APPROXIMATION OF SOLUTIONS OF INCLUSION PROBLEMS WITH APPLICATIONS TO HAMMERSTEIN EQUATION AND IMAGE RESTORATION

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CERTIFICATION

This is to certify that the thesis titled

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by Adamu, Abubakar

in the Department of Pure & Applied Mathematics.

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CERTIFICATE OF APPROVAL

Ph.D. Thesis

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ABSTRACT

Monotone operators and accretive operators are essential to modern optimization and fixed point theory. Monotone operators were first discovered by Minty in real Hilbert spaces to aid in the abstract study of electrical networks. Interest in the study of monotone operators stems mainly from their firm connection with optimization problems. Accretive operators were introduced independently by Browder and Kato. Interest in the study of accretive operators stems mainly form their firm connection with the existence theory for nonlinear evolution equations in Banach spaces.

This thesis provides an in-depth study of iterative methods for approximating solutions of nonlinear equations involving monotone and accretive operators in Banach spaces more general than Hilbert spaces. Our objectives are: to develop *new* theorems and iterative algorithms; apply these theorems to problems such as, Hammerstein equation, convex minimization problems, image restoration problems and, finally, conduct numerical experiments to show the efficiency of our algorithms.

Keywords

fixed point, zeros, Hammerstein equation, image restoration, inclusion problems, accretive, uniformly smooth, uniformly convex, q-uniformly smooth, quasi-bounded, subdifferential, α-inverse strongly accretive, m-accretive, maximal monotone, convex minimization, variational inequality problem.

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CHAPTER 1

GENERAL INTRODUCTION

The contents of this thesis fall within the general area of nonlinear functional analysis and its applications, a flourishing area of research for numerous mathematicians. In this thesis, we concentrate on the following three important topics, namely:

- Approximation of zeros of m-accretive and maximal monotone operators.
- Approximation of solutions of Hammerstein equations involving maximal monotone operators.
- Approximation of zeros of sum of accretive operators with an application to image restoration problems.

1.1 Background- zeros of m-accretive and maximal monotone operators

It is well known that many problems in the fields of science and engineering can be transformed as an equation of the form

$$Au = 0, \tag{1.1.1}$$

where A is either mapping a Banach space, E to itself (accretive case) or mapping E to its dual space, E^* (monotone case) and is, in general, a nonlinear map.

Example 1.1.1

Consider the evolution equation

$$\frac{du}{dt} + Au = 0, \tag{1.1.2}$$

where A is *accretive.* This equation describes the evolution of many physical phenomena. At equilibrium state,

$$\frac{du}{dt} = 0,$$

thus equation (1.1.2) reduces to equation (1.1.1). Therefore, a solution of equation (1.1.1) (i.e., a zero of A) corresponds to the equilibrium state of the system described in equation (1.1.2).

Example 1.1.2

In convex optimization theory, the problem of finding an optimizer (i.e., a minimizer or a maximizer) of a convex function, say f, when such optimizer exists is of interest. One of the classical methods of finding an optimizer of a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ is the following:

Step 1. Find the roots of the equation $f'(x) = 0$.

Step 2. If x^* is a root of $f'(x) = 0$ and $f''(x^*) > 0$, then x^* is a minimizer of f else, if $f''(x^*) < 0$, then x^* is a maximizer of f.

This is an explicit method of obtaining an optimizer of f. However, in certain applications, the functions obtained from some models are not differentiable. For example, the absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$ has a minimizer, which in fact is 0, but the absolute value function is *not* differentiable at 0. So, in a case where the operator under consideration is not differentiable, it becomes difficult to compute a minimizer with the above technique even when it exists.

However, if the function that is not differentiable is convex and lower semicontinuous, there is a tool (*subdifferential*) one can use to compute a minimizer of the function. In general, let E be a normed space and $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semicontinuous and convex function. The subdifferential map associated to f, $\partial f : E \to 2^{E^*}$, is defined by

$$\partial f(x) := \{u^* \in E^* : \langle y - x, u^* \rangle \leq f(y) - f(x), \quad \forall\, y \in E\}.$$

It is well-known that the subdifferential map ∂f is monotone on E and that $0 \in \partial f(x)$ *if and only if x is a minimizer of f* (see Appendix A. 10.1.1 for verification). Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$ is equivalent to solving for a minimizer of f. In the case where the operator A is single-valued, the inclusion $0 \in Au$ reduces to (1.1.1).

Part of our interest in this thesis is to improve and develop iterative algorithms for approximating solutions of the equation (1.1.1) and prove strong convergence theorems in Banach spaces more general than Hilbert spaces.

1.1.1 Approximation of zeros of m-accretive operators

Let E be a real normed space with dual space E^*. For $q > 1$, an operator $J_q : E \to 2^{E^*}$ defined by

$$J_q(x) := \{u^* \in E^* : \langle x, u^* \rangle = \|x\| \|u^*\|, \quad \|u^*\| = \|x\|^{q-1}\},$$

is called *generalized duality map on E.* If $q = 2$, J_2 is called *normalized duality map* and is denoted by J. In a real Hilbert space H, J is the identity map on H.

A set-valued mapping $A : E \to 2^E$ is said to be *accretive* if, $\forall x, y \in E$, $\eta \in Ax$, $\zeta \in Ay$ there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle \eta - \zeta, j_q(x - y) \rangle \geq 0.$$

Furthermore, if A is accretive then for $\eta \in Ax$, $\zeta \in Ay$, and $\lambda > 0$,

$$\|x - y\| \leq \|x - y + \lambda(\eta - \zeta)\| \tag{1.1.3}$$

(see Appendix A. 10.1.2 for verification).

The mapping A is said to be *maximal accretive* if, in addition, its graph is not properly included in the graph of any other accretive mapping. Also, the mapping A is said to be *m-accretive* (or *hypermaximal accretive* according to [Browder, 1968]) if, in addition to A being accretive, the following range condition holds: $R(I+\lambda A) = E$, for all $\lambda > 0$. It is also known that if A is m-accretive, then it is maximal accretive but not conversely, in general (see, e.g., [Calvert, 1970]). If E is a real Hilbert space, accretive mappings are called *monotone* and, following [Minty et al., 1962], A is m-accretive if and only if A is maximal accretive (see section 7 of [Reich, 1980] for more on this).

Interest in the study of accretive mappings stems from their usefulness in applications see e.g., [Browder, 1967b], [Chidume, 2009] and the references therein. In nonlinear functional analysis, accretive operators appear mainly in *two problems*, in elliptic differential problems and in evolution problems. In the case of an elliptic differential problem, we are solving an inclusion of the form $y \in Tx$, where the operator T may be decomposed into a sum of operators among which are accretive operators. In the case of an evolution problem we study a time-dependent differential inclusion which contains, in one of its terms, an operator T which may be decomposed into a sum of operators containing an accretive operator.

In general, a fundamental problem in the study of accretive operators in real Banach spaces is the following:

$$\text{find} \quad u \in E \quad \text{such that} \quad 0 \in Au. \tag{1.1.4}$$

Existence theorems have been proved for problem (1.1.4) (see, e.g., [Browder, 1963], [Martin, 1970]). Also, iterative algorithms for approximating solutions of the inclusion (1.1.4) have been studied extensively by numerous authors (see e.g., [Martin, 1970], [Browder, 1963], [Bruck and Reich, 1977], [Chidume et al., 2007] and the references contained in them). One of the classical methods for approximating solutions of the inclusion (1.1.4) is the celebrated *proximal point algorithm* introduced by [Martinet, 1970] and studied extensively by [Rockafellar, 1976] and a host of other authors (see, e.g., [Bruck and Reich, 1977] and [Nevanlinna and Reich, 1979]).

Let H be a real Hilbert space and $A : H \to 2^H$ be a monotone mapping. The PPA is based on the fact (see [Minty et al., 1962]) for each $z \in H$ and $\lambda > 0$ there is a unique $u \in H$ such that $(z - u) \in \lambda Au$. The operator $J_\lambda := \big(I + \lambda A)^{-1}$ is single-valued and nonexpansive (see Appendix A. 10.1.3 for verification). The operator J_λ is called *resolvent operator*.

The proximal point algorithm (PPA) for a maximal monotone operator, A, is an iterative procedure that starts at a point $u_1 \in H$, and generates iteratively a sequence $\{u_n\}$ in H by:

$$u_{n+1} = \Big(I + \frac{1}{\lambda_n}A\Big)^{-1}u_n,$$

where $\{\lambda_n\}$ is a sequence of positive real numbers. [Martinet, 1970] proved that the sequence $\{u_n\}$ converges *weakly* to a point $u^* \in H$ with $0 \in Au^*$. The question of whether the weak convergence established by [Martinet, 1970] can be improved to strong convergence remained open for many years. The answer is known to be affirmative if $A := \partial f$ with f quadratic (see, e.g., [Krasnosel'skii, 1960], [Kryanev, 1973]). Strong convergence of the PPA is also assured if $\{\lambda_n\}$ is bounded away from zero and A is *strongly monotone*.

[Rockafellar, 1976] proved that the PPA converges *weakly* starting from any point. He then posed the following question: "does the proximal point algorithm always converge strongly?".

[Güler, 1991] gave a negative answer to this question. He proved that in l_2, there exists a function f such that given any positive bounded sequence $\{\lambda_n\}$, there exists a starting point $u_1 \in D(f)$ (domain of f) and the PPA starting from u_1 converges *weakly,* but not strongly.

[Solodov and Svaiter, 2000], proposed a modification of the PPA which guarantees strong convergence in a real Hilbert space. Their algorithm is as follows:

choose any $x_0 \in H$ and $y_0 \in H$ such that $v_0 \in Ay_0$, and $\sigma \in [0, 1)$. At iteration k, having x_k, choose $\mu_k > 0$, and find (y_k, v_k), an inexact solution of $0 \in Ax+\mu_k(x-x_k)$, with tolerance σ. Define

$$C_k = \{z \in H : \big\langle z - y_k, v_k\big\rangle \le 0\},$$

$$Q_k = \{z \in H : \big\langle z - x_k, x_0 - x_k\big\rangle \le 0\}.$$

$$\text{Take} \qquad x_{k+1} = P_{C_k \cap Q_k} x_0, \quad k \ge 1.$$

where A is maximal monotone and P is the metric projection on H. The authors themselves noted (see [Solodov and Svaiter, 2000], p. 195) that "$\cdots$ at each iteration, there are two subproblems to be solved $\cdots$":

(*i*) find an inexact solution of the proximal point algorithm, and

(*ii*) find the projection of x_0 onto $C_k \cap Q_k$.

They also acknowledged that these two subproblems constitute a serious drawback in using their algorithm.

[Kamimura and Takahashi, 2002] extended and generalized this result of [Solodov and Svaiter, 2000] to uniformly convex and uniformly smooth real Banach spaces, where the operator A is maximal monotone. [Reich and Sabach, 2010] extended this result to reflexive Banach spaces.

[Lehdili and Moudafi, 1996], considered the technique of the proximal mapping and Tikhonov regularization to introduce and construct the so-called Prox-Tikhonov method which generates a sequence $\{x_n\}$ in a real Hilbert space H by the algorithm:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = J_{\lambda_n}^{A_n} x_n, \quad n \geq 0, \end{cases} \tag{1.1.5}$$

where $A_n := \mu_n A + A, \quad \mu_n > 0$ and $J_{\lambda_n}^{A_n} := \big(I + \frac{1}{\lambda_n} A_n\big)^{-1}$. Using the notion of variational distance, they proved strong convergence theorems for their algorithm and its perturbed version, under appropriate conditions on the parameters of their algorithm.

[Xu, 2002] also studied the recurrence relation (1.1.5). He used the technique of nonexpansive mappings to get convergence theorems for the perturbed version of the algorithm of (1.1.5), under much relaxed conditions on the parameters. Precisely, he studied the following algorithm: Let C be a nonempty closed and convex subset of a real Hilbert space, H. Choose an initial $x_0 \in C$ and define $\{x_n\} \subset H$ by

$$x_{n+1} := \alpha_n x_0 + (1 - \alpha_n)(I + \lambda_n A)^{-1} x_n + e_n, \quad n \geq 0. \tag{1.1.6}$$

[Xu, 2002] proved strong convergence of the sequence generated by (1.1.6) provided that the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ of real numbers and the sequence $\{e_n\}$ of errors are chosen appropriately.

[Xu, 2006] introduced and studied the following proximal-type algorithm:

Theorem 1.1.3 *Let E be a reflexive real Banach space that has a weakly continuous duality map J_φ with gauge φ and let A be an m-accretive operator in X such that $C = \overline{D(A)}$ (closure of the domain of A) is convex. Assume*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$, *(ii)* $\lim\limits_{n\to\infty} \lambda_n = \infty$.

Given $u, x_1 \in C$, let $\{x_n\}$ be the sequence generated by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\Big(I + \frac{1}{\lambda_n}A\Big)^{-1} x_n, \quad n \geq 1. \tag{1.1.7}$$

Then $\{x_n\}$ converges strongly to a zero of A.

[Qin and Su, 2007] extended and generalized the result of [Xu, 2006]. They proved strong convergence of the following theorem:

Theorem 1.1.4 *Let E be a uniformly smooth Banach space and A be an m-accretive operator in E such that $A^{-1}(0) := \{z \in D(A) : 0 \in Az\} \neq \emptyset$. Given a point $u \in C$ and given $\{\alpha_n\}$ in $(0,1)$ and $\{\beta_n\}$ in $[0,1]$, suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the conditions*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lambda_n \geq \epsilon, \ \forall\, n$ *and* $\beta_n \in [0, a)$, *for some* $\epsilon > 0$, $a \in (0,1)$;

(iii) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum\limits_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum\limits_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Let $\{x_n\}$ be the composite process defined by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)\Big(I + \frac{1}{\lambda_n}A\Big)^{-1} x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n. \end{cases} \tag{1.1.8}$$

Then $\{x_n\}$ converges strongly to a zero of A.

Remark 1.1.5 *The proximal point algorithm and its modifications listed above require either the computation of $\big(I + \frac{1}{\lambda_n}A\big)^{-1}(x_n)$ or, the construction of two closed convex non-empty subsets of the space and the projection of the initial vector onto the intersection of the two closed convex subsets constructed.*

Following this, [Chidume, 2016] posed the following question, "Can an iteration process be developed which will not involve the computation of $\big(I + \frac{1}{\lambda_n}A\big)^{-1}(x_n)$ or the construction of two closed convex subsets of the space and the projection of the initial vector onto the intersection of the two sets at each step of the iteration process, that will still guarantee strong convergence to a solution of $0 \in Au$?"

This question was resolved in the affirmative by [Chidume and Djitte, 2012]. They studied an iteration process introduced by [Chidume and Zegeye, 2004], which does not involve the computation of $\big(I + \frac{1}{\lambda_n}A\big)^{-1}(x_n)$ at any stage of the iteration process. [Chidume and Djitte, 2012] used the iteration process to approximate zeros of *bounded, m-accretive maps in 2-uniformly smooth real Banach spaces.* However, the following more general theorem has been proved by [Chidume, 2016].

Theorem 1.1.6 ([Chidume, 2016]) *Let E be a uniformly smooth real Banach space with modulus of smoothness ρ_E, and let $A : E \to 2^E$ be a set-valued* **bounded** *m-accretive operator with $D(A) = E$ such that the inclusion $0 \in Au$ has a solution. For arbitrary $u_1 \in E$ define a sequence $\{u_n\}$ by,*

$$u_{n+1} = u_n - \alpha_n \zeta_n - \alpha_n \beta_n (u_n - u_1), \quad \zeta_n \in Au_n, \quad n \geq 1, \tag{1.1.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1) satisfying the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$, *$\{\beta_n\}$ is decreasing;*

(ii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, $\qquad \sum_{n=1}^{\infty} \rho_E(\alpha_n M_1) < \infty$, *for some constant M_1 (sic);*

(iii) $\lim_{n\to\infty} \dfrac{\left(\frac{\beta_{n-1}-\beta_n}{\beta_n}\right)}{\alpha_n \beta_n} = 0.$

Assume that there exists a constant $\gamma_0 > 0$ such that $\frac{\rho_E(\alpha_n)}{\alpha_n} \leq \gamma_0 \beta_n$, then the sequence $\{u_n\}$ converges strongly to a zero of A.

In Chapter 3 of this thesis, a strong convergence theorem that does not require the boundedness restriction imposed in Theorem 1.1.6 is proved. Furthermore, it is obtained as an application that the algorithm studied converges *strongly* to a solution of *Hammerstein equation.*

1.1.2 Approximation of zeros of maximal monotone operators

Let H be a real Hilbert space. A set-valued map $A : H \to 2^H$ is called *monotone* if for each $u, v \in H$, $\eta \in Au$, $\gamma \in Av$, the following inequality holds:

$$\langle u - v, \eta - \gamma \rangle \geq 0. \tag{1.1.10}$$

Monotone maps were first introduced in real Hilbert spaces by [Minty, 1960] to aid the abstract study of electrical networks and later studied by [Browder, 1983] in the setting of partial differential equations and a host of other authors. Interest in the study of monotone operators stems mainly from their firm connection with optimization problems. [Pascall and Sburlan, 1978], p. 101 made the following remark:

Remark 1.1.7 *The monotone maps constitute the most manageable class, because of the very simple structure of the monotonicity condition. The monotone mappings appear in a rather wide variety of contexts, since they can be found in many functional equations. Many of them appear also in calculus of variations, as subdifferentials of convex functions.*

In general, a fundamental problem in the study of monotone maps in Banach spaces is the following:

$$\text{find} \;\; u \in E \;\; \text{such that} \;\; 0 \in Au. \tag{1.1.11}$$

This problem has been investigated in Hilbert spaces by numerous researchers. The PPA mentioned earlier has also been studied in an inexact form. Specifically, given $x_1 \in H$, the proximal point algorithm in its inexact form generates the next iterate by solving the following equation:

$$x_{n+1} = \Big(I + \frac{1}{\lambda_n}A\Big)^{-1} x_n + e_n, \quad n \geq 1, \tag{1.1.12}$$

where $\lambda_n > 0$ and e_n is an error vector. [Rockafellar, 1976] proved that if the sequence $\{\lambda_n\}$ is bounded above and the sequence of errors satisfies the condition $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then the resulting sequence $\{x_n\}$ of proximal point iterates converges *weakly* to a solution of (1.1.11), when $E = H$, provided that a solution exists. Several alternatives and modifications of the PPA have been proposed to obtain strong convergence under suitable conditions. For a brief review of these alternatives and modifications, see, e.g., section 1.1.1 of this thesis, above.

[Chidume et al., 2019] recently proved the following strong convergence theorem.

Theorem 1.1.8 ([Chidume et al., 2019]) *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $A : E \to 2^{E^*}$ be a maximal monotone and bounded mapping with $A^{-1}(0) \neq \emptyset$. For arbitrary $u_1 \in E$, define a sequence $\{u_n\}$ iteratively by:*

$$u_{n+1} = J^{-1}\big(Ju_n - \lambda_n \eta_n - \lambda_n \theta_n (Ju_n - Ju_1)\big), \quad \eta_n \in Au_n, \quad n \geq 1, \tag{1.1.13}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying certain conditions. Then, the sequence $\{u_n\}$ converges strongly to a point in $A^{-1}(0)$.

It is well known that the convergence of iterative algorithms for approximating zeros of monotone maps are generally slow. This is expected since monotone maps are generally not differentiable. Thus, fast converging algorithms such as the *Newton-Kantorovich* algorithm cannot be used. Consequently, a lot of effort is now being put in iterative algorithms for approximating zeros of maximal monotone maps that improve speed of convergence of known algorithms. One method that is now studied is to incorporate the *inertial extrapolation term* in algorithms.

The inertial extrapolation algorithm was first introduced by [Polyak, 1964], from the heavy ball experiment of two order time dynamical system, given by:

$$u^{''}(t) + \gamma u^{'}(t) + \nabla f(u(t)) = 0, \tag{1.1.14}$$

where $\gamma > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable functional. The dynamical system (1.1.14) is discretized such that, given x_{n-1} and x_n, the next term x_{n+1}, can be determined using

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} + \gamma \frac{x_n - x_{n-1}}{h} + \nabla f(u(t)) = 0, \quad n \geq 0 \tag{1.1.15}$$

where h is the step size. Equation (1.1.15) yields the following iterative algorithm:

$$x_{n+1} = x_n + \beta(x_n - x_{n-1}) - \alpha\nabla f(x_n), \quad n \geq 0, \tag{1.1.16}$$

where $\beta = 1 - \gamma h$, $\alpha = h^2$ and $\beta(x_n - x_{n-1})$ is called the *inertial extrapolation term*, which is intended to speed up the convergence of the sequence generated by equation (1.1.16). Following the idea of [Polyak, 1964], we have the following definition:

Definition 1.1.9 *An inertial-type algorithm is a two-step iterative process in which the next iterate is defined by making use of the previous two iterates.*

[Alvarez, 2004] studied the asymptotic *weak convergence* of three *inertial* implicit iterative methods for solving the inclusion $0 \in Au$, when A is a maximal monotone operator on a real Hilbert space, which generalizes the classical PPA. The motivation for the first of these three methods called *inertial proximal point algorithm* (IPPA) stems from a discretization of the equation for an oscillator with damping and conservative restoring force: $x^{''}(t) + \gamma x^{'}(t) + \nabla f(x(t)) = 0$, where $\gamma > 0$ and $f : H \to \mathbb{R}$ is differentiable. In the context of optimization problems, this dynamical system which is called *heavy ball with friction* (HBF) was first considered by [Polyak, 1964]. It has been known that the inertial nature of the HBF could be exploited in numerical computations to accelerate the trajectories and speed up convergence (see, e.g., [Antipin, 1994, Aluffi-Pentini et al., 1984]). Concerning asymptotic convergence, [Alvarez, 2000] showed that if f is differentiable and $(\nabla f)^{-1}(0) \neq \emptyset$, then, every trajectory of HBF converges *weakly* to some $x^* \in H$ with $(\nabla f)(x^*) = 0$. Considering the implicit discretization of the HBF, the following recursion formula, in terms of resolvents, has been obtained (see, e.g., [Alvarez, 2004], pp. 774):

$$x_{n+1} = J_\lambda^{\nabla f}(x_n + \alpha(x_n - x_{n-1})), \ n = 1, \ 2, \cdots, \tag{1.1.17}$$

where λ is a regularizing parameter that combines the damping factor of γ and the actual step size $h > 0$. Replacing ∇f with a maximal monotone operator A, and considering variable parameters $\lambda_n > 0$ and $\alpha_n \in [0, 1)$, the discussion above motivated the introduction of the *inertial-type* iteration:

$$\text{(IPPA)} \qquad x_{n+1} = J_{\lambda_n}^A(x_n + \alpha_k(x_n - x_{n-1})), \ n = 1, \ 2, \cdots. \tag{1.1.18}$$

The *inertial proximal point algorithm* (IPPA) was first considered in [Alvarez, 2000] for nonsmooth conservative operator $A = \partial f$, the subdifferential of a closed, proper and convex function $f : H \to \mathbb{R} \cup \{\infty\}$. [Alvarez, 2000] (Theorem 3.1) proved, under suitable conditions, that $\{x_n\}$ converges *weakly* to a minimizer of f. For the non-conservative case, a partial positive result for cocoercive operators was obtained in [Jules and Maingé, 2002], where comparisons with first-order-in-time methods are also given through numerical facts, showing improvements in the speed of convergence. For the case of arbitrary maximal monotone operators, see, for example, [Alvarez and Attouch, 2001].

From a different point of view, the following *relaxed proximal point algorithm* (RPPA) was proposed by [Eckstein and Bertsekas, 1992] to accelerate the standard PPA:

$$\text{(RPPA)} \qquad x_{n+1} = \left[(1 - \rho_n)I + \rho_n J_{\lambda_n}^A\right](x_n), \ n = 1, \ 2, \cdots, \tag{1.1.19}$$

where $\{\rho_n\} \subset (0,2)$ is a *relaxing factor* which is assumed to satisfy the following conditions: $\inf\limits_{n\geq 0}\rho_n > 0$ and $\sup\limits_{n\geq 0}\rho_n < 2$.

[Alvarez, 2004] coupled the IPPA and RPPA, two acceleration strategies, to propose the following iterative method:

$$\text{(RIPPA)} \qquad x_{n+1} = \left[(1-\rho_n)I + \rho_n J^A_{\lambda_n}\right](x_n + \alpha_n(x_n - x_{n-1})). \tag{1.1.20}$$

He proved *weak convergence* of the sequence $\{x_n\}$ to some $x^* \in A^{-1}(0)$.

We remark that each of the algorithms: IPPA, RPPA and RIPPA involves the resolvent operator, J^A_λ.

In Chapter 4 of this thesis, an *inertial iterative algorithm* is proposed for approximating a solution of a maximal monotone inclusion problem in a uniformly convex and uniformly smooth real Banach space. The sequence generated by the algorithm is proved to converge *strongly* to a solution of the inclusion. Moreover, the theorem proved is applied to approximate a solution of a convex optimization problem, and a solution of a Hammerstein equation. Furthermore, numerical experiments are given to compare, in terms of CPU time and number of iterations, the performance of the sequence generated by our algorithm with the sequences generated by IPPA, RPPA and RIPPA, respectively, for approximating a solution of a maximal monotone inclusion in Hilbert spaces. Finally, numerical examples are given to illustrate the implementability of our algorithm for approximating a solution of a convex optimization problem and for approximating a solution of a Hammerstein equation.

1.2 Background- Hammerstein equation

An *integral equation* is an equation in which the integrand contains the unknown function. A general integral equation for an unknown function u can be written as

$$h(x) = g(x)u(x) + \int_a^b \kappa(x,t)u(t)dt, \tag{1.2.1}$$

where h, g and κ are given functions.

Remark 1.2.1

We can classify the integral equation (1.2.1) in the following three ways:

1. The equation is said to be of the *first kind* if the unknown function only appears under the integral sign, i.e., if $g \equiv 0$, and otherwise it is said to be of the *second kind*.

2. The equation is said be a *Fredholm equation* if the integration limits a and b are constants and a *Volterra equation* if a and b are functions of x.

3. The equation is said to *homogeneous* if $h \equiv 0$, otherwise *inhomogeneous*.

For more details about integral equations, interested readers should see, e.g., [Kanwal, 2013] and the references therein.

A nonlinear integral equation of Hammerstein type is one of the form

$$u(x) + \int_\Omega \kappa(x,y) f(y,u(y)) d\mu(y) = h(x) \tag{1.2.2}$$

(see [Hammerstein et al., 1930]) where μ is a σ-finite measure on the measure space Ω, κ is a real-valued function defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear and h is a given a real-valued function on Ω. Defining a linear map K by

$$Kv(\cdot) := \int_\Omega \kappa(\cdot, y) v(y) d\mu(y)$$

on Ω and denoting by F the *superposition* or *Nemitskyi* operator corresponding to f, i.e., $Fu(y) := f(y, u(y))$, then the integral equation (1.2.2) can be put in operator theoretic form as follows:

$$u + KFu = 0, \tag{1.2.3}$$

where without loss of generality, we have have taken $h \equiv 0$. The equation (1.2.3) is called *Hammerstein equation*.

Numerous problems in differential equations, for instance, elliptic boundary value problems whose linear parts possess Greens functions can, as a rule, be transformed into an equation of Hammerstein type (see, e.g., [Pascall and Sburlan, 1978]).

Example 1.2.2 ([Pascall and Sburlan, 1978])

The amplitude of oscillation $v(t)$ is a solution of the problem

$$\begin{cases} \frac{d^2v}{dt^2} + a^2 \sin v(t) = z(t), & t \in [0,1] \\ v(0) = v(1) = 0, \end{cases} \tag{1.2.4}$$

where the driving force $z(t)$ is periodical and odd. The constant $a \neq 0$ depends on the length of the pendulum and on gravity. Since the Green's function of the problem

$$v''(t) = 0, \quad v(0) = v(1) = 0$$

is the function

$$k(t,x) = \begin{cases} t(1-x), & 0 \le t \le x \\ x(1-t), & x \le t \le 1, \end{cases}$$

problem (1.2.4) is equivalent to the nonlinear integral equation

$$v(t) = -\int_0^1 k(t,x)[z(x) - a^2 \sin v(x)] dx. \tag{1.2.5}$$

If $\int_0^1 k(t,x)z(x)dx = g(t)$ and $v(t) + g(t) = u(t)$, then (1.2.5) can be written as the Hammerstein integral equation

$$u(t) + \int_0^1 k(t,x)f(x,u(x))dx = 0,$$

where $f(x,u(x)) = a^2 \sin[u(x) - g(x)]$.

The Hammerstein equation also plays a crucial role in the theory of optimal control systems, in automation and in network theory. For example, in studying automatic control, [Narendra and Gallman, 1966], using a Hammerstein model, proposed an iterative method for the identification of nonlinear systems, for samples of inputs and outputs in the presence of noise. For more on problems in optimal control, automation and network system that can be modeled as Hammerstein equations, see, e.g., [Dolezale, 1979].

Remark 1.2.3 *It is obvious that the Hammerstein equation (1.2.3) is a special case of equation the $Au = 0$ in which $A := I + KF$. However, for the iterative approximation of solutions of the equation $Au = 0$, the monotonicity/accretivity of A is crucial and the Mann-type iteration scheme has successfully been employed. Attempts to apply this method to equation (1.2.3) have not provided satisfactory results (see, e.g., [Chidume, 2009]). Part of the difficulty is the fact that the composition of two monotone operators need not be monotone (see Appendix A. 10.1.4 for a counter example).*

Part of our interest in this thesis is to improve and develop iterative algorithms for approximating solutions of the Hammerstein equation (1.2.3) and prove strong convergence theorems in Banach spaces more general than Hilbert spaces.

1.2.1 Approximation of solutions of Hammerstein equations involving monotone maps

Several existence and uniqueness theorems have been proved for equations of Hammerstein type (see, e.g., [Brézis et al., 1974], [de Figueiredo and Gupta, 1973], [Browder and Gupta, 1969], [Brezis and Browder, 1975]).

In general, equations of Hammerstein type are nonlinear and thus, there is no closed-form solutions of such equations. Consequently, methods for approximating such solutions are of interest. Several attempts have been made to approximate solutions of equations of Hammerstein type. An early method was that introduced by [Brézis and Browder, 1975] in the case where one of the operators is *angle bounded*. They proved strong convergence of a suitably defined *Galerking approximation* to a solution of (1.2.3). They prove the following theorem:

Theorem 1.2.4 ([Brézis and Browder, 1975]) *Let X be a separable Banach space, $\{P_n\}$ a sequence of bounded linear mappings of X into X such that $P_n^2 = P_n$, each*

P_n has finite-dimensional range, and for each x in X, $P_n x \to x$ as $n \to \infty$. Let Y be a closed subspace of the conjugate space X^* of X, and suppose that for each y in Y, $P_n^* y \to y$ as $n \to \infty$. Consider the Hammerstein equation

$$(I + KF)u = y \tag{1.2.6}$$

for a given $y \in Y$, where K is a continuous bounded monotone mapping of X into Y, while F is a continuous monotone mapping of X^* into X such that F is angle-bounded and maps bounded set into weakly compact sets. If Y_n is the (finite-dimensional) range of P_n^*, we define the nth Galerkin approximation $\{u_n\}$ for our given Hammerstein equation (1.2.6) by setting

$$(I + K_n F_n)u_n = P_n^* y, \tag{1.2.7}$$

where $K_n = P_n^* K P_n$ maps the range X_n of P_n into Y_n, and $F_n = P_n F P_n^*$ maps Y_n into X_n. Then

(i) For each n, the Galerkin approximation (1.2.7) defines a unique element u_n of X_n.

(ii) As $n \to \infty$, u_n converges strongly in X^* to the unique solution u of the equation $(I + KF)u = y$.

Let E be a real Banach space and $F, K : E \to E$ be *accretive-type* mappings. Let $X := E \times E$. [Chidume et al., 2003] studied the mapping $T : X \to X$ defined by

$$T[u, v] = [Fu - v, Kv + u], \ \textit{for}\ [u, v] \in X.$$

We note that $T[u, v] = 0$ if and only if u solves (1.2.3) with $v = Fu$. With this definition, [Chidume et al., 2003] were able to prove *strong convergence* of an iterative algorithm defined in the cartesian product space X to a solution of the Hammerstein equation (1.2.3). Extensions of this early result of [Chidume et al., 2003] were obtained by several authors (see, e.g., [Shehu, 2014], [Minjibir and Mohammed, 2018], [Chidume and Zegeye, 2005], [Uba et al., 2017], [Chidume and Djitte, 2009b], [Chidume and Shehu, 2013], [Chidume and Djitte, 2009a], [Ofoedu and Onyi, 2014], [Chidume and Shehu, 2012], [Chidume and Ofoedu, 2011], [Ofoedu and Malonza, 2011], and the references contained in them).

[Djitte and Sene, 2013] proved strong convergence theorem for the following explicit iterative algorithm in uniformly smooth real Banach spaces.

Theorem 1.2.5 *Let E be a uniformly smooth real Banach space and $K, F : E \to E$ be bounded and accretive mappings with $R(F) = D(K) = E$. Let $\{u_n\}$ and $\{v_n\}$ be sequences in E defined iteratively from arbitrary points $u_1, v_1 \in E$ as follows:*

$$\begin{cases} u_{n+1} &= u_n - \lambda_n^2 (Fu_n - v_n) - \lambda_n \theta_n (u_n - u_1)\big), \\ v_{n+1} &= v_n - \lambda_n^2 (Kv_n + u_n) - \lambda_n \theta_n (v_n - v_1)\big), \end{cases} \tag{1.2.8}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0, 1)$ such that $\lambda_n = o(\theta_n)$; and $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$. Suppose that $u + KFu = 0$ has a solution u^* in E. Then, there exist positive real constants $\alpha, \beta > 0$, a set $K_{min} \subset E \times E$ such that if

(1) $\frac{\lambda_n}{\theta_n} < \frac{3}{16}$; *(2)* $\beta\theta_n < 1$;

(3) $\frac{\rho_E(\lambda_n)}{\lambda_n} < \frac{3r^2}{16\alpha\beta}$, $\forall n \geq 1$;

and $w^* := (u^*, v^*) \in K_{min}$, *where* $v^* := Fu^*$, *the sequence* $\{u_n\}$ *converges to* u^*.

[Chidume and Shehu, 2015] introduced a *new* explicit iterative algorithm in the setting of a real Hilbert space and proved the following strong convergence theorem.

Theorem 1.2.6 *Let H be a real Hilbert space. Let $K, F : H \to H$ be bounded, continuous and monotone mappings. Suppose that $u^* \in H$ is a solution of $u + KFu = 0$. Let $\{u_n\}$ and $\{v_n\}$ be sequences in H defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{cases} u_{n+1} = u_n - \beta_n^2(Fu_n - v_n) - \beta_n\alpha_n(u_n - u_1), \\ v_{n+1} = v_n - \beta_n^2(Kv_n + u_n) - \beta_n\alpha_n(v_n - v_1), \end{cases} \tag{1.2.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ such that $\sum_{n=1}^{\infty} \beta_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. Then, there exists a real constant $\epsilon_0 > 0$ such that if $\beta_n < \epsilon_0\alpha_n$, $\forall\ n \geq n_0$, for some $n_0 \in \mathbb{N}$, the sequence $\{u_n\}$ converges to u^*.

[Chidume and Bello, 2017], studied a coupled explicit algorithm different from relations (1.2.9) and proved the following strong convergence theorem in L_p spaces, $1 < p \leq 2$.

Theorem 1.2.7 *Let $E = L_p$, $1 < p \leq 2$. Let $F : E \to E^*$ and $K : E^* \to E$ be strongly monotone and bounded maps. For $(u_0, v_0) \in E \times E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^* respectively by*

$$\begin{cases} u_{n+1} &= J^{-1}(Ju_n - \alpha_n(Fu_n - v_n)), \quad n \geq 0, \\ v_{n+1} &= J(J^{-1}v_n - \alpha_n(Kv_n + u_n)), \quad n \geq 0, \end{cases} \tag{1.2.10}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} (\alpha_n)^{\frac{q}{q-1}} < \infty$, where q is such that $\frac{1}{q} + \frac{1}{p} = 1$. Assume that the equation $u + KFu = 0$ has a solution. Then there exists $\gamma_0 > 0$ such that if $\alpha_n \leq \gamma_0$ for all $n \geq 1$, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u and v, respectively, where u is the solution of $u + KFu = 0$ with $v = Fu$.

