VARIATIONAL INEQUALITY IN HILBERT SPACES AND THEIR APPLICATIONS

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By

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Certification

This is to certify that the thesis titled " VARIATIONAL INEQUALITY IN HILBERT SPACES AND THEIR APPLICATIONS" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research work carried out by Udeani Cyril Izuchukwu in the department of Pure and Applied Mathematics.

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A Thesis Presented to the Department of Pure and Applied Mathematics

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Dedication

This work is dedicated to God Almighty and my mother for her great support towards my success.

Abstract

The study of variational inequalities frequently deals with a mapping F from a vector space X or a convex subset of X into its dual $X^{'}$. Let H be a real Hilbert space and $a(u, v)$ be a real bilinear form on H. Assume that the linear and continuous mapping $A : H \longrightarrow H^{'}$ determines a bilinear form via the pairing $a(u, v) = \langle Au, v\rangle$. Given $K \subset H$ and $f \in H^{'}$. Then, Variational inequality(VI) is the problem of finding $u \in K$ such that $a(u, v - u) \geq \langle f, v - u\rangle$, for all $v \in K$. In this work, we outline some results in theory of variational inequalities. Their relationships with other problems of Nonlinear Analysis and some applications are also discussed

Keywords

Sobolev spaces, Variational inequalities,Hilbert Spaces, Elliptic variational inequalities

Contents

Introduction

In the study of variational inequalities, we are frequently concern with a mapping F from a vector space X or a convex subset of X into its dual $X^{'}$. Variational inequalities and Complementary problems are of fundamental importance in a wide range of mathematical and applied problems, such as programming, traffic engineering, economics and equilibrium problems. The idea and techniques of the variational inequalities are being applied in a variety of diverse areas in sciences and proved to be productive and innovative. It has been shown that this theory provides a simple, natural and unified framework for a general treatment of unrelated problems. The fixed point theory has played an important role in the development of various algorithms for solving variational inequalities. Using the projection operator technique, one usually establishes an equivalence between the variational inequalities and the fixed point problem. The alternative equivalent formulation was used by Lions and Stampacchia [8] to study the existence of a solution of the variational inequalities. Projction methods and its variant forms represent important tools for finding the approximate solution of variational inequalities. In this work, we intend to present the element of variational inequalities and free boundary problems with several examples and their applications.
The usual setting of the scalar variational inequality is the following:
Let K be a nonempty subset of $\mathbb{R}^n$ and $\langle ., . \rangle$ denote the scalar product in $\mathbb{R}^n$. Let an operator $F : K \to \mathbb{R}^n$ be given. The Stampacchia variational inequality problem [8] consist of finding $y \in K$ such that

$$\langle F(y), x - y \rangle \geq 0, \text{ for all } x \in K$$

while in the case of the Minty variational inequality [10], it is to find $y \in K$ such that

$$\langle F(y), y - x \rangle \geq 0, \text{ for all } x \in K.$$

Historically, Stampacchia variational inequality(SVI) was introduced by P.Hartman and G. Stampacchia, and was subsequently expanded by Stampacchia in several paper[8]. The study of the Minty variational inequality problem goes back to Minty, who studied the relationships of Stampachia variational inequality (SVI) and Minty variational inquality (MVI) in the case when F is a monotone operator, but without using the 'Variational Inequality' terminology[10]. New impetus has been given to the field by the recent paper of Giannessi. Readers interested in more details on the

Stampacchia Variational Inequality in finite dimension may consult the nice survey of Harker and Pang[7] for more motivations, examples, results and comprehensive bibliography.

Definition 0.0.1 *Let H be a real Hilbert space and $a(u, v)$ be a real bilinear form on H. Assume that the linear and continuous mapping $A : H \longrightarrow H'$ determines a bilinear form via the pairing*

$$a(u, v) = \langle Au, v\rangle.$$

Given $K \subset H$ and $f \in H'$. Then, Variational inequality(VI) is the problem of finding $u \in K$ such that

$$a(u, v - u) \geq \langle f, v - u\rangle, \text{ for all } v \in K.$$

The aim of this work is to obtain the solution of the differential equation

$$\begin{cases} -u'' + u = f \ \text{ on } I = (0, 1), \\ u(0) = \alpha, u(1) = \beta. \end{cases} \tag{0.0.1}$$

with $\alpha, \beta \in \mathbb{R}$ given and $f \in L_2(I)$ given using variational approach.
Suppose we multiply (0.0.1) by $\varphi \in C^1(I)$ and integrate by part; we obtain

$$\int_I u'\varphi' + \int_I u\varphi = \int_I f\varphi, \text{for all } \ \varphi \in C^1(I), \varphi(0) = \varphi(1) = 0. \tag{0.0.2}$$

We obtain the solution equation (0.0.2) using Stampacchia and Lax-Milgram Theorem[2] The first chapter is divided into three sections. The first section introduces Hilbert space, some examples and some of its properties. The second section briefly review the main function space used is this work and the final section discussed the general Sobolev spaces. Chapter two introduces the concept of variational inequality in $\mathbb{R}^N$, we stated the theorem about variational inequalities, some problems leading to variational inequality were discussed in connection with convex function. In chapter three, we started Stammpacchia and Lax-Milgram Theorems. Variational inequalities in Hilbert spaces were disscussed and their applications

CHAPTER 1

Linear Functional Analysis

The aim of this Chapter is to recall basic results from functional analysis and Distribution theory. The chapter is divided into three sections. The first section introduces Hilbert spaces and some basic properties of Hilbert spaces. The second section introduces basic concept of Distribution theory and the last section deals with basic results about Sobolev spaces that are of important in the remaining chapters.

1.1 Hilbert Spaces

Let us recall some definitions, theorems, and elementary properties on Hilbert spaces.

Definition 1.1.1 *Let E be a linear space over K, ($K = \mathbb{R}$ or $\mathbb{C}$). An inner product on E is a function*

$$\langle ., . \rangle : E \times E \longrightarrow K$$

such that the following are satisfied, for $x, y, z \in E; \lambda, \mu \in K$:

(i) $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$

(ii) $\langle x, y \rangle = \overline{\langle x, y \rangle}$

(iii) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$.

The pair $(E, \langle ., . \rangle)$ is called an inner product space.

Remark 1.1.2 *A complete inner product space is called a Hilbert space.*

Examples

1 Euclidean space: The space $\mathbb{R}^N$ is a Hilbert space with the inner product defined by

$$\langle x, y \rangle = \sum_{i=1}^{N} x_i y_i$$

where, $x = (x_1, x_2, ..., x_N)$ and $y = (y_1, y_2, ..., y_N)$.
We obtain that

$$\|x\| = \sqrt{\langle x, x\rangle} = (\sum_{i=1}^{N} x_i^2)^{\frac{1}{2}}.$$

2 Space $L_2(\Omega)$.

$$L_2(\Omega) := \{f : \Omega \longrightarrow \mathbb{R} : \text{ f is measurablbe and } \int_\Omega f^2 dx < \infty\}$$

where Ω is an open set in $\mathbb{R}^N$, is a Hilbert space with the inner product defined by

$$\langle f, g\rangle = \int_\Omega f(x)g(x)dx,$$

and

$$\|f\| = (\int_\Omega |f(x)|^2 dx)^{\frac{1}{2}}.$$

Proposition 1.1.3 *[2] (**Cauchy-Schwart's Inequality**) Let E be an inner product space. For arbitary $x, y \in E$ we have*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$$

Proof. Let $x, y \in E$ be arbitrary. Take $z \in \mathbb{C}$ with $|z| = 1$ and let $t \in \mathbb{R}$. Then,

$$\begin{aligned}
0 &\leq \langle tzx + y, tzx + y\rangle \\
&= \langle tzx, tzx\rangle + \langle tzx, y\rangle + \langle y, tzx\rangle + \langle y, y\rangle \\
&= t^2 z\overline{z}\langle x, x\rangle + tz\langle x, y\rangle + t\overline{z}\langle y, x\rangle + \langle y, y\rangle \\
&= t^2|z|\langle x, x\rangle + tz\langle x, y\rangle + t\overline{z\langle x, y\rangle} + \langle y, y\rangle \\
&= t^2\langle x, x\rangle + 2tRe(z\langle x, y\rangle) + \langle y, y\rangle \\
&\leq t^2\langle x, x\rangle + 2t|z\langle x, y\rangle| + \langle y, y\rangle \\
&= t^2\langle x, x\rangle + 2t|z||\langle x, y\rangle| + \langle y, y\rangle \\
&= t^2\langle x, x\rangle + 2t|\langle x, y\rangle| + \langle y, y\rangle. \qquad (1.1.1)
\end{aligned}$$

$t^2\langle x, x\rangle + 2t|\langle x, y\rangle| + \langle x, x\rangle$ is a quadratic function with variable $t \in \mathbb{R}$. Since,

$$t^2\langle x, x\rangle + 2t|\langle x, y\rangle| + \langle y, y\rangle \geq 0, \text{ for arbitrary } t \in \mathbb{R},$$

Hence,

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle, \text{ for all } x, y \in E.$$

Theorem 1.1.4 *[2] Let $\langle ., .\rangle$ be an inner product on E, then the mapping*

$$x \longmapsto \|x\| = \sqrt{\langle x, x\rangle}$$

is a norm on E.

Proof.
Let $x, y \in E$ be arbitrary. From the definition of the inner product, we have

$$\langle x, x\rangle = 0 \Leftrightarrow x = 0,$$

hence

$$\|x\|^2 = \langle x, x\rangle = 0 \Leftrightarrow x = 0.$$

Also,

$$\langle \lambda x, \lambda x\rangle = |\lambda|\langle x, x\rangle.$$

And

$$\begin{aligned} \|\lambda x\| &= \sqrt{\langle x, x\rangle} \\ &= \sqrt{|\lambda|^2\langle x, x\rangle} \\ &= |\lambda|\sqrt{\langle x, x\rangle} \\ &= \|x\|. \end{aligned}$$

Let $x, y \in E$. Then,

$$\begin{aligned} \|x+y\| &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + 2Re(\langle x, y\rangle) + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\sqrt{(\langle x, x\rangle\langle y, y\rangle)} + \|y\|^2 \\ &= \|x\|^2 + 2(\|x\| + \|y\|) + \|y\|^2 \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Definition 1.1.5 *Let E be a linear space and $F \subset E$ is said to be convex if for each $x, y \in F$ and $\lambda \in [0, 1]$ we have*

$$\lambda x + (1-\lambda)y \in E.$$

Proposition 1.1.6 *[4] (**Parallelogram Law**) Let E be an inner product space, then for $x, y \in E$*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Theorem 1.1.7 *[4] Let H denote a real Hilbert space and let K be a closed, convex subset of H. Then for each $x \in H$ there exists unique $y \in K$ such that*

$$\|x-y\| = inf\{\|x-\eta\| : \eta \in K\}. \tag{1.1.2}$$

Proof.
Let $\eta_k \in K$ be a minimizing sequence such that

$$\lim_{k\to\infty} \|\eta_k - x\| = d = inf_{\eta\in K}\|\eta - x\|.$$

Since H is a Hilbert Space, then by the Parallelogram Law, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \text{ for all } x, y \in H.$$

Thus,

$$\|\eta_k - \eta_p\|^2 + \|\eta_k + \eta_p\|^2 = 2(\|\eta_k\|^2 + \|\eta_p\|^2); \text{ for } \eta_k, \eta_p \in K.$$

By convexity of K, we have

$$\tfrac{1}{2}\eta_k + (1 - \tfrac{1}{2})\eta_p = \tfrac{1}{2}(\eta_k + \eta_p) \in K.$$

But $d = inf_{\eta \in K}\|x - \eta\| \leq \|x - \eta\|$, for all $\eta \in K$.

