle x^* - y^*, x - y\rangle \geq 0, \ \textit{whenever} \ (x, x^*), (y, y^*) \in A.$$

Definition 3.3.2 *[9] Let $T : X \longrightarrow X^*$ be a multivalued operator. Then, T is a monotone operator provided that for all $x, y \in X$,*

$$\langle x^* - y^*, x - y\rangle \geq 0, \ x^* \in Tx \ \textit{and} \ y^* \in Ty.$$

Furthermore, T is a monotone operator if and only if it graph, $G(T)$ is a monotone set.

Definition 3.3.3 *[9] A monotone set $B \subset X \times X^*$ is said to be maximal monotone if it is the maximal monotone set of all monotone subsets of $X \times X^*$. Thus, a monotone operator T is said to be maximal monotone if it graph, $G(T)$ is a maximal monotone set.*

Example 3.3.4 *[9] Let H be a Hilbert space and $T : H \longrightarrow H^*$ a linear map and single valued. Then, T is a monotone operator provided that*

$$\langle x, Tx\rangle \geq 0, \ \textit{for all} \ x \in H.$$

Example 3.3.5 *[9] Let $T : X \longrightarrow 2^{X^*}$ a multivalued map called the duality map and defined by*

$$Tx = \left\{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2, \ \textit{with} \ \|x\| = \|x^*\|\right\}.$$

Then, T is a monotone operator.
Indeed, for all $x, y \in X$ with $x^* \in Tx$ and $y^* \in Ty$. Then,

$$\begin{aligned}\langle x^* - y^*, x - y\rangle &= \langle x^*, x\rangle - \langle x^*, y\rangle - \langle y^*, x\rangle + \langle y^*, y\rangle \\ &= \|x\|^2 - \|x\|\|y\| - \|y\|\|x\| + \|y\|^2 \\ &= (\|x\| - \|y\|)^2\end{aligned}$$

Hence, we obtain that

$$\langle x^* - y^*, x - y\rangle \geq 0.$$

Example 3.3.6 *[9] The sub-differential operator $T : X \longrightarrow X^*$ defined by*

$$\partial T(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq T(y) - T(x), \ \textit{for all} \ y \in X\}.$$

is a monotone operator (in fact it is a maxiamal monotone operator).

Indeed, for all $x, y \in X$ and $x^* \in Tx$, $y^* \in Ty$. Then,

$$\begin{aligned}\langle x^* - y^*, x - y\rangle &= \langle x^*, x - y\rangle - \langle y^*, x - y\rangle \\ &\geq T(x) - T(y) - T(x) + T(y) = 0\end{aligned}$$

Hence, we obtain that

$$\langle x^* - y^*, x - y\rangle \geq 0.$$

Definition 3.3.7 *[8] Let $T : X \longrightarrow X^*$ be a single valued operator from $D(T) = X$. Then, T is said to be demi-continuous if $(u_n)_{n\geq 0} \subseteq X$ such that $u_n \longrightarrow u$ in X as $n \to \infty$, then $Tu_n \rightharpoonup Tu$ in X^* as $n \to \infty$.*

3.3.1 Yosida approximation

The Yosida approximations(or regularization) plays an important role in the smooth approximation of a maximal monotone operator, A.

Definition 3.3.8 *[9] Let X be a reflexive strictly convex Banach space with strictly convex dual X^* and let A be maximal monotone operator. Then, we define the Yosida approximation of A as the operator $A_\lambda : X \longrightarrow X^*$ such that*

$$A_\lambda x = \frac{1}{\lambda} J(x - J_\lambda x), \quad \lambda > 0, \quad x \in X.$$

Where $J : X \longrightarrow X^*$ is the duality mapping. The resolvent $J_\lambda x := x_\lambda$ is defined to be the unique solution of the resolvent equation $0 \in J(x_\lambda - x) + \lambda A x_\lambda$.

Proposition 3.3.9 *[8] The Yosida approximation A_λ is single-valued, maximal monotone, bounded on bounded subsets and demi-continuous operator from X to X^*.*

Proposition 3.3.10 *[8] Let $X = H$ a Hilbert space and A a maximal monotone operator. Then,*
(i). $J_\lambda = (I + \lambda A)^{-1}$
(ii). $\|A_\lambda x - A_\lambda y\| \leq \frac{1}{\lambda}\|x - y\|$, for all $x, y \in D(A)$, $\lambda > 0$.

Definition 3.3.11 *[2] Let D be a subset of a real vector space X and $f : D \longrightarrow \overline{\mathbb{R}}$ be a map. Then, f is said to be convex if D is a convex set and for each $t \in [0, 1]$ and for each $x_1, x_2 \in D$, we have that*

$$f(tx_1 + (1 - t)x_2) \leq tf(x_1) + (1 - t)f(x_2).$$

Furthermore, f is proper if there exists at least an $x_0 \in D$ such that $f(x_0) \neq \infty$

Definition 3.3.12 *[2] A function $f : X \longrightarrow \overline{\mathbb{R}}$ is called lower semi continuous at $x_0 \in X$ if and only if for all $\lambda \in \mathbb{R}$ such that $\lambda < f(x_0)$, there exists a neighbourhood of x_0, V, such that $f(x) > \lambda$ for all $x \in V$.*

Consider the elliptic variational problem given by

$$Tu + \partial\psi(u) = f, \;\; f \in X \tag{3.3.1}$$

Here, $\psi : X \longrightarrow \overline{\mathbb{R}}$ is a proper, lower semi continuous and convex function on X. Also, $\partial\psi \subset X \times X^*$ is the sub-differential of ψ.