[Uba et al., 2017] proved the following theorem

Theorem 1.2.8 *Let E be a uniformly convex and uniformly smooth real Banach space and $F : E \to E^*$, $K : E^* \to E$ be maximal monotone and bounded maps. For $u_1 \in E$, $v_1 \in E^*$ define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^* , respectively by*

$$\begin{cases} u_{n+1} &= J^{-1}\big(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)\big), \\ v_{n+1} &= J\big(J^{-1}v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(J^{-1}v_n - J^{-1}v_1)\big), \end{cases} \tag{1.2.11}$$

$n \geq 1$. Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of (1.2.3) with $v^* = Fu^*$, $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying some appropriate conditions.

Recently, [Uba et al., 2019], introduced a *new* coupled iterative algorithm and proved the following strong convergence theorem.

Theorem 1.2.9 *Let $E = L_p$, $1 < p \leq 2$. Let $F : E \to E^*$ and $K : E^* \to E$ be monotone and bounded maps. For $(u_0, v_0) \in E \times E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively by*

$$\begin{cases} u_{n+1} &= J^{-1}(Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n Ju_n), \quad n \geq 0, \\ v_{n+1} &= J(J^{-1}v_n - \alpha_n(Kv_n + u_n) - \alpha_n\theta_n J^{-1}v_n), \quad n \geq 0, \end{cases} \tag{1.2.12}$$

where $\{\alpha_n\}$ and θ_n are acceptably paired sequences in $(0, 1)$. Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

Remark 1.2.10 *We observe that in Theorems 1.2.6, 1.2.8 and 1.2.9, the operators K and F are required to be bounded. This is a drawback on the Theorems.*

In Chapter 5 of this thesis, we introduce a *new* recursion formula, and extend Theorem 1.2.6 to uniformly convex and uniformly smooth real Banach spaces and, at the same time, dispense with the requirement in Theorem 1.2.6 that the mappings K and F be continuous and bounded. Furthermore, using our new recursion formula, we are able to prove strong convergence without imposing any boundedness assumption on the mappings K and F. This makes our Theorem much more applicable than Theorem 1.2.8. Finally, we give numerical experiments to illustrate the convergence of the sequence of our theorem.

In Chapter 6 of this thesis, we extend Theorem 1.2.9 to uniformly convex and uniformly smooth real Banach spaces and, at the same time, dispense with the requirement in Theorem 1.2.9 that the mappings K and F be bounded. In particular, our Theorem is applicable in L_p spaces, $1 < p < \infty$, thereby providing an iterative algorithm which converges strongly to a solution of the Hammerstein equation (1.2.3) in L_p spaces, $1 < p < \infty$, and without requiring that K and F be bounded, as is imposed in Theorem 1.2.9.

1.3 Background- image restoration problems and zeros of sum of accretive operators

One of the main ways through which humans decipher information about the world is images. Images are produced to record or display useful information. Due to imperfections that may occur in the capturing process, the recorded images may invariably represent a degraded version of the original scene. The undoing of these imperfections is crucial to many of the subsequent image processing tasks. There are various forms of degradation that have to be taken into account. For example: noise, geometrical degradation (pin-cushion distortion), illumination and color imperfections (under or overexposure, saturation), blur and so on (see, e.g., [Kitkuan, 2019]). We shall introduce some basic degradation and illustrate their effect on some images.

Noise

In this context, we may define noise to be any degradation in the image signal, caused by external disturbance. If an image is being sent electronically from one place to another, via satellite or wireless transmission, or through networked cable, we may expect errors to occur in the image signal. These errors will appear on the image output in different ways depending on the type of disturbance in the signal. Thus, cleaning an image corrupted by noise is an important area of image restoration.

Types of noise

We will consider three (3) different types of noise and demonstrate how they appear on an image.

1. *Salt and pepper noise:* Also called *impulse noise, shot noise, or binary noise.* This degradation can be caused by a sudden disturbance in the image signal. Its appearance is randomly scattered white or black (or both) over pixels of the image. The amount of noise added defaults to 0.1; to add more or less noise we include an optional parameter, being a value between 0 and 1 indicating the fraction of pixels to be corrupted. Thus, for example, in Figure 1.1, we produce an image with 0.2 of its pixels corrupted by salt and pepper noise. The AUST mathematics institute image is shown in Figure 1.1(a) and the image with salt and pepper noise is shown in Figure 1.1(b).

(a) Original image

(b) Salt and pepper, σ=0.2

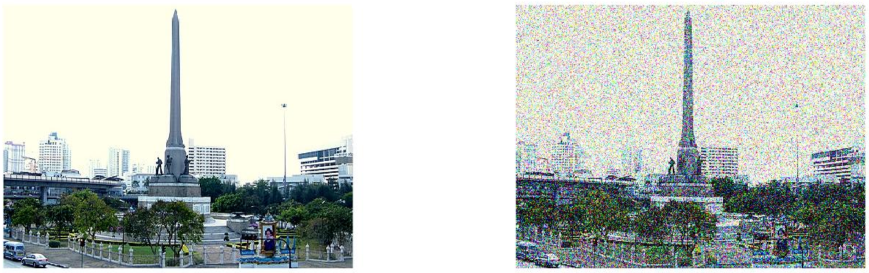
Figure 1.1: Salt and pepper noise on an image

2. *Gaussian noise:* Gaussian noise is an idealized form of *white noise,* which is caused by random fluctuations in the signal. We can observe white noise by watching a television which is slightly mistuned to a particular channel. Gaussian noise is a white noise which is normally distributed. As with salt and pepper noise, the "gaussian" parameter also can take optional values, giving the mean and variance of the noise. The default values are 0 and 0.01. An eiffel tower image is shown in Figure 1.2(a) and the image with Gaussian noise is shown in Figure 1.2(b).

(a) Original image (b) Gaussian, σ=0.08

Figure 1.2: Gaussian noise on an image

3. *Random noise:* Random noise is a noise generated by activities in the environment where seismic acquisition work is being carried out. In a construction site, random noise can be created by vehicles, people working in the survey area, wind, electrical power lines, and animal movement. This noise appears in a seismic record as spikes on the image. The victory monument image is shown in Figure 1.3(a) and the image with random noise is shown in Figure 1.3(b).



(a) Original image (b) Random, σ=0.3

Figure 1.3: Random noise on an image

Blurs

Blurring is a form of bandwidth reduction of an ideal image caused by imperfect image formation process. It can be caused by relative motion between the camera and the original scene or by an optical system that is out of focus. The blurred image is usually modeled as a convolution between the original image and a known point spread function (PSF). We will look at two different blurs, and demonstrate how they appear on an image.

1. *Motion blur:* Motion blur in images is usually modeled as the convolution of a PSF and the original image represented as pixel intensities. The knife-edge function can be used to model various types of motion blurs and hence

it allows for the construction of a PSF and an accurate estimation of the degradation function without knowledge of the specific degradation model. Thus, for example, Figure 1.4(b) would produce an image with PSF,

$$p = fspecial(motion, len, theta),$$

where "len" is linear motion of camera, specified as a numeric scalar, measured in pixels and "theta" is angle of camera motion, measured in degrees, in a counter-clockwise direction. A museum image (in Italy) is shown in Figure 1.4(a) and the image with motion blur is shown in Figure 1.4(b)

(a) Original image

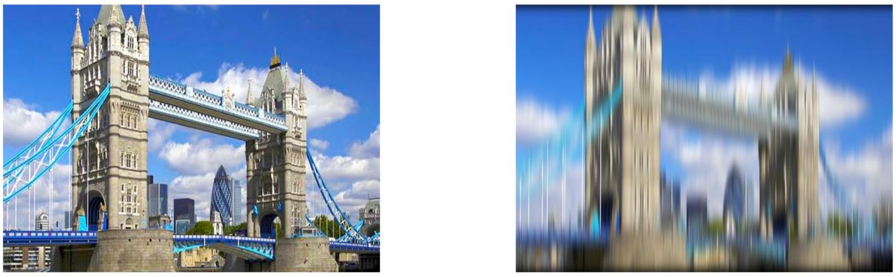
(b) Motion blur, len=30, θ=60

Figure 1.4: Motion blur on an image

2. *Gaussian blur:* The Gaussian (normal) distribution is used in weighting kernels so that the center pixel of the kernel has more influence over its final value than its neighbouring pixels. Using the Gaussian point spread function, one can create a new kernel. Thus, for example, Figure 1.5(b) would produce an image with PSF,

$$p = fspecial(gaussian, hsize, sigma),$$

where "hsize" is size of the filter, and "sigma" is the standard deviation (these constants are specified as a positive numbers). A bridge (in London) is shown in Figure 1.5(a) and the image with blur is shown in Figure 1.5(b).



(a) Original image (b) Gaussian blur, h=20, σ=10

Figure 1.5: Gaussian blur on an image

In this part of the thesis, we focus on using mathematical algorithms in the implementation of image processing tasks on computers. Precisely, we are interested in the classical problems of image restoration: image denoising and debluring. Assume we have a noisy image of dimension $n \times n$ with missing pixels, our objective is to find the closest image to the original image. General image restoration problem can be formulated by the inversion of the following observation model:

$$h = Ax + w,$$

where h is an observed image, x is an unknown image, w is a noise and A is a linear operator that depends on the concerned image recovery problem see Figure 1.6.

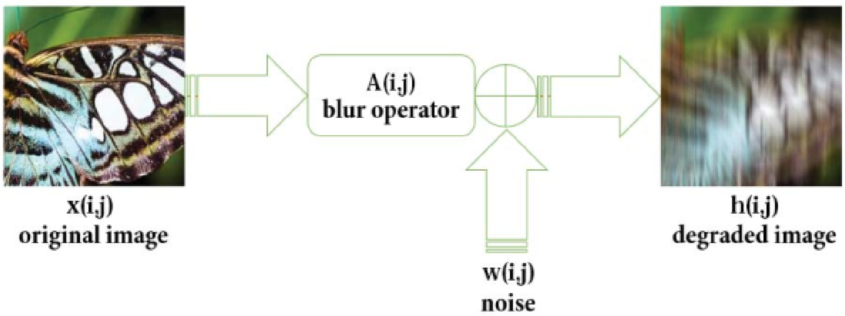


Figure 1.6: Degradation of an image

1.3.1 Approximation of zeros of sum of accretive operators

It is well known that problems arising from variational inequality, split feasibility, convex minimization, which have applications in machine learning; signal processing; linear inverse problems and image processing can be modeled as the following problem

$$\text{Find} \qquad u \in E \qquad \text{such that} \qquad 0 \in (A+B)u, \tag{1.3.1}$$

where $A : E \to E$ is a single valued operator and $B : E \to 2^E$ is a set valued operator.

Example 1.3.1

In fact, in a real Hilbert space H, if B is the subdifferential, $\partial f : H \to 2^H$ of a proper, lower semi-continuous and convex function $f : H \to \mathbb{R} \cup \{\infty\}$, then the inclusion (1.3.1) is equivalent to the following problem:

$$\text{Find} \qquad u \in H \qquad \text{such that} \qquad -Au \in Bu = \partial f(u), \tag{1.3.2}$$

i.e.,

$$f(y) - f(u) + \langle Au, y - u \rangle \geq 0, \quad \forall y \in H.$$

Observe that if f is the *indicator function* of a nonempty closed and convex subset, say C, of H, defined by

$$f(x) = i_C(x) := \begin{cases} 0, & x \in C, \\ \infty, & x \notin C, \end{cases}$$

then, problem (1.3.2) reduces to the variational inequality problem, i.e., find $u \in C$ such that

$$\langle Au, y - u \rangle \geq 0, \quad \forall y \in C.$$

Iterative algorithms for approximating solutions of the inclusion (1.3.1) have been studied extensively by numerous authors (see, e.g., [Combettes and Wajs, 2005], [Thong and Cholamjiak, 2019], [López et al., 2012], [Cholamjiak and Shehu, 2019a], [Mungkala and Kitkuan, 2019], [Tseng, 2000], [Sunthrayuth and Kumam, 2014], [Kitkuan et al., 2019a], [Kitkuan et al., 2020], [Kitkuan et al., 2019b], [Padcharoen et al., 2020]). Assuming existence of solution, one of the classical methods for approximating solution(s) of the inclusion problem (1.3.1) is the well-known *forward-backward splitting method* introduced independently by [Passty, 1979], and [Lions and Mercier, 1979] and studied extensively by [Mercier, 1980], [Gabay, 1983] and a host of other authors.

In a real Hilbert H, the forward-backward algorithm (FBA) for maximal monotone operators A and B is an iterative procedure that starts at a point $x_1 \in H$, and generates inductively the sequence $\{x_n\} \subset H$ by:

$$x_{n+1} = \big(I + \lambda_n B\big)^{-1}\big(I - \lambda_n A\big)x_n, \tag{1.3.3}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers. The FBA (1.3.3) as the name implies is based on an explicit forward step with respect to A followed by an implicit backward step with respect to B. Observe that the FBA (1.3.3) includes, in particular, the proximal point algorithm (when $A \equiv 0$). Mercier and Gabay proved that if A^{-1} is strongly monotone with modulus $\alpha > 0$, and $\{\lambda_n\} \subset (0, 2\alpha)$, then, the sequence $\{x_n\}$ converges *weakly* to a solution of (1.3.1). Furthermore, if, in addition, A is strongly monotone, then $\{x_n\}$ converges strongly to the unique solution (see, e.g., [Mercier, 1980]). [Chen and Rockafellar, 1997] showed that if A is Lipschitz and $(A+B)$ is strongly monotone then the sequence $\{x_n\}$ converges strongly.

[Takahashi et al., 2012] introduced and studied a generalization of the forward-backward splitting algorithm in real Hilbert spaces. They proved the following strong convergence theorem:

Theorem 1.3.2 *Let H be a real Hilbert space and let C be a closed convex subset of H. Let $\alpha > 0$. Let A be an α-inverse-strongly monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Suppose that $(A+B)^{-1}0 := \{x \in H : 0 \in (A+B)x\} \neq \emptyset$. Let $u \in C$, $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence ganerated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u + (1-\alpha_n)J_{\lambda_n}(x_n - \lambda_n A x_n)),$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0,\infty)$, $\{\beta_n\}$ and $\{\alpha_n\} \subset (0,1)$ satisfy

$$0 < a \leq \lambda_n \leq 2\alpha, \quad \lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0,$$

$$0 < c \leq \beta_n \leq d < 1, \quad \lim_{n\to\infty}\alpha_n = 0, \quad \sum_{n=1}^{\infty}\alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $(A+B)^{-1}0$.

In the sequel, κ_q is the best smoothness constant of the space E under consideration.

In the same year, [López et al., 2012] introduced and studied a new Halpern-type forward backward splitting algorithm in Banach spaces that are uniformly convex and q-uniformly smooth. They proved the following theorem.

Theorem 1.3.3 *Let X be a uniformly convex and q-uniformly smooth Banach space. Let $A : X \to X$ be an α-inverse strongly accretive mapping of other q and $B : X \to 2^X$ be an m-accretive operator. Assume that $S = (A+B)^{-1}0 \neq \emptyset$. We define a sequence $\{x_n\}$ by the iterative scheme*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(J_{r_n}(x_n - r_n(Ax_n + a_n)) + b_n),$$

where $u \in X$, $J_{r_n} = (I + r_n B)^{-1}$, $\{a_n\}, \{b_n\} \subset X$, $\{\alpha_n\} \subset (0,1]$, and $\{r_n\} \subset (0,\infty)$. Assume the following conditions are satisfied:

(i) $\displaystyle\sum_{n=1}^{\infty}\|a\| < \infty$ *and* $\displaystyle\sum_{n=1}^{\infty}\|b_n\| < \infty$, *or* $\displaystyle\lim_{n\to\infty}\frac{\|a_n\|}{\alpha_n} = \lim_{n\to\infty}\frac{\|b_n\|}{\alpha_n} = 0$;

(ii) $\displaystyle\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $\displaystyle 0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \left(\frac{\alpha q}{\kappa_q}\right)^{\frac{1}{q-1}}$

Then $\{x_n\}$ converges in norm to $z = Q(u)$, where Q is the sunny nonexpansive retraction of X onto S.

[Pholasaa et al., 2016] extended Theorem 1.3.2 from real Hilbert spaces to real Banach spaces that are uniformly convex and q-uniformly smooth. They introduced and studied the following algorithm:

Algorithm 1.3.4 Step 0. *Choose an arbitrary point $u, x_1 \in E$, and set $n = 1$.*

Step 1. *Compute*

$$y_n = \alpha_n u + (1-\alpha_n)\big(I + \lambda_n B\big)^{-1}(I - \lambda_n A)x_n.$$

Step 2. *Compute*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)y_n,$$

where $A : E \to E$ is α-inverse strongly accretive, $B : E \to 2^E$ is m-accretive and, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ $\{\lambda_n\} \subset (0,\infty)$ are sequences satisfying conditions C1-C3 below.

Step 3. *Update $n = n+1$ and go to Step 1.*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\limsup_{n\to\infty} \beta_n < 1$,

(C3) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \left(\frac{\alpha q}{\kappa_q}\right)^{\frac{1}{q-1}}$ (*sic*).

They proved that the sequence generated by Algorithm 1.3.4 converges strongly to a solution of problem (1.3.1).

Recently, [Kitkuan et al., 2019a] introduced and studied a generalized Halpern-type forward-backward splitting algorithm in a real Hilbert space. They proved the following theorem:

Theorem 1.3.5 *Let H be a real Hilbert space. Let $\alpha > 0$, $A : H \to H$ be an α-inverse strongly monotone mapping and $B : H \to 2^H$ be a maximal monotone operator. Let $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$ be the resolvent of B for $\lambda_n > 0$ (sic). Suppose that $\Omega := (A+B)^{-1}0 \neq \emptyset$. Let $u \in H$, $x_1 = x \in H$ and let $\{x_n\} \subset H$ be a sequence defined by*

$$\begin{aligned} z_n &= r_n x_n + (1-r_n)J^B_{\lambda_n}(I - \lambda_n A)x_n \\ y_n &= s_n x_n + (1-s_n)J^B_{\lambda_n}(I - \lambda_n A)z_n \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{r_n\}, \{s_n\}, \{\alpha_n\} \subset (0,1)$ satisfy the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\alpha$*;*

(iii) $\liminf_{n\to\infty} (1 - r_n)(1 - s_n) > 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $z \in \Omega$.

[Pan and Wang, 2019] introduced and studied an inertial viscosity type FBA in uniformly convex and uniformly smooth real Banach spaces. They studied the following algorithm:

Algorithm 1.3.6 Step 0. *Let* $x_0, x_1 \in E$ *be given starting points. Set* $n = 1$.

Step 1. *Compute*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = \delta_n w_n + (1 - \delta_n)\big(J^B_{\lambda_n}(w_n - \lambda_n A w_n)\big) \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n, \quad n \ge 1, \end{cases}$$

where $A : E \to E$ *is* α*-inverse strongly accretive,* $B : E \to 2^E$ *is m-accretive and* $f : E \to E$ *is a contraction and,* $\{\beta_n\} \subset (0,1)$*,* $\{\alpha_n\}, \{\delta_n\} \subset [0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$ *are sequences satisfying conditions C4-C7.*

Step 2. *Update* $n = n + 1$ *and go to Step 1.*

(C4) $\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\| < \infty$,

(C5) $\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty$,

(C6) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{2\alpha}{\kappa}$,

(C7) $\limsup_{n\to\infty} \delta_n < 1$.

They proved that the sequence $\{x_n\}$ generated by Algorithm 1.3.6 converges strongly to a solution of (1.1.1).

Recently, [Cholamjiak and Shehu, 2019b] introduced and studied an inertial version of the algorithm of Theorem 1.3.3. They studied the following algorithm in a uniformly convex and q-uniformly smooth real Banach space E:

Algorithm 1.3.7 **Step 0.** *Let $\beta \in [0,1)$ and $x_0, x_1 \in E$ be given starting points. Set $n = 1$.*

Step 1. *Given iterates x_{n-1} and x_n, $n \geq 1$, choose β_n such that $0 \leq \beta_n \leq \bar{\beta}_n$, where*

$$\bar{\beta}_n = \begin{cases} \min\left\{\beta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}, \\ \beta, & \text{otherwise.} \end{cases}$$

Step 2. *Compute*

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\big(J^B_{\lambda_n}(y_n - \lambda_n(Ay_n + a_n)) + b_n\big), \quad n \geq 1, \end{cases}$$

where $A : E \to E$ is α-inverse strongly accretive, $B : E \to 2^E$ is m-accretive and, $\{a_n\}, \{b_n\} \subset E$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ $\{\epsilon_n\}, \{\lambda_n\} \subset (0,\infty)$ are sequences satisfying conditions C8-C11.

Step 3. *Update $n = n+1$ and go to Step 1.*

(C8) $\lim\limits_{n\to\infty} \|a_n\|/\alpha_n = 0 \quad \lim\limits_{n\to\infty} \|b_n\|/\alpha_n = 0,$

(C9) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum\limits_{n=1}^{\infty} \alpha_n = \infty,$

(C10) $\liminf\limits_{n\to\infty} \lambda_n \leq \limsup\limits_{n\to\infty} \lambda_n < (\alpha q/\kappa_q)^{1/(q-1)},$

(C11) $\epsilon_n = o(\alpha_n)$, which means $\lim\limits_{n\to\infty} \epsilon_n/\alpha_n = 0.$

They proved that the sequence $\{x_n\}$ generated by Algorithm 1.3.7 converges strongly to a solution of (1.3.1).

The problem of finding a zero of monotone mappings which is also a fixed point of nonexpansive or nonexpansive-type mappings has been of interest over the years. Several hybrid algorithms have been proposed by many authors to solve this problem (see, e.g., [Takahashi et al., 2010], [Chidume et al., 2018], [Saewan and Kumam, 2011], [Cholamjiak et al., 2019], [Padcharoen et al., 2019]).

[Takahashi et al., 2010] introduced and studied a hybrid algorithm for approximating a zero of sum of two monotone mappings which is also a fixed point of a nonexpansive mapping, in real Hilbert spaces. They proved the following theorem:

Theorem 1.3.8 *Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let S be a nonexpansive mapping* *of C into itself, such that $F(S) \cap (A+B)^{-1}0 \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S\big(\alpha_n x + (1-\alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n)\big) \tag{1.3.4}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ satisfy

$$0 < a \leq \lambda_n \leq b < 2\alpha, \qquad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0, \qquad \lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point of $F(S) \cap (A+B)^{-1}0$ (sic).

In Chapter 7 of this thesis, we introduce a new hybrid viscosity-type forward-backward splitting method for approximating a solution of the inclusion (1.3.1) which is also a fixed point of a nonexpansive map in a uniformly convex and q-uniformly smooth real Banach space. We prove strong convergence of the sequence of our algorithm and apply the convergence result obtained to convex minimization and image restoration problems.

In Chapter 8 of this thesis, we introduce and study an inertial version of the algorithm studied in Theorem 1.3.2 of [Takahashi et al., 2012] and prove strong convergence of the sequence of our algorithm in real Banach spaces that are uniformly convex and q-uniformly smooth.

CHAPTER 2

PRELIMINARIES

In this chapter, we give some definitions, lemmas and examples of some nonlinear mappings used in this thesis; most of which could be found in standard monographs and papers of researchers working in this area of research, for example, [Alber and Ryazantseva, 2006], [Chidume, 2009], and [Cioranescu, 2012]. Throughout this chapter, except otherwise stated, we assume E is a real normed space with dual space E^* and C is a nonempty closed and convex subset of E.

2.1 Definition of terms

Definition 2.1.1 *A normed space E is said to be uniformly convex if and only if for all $\varepsilon \in (0,2]$, there exists a $\delta = \delta(\varepsilon) \in (0,1)$ such that for $x, y \in E$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, we have*

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Definition 2.1.2 *A normed space E is said to be strictly convex if and only if for all $x, y \in E, x \neq y, \|x\| = \|y\| = 1$, we have that $\|\lambda x + (1-\lambda)y\| < 1, \ \forall \lambda \in (0,1)$.*

Remark 2.1.3 *Every uniformly convex space is strictly convex. However the converse may not hold (see, e.g., [Chidume, 2009]).*

Remark 2.1.4 *Geometrically, a normed space E is uniformly convex if and only if the unit ball centred at the origin is "uniformly round". We list some examples of uniformly convex spaces.*

1. *Let E be the cartesian plane, $\mathbb{R}^2$, with the norm defined for each $x = (x_1, x_2) \in \mathbb{R}^2$ by $\|x\|_2 = \left[|x_1|^2 + |x_2|^2\right]^{\frac{1}{2}}$. Then $\mathbb{R}^2$ endowed with this norm is uniformly convex. But the space $\mathbb{R}^2$ endowed with the norms defined for each $x = (x_1, x_2) \in \mathbb{R}^2$ by $\|x\|_1 = |x_1| + |x_2|$ and $\|x\|_\infty = \max\{|x_1|, |x_2|\}$, respectively are not uniformly convex (see, e.g., [Nnyaba, 2019]).*

2. *Every real inner product space H is uniformly convex (see, e.g., [Chidume, 2009]).*

3. *L_p (or l_p or the Sobolev spaces $W_p^m(\Omega)$) spaces, $1 < p < \infty$, are uniformly convex (see, e.g., [Alber and Ryazantseva, 2006]).*

Definition 2.1.5 *The modulus of convexity of a normed space E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by*

$$\delta_E(\varepsilon) := \inf\Big\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1, \varepsilon = \|x - y\|\Big\}.$$

Definition 2.1.6 *Let E be a normed space and let $p > 1$ be a real number and $\delta_E : (0, 2] \to [0, 1]$ be the modulus of convexity of E. Then the space E is said to be p-uniformly convex if there exists a constant $c > 0$ such that*

$$\delta_E(\epsilon) \geq c\epsilon^p.$$

Remark 2.1.7 *It is well known (see, e.g., [Lindenstrauss and Tzafriri, 2013]) that L_p, l_p and the Sobolev spaces $W_p^m(\Omega)$, $1 < p < \infty$, are all p-uniformly convex and that the following estimates hold:*

$$\delta_{l_p}(\epsilon) = \delta_{L_p}(\epsilon) = \delta_{W_m^p(\Omega)}(\epsilon) = \begin{cases} \frac{p-1}{8}\epsilon^2 + o(\epsilon^2) > \frac{p-1}{8}\epsilon^2, & 1 < p < 2; \\ 1 - \Big[1 - (\frac{\epsilon}{2})^p\Big]^{\frac{1}{p}} > \frac{1}{p}\Big(\frac{\epsilon}{2}\Big)^p & p \geq 2. \end{cases}$$

Definition 2.1.8 *A normed space E is called smooth if and only if for all $x \in E$ with $\|x\| = 1$, there exists a unique $x^* \in E^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.*

Definition 2.1.9 *Let E be a normed linear space with $\dim(E) \geq 2$ (sic). The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by*

$$\begin{aligned} \rho_E(\tau) &:= \sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau\Big\} \\ &= \sup\Big\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = 1; \|y\| = 1\Big\}. \end{aligned}$$

Remark 2.1.10 *Recall that in any smooth space E, $\rho_E(\tau) \leq \tau$ for all $\tau \geq 0$, where $\rho_E : [0, \infty) \to [0, \infty)$ is the modulus of smoothness of E.*

Definition 2.1.11 *Let E be a normed linear space with $\dim(E) \geq 2$ (sic), the space E is called uniformly smooth if and only if*

$$\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0.$$

Definition 2.1.12 *For $q > 1$, a normed space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$, $\tau > 0$.*

Remark 2.1.13 *It is well known that L_p, l_p and Sobolev spaces $W_p^m(\Omega)$, $1 < p < \infty$, are all uniformly smooth and that the following estimates hold:*

$$\rho_{L_p}(\tau) = \rho_{l_p}(\tau) = \rho_{W_p^m}(\tau) = \begin{cases} (1+\tau^p)^{\frac{1}{p}} - 1 \leq \frac{1}{p}\tau^p, & 1 < p < 2; \\ \frac{p-1}{2}\tau^2 + o(\tau^2) \leq \frac{(p-1)}{2}\tau^2, & p \geq 2; \end{cases} \tag{2.1.1}$$

where $\tau \geq 0$ (see e.g., [Lindenstrauss and Tzafriri, 2013]). From (2.1.1), it is clear that if $p > 2$, then $E = L_p, l_p$ or $W_p^m(\Omega)$ is not 2- uniformly smooth.

Definition 2.1.14 *Let E be a Banach space and let $\mathcal{J} : E \to E^{**}$ be the canonical injection from E into E^{**}, that is $\langle f, \mathcal{J}(x)\rangle = \langle x, f\rangle$, $\forall\ x \in E, f \in E^*$. Then E is said to be reflexive if $\mathcal{J}$ is surjective, i.e., E is reflexive if $\mathcal{J}(E) = E^{**}$.*

Definition 2.1.15 *Let E be a normed space. For a multi-valued map $A : E \to 2^E$, the domain of A, the image of a subset S of E, $A(S)$, the range of A, $R(A)$ and the graph of A, $G(A)$ are defined as follows:*

$$D(A) := \{x \in E : Ax \neq \emptyset\} \quad A(S) := \cup\{Ax : x \in S\};$$

$$R(A) := A(E); \quad G(A) := \{[x,u] :\ x \in D(A),\ u \in Ax\}.$$

Definition 2.1.16 *Let E be a real normed linear space and $p > 1$, Then, the generalized duality map $J_p : E \to 2^{E^*}$ is defined by*

$$J_p(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|,\ \|x\| = \|x^*\|^{p-1}\}, \tag{2.1.2}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between elements of E and E^*.

In particular, for $p = 2$, we have from (2.1.2) that,

$$J_2(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|,\ \|x\| = \|x^*\|\}. \tag{2.1.3}$$

J_2 is called the normalized duality map and it is simply denoted as J.

Proposition 2.1.17 *Let E be a real normed space. Then, the duality map $J : E \to 2^{E^*}$ is well defined. That is, for every $x \in E$, $Jx \neq \emptyset$.*

Proof. Let $x \in E$. If $x = 0$, take $x^* = 0$ and the argument follows. Suppose $x \neq 0$, then $x\|x\| \neq 0$. By a consequence of Hahn Banach theorem, there exists $u^* \in E^*$ such that $\|u^*\| = 1$ and $\langle x\|x\|, u^*\rangle = \|x\|^2$. Now,

$$\langle x, \|x\|u^*\rangle = \langle x\|x\|, u^*\rangle = \|x\|^2. \tag{2.1.4}$$

Take $x^* = \|x\|u^*$. Then, $x^* \in Jx$. Hence, $Jx \neq \emptyset \quad \forall\, x \in E$.

Lemma 2.1.18 *Let E be a real Banach space and J be the normalized duality map on E. Then, $\forall \lambda \in \mathbb{R}$, $J(\lambda x) = \lambda J(x)$.*

Proof. For $\lambda = 0$, it is obvious that $J(0x) = 0Jx$. For $\lambda \neq 0$, let $f \in J(\lambda x)$. Then,
$$\langle \lambda x, f\rangle = \|\lambda x\|\|f\| \quad \text{and} \quad \|\lambda x\| = \|f\|. \quad \text{Thus,} \quad \langle \lambda x, f\rangle = \|f\|^2.$$

$$\text{Observe that } \langle x, \lambda^{-1}f\rangle = \lambda^{-1}\langle \lambda x, \lambda^{-1}f\rangle = \lambda^{-2}\langle \lambda x, f\rangle = \lambda^{-2}\|f\|^2 = \|\lambda^{-1}f\|^2$$
$$\text{implies } \langle x, \lambda^{-1}f\rangle = \|\lambda^{-1}f\|^2. \text{ But } \|\lambda x\| = \|f\| \Leftrightarrow \|x\| = \|\lambda^{-1}f\|.$$

Hence, $\lambda^{-1}f \in Jx$ which implies that $f \in \lambda Jx$. Thus, $J(\lambda x) \subset \lambda Jx$.
Conversely, let $f \in \lambda Jx$, then $\lambda^{-1}f \in Jx$. Now, $\lambda^{-1}f \in Jx$ implies
$$\langle x, \lambda^{-1}f\rangle = \|x\|\|\lambda^{-1}f\| \quad \text{and} \quad \|x\| = \|\lambda^{-1}f\|. \quad \text{Thus,} \quad \langle x, \lambda^{-1}f\rangle = \|x\|^2.$$

$$\text{Observe that } \langle \lambda x, f\rangle = \lambda\langle \lambda x, \lambda^{-1}f\rangle = \lambda^2\langle x, \lambda^{-1}f\rangle = \lambda^2\|x\|^2 = \|\lambda x\|^2$$
$$\text{implies } \langle \lambda x, f\rangle = \|\lambda x\|^2. \quad \text{But} \quad \|x\| = \|\lambda^{-1}f\| \Leftrightarrow \|\lambda x\| = \|f\|.$$

Hence, $f \in J(\lambda x)$. Thus, $\lambda Jx \subset J(\lambda x)$. This completes the proof. ■

Remark 2.1.19 *The following basic properties for Banach space E and for the normalized duality mapping J can be found in [Cioranescu, 2012]:*

(i) If E is an arbitrary Banach space, then J is monotone and bounded.

(ii) If E is strictly convex Banach space, then J is strictly monotone.

(iii) If E is a smooth Banach space, then J is single-valued and hemi-continuous, i.e., J is continuous from the strong topology of E to the weak star topology of E.

(iv) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.

(v) If E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping of E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$.

(vi) A Banach space E is uniformly smooth if and only if E^* is uniformly convex. If E is uniformly smooth, then it is smooth and reflexive.

(vii) If $E = L_p$ space $(2 \le p < \infty)$, then $J : L_p \to L_p^*$ is Lipschitz.

(viii) If $E = L_p$ space $(1 < p < 2)$, then $J : L_p \to L_p^*$ is Hölder continuous. i.e., $\forall\, x, y \in E$, $\|Jx - Jy\| \le M\|x-y\|^\alpha$, for some constants $M > 0$ and $\alpha \in (0, 1]$.

Furthermore, we have the following formulae, for J and J^{-1} in L_p, and l_p, $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ (see e.g., [Alber and Ryazantseva, 2006]; page 36):

$$\begin{aligned}
Ju &= \|u\|_{l_p}^{2-p} v \in l_q,\ v = \{|u_1|^{p-2}u_1,\ |u_2|^{p-2}u_2, \cdots\},\ u = \{u_1, u_2, \cdots\},\\
J^{-1}u &= \|u\|_{l_q}^{2-q} v \in l_p,\ v = \{|u_1|^{q-2}u_1,\ |u_2|^{q-2}u_2, \cdots\},\ u = \{u_1, u_2, \cdots\},\\
Ju &= \|u\|_{L_p}^{2-p}|u(s)|^{p-2}u(s) \in L_q(G),\ s \in G,\\
J^{-1}u &= \|u\|_{L_q}^{2-q}|u(s)|^{q-2}u(s) \in L_p(G),\ s \in G.
\end{aligned}$$

For example, let $\bar{x} = (1, \frac{1}{2}, \frac{1}{3}, ...) \in l_3$. Then,

$$J(\bar{x}) = \frac{1}{\Big(\sum\limits_{n=1}^{\infty} \frac{1}{n^3}\Big)^{\frac{1}{3}}} \Big(1, \frac{1}{4}, \frac{1}{9}, ...\Big).$$

In the sequel, except otherwise mentioned, E is a real normed space.

Definition 2.1.20 *A mapping T with domain $D(T)$ and range $R(T)$ in E is called a contraction if there exists a constant $k \in [0, 1)$ such that for all $x, y \in D(T)$,*

$$\|Tx - Ty\| \leq k\|x - y\|.$$

Remark 2.1.21 *If $0 \leq k \leq 1$, then T is called a nonexpansive mapping. Observe that every contraction is nonexpansive, but the converse is false.*

Definition 2.1.22 *Let X and Y be normed spaces. A map $T : X \to Y$ is bounded if T maps bounded subsets of X to bounded subsets of Y.*

Definition 2.1.23 *A map $A : D(A) \subset E \to 2^{E^*}$ is said to be monotone if $\forall\ x, y \in D(A),\ x^* \in Ax,\ y^* \in Ay$, we have*

$$\langle x - y, x^* - y^* \rangle \geq 0.$$

From the definition above, a single-valued map $A : D(A) \subset E \to E^*$ is monotone if $\forall\ x, y \in D(A)\ \langle x - y, Ax - Ay \rangle \geq 0$.

Definition 2.1.24 *Let H be a real Hilbert space. A nonlinear operator $A : H \to H$ is said to be angle-bounded with angle $\beta > 0$ if for any $x, y, z \in H$,*

$$\langle z - y, Ax - Ay \rangle \leq \beta \langle x - y, Ax - Ay \rangle. \tag{2.1.5}$$

For $y = z$, inequality (2.1.5) implies monotonicity of A.