Take $\eta = \frac{1}{2}(\eta_k + \eta_p)\epsilon$. Then,

$$\begin{aligned}
&d \leq \|x - \frac{1}{2}(\eta_k + \eta_p)\| \\
&\Rightarrow d^2 \leq \|x - \frac{1}{2}(\eta_k + \eta_p)\|^2 \\
&\Rightarrow -d^2 \geq -\|x - \frac{1}{2}(\eta_k + \eta_p)\|^2.
\end{aligned}$$

Now, using the Parallelogram Law and setting $y = x - \eta_k \in H$ and $x = x - \eta_p \in H$, we obtain that

$$\begin{aligned}
0 &\leq \|\eta_k - \eta_p\|^2 \\
&= 2\|x - \eta_k\|^2 + 2\|x - \eta_p\|^2 - \|2x - (\eta_k + \eta_p)\|^2 \\
&= 2\|x - \eta_k\|^2 + 2\|x - \eta_p\|^2 - 4\|x - \frac{1}{2}(\eta_k + \eta_p)\|^2 \\
&\leq 2\|x - \eta_k\|^2 + 2\|x - \eta_p\|^2 - 4d^2.
\end{aligned}$$

But $\lim_{k \to \infty}\|x - \eta_k\| = d$. Then,

$$2\|x - \eta_k\|^2 + 2\|x - \eta_p\|^2 - 4d^2 \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Consequently,

$$\begin{aligned}
&\|\eta_k - \eta_p\|^2 \longrightarrow 0 \text{ as } n \longrightarrow \infty \\
&\Rightarrow \|\eta_k - \eta_p\| \longrightarrow 0 \text{ as } n \longrightarrow \infty
\end{aligned}$$

Hence, $(\eta_k)_{k \geq 0}$ is Cauchy sequence in K. Since H is a Hilbert space, then there exists $\hat{x} \in H$ such that

$$\eta_k \longrightarrow \hat{x}.$$

But K is a closed subset of H and $\eta_k \in K$, thus $\hat{x} \in K$.
Therefore,

$$\|x - \hat{x}\| = \lim_{k \to \infty}\|x - \eta_k\| = d.$$

For uniqueness: Let $\hat{x}, \hat{y} \in K$ such that

$$\|x - \hat{x}\| = inf_{\eta \in K}\|x - \eta\|$$

and

$$\|x - \hat{y}\| = inf_{\eta \in K}\|x - \eta\|.$$

By the Parallelogram law and convexity of K, we obtain

$$\begin{aligned} 0 &\leq \|\hat{x} - \hat{y}\| \\ &= 2\|x - \hat{x}\|^2 + 2\|x - \hat{y}\|^2 - 4\|x - \frac{1}{2}(\hat{x} - \hat{y})\|^2 \\ &\leq 2d^2 + 2d^2 - 4d^2 = 0. \end{aligned}$$

Then $\|\hat{x} - \hat{y}\| = 0 \Leftrightarrow \hat{x} = \hat{y}$. Hence, there is a unique $y \in K$ such that

$$\|x - y\| = inf_{\eta \in K} \|x - \eta\|.$$

Remark 1.1.8 *The point $y \in H$ satisfying (1.1.2) is called a projection of x on K and $y = P_K x$.*

Corollary 1.1.9 *[2]*

Let K be a closed, convex subset of a Hilbert space H. Then the operator P_K is nonexpansive, that is

$$\|P_K x - P_K x^{'}\| \leq \|x - x^{'}\|, \text{ for all } x, x^{'} \in H.$$

Proof.

Let $x, x^{'} \in H$ such that $y = P_K x$, $y^{'} = P_K x^{'}$. Then, for $y, y^{'} \in K$ we have

$$\langle y, \eta - y \rangle \geq \langle x, \eta - y \rangle, \text{ for all } \eta \in K,$$

and

$$\langle y^{'}, \eta - y^{'} \rangle \geq \langle x^{'}, \eta - y^{'} \rangle \text{ , for all } \eta \in K.$$

Setting $\eta = y^{'}$ and $\eta = y$ in the first and second inequality respectively we obtain

$$\langle y, y^{'} - y \rangle \geq \langle x, y^{'} - y \rangle \text{ and } \langle y^{'}, y - y^{'} \rangle \geq \langle x^{'}, y - y^{'} \rangle.$$

Adding we obtain that,

$$\langle y, y^{'} - y \rangle + \langle y^{'}, y - y^{'} \rangle \geq \langle x, y^{'} - y \rangle + \langle x^{'}, y - y^{'} \rangle,$$

hence

$$\langle y, y^{'} - y \rangle - \langle y^{'}, y^{'} - y \rangle \geq \langle x, y^{'} - y \rangle - \langle x^{'}, y^{'} - y \rangle,$$

and

$$\langle y - y^{'}, y^{'} - y \rangle \geq \langle x - x^{'}, y^{'} - y \rangle.$$

Consequently,

$$-\langle y - y^{'}, y - y^{'} \rangle \geq \langle x - x^{'}, y^{'} - y \rangle.$$

Then,

$$\langle y - y^{'}, y - y^{'} \rangle \leq \langle x - x^{'}, y - y^{'} \rangle$$

$$\begin{aligned}\|y-y^{'}\|^2 &= \langle y-y^{'},y-y^{'}\rangle\\ &\leq \langle x-x^{'},y-y^{'}\rangle\\ &\leq |\langle x-x^{'},y-y^{'}\rangle|\\ &\leq \|x-x^{'}\|\|y-y^{'}\|,\end{aligned}$$

and thus

$$\|y-y^{'}\| \leq \|x-x^{'}\|.$$

Therefore,

$$\|P_Kx-P_Kx^{'}\| \leq \|x-x^{'}\|, \text{ for all } x,x^{'}\in H.$$

Theorem 1.1.10 *[8] Let K be a closed convex subset of a real Hilbert space H. Then $y=P_Kx$, the projection of x on K, if and only if $y\in K$ such that*

$$\langle y,\eta-y\rangle \geq \langle x,\eta-y\rangle, \text{ for all } \eta\in K.$$

Proof.
Let $x\in H$ and $y=P_Kx$. Since K is convex, then

$$t\eta+(1-t)y = y+t(\eta-y)\in K, \text{ for all } \eta\in K,\ 0\leq t\leq 1.$$

Set $\phi(t)=\|x-(y+t(\eta-y))\|^2$, $0\leq t\leq 1$.

$$\begin{aligned}\phi(t) &= \|x-y-t(\eta-y)\|^2\\ &= \|x-y\|^2-2tRe\langle x-y,\eta-y\rangle+t^2\|\eta-y\|^2\\ &= \|x-y\|^2-2t\langle x-y,\eta-y\rangle+t^2\|\eta-y\|^2.\end{aligned}$$

Then,

$$\phi^{'}(t) = -2\langle x-y,\eta-y\rangle+2t\|\eta-y\|^2,$$

thus, $\phi^{'}(0)=-2\langle x-y,\eta-y\rangle$. Therefore, the function ϕ attains its minimum at $t=0$. Thus,

$$\begin{aligned}\phi^{'}(0)\leq 0 &\Leftrightarrow \langle x-y,\eta-y\rangle\leq 0,\eta\in K\\ &\Leftrightarrow \langle x,\eta-y\rangle-\langle y,\eta-y>\leq 0\\ &\Leftrightarrow \langle y,\eta-y\rangle\geq\langle x,\eta-y\rangle,\eta\in K.\end{aligned}$$

Let $y\in K$. Then,

$$\langle y,\eta-y\rangle\geq\langle x,\eta-y\rangle,\ \eta\in K.$$

Thus,

$$\langle y,\eta-y\rangle-\langle x,\eta-y\rangle\geq 0$$

and

$$\langle y-x,\eta-y\rangle\geq 0$$

$$\begin{aligned} 0 &\leq \langle y - x, \eta - y \rangle \\ &= \langle y - x, (\eta - x) + (x - y) \rangle \\ &\leq -\|x - y\|^2 + \langle y - x, \eta - x \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \|y - x\|^2 &\leq \langle y - x, \eta - x \rangle \\ &\leq |\langle y - x, \eta - x \rangle| \\ &\leq \|y - x\| \|\eta - x\|. \end{aligned}$$

Thus,

$$\|y - x\| \leq \|\eta - x\|.$$

Hence for each $x \in H$ there exists $y \in K$ such that

$$\|y - x\| = inf_{\eta \in K} \|\eta - x\|.$$

Corollary 1.1.11 *[2] Let H be a real Hilbert space and K a closed subspace of H. Then, for arbitrary vector $x \in H$, there exist a unique vector $\tilde{y} \in K$ such that*

$$\|x - \tilde{y}\| \leq \|x - y\|, \textit{ for all } y \in K.$$

Theorem 1.1.12 *[2] (**Reiesz Theorem**) Let H be a Hilbert space. Then $H' = H$, where H' denote the dual of H.*

Theorem 1.1.13 *[4] (**Riesz Representation Theorem**) Let H be a Hilbert space and let f be a bounded linear functional on H. Then, there exists a unique vector of $x_0 \in H$ such that*

$$f(x) = \langle x, x_0 \rangle, \textit{ for each } x \in H.$$

Moreover, $\|f\| = \|x_0\|$.

1.2 Function spaces

We recall some definitions of function spaces used in this thesis

Definition 1.2.1 *An open connected set $\Omega \subset \mathbb{R}^N$ is called a domain. By $\overline{\Omega}$, we denote the closure of Ω; $\partial\Omega$ is the boundary and Ω° is the interior of Ω.*

$x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}$ is a multi-index.

$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_N$$

$$D^{\alpha} u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_N^{\alpha_N}}$$

$$\nabla u = (\partial_1 u, \partial_2 u, ..., \partial_N u)$$

$$|\nabla u| = (\sum_{j=1}^{N} |\partial_j u|^2)^{\frac{1}{2}}$$

Definition 1.2.2 *Let $f : \Omega \longrightarrow \mathbb{R}$ be continuous. We define support of f by*

$$supp(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

The function is said to be of compact support on Ω if the support is a compact set contained inside Ω. The space of test functions in Ω is denoted by $D(\Omega)$ and defined

$$\begin{aligned} D(\Omega) &:= \{f : \Omega \longrightarrow \mathbb{R}, C^\infty : \textit{support(f) is compact}\} \\ &= \{f \in C^\infty(\Omega) : \textit{supp(f) is compact}\} \end{aligned}$$

Definition 1.2.3 *Let $(\psi_n)_{n\geq 1}$ be a sequence in $D(\Omega)$ and $\psi \in D(\Omega)$. Then, $\psi_n \to \psi$ in $D(\Omega)$ if*

(i) there exists a compact set $K \subset \Omega$ such that, $supp(\psi_n) \subset K$, for all $n \geq 1$.

(ii) $D^\alpha \psi_n \to D^\alpha \psi$ uniformly on K as $n \to \infty$ and for all $\alpha \in \mathbb{N}^n$.

Definition 1.2.4 *A distribution on Ω is any continuous linear mapping $T : D(\Omega) \to \mathbb{R}$. The set of all distribution on Ω is denoted by $D^{'}(\Omega)$.*

Means that if

$$\psi_n \to 0 \text{ in } D(\Omega) \text{ , then } (T, \psi_n) \to 0 \text{ in } \mathbb{R}.$$

Example.

$$\text{The map } \delta : D(\mathbb{R}) \to \mathbb{R} \text{ defined by } \langle \delta, \psi \rangle = \delta(\psi) = \psi(0)$$

is a distribution. It is usually called Dirac distribution.
To see this, we have that δ is linear, since for $\psi_1, \psi_2 \in D(\Omega)$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned} \delta(\psi_1 + \alpha\psi_2) &= \langle \delta, \psi_1 + \alpha\psi_2 \rangle \\ &= (\psi_1 + \alpha\psi_2)(0) \\ &= \psi_1(0) + \alpha\psi_2(0) \\ &= \langle \delta, \psi_1 \rangle + \langle \delta, \psi_2 \rangle \\ &= \langle \delta, \psi_1 \rangle + \alpha \langle \delta, \psi_2 \rangle \\ &= \delta(\psi_1) + \delta(\psi_2). \end{aligned}$$

Hence, δ is linear.
Let $\{\psi_n\}_{n\geq 1} \subset D(\mathbb{R})$ such that $\psi_n \to 0$ as $n \to \infty$ on $D(\mathbb{R})$.
But $\psi_n \to 0$ on $D(\mathbb{R})$ implies that there exists a compact set $K \subset \mathbb{R}$ such that $supp(\psi_n) \subseteq K$ and for all $j \in \mathbb{N}$, $\psi_n^{(j)} \to 0$ uniformly on $\mathbb{R}$.
Thus, $0 \leq |\psi_n(0)| \leq sup_{x\in K}|\psi_n(x)| \to 0$ as $n \to \infty$. And then $\langle \delta, \psi_n \rangle = \psi_n(0) \to 0$ as $n \to \infty$. Therefore, δ is continuous and hence δ is a distribution.

Definition 1.2.5 *A funtion $f : \Omega \to \mathbb{R}$ is said to be locally integrable if for any compact set, $K \subset \Omega$, we have that*

$$\int_K |f(x)| dx < \infty.$$

The collection of all locally integrable funtionals is denoted by $L^1_{loc}(\Omega)$. *For any* $f \in L^1_{loc}(\Omega)$, *f gives a distribution* T_f *defined by*

$$(T_f, \psi) = \int_\Omega f(x)\psi(x)dx, \textit{ for all } \psi \in D(\Omega).$$

Remark 1.2.6 $L^1_{loc}(\Omega) \subseteq D'(\Omega)$.

Theorem 1.2.7 *[6] Let* $T \in D'(\Omega)$ *be a distribution on an open set* Ω *in* $\mathbb{R}^N$, *and* α, *a multi index. Then, for all* $\alpha \in \mathbb{N}^n$, $n \geq 1$

$$(D^\alpha T, \psi) = (-1)^{|\alpha|}(T, D^\alpha \psi),\ D^\alpha \psi \in D'(\Omega), \textit{ for all } \psi \in D(\Omega).$$

Definition 1.2.8 *By* $C^k(\overline{\Omega})$, *we denote the space of k times differentiable (real valued) functions on* $\overline{\Omega}$.

Definition 1.2.9 *[2] By* $C^{k,\lambda}(\overline{\Omega})$, $0 < \lambda < 1$, *we indicate the functions k times continuously differentiable in* $\overline{\Omega}$ *whose derivative of order k are continuous ,* $0 < \lambda < 1$.