Theorem 3.3.13 *[8] Let $T : X \longrightarrow X^*$ be a monotone, demi-continuous operator and $\psi : X \longrightarrow \overline{\mathbb{R}}$ a proper, lower semi continuous and convex function on X. Assume that there exists $u_0 \in D(\psi)$ such that*

$$\lim_{\|u\| \to \infty} \frac{(u - u_0, Tu) + \psi(u)}{\|u\|} = +\infty.$$

Then, problem (3.3.1) has atleast a solution. Moreover, the set of solutions is bounded, convex and closed in X and if the operator T is strictly monotone(That is, $\langle Tu - Tv, u - v\rangle = 0$ if and only if $u = v$), then the solution is unique.

3.4 Application to nonlinear elliptic partial differential equations

In this section, we propose to solve some nonlinear elliptic PDEs with homogeneous boundary conditions.

Consider the nonlinear elliptic partial differential equation with a homogeneous Dirichlet boundary condition given by

$$\begin{cases} \lambda u - div_x \beta(\nabla u) = f \quad \text{in} \quad \Omega, \;\; f \in L^2(\Omega) \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.4.1}$$

where Ω is smooth and open in $\mathbb{R}^n$. Here, $\beta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a maximal monotone operator.

Theorem 3.4.1 *[8] Suppose β is continuous and satisfies the following conditions:*
(i) $|\beta(r)| \leq c_1 \left(1 + |r|^{p-1}\right), \; for \; all \; r \in \mathbb{R}^n.$

(ii) $\beta(r).r \geq \omega |r|^p - c_2, \; for \; all \; r \in \mathbb{R}^n.$
where ω, c_1, c_2 are positive constants. Then, for each $f \in (W_0^{1,p}(\Omega))^*$ and $\lambda > 0$, there exists $u \in W_0^{1,p}(\Omega)$ that satisfies (3.4.1).

We solve problem (3.4.1) with the assumptions given in Theorem 3.4.1.

CONCLUSION

This thesis was devoted to the study of a space of functions on a subset Ω of $\mathbb{R}^n$ and its derivative called the Sobolev space with it applications to some elliptic partial differential equations.
In chapter one of this thesis, we discussed the L^p spaces and some of it important concepts relevant in this thesis.
Chapter two was devoted to the study of the Sobolev spaces. We also discussed the concept of embeddings(continuous and compact embeddings) of the Sobolev spaces with other spaces of functions like the Hölder space.
In chapter three, we were concerned with the applications to solve elliptic partial differential equations. We considered some Dirichlet and Neumann problems of linear elliptic partial differential equations. By applying the Lax-Milgram Theorem, we established the existence of a unique solution to these problems.
Also, we solve some nonlinear partial differential equations with Dirichlet and Neumann (respectively) boundary conditions as follows:

$$\begin{cases} \lambda u - div_x \beta(\nabla u) = f \quad \text{in} \quad \Omega, \quad f \in L^2(\Omega) \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.4.2}$$

$$\begin{cases} \lambda u - div_x \beta(\nabla u) = f \quad \text{in} \quad \Omega \\ \beta(\nabla u) = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.4.3}$$

We were able to establish the existence of a solution to these nonlinear problems by applying an appropriate theorem.
We note that further research can be done on the nonlinear problems.

BIBLIOGRAPHY

[1] H. Brezis; *Functional Analysis, Sobolev Spaces and Partial Differential Equation,* Springer, New york, (2010).

[2] C.E. Chidume; *Applicable Functional Analysis,* International centre for Theoretical Physics Trieste, Italy, July 2006.

[3] K. Ezzinbi; *Lecture Notes on Sobolev Spaces and Elliptic PDE,* AUST-Abuja, Nigeria, 2019.

[4] K. Ezzinbi; *Lecture Notes on Metric Spaces and Differential Calculus,* AUST-Abuja, Nigeria, 2018.

[5] K. Ezzinbi; *Lecture Notes on Ordinary Differential Equations,* AUST-Abuja, Nigeria, 2018.

[6] S. Shkoller; *Introduction to Sobolev Spaces,* May 2011.

[7] A. Bressan; *Lecture Notes on Sobolev Spaces,* February, 2012.

[8] V. Barbu; *Non-linear Differential Equations of Monotone types in Banach spaces,* Springer New york, 2010.

[9] R.R. Phelps; *Lectures Notes on Maximal Monotone Operator, given at Prague/Paseky Summer School, Czech Republic,* August, 1993.

[10] K.H. Karlsen; *Note on Weak Convergence* February, 2006.