Definition 2.1.25 *Let H be a real Hilbert space. A nonlinear operator $A : H \to H$ is said to be hemicontinuous if for any $x, y, z \in H$ and $t > 0$,*

$$\lim_{t \to 0^+} \langle z, A(x + ty) \rangle = \langle z, Ax \rangle.$$

Definition 2.1.26 *A map $A : D(A) \subset E \to 2^E$ is said to be accretive if $\forall\ x, y \in D(A),\ x^* \in Ax,\ y^* \in Ay$, there exists $j(x - y) \in J(x - y)$ such that*

$$\langle x^* - y^*,\ j(x - y) \rangle \geq 0.$$

A single-valued map $A : D(A) \subset E \to E$ is accretive if $\forall\ x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that $\langle Ax - Ay, j(x - y) \rangle \geq 0$.

Definition 2.1.27 *A map $A : D(A) \subset E \to E$ is called strongly accretive if there exists $k \in (0, 1)$ such that $\forall\ x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that*

$$\langle Ax - Ay, j(x - y) \rangle \geq k\|x - y\|^2.$$

Definition 2.1.28 *A map $A : D(A) \subset E \to E$ is γ-inverse strongly accretive of order q $(q > 1)$ if there exists $\gamma > 0$ such that $\forall\ x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that*

$$\langle Ax - Ay, j(x - y) \rangle \geq \gamma\|Ax - Ay\|^q,$$

Remark 2.1.29 *If $q = 2$, we simply say A is γ-inverse strongly accretive.*

2.2 Some useful tools

In the sequel, we shall need the following definitions and results. Let E be a smooth real Banach space with dual E^*. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall\, x,y \in E, \tag{2.2.1}$$

where J is the normalized duality mapping from E into E^* will play a central role in the sequel. It was introduced by Alber and has been studied by a host of authors. If $E = H$, a real Hilbert space, equation (2.2.1) reduces to $\phi(x,y) = \|x-y\|^2$, $\forall\, x,y \in H$. From the definition of the function ϕ, we have that

$$\begin{aligned}\phi(x,y) &= \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2,\end{aligned}$$

also, we have that

$$\begin{aligned}\phi(x,y) &= \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \\ &\ge \|x\|^2 - 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| - \|y\|)^2.\end{aligned}$$

Then, combining the two inequalities, we obtain that

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2 \quad \forall\, x,y \in E. \tag{2.2.2}$$

Furthermore,

$$\phi(v,u) = \phi(u,v) - 2\langle u+v, Ju - Jv\rangle + 2(\|u\|^2 - \|v\|^2). \tag{2.2.3}$$

To see this, by definition,

$$\begin{aligned}\phi(v,u) &= \|v\|^2 - 2\langle v, Ju\rangle + \|u\|^2 \\ &= \|v\|^2 - 2\langle u, Jv\rangle + \|u\|^2 + 2\langle u, Jv\rangle - 2\langle v, Ju\rangle \\ &= \phi(u,v) - 2(\langle v, Ju\rangle - \langle u, Jv\rangle).\end{aligned} \tag{2.2.4}$$

But,

$$\langle u+v, Ju - Jv\rangle = \|u\|^2 - \langle u, Jv\rangle + \langle v, Ju\rangle - \|v\|^2,$$

so that

$$\langle v, Ju\rangle - \langle u, Jv\rangle = \langle u+v, Ju - Jv\rangle + \|v\|^2 - \|u\|^2;$$

and substituting this equation in (2.2.4), we obtain equation (2.2.3). Define a map $V : X \times X^* \to \mathbb{R}$ by

$$V(x,x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2. \tag{2.2.5}$$

From this definition, we obtain

$$\begin{aligned}V(x,x^*) &= \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \\ &= \|x\|^2 - \langle x, J(J^{-1}x^*)\rangle + \|J^{-1}x^*\|^2 \\ &= \phi(x, J^{-1}(x^*)).\end{aligned}$$

Thus,

$$V(x,x^*) = \phi(x, J^{-1}(x^*)) \quad \forall \;\; x \in X, \; x^* \in X^*.$$

Lemma 2.2.1 *(see e.g., [Alber and Ryazantseva, 2006], p.36) Let E be a reflexive strictly convex Banach space with strictly convex dual space E^*. If $J_p : E \to E^*$ and $J_q^* : E^* \to E$ are the duality mappings on E and E^*, respectively, where $p, q \in (1, \infty)$ and are such that $\frac{1}{p} + \frac{1}{q} = 1$, then $J_p^{-1} = J_q^*$.*

Lemma 2.2.2 *Let $f : E \to R \cup \{+\infty\}$ be a function defined by*

$$f(x) = \frac{1}{2}\|x\|^2 \quad \forall x \in E.$$

Then, for each $x \in E$, $\partial f(x) = J(x)$, where J is the duality map on E.

Proof. For $x \in E$, let $x^* \in J(x)$. Then, for any $y \in E$, we have that

$$\begin{aligned}
\langle y - x, x^* \rangle &= \langle y, x \rangle - \|x\|^2 \\
&\leq \|y\|\|x\| - \|x\|^2 \\
&\leq \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \\
&= f(y) - f(x).
\end{aligned}$$

Thus, we have $x^* \in \partial f(x)$.

Conversely, for $x^* \in \partial f(x)$, we have

$$\langle y - x, x^* \rangle \leq f(y) - f(x) \quad \forall y \in E.$$

For $t \in (0,1)$, set $y_t = x + ty$, then we have that

$$\begin{aligned}
\langle y, x^* \rangle &\leq \frac{1}{2t}(\|x + ty\|^2 - \|x\|^2) \\
&\leq \frac{1}{2t}\big(\|x\|^2 + 2t\langle y, J(x+y)\rangle - \|x\|^2\big) \\
&\leq \frac{1}{2t}\big(2t\|y\|\|x + ty\|\big) \\
&\leq \|x\|\|y\| + t\|y\|^2.
\end{aligned}$$

As $t \to 0^+$, we have $\langle y, x^* \rangle \leq \|x\|\|y\|$. Replacing y with $-y$ gives $-\|x\|\|y\| \leq \langle y, x^* \rangle$. Thus, $|\langle y, x^* \rangle| \leq \|x\|\|y\|$ which implies $\|x^*\| \leq \|x\|$. Also, using the fact that $x^* \in \partial f(x)$ and setting $y = x - tx$, $t \in (0,1)$, we have

$$2t\langle -x, x^* \rangle \leq \|x - tx\|^2 - \|x\|^2 = (t^2 - 2t)\|x\|^2.$$

So, we have $(2-t)\|x\|^2 \leq 2\langle x, x^* \rangle$. Now, as $t \to 0^+$ we obtain

$$\|x\|^2 \leq \langle x, x^* \rangle \leq \|x\|\|x^*\|,$$

which implies $\|x\| \leq \|x^*\|$. Thus, $\|x\| = \|x^*\|$ and $\langle x, x^* \rangle = \|x\|^2$. Therefore, $x^* \in J(x)$. Hence, $\partial f(x) = J(x)$.

Lemma 2.2.3 ([Alber and Ryazantseva, 2006]) *Let X be a reflexive stricltly convex and smooth Banach space with X^* as its dual. Then for all $x \in X$ and $x^*, y^* \in X^*$,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \le V(x, x^* + y^*). \tag{2.2.6}$$

Proof. For arbitrary $x \in E$ and $x^*, y^* \in E^*$, we have that

$$\begin{aligned} V(x, x^*) &= \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \\ &= \|x\|^2 - 2\langle x, x^* + y^*\rangle + \|x^* + y^*\|^2 + \|x^*\|^2 - \|x^* + y^*\|^2 + 2\langle x, y^*\rangle \\ &= V(x, x^* + y^*) + \|x^*\|^2 - \|x^* + y^*\|^2 + 2\langle x, y^*\rangle. \end{aligned}$$

Using the subdifferential inequality and the fact that $\partial(\frac{1}{2}\|\cdot\|_*^2) = J_* = J^{-1}$ (see Lemmas 2.2.1 and 2.2.2), where $\|\cdot\|_*$ and J_* are the norm and the normalized duality map on E^*, respectively, then we have that Thus, we have

$$\begin{aligned} V(x, x^*) &\le V(x, x^* + y^*) - 2\langle J^{-1}x^*, y^*\rangle - 2\langle -x, y^*\rangle \\ &= V(x, x^* + y^*) - 2\langle J^{-1}x^* - x, y^*\rangle. \end{aligned}$$

Hence, this completes the proof. ■

Lemma 2.2.4 ([Alber and Ryazantseva, 2006], p.50) *Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Let $W : E \times E \to \mathbb{R}$ be defined by $W(x, y) = \frac{1}{2}\phi(y, x)$. Then, for all x, y, $z \in E$,*

$$W(x, y) - W(z, y) \ge \langle z - y, Jx - Jz\rangle,$$

i.e., $$\phi(y, x) - \phi(y, z) \ge 2\langle z - y, Jx - Jz\rangle,$$

and also $$W(x, y) \le \langle x - y, Jx - Jy\rangle.$$

Lemma 2.2.5 ([Alber and Ryazantseva, 2006], p.45) *Let E be a uniformly convex Banach space. Then, for any $R > 0$ and any x, $y \in E$ such that $\|x\| \le R$, $\|y\| \le R$, the following inequality holds:*

$$\langle x - y, Jx - Jy\rangle \ge (2L)^{-1}\delta_E(c_2^{-1}\|x - y\|),$$

where $c_2 = 2max\{1, R\}$, $1 < L < 1.7$.

Define

$$\mathcal{K} := 4RLsup\{\|Jx - Jy\| : \|x\| \le R,\ \|y\| \le R\} + 1. \tag{2.2.7}$$

Lemma 2.2.6 ([Alber and Ryazantseva, 2006], p.46) *Let E be a uniformly smooth and strictly convex Banach space. Then for any $R > 0$ and any x, $y \in E$ such that $\|x\| \le R$, $\|y\| \le R$ the following inequality holds:*

$$\langle x - y, Jx - Jy\rangle \ge (2L)^{-1}\delta_{E^*}(c_2^{-1}\|Jx - Jy\|),$$

where $c_2 = 2max\{1, R\}$, $1 < L < 1.7$.

Lemma 2.2.7 ([Reich, 1979] p. 342) *Let E^* be a real strictly convex dual space with a Fréchet differentiable norm, and let A be a maximal monotone operator from E to E^* such that $A^{-1}0 \neq \emptyset$. Let $s \in E^*$ be arbitrary but fixed. For each $\lambda > 0$ there exists a unique $u_\lambda \in E$ such that $Ju_\lambda + \lambda Au_\lambda \ni s$. Furthermore, u_λ converges strongly to a unique point $p \in A^{-1}0$ (sic).*

Remark 2.2.8 *From Lemma 2.2.7, setting $\lambda_n := \frac{1}{\theta_n}$, where $\theta_n \to 0$, as $n \to \infty$, $\theta_n \leq \theta_{n-1}, \forall\ n \geq 1$, $s = Jh$, for some $h \in E$, $y_n := \left(J + \frac{1}{\theta_n}A\right)^{-1} s$ and $v_n \in Ay_n$. Observe that*

$$y_n = \left(J + \frac{1}{\theta_n}A\right)^{-1} s \quad \Leftrightarrow \quad Jy_n + \frac{1}{\theta_n}v_n = s.$$

Thus,

$$v_n = \theta_n(s - Jy_n) \quad i.e., \quad v_n = \theta_n(Jh - Jy_n), \tag{2.2.8}$$

where $A : E \to 2^{E^*}$ is a maximal monotone operator. By Lemma 2.2.7, $y_n \to y^* \in A^{-1}(0)$.

Remark 2.2.9 *Let $R > 0$ such that $\|v\| \leq R, \|y_n\| \leq R$, for all $n \geq 1$. The following estimates will be needed in the sequel (see [Chidume and Idu, 2016], pages 8 and 9).*

$$\|y_{n-1} - y_n\| \leq c_2\delta_E^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\mathcal{K}\right), \tag{2.2.9}$$

$$\|Jy_{n-1} - Jy_n\| \leq c_2\delta_{E^*}^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\mathcal{K}\right), \tag{2.2.10}$$

where $\mathcal{K}$ is the constant appearing in (2.2.7).

Remark 2.2.10 *From Lemma 2.2.7, setting $\lambda_n = \frac{\beta_n}{\alpha_n}$, where $\frac{\alpha_n}{\beta_n} \to 0$, as $n \to \infty, s = Jv$ for some $v \in E$, and $y_n := \left(J + \frac{\beta_n}{\alpha_n}A\right)^{-1}s$, we obtain*

$$Ay_n = \frac{\alpha_n}{\beta_n}(Jv - Jy_n), \tag{2.2.11}$$

$y_n \to y^* \in A^{-1}0$, where $A : E \to E^*$ is maximal monotone.

Remark 2.2.11 *Let $R > 0$ such that $\|v\| \leq R, \|y_n\| \leq R$, for all $n \geq 1$. We observe that equation (2.2.11),*

$$Jv - Jy_n - \frac{\beta_n}{\alpha_n}Ay_n = 0. \tag{2.2.12}$$

Similarly,

$$\begin{aligned} Jv - Jy_{n-1} - \frac{\beta_{n-1}}{\alpha_{n-1}}Ay_{n-1} &= 0 \\ \Rightarrow \frac{\alpha_{n-1}}{\beta_{n-1}}(Jv - Jy_{n-1}) - Ay_{n-1} &= 0 \\ \Rightarrow \frac{\alpha_{n-1}\beta_n}{\alpha_n\beta_{n-1}}(Jv - Jy_{n-1}) - \frac{\beta_n}{\alpha_n}Ay_{n-1} &= 0. \end{aligned} \tag{2.2.13}$$

Equating (2.2.12) and (2.2.13), we obtain that

$$\begin{aligned}
Jv - Jy_n - \frac{\beta_n}{\alpha_n}Ay_n &= \frac{\alpha_{n-1}\beta_n}{\alpha_n\beta_{n-1}}(Jv - Jy_{n-1}) - \frac{\beta_n}{\alpha_n}Ay_{n-1} \\
Jy_{n-1} - Jy_n + \frac{\beta_n}{\alpha_n}(Ay_{n-1} - Ay_n) &= Jy_{n-1} - Jv + \frac{\alpha_{n-1}\beta_n}{\alpha_n\beta_{n-1}}(Jv - Jy_{n-1}) \\
Jy_{n-1} - Jy_n + \frac{\beta_n}{\alpha_n}(Ay_{n-1} - Ay_n) &= \Big(\frac{\alpha_{n-1}\beta_n}{\alpha_n\beta_{n-1}} - 1\Big)(Jv - Jy_{n-1}),
\end{aligned}$$

which yields

$$Jy_{n-1} - Jy_n + \frac{\beta_n}{\alpha_n}(Ay_{n-1} - Ay_n) = \varepsilon_n(Jv - Jy_{n-1}), \tag{2.2.14}$$

where $\varepsilon_n = \left(\frac{\alpha_{n-1}\beta_n - \alpha_n\beta_{n-1}}{\alpha_n\beta_{n-1}}\right)$.

Taking the duality pairing of both sides of equation (2.2.14) with $(y_{n-1} - y_n)$ *and using the fact that* A *is monotone, we obtain that*

$$\begin{aligned}
\langle y_{n-1} - y_n, Jy_{n-1} - Jy_n\rangle &= -\frac{\beta_n}{\alpha_n}\langle y_{n-1} - y_n, Ay_{n-1} - Ay_n\rangle \\
&\quad + \varepsilon_n\langle y_{n-1} - y_n, Jv - Jy_{n-1}\rangle \\
&\le \varepsilon_n\langle y_{n-1} - y_n, Jv - Jy_{n-1}\rangle \\
&\le \varepsilon_n\|y_{n-1} - y_n\|\|Jv - Jy_{n-1}\|.
\end{aligned}$$

It follows that if E *is uniformly convex and uniformly smooth, using Lemma 2.2.5, then*

$$\begin{aligned}
(2L)^{-1}\delta_E(c_2^{-1}\|y_{n-1} - y_n\|) &\le \varepsilon_n\|Jv - Jy_{n-1}\|\|y_{n-1} - y_n\| \\
&\le 2R \sup_{\|v\|\le R, \|y_n\|\le R}\big\{\|Jv - Jy_{n-1}\|\big\}\varepsilon_n.
\end{aligned} \tag{2.2.15}$$

This implies that

$$\begin{aligned}
\delta_E(c_2^{-1}\|y_{n-1} - y_n\|) &\le 4LR \sup_{\|v\|\le R, \|y_n\|\le R}\big\{\|Jv - Jy_{n-1}\|\big\}\varepsilon_n \\
&\le \mathcal{K}\varepsilon_n,
\end{aligned}$$

where $\mathcal{K}$ *is the constant appearing in (2.2.7). Thus,*

$$\|y_{n-1} - y_n\| \le c_2\delta_E^{-1}(\mathcal{K}\varepsilon_n). \tag{2.2.16}$$

Similarly, using Lemma 2.2.6, we obtain

$$\|Jy_{n-1} - Jy_n\| \le c_2\delta_{E^*}^{-1}(\mathcal{K}\varepsilon_n). \tag{2.2.17}$$

Lemma 2.2.12 ([Chidume and Idu, 2016]) *Let* X *and* Y *be real uniformly convex and uniformly smooth spaces. Let* $E = X \times Y$ *with the norm* $\|z\|_E = \big(\|u\|_X^q +$

$\|v\|_Y^q)^{\frac{1}{q}}$, *for arbitrary* $z = [u, v] \in E$. *Let* $E^* = X^* \times Y^*$ *denote the dual space of* E. *For arbitrary* $x = [x_1, x_2] \in E$, *define the map* $J_q^E : E \to E^*$ *by*

$$J_q^E(x) = J_q^E[x_1, x_2] := \big[J_q^X x_1, J_q^Y x_2\big],$$

so that for arbitrary $z_1 = [u_1, v_1], z_2 = [u_2, v_2]$ *in* E, *the duality pairing* $\langle\cdot,\cdot\rangle$ *is given by*

$$\langle z_1, J_q^E z_2\rangle := \langle u_1, J_q^X u_2\rangle + \langle v_1, J_q^Y v_2\rangle.$$

Then

(i) *E is uniformly smooth and uniformly convex,*

(ii) *J_q^E is single valued.*

Lemma 2.2.13 ([Browder, 1967a]) *Let X be a strictly convex reflexive Banach space with a strictly convex dual space X^*, T_1 a maximal monotone mapping from X to X^*, T_2 a hemicontinuous monotone mapping of all of X into X^* which carries bounded subsets of X into bounded subsets of X^*. Then the mapping $T = T_1 + T_2$ is a maximal monotone map of X into X^*.*

Lemma 2.2.14 *Let E be a real Banach space. Let $F : E \to E^*$ and $K : E^* \to E$ be maximal monotone mappings. Let $A : E \times E^* \to E^* \times E$ be defined by*

$$A[u, v] := [Fu - v, Kv + u], \quad \forall (u, v) \in E \times E^*,$$

then A is maximal monotone.

Proof. Let $S, T : E \times E^* \to E^* \times E$ be define as

$$S[u, v] := [Fu, Kv], \quad T[u, v] = [-v, u].$$

Then $A = S + T$. It suffices to show that S and T are maximal monotone. Now, let $[x_1, y_1], [x_2, y_2] \in E \times E^*$. Then,

$$\begin{aligned}\big\langle [x_1, y_1] - [x_2, y_2], S[x_1, y_1] - S[x_2, y_2]\big\rangle &= \big\langle [x_1 - x_2, y_1 - y_2], [Fx_1, Ky_1] - [Fx_2, Ky_2]\big\rangle \\ &= \big\langle [x_1 - x_2, y_1 - y_2], [Fx_1 - Fx_2, Ky_1 - Ky_2]\big\rangle \\ &= \big\langle x_1 - x_2, Fx_1 - Fx_2\big\rangle + \big\langle Ky_1 - Ky_2, y_1 - y_2\big\rangle \geq 0,\end{aligned}$$

establishing the monotonicity of S. Let $h = [h_1, h_2] \in E^* \times E$. Since F and K are maximal monotone, take $u = \big(J_E + \lambda F\big)^{-1} h_1$ and $v = \big(J_E^{-1} + \lambda K\big)^{-1} h_2$. Then $(J_{E\times E^*} + \lambda S)w = h$, where $w = [u, v]$. Hence, S is maximal monotone.

Clearly, from the definition of T, it will map bounded subsets of $E \times E^*$ to bounded subsets of $E^* \times E$. Thus, T is bounded. Next, we show that T is monotone. Let $[x_1, y_1], [x_2, y_2] \in E \times E^*$. Then,

$$\begin{aligned}\big\langle [x_1, y_1] - [x_2, y_2], T[x_1, y_1] - T[x_2, y_2]\big\rangle &= \big\langle [x_1 - x_2, y_1 - y_2], [-y_1, x_1] - [-y_2, x_2]\big\rangle \\ &= \big\langle [x_1 - x_2, y_1 - y_2], [-y_1 + y_2, x_1 - x_2]\big\rangle \\ &= -\big\langle x_1 - x_2, y_1 - y_2\big\rangle + \big\langle x_1 - x_2, y_1 - y_2\big\rangle = 0,\end{aligned}$$

establishing the monotonicity of T. Furthermore, for any sequence $\{[x_n, y_n]\}$ in $E \times E^*$ such that

$$\lim_{n\to\infty}[x_n, y_n] = [x, y] \in E \times E^*,$$

$$\lim_{n\to\infty} T[x_n, y_n] = \lim_{n\to\infty}[-y_n, x_n] = [-y, x] = T[x, y].$$

Hence, T is continuous and thus, it is hemicontinuous. Therefore, by Lemma 2.2.13, $A = S + T$ is maximal monotone. ■

Remark 2.2.15 *From Lemma 2.2.7, setting $\lambda_n := \frac{\beta_n}{\alpha_n}$ where $\frac{\alpha_n}{\beta_n} \to 0$, as $n \to \infty$, $z = [z_1, z_2] = J_{E\times E^*}[u_1, v_1]$, for some $[u_1, v_1] \in E \times E^*$, and $[y_n, y_n^*] := (J_{E\times E^*} + \frac{\beta_n}{\alpha_n}A)^{-1}[z_1, z_2]$, we obtain that:*

$$\begin{aligned}
[y_n, y_n^*] &= \big(J_{E\times E^*} + \frac{\beta_n}{\alpha_n}A\big)^{-1}[z_1, z_2] \\
&\Leftrightarrow J_{E\times E^*}[y_n, y_n^*] + \frac{\beta_n}{\alpha_n}A[y_n, y_n^*] = [z_1, z_2] \\
&\Leftrightarrow [J_E y_n, J_E^{-1}y_n^*] + \frac{\beta_n}{\alpha_n}[Fy_n - y_n^*, Ky_n^* + y_n] = [z_1, z_2] \\
&\Leftrightarrow [J_E y_n + \frac{\beta_n}{\alpha_n}(Fy_n - y_n^*), J_E^{-1}y_n^* + \frac{\beta_n}{\alpha_n}(Ky_n^* + y_n)] = [z_1, z_2].
\end{aligned}$$

Thus,

$$J_E y_n + \frac{\beta_n}{\alpha_n}(Fy_n - y_n^*) = z_1, \quad \forall n \geq 1, \quad \textit{and} \tag{2.2.18}$$

$$J_E^{-1}y_n^* + \frac{\beta_n}{\alpha_n}(Ky_n^* + y_n) = z_2, \quad \forall n \geq 1, \tag{2.2.19}$$

Remark 2.2.16 *The following estimates (see, [Uba et al., 2019], Remark 2) will be needed in the sequel.*

$$Jy_n + \frac{1}{\theta_n}(Fy_n - y_n^*) = 0, \quad \forall n \geq 1, \quad \textit{and} \tag{2.2.20}$$

$$J^{-1}y_n^* + \frac{1}{\theta_n}(Ky_n^* + y_n) = 0, \quad \forall n \geq 1, \tag{2.2.21}$$

2.3 Important Lemmas

Lemma 2.3.1 ([Pascall and Sburlan, 1978]) *Let E be a real normed space and $A : E \to 2^{E^*}$ be a monotone map with $0 \in Int(D(A))$. Then, A is quasi-bounded, i.e., for any $M > 0$, there exists $C > 0$ such that:*

(i) $(y, v) \in G(A)$; (meaning $v \in Ay$) (ii) $\langle y, v\rangle \leq M\|y\|$; and (iii) $\|y\| \leq M$, imply $\|v\| \leq C$.

Lemma 2.3.2 ([Chidume, 2009]) *For $q > 1$, let J_q be the generalized duality mapping, then for all $x, y \in E$ there exists $j_q(x + y) \in J_q(x + y)$ such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle.$$

Lemma 2.3.3 ([Reich, 1979]) *Let E be a uniformly smooth real Banach space. Then, there exists a nondecreasing function $\rho : [0, \infty) \to [0, \infty)$ satisfying the following condition: $\forall\ x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + max\{\|x\|, 1\}\|y\|\rho(\|y\|).$$

Lemma 2.3.4 ([Xu, 1991]) *Let E be a uniformly convex real Banach space and let $q > 1$ and $r > 0$. Then there exist strictly increasing continuous and convex functions $\phi, \psi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\psi(0) = 0$ such that for all $x, y \in B(0, r) := \{x \in E : \|x\| \leq r\}$,*

(i) $\|\lambda x + (1 - \lambda)y\|^q \leq \lambda\|x\|^q + (1 - \lambda)\|y\|^q - \lambda(1 - \lambda)\phi(\|x - y\|)$,

(ii) $\psi(\|x - y\|) \leq \|x\|^q - q\langle x, j_q(y)\rangle + (q - 1)\|y\|^q$,

where $j_q(y) \in J_q(y)$.

Lemma 2.3.5 ([Cho et al., 2004]) *Let E be a uniformly convex real Banach space and, $q > 1$ and $r > 0$. Then there exists a strictly increasing continuous and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for all $x, y, z \in B(0, r)$, and $\tau, \sigma, \mu \in [0, 1]$ with $\tau + \sigma + \mu = 1$*

$$\|\tau x + \sigma y + \mu z\|^q \leq \tau\|x\|^q + \sigma\|y\|^q + \mu\|z\|^q - \tau\sigma g(\|x - y\|).$$

Lemma 2.3.6 ([Rockafellar et al., 1969]) *A monotone mapping $T : E \to 2^{E^*}$ is locally bounded at the interior points of its domain.*

Lemma 2.3.7 ([Fitzpatrick et al., 1972]) *Let E be a real reflexive Banach space and let $A : D(A) \subset E \to E$ be an accretive mapping. Then A is locally bounded at any interior point of $D(A)$.*

Lemma 2.3.8 ([Kamimura and Takahashi, 2002]) *Let X be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Remark 2.3.9 *Let E be a real Banach space and let $A : E \to 2^E$ be an m-accretive map the resolvent $J_\lambda : E \to E$ of A is defined by $J_\lambda^A x := \{u \in E : x \in (u + \lambda Au)\}$. It is well-known that J_λ^A is single valued with $F(J_\lambda^A) = A^{-1}0$ (see, e.g., [López et al., 2012]) and J_λ is firmly nonexpansive (see, e.g., [Minty et al., 1962], [Bauschke et al., 2008]) i.e.,*

$$\|J_\lambda^A x - J_\lambda^A y\|^q \leq \langle x - y, j_q(J_\lambda^A x - J_\lambda^A y)\rangle.$$

If the notation is clear, we shall write J_λ instead of J_λ^A to mean the resolvent operator of A.
In the sequel we shall adopt the following notation:

$$W_\lambda^{A,B} := J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A), \quad \lambda > 0.$$

Then (see, e.g., [López et al., 2012])

(i) for $\lambda > 0$, $F(W_\lambda^{A,B}) = (A+B)^{-1}0$,

(ii) for $0 < \lambda \leq \epsilon$ and $x \in E$, $\|x - W_\lambda^{A,B}x\| \leq 2\|x - W_\epsilon^{A,B}x\|$.

Lemma 2.3.10 (**[López et al., 2012]**) *Let E be a uniformly convex and q-uniformly smooth real Banach space and let $A : E \to E$ be an α-inverse strongly accretive (α-isa) mapping of order q and $B : E \to 2^E$ be an m-accretive mapping. Then given $r > 0$, there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for all $x, y \in B(0,r)$,*

$$\|W_\lambda^{A,B}x - W_\lambda^{A,B}y\|^q \leq \|x-y\|^q - \lambda\big(\alpha q - \lambda^{q-1}\kappa_q\big)\|Ax - Ay\|^q \\ - g\big(\|(I - J_\lambda)(I - \lambda A)x - (I - J_\lambda)(I - \lambda A)y\|\big),$$

where κ_q is the best smoothness constant of E.

Lemma 2.3.11 (**[Xu, 2002]**) *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n b_n + c_n, \quad n \geq 1,$$

where $\{\sigma_n\}, \{b_n\}$ and $\{c_n\}$ satisfy the conditions:

(i) $\{\sigma_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \sigma_n = \infty$; *(ii)* $\limsup_{n\to\infty} b_n \leq 0$; *(iii)* $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3.12 (**[He and Yang, 2013]**) *Let $\{d_n\}$ be a sequence of nonnegative real numbers such that*

$$d_{n+1} \leq (1-\alpha_n)d_n + \alpha_n\theta_n \quad \text{and} \quad d_{n+1} \leq d_n - \eta_n + \beta_n,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of of nonnegative real numbers, $\{\theta_n\}$ and $\{\beta_n\}$ are real sequences such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, *(ii)* $\lim_{n\to\infty} \beta_n = 0$,

(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \theta_{n_k} \leq 0$, *for any subsequence $\{n_k\}$ of the sequence $\{n\}$.*

Then, $\lim_{n\to\infty} d_n = 0$.

Lemma 2.3.13 (**[López et al., 2012]**) *Let E be a q-uniformly smooth real Banach space and let $A : C \to E$ be an α-isa of order q. Then the following inequality holds for all $x, y \in C$*

$$\|(I-\lambda A)x - (I-\lambda A)y\|^q \leq \|x-y\|^q - \lambda(\alpha q - \kappa_q\lambda^{q-1})\|Ax - Ay\|^q,$$

where $\kappa_q > 0$ is the q-uniform smoothness coefficient of E. In particular, if $0 < \lambda < (\alpha q - \kappa_q\lambda^{q-1})$ then $(I - \lambda A)$ is nonexpansive.

Lemma 2.3.14 ([Cai and Bu, 2013]) *Let E be a uniformly smooth real Banach space, C be a closed convex subset of E and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$ and $f : C \to C$ be a contraction with coefficient $k \in (0,1)$. Let $\{z_t\}$ be a net sequence defined by*

$$z_t = tf(z_t) + (1-t)Sz_t, \quad \forall t \in (0,1).$$

Then, $\{z_t\}$ converges strongly as $t \to 0$ to a point $x^* \in F(S)$.

Lemma 2.3.15 ([F.E.Browder, 1965]) *Let C be a nonempty, closed and convex subset of a uniformly convex real Banach space, E and let $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero (i.e., if $\{x_n\}$ converges weakly to x and $\{x_n - Sx_n\}$ converges strongly to zero, we have $Sx = x$).*

Lemma 2.3.16 ([Reich, 1980]) *Let E be a uniformly smooth real Banach space, and let $A : E \to 2^E$ be m-accretive. Let $J_\lambda x := (I + \lambda A)^{-1}x, \quad \lambda > 0$ be the resolvent of A, and assume that $A^{-1}(0)$ is nonempty. Then, for each $x \in E$, $\lim\limits_{\lambda \to \infty} J_\lambda x$ exists and belongs to $A^{-1}(0)$.*

Lemma 2.3.17 ([Chidume et al., 2003]) *For $q > 1$, let E be a q-uniformly smooth real Banach space and let $F : E \to E$ be a continuous α-strongly accretive mapping and $K : E \to E$ be a continuous β-strongly accretive mapping such that $\alpha > \frac{d_q - 1}{q}$ and $\beta > \frac{1}{q}$, for some $d_q > 1$. Then, $A : E \times E \to E \times E$ be defined by $A[u,v] := [Fu - v, Kv + u]$, is continuous γ-strongly accretive, where $\gamma = \min\{\alpha - \frac{d_q - 1}{q}, \beta - \frac{1}{q}\}$.*

Lemma 2.3.18 ([Rockafellar, 1970]) *Let E be a Banach space and let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper, convex and lower semi-continuous function. Then, the subdifferential of f; ∂f is maximal monotone Furthermore, $0 \in \partial f(u^*)$ if and only if u^* is a minimizer of f.*

CHAPTER 3

ON THE STRONG CONVERGENCE OF THE PROXIMAL POINT ALGORITHM WITH AN APPLICATION TO HAMMERSTEIN EQUATIONS

3.1 Introduction

In this chapter, a new notion of *quasi-boundedness* for operators, $A : E \to 2^E$, and the following general important result is proved: *an accretive operator with zero in the interior of its domain is quasi-bounded.* Using this result, a new strong convergence theorem for approximating a zero of an m-accretive operator is proved in a uniformly smooth real Banach space. This result complements the celebrated *proximal point algorithm* for approximating solutions of $0 \in Au$ in a real Hilbert space, where A is a *maximal monotone operator.* Furthermore, as an application of our theorem, a new strong convergence theorem for approximating a solution of a Hammerstein equation is proved. Finally, several numerical experiments are presented to illustrate the strong convergence of the sequence generated by our algorithm and the results obtained are compared with those obtained using some recent important algorithms.

3.2 Main results

Definition 3.2.1 *Let E be a real normed space. A mapping $A : E \to 2^E$ is called quasi-bounded if for any $M > 0$ there exists $C_M > 0$ such that whenever $\langle \zeta, jx - j(x-y) \rangle \le M(2\|x\| + \|y\|)$ and $\|y\| \le M$, $\|x\| \le M$, for some $jx \in Jx$ and $j(x-y) \in J(x-y)$, then $\|\zeta\| \le C_M$, $\zeta \in Ay$.*

Remark 3.2.2 *A notion of quasi-boundedness for maps $A : E \to 2^{E^*}$, where E^* is the dual space of E, is already defined (see e.g., [Cioranescu, 2012], p. 176, Exercise 9). If $x = 0$ and E is a real Hilbert space, Definition 3.2.1 and that given for maps from $E \to 2^{E^*}$ coincide.*

Theorem 3.2.3 *Let E be a smooth and reflexive real Banach space. Any accretive mapping $A : D(A) \subset E \to 2^E$ with $0 \in int\ D(A)$ is quasi-bounded.*

Proof. By Lemma 2.3.7, A is locally bounded at 0. This implies that there exist $r > 0,\ M^* > 0$ such that $B_E(0,r) := \{x \in E : \|x\| \leq r\} \subset int\ D(A)$ and

$$\forall\ x \in B_E(0,r), \quad \eta \in Ax, \quad \|\eta\| \leq M^*.$$

Let $M > 0$, $x \in B_E(0,r)$ and $y \in D(A)$. Assume that $\|y\| \leq M$ and $\zeta \in Ay$ such that

$$\langle \zeta, Jx - J(x-y)\rangle \leq M(2\|x\| + \|y\|).$$

By the accretivity of A, $\langle \zeta - \eta, J(y-x)\rangle \geq 0$, $\forall \eta \in Ax$. This implies that

$$\langle \zeta, J(x-y)\rangle \leq \langle \eta, J(x-y)\rangle \leq M^*(\|y\| + r).$$

Furthermore,

$$\begin{aligned}\langle \zeta, Jx\rangle &= \langle \zeta, J(x-y)\rangle + \langle \zeta, Jx - J(x-y)\rangle \\ &\leq M^*(\|y\| + r) + M(2\|x\| + \|y\|) \\ &\leq M^*(M + r) + M(2r + M).\end{aligned}$$

Replacing x by $-x$ and using Lemma 2.1.18, we obtain that

$$-\langle \zeta, Jx\rangle \leq M^*(M + r) + M(2r + M).$$

This implies that

$$|\langle \zeta, Jx\rangle| \leq M^*(M + r) + M(2r + M), \qquad \forall x \in B_E(0,r). \tag{3.2.1}$$

We recall that for $r > 0$, $G = \{g \in E : \|g\| < r\} = r\widehat{G}$, where $\widehat{G} = \{g \in E : \|g\| < 1\}$. To see this, let $g \in r\widehat{G}$. Then, there exists $\widehat{g} \in \widehat{G}$ such that $g = r\widehat{g}$. We show that $g \in G$. Now, $\|g\| = \|r\widehat{g}\| = r\|\widehat{g}\| < r$. Hence, $g \in G$. Thus, $r\widehat{G} \subset G$.