1.3 Sobolev spaces

Definition 1.3.1 *Let* Ω *be open set in* $\mathbb{R}^N$. *Let* $p \in \mathbb{R}$ *with* $1 \leq p < +\infty$.

$$L_P(\Omega) := \{f : \Omega \longrightarrow R; measurable : \int_\Omega |f|^p d\lambda < +\infty\}$$

where λ *is a measure on* Ω *and*

$$\|f\|_p := (\int_\Omega |f|^p d\lambda)^{\frac{1}{p}}$$

$L_\infty(\Omega) := \{f : \Omega \longrightarrow R : f \textit{ is essentially bounded}\}$, *i.e* $f \in L_\infty(\Omega) \Leftrightarrow$ *there exists* $c > 0$ *such that* $|f(x) \leq c$ *a.e on* Ω *and* $\|f\|_\infty = inf\{c > 0\text{: } |f(x)| \leq c \text{ a.e on } \Omega\}$.

Theorem 1.3.2 *[6]* $L_P(\Omega)$*is a Banach space for* $1 \leq p \leq \infty$.

Proposition 1.3.3 *[6]* ***(Holder's Inequality)*** *Let* $f \in L_p(\Omega), g \in L_q(\Omega)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $fg \in L_1(\Omega)$. *Moreover,*

$$\int_\Omega |fg| \leq (\int_\Omega |f|^p)^{\frac{1}{p}} (\int_\Omega |g|^q)^{\frac{1}{q}} = \|f\|_p \| \|g\|_q. \tag{1.3.1}$$

Definition 1.3.4 *The space* $H^m(\Omega)$ *is called Sobolev space of order m and it is defined as*

$$H^m(\Omega) := \{f \in L_2(\Omega) : D^\alpha f \in L_2(\Omega), |\alpha| \leq m\},$$

endowed with the inner product

$$< f, g >_{H^m(\Omega)} = < f, g >_{L_2(\Omega)} + \Sigma_{\alpha| \leq m} < D^\alpha f, D^\alpha g >_{L_2(\Omega)} . \tag{1.3.2}$$

and

$$\|f\|_{H^m(\Omega)} = (\|f\|_{L_2(\Omega)} + \Sigma_{|\alpha|\leq m}|D^\alpha f|^2)^{\frac{1}{2}}, \text{ for all } f, g \in H^m(\Omega).$$

Theorem 1.3.5 *[6] The spaces $H^m(\Omega)$, $m \geq 0$ endowed with the inner product (1.1.1) are Hilbert spaces.*

Proof.
Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $H^m(\Omega)$. Let $\epsilon > 0$ be given. Then, there exists $n_o \in \mathbb{N}$ such that

$$\|f_n - f_m\|_{H^m(\Omega)} \text{ for all } n, m \geq n_o.$$

Thus

$$\|f_n - f_m\|^2_{L_2(\Omega)} + \sum_{|\alpha|\leq m}\|D^\alpha f_n - D^\alpha f_m\|^2 < \epsilon^2, \text{ for all } n, m \geq n_o$$

which implies

$$\|f_n - f_m\|^2_{L_2(\Omega)} < \epsilon^2 \text{ and } \sum_{|\alpha|\leq m}\|D^\alpha f_m - D^\alpha f_m\|^2 < \epsilon^2, \text{ for all } n, m \geq n_o.$$

then

$$\|f_n - f_m\| < \epsilon \text{ and } \sum_{|\alpha|\leq m}\|D^\alpha f_n - D^\alpha f_m\| \leq \epsilon^2, \text{ for all } n, m \geq n_o.$$

Thus, we obtain that $(f_n)_{n\geq 1}$ is a Cauchy sequence in $L_2(\Omega)$ and $(D^\alpha(f_n))_n$ is a Cauchy sequence in $L_2(\Omega)$. Since $L_2(\Omega)$ is complete, then there exist $f, f_i \in L_2(\Omega)$ such that

$$f_n \longrightarrow f \text{ in } L_2(\Omega) \text{ as } n \longrightarrow \infty \text{ and } D^\alpha f_n \longrightarrow f_i \text{ in } L_2(\Omega) \text{ as } n \longrightarrow \infty.$$

Since $L_2(\Omega) \subset D'(\Omega)$ we obtain that

$$f_n \longrightarrow f \text{ in } D'(\Omega) \text{ and } D^\alpha f_n \longrightarrow f_i.$$

But $D^\alpha f \longrightarrow D^\alpha f$ in $D'(\Omega)$ as $n \longrightarrow \infty$. By uniqueness of limit we obtain that $D^\alpha f = f_i$ in $D'(\Omega)$. Thus,

$$f_n \longrightarrow f \text{ in } L_2(\Omega) \text{ and } D^\alpha f_n \longrightarrow D^\alpha f \text{ in } L_2(\Omega), |\alpha| \leq m.$$

Thus $f \in H^m(\Omega)$ with

$$\|f_n - f\|_{L_2(\Omega)} \longrightarrow 0 \text{ and } \sum_{|\alpha|\leq m}\|D^\alpha f_n - D^\alpha f\|_{L_2(\Omega)} \longrightarrow 0 \text{ an } n \longrightarrow \infty.$$

Hence,

$$f \in H^m(\Omega) \text{ and } \|f_n - f\|_{H^m(\Omega)} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore, $H^m(\Omega)$ is a Hilbert space.

For $m = 1$ we obtain

$$H^1(\Omega) := \{f \in L_2(\Omega) : \tfrac{\partial f}{\partial x_i} \in L_2(\Omega), i = 1, 2, ..., N\}$$

and on $H^1(\Omega)$ we have the following inner product

$$\begin{aligned}\langle f, g\rangle_{H^1(\Omega)} &= \langle f, g\rangle_{L_2(\Omega)} + \sum_{i=1}^{N}\langle \frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i}\rangle_{L_2(\Omega)} \\ &= \int_\Omega fg + \sum_{i=1}^{N}\int_\Omega \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i}\end{aligned}$$

and

$$\begin{aligned}\|f\|_{H^1(\Omega)} &= \sqrt{\langle f, g\rangle_{H^1(\Omega)}}, \text{for all } f \in H^1(\Omega) \\ &= \sqrt{\|f\|^2_{L_2(\Omega)} + \sum_{i=1}^{N}\|\frac{\partial f}{\partial x_i}\|_{L_2(\Omega)}}.\end{aligned}$$

Definition 1.3.6 *We define* $H_0^1(\Omega) := \overline{D(\Omega)} \mid_{H^1(\Omega)}$.

$H_0^1(\Omega)$ is a Hilbert space with the norm $\|.\|_{H^1(\Omega)}$ and we define the norm on $H_0^1(\Omega)$ by

$$\|u\|_{H_0^1(\Omega)} := \sqrt{\int_\Omega |\nabla u|^2}.$$

Theorem 1.3.7 *[6] (**Poincare's inequality**) Suppose Ω is bounded. Then, there exists $c > 0$ such that*

$$\int_\Omega u^2 dx \leq c^2 \int_\Omega |\nabla u|^2 dx, \textit{ for all } u \in H_0^1(\Omega).$$

Proof.
Since Ω is bounded. Then $\Omega \subseteq \Pi_{i=1}^N [a_i, b_i]$. We proceed this way, we first prove it in $D(\Omega)$.
Let $\varphi \in D(\Omega)$ such that $\varphi(t, x) = \varphi(t, x_2, x_3, ..., x_N)$.
But $\int_{a_1}^t \frac{\partial}{\partial s}\varphi(s, x)ds = \varphi(t, x) - \varphi(a_1, x) = \varphi(t, x)$. Then using Cauchy Schwartz inequality, we obtain that

$$\begin{aligned}\varphi^2(t, x) &= (\int_{a_1}^t \frac{\partial}{\partial s}\varphi(s, x)ds)^2 \\ &\leq (t - a_1)\int_{a_1}^t (\frac{\partial}{\partial s}\varphi(s, x))^2 ds.\end{aligned}$$

Integrating, we obtain that

$$\begin{aligned}
\int_\Omega \varphi^2(t,x)dtdx &\le \int_\Omega ((t-a_1)\int_{a_1}^{t}(\frac{\partial}{\partial s}\varphi(s,x))^2)ds)dtdx \\
&\le \int_\Omega \int_{a_1}^{b_1}(t-a_1)(\frac{\partial}{\partial s}\varphi(s,x))^2)dsdtdx \\
&= \int_{a_1}^{b_1}(t-a_1)dt\int_\Omega(\frac{\partial}{\partial s}\varphi(s,x))^2dsdx \\
&= \frac{1}{2}(b_1-a_1)^2\int_\Omega(\frac{\partial}{\partial s}\varphi(s,x))^2dsdx \\
&\le \frac{1}{2}(b_1-a_1)^2\int_\Omega|\nabla\varphi|^2.
\end{aligned}$$

Choose $c^2 = \frac{1}{2}(b_1-a_1)^2 > 0$. Then

$$\int_\Omega \varphi^2 \le c^2\int_\Omega|\nabla\varphi|^2, \text{for all } \varphi\in D(\Omega). \tag{1.3.3}$$

Now, let $u \in H_0^1(\Omega) = \overline{D(\Omega)}\;|_{H^1(\Omega)}$. Then, there exist $(\varphi_p)_{p\ge 1}\subset D(\Omega)$ such that

$$\|\varphi_p - u\| \to 0 \text{ as } p\to\infty.$$

We have

$$\int_\Omega \varphi_p^2 \le c^2\int_\Omega|\nabla\varphi_p|^2. \tag{1.3.4}$$

$$\int_\Omega|\varphi_p-u|^2+\int_\Omega|\nabla(\varphi_p-u)|^2\to 0, \text{ as } n\to\infty.$$

And thus,

$$\int_\Omega\varphi_p^2\to\int_\Omega u^2 \text{and} \int_\Omega|\nabla\varphi_p|^2\to\int_\Omega|\nabla u|^2.$$

Letting $p\to\infty$ in equation (1.3.4), we obtain that

$$\int_\Omega u^2\le c^2\int_\Omega|\nabla u|^2, \text{ for all } u\in H_0^1(\Omega).$$

Theorem 1.3.8 *[6] (**Poincare-Wirtinger's inequality**) Suppose Ω is smooth and connected, then for any $u\in H^1(\Omega)$ there exists $c>0$ such that*

$$\int_\Omega|u-\hat{u}|^2\le c^2\int_\Omega|\nabla u|^2, \textit{ for all } u\in H^1(\Omega),$$

where

$$\hat{u} = \frac{1}{mes(\Omega)}\int_\Omega u.$$

Corollary 1.3.9 *The norm $\|.\|_{H_0^1(\Omega)}$ is equivalent to $\|.\|_{H^1(\Omega)}$.*

Proof of Corollary.
Let $u \in H_0^1(\Omega)$. Then,

$$\begin{aligned}\|u\|_{H^1(\Omega)}^2 &= \|u\|_{L_2(\Omega)}^2 + \int_\Omega |\nabla u|^2 \\ &\geq \int_\Omega |\nabla u|^2 \\ &= \|u\|_{H_0^1(\Omega)}^2.\end{aligned}$$

Thus,

$$\|u\|_{H_o^1(\Omega)} \leq \|u\|_{H^1(\Omega)}. \tag{1.3.5}$$

Now using Poincare's inequality we obtain that

$$\begin{aligned}\|u\|_{H^1(\Omega)}^2 &= \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx \\ &\leq c^2 \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla u|^2 \\ &= (c^2+1)\int_\Omega |\nabla u|^2 \\ &= (c^2+1)\|u\|_{H_0^1(\Omega)}^2,\end{aligned}$$

which implies
$\|u\|_{H^1(\Omega)} \leq (c^2+1)\|u\|_{H_0^1(\Omega)}$ and thus

$$\beta\|u\|_{H^1(\Omega)} \leq \|u\|_{H_0^1(\Omega)}, \beta = \frac{1}{(c^2+1)}. \tag{1.3.6}$$

Therefore, from equation (1.3.5) and (1.3.6) we obtain

$$\beta\|u\|_{H^1(\Omega)} \leq \|u\|_{H_0^1(\Omega)} \leq \|u\|_{H^1(\Omega)}.$$

Therefore, the two norms are equivalent on $H_0^1(\Omega)$.

Theorem 1.3.10 *[6] Let Ω be smooth in $\mathbb{R}^N, N \geq 2$ and $D(\overline{\Omega}) = \{u \mid_\Omega : u \in D(\mathbb{R}^N)\}$.*

$$\overline{D(\overline{\Omega})} = H^1(\Omega).$$

Then, $\gamma : D(\overline{\Omega}) \to L_2(\partial\Omega)$ is continuous with the $H^1(\Omega)$ norm. Hence, γ is extensible by continuity over $H^1(\Omega)$. i.e

$$\begin{gathered}\gamma : H^1(\Omega) \to L_2(\partial\Omega) \text{ is continuous} \\ u \to \partial u = u \mid_{\partial\Omega}.\end{gathered}$$

Moreover, there exists $\alpha > 0$ such that

$$\int_{\partial\Omega} u^2 d\sigma \leq \alpha^2 \|u\|_{H^1(\Omega)}, \text{ for all } u \in H^1(\Omega).$$

Application
Let Ω be smooth and connected in $\mathbb{R}^N$. Define

$$\hat{V} = \{u \in H^1(\Omega) : \int_{\partial\Omega} u d\sigma = 0\}.$$

Then $\hat{V}$ is closed in $H^1(\Omega)$.
To see this, let $(u_n)_{n\geq 1}$ be a sequence in $\hat{V}$ such that $u_n \to u$ in $H^1(\Omega)$.
Since $(u_n)_{n\geq 1} \subset \hat{V}$, then

$$\int_{\partial\Omega} u_n d\sigma = 0.$$

Thus, we obtain that

$$\int_{\partial\Omega} |u_n - u|^2 d\sigma \leq \alpha^2 \|u_n - u\|_{H^1(\Omega)}.$$

But $u_n \to u$ in $H^1(\Omega)$, thus $\|u_n - u\|_{H^1(\Omega)} \to 0$ as $n \to \infty$. Which implies

$$\int_{\partial\Omega} |u_n - u|^2 d\sigma \to 0 \text{ as } n \to \infty.$$

But

$$\int_{\partial\Omega} |u_n - u| d\sigma \leq (\int_{\partial\Omega} |u_n - u|^2 d\sigma)^{\frac{1}{2}} (mes(\partial\Omega))^{\frac{1}{2}}.$$

Hence, since $mes(\partial\Omega) > 0$ we have

$$\int_{\partial\Omega} |u_n - u| d\sigma \to 0 \text{ in } L_1(\Omega).$$

And we obtan that

$$\int_{\partial\Omega} u_n d\sigma \to \int_{\partial\Omega} u d\sigma.$$

But

$$\int_{\partial\Omega} u_n d\sigma = 0.$$

Then by uniqueness of limit we obtain that

$$\int_{\partial\Omega} u d\sigma = 0.$$

Hence $u \in \hat{V}$ and therefore $\hat{V}$ is closed.