Let $g \in G$. Then, $s = \frac{g}{r+1} \in G$. Set $\widehat{g} = \frac{g}{r+1}$. Clearly, $\widehat{g} \in \widehat{G}$ and thus, $s \in \widehat{G}$. Therefore, $G \subset \widehat{G}$. Thus, $G = r\widehat{G}$.

For $f \in B_{E^*}(0,1)$, by the reflexivity and smoothness of E, there exists $x \in B_E(0,r)$ such that $Jx = rf$. So, using these facts, and inequality (3.2.1), we obtain that

$$\sup_{\|f\| \leq 1} |\langle \zeta, f\rangle| \leq \frac{1}{r}\Big(M^*(M + r) + M(2r + M)\Big).$$

The quasi-boundedness of A follows. ■

Lemma 3.2.4 *Let E be a uniformly smooth real Banach space and let $A : E \to 2^E$ be a set-valued m-accretive mapping such that the inclusion $0 \in Au$ has a solution. For arbitrary $u_1 \in E$, define inductively a sequence $\{u_n\}$ by*

$$u_{n+1} = (1 - \alpha_n\beta_n)u_n - \alpha_n\zeta_n, \qquad \zeta_n \in Au_n,\ n \geq 1, \tag{3.2.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. There exists constants $M_0,\ \gamma_0 > 0$ such that if $\frac{\rho_E(\alpha_n M_0)}{\alpha_n M_0} \leq \gamma_0\beta_n$ then the sequence $\{u_n\}$ is bounded.

Proof. Let u^* be a solution of the inclusion $0 \in Au$, i.e., $0 \in Au^*$ and let $u_1 \in E$. Then, there exists $r > 0$ such that $\|u^*\| \leq \frac{r}{2}$ and $\|u_1 - u^*\| \leq \frac{r}{2}$. Define $B = B(u^*, r) := \{u \in E : \|u - u^*\| < r\}$. Let $u \in B$, then $\|u\| \leq \|u^*\| + r$. Since A is locally bounded at $0 \in B$, there exist $m_1 > 0$, with $m_1 < r$ and $k_1 > 0$ such that

$$\|\eta\| \leq k_1, \quad \eta \in Av, \quad \forall v \in B_1(0, m_1).$$

Let $v \in B_1(0, m_1)$ such that $\omega_J(\|v\|) < m_1$, where ω_J is the modulus of continuity of J. By the accretivity of A, we have that

$$\langle \zeta, Ju \rangle \geq \langle \eta, J(u - v) \rangle + \langle \zeta, Ju - J(u - v) \rangle, \quad \zeta \in Au.$$

Using Lemma 2.1.18, this implies that

$$\langle \zeta, J(-u) \rangle \leq \langle \eta, J(v - u) \rangle + \langle \zeta, J(u - v) + J(-u) \rangle.$$

Let $z = -u$. Then,

$$\begin{aligned} \langle \zeta, Jz \rangle &\leq \langle \eta, J(v + z) \rangle + \langle \zeta, Jz - J(z + v) \rangle \\ &\leq \|\eta\|(\|v\| + \|z\|) + \|\zeta\|\|Jz - J(z + v)\| \\ &\leq k_1(m_1 + \|u^*\| + r) + \|\zeta\| m_1. \end{aligned} \tag{3.2.3}$$

Similarly, replacing z by $-z$ in (3.2.3) and using Lemma 2.1.18, we obtain that

$$-\langle \zeta, Jz \rangle \leq k_1(m_1 + \|u^*\| + r) + \|\zeta\| m_1.$$

Thus,

$$|\langle \zeta, Jz \rangle| \leq k_1(m_1 + \|u^*\| + r) + \|\zeta\| m_1.$$

This implies that

$$\sup_{\|z\| = \|Jz\| \leq \|u^*\| + r} |\langle \zeta, Jz \rangle| = (\|u^*\| + r) \sup_{\|Jz\| \leq 1} |\langle \zeta, Jz \rangle| \leq k_1(m_1 + \|u^*\| + r) + \|\zeta\| m_1,$$

so that

$$\|\zeta\| \leq \frac{k_1(m_1 + \|u^*\| + r)}{\|u^*\| + r - m_1} := k_2.$$

Setting $M = \max\{k_2, \|u^*\| + r\}$, we have that

$$\langle \zeta, Jv - J(v - u) \rangle \leq M(2\|v\| + \|u\|), \quad \|u\| \leq M \quad \textit{and} \quad \|v\| \leq M.$$

By Theorem 3.2.3, A is quasi-bounded. Thus, there exists $k > 0$ such that

$$\|\zeta\| \leq k, \quad \forall u \in B.$$

Define $M_0 := \sup_{u \in B,\ \theta \in (0,1)} \Big\{\|\theta u + \zeta\|\Big\} + 1,\ \zeta \in Au,\ \gamma_0 := \min\Big\{1, \dfrac{r^2}{8(r+1)M_0^2}\Big\}.$

The quasi-boundedness of A and $u \in B$ guarantee that M_0 is well defined.

Now, it suffices to show that $u_n \in B$, $\forall\ n \geq 1$. We show this by induction. For $n = 1$, by construction, $\|u_1 - u^*\| \leq r$. Assume $\|u_n - u^*\| \leq r$, for some $n \geq 1$. We show that $\|u_{n+1} - u^*\| \leq r$. For contradiction, suppose $r < \|u_{n+1} - u^*\|$. Now, using recursion formula (3.2.2), Lemma 2.3.3, and the condition that $\frac{\rho_E(\alpha_n M_0)}{\alpha_n M_0} \leq \gamma_0 \beta_n$, we have:

$$\begin{aligned}
r^2 < \|u_{n+1} - u^*\|^2 &= \|(1 - \alpha_n\beta_n)u_n - \alpha_n\zeta_n - u^*\|^2 \\
&\leq \|u_n - u^*\|^2 - 2\alpha_n\langle \zeta_n + \beta_n u_n, J(u_n - u^*)\rangle \\
&\quad + \max\{\|u_n - u^*\|, 1\}\alpha_n\|\zeta_n + \beta_n u_n\|\rho_E(\alpha_n\|\zeta_n + \beta_n u_n\|) \\
&\leq \|u_n - u^*\|^2 - 2\alpha_n\langle \zeta_n, J(u_n - u^*)\rangle \\
&\quad - 2\alpha_n\beta_n\langle u_n - u^*, J(u_n - u^*)\rangle \\
&\quad - 2\alpha_n\beta_n\langle u^*, J(u_n - u^*)\rangle + (r+1)M_0\rho_E(\alpha_n\|\zeta_n + \beta_n u_n\|) \\
&\leq \|u_n - u^*\|^2 - 2\alpha_n\beta_n\|u_n - u^*\|^2 \\
&\quad + \alpha_n\beta_n(\|u^*\|^2 + \|u_n - u^*\|^2) \\
&\quad + (r+1)M_0\rho_E(\alpha_n\|\zeta_n + \beta_n u_n\|) \\
&\leq (1 - \alpha_n\beta_n)\|u_n - u^*\|^2 + \alpha_n\beta_n\|u^*\|^2 \\
&\quad + (r+1)M_0\frac{\rho_E(\alpha_n M_0)}{\alpha_n M_0}\alpha_n M_0 \\
&\leq (1 - \alpha_n\beta_n)\|u_n - u^*\|^2 + \alpha_n\beta_n\|u^*\|^2 + (r+1)M_0^2\gamma_0\beta_n\alpha_n \\
&\leq \Big(1 - \frac{5\alpha_n\beta_n}{8}\Big)r^2 < r^2.
\end{aligned}$$

This is a contradiction. Hence, $\|u_{n+1} - u^*\| \leq r$. Therefore, $\{u_n\}$ is bounded. ■

We now state and prove our strong convergence theorem.

Theorem 3.2.5 *Let E be a uniformly smooth real Banach space and let $A : E \to 2^E$ be a set-valued m-accretive mapping such that the inclusion $0 \in Au$ has a solution. For arbitrary $u_1 \in E$, define inductively a sequence $\{u_n\}$ by*

$$u_{n+1} = (1 - \alpha_n\beta_n)u_n - \alpha_n\zeta_n, \qquad \zeta_n \in Au_n,\ n \geq 1, \tag{3.2.4}$$

where $\{\alpha_n\}$ $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$, *$\{\beta_n\}$ is decreasing;*

(ii) $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$;

(iii) there exists a constant $M_0 > 0$ such that $\sum_{n=1}^{\infty} \rho_E(\alpha_n M_0) < \infty$;

(iv) $\lim_{n\to\infty} \dfrac{\left(\frac{\beta_{n-1}}{\beta_n} - 1\right)}{\alpha_n\beta_n} = 0.$

There exists constants $M_0, \gamma_0 > 0$ such that if $\frac{\rho_E(\alpha_n M_0)}{\alpha_n M_0} \leq \gamma_0 \beta_n$. Then, the sequence $\{u_n\}$ converges strongly to a zero of A.

Proof. We observe that the recurrence relation (3.2.4) is the same as the recurrence relation (1.1.9) in which $u_1 \equiv 0$. This is possible since u_1 is an arbitrary element in domain of A which, in this case, is E. By Lemma 3.2.4, the sequence $\{u_n\}$ is bounded. The rest of the argument now follows exactly as in the proof of Theorem 1.1.6 (see, [Chidume, 2016]). However, for completeness, we repeat the proof here.

By Lemma 2.3.16, there exists a sequence $\{y_n\}$ defined by $y_n := J_{\lambda_n} u_1$ where u_1 is an arbitrary point in E; set $\lambda_n = \beta_n^{-1}$, $\forall\ n \geq 1$ and observe that with this λ_n, the sequence $\{y_n\}$ satisfies the following condition: there exists $z_n \in Ay_n$ such that

$$\beta_n(y_n - u_1) + z_n = 0, \quad n \geq 1. \tag{3.2.5}$$

$$y_n \to y^* \in A^{-1}0. \tag{3.2.6}$$

From equation (3.2.5), in particular, setting $u_1 = 0 \in E$ we obtain that

$$\beta_n y_n + z_n = 0. \tag{3.2.7}$$

We now prove that $\|u_{n+1} - y_n\| \to 0, \quad as \quad n \to \infty$.

By Lemma 3.2.4 $\{x_n\}$ is bounded. Also, $\{y_n\}$ is bounded as a convergent sequence. Hence, using Lemma 2.3.3, there exists a constant $M_1 > 0$ such that

$$\begin{aligned}
\|u_{n+1} - y_n\|^2 &= \|u_n - y_n - \alpha_n(\zeta_n + \beta_n u_n)\|^2 \\
&\leq \|u_n - y_n\|^2 - 2\alpha_n\langle \zeta_n + \beta_n u_n, J(u_n - y_n)\rangle \\
&\quad + M_1 \rho_E(\alpha_n M_0).
\end{aligned}$$

Furthermore, using the fact that A is accretive, we obtain, for $z_n \in Ay_n$, using equation (3.2.7), that

$$\begin{aligned}
\langle \zeta_n + \beta_n u_n, J(u_n - y_n)\rangle &= \langle \zeta_n - z_n, J(u_n - y_n)\rangle + \beta_n\|u_n - y_n\|^2 \\
&\quad + \langle z_n + \beta_n y_n, J(u_n - y_n)\rangle \\
&\geq \beta_n \big\|u_n - y_n\big\|^2.
\end{aligned}$$

Therefore,

$$\|u_{n+1} - y_n\|^2 \leq (1 - \alpha_n\beta_n)\|u_n - y_n\|^2 + M_1\rho_E(\alpha_n M_0). \tag{3.2.8}$$

Using again the fact that A is accretive (accretivity defined by inequality (1.1.3)), we have

$$\|y_{n-1} - y_n\| \leq \Big\|y_{n-1} - y_n + \frac{1}{\beta_n}(z_{n-1} - z_n)\Big\|. \tag{3.2.9}$$

Observing from equation (3.2.7) that

$$y_{n-1} - y_n + \frac{1}{\beta_n}(z_{n-1} - z_n) = \left(\frac{\beta_n - \beta_{n-1}}{\beta_n}\right) y_{n-1},$$

it follows from inequality (3.2.9) that

$$\|y_{n-1} - y_n\| \leq \left(\frac{\beta_n - \beta_{n-1}}{\beta_n}\right)\|y_{n-1}\|. \tag{3.2.10}$$

By Lemma 2.3.2 we have

$$\begin{aligned}
\|u_n - y_n\|^2 &= \|(u_n - y_{n-1}) + (y_{n-1} - y_n)\|^2 \\
&\leq \|u_n - y_{n-1}\|^2 + 2\langle y_{n-1} - y_n, J(u_n - y_n)\rangle \\
&\leq \|u_n - y_{n-1}\|^2 + 2\|y_{n-1} - y_n\|\|u_n - y_n\|.
\end{aligned}$$

Using and the fact that $\{u_n\}$ and $\{y_n\}$ are bounded, we have,

$$\begin{aligned}
\|u_{n+1} - y_n\|^2 &\leq (1 - \alpha_n\beta_n)\|u_n - y_{n-1}\|^2 + M^*\left(\frac{\beta_n - \beta_{n-1}}{\beta_n}\right) + M_1\rho_E(\alpha_n M_0) \\
&= (1 - \alpha_n\beta_n)\|u_n - y_{n-1}\|^2 + (\alpha_n\beta_n)b_n + c_n,
\end{aligned}$$

for some constant $M^* > 0$ where,

$$b_n := \frac{M^*\left(\frac{\beta_n - \beta_{n-1}}{\beta_n}\right)}{\alpha_n\beta_n} = M^*\left(\frac{\frac{\beta_{n-1}}{\beta_n} - 1}{\alpha_n\beta_n}\right), \quad c_n := M_1\rho_E(\alpha_n M_0).$$

Thus, by Lemma 2.3.11, $\lim\limits_{n\to\infty}(x_n - y_{n-1}) = 0$. Using (3.2.6), it follows that $\lim\limits_{n\to\infty} x_n = y^*$ and $0 \in Ay^*$. This completes the proof.

■

3.3 Application to Hammerstein Equations

In this section, we shall apply Theorem 3.2.5 to approximate a solution of the Hammerstein equation (1.2.3).

Theorem 3.3.1 *For $q > 1$, let X be a q-uniformly smooth real Banach space and let $F, K : X \to X$ be continuous and strongly accretive mappings with constants λ and θ respectively, satisfying $\lambda > \frac{d_q - 1}{q}$, $\theta > \frac{1}{q}$. Let $\gamma := \min\{\lambda - \frac{d_q - 1}{q}, \theta - \frac{1}{q}\}$, for some $d_q > 1$. Define $E := X \times X$ and let $A : E \to E$ be defined by $A([u, v]) = [Fu - v, Kv + u]$. For $(u_1, v_1) \in E$, define the sequences $[\{u_n\}, \{v_n\}] \in E$, by*

$$\begin{cases} u_{n+1} = (1 - \alpha_n\beta_n)u_n - \alpha_n(Fu_n - v_n), & n \geq 1, \\ v_{n+1} = (1 - \alpha_n\beta_n)v_n - \alpha_n(Kv_n + u_n), & n \geq 1. \end{cases} \tag{3.3.1}$$

Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

Proof. By Lemma 2.2.12 and Lemma 2.3.17, E is q-uniformly smooth and A is γ-strongly accretive, respectively. Thus, E is uniformly smooth and A is m-accretive. Hence, the conclusion follows from Theorem 3.2.5. ■

3.4 Numerical Illustration

In this section, we present numerical examples to compare the convergence of the sequence generated by our algorithm: (Algorithm 3.2.4 of this chapter), with respect to CPU time and number of iterations with the following algorithms,

(a) Algorithm 1.1.7 (Algorithm of [Xu, 2006]),

(b) Algorithm 1.1.8 (Algorithm of [Qin and Su, 2007]) and

(c) Algorithm 1.1.9 (Algorithm of [Chidume, 2016]).

First, in examples 3.4.1 and 3.4.2, we compare the convergence of the sequence of algorithm (3.2.4) and algorithm (1.1.9). In these examples, we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorems 1.1.6 and 3.2.5. Furthermore, we use a tolerance of 10^{-8} and set maximum number of iterations $n = 5,000$.

Example 3.4.1

In Theorems 1.1.6 and 3.2.5, set $E = \mathbb{R}^2$. Consider the mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A(u, v) = (u + v + \sin u, -u + v + \sin v)$. It is easy to see that A is accretive and $(0, 0)$ is a solution of the problem $A(u, v) = (0, 0)$. See Table 3.1 (a) and Figure 3.1 (a) for the numerical results.

Example 3.4.2

Let $f : \mathbb{R} \to \mathbb{R}$ be monotone nondecreasing. It is well known that the mapping $A_f : \mathbb{R} \to 2^{\mathbb{R}}$ defined by $A_f x := \left[\lim_{t\to x^-} f(t), \lim_{t\to x^+} f(t)\right]$ is maximal monotone, (see, e.g., [Pascall and Sburlan, 1978]). Now, in Theorems 1.1.6 and 3.2.5, set $E = \mathbb{R}$. Consider the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ 1, & -2 < x < 0 \\ x + 1, & x \leq -2 \end{cases} \quad \text{then,} \qquad A_f x = \begin{cases} \{x^2 + 1\}, & x \geq 0 \\ \{1\}, & -2 < x < 0 \\ [-1, 1], & x = -2 \\ \{x + 1\}, & x < -2. \end{cases}$$

It is easy to see that f is accretive (monotone) and thus A is accretive. Furthermore, -2 is the unique solution of the inclusion $0 \in Au$. See Table 3.1 (b) and Figure 3.1(b) for the numerical results.

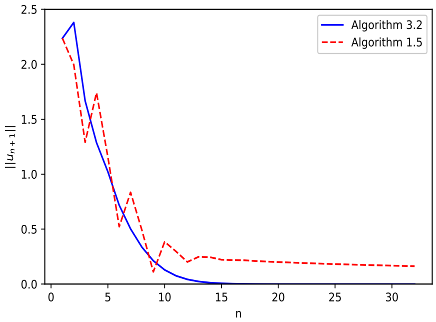
	Alg. (1.1.9)	Alg. (3.2.4)
n	$\|u_{n+1}\|$	$\|u_{n+1}\|$
1	0.2236	0.2236
5	0.1156	0.1022
10	0.3843	0.1289
15	0.2213	$6.297\times e^{-3}$
20	0.2002	$1.685\times e^{-4}$
25	0.1817	$3.191\times e^{-6}$
30	0.1675	$4.978\times e^{-8}$
32	0.1627	$9.127\times e^{-9}$

(a) Table of values choosing $u_1 = (1, 2)$

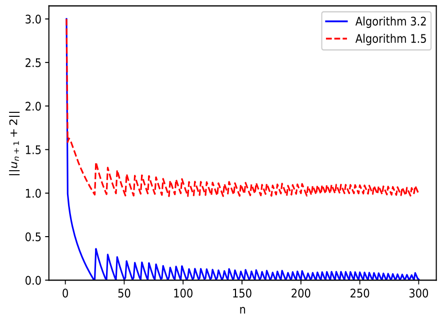
	Alg. (1.1.9)	Alg. (3.2.4)
n	$\|u_{n+1}+2\|$	$\|u_{n+1}+2\|$
1	3	3
5	1.5810	0.6841
10	1.3902	0.4209
100	1.1198	0.1199
500	0.9768	0.0231
1000	1.0059	0.0059
3000	0.9990	0.0009
4999	1.0124	0.0124

(b) Table of values choosing $u_1 = 1$

Table 3.1: Numerical results for Examples 3.4.1 and 3.4.2



(a) Graph of the first 32 iterates of algorithms (1.1.9) and (3.2.4)



(b) Graph of the first 14 iterates of algorithms (1.1.7) and (3.2.4)

Figure 3.1: Graphical illustration of the data in Table 3.1

Next, in example 3.4.3 we compare the convergence of the sequence of algorithm (3.2.4) and algorithms (1.1.7) and (1.1.8), and in example 3.4.4, we compare the convergence of the sequence of algorithm (3.2.4) and algorithm (1.1.8). In these examples, we consider the $L_p([0,1])$ spaces, $1 < p < \infty$, with inner product and norm defined by

$$\langle x, y\rangle := \int_0^1 x(t)y(t)dt, \qquad \|x\|_p := \left(\int_0^1 |x(t)|^p dt\right)^{\frac{1}{p}} \quad \forall x, y \in L_p([0,1]),$$

respectively and we choose the operator A such that the resolvent can be computed easily.

Example 3.4.3

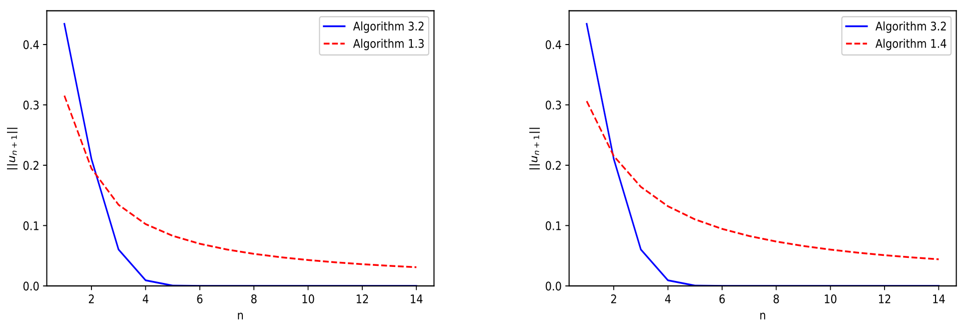
In Theorems 1.1.3, 1.1.4 and 3.2.5, set $E = L_2([0,1])$. Consider the mapping $A : L_2([0,1]) \to L_2([0,1])$ defined by

$$(Au)(t) := (t+1)u(t) \quad \text{then,} \quad J_\lambda u(t) = \frac{u(t)}{1+\lambda(t+1)}.$$

It is easy to see that A is accretive and the function $u(t) = 0,\ \forall\, t \in [0,1]$ is the only solution of the equation $Au(t) = 0$. In algorithm (1.1.7), we take $\alpha_n = \frac{1}{n+1}$, $\lambda_n = n$, in algorithm (1.1.8), we take $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.25$, $\lambda_n = 5$, and in algorithm (3.2.4), we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorems 1.1.3, 1.1.4 and 3.2.5. Furthermore, we use a tolerance of 10^{-8} and set maximum number of iterations $n = 20$.

	Alg. (1.1.7) Time= 0.041	Alg. (1.1.8) Time=21.21	Alg. (3.2.4) Time= 0.21s
n	$\|u_{n+1}\|$	$\|u_{n+1}\|$	$\|u_{n+1}\|$
1	0.3152	0.3061	0.4342
2	0.1945	0.2151	0.2103
3	0.1344	0.1641	0.06
5	0.0829	0.1102	$5.21 \times e^{-4}$
10	0.0429	0.0601	$1.16 \times e^{-7}$
14	0.0309	0.0441	$9.27 \times e^{-9}$
20	0.0219	0.0315	successful

Table 3.2: Numerical results for Example 3.4.3



(a) Graph of the first 14 iterates of algorithms (1.1.7) and (3.2.4)

(b) Graph of the first 14 iterates of algorithms (1.1.8) and (3.2.4)

Figure 3.2: Graphical illustration of the data in Table 3.2

Example 3.4.4

In Theorems 1.1.6 and 3.2.5, set $E = L_3([0,1])$. Consider the mapping $A : L_3([0,1]) \to L_3([0,1])$ defined by

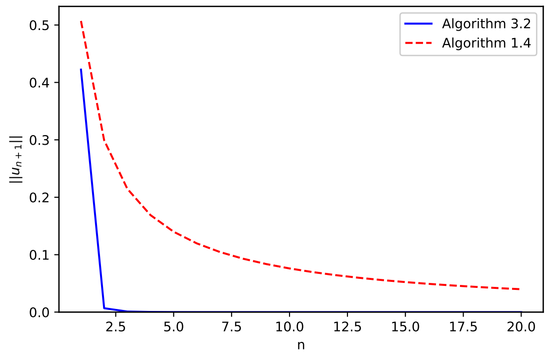
$$(Au)(t) := u(t) \quad \text{then} \quad J_\lambda u(t) = \frac{u(t)}{1+\lambda}.$$

It is easy to see that A is accretive and the function $u(t) = 0,\ \forall\, t \in [0,1]$ is the only solution of the equation $Au(t) = 0$. In algorithm (1.1.8), we take $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.25$, $\lambda_n = 5$, and in algorithm (3.2.4), we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n =$

$1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorems 1.1.4 and 3.2.5, respectively. Furthermore, we use a tolerance of 10^{-8} and set maximum number of iterations $n = 20$.

Table of values choosing
$u_1(t) = t^2 + 1$

	Alg.(1.1.8) Time= 14.44	Alg. (3.2.4) Time= 7.41s
n	$\|u_{n+1}\|$	$\|u_{n+1}\|$
1	0.507	0.4223
2	0.2993	$6.77\times e^{-3}$
3	0.2147	$9.92\times e^{-4}$
5	0.1399	$8.32\times e^{-5}$
10	0.076	$1.81\mathrm{E}\times e^{-6}$
15	0.0522	$1.31\times e^{-7}$
20	0.0398	$1.64\times e^{-8}$



Graph of the first 20 iterates of algorithms (1.1.8) and (3.2.4).

Remark 3.4.5 *From the numerical comparisons above, we observe that the proposed method (algorithm (3.2.4)) converges faster in terms of number of iteration and CPU time in all the examples considered. Thus, the proposed method which does not require the boundedness of the operator A or computation of the resolvent of A, would, perhaps, be a preferable alternative to the proximal and proximal type algorithms in any possible application.*

Example 3.4.6

In Theorem 3.3.1, set $E = \mathbb{R}^2$. Consider the mapping $F, K : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(u_1, u_2) = (u_1 + u_2 + \sin u_1, -u_1 + u_2 + \sin u_2), \quad K(v_1, v_2) = (v_1 + v_2, v_1 + v_2).$$

It is easy to see that F and K are accretive and the vector $[u, v] = [\mathbf{0}, \mathbf{0}]$ is the only solution of the equation $u + KFu = 0$ with $v = Fu$. In algorithm (3.3.1) we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorem 3.3.1. Furthermore, we use a tolerance of 10^{-8} and set maximum number of iterations $n = 100$. See Table 3.3 (a) and Figure 3.3(a) for the numerical results.

Example 3.4.7

In Theorem 3.3.1, set $E = L_{1.5}([0, 1])$. Consider the mapping $F, K : L_{1.5}([0, 1]) \to L_{1.5}([0, 1])$ defined by

$$(Fu)(t) = tu(t) \qquad \text{and} \qquad (Ku)(t) = u(t).$$

It is easy to see that F and K are accretive and the function $[u(t), v(t)] = [0, 0]\ \forall\ t \in [0, 1]$ is the only solution of the equation $u(t) + KFu(t) = 0$ with $v(t) = Fu(t)$. In algorithm (3.3.1) $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorem 3.3.1. Furthermore, we use a tolerance of 10^{-8} and set maximum number of iterations $n = 15$. See Table 3.3 (b) and Figure 3.3(b) for the numerical results.

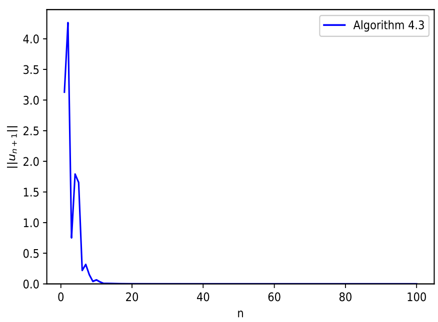
	Alg. (3.3.1)	Alg. (3.3.1)
n	$\|u_{n+1}\|$	$\|v_{n+1}\|$
1	4.2426	3.1301
5	1.4027	1.6546
10	0.0687	0.0651
20	$7.547\times e^{-3}$	$8.35\times e^{-3}$
30	$7.25320\times e^{-5}$	$1.1242\times e^{-5}$
60	$9.771\times e^{-7}$	$1.5105\times e^{-6}$
90	$4.4384\times e^{-8}$	$6.8612\times e^{-8}$
100	$1.8315\times e^{-8}$	$2.8312\times e^{-8}$

(a) Table of values choosing $u_1 = (0, 5)^T$, $v_1 = (-1, 1)^T$

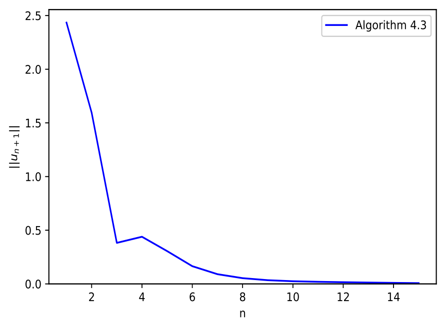
	Alg. (3.3.1)	Alg. (3.3.1)
n	$\|u_{n+1}\|$	$\|v_{n+1}\|$
1	2.8288	2.4342
4	0.4413	0.4391
6	0.1327	0.165
8	0.0598	0.0533
10	0.0325	0.0246
12	0.016	0.0157
14	0.0072	0.0093
15	0.0048	0.0069

(b) Table of values choosing $u_1(t) = e^t$, $v_1(t) = 4$

Table 3.3: Numerical results for Examples 3.4.6 and 3.4.7



(a) Graph of the first 100 iterates of algorithm (3.3.1)



(b) Graph of the first 15 iterates of algorithm (3.3.1) choosing $u_1(t) = e^t$, $v_1(t) = 4$

Figure 3.3: Graphical illustration of the data in Table 3.3

Conclusion. In this chapter, a significant improvement of Theorem 1.1.6 is proved by dispensing with the restriction that A be bounded imposed in the theorem. This is achieved by first introducing a new notion of *quasi-boundedness* for operators $A : E \to 2^E$ and then proving a general theorem of independent interest on accretive operators, that: *an accretive operator with zero in the interior of its domain is quasi-bounded.* Using this result, a strong convergence theorem for approximating a solution of $0 \in Au$ is proved. Furthermore, as an application of our theorem, a strong convergence theorem for approximating a solution a Hammerstein equation is proved. Finally, several numerical experiments are presented to illustrate the strong convergence of the sequence of our algorithm and the results obtained are compared with those obtained using some recent important algorithm.

The results obtained in this chapter have gained publication, and appeared in the following paper: C.E. Chidume, **A. Adamu**, M.S. Minjibir and U.V. Nnyaba; *On the Strong convergence of the proximal point algorithm with application to Hammerstein equations*, **Journal of Fixed Point Theory and Applications** (2020) 22:61 https://doi.org/10.1007/s11784-020-00793-6.

CHAPTER 4

STRONG CONVERGENCE OF AN INERTIAL ALGORITHM NOT INVOLVING THE RESOLVENT FOR MAXIMAL MONOTONE INCLUSIONS WITH APPLICATIONS

4.1 Introduction

In this chapter, an inertial algorithm which does not involve the reslovent operator is proposed for approximating a solution of a maximal monotone inclusion in a uniformly convex and uniformly smooth real Banach space. The sequence generated by the algorithm is proved to converge *strongly* to a solution of the inclusion. Moreover, the theorem proved is applied to approximate a solution of a convex optimization problem and a solution of a Hammerstein integral equation. Furthermore, numerical experiments are given to compare, in terms of CPU time and number of iterations, the performance of the sequence generated by our algorithm with the performance of the sequences generated by three recent inertial type algorithms for approximating zeros of maximal monotone operators. In addition, the performance of the sequence generated by our algorithm is compared with the performance of a sequence generated by another recent algorithm for approximating a solution of a Hammerstein integral equation. Finally, a numerical example is given to illustrate the implementability of our algorithm for approximating a solution of a convex optimization problem.

4.2 Main Results

The following conditions are required in proofs of Lemma 4.2.1 and Theorem 4.2.2 below.

(i) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$, (ii) $\delta_E^{-1}(\lambda_n \mathcal{K}) \leq \theta_n^2 \gamma_0$, (iii) $\delta_{E^*}^{-1}(\lambda_n \mathcal{K}) \leq \theta_n^2 \gamma_0$,

(iv) $\omega_J(\beta_n \mathcal{K}) \leq \lambda_n^4 \theta_n \gamma_0$, (v) $\delta_E^{-1}(\eta_n) \to 0$, (vi) $\delta_{E^*}^{-1}(\eta_n) \to 0$,

(vii) $\frac{\delta_E^{-1}(\eta_n)}{\lambda_n\theta_n} \to 0$; $\quad$ $(viii)$ $\frac{\delta_{E^*}^{-1}(\eta_n)}{\lambda_n\theta_n} \to 0$, $\quad$ (ix) $\lambda_n \le \theta_n\gamma_0$,

where $\eta_n = \left(\frac{\theta_{n-1}}{\theta_n} - 1\right)\mathcal{K}$, for some constants $\gamma_0 > 0$, $\mathcal{K} > 0$; and δ_E is the modulus of convexity of E, ω_J is the modulus of continuity of J.

Lemma 4.2.1 *Let E be a uniformly smooth and uniformly convex real Banach space and $A : E \to 2^{E^*}$ be a maximal-monotone operator with $D(A) = E$ such that the inclusion $0 \in Az$ has a solution. For arbitrary $z_0, z_1 \in E$, define a sequence $\{z_n\}$ by*

$$\begin{cases} w_n = z_n + \beta_n(z_n - z_{n-1}), \\ z_{n+1} = J^{-1}\left(Jw_n - \lambda_n\mu_n - \lambda_n\theta_n Jw_n\right),\ \mu_n \in Aw_n, \quad n \ge 1. \end{cases} \tag{4.2.1}$$

Then, the sequence $\{z_n\}$ is bounded.

Proof. We show that the sequence $\{z_n\}$ is bounded.

Let z^* be a solution of $0 \in Az$, i.e., $0 \in Az^*$. Then, there exists $r > 0$ such that

$$r > \max\{4\|z^*\|^2, \phi(z^*, z_1)\}. \tag{4.2.2}$$

Define $B(z^*, r) = B := \{z \in E : \phi(z^*, z) < r\}$. Clearly, $B \subset intD(A)$. It suffices to show that $\{\phi(z^*, z_n)\}$ is bounded, i.e., we show that $\{z_n\}$ is in B. We proceed by induction. For $n = 1$, by construction, we have that $\phi(z^*, z_1) < r$. Assume that $\phi(z^*, z_n) < r$, for some $n \ge 1$. Using inequality (2.2.2), we have that $\|z_n\| < \|z^*\| + \sqrt{r}$. Now, we show that $\phi(z^*, z_{n+1}) < r$. Suppose for contradiction that $\phi(z^*, z_{n+1}) < r$ does not hold. Then, $\phi(z^*, z_{n+1}) \ge r$.

Since A is locally bounded at 0, there exist $h_0 > 0$, with $h_0 < \sqrt{r}$, $m_0 > 0$ such that for all $x \in B_{h_0}(0)$, $u \in Ax$,

$$\|u\| \le m_0,$$

where $B_{h_0}(0) := \{x \in E : \|x\| < h_0\}$. Let $y \in B$, $v \in Ay$, $x \in B_{h_0}(0)$ and $u \in Ax$. By the monotonicity of A, we have that:

$$\begin{aligned} \langle y, v\rangle &\ge \langle y - x, u\rangle + \langle x, v\rangle, \\ \text{which implies that} \quad \langle -y, v\rangle &\le \langle x - y, u\rangle + \langle -x, v\rangle \end{aligned}$$

Setting $s = -y$, we have that

$$\begin{aligned} \langle s, v\rangle &\le \langle x + s, u\rangle + \langle -x, v\rangle \\ &\le \|u\|(\|x\| + \|s\|) + \|x\|\|v\| \\ &\le m_0\left(h_0 + \|z^*\| + \sqrt{r}\right) + h_0\|v\|. \end{aligned} \tag{4.2.3}$$

Similarly, replacing s with $-s$ in inequality (4.2.3), we obtain that

$$-\langle s, v\rangle \le m_0\left(h_0 + \|z^*\| + \sqrt{r}\right) + h_0\|v\|.$$

Hence,

$$|\langle s, v\rangle| \le m_0\left(h_0 + \|z^*\| + \sqrt{r}\right) + h_0\|v\|.$$

This implies that:

$$\sup_{\|s\|\leq(\|z^*\|+\sqrt{r})} |\langle s,v\rangle| \leq m_0\left(h_0+\|z^*\|+\sqrt{r}\right)+h_0\|v\|. \tag{4.2.4}$$

Thus,

$$(\|z^*\|+\sqrt{r})\|v\| \leq m_0\left(h_0+\|z^*\|+\sqrt{r}\right)+h_0\|v\|.$$

So that,

$$\|v\| \leq \frac{m_0\left(h_0+\|z^*\|+\sqrt{r}\right)}{\|z^*\|+\sqrt{r}-h_0} := M_0, \quad \forall\ y\in B.$$

Define $M := \max\{M_0,\ \|z^*\|+\sqrt{r}\}$. Then, $\langle v,y\rangle \leq M\|y\|$ and $\|y\| \leq M$.

By Lemma 2.3.1, there exists $C>0$ such that $\|v\|\leq C, \quad \forall\ y\in B$.

Define:

$$\begin{aligned}
M_1 &= \sup_{w\in B}\left\{\|\mu\ +\ \theta Jw\|,\quad\right\}+1;\\
M_2 &= \sup_{w\in B}\left\{\|J^{-1}\left(Jw-\lambda\mu-\lambda\theta Jw\right)\|\right\}+1,
\end{aligned}$$

where $\mu\in Aw$, $\theta\in(0,1)$. The quasi-boundedness of A on B and the fact that J and J^{-1} are bounded on bounded sets in uniformly convex and uniformly smooth spaces guarantee the existence of these suprema.

From the recursion formula (4.2.1) and Lemma 2.2.5 we have that:

$$\|Jz_{n+1}-Jw_n\| \leq \lambda_n M_1 \ \text{ and } \ \|z_{n+1}-w_n\| \leq c_2\delta_E^{-1}(\lambda_n M^*), \tag{4.2.5}$$

for some $M^*>0$.

Define:

$$\gamma_0 := \min\left\{1, \frac{r}{32K^*}\right\}, \tag{4.2.6}$$

where $K^*=\max\{M^*,M_1,M_2,M_1M_2,M,c_2M_1\}$.

Using Lemma 2.2.3, we compute as follows:

$$\begin{aligned}
\phi(z^*,z_{n+1}) &= V(z^*,Jw_n-\lambda_n\mu_n-\lambda_n\theta_n w_n)\\
&\leq V(z^*,Jw_n)-2\lambda_n\langle z_{n+1}-z^*,\mu_n+\theta_n Jw_n\rangle\\
&= \phi(z^*,w_n)-2\lambda_n\langle z_{n+1}-w_n,\mu_n+\theta_n Jw_n\rangle\\
&\quad -2\lambda_n\langle w_n-z^*,\mu_n+\theta_n Jw_n\rangle\\
&\leq \phi(z^*,w_n)+2c_2\lambda_n\delta_E^{-1}(\lambda_n M^*)M_1\ -\ 2\lambda_n\langle w_n-z^*,\mu_n\rangle\\
&\quad -\ 2\theta_n\lambda_n\langle w_n-z^*,Jw_n\rangle\\
&\leq \phi(z^*,w_n)+2c_2\lambda_n\delta_E^{-1}(\lambda_n M^*)M_1\ -2\theta_n\lambda_n\langle w_n-z^*,Jz_{n+1}\rangle\\
&\quad -\ 2\theta_n\lambda_n\langle w_n-z^*,Jw_n-Jz_{n+1}\rangle.
\end{aligned} \tag{4.2.7}$$

By Lemma 2.2.4, we have that

$$
\begin{aligned}
-2\lambda_n\theta_n\langle w_n - z^*, Jz_{n+1}\rangle &\leq \lambda_n\theta_n\|z^*\|^2 + 2M\lambda_n\theta_n\|w_n - z_{n+1}\| \\
&\quad -\lambda_n\theta_n\phi(z^*, z_{n+1}).
\end{aligned}
$$

Again, by Lemma 2.2.4, we have that

$$
\begin{aligned}
\phi(z^*, w_n) &\leq \phi(z^*, z_n) - 2\langle w_n - z^*, Jz_n - Jw_n\rangle \\
&\leq \phi(z^*, z_n) + 2\|w_n - z^*\|\|Jz_n - Jw_n\| \\
&\leq \phi(z^*, z_n) + 2M_2\omega_J(\beta_n M).
\end{aligned}
$$

Substituting these inequalities in inequality (4.2.7), we have that

$$
\begin{aligned}
r \leq \phi(z^*, z_{n+1}) &\leq \phi(z^*, z_n) + 2M_2\omega_J(\beta_n M) + 2c_2\lambda_n\delta_E^{-1}(\lambda_n M^*)M_1 \\
&\quad +\lambda_n\theta_n\|z^*\|^2 + 2\lambda_n\theta_n M\|Jw_n - Jz_{n+1}\| \\
&\quad +2\lambda_n\theta_n M\|w_n - z_{n+1}\| - \lambda_n\theta_n\phi(z^*, z_{n+1}) \\
&\leq \phi(z^*, z_n) + 2M_2\omega_J(\beta_n M) + 2c_2\lambda_n\delta_E^{-1}(\lambda_n M^*)M_1 \\
&\quad +\lambda_n\theta_n\|z^*\|^2 + 2\lambda_n^4\theta_n MM_1 + 2c_2\lambda_n\theta_n\delta_E^{-1}(\lambda_n M^*)M \\
&\quad -\lambda_n\theta_n\phi(z^*, z_{n+1}) \\
&< r + 2M_2\lambda_n\theta_n\gamma_0 + 2c_2\lambda_n\theta_n M_1\gamma_0 + \lambda_n\theta_n\frac{r}{4} \\
&\quad +2\lambda_n\theta_n MM_1\gamma_0 + 2\lambda_n\theta_n\gamma_0 M - \lambda_n\theta_n r \\
&< r + \lambda_n\theta_n\frac{r}{2} - \lambda_n\theta_n r < r.
\end{aligned}
$$

This is a contradiction. Hence, $\phi(z^*, z_{n+1}) < r$. Therefore, $\phi(z^*, z_n) < r$, for all $n \geq 1$. ■

Theorem 4.2.2 *Let E be a uniformly smooth and uniformly convex real Banach space. Let $A : E \to 2^{E^*}$ be a maximal monotone operator with $D(A) = E$ such that the inclusion $0 \in Az$ has a solution. For arbitrary $z_0, z_1 \in E$, define a sequence $\{z_n\}$ by*

$$
\begin{cases}
w_n = z_n + \beta_n(z_n - z_{n-1}), \\
z_{n+1} = J^{-1}\left(Jw_n - \lambda_n\mu_n - \lambda_n\theta_n Jw_n\right), \; \mu_n \in Aw_n, \quad n \geq 1.
\end{cases} \tag{4.2.8}
$$

Then, the sequence $\{z_n\}$ converges strongly to a zero of A.

Proof. Let y_n be as defined in Remark 2.2.8. Using Lemma 2.2.3 and equation (2.2.4), we have

$$
\begin{aligned}
\phi(y_n, z_{n+1}) &= V(y_n, Jw_n - \lambda_n\mu_n - \lambda_n\theta_n Jw_n) \\
&\leq V(y_n, Jw_n) - 2\lambda_n\langle z_{n+1} - y_n, \mu_n + \theta_n Jw_n\rangle \\
&= \phi(w_n, y_n) + 2\langle w_n, Jy_n\rangle - 2\langle y_n, Jw_n\rangle \\
&\quad -2\lambda_n\langle z_{n+1} - y_n, \mu_n + \theta_n Jw_n\rangle.
\end{aligned} \tag{4.2.9}
$$

Observe that

$$\begin{aligned}
\phi(w_n, y_n) &= V(w_n, Jy_n) = V(w_n, Jy_{n-1} + Jy_n - Jy_{n-1}) \\
&\leq V(w_n, Jy_{n-1}) - 2\langle y_n - w_n, Jy_{n-1} - Jy_n\rangle. \qquad (4.2.10)
\end{aligned}$$

Thus, from inequalities (4.2.9), (4.2.10) and the fact that $v_n \in Ay_n$, we obtain

$$\begin{aligned}
\phi(y_n, z_{n+1}) &\leq V(w_n, Jy_{n-1}) - 2\langle y_n - w_n, Jy_{n-1} - Jy_n\rangle + 2\langle w_n, Jy_n\rangle \\
&\quad -2\langle y_n, Jw_n\rangle - 2\lambda_n\langle z_{n+1} - y_n, \mu_n + \theta_n Jw_n\rangle \\
&= \phi(y_{n-1}, w_n) + 2\langle y_{n-1}, Jw_n\rangle - 2\langle w_n, Jy_{n-1}\rangle \\
&\quad -2\langle y_n - w_n, Jy_{n-1} - Jy_n\rangle + 2\langle w_n, Jy_n\rangle \\
&\quad -2\langle y_n, Jw_n\rangle - 2\lambda_n\langle z_{n+1} - y_n, \mu_n + \theta_n Jw_n\rangle \\
&= \phi(y_{n-1}, w_n) + 2\langle y_{n-1} - y_n, Jw_n\rangle + 2\langle w_n, Jy_n - Jy_{n-1}\rangle \\
&\quad -2\langle y_n - w_n, Jy_{n-1} - Jy_n\rangle - 2\lambda_n\langle z_{n+1} - y_n, \mu_n + \theta_n Jw_n\rangle \\
&= \phi(y_{n-1}, w_n) + 2\|y_{n-1} - y_n\|\|w_n\| + 2\|w_n\|\|Jy_n - Jy_{n-1}\| \\
&\quad +2\|y_n - w_n\|\|Jy_{n-1} - Jy_n\| + 2\lambda_n\|z_{n+1} - w_n\|M_1 \\
&\quad -2\lambda_n\langle w_n - y_n, \mu_n - v_n\rangle - \underline{2\lambda_n\langle w_n - y_n, v_n\rangle} \\
&\quad -\underline{2\lambda_n\theta_n\langle w_n - y_n, Jw_n\rangle} \qquad (4.2.11)
\end{aligned}$$

Observe that

$$\begin{aligned}
-\underline{2\lambda_n\theta_n\langle w_n - y_n, Jw_n\rangle} &= 2\lambda_n\theta_n\langle w_n - y_{n-1}, Jy_{n-1} - Jw_n\rangle \\
&\quad -2\lambda_n\theta_n\langle y_{n-1} - y_n, Jw_n - Jy_{n-1}\rangle \\
&\quad -2\lambda_n\theta_n\langle w_n - y_n, Jy_{n-1} - Jy_n\rangle \\
&\quad -2\lambda_n\theta_n\langle w_n - y_n, Jy_n\rangle \\
&\leq -\lambda_n\theta_n\phi(y_{n-1}, w_n) + 2\lambda_n\theta_n\|y_{n-1} - y_n\|M \\
&\quad -\underline{2\lambda_n\theta_n\langle w_n - y_n, Jy_n\rangle} \\
&\quad +2\lambda_n\theta_n\|Jy_{n-1} - Jy_n\|M. \qquad (4.2.12)
\end{aligned}$$

Also, from Remark 2.2.8 (with $s = 0$), we obtain that

$$-\underline{2\lambda_n\langle w_n - y_n, v_n\rangle} - \underline{2\lambda_n\theta_n\langle w_n - y_n, Jy_n\rangle} = -2\lambda_n\langle w_n - y_n, v_n + \theta_n Jy_n\rangle = 0. \quad (4.2.13)$$

Hence, substituting inequality (4.2.12) and equation (4.2.13) in inequality (4.2.11), we have that:

$$\begin{aligned}
\phi(y_n, z_{n+1}) &\le (1-\lambda_n\theta_n)\phi(y_{n-1}, w_n) + 2\|y_{n-1} - y_n\|M + 4\|Jy_{n-1} - Jy_n\|M \\
&\quad +2\lambda_n\|z_{n+1} - w_n\|M + 2\lambda_n\theta_n\|y_{n-1} - y_n\|M + 2\lambda_n\theta_n\|Jy_{n-1} - Jy_n\|M \\
&\le (1-\lambda_n\theta_n)\phi(y_{n-1}, w_n) + 2M\Big(\delta_E^{-1}(\eta_n) + \delta_{E^*}^{-1}(\eta_n)\Big) + 2\lambda_n\delta_E^{-1}(\lambda_n M)M \\
&\quad +2\lambda_n\theta_n M\Big(\delta_E^{-1}(\eta_n) + \delta_{E^*}^{-1}(\eta_n)\Big) \\
&\le (1-\lambda_n\theta_n)\phi(y_{n-1}, z_n) + 2M_2\omega_J(\beta_n M) + 2M\lambda_n\theta_n^2\gamma_0 \\
&\quad +2\lambda_n\theta_n\left(\frac{\delta_E^{-1}(\eta_n)}{\lambda_n\theta_n} + \frac{\delta_{E^*}^{-1}(\eta_n)}{\lambda_n\theta_n} + \delta_E^{-1}(\eta_n) + \delta_{E^*}^{-1}(\eta_n)\right)M \\
&\le (1-\lambda_n\theta_n)\phi(y_{n-1}, z_n) + 2M\lambda_n^4\theta_n\gamma_0 + \\
&\quad 2\lambda_n\theta_n\Big(\frac{\delta_E^{-1}(\eta_n)}{\lambda_n\theta_n} + \frac{\delta_{E^*}^{-1}(\eta_n)}{\lambda_n\theta_n} + \theta_n\gamma_0 + \delta_E^{-1}(\eta_n) + \delta_{E^*}^{-1}(\eta_n)\Big)M
\end{aligned} \tag{4.2.14}$$

Set $a_n := \Phi(y_{n-1}, z_n)$, $\sigma_n := \lambda_n\theta_n$, $c_n := \lambda_n^4\theta_n$ and

$$b_n : = M\Big(\frac{\delta_E^{-1}(\eta_n)}{\lambda_n\theta_n} + \frac{\delta_{E^*}^{-1}(\eta_n)}{\lambda_n\theta_n} + \delta_E^{-1}(\eta_n) + \delta_{E^*}^{-1}(\eta_n) + \theta_n\gamma_0\Big).$$

Hence, inequality (4.2.14) becomes $a_{n+1} \le (1-\sigma_n)a_n + \sigma_n b_n + c_n, n \ge 1$. It follows from Lemma 2.3.11 that $\lim\limits_{n\to\infty}\phi(y_{n-1}, z_n) = 0$. By Lemma 2.3.8, we have $\lim\limits_{n\to\infty}\|z_n - y_{n-1}\| = 0$. Since $\lim\limits_{n\to\infty} y_n = y^* \in A^{-1}0$, we have that $\{z_n\}$ converges to $y^* \in A^{-1}0$. This completes the proof. ■

4.3 Applications

In this section, we shall apply Theorem 4.2.2 to convex optimization problem and Hammerstein equation.

4.3.1 Application to a convex optimization problem

Theorem 4.3.1 *Let E be a uniformly convex and uniformly smooth real Banach space with dual E^*. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semi-continuous, and convex function such that $(\partial f)^{-1}0 \neq \emptyset$. For given $z_0, z_1 \in E$, let $\{z_n\}$ be generated by the algorithm*

$$\begin{cases} w_n = z_n + \beta_n(z_n - z_{n-1}), \\ z_{n+1} = J^{-1}\left(Jw_n - \lambda_n\mu_n - \lambda_n\theta_n Jw_n\right), \quad \mu_n \in \partial f(w_n) \quad n \ge 1. \end{cases} \tag{4.3.1}$$

Then, the sequence $\{z_n\}$ converges strongly to a minimizer of f.

Proof. By Lemma 2.3.18, ∂f is maximal monotone. The conclusion follows from Theorem 4.2.2. ■

4.3.2 Applications to Hammerstein integral equations

In this subsection, we shall apply Theorem 4.2.2, for the case where the map A is single-valued, to approximate a solution of equation (1.2.3).
Let the sequences $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ be sequences in $(0,1)$ and satisfy the conditions as given in Theorem 4.2.2.

Theorem 4.3.2 *Let E be a uniformly smooth and uniformly convex real Banach space with dual space E^*. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone maps with $R(F) = D(K)$, where $R(F)$ is the range of F and $D(K)$ is the domain of K. For arbitrary*
(u_0, v_0), $(u_1, v_1) \in E \times E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in $E \times E^*$, by

$$\begin{cases} c_n = u_n + \beta_n(u_n - u_{n-1}),\ d_n = v_n + \beta_n(v_n - v_{n-1}), \\ u_{n+1} = J^{-1}\left(Jc_n - \lambda_n(Fc_n - d_n) - \lambda_n\theta_n Jc_n\right), \quad n \geq 1, \\ v_{n+1} = J\left(J^{-1}d_n - \lambda_n(Kd_n + c_n) - \lambda_n\theta_n J^{-1}d_n\right), \quad n \geq 1. \end{cases} \tag{4.3.2}$$

Assume that the equation $u + KFu = 0$ has a solution, then, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of $u + KFu = 0$, with $v^* = Fu^*$.

Proof. By Lemma 2.2.12, E is uniformly smooth and uniformly convex, also, by Lemma 2.2.14, A maximal monotone. Therefore, the conclusion follows from Theorems 4.2.2. ■

4.4 Numerical Illustration

In this section, we present numerical examples to compare the convergence of the sequence of our inertial algorithms and some important algorithms. First, we compare the convergence of the sequence of Inertial Algorithm (4.2.8) with (1.1.13), (1.1.18), (1.1.19) and (1.1.20), respectively. Also, we present numerical examples to compare the convergence of the sequence of Algorithm (4.3.2) with Algorithms (1.2.11) and (6.2.10), respectively. Finally, we present numerical example to illustrate the implementability of Algorithm (4.3.1) whose sequence approximates a solution of a convex optimization problem.

Example 4.4.1

In Theorem 1.1.8, algorithms, (1.1.18), (1.1.19), (1.1.20), and Theorem 4.2.2 set $E = L_2([0,1])$. Consider the map $A : E \to E$ defined by

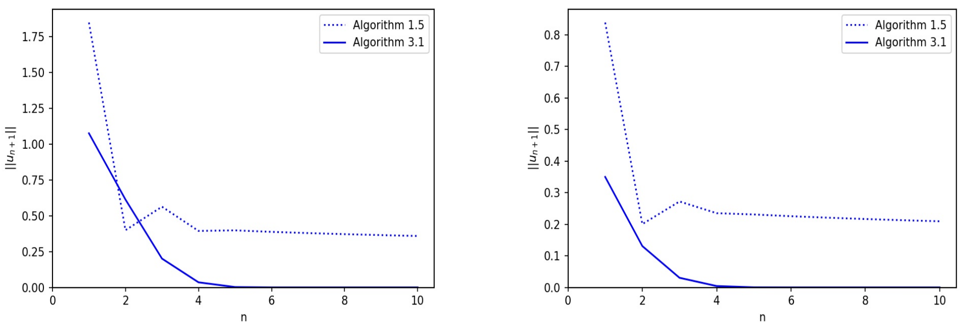
$$(Au)(t) := (t+1)u(t).$$

Then, it is easy to see that A is maximal monotone. Furthermore, the function $u(t) = 0,\ \forall\ t \in [0,1]$ is the solution of the equation $Au(t) = 0$. In Theorems 1.1.8, we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, in Algorithm (1.1.18), take $\lambda_n = \frac{n}{n+1}$, $\alpha_n = \frac{1}{(n+1)^2}$, in Algorithm (1.1.19), take $\lambda_n = \frac{n}{n+1} = \rho_n$, in Algorithm (1.1.20), take

$\lambda_n = \frac{n}{n+1} = \rho_n$, $\alpha_n = \frac{1}{(n+1)^2}$ and in Theorem 4.2.2, we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\beta_n = \frac{1}{(n+1)^2}$, $n = 1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of the theorems , respectively. Setting a tolerance of 10^{-6} and maximum number of iterations $n = 10$, we obtain the following iterates:

Table 4.1: Numerical results for Example 4.4.1

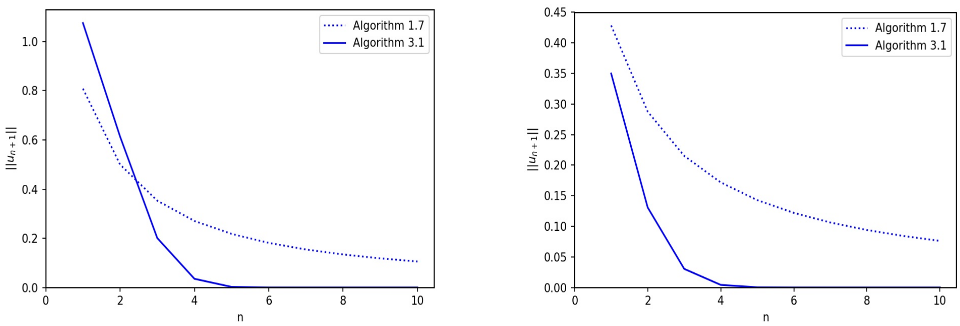
Algorithm (1.1.13)				Algorithm (1.1.18)				Algorithm (4.2.8)			
IP	n	$\|u_{n+1}\|$	T (sec)	IP	n	$\|u_{n+1}\|$	T (sec)	IP	n	sol.	T (sec)
$u_1(t) = t^2 + 1$	10	0.3587	0.032	$u_1(t) = t^2 + 1$	10	0.0762	0.081	$u_0(t) = 2t$ $u_1(t) = t^2 + 1$	10	1.999E-6	15.69
$u_1(t) = \frac{1}{t+1}$	10	0.2093	0.058	$u_1(t) = \frac{1}{t+1}$	10	0.1056	0.082	$u_0(t) = 2t$ $u_1(t) = \frac{1}{t+1}$	10	1.87E-6	17.65
$u_1(t) = te^t$	10	0.2984	0.056	$u_1(t) = te^t$	10	0.0552	0.095	$u_0(t) = 2t$ $u_1(t) = te^t$	8	1.89E-6	92.44



(a) Graph of some iterates of Algorithms (1.1.13) and (4.2.8) with $u_1(t) = t^2 + 1$

(b) Graph of some iterates of Algorithms (1.1.13) and (4.2.8) with $u_1(t) = \frac{1}{t+1}$

Figure 4.1: Graphical illustration of the data in Table 4.1



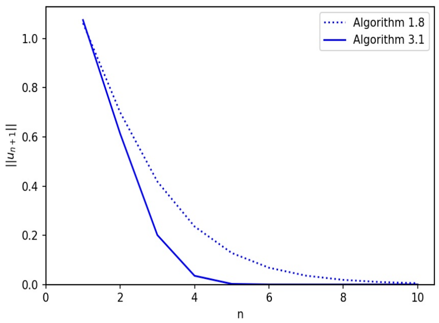
(a) Graph of some iterates of Algorithms (1.1.18) and (4.2.8) with $u_1(t) = t^2 + 1$

(b) Graph of some iterates of Algorithms (1.1.18) and (4.2.8) with $u_1(t) = \frac{1}{t+1}$

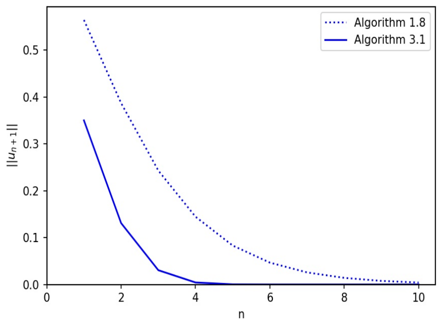
Figure 4.2: Graphical illustration of the data in Table 4.1

Table 4.2: Numerical results for Example 4.4.1

Algorithm (1.1.19)				Algorithm (1.1.20)				Algorithm (4.2.8)			
IP	n	$\|u_{n+1}\|$	T (sec)	IP	n	$\|u_{n+1}\|$	T (sec)	IP	n	sol.	T (sec)
$u_1(t) = t^2 + 1$	10	0.005	0.025	$u_1(t) = t^2 + 1$	10	0.0051	16.68	$u_0(t) = 2t$ $u_1(t) = t^2 + 1$	10	1.999E-6	15.69
$u_1(t) = \frac{1}{t+1}$	10	0.0041	0.0381	$u_1(t) = \frac{1}{t+1}$	10	0.0042	17.95	$u_0(t) = 2t$ $u_1(t) = \frac{1}{t+1}$	10	1.87E-6	17.65
$u_1(t) = te^t$	10	0.0021	0.0392	$u_1(t) = te^t$	10	0.0017	21.13	$u_0(t) = 2t$ $u_1(t) = te^t$	8	1.89E-6	92.44

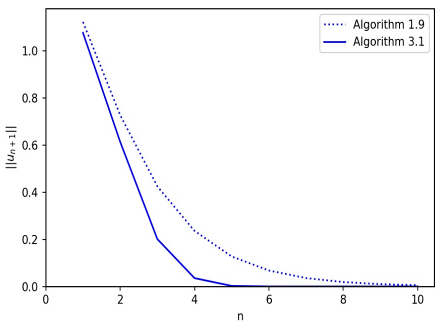


(a) Graph of some iterates of Algorithms (1.1.19) and (4.2.8) with $u_1(t) = t^2 + 1$

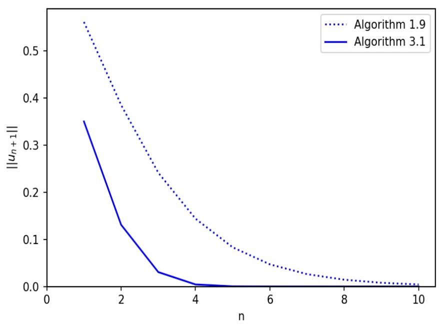


(b) Graph of some iterates of Algorithms (1.1.19) and (4.2.8) with $u_1(t) = \frac{1}{t+1}$

Figure 4.3: Graphical illustration of the data in Table 4.2



(a) Graph of some iterates of Algorithms (1.1.20) and (4.2.8) with $u_1(t) = t^2 + 1$



(b) Graph of some iterates of Algorithms (1.1.20) and (4.2.8) with $u_1(t) = \frac{1}{t+1}$

Figure 4.4: Graphical illustration of the data in Table 4.2

Example 4.4.2

In Theorems 1.2.8, 6.2.2 and 4.3.2, respectively, set $E = L_5([0,1])$, then, $E^* = L_{\frac{5}{4}}([0,1])$ and $F : L_5([0,1]) \to L_{\frac{5}{4}}([0,1])$ is defined by

$$(Fu)(t) = Ju(t).$$

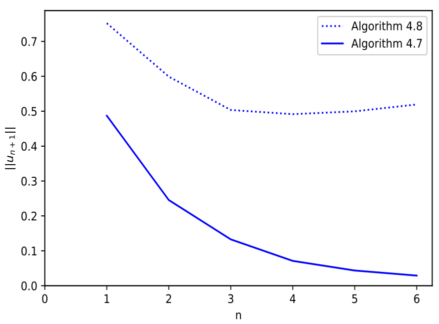
Then, it is to see that F is maximal monotone. Let $K : L_{\frac{5}{4}}([0,1]) \to L_5([0,1])$ be defined by

$$(Kv)(t) = tv(t).$$

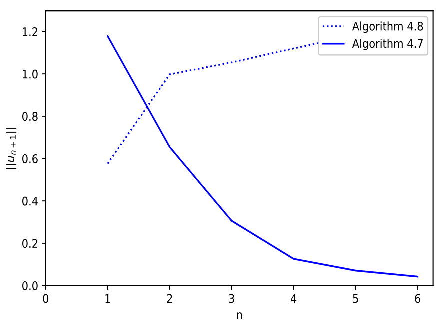
Observe that K is linear. Furthermore, it is easy to see that K maximal monotone and the function $u^*(t) = 0, \forall\, t \in [0,1]$ is the only solution of the equation $u + KFu = 0$. In algorithm (1.2.11), we take $\lambda_n = \theta_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, in algorithm (6.2.10), we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \cdots$, and in algorithm (4.3.2), we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\beta_n = \frac{1}{(n+1)^2}$, $n = 1, 2, \cdots$, as our parameters and fixed $u_0(t) = t$ and $v_0(t) = t + 1$. Clearly, these parameters satisfy the hypotheses of these Theorems, respectively. Setting a tolerance of 10^{-6} and maximum number of iterations $n = 6$, we obtain the following iterates:

Table 4.3: Numerical results for Example 4.4.2

Algorithm (1.2.11)				Algorithm (6.2.10)				Algorithm (4.3.2)			
IP	n	$\|u_{n+1}\|$	T (sec)	IP	n	$\|u_{n+1}\|$	T (sec)	IP	n	sol.	T (sec)
$u_1(t) = \sin t$ $v_1(t) = \cos t$	6	0.5193	41.56	$u_1(t) = \sin t$ $v_1(t) = \cos t$	6	0.0337	92.78	$u_1(t) = \sin t$ $v_1(t) = \cos t$	6	0.0291	4129.97
$u_1(t) = t^2 - 2$ $v_1 = e^t - 1$	6	1.2381	244.31	$u_1 = t^2 - 2$ $v_1 = e^t - 1$	6	0.0463	28.55	$u_1 = t^2 - 2$ $v_1 = e^t - 1$	6	0.0424	884.05
$u_1(t) = 2t^3 - 2$ $v_1 = te^t + 2t$	6	1.4154	647.69	$u_1 = 2t^3 - 2$ $v_1 = te^t + 2t$	6	0.0720	57.03	$u_1 = 2t^3 - 2$ $v_1 = te^t + 2t$	6	0.0519	2268.58

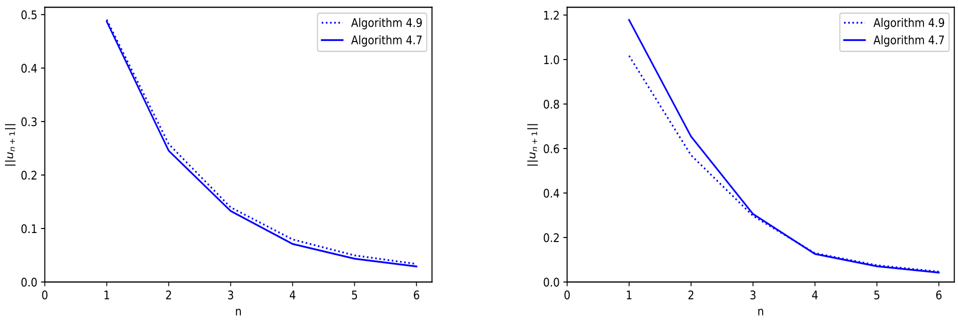


(a) Graph of some iterates of Algorithms (1.2.11) and (4.3.2) with $u_1(t) = \sin t$ and $v_1(t) = \cos t$



(b) Graph of some iterates of Algorithms (1.2.11) and (4.3.2) with $u_1(t) = t^2 - 2$ and $v_1(t) = e^t - 1$

Figure 4.5: Graphical illustration of the data in Table 4.3



(a) Graph of some iterates of Algorithms (6.2.10) and (4.3.2) with $u_1(t) = \sin t$ and $v_1(t) = \cos t$

(b) Graph of some iterates of Algorithms (6.2.10) and (4.3.2) with $u_1(t) = t^2 - 2$ and $v_1(t) = e^t - 1$

Figure 4.6: Graphical illustration of the data in Table 4.3

Example 4.4.3

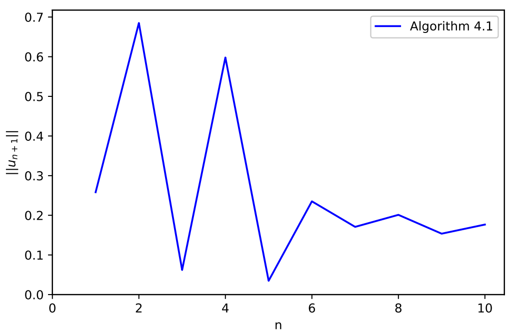
In Theorem 4.3.1, set set $E = L_2([0,1])$. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be defined by

$$f(z) = \|z\| \quad \text{then,} \quad \partial f(z) = \begin{cases} \frac{z(t)}{\|z\|}, & z(t_0) \neq 0, \quad t_0 \in [0,1]; \\ B(0,1), & z(t) = 0, \quad \forall t \in [0,1]. \end{cases} \tag{4.4.1}$$

Then it is easy to see that ∂f is maximal monotone. Furthermore, the function $u(t) = 1, \forall\, t \in [0,1]$ is the solution of the equation $\partial f z(t) = 0$. We take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\beta_n = \frac{1}{(n+1)^2}$, $n = 1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of the Theorem 4.3.1. Setting a tolerance of 10^{-6} and maximum number of iterations $n = 10$, we obtain the following iterates:

Numerical results for Example 3

	Algorithm 4.3.1
n	$\|z_{n+1}\|$
1	0.2581
2	0.685
3	0.6209
4	0.5979
5	0.0347
6	0.2352
7	0.1707
8	0.2011
9	0.1536
10	0.1766



Graph of some iterates of Algorithm (4.3.1) $z_1(t) = t + \sin t$.

Conclusion. An inertial iterative algorithm which does not involve the reslovent operator is proposed for approximating a solution of a maximal monotone inclusion in a uniformly convex and uniformly smooth real Banach space. The sequence generated by the algorithm is proved to converge *strongly* to a solution of the inclusion. Moreover, the theorem proved is applied to approximate a solution of a convex optimization problem, and a solution of a Hammerstein integral equation. Furthermore, numerical experiments are given to compare, in terms of CPU time and number of iterations, the performance of the sequence generated by our algorithm with the performance of the sequences generated by IPPA, RPPA and RIPPA, respectively. Finally, a numerical example is given to illustrate the implementability of our algorithm for approximating a solution of a convex optimization problem and to approximate a solution of a Hammerstein integral Equation.

The results obtained in this chapter have gained publication, and appeared in the following paper: C.E Chidume, **A. Adamu**, M.O. Nnakwe, *Strong convergence of an inertial algorithm not involving the resolvent for maximal monotone inclusions with applications*, **Fixed Point Theory and Applications**, https://doi.org/10.1186/s13663-020-00680-2.

CHAPTER 5

ITERATIVE ALGORITHMS FOR SOLUTIONS OF HAMMERSTEIN EQUATIONS IN REAL BANACH SPACES

5.1 Introduction

In this chapter, an iterative algorithm is constructed and the sequence of the algorithm is proved to converge strongly to a solution of the Hammerstein equation $u + KFu = 0$. This theorem is a significant improvement of some important recent results which were proved in real Hilbert spaces under the assumption that F and K are maximal monotone *continuous* and *bounded*. The continuity and boundedness restrictions on K and F have been dispensed with, using our new method, even in the more general setting considered in our theorems. Finally, numerical experiments are presented to illustrate the convergence of the sequence of our algorithm.

5.2 Main result

Definition 5.2.1 *A mapping $A : E \to E^*$ is quasi-bounded if for any $\sigma > 0$ there exists $\tau > 0$ such that whenever $\langle y, Ay\rangle \le \sigma\|y\|$ and $\|y\| \le \sigma$ then $\|Ay\| \le \tau$.*

We first prove the following Lemma (see, [Pascall and Sburlan, 1978], Chapter III, Lemma 3.6).

Lemma 5.2.2 *Let E be a real normed space with dual space E^*. Any monotone map $A : D(A) \subset E \to E^*$ with $0 \in IntD(A)$ is quasi-bounded.*

Proof. By Lemma 2.3.6 A is locally bounded at 0. Now, A is locally bounded at 0 means that there exists $r > 0$ and $\sigma > 0$ such that $\forall x \in B_r(0) := \{x \in E : \|x\| < r\}$, $\|Ax\| \le \sigma$. Using this $\sigma > 0$, suppose $\langle y, Ay\rangle \le \sigma\|y\|$ and $\|y\| \le \sigma$. Let $x \in B(0, r)$. Then, by monotonicity of A, we have

$$\langle y, Ay\rangle \ge \langle x, Ay\rangle + \langle y - x, Ax\rangle. \tag{5.2.1}$$

Observe that

$$\begin{aligned}\langle y-x, Ax\rangle &\le |\langle y-x, Ax\rangle| \\ &\le \|Ax\|(\|y\|+\|x\|) \le \sigma(\|y\|+r). \qquad (5.2.2)\end{aligned}$$

From inequality (5.2.1) and using inequality (5.2.2), we deduce that

$$\begin{aligned}\langle x, Ay\rangle &\le \langle y, Ay\rangle + \langle x-y, Ax\rangle \qquad (5.2.3)\\ &\le \sigma\|y\| + \sigma(\|y\|+r).\end{aligned}$$

Similarly, replacing x with $-x$ in inequality (5.2.3), we obtain that

$$-\langle x, Ay\rangle \le \sigma\|y\| + \sigma(\|y\|+r).$$

Thus,

$$|\langle x, Ay\rangle| \le \sigma\|y\| + \sigma(\|y\|+r), \quad \forall x \in B(0,r).$$

$$\text{Hence,} \quad \sup_{\|x\|\le r} |\langle x, Ay\rangle| \le \sigma\|y\| + \sigma(\|y\|+r).$$

$$\text{Therefore,} \quad \|Ay\| \le \frac{\sigma}{r}\|y\| + \frac{\sigma}{r}(\|y\|+r),$$

where $\|Ay\| = \sup_{\|x\|\le 1} \langle x, Ay\rangle$. This implies the quasi-boundedness of A. ■

In Lemma 5.2.3 below, the sequences $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$,

(ii) there exist constants $M_0, M_0^*, S^*, \gamma_0 > 0$ such that for all $n \ge 1$,
$\delta_E^{-1}(\beta_n M_0) \le \alpha_n\gamma_0$; $\delta_{E^*}^{-1}(\beta_n M_0^*) \le \alpha_n\gamma_0$; $\beta_n S^* \le \alpha_n\gamma_0$, $\forall\, n \ge 1$

Lemma 5.2.3 *Let E be a uniformly convex and uniformly smooth real Banach space. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone mappings. For arbitrary $x \in E$, $y \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively by: $u_1 \in E$, $v_1 \in E^*$,*

$$u_{n+1} = J^{-1}\big(Ju_n - \beta_n^2(Fu_n - v_n) - \alpha_n\beta_n(Ju_n - Jx)\big), \qquad (5.2.4)$$

$$v_{n+1} = J\big(J^{-1}v_n - \beta_n^2(Kv_n + u_n) - \alpha_n\beta_n(J^{-1}v_n - J^{-1}y)\big), \qquad (5.2.5)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying conditions (i) and (ii). Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ are bounded.