Theorem 1.3.11 *[6] (**Rellich Theorem**) If Ω is smooth, then $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact. Moreover, if $(u_n)_{n\geq 1}$ is a bounded sequence in $H^1(\Omega)$, then there exists a subsequence $(u_{n_k})_{k\geq 1}$ of $(u_n)_{n\geq 1}$ such that $(u_{n_k})_{k\geq 1}$ converges in $L_2(\Omega)$.*

Application
Let Ω be smooth and connected in $\mathbb{R}^N$. Define

$$V = \{u \in H^1(\Omega) : \int_\Omega u = 0\}.$$

Then, Poincare's inequality is true on V. Thus, there exists $c > 0$ such that

$$\int_\Omega u^2 \leq \int_\Omega |\nabla u|^2, \text{ for all } u \in V.$$

We proceed by contradiction. Suppose for all $n \in \mathbb{N}$ there exists $(u_n) \in V$ such that

$$\int_\Omega u_n^2 > (\sqrt{n})^2 \int_\Omega |\nabla u_n|^2,$$

thus

$$\int_\Omega u_n^2 > n \int_\Omega |\nabla u_n|^2.$$

But

$$\int_\Omega u_n^2 + \int_\Omega |\nabla u_n|^2 \geq \int_\Omega u_n^2 > n \int_\Omega |\nabla u_n|^2.$$

Hence,

$$\|u_n\|_{H^1(\Omega)} > n \int_\Omega |\nabla u_n|^2. \tag{1.3.7}$$

Let $v_n = \dfrac{u_n}{\|u_n\|_{H^1(\Omega)}}$. Then $\|v_n\|_{H^1(\Omega)} = 1$, for all $n \geq 1$.
Since $(v_n)_{n\geq 1}$ is bounded in $H^1(\Omega)$. Then by Rellich Theorem, there exists a subsequence $(v_{n_k})_{k\geq 1}$ of $(v_n)_{n\geq 1}$ and $f \in L_2(\Omega)$ such that $v_{n_k} \to f$ in $L_2(\Omega)$.
Multiplying equation (1.3.7) by $\dfrac{1}{\|u_n\|^2_{H^1(\Omega)}}$, we obtain that

$$n \frac{1}{\|u_n\|^2_{H^1(\Omega)}} \int_\Omega |\nabla u_n|^2 < 1, \text{ for all } n \geq 1.$$

Which implies that

$$\int_\Omega |\nabla v_n|^2 < \frac{1}{n}, \text{ for all } n \geq 1.$$

Thus

$$\int_\Omega |\nabla v_n|^2 \to 0 \text{ as} n \to \infty.$$

And hence

$$\int_\Omega |\nabla v_{n_k}|^2 \to 0 \text{ as } k \to \infty.$$

Then, for all $i = 1, 2, ..., N$ $\dfrac{\partial v_{n_k}}{\partial x_i} \to 0$ in $L_2(\Omega)$.
But $v_{n_k} \to f$ in $L_2(\Omega)$ as $k \to \infty$ and since $L_2(\Omega) \subseteq D'(\Omega)$, we obtain that

$$v_{n_k} \to f \text{ in } D'(\Omega) \text{ and } \frac{\partial v_{n_k}}{\partial x_i} \to 0 \text{ in } D'(\Omega).$$

And by convergence in $D'(\Omega)$, we have that $\dfrac{\partial v_{n_k}}{\partial x_i} \to \dfrac{\partial f}{\partial x_i} \to$ in $D'(\Omega)$. Thus, by uniqueness of limits $\dfrac{\partial f}{\partial x_i} = 0$, for all $i = 1, 2, ..., N$. And therefore f is constant.
Thus $f = \tilde{c}$ and by the above argument, we have that $v_{n_k} \to \tilde{c}$ in $H^1(\Omega)$.
But V is closed and $v_{n_k} \in V$, then $\tilde{c} \in V$. It implies that

$$\int_\Omega \tilde{c}dx = 0$$

and thus $\tilde{c} = 0$. Hence, $v_{n_k} \to 0$ in $H^1(\Omega)$ as $k \to \infty$.
But $\|v_{n_k}\|_{H^1(\Omega)} = 1$, a contracdiction. Therefore, the claim is true.

Definition 1.3.12 *More generally, we define for every $1 \leq p < \infty$ and for $m \geq 0$, the Sobolev spaces*

$$W^{m,p}(\Omega) = \{f \in L_p(\Omega) : D^\alpha f \in L_p(\Omega), |\alpha| \leq m\}$$

endowed with the following norm

$$\|f\|_{W^{m,p}(\Omega)} = \|f\|^p_{L^p(\Omega)} + (\Sigma_{|\alpha|\leq m} \|D^\alpha f\|^p_{L_P(\Omega)})^{\frac{1}{p}}.$$

We define

$$W_0^{m,q} = \overline{D(\Omega)}\ |_{W^{m,q}(\Omega)}.$$

Thus, $W_0^{m,q}$ is the closure of $D(\Omega)$ with respect to the norm $\|.\|_{W^{m,q}(\Omega)}$.
When $q = 2$, we write $H^m(\Omega) = W^{m,2}(\Omega)$ and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.
For $m = 0$ we have that

$$W^{0,q}(\Omega) = L_q(\Omega).$$

Theorem 1.3.13 *[2] Suppose Ω is smooth, then*

$$W_0^{m,q}(\Omega) := \{f \in W^{m,q}(\Omega) : f = Df = ... = ...D^{m-1}f = 0 \text{ on } \partial\Omega\}.$$

For $p = 2$, we obtain that

$$W_0^{m,2}(\Omega) := \{f \in W^{m,2}(\Omega) : f = Df = ... = D^{m-1}f = 0 \text{ on } \partial\Omega\}.$$

Theorem 1.3.14 *[6] $W^{m,p}(\Omega)$ is Banach space.*

Proof.
Let $(f_n)_{n\geq 1}$ be a Cauchy in $W^{m,q}(\Omega)$. Let $\epsilon > 0$ be given, then there exists $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_k\|_{W^{m,q}(\Omega)} < \epsilon, \text{ for all } n, k \geq n_0.$$

Then,

$$(\|f_n - f_k\|^q_{L_q(\Omega)} + \textstyle\sum_{|\alpha|\leq m} \|D^\alpha f_n - D^\alpha f_k\|^q_{L_q(\Omega)})^{\frac{1}{q}} < \epsilon, \text{ for all } n, k \geq n_0.$$

And

$$\|f_n - f_k\|^q_{L_q(\Omega)} + \textstyle\sum_{|\alpha|\leq m} \|D^\alpha f_n - D^\alpha f_k\|^q_{L_q(\Omega)} < \epsilon^q, \text{ for all } n, k \geq n_0.$$

Consequently,

$$\|f_n - f_k\|^q_{L_q(\Omega)} < \epsilon^q \text{ and } \textstyle\sum_{|\alpha|\leq m} \|D^\alpha f_n - D^\alpha f_k\|^q_{L_q(\Omega)} < \epsilon^q, \text{ for all } n, k \geq n_0,$$

thus

$$\|f_n - f_k\|_{L_q(\Omega)} < \epsilon \text{ and } \|D^\alpha f_n - D^\alpha f_k\|_{L_q(\Omega)} < \epsilon, \text{ for all } n, k \geq n_0.$$

Hence, $(f_n)_{n\geq 1}$ and $(D^\alpha f_n)_n$ are Cauchy sequences in $L_q(\Omega)$ and since $L_q(\Omega)$ is complete, then there exists $f, f_i \in L_q(\Omega)$ such that

$$f_n \longrightarrow f \text{ in } L_q(\Omega) \text{ as } n \longrightarrow \infty \text{ and } D^\alpha f_n \longrightarrow f_i \text{ in } L_q(\Omega) \text{ as } n \longrightarrow \infty.$$

But $L_q(\Omega) \subset D'(\Omega)$, we obtain that

$$f_n \longrightarrow f \text{ in } D'(\Omega) \text{ as } n \longrightarrow \infty \text{ and } D^\alpha f_n \longrightarrow D^\alpha f \text{ as } n \longrightarrow \infty \text{ in } D'(\Omega)$$

By uniqueness of limit we obtain that $D^\alpha f = f_i$ in $D'(\Omega)$. Thus
$f_n \longrightarrow f$ as $n \longrightarrow \infty$ in $L_q(\Omega)$ and $D^\alpha f_n \longrightarrow D^\alpha f$ as $n \longrightarrow \infty$ in $L_q(\Omega)$, $|\alpha| \leq m$. Hence

$$\|f_n - f_k\|_{L_q(\Omega)} \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ and } \textstyle\sum_{|\alpha|\leq m} \|D^\alpha f_n - D^\alpha f\|_{L_q(\Omega)} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

which implies that $\|f_n - f\|_{W^{m,q}(\Omega)} \longrightarrow 0$ as $n \longrightarrow \infty$. Thus,

$$f \in W^{m,q}(\Omega) \text{ and } \|f_n - f\|_{W^{m,q}(\Omega)} \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ in } W^{m,q}(\Omega).$$

Therefore, $W^{m,q}(\Omega)$ is a Banach space.

Theorem 1.3.15 *[6] (**Green's Formula**)*
Let Ω be smooth in $\mathbb{R}^n$, $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$. Then,

$$\int_\Omega \nabla u \nabla v = -\int_\Omega \Delta u v + \int_{\partial(\Omega)} \frac{\partial u}{\partial n} v d\sigma, n \geq 2$$

where $\frac{\partial u}{\partial n}$ *denotes the normal derivatives defined by* $\frac{\partial u}{\partial n} = \nabla u.\vec{n}$ *and* $\vec{n}$ *denote the normal vector.*

Definition 1.3.16 *The bilinear form* $a : H \times H \to \mathbb{R}$ *is coercive on* H *if there exists* $\alpha > 0$ *such that*

$$a(v, v) \geq \alpha \|v\|^2, \textit{ for all } v \in H$$

Example
Let Ω be smooth and connected with $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Define

$$H = \{u \in H^1(\Omega) : u \mid_{\Gamma_0} = 0\}.$$

Then, the bilinear form

$$a(u, v) = \int_\Omega \nabla u \nabla v$$

is coercive on H.
To see this, we proceed by contradiction. suppose it is not coercive then for all $n \geq 1$ there exists $(u_n)_n \in H$ such that

$$a(u_n, u_n) < \tfrac{1}{n} \|u_n\|^2_{H^1(\Omega)}.$$

Thus,

$$\int_\Omega |\nabla u_n|^2 < \frac{1}{n} \|u_n\|^2_{H^1(\Omega)}. \tag{1.3.8}$$

Let $v_n = \frac{u_n}{\|u_n\|_{H^1(\Omega)}}$. Then, $\|v_n\|_{H^1(\Omega)} = 1$ Multiplying equation (1.3.8) by $\frac{1}{\|u_n\|}$, we obtain that

$$\int_\Omega |\nabla v_n|^2 < \frac{1}{n}.$$

Which implies that

$$\int_\Omega |\nabla v_n|^2 \to 0$$

in $L_2(\Omega)$. But $|v_n| = 1$, hence $(v_n)_{n\geq 1}$ is bounded and by Rellich theorem there exists a subsequence $(v_{n_k})_{k\geq 1} \subseteq (v_n)_{n\geq 1}$ such that $(v_{n_k}) \to g$ in $L_2(\Omega)$. Thus, $(v_{n_k}) \to g$ in $D'(\Omega)$ and $\frac{\partial v_{n_k}}{\partial x_i} \to \frac{\partial g}{\partial x_i}$ in $D^{'}(\Omega)$. By uniquness of limit in $D^{'}(\Omega)$. Thus, $\frac{\partial g}{\partial x_i} = 0$. Since Ω is connected we have that $g = \hat{c}$, a constant. Thus, $v_{n_k} \to \hat{c}$ in $H^1(\Omega)$. By Trace theorem we obtain that

$$v_{n_k} \mid_{\partial\Omega} \to \hat{c} \text{ in } L_2(\partial\Omega).$$

Thus

$$v_{n_k} \mid_{\Gamma_0} \to \hat{c} \text{ in } L_2(\Gamma_0).$$

Hence

$$\int_{\Gamma_0} |v_{n_k}|^2 d\sigma \to (\hat{c})^2 mes(\Gamma_0).$$

But

$$\int_{\Gamma_0} |v_{n_k}|^2 d\sigma \to 0.$$

Therefore, $(\hat{c})^2 mes(\Gamma_0) = 0$. Since $mes(\Gamma_0) > 0$, then $\hat{c} = 0$. And hence $v_{n_k} \to 0$ in $H^1(\Omega)$.
But $\|v_{n_k}\|_{H^1(\Omega)} = 1$, a contradiction. Therefore the bilinear form is coercive on H.