Proof. Now, to show that the sequences $\{u_n\}$ and $\{v_n\}$ are bounded, set $w_n = (u_n, v_n), w^* = (u^*, v^*) \in W = E \times E^*$, where u^* is a solution of the Hammerstein equation $u + KFu = 0$, with $v^* = Fu^*$.

Define $\Phi : W \times W \to \mathbb{R}$ by

$$\Phi(w_1, w_2) = \phi(u_1, u_2) + \phi(v_1, v_2),$$

where $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$.

Remark 5.2.4 *Since E^* is uniformly smooth, the functional $\phi : E^* \times E^* \to \mathbb{R}$ is define as*

$$\phi(v_1, v_2) := \|v_1\|_{E^*}^2 - 2\langle J^{-1}v_2, v_1\rangle + \|v_2\|_{E^*}^2.$$

Let W be endowed with norm

$$\|(u,v)\|_W = \left(\|u\|_E^2 + \|v\|_{E^*}^2\right)^{\frac{1}{2}}.$$

It suffices to show that $\{w_n\}$ is bounded. We show this by induction. Let $w_1 \in W$. Then there exists $r > 0$ such that $\Phi(w^*, w_1) \leq \frac{r}{8}$ and $B := \big\{w = (u,v) \in W : \Phi(w^*, w) \leq r\big\}$. It suffices to show that $\Phi(w^*, w_n) \leq r$, for all $n \geq 1$. Let $w, w_1 \in B$ and $\alpha, \beta \in (0,1)$. Then,

$$\Phi(w^*, w) \leq r \qquad i.e., \quad \phi(u^*, u) + \phi(v^*, v) \leq r.$$

$$\text{Therefore,} \qquad \phi(u^*, u) \leq r \qquad \text{and} \qquad \phi(v^*, v) \leq r.$$

Now, using inequality (2.2.2),

$$\phi(u^*, u) \leq r \quad \Rightarrow \quad \|u\| \leq \|u^*\| + \sqrt{r}.$$

Since F is also locally bounded at $0 \in E$, there exist $h_0 > 0$ with $h_0 < \sqrt{r}$, $m_0 > 0$ such that $\|Fx\| \leq m_0$, $\forall \ \ x \in B_{h_0}(0) := \{x \in E : \|x\| < h_0\}$. Let $(u,v) \in B$, then $u \in E$, and let $x \in B_{h_0}(0)$. By monotonicity of F, we have that:

$$\langle u, Fu\rangle \geq \langle u - x, Fx\rangle + \langle x, Fu\rangle$$

implies

$$\langle -u, Fu\rangle \leq \langle x - u, Fx\rangle - \langle x, Fu\rangle.$$

Setting $z = -u$, we have that

$$\begin{aligned} \langle z, Fu\rangle &\leq \langle x + z, Fx\rangle - \langle x, Fu\rangle \\ &\leq \|Fx\|(\|x\| + \|z\|) + \|Fu\|\|x\| \\ &\leq m_0(h_0 + \|u^*\| + \sqrt{r}) + h_0\|Fu\|. \end{aligned} \tag{5.2.6}$$

Similarly, replacing z by $-z$ in inequality (5.2.6), we obtain that

$$-\langle z, Fu\rangle \leq m_0(h_0 + \|u^*\| + \sqrt{r}) + h_0\|Fu\|.$$

Thus,

$$|\langle z, Fu\rangle| \leq m_0(h_0 + \|u^*\| + \sqrt{r}) + h_0\|Fu\|.$$

Hence,

$$\sup_{\|z\| \leq \sqrt{r} + \|u^*} |\langle z, Fu\rangle| \leq m_0(h_0 + \|u^*\| + \sqrt{r}) + h_0\|Fu\|$$

$$(\sqrt{r} + \|u^*\|)\|Fu\| \leq m_0(h_0 + \|u^*\| + \sqrt{r}) + h_0\|Fu\|$$

$$\|Fu\| \leq \frac{m_0\,(h_0 + \|u^*\| + \sqrt{r})}{\|u^*\| + \sqrt{r} - h_0} := k_1, \quad \forall\ u \in E \ \text{ such that } (u,v) \in B.$$

Define $\sigma := \max\{k_1, \|u^*\| + \sqrt{r}\}$. Hence, $\langle u, Fu\rangle \leq \sigma\|u\|$ and $\|u\| \leq \sigma$. By Lemma 5.2.2, F is quasi-bounded. Thus, there exists $\tau_1 > 0$ such that

$$\|Fu\| \leq \tau_1, \quad \forall u \in E, \text{such that } (u, v) \in B.$$

Similarly, there exists $\tau_2 > 0$ such that

$$\|Kv\| \leq \tau_2, \quad \forall v \in E^*, \text{such that} (u, v) \in B.$$

Define:

$$S^* := \max \left\{ \begin{array}{c} \sup\Big\{\|\beta(Fu - v) + \alpha(Ju - Jx)\|\Big\} + 1, \\ \sup\Big\{\|Ju - Jx\|\Big\} + 1, \sup\Big\{\|J^{-1}v - J^{-1}y\|\Big\} + 1, \\ \sup\Big\{\|J^{-1}\big(Ju - \beta^2(Fu - v) - \alpha\beta(Ju - Jx)\big) - u\|\Big\} + 1, \\ \sup\Big\{\|\beta(Kv + u) + \alpha(J^{-1}v - J^{-1}y)\|\Big\} + 1, \\ \sup\Big\{\|J\big(J^{-1}v - \beta^2(Kv + u) - \alpha\beta(J^{-1}v - J^{-1}y)\big) - v\|\Big\} + 1, \end{array} \right\}$$

where all the sup are taken over $(u, v) \in B$. The quasi-boundedness of K and F on B and the fact that J and J^{-1} are on bounded subsets of a uniformly convex and uniformly real Banach space guarantee that S^* is well defined. Let $M := \max\big\{c_2S^*, S^*\big\}$, $M_0 = M_0^* := 2L(S^*)^2$ (where L is the constant appearing in Lemma 2.2.5) $\gamma_0 := \min\Big\{1, \frac{r}{32M}\Big\}$. Then, for $n = 1$, by construction $\Phi(w^*, w_1) \leq r$. Assume $\Phi(w^*, w_n) \leq r$, for some $n \geq 1$, i.e.,

$$\phi(u^*, u_n) + \phi(v^*, v_n) \leq r.$$

We show that $\Phi(w^*, w^{n+1}) \leq r$. For contradiction, suppose $r < \Phi(w^*, w_{n+1})$. Using Lemma 2.2.5 and recurrence relation (5.2.4), we have

$$\begin{aligned} (2L)^{-1}\delta_E(c_2^{-1}\|u_{n+1} - u_n\|) &\leq \langle Ju_{n+1} - Ju_n, u_{n+1} - u_n\rangle \\ &\leq \|Ju_{n+1} - Ju_n\|\|u_{n+1} - u_n\| \\ &\leq \beta_n\Big[\sup\Big\{\|\beta_n(Fu_n - v_n) + \alpha_n(Ju_n - Jx)\|\Big\} \\ &\quad + 1\Big]\|u_{n+1} - u_n\|. \end{aligned}$$

Thus,

$$\|u_{n+1} - u_n\| \leq c_2\delta_E^{-1}(\beta_n M_0). \tag{5.2.7}$$

Similarly, using a result of Lemma 2.2.5, we obtain

$$\|v_{n+1} - v_n\| \leq c_2\delta_{E^*}^{-1}(\beta_n M_0^*). \tag{5.2.8}$$

Now, using recurrence relation (5.2.4), Lemma 2.2.3, and inequality (5.2.7), we have

$$\begin{aligned}\phi(u^*, u_{n+1}) &= V\big(u^*, Ju_n - \beta_n^2(Fu_n - v_n) - \alpha_n\beta_n(Ju_n - Jx)\big)\\ &\le V(u^*, Ju_n) - 2\langle u_{n+1} - u^*, \beta_n^2(Fu_n - v_n) + \alpha_n\beta_n(Ju_n - Jx)\rangle\\ &= \phi(u^*, u_n) - 2\langle u_n - u^*, \beta_n^2(Fu_n - v_n) + \alpha_n\beta_n(Ju_n - Jx)\rangle\\ &\quad - 2\langle u_{n+1} - u_n, \beta_n^2(Fu_n - v_n) + \alpha_n\beta_n(Ju_n - Jx)\rangle\\ &\le \phi(u^*, u_n) - 2\langle u_n - u^*, \beta_n^2(Fu_n - v_n) + \alpha_n\beta_n(Ju_n - Jx)\rangle\\ &\quad + 2\beta_n\|u_{n+1} - u_n\|\|\beta_n(Fu_n - v_n) + \alpha_n(Ju_n - Jx)\|\\ &\le \phi(u^*, u_n) - 2\beta_n^2\langle u_n - u^*, Fu_n - v_n\rangle - 2\alpha_n\beta_n\langle u_n - u^*, Ju_n - Jx\rangle\\ &\quad + 2\beta_n c_2 S^*\delta_E^{-1}(\beta_n M_0). \end{aligned} \tag{5.2.9}$$

Observe that by monotonicity of F and the fact that $v^* = Fu^*$, we have

$$\langle u_n - u^*, Fu_n - v_n\rangle \ge \langle u_n - u^*, v^* - v_n\rangle.$$

Thus, substituting this in inequality (5.2.9), we have

$$\begin{aligned}\phi(u^*, u_{n+1}) \le \phi(u^*, u_n) &- 2\beta_n^2\langle u_n - u^*, v^* - v_n\rangle\\ &- 2\alpha_n\beta_n\langle u_n - u_{n+1}, Ju_n - Jx\rangle\\ &- 2\alpha_n\beta_n\langle u_{n+1} - u^*, Ju_n - Ju_{n+1}\rangle\\ &- 2\alpha_n\beta_n\langle u_{n+1} - u^*, Ju_{n+1} - Jx\rangle\\ &+ 2\beta_n c_2 S^*\delta_E^{-1}(\beta_n M_0). \end{aligned} \tag{5.2.10}$$

Using Lemma 2.2.4, we have

$$-2\alpha_n\beta_n\langle u_{n+1} - u^*, Ju_{n+1} - Jx\rangle \le \alpha_n\beta_n\phi(u^*, x) - \alpha_n\beta_n\phi(u^*, u_{n+1}).$$

Substituting this in inequality (5.2.10), we obtain

$$\begin{aligned}\phi(u^*, u_{n+1}) &\le \phi(u^*, u_n) - \alpha_n\beta_n\phi(u^*, u_{n+1}) + \alpha_n\beta_n\phi(u^*, x)\\ &\quad - 2\beta_n^2\langle u_n - u^*, v^* - v_n\rangle - 2\alpha_n\beta_n\langle u_n - u_{n+1}, Ju_n - Jx\rangle\\ &\quad - 2\alpha_n\beta_n\langle u_{n+1} - u^*, Ju_n - Ju_{n+1}\rangle + 2\beta_n c_2 S^*\delta_E^{-1}(\beta_n M_0)\\ &\le \phi(u^*, u_n) - \alpha_n\beta_n\phi(u^*, u_{n+1}) + \alpha_n\beta_n\phi(u^*, x)\\ &\quad - 2\beta_n^2\langle u_n - u^*, v^* - v_n\rangle + 2\alpha_n\beta_n\|u_n - u_{n+1}\|\|Ju_n - Jx\|\\ &\quad + 2\alpha_n\beta_n\|u_{n+1} - u^*\|\|Ju_n - Ju_{n+1}\| + 2\beta_n c_2 S^*\delta_E^{-1}(\beta_n M_0)\\ &\le \phi(u^*, u_n) - \alpha_n\beta_n\phi(u^*, u_{n+1}) + \alpha_n\beta_n\phi(u^*, x)\\ &\quad - 2\beta_n^2\langle u_n - u^*, v^* - v_n\rangle + 2\alpha_n\beta_n c_2 S^*\delta_E^{-1}(\beta_n M_0)\\ &\quad + 2\alpha_n\beta_n(\beta_n S^*)S^* + 2\beta_n c_2 S^*\delta_E^{-1}(\beta_n M_0) \end{aligned} \tag{5.2.11}$$

Similarly, using recurrence relation (5.2.5), Lemma 2.2.3, inequality (5.2.8), monotonicity of K, the fact that $Kv^* = -u^*$ and Lemma 2.2.4, we obtain

$$\begin{aligned}\phi(v^*, v_{n+1}) \le \phi(v^*, v_n) &- \alpha_n\beta_n\phi(v^*, v_{n+1}) + \alpha_n\beta_n\phi(v^*, y)\\ &- 2\beta_n^2\langle v_n - v^*, u_n - u^*\rangle + 2\alpha_n\beta_n c_2 S^*\delta_{E^*}^{-1}(\beta_n M_0^*)\\ &+ 2\alpha_n\beta_n(\beta_n S^*)S^* + 2\beta_n c_2 S^*\delta_{E^*}^{-1}(\beta_n M_0^*). \end{aligned} \tag{5.2.12}$$

Thus, adding inequalities (5.2.11) and (5.2.12), we obtain

$$
\begin{aligned}
r < \Phi(w^*, w^{n+1}) &= \phi(u^*, u_{n+1}) + \phi(v^*, v_{n+1}) \\
&\leq \Phi(w^*, w^n) - \alpha_n\beta_n\Phi(w^*, w^{n+1}) + \alpha_n\beta_n\Phi(w^*, w^1) \\
&\quad + 2\alpha_n\beta_n c_2 S^* \delta_E^{-1}(\beta_n M_0) + 2\alpha_n\beta_n(\beta_n S^*)S^* \\
&\quad + 2\beta_n c_2 S^* \delta_E^{-1}(\beta_n M_0) + 2\alpha_n\beta_n c_2 S^* \delta_{E^*}^{-1}(\beta_n M_0^*) \\
&\quad + 2\alpha_n\beta_n(\beta_n S^*)S^* + 2\beta_n c_2 S^* \delta_{E^*}^{-1}(\beta_n M_0^*) \\
&\leq \Phi(w^*, w^n) - \alpha_n\beta_n\Phi(w^*, w^{n+1}) + \alpha_n\beta_n\Phi(w^*, w^1) \\
&\quad + 2\alpha_n^2\beta_n c_2 S^*\gamma_0 + 2\alpha_n^2\beta_n S^*\gamma_0 + 2\alpha_n\beta_n c_2 S^*\gamma_0 \\
&\quad + 2\alpha_n^2\beta_n c_2 S^*\gamma_0 + 2\alpha_n^2\beta_n S^*\gamma_0 + 2\alpha_n\beta_n c_2 S^*\gamma_0 \\
&\quad \leq \Phi(w^*, w^n) - \alpha_n\beta_n\Phi(w^*, w^{n+1}) + \alpha_n\beta_n\Phi(w^*, w^1) \\
&\quad + 2\alpha_n\beta_n M\gamma_0 + 2\alpha_n\beta_n M\gamma_0 + 2\alpha_n\beta_n M\gamma_0 \\
&\quad + 2\alpha_n\beta_n M\gamma_0 + 2\alpha_n\beta_n M\gamma_0 + 2\alpha_n\beta_n M\gamma_0 \\
&\leq \Phi(w^*, w^n) - \alpha_n\beta_n\Phi(w^*, w^{n+1}) + \frac{r\alpha_n\beta_n}{8} + \frac{3r\alpha_n\beta_n}{8} \\
&\leq r - r\alpha_n\beta_n + \frac{r\alpha_n\beta_n}{2} = r - \frac{r\alpha_n\beta_n}{2} < r.
\end{aligned}
$$

This is a contradiction. Hence, $\Phi(w^*, w^{n+1}) \leq r$. Thus, $\Phi(w^*, w^n) \leq r$, for all $n \geq 1$. Consequently, we have $\phi(u^*, u_n) \leq r$ and $\phi(v^*, v_n) \leq r$, for all $n \geq 1$. Therefore, using inequality (2.2.2), we deduce that $\{u_n\}$ and $\{v_n\}$ are bounded. ■

In Theorem 5.2.5 below, the sequences $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$,

(ii) there exist constants $M_0, M_0^*, S^*, \gamma_0 > 0$, such that $\delta_E^{-1}(\beta_n M_0) \leq \alpha_n\gamma_0$; $\delta_{E^*}^{-1}(\beta_n M_0^*) \leq \alpha_n\gamma_0$; $\beta_n S^* \leq \alpha_n\gamma_0$, $\forall\, n \geq 1$,

(iii) $\delta_E^{-1}(\eta_n) \to 0$; $\delta_{E^*}^{-1}(\eta_n) \to 0$, (iv) $\frac{\delta_E^{-1}(\eta_n)}{\alpha_n\beta_n} \to 0$; $\frac{\delta_{E^*}^{-1}(\eta_n)}{\alpha_n\beta_n} \to 0$,

$\eta_n = \dfrac{\alpha_{n-1}\beta_n - \alpha_n\beta_{n-1}}{\alpha_n\beta_{n-1}}\mathcal{K}$, where $\mathcal{K}$ is the constant appearing in Remark 2.2.11.

Theorem 5.2.5 *Let E be a uniformly convex and uniformly smooth real Banach space. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone mappings. For arbitrary $x \in E$, $y \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively by $u_1 \in E$, $v_1 \in E^*$,*

$$
\begin{cases}
u_{n+1} = J^{-1}\big(Ju_n - \beta_n^2(Fu_n - v_n) - \alpha_n\beta_n(Ju_n - Jx)\big), \\
v_{n+1} = J\big(J^{-1}v_n - \beta_n^2(Kv_n + u_n) - \alpha_n\beta_n(J^{-1}v_n - J^{-1}y)\big),
\end{cases}
\tag{5.2.13}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying conditions (i)-(iv). Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of $u + KFu = 0$ with $v^* = Fu^*$.

Proof. Let y_n be as defined in Remark 2.2.10. Using Lemma 2.2.3 and equation (2.2.3), we have

$$\begin{aligned}
\phi(y_n, u_{n+1}) &= V\big(y_n, Ju_n - \beta_n^2(Fu_n - v_n) - \alpha_n\beta_n(Ju_n - Jx)\big) \\
&\leq V(y_n, Ju_n) - 2\langle u_{n+1} - y_n, \beta_n^2(Fu_n - v_n) + \alpha_n\beta_n(Ju_n - Jx)\rangle \\
&= \phi(y_n, u_n) - 2\beta_n^2\langle u_{n+1} - y_n, Fu_n - v_n\rangle \\
&\quad - 2\alpha_n\beta_n\langle u_{n+1} - y_n, Ju_n - Jx\rangle \\
&= \phi(u_n, y_n) - 2\langle u_n + y_n, Ju_n - Jy_n\rangle + 2(\|u_n\|^2 - \|y_n\|^2) \\
&\quad - 2\beta_n^2\langle u_{n+1} - y_n, Fu_n - v_n\rangle \\
&\quad - 2\alpha_n\beta_n\langle u_{n+1} - y_n, Ju_n - Jx\rangle
\end{aligned} \tag{5.2.14}$$

Observe that

$$\begin{aligned}
\phi(u_n, y_n) = V(u_n, Jy_n) &= V(u_n, Jy_{n-1} + Jy_n - Jy_{n-1}) \\
&\leq V(u_n, Jy_{n-1}) - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle.
\end{aligned}$$

Thus, substituting this in inequality (5.2.14), and using equation (2.2.3), we obtain

$$\begin{aligned}
\phi(y_n, u_{n+1}) &\leq V(u_n, Jy_{n-1}) - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \\
&\quad - 2\langle u_n + y_n, Ju_n - Jy_n\rangle + 2(\|u_n\|^2 - \|y_n\|^2) \\
&\quad - 2\beta_n^2\langle u_{n+1} - y_n, Fu_n - v_n\rangle - 2\alpha_n\beta_n\langle u_{n+1} - y_n, Ju_n - Jx\rangle \\
&= \phi(y_{n-1}, u_n) - 2\langle y_{n-1} + u_n, Jy_{n-1} - Ju_n\rangle + 2(\|y_{n-1}\|^2 - \|u_n\|^2) \\
&\quad - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle - 2\langle u_n + y_n, Ju_n - Jy_n\rangle \\
&\quad + 2(\|u_n\|^2 - \|y_n\|^2) - 2\beta_n^2\langle u_{n+1} - y_n, Fu_n - v_n\rangle \\
&\quad - 2\alpha_n\beta_n\langle u_{n+1} - y_n, Ju_n - Jx\rangle \\
&= \phi(y_{n-1}, u_n) + 2(\|y_{n-1}\|^2 - \|y_n\|^2) \\
&\quad + 2\langle y_{n-1} + u_n, Ju_n - Jy_{n-1}\rangle - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \\
&\quad - 2\langle u_n + y_n, Ju_n - Jy_n\rangle - 2\beta_n^2\langle u_{n+1} - y_n, Fu_n - v_n\rangle \\
&\quad \boxed{-2\alpha_n\beta_n\langle u_{n+1} - y_n, Ju_n - Jx\rangle}.
\end{aligned} \tag{5.2.15}$$

Now, we estimate the boxed term using Lemma 2.2.4.

$$\begin{aligned}
&-2\alpha_n\beta_n\langle u_{n+1} - y_n, Ju_n - Jx\rangle \\
&\quad= -2\alpha_n\beta_n\langle u_{n+1} - u_n, Ju_n - Jx\rangle - 2\alpha_n\beta_n\langle u_n - y_{n-1}, Ju_n - Jy_{n-1}\rangle \\
&\qquad - 2\alpha_n\beta_n\langle u_n - y_{n-1}, Jy_{n-1} - Jx\rangle - 2\alpha_n\beta_n\langle y_{n-1} - y_n, Ju_n - Jx\rangle \\
&\quad\leq -2\alpha_n\beta_n\langle u_{n+1} - u_n, Ju_n - Jx\rangle - \alpha_n\beta_n\phi(y_{n-1}, u_n) \\
&\qquad - 2\alpha_n\beta_n\langle u_n - y_{n-1}, Jy_{n-1} - Jx\rangle - 2\alpha_n\beta_n\langle y_{n-1} - y_n, Ju_n - Jx\rangle
\end{aligned}$$

Therefore, substituting this in inequality (5.2.15), we have

$$\begin{aligned}
\phi(y_n, u_{n+1}) \leq\ & \phi(y_{n-1}, u_n) + 2(\|y_{n-1}\|^2 - \|y_n\|^2) + 2\langle y_{n-1} + u_n, Ju_n - Jy_{n-1}\rangle \\
& - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle - 2\langle u_n + y_n, Ju_n - Jy_n\rangle \\
& - 2\beta_n^2\langle u_{n+1} - y_n, Fu_n - v_n\rangle - 2\alpha_n\beta_n\langle u_{n+1} - u_n, Ju_n - Jx\rangle \\
& - \alpha_n\beta_n\phi(y_{n-1}, u_n) - 2\alpha_n\beta_n\langle u_n - y_{n-1}, Jy_{n-1} - Jx\rangle \\
& - 2\alpha_n\beta_n\langle y_{n-1} - y_n, Ju_n - Jx\rangle \\
=\ & \phi(y_{n-1}, u_n) + 2(\|y_{n-1}\|^2 - \|y_n\|^2) + 2\langle y_{n-1} - y_n, Ju_n - Jy_{n-1}\rangle \\
& + 2\langle u_n + y_n, Ju_n - Jy_{n-1}\rangle - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \\
& - 2\langle u_n + y_n, Ju_n - Jy_n\rangle - 2\beta_n^2\langle u_{n+1} - y_n, Fu_n - v_n\rangle \\
& - 2\alpha_n\beta_n\langle u_{n+1} - u_n, Ju_n - Jx\rangle - \alpha_n\beta_n\phi(y_{n-1}, u_n) \\
& - 2\alpha_n\beta_n\langle u_n - y_{n-1}, Jy_{n-1} - Jx\rangle - 2\alpha_n\beta_n\langle y_{n-1} - y_n, Ju_n - Jx\rangle \\
=\ & \phi(y_{n-1}, u_n) - \alpha_n\beta_n\phi(y_{n-1}, u_n) + 2(\|y_{n-1}\|^2 - \|y_n\|^2) \\
& + 2\langle y_{n-1} - y_n, Ju_n - Jy_{n-1}\rangle - 2\langle u_n + y_n, Jy_{n-1} - Jy_n\rangle \\
& - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \boxed{-2\beta_n^2\langle u_{n+1} - y_n, Fu_n - v_n\rangle} \\
& - 2\alpha_n\beta_n\langle u_{n+1} - u_n, Ju_n - Jx\rangle \boxed{-2\alpha_n\beta_n\langle u_n - y_{n-1}, Jy_{n-1} - Jx\rangle} \\
& - 2\alpha_n\beta_n\langle y_{n-1} - y_n, Ju_n - Jx\rangle
\end{aligned} \tag{5.2.16}$$

We now estimate the boxed terms. Using equation (2.2.18) and the fact that F is monotone, we obtain

$$\begin{aligned}
-2\beta_n^2\langle u_{n+1} - y_n, Fu_n - v_n\rangle - 2\alpha_n\beta_n\langle u_n - y_{n-1}, Jy_{n-1} - Jx\rangle \hspace{-6cm} & \\
&= -2\beta_n^2\langle u_{n+1} - u_n, Fu_n - v_n\rangle \\
&\quad - 2\beta_n^2\langle u_n - y_n, Fu_n - v_n\rangle \\
&\quad - 2\alpha_n\beta_n\langle y_n - y_{n-1}, Jy_{n-1} - Jx\rangle \\
&\quad - 2\alpha_n\beta_n\langle u_n - y_n, Jy_{n-1} - Jy_n\rangle \\
&\quad + 2\beta_n\langle u_n - y_n, -\alpha_n(Jy_n - Jx)\rangle \\
&= -2\beta_n^2\langle u_{n+1} - u_n, Fu_n - v_n\rangle \\
&\quad + 2\beta_n^2\langle u_n - y_n, v_n - Fu_n\rangle \\
&\quad - 2\alpha_n\beta_n\langle y_n - y_{n-1}, Jy_{n-1} - Jx\rangle \\
&\quad - 2\alpha_n\beta_n\langle u_n - y_n, Jy_{n-1} - Jy_n\rangle \\
&\quad + 2\beta_n^2\langle u_n - y_n, Fy_n - y_n^*\rangle \\
&\leq -2\beta_n^2\langle u_{n+1} - u_n, Fu_n - v_n\rangle \\
&\quad - 2\alpha_n\beta_n\langle u_n - y_n, Jy_{n-1} - Jy_n\rangle \\
&\quad - 2\alpha_n\beta_n\langle y_n - y_{n-1}, Jy_{n-1} - Jx\rangle \\
&\quad + 2\beta_n^2\langle u_n - y_n, v_n - y_n^*\rangle.
\end{aligned}$$

Thus, substituting this in inequality (5.2.16), and using inequalities (2.2.16), (2.2.17) and (5.2.7), we obtain

$$
\begin{aligned}
\phi(y_n,u_{n+1}) \leq{}& \phi(y_{n-1},u_n) - \alpha_n\beta_n\phi(y_{n-1},u_n) + 2(\|y_{n-1}\|^2 - \|y_n\|^2)\\
&+ 2\langle y_{n-1}-y_n, Ju_n - Jy_{n-1}\rangle - 2\langle u_n + y_n, Jy_{n-1} - Jy_n\rangle\\
&- 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle - 2\alpha_n\beta_n\langle u_{n+1}-u_n, Ju_n - Jx\rangle\\
&- 2\alpha_n\beta_n\langle y_{n-1}-y_n, Ju_n - Jx\rangle - 2\beta_n^2\langle u_{n+1}-u_n, Fu_n - v_n\rangle\\
&- 2\alpha_n\beta_n\langle u_n - y_n, Jy_{n-1} - Jy_n\rangle - 2\alpha_n\beta_n\langle y_n - y_{n-1}, Jy_{n-1} - Jx\rangle\\
&+ 2\beta_n^2\langle u_n - y_n, v_n - y_n^*\rangle\\
\leq{}& (1-\alpha_n\beta_n)\phi(y_{n-1},u_n) + 2N_1(\|y_{n-1}-y_n\| + \|Jy_{n-1} - Jy_n\|)\\
&+ 2\alpha_n\beta_nN_2(\|u_{n+1}-u_n\| + \|y_{n-1}-y_n\| + \|Jy_{n-1}-Jy_n\|)\\
&+ 2\beta_n^2N_3\|u_{n+1}-u_n\| + 2\beta_n^2\langle u_n - y_n, v_n - y_n^*\rangle\\
\leq{}& (1-\alpha_n\beta_n)\phi(y_{n-1},u_n) + 2N_1\Big(c_2\delta_E^{-1}(\eta_n) + c_2\delta_{E^*}^{-1}(\eta_n)\Big)\\
&+ 2\alpha_n\beta_nN_2\Big(c_2\delta_E^{-1}\big(\beta_nM_0\big) + c_2\delta_E^{-1}(\eta_n) + c_2\delta_{E^*}^{-1}(\eta_n)\Big)\\
&+ 2\beta_n^2N_3c_2\delta_E^{-1}(\beta_nM_0) + 2\beta_n^2\langle u_n - y_n, v_n - y_n^*\rangle\\
\leq{}& (1-\alpha_n\beta_n)\phi(y_{n-1},u_n) + \alpha_n\beta_n\widehat{N}\Bigg(c_2\delta_E^{-1}\big(\beta_nM_0\big) + c_2\delta_E^{-1}(\eta_n) + c_2\delta_{E^*}^{-1}(\eta_n)\\
&+ c_2\frac{\delta_E^{-1}\big(\beta_nM_0\big)}{\alpha_n} + c_2\frac{\delta_E^{-1}(\eta_n)}{\alpha_n\beta_n} + c_2\frac{\delta_{E^*}^{-1}(\eta_n)}{\alpha_n\beta_n}\Bigg)\\
&+ 2\beta_n^2\langle u_n - y_n, v_n - y_n^*\rangle,
\end{aligned}
\tag{5.2.17}
$$

for some $N_1, N_2, N_3 > 0$, and $\widehat{N} = \max\{N_1, N_2, N_3\}$.

Similarly, using Lemma 2.2.3, equation 2.2.3, Lemma 2.2.4 and equation (2.2.19), we obtain

$$
\begin{aligned}
\phi\big(y_n^*, v_{n+1}\big) \leq{}& \phi\big(y_{n-1}^*, v_n\big) - \alpha_n\beta_n\phi(y_{n-1}^*, v_n) + 2(\|y_{n-1}^*\|^2 - \|y_n^*\|^2)\\
&+ 2\langle y_{n-1}^* - y_n^*, J^{-1}v_n - J^{-1}y_{n-1}\rangle + 2\langle y_n^* + v_n, J^{-1}y_n - J^{-1}y_{n-1}^*\rangle\\
&- 2\langle y_n^* - v_n, J^{-1}y_{n-1}^* - J^{-1}y_n^*\rangle - 2\alpha_n\beta_n\langle v_{n+1} - v_n, J^{-1}v_n - J^{-1}y\rangle\\
&- 2\alpha_n\beta_n\langle y_{n-1}^* - y_n^*, J^{-1}v_n - J^{-1}y\rangle - 2\beta_n^2\langle v_{n+1} - v_n, Kv_n + u_n\rangle\\
&- 2\alpha_n\beta_n\langle v_n - y_n^*, J^{-1}y_{n-1} - J^{-1}y_n^*\rangle\\
&- 2\alpha_n\beta_n\langle y_{n-1}^* - y_n^*, J^{-1}y_{n-1} - J^{-1}y\rangle + 2\beta_n^2\langle v_n - y_n^*, y_n - u_n\rangle.
\end{aligned}
$$

Since E^* is uniformly convex and uniformly smooth, using inequalities (2.2.16), (2.2.17) and (5.2.8), we obtain that for some $\widehat{N}^* > 0$,

$$
\begin{aligned}
\phi(y_n^*, v_{n+1}) \leq{}& (1-\alpha_n\beta_n)\phi(y_{n-1}^*, v_n) + \alpha_n\beta_n\widehat{N}^*\Bigg(c_2\delta_{E^*}^{-1}(\beta_nM_0^*) + 2c_2\delta_E^{-1}(\eta_n)\\
&+ c_2\frac{\delta_{E^*}^{-1}\big(\beta_nM_0\big)}{\alpha_n} + c_2\frac{\delta_{E^*}^{-1}(\eta_n)}{\alpha_n\beta_n} + c_2\frac{\delta_E^{-1}(\eta_n)}{\alpha_n\beta_n}\Bigg)\\
&+ 2\beta_n^2\langle v_n - y_n^*, y_n - u_n\rangle.
\end{aligned}
\tag{5.2.18}
$$

Let $p_n = (y_n, y_n^*)$, adding inequalities (5.2.17) and (5.2.18) we obtain

$$\begin{aligned}\Phi(p_n, w_{n+1}) \leq (1-\alpha_n\beta_n)\Phi(p_{n-1}, w_n) + \alpha_n\beta_n N\Big(& c_2\delta_E^{-1}\big(\beta_n M_0\big) + 2c_2\delta_E^{-1}(\eta_n) \\ & + 2c_2\delta_{E^*}^{-1}(\eta_n) + c_2\frac{\delta_E^{-1}\big(\beta_n M_0\big)}{\alpha_n} + c_2\frac{\delta_{E^*}^{-1}\big(\beta_n M_0\big)}{\alpha_n} + 2c_2\frac{\delta_E^{-1}(\eta_n)}{\alpha_n\beta_n} \\ & + 2c_2\frac{\delta_{E^*}^{-1}(\eta_n)}{\alpha_n\beta_n} + c_2\delta_{E^*}^{-1}(\beta_n M_0^*)\Big),\end{aligned} \tag{5.2.19}$$

where $N = \max\{\widehat{N}, \widehat{N}^*\}$. Now, setting

$$a_n := \Phi(p_{n-1}, w_n); \qquad \sigma_n := \alpha_n\beta_n \qquad c_n \equiv 0,$$

and

$$\begin{aligned}b_n := N\Big(& c_2\delta_E^{-1}\big(\beta_n M_0\big) + 2c_2\delta_E^{-1}(\eta_n) + 2c_2\delta_{E^*}^{-1}(\eta_n) + c_2\frac{\delta_E^{-1}\big(\beta_n M_0\big)}{\alpha_n} \\ & + c_2\frac{\delta_{E^*}^{-1}\big(\beta_n M_0\big)}{\alpha_n} + 2c_2\frac{\delta_E^{-1}(\eta_n)}{\alpha_n\beta_n} + 2c_2\frac{\delta_{E^*}^{-1}(\eta_n)}{\alpha_n\beta_n} + c_2\delta_{E^*}^{-1}(\beta_n M_0^*)\Big).\end{aligned}$$

Inequality (5.2.19) becomes

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n b_n + c_n, \quad n \geq 1.$$

It follows from Lemma 2.3.11, that $a_n \to 0$ as $n \to \infty$, which implies that $\Phi(p_{n-1}, w_n) \to 0$, as $n \to \infty$. By Lemma 2.3.8, we have $\|w_n - p_{n-1}\|_W \to 0$. Consequently, $\|u_n - y_{n-1}\| \to 0$. Furthermore, since $[y_n, y_n^*] \to [u^*, v^*] \in A^{-1}0$, we have that $\{u_n\}$ converges strongly to a solution of the Hammerstein equation $u + KFu = 0$ with $v^* = Fu^*$. This completes the proof. ■

Corollary 5.2.6 *Let $E = L_p\big(or W_p^m(\Omega)\big)$, $1 < p < \infty$. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone mappings. For arbitrary $x \in E$, $y \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively by: $u_1 \in E$, $v_1 \in E^*$,*

$$\begin{aligned} u_{n+1} &= J^{-1}\big(Ju_n - \beta_n^2(Fu_n - v_n) - \alpha_n\beta_n(Ju_n - Jx)\big), \\ v_{n+1} &= J\big(J^{-1}v_n - \beta_n^2(Kv_n + u_n) - \alpha_n\beta_n(J^{-1}v_n - J^{-1}y)\big), \end{aligned} \tag{5.2.20}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying conditions (i)-(iv). Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of $u + KFu = 0$ with $v^* = Fu^*$.