Definition 1.3.17 *A bilinear form $a : H \times H \longrightarrow R$ is said to be continuous if there exists a constant $c > 0$ such that*

$$|a(u,v)| \leq c\|u\|\|v\|, \text{ for } u, v \in H$$

Example
The bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$a(u,v) = \int_\Omega \nabla u \nabla v + \int_{\partial\Omega} \lambda(x)u(x)v(x)d\sigma \text{ is continuous, } \lambda \in L_\infty(\partial\Omega).$$

To see this we apply Cauchy schwartz inequality. Let $u, v \in H^1(\Omega)$, thus

$$\begin{aligned}
|a(u,v)| &= |\int_\Omega \nabla u \nabla v + \int_{\partial\Omega} \lambda(x)u(x)v(x)d\sigma| \\
&\leq \int_\Omega |\nabla u \nabla v| + \int_{\partial\Omega} |\lambda(x)u(x)v(x)|d\sigma \\
&\leq (\int_\Omega |\nabla u|^2)^{\frac{1}{2}} (\int_\Omega |\nabla v|^2)^{\frac{1}{2}} + |\lambda| \int_{\partial\Omega} |u(x)v(x)|d\sigma \\
&\leq \|u\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} + \|\lambda\|_\infty \|u\|_{L_2(\partial\Omega)}\|v\|_{L_2(\partial\Omega)} \\
&\leq \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} + \alpha^2\|\lambda\|_\infty \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \\
&= (1 + \alpha^2\|\lambda\|_\infty)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.
\end{aligned}$$

Take $c = (1 + \alpha^2\|\lambda\|_\infty)$, then

$$|a(u,v)| \leq c\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

Therefore, a is continuous.

CHAPTER 2

Variational Inequalities in $\mathbb{R}^N$

Given $K \subset \mathbb{R}^N$ and $F : K \longrightarrow \mathbb{R}^N$, a continuous mapping. Then, the Variational inequalities(VI) is the problem of finding a point $u \in K$ such that

$$\langle F(u), v - u \rangle \geq 0, v \in K. \tag{2.0.1}$$

Variational inequalities(VI) are closely related with many general problems of nonlinear Analysis such as complementary, fixed point and optimization problem. The simplest examples of variational inequalites is the problem of solving a system of equation. Here, we intend to discuss variational inequalities in $\mathbb{R}^N$, fixed point and some elementary problem that are associated to variational inequality. In particular,we discuss the connection between variational inequalities and convex funtions.

2.1 Basic Theorems and Definition about Fixed point

Definition 2.1.1 *Let S be a metric space with metric d. A mapping $F : S \longrightarrow S$ is said to be a strictly contraction map if there exists $\alpha \in [0, 1[$*

$$d(F(x), F(y)) \leq \alpha d(x, y) \text{ , for all } x, y \in S.$$

Remark 2.1.2 *if $\alpha = 1$, then F is nonexpansive.*

Theorem 2.1.3 *[3] (**Banach's fixed point Theorem**) Let S be a complete metric space and let $F : S \longrightarrow S$ be a strict contraction mapping. Then, there exist a unique fixed point of F.*

Theorem 2.1.4 *[3] (**Brouwer's fixed point Theorem**) Let F be a continuous mapping from a closed ball $G \subset \mathbb{R}^N$ into itself. Then, F admit at least one fixed point in G.*

Theorem 2.1.5 *[3] (**Schauder's fixed point Theorem**) Let G be a compact convex subset of $\mathbb{R}^N$ and F be a continuous mapping from G into itself. Then, F admits a fixed point in G.*

2.2 First Theorem about variational inequalities

Theorem 2.2.1 *[8] Let K be compact and convex set in $\mathbb{R}^N$ and let $F : K \longrightarrow \mathbb{R}^N$ be continuous. Then, there exists $x \in K$ such that*

$$\langle F(x), y - x\rangle \geq 0, \textit{ for all } y \in K.$$

Proof. Let $\Pi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be the identication and $(.,.)$ be the scalar product on $\mathbb{R}^N$. Let $P_K(I - \Pi F) : K \longrightarrow K$ be continuous, where $Ix = x$. Then by Schauder fixed point Theorem, $P_K(I - \Pi F)$ admits a fixed point. Thus there exists $x \in K$ such that

$$P_K(I - \Pi F)x = x.$$

By the characterisation of projection Theorem we obtain that

$$\begin{aligned}(x, y - x) &\geq ((I - \Pi F)x, y - x), \text{for all} x, y \in k\\ &= (x - \Pi F(x), y - x)\\ &= (x, y - x) - \Pi(F(x), y - x), \text{for all} x, y \in K.\end{aligned}$$

Then,

$$\Pi(F(x), y - x) \geq (x, y - x) - (x, y - x) = 0, \text{ for all } x, y \in K,$$

namely

$$(F(x), y - x) \geq 0, \text{ for all } y \in K.$$

Therefore, there exists $x \in K$ such that

$$\langle F(x), y - x\rangle \geq 0, \text{ for all } y \in K.$$

Applications
Variational Inequality theory provides us with a tool for: formulating a variety of equilibrium problems; qualitatively analysing the problem in terms of existence and uniquness of solutions and stability. Many of the applications explored to date that have been formulated, studied and solved as variational inequality problems are in fact, network problems. Indeed, many mathematical problems can be formulated as variational inequality problems and several examples applicable to equilibrium analysis follows thus
Systems Equations
Many classical economic equilibrium problems have been formulated as systems of equation, since market clearing conditions necessarily equate the total supply with the total demand. In terms of variational inequality problem, the formulation of a system of equation is as follow.

Proposition 2.2.2 *[9] Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a mapping. Then for any $x \in \mathbb{R}^N$ we have that*

$$\langle F(x), y - x\rangle \geq 0, \textit{ for all } y \in \mathbb{R}^N$$

if and only if $F(x) = 0$.

Proof.
Suppose $x \in \mathbb{R}^N$ such that $F(x) = 0$, then x is a solution to the variational inequality. Let $x \in \mathbb{R}^N$. Then for any $\nu \in \mathbb{R}^N$, there exists $\epsilon \geq 0$ and $y \in \mathbb{R}^N$ such that

$$\nu = \epsilon(y - x).$$

Then, for all $\nu \in \mathbb{R}^N$

$$\begin{aligned}\langle F(x), \nu\rangle &= \langle F(x), \epsilon(y - x)\rangle \geq 0 \\ &= \epsilon\langle F(x), y - x\rangle \geq 0.\end{aligned}$$

Therefore $\langle F(x), \nu\rangle \geq 0$, for all $\nu \in \mathbb{R}^N$. Hence $F(x) = 0$, for all $x \in \mathbb{R}^N$.
An example (market equilibrium with equalities)
As an illustration, we present an example of a system of equation. Consider m consumers, with a typical consumer denoted by j, and n commodities, with a typical commodities i. Let p denote the N-dimensional vector of the commodity price with commodities $\{p_1, p_2, ..., p_N\}$. Assume that the demand for commodity i, d_i, may in general depend on the price of all the commodities, that is

$$d_i(p) = \sum_{j=1}^{m} d_i^j(p)$$

where $d_i^j(p)$ denotes the demand for commodity i by consumer j at the price vector p. Similarly, assume that the supply of a commodity i, s_i, may in general depend on the price of all the commodities ,that is

$$s_i(p) = \Sigma_{j=i}^{m}$$

where $s_i^j(p)$ denote the supply of commodity$i$ of consumer j at the price vector p. We then argue the aggregate demands for the commodities into the n-dimensional column vector d with component: $\{d_1, d_{2,...,d_N}\}$ and the aggregate supplies of the commodities into the N-dimensional column vectors with component: $\{s_1, s_2, ..., s_N\}$. But the market equilibrium conditions requires that the supply of each commodity must be equal to the demands for each commodity at the equilibrium price p^*, thus we obtain the following system of equation

$$s(p^*) - d(p^*) = 0.$$

Now, suppose we define the vectors $x \equiv p$ and $F(x) = s(p) - d(p)$, we obtain a nonlinear equation.
Problem 1
Given K closed and convex subset of $\mathbb{R}^N$ and $F : K \longrightarrow \mathbb{R}^N$ continuous, find $x \in K$ such that

$$\langle F(x), y - x\rangle \geq 0, \text{ for all } y \in K.$$

The problem (1) does not always have a solution. If K is bounded, then the problem admits a solution guaranteed by Theorem 2.2.1. If K is not bounded, then the problem does not admits a solution. For instance, take $K = \mathbb{R}$ and $f : K \longrightarrow \mathbb{R}'$ defined by

$$f(x) = \exp(x).$$

Then,

$$f(x)(y-x) \geq 0, \text{ for all } y \in \mathbb{R},$$

has no solution.
To see this, take $y = -1 + x$, then

$$0 \leq f(x)(y-x) = \exp(x)(-1+x-x) = -1\exp(x).$$

$\Rightarrow -1\exp(x) \geq 0$, impossible.
Now, given a convex set K, we set $K_R = K \cap \overline{B}(0, R)$. Returning to $F : K \to \mathbb{R}^N$,we have that there exists at least one $x_R \in K_R$ such that

$$\langle F(x_R), y - x_R \rangle \geq 0, \text{for all } y \in K_R \tag{2.2.1}$$

whenever $K_R \neq \emptyset$ by the previous theorem.

Theorem 2.2.3 *[8] Let $K \subset \mathbb{R}^N$ be closed and convex. Let $F : K \longrightarrow \mathbb{R}^N$ be continuous. Set $K_R = K \cap \overline{B}(0, R)$. A necessary and sufficient condition that there exists a solution to problem (1) is that there exists $x_R \in K_R$ such that*

$$\langle F(x_R), y - x_R \rangle \geq 0, \textit{ for all } y \in K$$

and

$$|x_R| < R.$$

Proof.
Let $K_R = K \cap \overline{B}(0, R)$, $0 \in \mathbb{R}^N$ and $x \in K$. Then,

$$\langle F(x), y - x \rangle \geq 0, \ y \in K.$$

$|x| < R \Rightarrow x \in B(0, R) \subset \overline{B}(0, R)$, thus $x \in K_R$ such that
Hence $x \in K_R$ such that

$$\langle F(x_R), y - x_R \rangle \geq 0, \text{ for all } y \in K_R$$

Suppose that $x_R \in K_R$ such that $|x_R| < R$. Since $|x_R| < R$ then given $y \in K$, take $\epsilon \geq 0$ sufficiently small such that $\omega = x_R + \epsilon(y - x_R) \in K_R$. Thus $x_R \in K_R$:

$$\begin{aligned} 0 &\leq \langle F(x_R), \omega - x_R \rangle \\ &= \langle F(x_R), x_R + \epsilon(y - x_R) - x_R \rangle, \text{for all} y \in K \\ &\langle F(x_R), \epsilon(y - x_R) \rangle \\ &= \epsilon \langle F(x_R), y - x_R \rangle, \end{aligned}$$

which implies $\epsilon\langle F(x_R), y - x_R \rangle \geq 0$, for all $y \in K$
Hence,

$$\langle F(x_R), y - x_R \rangle \geq 0, \text{ for all } y \in K.$$

Corollary 2.2.4 *[8]*
Let $F : K \longrightarrow \mathbb{R}^N$ be a mapping. Suppose there exists $x_0 \in K$ such that

$$\frac{\langle F(x) - F(x_0), x - x_0 \rangle}{|x - x_0|} \longrightarrow +\infty \textit{ as } |x| \longrightarrow +\infty.$$

Then, problem (1) has a solution in K.

Proof.
Let $x \in K$. Choose $M > |F(x_0)|$ and $R > |x_0$ such that

$$\langle F(x) - F(x_0), x - x_0 \rangle \geq M|x - x_0|, \text{ for } |x| \geq R \text{ and } x \in K.$$

Then,

$$\begin{aligned} M|x - x_0| &\leq \langle F(x) - F(x_0), x - x_0 \rangle \\ &= \langle F(x), x - x_0 \rangle - \langle F(x_0), x - x_0 \rangle, \end{aligned}$$

which implies $\langle F(x), x - x_0 \rangle \geq M|x - x_0| + \langle F(x_0), x - x_0 \rangle$. But

$$\begin{aligned} \langle F(x_0), x - x_0 \rangle &\leq |\langle F(x_0), x - x_0 \rangle| \\ &\leq |F(x)||x - x_0| \end{aligned}$$

and $-|F(x_0)||x - x_0| \leq -\langle F(x_0), x - x_0 \rangle$. Then,

$$\begin{aligned} \langle F(x), x - x_0 \rangle &\geq M|x - x_0| + \langle F(x_0), x - x_0 \rangle \\ &= M|x - x_0| - \langle F(x_0), x_0 - x \rangle \\ &\geq M|x - x_0| - |F(x_0)||x - x_0| \\ &= (M - |F(x_0)|)|x - x_0 \\ &\geq (M - |F(x_0)|)|x| - |x_0|, |x - x_0| \geq |x| - |x_0|. \end{aligned}$$

Since $M > |F(x_0)|$, then

$$(M - |F(x_0)|)(|x| - |x_0|) > 0, \text{ for } |x| = R$$

therefore $\langle F(x), x - x_0 \rangle \leq 0$.

2.3 Some problems leading to variational inequality

Theorem 2.3.1 *[8] Let $K \subset \mathbb{R}^N$ be a closed convex set and $f \in C^1(\mathbb{R}^N)$. Set $F(x) = gradf(x)$. Suppose there exists $x \in K$ such that*

$$f(x) = min_{y \in K} f(y).$$

Then, x is a solution of the following variational inequality;

$$(F(x), y - x) \geq 0, \textit{ for all } y \in K.$$

Proof.
Let $x \in K$ be such that $f(x) = min_{y \in K} f(y)$. Since K is convex, then

$$x + t(y - x) = ty + (1 - t)x \in K \text{ , for } y \in K \text{ and } 0 \leq t \leq 1$$

Set $\phi(t) = f(x + t(y - x)), 0 \leq t \leq 1$. If $t = 0$, then $\phi(0) = f(x) = min_{y \in K} f(y)$. Hence ϕ attains its minimum when $t = 0$ and

$$\phi^{'}(t) = f^{'}(x + t(y - x))(y - x).$$

Then,

$$\begin{aligned} 0 &\leq \phi^{'}(0) \\ &= f^{'}(x)(y-x) \\ &= (f^{'}(x), y-x) \\ &= (gradf(x), y-x) \\ &= (F(x), y-x). \end{aligned}$$

Therefore,

$$(F(x), y-x) \geq 0, \text{ for all } y \in K.$$

Remark 2.3.2 *In general, the converse does not hold.*

Definition 2.3.3 ***(Convex and Concave function)***
Let K be a subset of a real vector space and $f : K \to \mathbb{R} \cup \{+\infty\}$ be a map. Then,

(a) f is said to be convex if

  (i) K is a convex set, and

  (ii) For each $t \in [0,1]$ and for each $x_1, x_2 \in K$ we have that

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

(b) f is concave if

  (i) K is a convex set, and

  (ii) For each $t \in [0,1]$ and for each $x_1, x_2 \in K$, we have that

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2).$$

Remark 2.3.4 *f is convex if and only if $(-f)$ is concave,and f is concave if and only if $(-f)$ is convex.*

Properties of convex functions

Definition 2.3.5 *Let X be a normed space and $K \subset X$. Let $f : K \to \mathbb{R} \cup +\infty$ be a map. The epigragraph of f is the set defined by*

$$epi(f) := \{(x, \alpha) \in X \times \mathbb{R} : x \in D(f) \text{ and } f(x) \leq \alpha\}.$$

Theorem 2.3.6 *[4] Let X be a real normed space. A mapping $f : X \to \mathbb{R} \cup +\infty$ is convex if and only if epi(f) is convex.*

Proof.
$(\Rightarrow)$ Suppose that f is convex. Let $(x_1, \alpha_1), (x_1, \alpha_2) \in epi(f)$ and $t \in [0,1]$. Then, $f(x_1) \leq \alpha_1$ and $f(x_2) \leq \alpha_2$. Now, since f is convex,

$$\begin{aligned} f(tx_1 + (1-t)x_2) &\leq tf(x_1) + (1-t)f(x_2) \\ &\leq t\alpha_1 + (1-t)\alpha_2. \end{aligned}$$

Thus,

$$(tx_1 + (1-t)x_2, t\alpha_1 + (1-t)\alpha_2) \in epi(f).$$

But

$$(tx_1 + (1-t)x_2, t\alpha_1 + (1-t)\alpha_2) = t(x_1, \alpha_1) + (1-t)(x_2, \alpha_2).$$

Hence, $epi(f)$ is convex.
($\Leftarrow$) Suppose $epi(f)$ is convex. Let $x_1, x_2 \in D(f)$ and $t \in [0, 1]$. Then $x_1, x_2 \in D(f)$ implies that $f(x_1), f(x_2) \in (R)$. Thus, $(x_1, f(x_1)), (x_2, f(x_2)) \in epi(f)$. But $epi(f)$ is convex. Hence,

$$t(x_1, f(x_1)) + (1-t)(x_2, f(x_2)) \in epi(f)$$

and

$$(tx_1 + (1-t)x_2, tf(x_1) + (1-t)f(x_2)) \in epi(f).$$

Consequently,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Therefore, f is convex.

Definition 2.3.7 *A mapping $f : X \to \mathbb{R} \cup +\infty$ is called strictly convex if for each $x_1, x_2 \in D(f), x_1 \neq x_2$, and for each $t \in (0, 1)$ we have*

$$f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2).$$

Moreover, f is strictly concave if $(-f)$ is strictly convex.

Definition 2.3.8 *[4] Let X be normed vector space, $\lambda \in \mathbb{R}$ and $f : X \to \mathbb{R}$ be a map. The section of f; $S_{f,\lambda}$ is defined by*

$$S_{f,\lambda} := \{x \in D(f) : f(x) \leq \lambda\}.$$

Theorem 2.3.9 *[4] Suppose f is a convex function, then the sections of f are convex*

Proof.
Let $x_1, x_2 \in S_{f,\lambda}$ and $t \in [0, 1]$. Then, $f(x_1) \leq \lambda$ and $f(x_2) \leq \lambda$.
By the convexity of f we obtain

$$\begin{aligned} f(tx_1 + (1-t)x_2) &\leq tf(x_1) + (1-t)f(x_2) \\ &\leq t\lambda + (1-t)\lambda = \lambda. \end{aligned}$$

Thus,

$$f(tx_1 + (1-t)x_2) \leq \lambda.$$

And consequently,

$$tx_1 + (1-t)x_2 \in S_{f,\lambda}.$$

Therefore, $S_{f,\lambda}$ is convex.

Theorem 2.3.10 *Let f_1 and f_2 be convex functions. Then $f_1 + f_2$ is convex.*

Examples of convex functions.
Example 1
In $\mathbb{R}$, the following functions are convex $f(x) = x^2, f(x) = x^4$ and $f(x) = |x|$
Example 2
Let $g^{'} \in \mathbb{R}^n$ and define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \langle x, g^{'} \rangle.$$

Then, f is convex.
To see this, let $x_1, x_2 \in \mathbb{R}^n$ and $t \in [0, 1]$. Then,

$$\begin{aligned} f(tx_1 + (1-t)x_2) = \langle tx_1 + (1-t)x_2, g^{'} \rangle & \\ &= \langle tx_1, g^{'} \rangle + \langle (1-t)x_2, g^{'} \rangle \\ &= t\langle x_1, g^{'} \rangle + (1-t)\langle x_2, g^{'} \rangle \\ &= tf(x_1) + (1-t)f(x_2). \end{aligned}$$

Therefore, f is convex.
Example 3
Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear map. Suppose for each $x \in \mathbb{R}^n$, $a(x, x) \geq 0$. Then $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = a(x, x)$$

is convex. To see this, let $x_1, x_2 \in \mathbb{R}^n$ and $t \in [0, 1]$. Then,

$$\begin{aligned} f(tx_1 + (1-t)x_2) - tf(x_1) - (1-t)f(x_2) &= a(tx_1 + (1-t)x_2, tx_1 + (1-t)x_2) - ta(x_1, x_1) - (1-t) \\ &= a(tx_1, tx_1) + a(tx_1, (1-t)x_2) + a((1-t)x_2, tx_1) + a((1 \\ &= t^2 a(x_1, x_1) + t(1-t)a(x_1, x_2) + t(1-t)a(x_2, x_1) + (1 - \\ &= (t^2 - t)a(x_1, x_1) + t(1-t)[a(x_1, x_2) + a(x_2, x_1)] + ((1 - \\ &= (t^2 - t)a(x_1, x_1) + t(1-t)[a(x_1, x_1) + a(x_2, x_2) - a(x_1 - \\ &= -t(1-t)a(x_1 - x_2, x_1 - x_2) \leq 0. \end{aligned}$$

Thus,

$$f(tx_1 + (1-t)x_2) - tf(x_1) - (1-t)f(x_2) \leq 0,$$

and consequently,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) - (1-t)f(x_2).$$

Therefore, f is convex.

Theorem 2.3.11 *[1] Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $x \in K$ satisfying*

$$(F(x), y - x) \geq 0, \text{ for all } y \in K.$$

Then,

$$f(x) = min_{y \in K} f(y).$$

Proof.
Let $x, y \in K$ such that $(F(x), y - x) \geq 0$. Since K is convex, then $yt + (1-t)x = x + (y-x)t \in K$. By convexity of f, we have

$$f(x+(y-x)t) \leq f(x) + tf(y-x).$$

Then,

$$\begin{aligned} f^{'}(x+t(y-x))(y-x) &\leq f(y-x) \\ &= f(x-y). \end{aligned}$$

Thus, $f(y) \geq f(x) + f^{'}(x+t(y-x))(y-x)$. If $t=0$, we obtain that

$$\begin{aligned} f(y) &\geq f(x) + f^{'}(y-x) \\ &= f(x) + (F(x), y-x) \\ &\geq f(x). \end{aligned}$$

Then, $f(x) \leq f(y)$, for all $y \in K$. Therefore,$f(x) = min_{y \in K} f(y)$

Theorem 2.3.12 *[5] Let $f : K \longrightarrow \mathbb{R}^{'}, K \subset \mathbb{R}^N$, be a continuously differentiable convex function. Then, $F(x) = gradf(x)$ is monotone.*

Proof.
Let $x, x^{'} \in K$. Then,

$$\langle F(x), y-x \rangle \geq 0, \text{ for all } y \in K$$

and

$$\langle F(x^{'}), y-x^{'} \rangle \geq 0, \text{ for all } y \in K.$$

Since f is convex, then we have that for all $y \in K$

$$f(y) \geq f(x) + (F(x), y-x), \tag{2.3.1}$$

and

$$f(y) \geq f(x^{'}) + (F(x^{'}), y-x^{'}). \tag{2.3.2}$$

Set $x = y$ and $x^{'}$ in the variational inequality of x and $x^{'}$ respectively and adding, we obtain that

$$f(x^{'}) + f(x) \geq f(x) + f(x^{'}) + (F(x^{'}), x-x^{'}) + (F(x), x^{'}-x).$$

Then,

$$\begin{aligned} &\Rightarrow (F(x), x^{'}-x) + (F(x^{'}), x-x^{'}) \leq 0 \\ &\Rightarrow (F(x), x^{'}-x) - (F(x^{'}), x^{'}-x) \leq 0 \\ &\Rightarrow (F(x) - F(x^{'}), x^{'}-x) \leq 0 \\ &\Rightarrow -(F(x^{'}) - F(x), x^{'}-x) \leq 0 \\ &\Rightarrow (F(x^{'}) - F(x), x^{'}-x) \geq 0. \end{aligned}$$

Hence, $F(x) = gradf(x)$ is monotone.
Problem 2
Let $\mathbb{R}^{\mathbb{N}}_{+} = \{x = (x_1, x_2, ..., x_N) \in \mathbb{R}^{\mathbb{N}} : x_i \geq 0, i = 1, 2, ..., N\}$ be closed and convex subset of $\mathbb{R}^{\mathbb{N}}$. Let $F : \mathbb{R}^{\mathbb{N}}_{+} \longrightarrow \mathbb{R}^{\mathbb{N}}$ be a mapping. Find $x_0 \in \mathbb{R}^{\mathbb{N}}_{+}$ such that

$$F(x_0) \in \mathbb{R}^{\mathbb{N}}_{+} \text{ and } (F(x_0), x_0) = 0.$$

Theorem 2.3.13 *[11] The point $x_0 \in \mathbb{R}^{\mathbb{N}}_{+}$ is a solution to problem (2) if and only if $x_0 \in \mathbb{R}^{\mathbb{N}}_{+}$ such that*

$$(F(x_0), y - x_0) \geq 0, \textit{ for all } y \in \mathbb{R}^{\mathbb{N}}_{+}.$$

Proof.
Suppose $x_0 \in \mathbb{R}^{\mathbb{N}}_{+}$ is a solution to problem (2), then

$$F(x_0) \in \mathbb{R}^{\mathbb{N}}_{+} \text{ and } (F(x_0), x_0) = 0.$$

Thus $(F(x_0), y) \geq 0$, for all $y \in \mathbb{R}^{\mathbb{N}}_{+}$

$$\begin{aligned}(F(x_0), y - x_0) &= (F(x_0), y) - (F(x_0), x_0) \\ &= (F(x_0), y) \geq 0.\end{aligned}$$

Therefore, $x_0 \in \mathbb{R}^{\mathbb{N}}_{+}$: $(F(x_0), y - x_0) \geq 0$, for all $y \in \mathbb{R}^{\mathbb{N}}_{+}$. Now suppose $x_0 \in \mathbb{R}^{\mathbb{N}}_{+}$:

$$(F(x_0), y - x_0) \geq 0, \text{ for all } y \in \mathbb{R}^{\mathbb{N}}_{+}.$$

Let $y = x_0 + e_i$, where $e_i = (0, 0, ..., 0, 1, 0, ..., 0)$

$$\begin{aligned}0 &\leq (F(x_0), y - x_0) \\ &= (F(x_0), x_0 + e_i - x_0) \\ &= (F(x_0), e_i) \\ &= F_i(x_0) \\ &= F(x_0),\end{aligned}$$

thus,$F(x_0) \in \mathbb{R}^{\mathbb{N}}_{+}$. Since $x_0 \in \mathbb{R}^{\mathbb{N}}_{+}$ such that $(F(x_0), y - x_0) \geq 0$, for all $y \in \mathbb{R}^{\mathbb{N}}_{+}$, then for $y = 0$

$$\begin{aligned}(F(x_0), 0 - x_0) \geq 0& \\ &\Rightarrow (F(x_0), -x_0) \geq 0 \\ &\Rightarrow -(F(x_0), x_0) \geq 0 \\ &\Rightarrow (F(x_0), x_0) \leq 0.\end{aligned}$$

But $(F(x_0), x_0) \geq 0$, for any $x_0 \in \mathbb{R}^{\mathbb{N}}_{+}$ and $F(x_0) \in \mathbb{R}^{\mathbb{N}}_{+}$. Therefore $(F(x_0), x_0) = 0$.