Proof. $L_p\big(or W_p^m(\Omega)\big)$, $1 < p < \infty$ are uniformly convex and uniformly smooth. Hence, the conclusion follows from Theorem 5.2.5. ■

Corollary 5.2.7 *Let H be a real Hilbert space. Let $F : H \to H$, $K : H \to H$ be maximal monotone mappings. For arbitrary $x, y \in H$, define the sequences $\{u_n\}$ and $\{v_n\}$ in H, by: $u_1, v_1 \in H$,*

$$\begin{aligned} u_{n+1} &= u_n - \beta_n^2(Fu_n - v_n) - \alpha_n\beta_n(u_n - x), \\ v_{n+1} &= v_n - \beta_n^2(Kv_n + u_n) - \alpha_n\beta_n(v_n - y), \end{aligned} \tag{5.2.21}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying conditions (i)-(iv). Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of $u + KFu = 0$ with $v^* = Fu^*$.

Remark 5.2.8 *If $E = L_p$ $(1 < p < \infty)$, real sequences that satisfy the hypothesis of the theorem and corollaries above are $\alpha_n = (n+1)^{-a}$ and $\beta_n = (n+1)^{-b}$ with $0 < b < \frac{1}{r}a$ and $a + b < \frac{1}{r}$, where $r = \max\{p, q\}$. In particular, without loss of generality, let $r = p$. Then, one can choose $a := \frac{1}{p+1}$ and $b := \min\{\frac{1}{2\mathcal{K}}, \frac{1}{2p(p+1)}\}$, where $\mathcal{K}$ is the constant appearing in Remark 2.2.11 (see Appendix A. 10.1.8 for verification).*

5.3 Numerical Illustration

In this section, we present numerical examples to compare the convergence of the sequence generated by our algorithm, algorithm (5.2.13) and algorithms (1.2.10) and (1.2.11).

Example 5.3.1

In Theorems 1.2.7, 1.2.8 and 5.2.5, set $E = \mathbb{R}^2$ then $E^* = \mathbb{R}^2$. Let

$$Fu = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad Kv = \begin{pmatrix} 7 & 2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then, it is easy to see that F and K are continuous and strongly monotone. Thus, K and F are maximal monotone and the vector $u^* = (0,0)^T$ is the only solution of the equation $u + KFu = 0$. In Theorem 1.2.7, we take $\alpha_n = \frac{1}{n}$, in Theorem 1.2.8, we take $\lambda_n = \theta_n = \frac{1}{(n+1)^{\frac{1}{2}}}$ and in Theorem 5.2.5, we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorems 1.2.7, 1.2.8 and 5.2.5, respectively. Setting a tolerance of 10^{-8} and maximum number of iterations $n = 1{,}000$ and choosing $u_1 = (1,0)^T$, $v_1 = (2,1)^T$, we obtain the following results (see Table 5.1 and Figure 5.1 (a)).

Example 5.3.2

In Theorems 1.2.7, 1.2.8 and 5.2.5, set $E = \mathbb{R}^2$ then $E^* = \mathbb{R}^2$. Let

$$Fu = (3u_1 - u_2 + \sin u_1, u_1 + 7u_2 + \sin u_2), \qquad Kv = (5v_1 - 5v_2, 3v_1 + 6v_2).$$

Then, it is easy to see that F and K are continuous and strongly monotone. Thus, K and F are maximal monotone and the vector $u^* = (0,0)^T$ is the only solution of the equation $u + KFu = 0$. In Theorem 1.2.7, we take $\alpha_n = \frac{1}{n}$, in Theorem 1.2.8, we take $\lambda_n = \theta_n = \frac{1}{(n+1)^{\frac{1}{2}}}$ and in Theorem 5.2.5, we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorems 1.2.7, 1.2.8 and 5.2.5, respectively. Setting a tolerance of

10^{-8} and maximum number of iterations $n = 1,000$ and choosing $u_1 = (0, -\frac{1}{4})^T$, $v_1 = (-\frac{1}{2}, 1)^T$ we obtain the following results (see Table 5.2 and Figure 5.1 (b)).

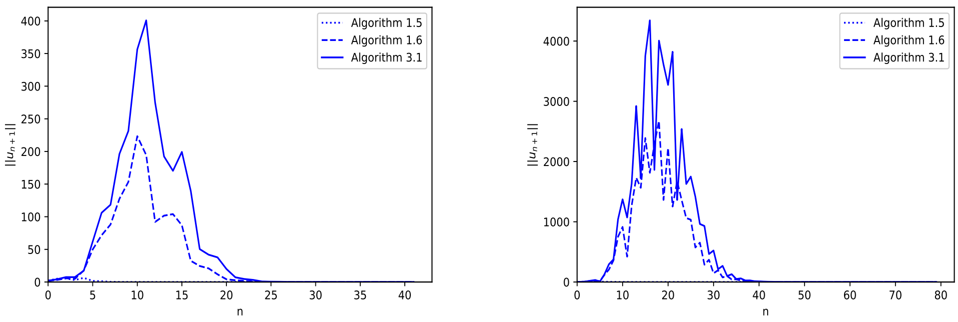
Table 5.1: Numerical results for Example 5.3.1

	Algorithm (1.2.10)		Algorithm (1.2.11)		Algorithm (5.2.13)	
n	$\|u_{n+1}\|$	$\|v_{n+1}\|$	$\|u_{n+1}\|$	$\|v_{n+1}\|$	$\|u_{n+1}\|$	$\|v_{n+1}\|$
1	1	2	1	2	1	2
2	5.0	9.0	3.2426	6.0711	4.138	6.665
3	5.5	24.0	5.7301	21.5393	7.464	24.944
10	0.0331	0.1479	153.338	562.409	231.37	998.396
20	0.0002	0.0007	4.1488	17.0121	37.730	73.156
41	$4.780\times e^{-6}$	$6.6\times e^{-6}$	0.0862	0.0571	$3.323\times e^{-8}$	$1.064\times e^{-7}$
50	$2.377\times e^{-6}$	$2.387\times e^{-6}$	0.0784	0.0519	successful	
100	$1.703\times e^{-7}$	$2.438\times e^{-8}$	0.0561	0.0371	successful	
116	$9.703\ \times e^{-8}$	$2.043\times e^{-8}$	0.0519	0.0343	successful	
200	successful		0.03993	0.0263	successful	
500	successful		0.02539	0.0167	successful	
1,000	successful		0.0179	0.0118	successful	

Table 5.2: Numerical results for Example 5.3.1

	Algorithm (1.2.10)		Algorithm (1.2.11)		Algorithm (5.2.13)	
n	$\|u_{n+1}\|$	$\|v_{n+1}\|$	$\|u_{n+1}\|$	$\|v_{n+1}\|$	$\|u_{n+1}\|$	$\|v_{n+1}\|$
1	0.25	1	0.25	1	0.25	1
2	2.747	7.0	1.869	4.8033	2.018	5.101
3	8.31	18.25	7.7957	16.312	8.738	19.059
10	0.683	4.419	914.49	2771.05	1040.71	3389.62
20	0.0224	0.0409	1362.57	8426.81	3612.04	13675.89
50	$6.307\times e^{-5}$	0.0004	0.0929	0.2106	0.0718	0.2818
80	$1.678\times e^{-5}$	$2.23\times e^{-5}$	0.0015	0.0161	$9.382\times e^{-8}$	$2.875\times e^{-8}$
383	$1.106\ \times e^{-8}$	$5.064\times e^{-9}$	0.0007	0.0075	successful	
500	successful		0.0006	0.0065	successful	
1,000	successful		0.0004	0.0046	successful	

In all the graphs sketched below, the y-axis represents the values of $\|u_{n+1} - \mathbf{0}\|$ while the x-axis represents the number of iterations n.



(a) Graph of the first 41 iterates of algorithms (1.2.10), (1.2.11) and (5.2.13), choosing $u_1 = (1, 0)^T$, $v_1 = (2, 1)^T$

(b) Graph of the first 80 iterates of algorithms (1.2.10), (1.2.11) and (5.2.13), choosing $u_1 = (0, -\frac{1}{4})^T$, $v_1 = (-\frac{1}{2}, 1)^T$

Figure 5.1: Graphical illustration of the data in Tables 5.1 and 5.2

In Theorems 1.2.8 and 5.2.5, set $E = L_p([0, 1])$, $\quad E^* = L_q([0, 1])$, $\quad 1 < p < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

$$\begin{aligned} F : L_p([0, 1]) &\to L_q([0, 1]) \\ u \quad &\longmapsto (Fu)(t) = Ju(t). \end{aligned}$$

It is well known that the normalized duality map J is monotone and uniformly continuous on bounded subsets of L_p and thus, maximal monotone. Let

$$\begin{aligned} K : L_q([0, 1]) &\to L_p([0, 1]) \\ v \quad &\longmapsto (Kv)(t) = tv(t). \end{aligned}$$

Since $L_q([0, 1]) \subset L_p([0, 1])$, K is well-defined. Observe that by definition K is linear. Next we show that K is monotone. Let $v, w \in L_q([0, 1])$, then

$$\begin{aligned} \big\langle (Kv)(t) - (Kw)(t), v(t) - w(t) \big\rangle &= \int_0^1 \big(tv(t) - tw(t)\big)\big(v(t) - w(t)\big)dt \\ &= \int_0^1 t\big(v(t) - w(t)\big)^2 dt \geq 0. \end{aligned}$$

Hence, K is monotone. Furthermore, since K is continuous, K is maximal monotone and the function $u^*(t) = (0, 0)^T$ is the only solution of the equation $u + KFu = 0$. For the numerical experiments, in Examples 3 and 4 below, in Theorem 1.2.8, we take $\lambda_n = \theta_n = \frac{1}{(n+1)^{\frac{1}{2}}}$ and in Theorem 5.2.5, we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorems 1.2.8 and 5.2.5, respectively. We set a tolerance of 10^{-2} and maximum number of iterations $n = 10$.

Example 5.3.3 *Taking $p = 1.5$, we have $E = L_{1.5}([0, 1])$, $\quad E^* = L_3([0, 1])$*

$$\begin{aligned} F : L_{1.5}([0, 1]) &\to L_3([0, 1]) \\ u \quad &\longmapsto (Fu)(t) = Ju(t), \end{aligned} \qquad \begin{aligned} K : L_3([0, 1]) &\to L_{1.5}([0, 1]) \\ v \quad &\longmapsto (Kv)(t) = tv(t). \end{aligned}$$

Choosing $u_1(t) = t^2 + 1$ and $v_1(t) = \cos t \exp(-t)$, we obtain the following results (see Table 5.3 and Figure 5.2 (a)).

Example 5.3.4 *Taking $p = \frac{5}{3}$, we have $E = L_{\frac{5}{3}}([0, 1])$, $E^* = L_{2.5}([0, 1])$*

$$\begin{aligned} F : L_{\frac{5}{3}}([0,1]) &\to L_{2.5}([0,1]) \\ u &\longmapsto (Fu)(t) = Ju(t) \end{aligned} \qquad \begin{aligned} K : L_{2.5}([0,1]) &\to L_{\frac{5}{3}}([0,1]) \\ v &\longmapsto (Kv)(t) = tv(t). \end{aligned}$$

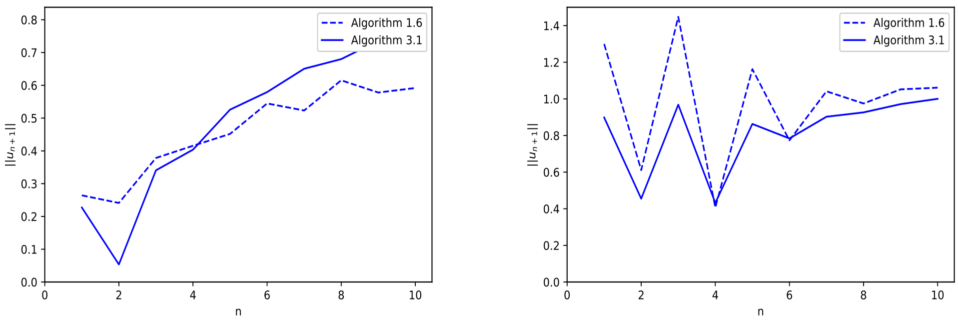
Choosing $u_1(t) = \frac{1}{1+x\sin x}$ and $v_1(t) = \exp(t)$, we obtain the following results (see Table 5.4 and Figure 5.2 (b)).

Table 5.3: Numerical results for Example 5.3.3

	Algorithm (1.2.11) Time=38,168sec		Algorithm (5.2.13) Time=33,342sec	
n	$\|u_{n+1}\|$	$\|v_{n+1}\|$	$\|u_{n+1}\|$	$\|v_{n+1}\|$
1	0.2644	0.4005	0.2273	0.1152
2	0.2411	0.4674	0.0535	0.3433
3	0.3784	0.3742	0.3407	0.2467
4	0.4156	0.4267	0.4038	0.3151
5	0.4518	0.3939	0.5258	0.3004
6	0.5447	0.4664	0.5794	0.3319
7	0.5232	0.3954	0.6505	0.3273
8	0.6153	0.4914	0.6800	0.3405
9	0.5780	0.3874	0.7354	0.3504
10	0.5918	0.3074	0.8007	0.2724

Table 5.4: Numerical results for Example 5.3.4

	Algorithm (1.2.11) Time=17,700sec		Algorithm (5.2.13) Time=14,223sec	
n	$\|u_{n+1}\|$	$\|v_{n+1}\|$	$\|u_{n+1}\|$	$\|v_{n+1}\|$
1	1.2997	0.8496	0.8985	0.7560
2	0.6109	1.8814	0.4550	1.5637
3	1.4480	0.4603	0.9682	0.2581
4	0.4061	1.3936	0.4318	1.0630
5	1.1624	0.7364	0.8631	0.7177
6	0.7735	1.1298	0.7839	0.9367
7	1.0413	0.9851	0.9025	0.9110
8	0.9748	1.0935	0.9261	0.9713
9	1.0519	1.0873	0.9713	0.9977
10	1.0611	1.1242	1.0002	1.0301



(a) Graph of the first 10 iterates of algorithms (1.2.11) and (5.2.13), choosing $u_1(t) = t^2 + 1$ and $v_1(t) = \cos t \exp(-t)$

(b) Graph of the first 10 iterates of algorithms (1.2.11) and (5.2.13), choosing $u_1(t) = \frac{1}{1+x \sin x}$ and $v_1(t) = \exp(t)$

Figure 5.2: Graphical illustration of the data in Tables 5.3 and 5.4

Remark 5.3.5 *From the numerical experiments above, we see that the algorithm (5.2.13) is more robust and efficient than algorithms (1.2.10) and (1.2.11), and converges faster in terms of number of iterations and CPU time in all the problems tested.*

Conclusion. In this Chapter, an iterative algorithm that extends the results of [Chidume and Shehu, 2015], and complements the results of [Uba et al., 2017] is constructed. Strong convergence of the sequence generated by the algorithm is proved in a uniformly convex and uniformly smooth real Banach space. The theorem proved is a significant improvement of the results of [Chidume and Shehu, 2015] which was proved in real Hilbert spaces under the assumption that F and K are *continuous* and *bounded*. These restrictions on K and F have been dispensed with even in the more general setting considered here. Finally, numerical experiments are presented to demonstrate the convergence of the sequence of the proposed algorithm.

The results obtained in this chapter have gained publication, and appeared in the following paper: C.E. Chidume, **A. Adamu** and L.O. Chinwendu, *Iterative algorithms for solutions of Hammerstein equations in real Banach spaces,* **Fixed Point Theory and Applications,** https://doi.org/10.1186/s13663-020-0670-7.

CHAPTER 6

APPROXIMATION OF SOLUTIONS OF HAMMERSTEIN EQUATIONS WITH MONOTONE MAPPINGS IN REAL BANACH SPACES

6.1 Introduction

In this chapter, an iterative algorithm is constructed and the sequence of the algorithm is proved to converge strongly to a solution of the Hammerstein equation $u + KFu = 0$. This theorem is a significant improvement of some important recent results which were proved in L_p spaces, $1 < p \leq 2$ under the assumption that F and K are *bounded*. This restriction on F and K have been dispensed with even in the more general setting considered here. Finally, a numerical experiment is presented to illustrate the convergence of the sequence of the algorithm which is found to be much faster, in terms of the number of iterations and the computational time than the convergence obtained with existing algorithms.

6.2 Main Results

In Theorem 6.2.1 below, the sequences $\{\alpha_n\}$ and $\{\theta_n\}$ are in $(0,1)$ and are assumed to satisfy the following conditions:

(i) $\delta_E^{-1}(\alpha_n M_0) \leq \theta_n \gamma_0; \quad \alpha_n M_1 \leq \theta_n \gamma_0,$

(ii) $\delta_{E^*}^{-1}(\alpha_n M_0^*) \leq \theta_n \gamma_0; \quad \alpha_n M_1^* \leq \theta_n \gamma_0,$

for all $n \geq 1$ and for some constants, $M_0, M_0^*, M_1, M_1^*, \gamma_0 > 0$.

Theorem 6.2.1 *Let E be a uniformly convex and uniformly smooth real Banach space. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone mappings. For $u_1 \in E$, $v_1 \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively by*

$$u_{n+1} = J^{-1}\Big(Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n Ju_n\Big), \tag{6.2.1}$$

$$v_{n+1} = J\Big(J^{-1}v_n - \alpha_n(Kv_n + u_n) - \alpha_n\theta_n J^{-1}v_n\Big). \tag{6.2.2}$$

Assume that the equation $u + KFu = 0$ has a solution u^*, with $v^* = Fu^*$. Then, the sequences $\{u_n\}$ and $\{v_n\}$ are bounded.

Proof. To show that the sequences $\{u_n\}$ and $\{v_n\}$ are bounded, set $w_n = (u_n, v_n), w^* = (u^*, v^*) \in W = E \times E^*$, where u^* is a solution of the Hammerstein equation $u + KFu = 0$, with $v^* = Fu^*$.

Define $\Phi : W \times W \to \mathbb{R}$ by

$$\Phi(w_1, w_2) = \phi(u_1, u_2) + \phi(v_1, v_2),$$

where $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$.

Let W be endowed with norm

$$\|(u, v)\|_W = \left(\|u\|_E^2 + \|v\|_{E^*}^2\right)^{\frac{1}{2}}.$$

It suffices to show that $\{w_n\}$ is bounded. We show this by induction. Let $w_1 \in W$. Then there exists $r > 0$ such that $\|w^*\|_W \leq \frac{r}{8}$, $\Phi(w^*, w_1) \leq \frac{r}{4}$ and $B := \big\{w = (u, v) \in W : \Phi(w^*, w) \leq r\big\}$. It suffices to show that $\Phi(w^*, w_n) \leq r$, for all $n \geq 1$. Let $w, w_1 \in B$ and $\alpha, \beta \in (0, 1)$. Then,

$$\Phi(w^*, w) \leq r \qquad i.e., \quad \phi(u^*, u) + \phi(v^*, v) \leq r.$$

$$\text{Therefore,} \qquad \phi(u^*, u) \leq r \qquad \text{and} \qquad \phi(v^*, v) \leq r.$$

Now, using inequality (2.2.2),

$$\phi(u^*, u) \leq r \quad \Rightarrow \quad \|u\| \leq \|u^*\| + \sqrt{r}.$$

Since F is also locally bounded at $0 \in E$, there exist $h_0 > 0$ with $h_0 < \sqrt{r}$, $m_0 > 0$ such that $\|Fx\| \leq m_0$, $\forall\ x \in B_{h_0}(0) := \{x \in E : \|x\| < h_0\}$. Let $(u, v) \in B$, then $u \in E$, and let $x \in B_{h_0}(0)$. By monotonicity of F, we have that:

$$\langle u, Fu \rangle \geq \langle u - x, Fx \rangle + \langle x, Fu \rangle$$

implies

$$\langle -u, Fu \rangle \leq \langle x - u, Fx \rangle - \langle x, Fu \rangle.$$

Setting $z = -u$, we have that

$$\begin{aligned} \langle z, Fu \rangle &\leq \langle x + z, Fx \rangle - \langle x, Fu \rangle \\ &\leq \|Fx\|(\|x\| + \|z\|) + \|Fu\|\|x\| \\ &\leq m_0(h_0 + \|u^*\| + \sqrt{r}) + h_0\|Fu\|. \end{aligned} \tag{6.2.3}$$

Similarly, replacing z by $-z$ in inequality (6.2.3), we obtain that

$$-\langle z, Fu \rangle \leq m_0(h_0 + \|u^*\| + \sqrt{r}) + h_0\|Fu\|.$$

Thus,

$$|\langle z, Fu \rangle| \leq m_0(h_0 + \|u^*\| + \sqrt{r}) + h_0\|Fu\|.$$

Hence,

$$\sup_{\|z\|\le\sqrt{r}+\|u^*} |\langle z, Fu\rangle| \le m_0(h_0+\|u^*\|+\sqrt{r})+h_0\|Fu\|$$
$$(\sqrt{r}+\|u^*\|)\|Fu\| \le m_0(h_0+\|u^*\|+\sqrt{r})+h_0\|Fu\|$$
$$\|Fu\| \le \frac{m_0\,(h_0+\|u^*\|+\sqrt{r})}{\|u^*\|+\sqrt{r}-h_0} := k_1, \quad \forall\ u\in E \ \text{ such that } (u,v)\in B.$$

Define $\sigma := \max\{k_1, \|u^*\|+\sqrt{r}\}$. Hence, $\langle u, Fu\rangle \le \sigma\|u\|$ and $\|u\|\le\sigma$. By Lemma 5.2.2, F is quasi-bounded. Thus, there exists $\tau_1>0$ such that

$$\|Fu\| \le \tau_1, \quad \forall u\in E, \text{such that } (u,v)\in B.$$

Similarly, there exists $\tau_2>0$ such that

$$\|Kv\| \le \tau_2, \quad \forall v\in E^*, \text{such that} (u,v)\in B.$$

Define:

$$M_1 = \sup\Big\{\|Fu-v+\theta Ju\|,\ \|Ju-\alpha(Fu-v+\theta Ju)\|\Big\}+1;$$
$$M_1^* = \sup\Big\{\|Kv+u+\theta J^{-1}v\|,\ \|Jv-\alpha(Kv+u+\theta J^{-1}v)\|\Big\}+1;$$
$$M_2 = \sup\Big\{\|u-u^*\|\Big\}+1; \qquad M_2^* = \sup\Big\{\|v-v^*\|\Big\}+1;$$

Let $M := \max\{c_2M_1, c_2M_1^*, M_2, M_2^*\}$, $\gamma_0 := \min\Big\{1, \frac{r}{16M}\Big\}$. Then, for $n=1$, by construction $\Phi(w^*,w_1)\le r$. Assume $\Phi(w^*,w_n)\le r$, for some $n\ge 1$, i.e., $\phi(u^*,u_n)+\phi(v^*,v_n)\le r$, $\ \ for\ some\ n\ge 1$. We show that $\Phi(w^*,w_{n+1})\le r$. For contradiction, suppose $r<\Phi(w^*,w_{n+1})$. Observe that

$$\|u_{n+1}-u_n\| = \|J^{-1}\Big(Ju_n-\alpha_n(Fu_n-v_n)-\alpha_n\theta_nJu_n\Big)-J^{-1}(Ju_n)\|.$$

Now, using Lemma 2.2.5 and recurrence relation (6.2.1), we have

$$\begin{aligned}(2L)^{-1}\delta_E(c_2^{-1}\|u_{n+1}-u_n\|) &\le \langle Ju_{n+1}-Ju_n, u_{n+1}-u_n\rangle\\ &\le \|Ju_{n+1}-Ju_n\|\|u_{n+1}-u_n\| \le \alpha_nM_1\|u_{n+1}-u_n\|.\end{aligned}$$

Thus,

$$\|u_{n+1}-u_n\| \le c_2\delta_E^{-1}(\alpha_nM_0), \quad for\ some\ M_0>0. \tag{6.2.4}$$

Similarly, using Lemma 2.2.6 and recurrence relation (6.2.2), we obtain

$$\|v_{n+1}-v_n\| \le c_2\delta_{E^*}^{-1}(\alpha_nM_0^*), \quad for\ some\ M_0^*>0. \tag{6.2.5}$$

Now, using recurrence relation (6.2.1), Lemma 2.2.3, and inequality (6.2.4), we have

$$\begin{aligned}\phi(u^*, u_{n+1}) &= V(u^*, Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n Ju_n) \\ &\leq V(u^*, Ju_n) - 2\langle u_{n+1} - u^*, \alpha_n(Fu_n - v_n) + \alpha_n\theta_n Ju_n\rangle \\ &= \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, Fu_n - v_n + \theta_n Ju_n\rangle \\ &\quad - 2\alpha_n\langle u_{n+1} - u_n, Fu_n - v_n + \theta_n Ju_n\rangle \\ &\leq \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, Fu_n - v_n + \theta_n Ju_n\rangle \\ &\quad + 2\alpha_n\|u_{n+1} - u_n\|\|Fu_n - v_n + \theta_n Ju_n\| \\ &\leq \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, Fu_n - v_n + \theta_n Ju_n\rangle \\ &\quad + 2\alpha_n c_2 M_1 \delta_E^{-1}(\alpha_n M_0). \end{aligned} \tag{6.2.6}$$

Observe that by monotonicity of F and the fact that $v^* = Fu^*$, we have

$$\langle u_n - u^*, Fu_n - v_n + \theta_n Ju_n\rangle \geq \langle u_n - u^*, v^* - v_n + \theta_n Ju_n\rangle.$$

Thus, substituting this in inequality (6.2.6), we have

$$\begin{aligned}\phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, v^* - v_n + \theta_n Ju_n\rangle + 2\alpha_n c_2 M_1\delta_E^{-1}(\alpha_n M_0) \\ &= \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, v^* - v_n\rangle - 2\alpha_n\theta_n\langle u_n - u^*, Ju_n - Ju_{n+1}\rangle \\ &\quad - 2\alpha_n\theta_n\langle u_n - u^*, Ju_{n+1}\rangle + 2\alpha_n c_2 M_1\delta_E^{-1}(\alpha_n M_0) \\ &= \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, v^* - v_n\rangle - 2\alpha_n\theta_n\langle u_n - u^*, Ju_n - Ju_{n+1}\rangle \\ &\quad - 2\alpha_n\theta_n\langle u_{n+1} - u^*, Ju_{n+1}\rangle - 2\alpha_n\theta_n\langle u_n - u_{n+1}, Ju_{n+1}\rangle \\ &\quad + 2\alpha_n c_2 M_1\delta_E^{-1}(\alpha_n M_0). \end{aligned} \tag{6.2.7}$$

Using Lemma 2.2.4 (with $z = u_{n+1},\ y = u^*$ and $x = 0$), we have that

$$-2\alpha_n\theta_n\langle u_{n+1} - u^*, Ju_{n+1}\rangle \leq \alpha_n\theta_n\|u^*\|^2 - \alpha_n\theta_n\phi(u^*, u_{n+1}).$$

Substituting this in inequality (6.2.7), we obtain

$$\begin{aligned}\phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) - 2\alpha_n\langle u_n - u^*, v^* - v_n\rangle - 2\alpha_n\theta_n\langle u_n - u^*, Ju_n - Ju_{n+1}\rangle \\ &\quad - 2\alpha_n\theta_n\langle u_n - u_{n+1}, Ju_{n+1}\rangle + \alpha_n\theta_n\|u^*\|^2 - \alpha_n\theta_n\phi(u^*, u_{n+1}) \\ &\quad + 2\alpha_n c_2 M_1\delta_E^{-1}(\alpha_n M_0) \\ &\leq \phi(u^*, u_n) - \alpha_n\theta_n\phi(u^*, u_{n+1}) + \alpha_n\theta_n\|u^*\|^2 + 2\alpha_n\theta_n\|u_n - u^*\| \\ &\quad \times \|Ju_n - Ju_{n+1}\| + 2\alpha_n\theta_n\|u_n - u_{n+1}\|\|Ju_{n+1}\| \\ &\quad + 2\alpha_n c_2 M_1\delta_E^{-1}(\alpha_n M_0) - 2\alpha_n\langle u_n - u^*, v^* - v_n\rangle \\ &\leq \phi(u^*, u_n) - \alpha_n\theta_n\phi(u^*, u_{n+1}) + \alpha_n\theta_n\|u^*\|^2 + 2\alpha_n\theta_n M_2(\alpha_n M_1) \\ &\quad + 2\alpha_n\theta_n c_2 M_1\delta_E^{-1}(\alpha_n M_0) + 2\alpha_n c_2 M_1\delta_E^{-1}(\alpha_n M_0) \\ &\quad - 2\alpha_n\langle u_n - u^*, v^* - v_n\rangle. \end{aligned} \tag{6.2.8}$$

Similarly, using recurrence relation (6.2.2), Lemma 2.2.3, inequality (6.2.5), monotonicity of K, the fact that $Kv^* = -u^*$ and Lemma 2.2.4, we obtain

$$\begin{aligned}\phi(v^*, v_{n+1}) &\leq \phi(v^*, v_n) - \alpha_n\theta_n\phi(v^*, v_{n+1}) + \alpha_n\theta_n\|v^*\|^2 + 2\alpha_n\theta_n M_2^*(\alpha_n M_1^*) \\ &\quad + 2\alpha_n\theta_n c_2 M_1^*\delta_{E^*}^{-1}(\alpha_n M_0^*) + 2\alpha_n c_2 M_1^*\delta_{E^*}^{-1}(\alpha_n M_0^*) \\ &\quad - 2\alpha_n\langle v_n - v^*, u_n - u^*\rangle. \end{aligned} \tag{6.2.9}$$

Thus, adding inequalities (6.2.8) and (6.2.9), we obtain

$$\begin{aligned}
r < \Phi(w^*, w_{n+1}) &= \phi(u^*, u_{n+1}) + \phi(v^*, v_{n+1}) \\
&\leq \Phi(w^*, w_n) - \alpha_n\theta_n\Phi(w^*, w_{n+1}) + \alpha_n\theta_n\|w^*\|_W^2 \\
&\quad + 2\alpha_n\theta_n M_2(\alpha_n M_1) + 2\alpha_n c_2 M_1\delta_E^{-1}(\alpha_n M_0) \\
&\quad + 2\alpha_n\theta_n c_2 M_1\delta_E^{-1}(\alpha_n M_0) + 2\alpha_n\theta_n M_2^*(\alpha_n M_1^*) \\
&\quad + 2\alpha_n\theta_n c_2 M_1^*\delta_{E^*}^{-1}(\alpha_n M_0^*) + 2\alpha_n c_2 M_1\delta_E^{-1}(\alpha_n M_0^*) \\
&\leq \Phi(w^*, w_n) - \alpha_n\theta_n\Phi(w^*, w_{n+1}) + \alpha_n\theta_n\|w^*\|_W^2 \\
&\quad + 2\alpha_n\theta_n^2 M_2\gamma_0 + 2\alpha_n\theta_n\gamma_0 c_2 M_1 + 2\alpha_n\theta_n^2 c_2 M_1\gamma_0 \\
&\quad + 2\alpha_n\theta_n^2 M_2^*\gamma_0 + 2\alpha_n\theta_n^2 c_2 M_1^*\gamma_0 + 2\alpha_n\theta_n\gamma_0 c_2 M_1^* \\
&\leq \Phi(w^*, w_n) - \alpha_n\theta_n\Phi(w^*, w_{n+1}) + \alpha_n\theta_n\|w^*\|_W^2 \\
&\quad + 2\alpha_n\theta_n M\gamma_0 + 2\alpha_n\theta_n M\gamma_0 + 2\alpha_n\theta_n M\gamma_0 \\
&\quad + 2\alpha_n\theta_n M\gamma_0 + 2\alpha_n\theta_n M\gamma_0 + 2\alpha_n\theta_n M\gamma_0 \\
&\leq r - \alpha_n\theta_n r + \frac{7}{8}\alpha_n\theta_n r = r - \frac{1}{8}\alpha_n\theta_n r < r.
\end{aligned}$$

This is a contradiction. Hence, $\Phi(w^*, w_{n+1}) \leq r$. Thus, $\Phi(w^*, w_n) \leq r$, for all $n \geq 1$. Consequently, we have $\phi(u^*, u_n) \leq r$ and $\phi(v^*, v_n) \leq r$, for all $n \geq 1$. Therefore, using inequality (2.2.2), we deduce that $\{u_n\}$ and $\{v_n\}$ are bounded.

■

In Theorem 6.2.2 below, $\{\alpha_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty}\alpha_n\theta_n = \infty$, (ii) $\delta_E^{-1}(\alpha_n M_0) \leq \theta_n^2\gamma_0$, (iii) $\delta_{E^*}^{-1}(\alpha_n M_0^*) \leq \theta_n^2\gamma_0$,

(iv) $\delta_E^{-1}(\eta_n) \to 0$; $\delta_{E^*}^{-1}(\eta_n) \to 0$, (v) $\frac{\delta_E^{-1}(\eta_n)}{\alpha_n\theta_n} \to 0$; $\frac{\delta_{E^*}^{-1}(\eta_n)}{\alpha_n\theta_n} \to 0$,

where $\eta_n = \left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\mathcal{K}\right)$, where $\mathcal{K}$ is the constant appearing in Remark 2.2.11.

Theorem 6.2.2 *Let E be a uniformly convex and uniformly smooth real Banach space. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone mappings. For $u_1 \in E$, $v_1 \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively by*

$$\begin{aligned}
u_{n+1} &= J^{-1}\Big(Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n Ju_n\Big), \\
v_{n+1} &= J\Big(J^{-1}v_n - \alpha_n(Kv_n + u_n) - \alpha_n\theta_n J^{-1}v_n\Big),
\end{aligned} \tag{6.2.10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying conditions (i)-(v). Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of $u + KFu = 0$ with $v^* = Fu^*$.