CHAPTER 3

Variational Inequality in Hilbert Spaces

Here, we study variational inequalities in Hilbert space. Some basic theorems and proofs are presented in this chapter. This will be used in obtaining our main existence and uniqueness theorem. The study of variational inequalites started being considered around nintheenth century. Many differential equations that arise from different kind of application are solved by a very simple calculation. This approach does not give the existence and uniqueness of classical and weak solutions. Hence, the concept of Variational approach is paramount.

Let H be a real Hilbert space and $a(u, v)$ be a real bilinear form on H. Assume that the linear and continuous mapping $A : H \longrightarrow H^{'}$ determines a bilinear form via the pairing

$$a(u, v) = \langle Au, v\rangle.$$

3.1 Problem

Let H be a real Hilbert space and $f \in H^{'}$. Let $K \subset H$ be closed and convex. Find $u \in K$ such that

$$a(u, v - u) \geq \langle f, v - u\rangle, \text{for all } v \in K. \tag{3.1.1}$$

Theorem 3.1.1 *[2](**Stampacchia Theorem**) Let $a(u, v)$ be a continuous coercive bilinear form on H. Let $K \subset H$ be a nonempty closed and convex with $f \in H^{'}$. Then there exists a unique solution to problem (3.1.1).*
Moreover, if $u_1, u_2 \in K$ are solutions to problem (3.1.1) corresponding to $f_1, f_2 \in H^{'}$, then

$$\|u_1 - u_2\|_H \leq \frac{1}{\alpha}\|f_1 - f_2\|_{H^{'}}, \alpha > 0. \tag{3.1.2}$$

Proof.
Let $u_1, u_2 \in H$ be solutions of the variational inequality $u_i \in K$ such that

$$a(u_i, v - u_i) \geq \langle f_i, v - u_i\rangle,\ v \in K,\ i = 1, 2.$$

Thus,

$$a(u_1, v - u_1) \geq \langle f_1, v - u_1 \rangle \text{ ,for all } v \in K$$

and

$$a(u_2, v - u_2) \geq \langle f_2, v - u_2 \rangle, \text{ for all } v \in K.$$

Setting $v = u_1$ and $v = u_2$ in the variational inequality of u_1 and u_2 respectively, we obtain

$$a(u_1, u_2 - u_1) \geq \langle f_1, u_2 - u_1 \rangle \text{ and } a(u_2, u_1 - u_2) \geq \langle f_2, u_1 - u_2 \rangle.$$

Adding gives

$$\begin{aligned}
&a(u_1, u_2 - u_1) + a(u_2, u_1 - u_2) \geq \langle f_1, u_2 - u_1 \rangle + \langle f_2, u_1 - u_2 \rangle \\
&\Rightarrow a(u_2, u_1 - u_2) - a(u_1, u_1 - u_2) \geq \langle 2, u_1 - u_2 \rangle - \langle f_1, u_1 - u_2 \rangle \\
&\Rightarrow a(u_2 - u_1, u_1 - u_2) \geq \langle f_2 - f_1, u_1 - u_2 \rangle \\
&\Rightarrow -a(u_1 - u_2, u_1 - u_2) \geq \langle f_2 - f_1, u_1 - u_2 \rangle \\
&\Rightarrow a(u_1 - u_2, u_1 - u_2) \leq \langle f_2 - f_1, u_1 - u_2 \rangle.
\end{aligned}$$

But a is coercive, then there exists $\alpha > 0$ such that

$$\begin{aligned}
\alpha \|u_1 - u_2\|^2 &\leq a(u_1 - u_2, u_1 - u_2) \\
&\leq \langle f_1 - f_2, u_1 - u_2 \rangle \\
&\leq |\langle f_1 - f_2, u_1 - u_2 \rangle| \\
&\leq \|f_1 - f_2\| \|u_1 - u_2\|.
\end{aligned}$$

Then,

$$\alpha \|u_1 - u_2\| \leq \|f_1 - f_2\| \|u_1 - u_2\|,$$

and therefore $\|u_1 - u_2\| \leq \frac{1}{\alpha} \|f_1 - f_2\|$.
Let a be symmetric and define $I : H \longrightarrow \mathbb{R}$ by

$$I(u) = a(u, u) - 2\langle f, u \rangle, \text{ for all } u \in H.$$

Let $d = inf_{u \in\ K} I(u)$. Then $d > -\infty$. To see this we use the fact that $f \in H^{'}$ and a is corecive.

$$\begin{aligned}
I(u) &= a(u, u) - 2\langle f, u \rangle \\
&\geq \alpha \|u\|_H^2 - 2\|f\|_{H'} \|u\|_H \\
&\geq \alpha \|u\|_H^2 - \frac{1}{\alpha} \|f\|_{H'}^2 - \alpha \|u\|_H^2 \\
&= -\frac{1}{\alpha} \|f\|_{H'}^2.
\end{aligned}$$

thus, $d \geq -\frac{1}{\alpha} \|f\|_{H'}^2 > -\infty$, $\alpha > 0$.
Now, $d = inf_{u \in K} I(u) \Rightarrow$ for all $n \in \mathbb{N}$, there exists $u_n \in K$ such that

$$d \leq I(u_n) < d + \tfrac{1}{n}.$$

Since a is coercive, then there exists $\alpha > 0$ such that

$$
\begin{aligned}
\alpha\|u_n - u_m\|^2 &\leq a(u_n - u_m, u_n - u_m) \\
&= a(u_n, u_m) - 2a(u_n, u_m) + a(u_m, u_m) \\
&= a(u_n, u_n) + a(u_m, u_m) + a(u_n, u_n) + a(u_m, u_m) - a(u_n, u_n) - a(u_m, u_m) - 2a(u_n, u_m) \\
&= 2a(u_n, u_n) + 2a(u_m, u_m) - \{a(u_n, u_n) + a(u_m, u_m) + 2a(u_n, u_m)\} \\
&= 2a(u_n, u_n) + 2a(u_m, u_m) - \{a(u_n + u_m, u_n + u_m)\} \\
&= 2a(u_n, u_n) + 2a(u_m, u_m) - 4a(\frac{1}{2}(u_n + u_m), \frac{1}{2}(u_n + u_m). \qquad (3.1.1)
\end{aligned}
$$

But $I(u_n) = a(u_n, u_n) - 2\langle f, u_n\rangle$.Thus

$$
\begin{aligned}
2I(u_n) &= 2a(u_n, u_n) - 4\langle f, u_n\rangle, \text{ for all } f \in H' \\
\Rightarrow 4I(\tfrac{1}{2}(u_n + u_m) &= 4a(\tfrac{1}{2}(u_n + u_m), \tfrac{1}{2}(u_n + u_m)) - 8\langle f, \tfrac{1}{2}(u_n + u_m)\rangle.
\end{aligned}
$$

Then,

$$
\begin{aligned}
2I(u_n) + 2(u_m) - 4I(\frac{1}{2}(u_n + u_m)) &= 2a(u_n, u_n) + 2a(u_m, u_m) - 4\langle f, u_n\rangle - 4\langle f, u_m\rangle - 4a(\frac{1}{2}(u_n + u_m), \\
&= 2a(u_n, u_n) + 2a(u_m, u_m) - 4a(\frac{1}{2}(u_n + u_m), \frac{1}{2}(u_n + u_m)) - 4\{\langle f, \\
&= 2a(u_n + u_n) + 2a(u_m + u_m) - 4a(\frac{1}{2}(u_n + u_m), \frac{1}{2}(u_n + u_m)) - 4\langle \\
&= 2a(u_n, u_n) + 2a(u_m, u_m) - 4a(\frac{1}{2}(u_n + u_m), \frac{1}{2}(u_n + u_m)).
\end{aligned}
$$

Using characterisation of infimum we obtain

$$
\begin{aligned}
\alpha\|u_n - u_m\|^2 &\leq 2I(u_n) + 2I(u_m) - 4I(\frac{1}{2}(u_n + u_m)) \\
&\leq 2(d + \frac{1}{2}) + 2(d + \frac{1}{m}) - 4d \\
&= 2(\frac{1}{n} + \frac{1}{m}) \longrightarrow 0, \text{as n,m} \longrightarrow \infty.
\end{aligned}
$$

Hence $(u_n)_{n\geq 1}$ is a Cauchy sequence and since H is complete,then there exists $u \in H$ such that $u_n \longrightarrow u$. But K is closed, then $u \in K$.
By continuity of I we obtain $u_n \longrightarrow u \Rightarrow I(u_n) \longrightarrow I(u)$ as $n \longrightarrow \infty$, then $I(u) = d$.
Now, let $v \in K$. Since K is convex we obtain $u + t(v - u) \in K, 0 \leq t \leq 1$.
Thus, $d = I(u) \leq I(u + t(v - u)) \Rightarrow \frac{d}{dt}(I(u + t(v - u)))|_{t=o} \geq 0$ i.e $(v - u)I'(u) \geq 0$

$$
\begin{aligned}
I(u) &\leq I(u + t(v - u)) \\
&= a(u + t(v - u), u + t(v - u)) - 2\langle f, u + t(v - u)\rangle \\
&= a(u, u) + 2a(u, t(v - u)) + t^2a(v - u, v - u) - 2t\langle f, v - u\rangle - 2\langle f, u\rangle \\
&= a(u, u) + 2ta(u, v - u) + t^2a(v - u, v - u) - 2t\langle f, v - u\rangle - 2\langle f, u\rangle. \\
&= I(u) + 2ta(u, v - u) + t^2a(v - u, v - u) - 2t\langle f, v - u\rangle.
\end{aligned}
$$

Thus, $2ta(u, v - u) + t^2a(v - u, v - u) - 2t\langle f, v - u\rangle \geq 0$ and
hence $a(u, v - u) + \frac{t}{2}a(v - u, v - u) \geq \langle f, v - u\rangle$, for all $v \in K$ and for all $t \in [0, 1]$.
Letting $t = 0$ we obtain

$$a(u, v-u) \geq \langle f, v-u\rangle, \text{ for all } v \in K.$$

Hence, $u \in K$ is a solution to the variational inequality.

Corollary 3.1.2 *[2]**(Lax-Milgram Theorem)** Assume that $a(u,v)$ is a continuous coercive bilinear form on H. Then, for any $f \in H'$, there exists a unique $u \in H$ such that*

$$a(u,v) = \langle f, v\rangle, \textit{ for all } v \in H.$$

Moreover, if a is symmetric, then u is characterised by the property

$$u \in H \textit{ and } \tfrac{1}{2}a(u,u) - \langle f, u\rangle = min_{v\in H}\{\tfrac{1}{2}a(v,v) - \langle f, v\rangle\}.$$

Example 3.1.3 *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be measurable and choose $\varphi \in L_2(\Omega)$. Define*

$$K = \{v \in L_2(\Omega) : v \geq 0, a.e\ in \Omega\}$$

Let

$$a(u,v) = \int_\Omega u(x)v(x)dx$$

be the scalar product on $L_2(\Omega)$. Then, for any $f \in L_2(\Omega)$, there exists a unique $u \in K$:

$$\int_\Omega u(v-u)dx \geq \int_\Omega f(v-u)dx, \textit{ for all } v \in K$$

.

Claim 1: K is closed and convex.
To see this, let $(v_n)_{n\geq 1} \subset K$ such that $v_n \longrightarrow v$ as $n \longrightarrow \infty$. But

$$\begin{aligned} v_n \in K &\Rightarrow v_n \geq \varphi, \text{a.e in } \Omega, \text{for some } \varphi \in L_2(\Omega) \\ &\Rightarrow v = \lim_{n\to\infty} v_n \geq \varphi, \text{a.e in } \Omega \\ &\Rightarrow v \in K. \end{aligned}$$

Hence, K is closed.
For Converxity: Let $v, u \in K$ and $0 \leq t \leq 1$.

$v \in K \Rightarrow v \geq \varphi_1$ a.e in Ω for some $\varphi_1 \in L_2(\Omega)$, thus $tv \geq t\varphi_1$ a.e in $\Omega, 0 \leq t \leq 1$.

Also

$u \in K \Rightarrow u \geq \varphi_2$ a.e in Ω, for some $\varphi_2 \in L_2(\Omega)$, then $(1-t)u \geq (1-t)\varphi_2$ a.e in Ω, $0 \leq t \leq 1$.

Now,

$$\begin{aligned} tv + (1-t)u &\geq t\varphi_1 + (1-t)\varphi_2 \\ &= \varphi_2 + (\varphi_1 - \varphi_2)t \\ &=: \varphi_3 \in L_{2(\Omega)}. \end{aligned}$$

then $tv + (1-t)u \in K$. Therefore, K is convex. We have that $a(u,v)$ is coercive since it is an inner produvt on $L_2(\Omega)$. And therefore, by Theorem 3.1.1 there exists $u \in K$ such

$$\int_\Omega u(v-u)dx \geq \int_\Omega f(v-u)dx, \text{ for all } v \in K.$$

Claim 2: $u = max\{\varphi, f\}$

then, $max\{\varphi, f\} = f(x)$, if $\varphi(x) \leq f(x)$ and $max\varphi, f = \varphi(x)$, if $f(x) \leq \varphi(x)$.

Now

$$\begin{aligned}\int_\Omega u(v-u)dx &= \int_{(f<\varphi)} \varphi(v-\varphi)dx + \int_{(\varphi\leq f)} f(v-f)dx \\ &\geq \int_{(f\leq\varphi)} f(v-\varphi) + \int_{(\varphi\leq f)} f(v-f)dx \\ &\geq \int_\Omega f(v-f)dx, v-\varphi \geq 0, \text{for all } v \in K.\end{aligned}$$

Hence, u is a solution to the variational inequality.

3.2 Application

Example 3.2.1 *Consider the following Problem*

$$\begin{cases} -u'' + u = f \quad on\ I = (0,1), \\ u(0) = \alpha, u(1) = \beta. \end{cases} \tag{3.2.1}$$

with $\alpha, \beta \in \mathbb{R}$ *given and* $f \in L_2(I)$ *given.*
We proceed as follows:
Defined in the space $H^1(I)$ *the set* K *by*

$$K := \{v \in H^1(I) : v(0) = \alpha\ and\ v(1) = \beta\}.$$

Claim. K *is nonempty, closed and convex.*
To see this, consider the mapping $v : I \longrightarrow \mathbb{R}$ *defined by*

$$v(x) = \alpha,\ if \in [0,1)\ and\ v(x) = \beta,\ if\ x = 1.$$

We have that $v \in K$ *since* $v \in H^1(I)$, $v(0) = \alpha$ *and* $v(1) = \beta$.
Let $(v_n)_{n\geq 1} \subset K$ *such that* $v_n \longrightarrow v$ *as* $n \longrightarrow \infty$. *But* $v_n \in K \Rightarrow v_n(0) = \alpha$ *and* $v_n(1) = \beta$.
Thus,

$$v(0) = \lim_{n\to\infty} v_n(0) = \alpha\ and\ v(1) = \lim_{n\to\infty} v_n(1) = \beta.$$

Hence, $v \in K$ *since* $v(0) = \alpha$ *and* $v(1) = \beta$. *Therefore,* K *is closed.*
For convexity.
Let $v, u \in K$ *and* $0 \leq t \leq 1$.
But $u \in K \Rightarrow u(0) = \alpha$ *and* $u(1) = \beta$. *Thus,*

$$tu(0) = t\alpha\ and\ tu(1) = t\beta.$$

Also, $v \in K \Rightarrow v(0) = \alpha$ *and* $v(1) = \beta$. *Thus,*

$$(1-t)v(0) = (1-t)\alpha \text{ and } (1-t)v(1) = (1-t)\beta.$$

Now,

$$\begin{aligned}(tu + (1-t)v)(0) &= tu(0) + (1-t)v(0)\\ &= t\alpha + (1-t)\alpha\\ &= \alpha.\end{aligned}$$

Consequently,

$$\begin{aligned}(tu + (1-t)v)(1) &= tu(1) + (1-t)v(1)\\ &= t\beta + (1-t)\beta\\ &= \beta.\end{aligned}$$

Hence, $tu + (1-t)v \in K$. *Therefore,* K *is convex.*
We assume that u *is a classical solution of (3.2.1), then multiplying (3.2.1) by* $v \in D(I)$ *and integrating by part, we obtain*

$$\int_I u'(v-u)' + \int_I u(v-u) = \int_I f(v-u), \text{ for all } v \in K. \tag{3.2.2}$$

Hence,

$$\int_I u'(v-u)' + \int_I u(v-u) \geq \int_I f(v-u), \text{ for all } v \in K. \tag{3.2.3}$$

Set

$$a(u, v-u) = \int_I u'(v-u)' + \int_I u(v-u).$$

But

$$a(u,v) = \int_I u'v' + \int_I uv$$

is continuous and coercive. Therefore by Stampacchia Theorem there exist a unique function $u \in K$ *satisfying(3.2.3).*
We proceed to recover the classical solution of (3.2.1). Let $w \in H_0^1(I)$, *setting* $v = u + w$ *in the equation (3.2.3) we obtain*

$$\int_I u'w' + \int_I uw = \int_I fw, \text{ for all } w \in K. \tag{3.2.4}$$

Since $f \in L_2(I)$ *and* $u \in H_0^1(I)$ *is a weak solution of the problem, then* $u \in H^2(I)$.*Suppose* $f \in C(I)$, *then the weak solution* u *belongs to* $C^2(I)$. *To see this, we obtain from the assumption that* $f - u \in C(I)$. *Thus* $(u')' \in C(I)$ *and hence* $u \in C^2(I)$. *Therefore,* u *is a classical solution of the problem.*

Example 3.2.2

$$\begin{cases} -\Delta u + u = f \quad on\, \Omega, f \in L_2(\Omega) \\ u = g \ on\, \Gamma. \end{cases} \tag{3.2.5}$$

We proceed thus:
Suppose that there exist a function $\overline{g} \in H^1(\Omega) \cap C(\Omega)$ *such that* $\overline{g} = g$ *on* Γ. *Define in the space* $H^1(\Omega)$ *the set* K *by*

$$K := \{v \in H^1(\Omega) : v - \overline{g} = 0 \text{ on } \Gamma\}.$$

K is nonempty, closed and convex.
Assume that $u \in K$ is a weak solution of the problem, then

$$\int_\Omega \nabla u(\nabla v - \nabla u) + \int_\Omega u(v-u) \geq \int_\Omega f(v-u), \text{ for all } v \in K. \tag{3.2.6}$$

Set

$$a(u, v-u) = \int_\Omega \nabla u(\nabla v - \nabla u) + \int_\Omega u(v-u).$$

But

$$a(u,v) = \int_\Omega \nabla u \nabla v + \int_\Omega uv$$

is continuous and coercive. Therefore by Stampacchia Theorem there exist a unique function $u \in K$ satisfying the equation (3.2.6).
The classical solution is recovered following the same approach in the previous example.

Let $w \in H_0^1(\Omega)$, setting $v = u \pm w$ we obtain

$$\int_\Omega \nabla u \nabla w + \int_\Omega uw = \int_\Omega fw, \text{ for all } w \in K. \tag{3.2.7}$$

Since $f \in L_2(\Omega)$ and $u \in H_0^1(\Omega)$ is a weak solution of (3.2.5), then $u \in H^2(\Omega)$.Suppose $f \in C(\Omega)$, then the weak solution u belongs to $C^2(\Omega)$. To see this, we obtain from the assumption that $f - u \in C(\Omega)$. Thus $(u')' \in C(\Omega)$ and hence $u \in C^2(\Omega)$. Therefore, u is a classical solution of (3.2.5) .

Example 3.2.3 (Application Problem) *Let Ω be smooth on $\mathbb{R}^n$, $n \geq 2$. Consider the problem*

$$\begin{cases} -\Delta u + u = f \quad \text{on } \Omega, f \in L_2(\Omega) \\ u \mid_{\partial\Omega} = 0. \end{cases} \tag{3.2.8}$$

We proceed thus:
Assume that the problem has a solution. Define the space $H_0^1(\Omega)$ by

$$H_0^1(\Omega) := \{u \in H^1(\Omega) : u \mid_{\partial\Omega} = 0\}$$

Multiplying the equation (3.2.8) by $v \in H_0^1(\Omega)$ and integrating on Ω we have

$$-\int_\Omega \Delta uv + \int_\Omega uv = \int_\Omega fv \tag{3.2.9}$$

We then use Green Formula to obtain

$$\int_\Omega \nabla u \nabla v + \int_\Omega uv = \int_\Omega fv, \text{for all } H_0^1(\Omega). \tag{3.2.10}$$

The bilinear form $a(u,v) = \langle u, v\rangle_{H^1(\Omega)}$ is coercive and continuous and f is in the dual of $H_0^1(\Omega)$. Therefore by Lax-Milgram theorem, there exist a unique $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \langle f, v\rangle, \text{ for all } v \in H_0^1(\Omega).$$

Thus

$$\int_\Omega \nabla u \nabla v + \int_\Omega uv = \int_\Omega fv, \text{for all } H_0^1(\Omega). \tag{3.2.11}$$

Let $v = \varphi \in D(\Omega)$*, we have*

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega u\varphi = \int_\Omega f\varphi.$$

Thus

$$\int_\Omega \nabla u \nabla \varphi + \langle u, \varphi\rangle = \langle f, \varphi\rangle,$$

and consequently

$$-\langle \Delta u, \varphi\rangle + \langle u, \varphi\rangle = \langle f, \varphi\rangle. \text{ for all } \varphi \in D(\Omega).$$

Hence, $-\langle \Delta u + u, \varphi\rangle = \langle f, \varphi\rangle$*. for all* $\varphi \in D(\Omega)$*. We have*

$$-\Delta u + u = f \text{ on } D^{'}(\Omega).$$

But by regularity property we have $\Delta u = u - f \in L_2(\Omega)$*,then* $u \in H^2(\Omega)$*. Therefore the problem has a unique solution on* $H_0^1(\Omega) \cap H^2(\Omega)$*.*

CHAPTER 4

CONCLUSION

In this work, we studied variational inequalities in Hilbert space. Some basic theorems and proofs were presented. We studied and obtained existence and uniqueness theorems for variational inequalities. Many differential equations that arise from different kind of application were solved by a very simple calculation. We discovered that this approach does not give the existence and uniqueness of classical and weak solutions. Hence, the concept of Variational approach is paramount. we established the existence and uniqueness of solutions of variational inequalities. This was achieved through the use of Stampacchia theorem and Lax-Milgram theorem. And its applications.
We Considered the following Problem

$$\begin{cases} -u'' + u = f \quad \text{on } I = (0,1), \\ u(0) = \alpha, u(1) = \beta. \end{cases} \tag{4.0.1}$$

with $\alpha, \beta \in \mathbb{R}$ given and $f \in L_2(I)$ given.
And obtained its solution using variational approach via Stammpacchia Theorem. We also looked at its application in $\mathbb{R}^n$ and more generally in Hilbert Space.
We also considered the problem of the form

$$\begin{cases} -\Delta u + u = f \quad \text{on } \Omega, f \in L_2(\Omega) \\ u = g \text{ on } \Gamma. \end{cases} \tag{4.0.2}$$

We obtained its solution using variational approach by applying Stammpacchia theorem.

Bibliography

[1] Blum, E. From Optimization and Variational Inequalities to Equilibrium Problems *Math.Student*, pp. 123-145 Vol.63, 1994.

[2] Brezis, H. Functional Analysis, Sobolev Spaces and Partial Differential Equation *Spring Science and Business Media*, 2010.

[3] Browder,F. E. Fixed Point Theory and Nonlinear Problems *Proc. Sym. Pure. Math*, pp. 49-88, Vol.39, 1983.

[4] Chidume, C. E. Applicable Functional Analysis *University of Ibadan, Press*, 2014.

[5] Cottle, R. W., Giannessi, F., and Lions, J. L. Variational Inequalities and Complementarity Problems: Theory and Applications *John Wiley and Sons*, 1980.

[6] Ezzinbi, K. Lecture Notes on Distribution Theory, Sobolev Spaces and Elliptic Partial Differential Equation *African University of Science and Technology, Abuja*, 2018.

[7] Harker, P. T. and Pang, J. S. Finite-Dimensional Variational Inequalities and Nonlinear Complementarity Problems: a Survey of Theory, Algorithms and Applications *Mathematical Programming*, Vol.48(1-3), pp. 161-220, 1990.

[8] Kinderlehrer, D. and Stampachia, G. An Introduction to Variational Inequalities and their Applications *SIAM*, Vol.31, 1980.

[9] Konnov, I. V. and Laitinen, E. Theory and Applications of Variational Inequalities *University of Oulu, Department of Mathematical Sciences*, 2002.

[10] Minty, G. J. On the Generalization of a Direct Method of the Calculus of Variations *Bulletin of American Mathematical Society*, Vol.73(3), pp. 315-321, 1967.

[11] Patriksson, M. Nonlinear Programming and Variational Inequalities Problems: a Unified Apploach *Spring Science and Business Media*, Vol.23, 2013.