Proof. Let y_n be as defined in Remark 2.2.8. Using Lemma 2.2.3 and equation 2.2.3, we have

$$\begin{aligned}
\phi(y_n, u_{n+1}) &= V(y_n, Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n Ju_n) \\
&\leq V(y_n, Ju_n) - 2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n + \theta_n Ju_n\rangle \\
&= \phi(u_n, y_n) - 2\langle u_n + y_n, Ju_n - Jy_n\rangle + 2(\|u_n\|^2 - \|y_n\|^2) \\
&\quad - 2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n + \theta_n Ju_n\rangle
\end{aligned} \tag{6.2.11}$$

Observe that

$$\begin{aligned}
\phi(u_n, y_n) = V(u_n, Jy_n) &= V(u_n, Jy_{n-1} + Jy_n - Jy_{n-1}) \\
&\leq V(u_n, Jy_{n-1}) - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle.
\end{aligned}$$

Thus, substituting this in inequality (6.2.11), and using equation (2.2.3) and Lemma 2.2.3 we obtain

$$\begin{aligned}
\phi(y_n, u_{n+1}) &\leq V(u_n, Jy_{n-1}) - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \\
&\quad - 2\langle u_n + y_n, Ju_n - Jy_n\rangle + 2(\|u_n\|^2 - \|y_n\|^2) \\
&\quad - 2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n + \theta_n Ju_n\rangle \\
&= \phi(y_{n-1}, u_n) + 2\langle y_{n-1} + u_n, Ju_n - Jy_{n-1}\rangle \\
&\quad + 2(\|y_{n-1}\|^2 - \|y_n\|^2) - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \\
&\quad - 2\langle u_n + y_n, Ju_n - Jy_n\rangle - 2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n\rangle \\
&\quad - 2\alpha_n\theta_n\langle u_{n+1} - u_n, Ju_n\rangle - 2\alpha_n\theta_n\langle u_n - y_{n-1}, Ju_n - Jy_{n-1}\rangle \\
&\quad - 2\alpha_n\theta_n\langle u_n - y_{n-1}, Jy_{n-1}\rangle - 2\alpha_n\theta_n\langle y_{n-1} - y_n, Ju_n\rangle \\
&\leq \phi(y_{n-1}, u_n) + 2\langle y_{n-1} + u_n, Ju_n - Jy_{n-1}\rangle + 2(\|y_{n-1}\|^2 - \|y_n\|^2) \\
&\quad - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle - 2\langle u_n + y_n, Ju_n - Jy_n\rangle \\
&\quad - 2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n\rangle - 2\alpha_n\theta_n\langle u_{n+1} - u_n, Ju_n\rangle \\
&\quad - \alpha_n\theta_n\phi(y_{n-1}, u_n) - 2\alpha_n\theta_n\langle u_n - y_{n-1}, Jy_{n-1}\rangle \\
&\quad - 2\alpha_n\theta_n\langle y_{n-1} - y_n, Ju_n\rangle \\
&= (1 - \alpha_n\theta_n)\phi(y_{n-1}, u_n) + 2(\|y_{n-1}\|^2 - \|y_n\|^2) \\
&\quad + 2\langle y_{n-1} - y_n, Ju_n - Jy_{n-1}\rangle - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \\
&\quad - 2\langle u_n + y_n, Jy_{n-1} - Jy_n\rangle \underline{-2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n\rangle} \\
&\quad - 2\alpha_n\theta_n\langle u_{n+1} - u_n, Ju_n\rangle \underline{-2\alpha_n\theta_n\langle u_n - y_{n-1}, Jy_{n-1}\rangle} \\
&\quad - 2\alpha_n\theta_n\langle y_{n-1} - y_n, Ju_n\rangle
\end{aligned} \tag{6.2.12}$$

We now estimate the underlined terms. Using equation (2.2.20) and the fact that F is monotone, we obtain

$$\begin{aligned}
&-2\alpha_n\langle u_{n+1} - y_n, Fu_n - v_n\rangle - 2\alpha_n\theta_n\langle u_n - y_{n-1}, Jy_{n-1}\rangle \\
&\qquad = -2\alpha_n\langle u_{n+1} - u_n, Fu_n - v_n\rangle - 2\alpha_n\langle u_n - y_n, Fu_n - v_n\rangle \\
&\qquad\quad - 2\alpha_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1}\rangle - 2\alpha_n\theta_n\langle u_n - y_n, Jy_{n-1} - Jy_n\rangle \\
&\qquad\quad + 2\alpha_n\langle u_n - y_n, Fy_n - y_n^*\rangle \\
&\qquad \leq -2\alpha_n\langle u_{n+1} - u_n, Fu_n - v_n\rangle - 2\alpha_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1}\rangle \\
&\qquad\quad - 2\alpha_n\theta_n\langle u_n - y_n, Jy_{n-1} - Jy_n\rangle + 2\alpha_n\langle u_n - y_n, v_n - y_n^*\rangle.
\end{aligned}$$

Thus, substituting this in inequality (6.2.12), and using inequalities (2.2.9), (2.2.10) and (6.2.4), we obtain

$$\begin{aligned}
\phi(y_n, u_{n+1}) &\leq (1-\alpha_n\theta_n)\phi(y_{n-1}, u_n) + 2(\|y_{n-1}\|^2 - \|y_n\|^2) \\
&\quad + 2\langle y_{n-1} - y_n, Ju_n - Jy_{n-1}\rangle - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \\
&\quad - 2\langle u_n + y_n, Jy_{n-1} - Jy_n\rangle - 2\alpha_n\theta_n\langle u_{n+1} - u_n, Ju_n\rangle \\
&\quad - 2\alpha_n\theta_n\langle y_{n-1} - y_n, Ju_n\rangle - 2\alpha_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1}\rangle \\
&\quad - 2\alpha_n\theta_n\langle u_n - y_n, Jy_{n-1} - Jy_n\rangle - 2\alpha_n\langle u_{n+1} - u_n, Fu_n - v_n\rangle \\
&\quad + 2\alpha_n\langle u_n - y_n, v_n - y_n^*\rangle \\
&\leq (1-\alpha_n\theta_n)\phi(y_{n-1}, u_n) + 2N_1(\|y_{n-1} - y_n\| + \|Jy_{n-1} - Jy_n\|) \\
&\quad + 2\alpha_n\theta_n N_2(\|u_{n+1} - u_n\| + \|y_{n-1} - y_n\| + \|Jy_{n-1} - Jy_n\|) \\
&\quad + 2\alpha_n N_3\|u_{n+1} - u_n\| + 2\alpha_n\langle u_n - y_n, v_n - y_n^*\rangle \\
&\leq (1-\alpha_n\theta_n)\phi(y_{n-1}, u_n) + 2N_1\Big(c_2\delta_E^{-1}(\eta_n) + c_2\delta_{E^*}^{-1}(\eta_n)\Big) \\
&\quad + 2\alpha_n\theta_n N_2\Big(c_2\delta_E^{-1}\big(\alpha_n M_0\big) + c_2\delta_E^{-1}(\eta_n) + c_2\delta_{E^*}^{-1}(\eta_n)\Big) \\
&\quad + 2\alpha_n N_3 c_2\delta_E^{-1}(\alpha_n M_0) + 2\alpha_n\langle u_n - y_n, v_n - y_n^*\rangle \\
&\leq (1-\alpha_n\theta_n)\phi(y_{n-1}, u_n) + \alpha_n\theta_n\widehat{N}\Bigg(c_2\delta_E^{-1}\big(\alpha_n M_0\big) + c_2\delta_E^{-1}(\eta_n) \\
&\quad + c_2\delta_{E^*}^{-1}(\eta_n) + c_2\frac{\delta_E^{-1}(\alpha_n M_0)}{\theta_n} + c_2\frac{\delta_E^{-1}(\eta_n)}{\alpha_n\theta_n} + c_2\frac{\delta_{E^*}^{-1}(\eta_n)}{\alpha_n\theta_n}\Bigg) \\
&\quad + 2\alpha_n\langle u_n - y_n, v_n - y_n^*\rangle,
\end{aligned} \tag{6.2.13}$$

for some $N_1, N_2, N_3 > 0$, and $\widehat{N} = \max\{N_1, N_2, N_3\}$. Similarly, using Lemma 2.2.3 equation 2.2.3, Lemma 2.2.4 and equation (2.2.21), we obtain

$$\begin{aligned}
\phi(y_n^*, v_{n+1}) &\leq (1-\alpha_n\theta_n)\phi(y_{n-1}^*, v_n) + 2(\|y_{n-1}^*\|^2 - \|y_n^*\|^2) \\
&\quad + 2\langle y_{n-1}^* - y_n^*, J^{-1}v_n - J^{-1}y_{n-1}\rangle + 2\langle y_n^* + v_n, J^{-1}y_n - J^{-1}y_{n-1}^*\rangle \\
&\quad - 2\langle y_n^* - v_n, J^{-1}y_{n-1}^* - J^{-1}y_n^*\rangle - 2\alpha_n\theta_n\langle v_{n+1} - v_n, J^{-1}v_n\rangle \\
&\quad - 2\alpha_n\theta_n\langle y_{n-1}^* - y_n^*, J^{-1}v_n\rangle - 2\alpha_n\theta_n\langle y_n^* - y_{n-1}^*, J^{-1}y_{n-1}^*\rangle \\
&\quad - 2\alpha_n\theta_n\langle v_n - y_n^*, J^{-1}y_{n-1} - J^{-1}y_n^*\rangle - 2\alpha_n\langle v_{n+1} - v_n, Kv_n + u_n\rangle \\
&\quad + 2\alpha_n\langle v_n - y_n^*, y_n - u_n\rangle.
\end{aligned}$$

Thus, using inequalities (2.2.9), (2.2.10) and (6.2.5), we obtain

$$\begin{aligned}
\phi(y_n^*, v_{n+1}) &\leq (1-\alpha_n\theta_n)\phi(y_{n-1}^*, v_n) + \alpha_n\theta_n\widehat{N}^*\Bigg(c_2\delta_{E^*}^{-1}(\alpha_n M_0^*) + c_2\delta_{E^*}^{-1}(\eta_n) \\
&\quad + c_2\delta_E^{-1}(\eta_n) + c_2\frac{\delta_{E^*}^{-1}(\alpha_n M_0)}{\theta_n} + c_2\frac{\delta_{E^*}^{-1}(\eta_n)}{\alpha_n\theta_n} + c_2\frac{\delta_E^{-1}(\eta_n)}{\alpha_n\theta_n}\Bigg) \\
&\quad + 2\alpha_n\langle v_n - y_n^*, y_n - u_n\rangle,
\end{aligned} \tag{6.2.14}$$

for some $\widehat{N}^* > 0$. Let $p_n = (y_n, y_n^*)$, adding inequalities (6.2.13) and (6.2.14) we obtain

$$
\begin{aligned}
\Phi(p_n, w_{n+1}) \le (1-\alpha_n\theta_n)\Phi(p_{n-1}, w_n) + \alpha_n\theta_n N\Big(& c_2\delta_E^{-1}\big(\alpha_n M_0\big) + 2c_2\delta_E^{-1}(\eta_n) \\
& + c_2\frac{\delta_E^{-1}(\alpha_n M_0)}{\theta_n} + 2c_2\delta_{E^*}^{-1}(\eta_n) + c_2\frac{\delta_{E^*}^{-1}(\alpha_n M_0)}{\theta_n} + \frac{\delta_E^{-1}(\eta_n)}{\alpha_n\theta_n} \\
& + 2c_2\frac{\delta_{E^*}^{-1}(\eta_n)}{\alpha_n\theta_n} + c_2\delta_{E^*}^{-1}(\alpha_n M_0^*)\Big), \qquad (6.2.15)
\end{aligned}
$$

where $N = \max\{\widehat{N}, \widehat{N}^*\}$. Now, setting $a_n = \Phi(p_{n-1}, w_n)$; $\sigma_n = \alpha_n\beta_n$; $c_n \equiv 0$ and

$$
\begin{aligned}
b_n :=N\Big(& c_2\delta_E^{-1}\big(\alpha_n M_0\big) + 2c_2\delta_E^{-1}(\eta_n) + c_2\frac{\delta_E^{-1}(\alpha_n M_0)}{\theta_n} + 2c_2\delta_{E^*}^{-1}(\eta_n) \\
& + c_2\frac{\delta_{E^*}^{-1}(\alpha_n M_0)}{\theta_n} + \frac{\delta_E^{-1}(\eta_n)}{\alpha_n\theta_n} + 2c_2\frac{\delta_{E^*}^{-1}(\eta_n)}{\alpha_n\theta_n} + c_2\delta_{E^*}^{-1}(\alpha_n M_0^*)\Big),
\end{aligned}
$$

inequality (6.2.15) becomes

$$
a_{n+1} \le (1-\sigma_n)a_n + \sigma_n b_n + c_n, \quad n \ge 1.
$$

It follows from Lemma 2.3.11 that $\Phi(p_{n-1}, w_n) \to 0$, as $n \to \infty$. By Lemma 2.3.8, we have $\|w_n - p_{n-1}\|_W \to 0$. Consequently, $\|u_n - y_{n-1}\| \to 0$. Furthermore, since $[y_n, y_n^*] \to [u^*, v^*] \in A^{-1}0$, we have that $\{u_n\}$ converges to a solution of the Hammerstein equation $u + KFu = 0$, with $v^* = Fu^*$. This completes the proof. ∎

Remark 6.2.3 *Real sequences that satisfy the hypothesis of above theorem are* $\alpha_n = (n+1)^{-a}$ *and* $\theta_n = (n+1)^{-b}$ *with* $0 < b < a$ *and* $a + b < 1$.

6.3 Numerical Illustration

In this section, we present a numerical example to compare the convergence of sequences generated algorithms (1.2.11) and (1.2.8), and our algorithm, algorithm (6.2.10).

Example 6.3.1

In Theorems 1.2.8, 1.2.5 and 6.2.2 set $E = \mathbb{R}^2$, $E^* = \mathbb{R}^2$,

$$
Fu = (u_1 + u_2 + \sin u_1, -u_1 + u_2 + \sin u_2), \qquad Kv = (v_1 + v_2, v_1 + v_2).
$$

Then, it is easy to see that F and K are monotone and the vector $u^* = (0,0)$ is the only solution of the equation $u + KFu = 0$. In algorithms (1.2.11) and (6.2.10), we take $\alpha_n = \lambda_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{5}}}$, $n = 1, 2, \cdots$, and in algorithm (1.2.8), we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \cdots$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorems 1.2.8, 1.2.5 and 6.2.2, respectively. Choosing $u_1 = (1,0)$, $v_1 = (1,1)$, $n = 5000$ and using a tolerance of 10^{-8} we obtain the following iterates. And in the graph below, y-axis represents the values of $\|u_{n+1} - \mathbf{0}\|$ while the x-axis represents the number of iterations n.

Table 6.1: Numerical results for Example 6.3.1

CPU time	Algorithm (1.2.11) 0.43 sec	Algorithm (1.2.8) 0.49 sec	Algorithm (6.2.10) 0.21 sec
No. iter.	$\|u_{k+1}\|$	$\|u_{k+1}\|$	$\|u_{k+1}\|$
1	1.6817	1.4142	1.6817
10	1.2912	0.1825	1.2932
30	0.1811	0.1566	0.0008
52	0.1656	0.1385	8.8×10^{-9}
5000	0.0703	0.0465	-

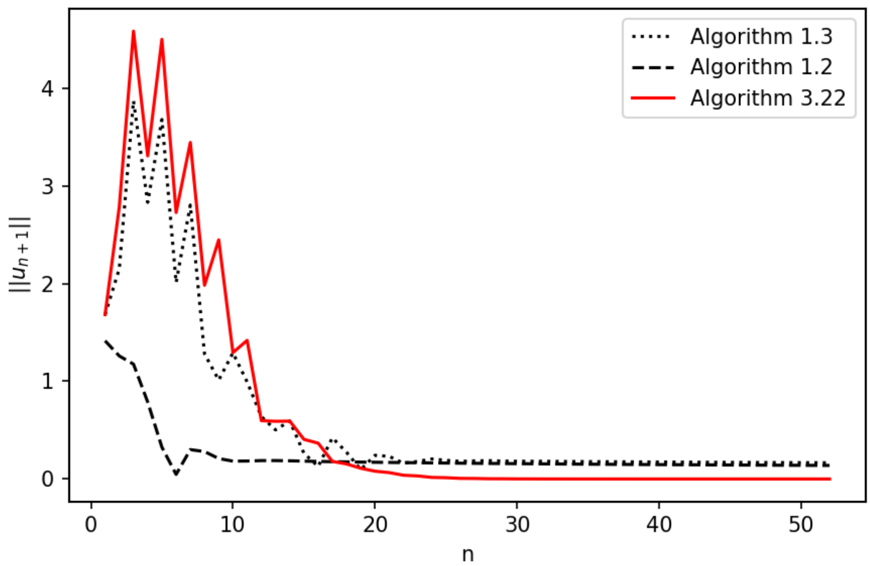


Figure 6.1: Graphical illustration of the data in Table 6.1

Conclusion. Observe that in this experiment and with the specified tolerance, the sequence of our iteration process converges after 52 iterations, whereas, after 5,000 iterations the sequences of algorithms (1.2.11) and (1.2.8), with this given tolerance, are yet to converge. From the results obtained, Algorithm (6.2.10) would, perhaps, be preferred to either Algorithm (1.2.11) or Algorithm (1.2.8) in any possible application.

The results obtained in this chapter have gained publication and appeared following paper: C.E. Chidume, **A. Adamu** and L.O. Chinwendu, *Approximation of solutions of Hammerstein equations with monotone mappings in real Banach spaces,* **Carpathian Journal of Mathematics,** 35 (2019), No. 3, 305 - 316.

CHAPTER 7

GENERALIZED HYBRID VISCOSITY-TYPE FORWARD-BACKWARD SPLITTING METHOD WITH APPLICATION TO CONVEX MINIMIZATION AND IMAGE RESTORATION PROBLEMS

7.1 Introduction

In this chapter, a viscosity-type forward-backward splitting iterative method for approximating a zero of sum of two operators $(A+B)$ which is also a fixed point of an operator S is studied. Strong convergence theorem of the method is proved under suitable conditions. Furthermore, the convergence result obtained is applied to convex minimization and image restoration problems. Finally, numerical illustrations are presented to compare the convergence of the sequence of our algorithm and that of some recent important algorithms.

7.2 Main result

The following assumptions are central in the proof of Lemma 7.2.4 and Theorem 7.2.5.

Assumption 7.2.1 *The space E is a uniformly convex and q-uniformly smooth real Banach space.*

Assumption 7.2.2 *The operator $A : E \to E$ is α-isa of order q and $B : E \to 2^E$ is a set-valued m-accretive operator, the mapping $f : E \to E$ is a contraction with constant $k \in (0,1)$, $S : E \to E$ is a nonexpansive mapping and the solution set $\Omega := F(S) \cap (A+B)^{-1}0 \neq \emptyset$.*

We now give our algorithm.

Algorithm 7.2.3 *Hybrid viscosity-type forward-backward splitting algorithm.*

Step 0. *(Initialization) choose an arbitrary point $x_1 \in E$, and set $n = 1$,*

Step 1. *Compute*

$$z_n = \gamma_n x_n + (1-\gamma_n) J^B_{\lambda_n}(I - \lambda_n A)x_n,$$

Step 2. *Compute*

$$y_n = s_n x_n + (1-s_n) J^B_{\lambda_n}(I - \lambda_n A)z_n,$$

Step 3. *Compute*

$$x_{n+1} = \tau_n f(x_n) + \sigma_n x_n + \mu_n S y_n,$$

where $\{\tau_n\}, \{\sigma_n\}$ and $\{\mu_n\}$ are sequences in $(0,1)$ such that $\tau_n + \sigma_n + \mu_n = 1$, $\{\gamma_n\}, \{s_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ with $0 < \lambda \leq \kappa_q \lambda_n^{q-1} < \alpha q$.

Step 4. *Update $n = n+1$ and go to Step 1.*

Lemma 7.2.4 *Let $\{x_n\}$ be the sequence generated by Algorithm 7.2.3, then $\{x_n\}$ is bounded.*

Proof. Using the fact that the resolvent operator is firmly nonexpansive and Lemma 2.3.13, we obtain

$$\begin{aligned}
\|W^{A,B}_{\lambda_n} x - W^{A,B}_{\lambda_n} y\|^q &= \|J^B_{\lambda_n}(I - \lambda_n A)x - J^B_{\lambda_n}(I - \lambda_n A)y\|^q \\
&\leq \|(I - \lambda_n A)x - (I - \lambda_n A)y\|^q \\
&\leq \|x - y\|^q - \lambda_n(\alpha q - \kappa_q \lambda_n^{q-1})\|Ax_n - Ay\|^q \\
&\leq \|x - y\|^q.
\end{aligned}$$

Thus, $W^{A,B}_{\lambda_n}$ is nonexpansive.

Let $z \in \Omega$. Then, using Remark 2.3.9 (i) and the nonexpansivity of $W^{A,B}_{\lambda_n}$, we have that

$$\begin{aligned}
\|z_n - z\| &= \|\gamma_n x_n + (1-\gamma_n) W^{A,B}_{\lambda_n} x_n - z\| \\
&\leq \gamma_n \|x_n - z\| + (1-\gamma_n)\|W^{A,B}_{\lambda_n} x_n - W^{A,B}_{\lambda_n} z\| \\
&\leq \gamma_n \|x_n - z\| + (1-\gamma_n)\|x_n - z\| = \|x_n - z\|. \qquad (7.2.1)
\end{aligned}$$

Similarly, using Remark 2.3.9 (i), the nonexpansivity of $W^{A,B}_{\lambda_n}$ and inequality (7.2.1), we have

$$\begin{aligned}
\|y_n - z\| &= \|s_n x_n + (1-s_n) W^{A,B}_{\lambda_n} z_n - z\| \\
&\leq s_n \|x_n - z\| + (1-s_n)\|W^{A,B}_{\lambda_n} z_n - W^{A,B}_{\lambda_n} z\| \\
&\leq s_n \|x_n - z\| + (1-s_n)\|z_n - z\| \\
&\leq s_n \|x_n - z\| + (1-s_n)\|x_n - z\| = \|x_n - z\|. \qquad (7.2.2)
\end{aligned}$$

Now, using the fact that S is nonexpansive, f is a contraction and inequality (7.2.2), we obtain that

$$\begin{aligned}
\|x_{n+1}-z\| &= \|\tau_n f(x_n)+\sigma_n x_n+\mu_n Sy_n-z\| \\
&\le \tau_n\|f(x_n)-z\|+\sigma_n\|x_n-z\|+\mu_n\|Sy_n-z\| \\
&\le \tau_n\|f(x_n)-f(z)\|+\tau_n\|f(z)-z\|+\sigma_n\|x_n-z\|+\mu_n\|y_n-z\| \\
&\le k\tau_n\|x_n-z\|+(1-\tau_n)\|x_n-z\|+\tau_n\|f(z)-z\| \\
&\le \big(1-(1-k)\tau_n\big)\|x_n-z\|+\tau_n\|f(z)-z\| \\
&\le \max\Big\{\|x_n-z\|, \frac{1}{1-k}\|f(z)-z\|\Big\}.
\end{aligned}$$

It follows inductively, that

$$\|x_n-z\| \le \max\Big\{\|x_1-z\|,\ \frac{1}{1-k}\|f(z)-z\|\Big\},\quad \forall\ n\ge 1.$$

Thus, $\{x_n\}$ is bounded. Furthermore, it is easy to show that $\{z_n\},\{y_n\},\{Sy_n\}$ and $\{Ax_n\}$ are bounded.

■

In the proof of Theorem 7.2.5 below, the sequences $\{\tau_n\}$ and $\{\lambda_n\}$ are assumed to satisfy the following conditions:

(C1) $\lim\limits_{n\to\infty}\tau_n=0$ and $\sum\limits_{n=1}^{\infty}\tau_n=\infty$,

(C2) $0<\lambda\le \kappa_q\lambda_n^{q-1}<\alpha q$.

Theorem 7.2.5 *Let $\{x_n\}$ be the sequence generated by algorithm 7.2.3. Then $\{x_n\}$ converges strongly to $z\in\Omega$.*

Proof. Let $z\in\Omega$. Using Remark 2.3.9 (i), the fact that J_λ is firmly nonexpansive, Lemma 2.3.4 (ii) and Lemma 2.3.13, we have

$$\begin{aligned}
\|W_{\lambda_n}^{A,B}x_n-z\|^q &= \|J_{\lambda_n}^B(I-\lambda_nA)x_n-J_{\lambda_n}^B(I-\lambda_nA)z\|^q \\
&\le \langle x_n-\lambda_nAx_n-(z-\lambda_nAz), j_q(W_{\lambda_n}^{A,B}x_n-z)\rangle \\
&\le \frac{1}{q}\Big(\|(I-\lambda_nA)x_n-(I-\lambda_nA)z\|^q+(q-1)\|W_{\lambda_n}^{A,B}x_n-z\|^q \\
&\quad -\psi_1\big(\|x_n-\lambda_n(Ax_n-Az)-W_{\lambda_n}^{A,B}x_n\|\big)\Big) \\
&= \|(I-\lambda_nA)x_n-(I-\lambda_nA)z\|^q \\
&\quad -\psi_1(\|x_n-\lambda_n(Ax_n-Az)-W_{\lambda_n}^{A,B}x_n\|) \\
&\le \|x_n-z\|^q-\lambda_n(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Az\|^q \\
&\quad -\psi_1(\|x_n-\lambda_n(Ax_n-Az)-W_{\lambda_n}^{A,B}x_n\|). \qquad (7.2.3)
\end{aligned}$$

By Lemma 2.3.4 (i) and inequality (7.2.3), we obtain

$$\begin{aligned}
\|z_n - z\|^q &= \|\gamma_n x_n + (1-\gamma_n) W_{\lambda_n}^{A,B} x_n - z\|^q \\
&\le \gamma_n \|x_n - z\|^q + (1-\gamma_n)\|W_{\lambda_n}^{A,B} x_n - z\|^q \\
&\le \gamma_n \|x_n - z\|^q + (1-\gamma_n)\Big(\|x_n - z\|^q - \lambda_n(\alpha q - \kappa_q \lambda_n^{q-1})\|Ax_n - Az\|^q \\
&\quad - \psi_1(\|x_n - \lambda_n(Ax_n - Az) - W_{\lambda_n}^{A,B} x_n\|)\Big) \\
&= \|x_n - z\|^q - \lambda_n(1-\gamma_n)(\alpha q - \kappa_q \lambda_n^{q-1})\|Ax_n - Az\|^q \\
&\quad - (1-\gamma_n)\psi_1(\|x_n - \lambda_n(Ax_n - Az) - W_{\lambda_n}^{A,B} x_n\|). \qquad (7.2.4)
\end{aligned}$$

Similarly, using inequality (7.2.4), we have

$$\begin{aligned}
\|y_n - z\|^q &= \|s_n x_n + (1-s_n) W_{\lambda_n}^{A,B} z_n - z\|^q \\
&\le s_n \|x_n - z\|^q + (1-s_n)\|W_{\lambda_n}^{A,B} z_n - z\|^q \\
&\le s_n \|x_n - z\|^q + (1-s_n)\Big(\|z_n - z\|^q \\
&\quad - \lambda_n(\alpha q - \kappa_q \lambda_n^{q-1})\|Az_n - Az\|^q \\
&\quad - \psi_2(\|z_n - \lambda_n(Az_n - Az) - W_{\lambda_n}^{A,B} z_n\|)\Big) \\
&= s_n \|x_n - z\|^q + (1-s_n)\|z_n - z\|^q \\
&\quad - \lambda_n(1-s_n)(\alpha q - \kappa_q \lambda_n^{q-1})\|Az_n - Az\|^q \\
&\quad - (1-s_n)\psi_2(\|z_n - \lambda_n(Az_n - Az) - W_{\lambda_n}^{A,B} z_n\|) \\
&\le s_n \|x_n - z\| + (1-s_n)\Big(\|x_n - z\|^q \\
&\quad - \lambda_n(1-\gamma_n)(\alpha q - \kappa_q \lambda_n^{q-1})\|Ax_n - Az\|^q \\
&\quad - (1-\gamma_n)\psi_1(\|x_n - \lambda_n(Ax_n - Az) - W_{\lambda_n}^{A,B} x_n\|)\Big) \\
&\quad - \lambda_n(1-s_n)(\alpha q - \kappa_q \lambda_n^{q-1})\|Az_n - Az\|^q \\
&\quad - (1-s_n)\psi_2(\|z_n - \lambda_n(Az_n - Az) - W_{\lambda_n}^{A,B} z_n\|) \\
&= \|x_n - z\|^q - \lambda_n(1-s_n)(1-\gamma_n)(\alpha q - \kappa_q \lambda_n^{q-1})\|Ax_n - Az\|^q \\
&\quad - (1-s_n)(1-\gamma_n)\psi_1(\|x_n - \lambda_n(Ax_n - Az) - W_{\lambda_n}^{A,B} x_n\|) \\
&\quad - \lambda_n(1-s_n)(\alpha q - \kappa_q \lambda_n^{q-1})\|Az_n - Az\|^q \\
&\quad - (1-s_n)\psi_2(\|z_n - \lambda_n(Az_n - Az) - W_{\lambda_n}^{A,B} z_n\|). \qquad (7.2.5)
\end{aligned}$$

Now, using Lemma 2.3.2 and Lemma 2.3.5, and inequality (7.2.5), we obtain

$$
\begin{aligned}
\|x_{n+1}-z\|^q &= \|\tau_n f(x_n)+\sigma_n x_n+\mu_n Sy_n - z\|^q \\
&= \|\tau_n(f(x_n)-f(z))+\sigma_n(x_n-z)+\mu_n(Sy_n-z)+\tau_n(f(z)-z)\|^q \\
&\le \|\tau_n(f(x_n)-f(z))+\sigma_n(x_n-z)+\mu_n(Sy_n-z)\|^q \\
&\quad + q\tau_n\langle f(z)-z, j_q(x_{n+1}-z)\rangle \\
&\le \tau_n k\|x_n-z\|^q+\sigma_n\|x_n-z\|^q+\mu_n\|y_n-z\|^q-\sigma_n\mu_n\phi(\|x_n-Sy_n\|) \\
&\quad + q\tau_n\langle f(z)-z, j_q(x_{n+1}-z)\rangle \\
&\le \tau_n k\|x_n-z\|^q+\sigma_n\|x_n-z\|^q+\mu_n\Big(\|x_n-z\|^q \\
&\quad -\lambda_n(1-s_n)(1-\gamma_n)(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Az\|^q \\
&\quad -(1-s_n)(1-\gamma_n)\psi_1(\|x_n-\lambda_n(Ax_n-Az)-W_{\lambda_n}^{A,B}x_n\|) \\
&\quad -\lambda_n(1-s_n)(\alpha q-\kappa_q\lambda_n^{q-1})\|Az_n-Az\|^q \\
&\quad -(1-s_n)\psi_2(\|z_n-\lambda_n(Az_n-Az)-W_{\lambda_n}^{A,B}z_n\|)\Big) \\
&\quad -\sigma_n\mu_n\phi(\|x_n-Sy_n\|)+q\tau_n\langle f(z)-z, j_q(x_{n+1}-z)\rangle \\
&\le \tau_n k\|x_n-z\|^q+(1-\tau_n)\|x_n-z\|^q \\
&\quad -\lambda_n\mu_n(1-s_n)(1-\gamma_n)(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Az\|^q \\
&\quad -\mu_n(1-s_n)(1-\gamma_n)\psi_1(\|x_n-\lambda_n(Ax_n-Az)-W_{\lambda_n}^{A,B}x_n\|) \\
&\quad -\mu_n\lambda_n(1-s_n)(\alpha q-\kappa_q\lambda_n^{q-1})\|Az_n-Az\|^q \\
&\quad -\mu_n(1-s_n)\psi_2(\|z_n-\lambda_n(Az_n-Az)-W_{\lambda_n}^{A,B}z_n\|) \\
&\quad -\sigma_n\mu_n\phi(\|x_n-Sy_n\|)+q\tau_n\langle f(z)-z, j_q(x_{n+1}-z)\rangle \\
&= \big(1-(1-k)\tau_n\big)\|x_n-z\|^q \\
&\quad -\lambda_n\mu_n(1-s_n)(1-\gamma_n)(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Az\|^q \\
&\quad -\mu_n(1-s_n)(1-\gamma_n)\psi_1(\|x_n-\lambda_n(Ax_n-Az)-W_{\lambda_n}^{A,B}x_n\|) \\
&\quad -\mu_n\lambda_n(1-s_n)(\alpha q-\kappa_q\lambda_n^{q-1})\|Az_n-Az\|^q \\
&\quad -\mu_n(1-s_n)\psi_2(\|z_n-\lambda_n(Az_n-Az)-W_{\lambda_n}^{A,B}z_n\|) \\
&\quad -\sigma_n\mu_n\phi(\|x_n-Sy_n\|)+q\tau_n\langle f(z)-z, j_q(x_{n+1}-z)\rangle \qquad (7.2.6) \\
&\le \big(1-(1-k)a_n\big)\|x_n-z\|^q+q\tau_n\langle f(z)-z, j_q(x_{n+1}-z)\rangle.
\end{aligned}
$$

From inequality (7.2.6), we deduce that

$$
\begin{aligned}
\|x_{n+1}-z\|^q \le \|x_n-z\|^q &-\lambda_n\mu_n(1-s_n)(1-\gamma_n)(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Az\|^q \\
&-\mu_n(1-s_n)(1-\gamma_n)\psi_1(\|x_n-\lambda_n(Ax_n-Az)-W_{\lambda_n}^{A,B}x_n\|) \\
&-\mu_n\lambda_n(1-s_n)(\alpha q-\kappa_q\lambda_n^{q-1})\|Az_n-Az\|^q \\
&-\mu_n(1-s_n)\psi_2(\|z_n-\lambda_n(Az_n-Az)-W_{\lambda_n}^{A,B}z_n\|) \\
&-\sigma_n\mu_n\phi(\|x_n-Sy_n\|)+q\tau_n\langle f(z)-z, j_q(x_{n+1}-z)\rangle.
\end{aligned}
$$

Setting

$$
d_n=\|x_n-z\|^q, \qquad \alpha_n=(1-k)\tau_n, \qquad \beta_n=\tau_n\langle f(z)-z, j_q(x_{n+1}-z)\rangle,
$$

$$\theta_n = \frac{q}{1-k}\langle f(z) - z, j_q(x_{n+1} - z)\rangle,$$

and

$$\begin{aligned}\eta_n &= \lambda_n\mu_n(1-s_n)(1-\gamma_n)(\alpha q - \kappa_q\lambda_n^{q-1})\|Ax_n - Az\|^q \\ &+ \mu_n(1-s_n)(1-\gamma_n)\psi_1(\|x_n - \lambda_n(Ax_n - Az) - W_{\lambda_n}^{A,B}x_n\|) \\ &+ \mu_n\lambda_n(1-s_n)(\alpha q - \kappa_q\lambda_n^{q-1})\|Az_n - Az\|^q \\ &+ \mu_n(1-s_n)\psi_2(\|z_n - \lambda_n(Az_n - Az) - W_{\lambda_n}^{A,B}z_n\|) + \sigma_n\mu_n\phi(\|x_n - Sy_n\|),\end{aligned}$$

we obtain that

$$d_{n+1} \le (1-\alpha_n)d_n + \alpha_n\theta_n, \quad \text{and} \quad d_{n+1} \le d_n - \eta_n + \beta_n, \quad n \ge 1.$$

Since $\sum_{n=1}^{\infty}\tau_n = \infty$, $\sum_{n=1}^{\infty}\alpha_n = \infty$. By the boundedness of $\{x_n\}$, the fact that $\lim_{n\to\infty}\tau_n = 0$ and the uniform continuity of j_q on bounded sets, we have $\lim_{n\to\infty}\beta_n = 0$. In order to complete the proof, by using Lemma 2.3.12 it remains to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\theta_{n_k} \le 0$, for any subsequence $\{n_k\} \subset \{n\}$. Indeed, if $\{n_k\}$ is a subsequence of $\{n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$, then by hypothesis of Lemma 2.3.4, we can deduce that

$$\lim_{k\to\infty}\|Ax_{n_k} - Az\| = 0, \quad \lim_{k\to\infty}\|Az_{n_k} - Az\| = 0, \quad \lim_{k\to\infty}\|x_{n_k} - Sy_{n_k}\| = 0, \tag{7.2.7}$$

$$\lim_{k\to\infty}\|x_{n_k} - \lambda_{n_k}(Ax_{n_k} - Az) - W_{\lambda_{n_k}}^{A,B}x_{n_k}\| = 0 \quad \text{and}$$

$$\lim_{k\to\infty}\|z_{n_k} - \lambda_{n_k}(Az_{n_k} - Az) - W_{\lambda_{n_k}}^{A,B}z_{n_k}\| = 0.$$

Thus, by the triangle inequality, we have

$$\lim_{k\to\infty}\|W_{\lambda_{n_k}}^{A,B}x_{n_k} - x_{n_k}\| = 0 \qquad \text{and} \qquad \lim_{k\to\infty}\|W_{\lambda_{n_k}}^{A,B}z_{n_k} - z_{n_k}\| = 0.$$

Observe that

$$\begin{aligned}\|W_{\lambda_{n_k}}^{A,B}z_{n_k} - x_{n_k}\| &\le \|W_{\lambda_{n_k}}^{A,B}z_{n_k} - z_{n_k}\| + \|z_{n_k} - x_{n_k}\| \\ &= \|W_{\lambda_{n_k}}^{A,B}z_{n_k} - z_{n_k}\| + \|\gamma_{n_k}x_{n_k} + (1-\gamma_{n_k})W_{\lambda_{n_k}}^{A,B}x_{n_k} - x_{n_k}\| \\ &= \|W_{\lambda_{n_k}}^{A,B}z_{n_k} - z_{n_k}\| + (1-\gamma_{n_k})\|W_{\lambda_{n_k}}^{A,B}x_{n_k} - x_{n_k}\|.\end{aligned}$$

implies

$$\lim_{k\to\infty}\|W_{\lambda_{n_k}}^{A,B}z_{n_k} - x_{n_k}\| = 0.$$

Also,

$$\|x_{n_k} - z_{n_k}\| \le \|W_{\lambda_{n_k}}^{A,B}z_{n_k} - x_{n_k}\| + \|z_{n_k} - W_{\lambda_{n_k}}^{A,B}z_{n_k}\| \quad \Rightarrow \quad \lim_{k\to\infty}\|x_{n_k} - z_{n_k}\| = 0.$$

Moreover,

$$\|y_{n_k} - x_{n_k}\| = (1 - s_{n_k})\|W^{A,B}_{\lambda_{n_k}} z_{n_k} - x_{n_k}\| \quad \Rightarrow \quad \lim_{k\to\infty} \|y_{n_k} - x_{n_k}\| = 0.$$

Furthermore,

$$\begin{aligned} \|x_{n_k} - Sx_{n_k}\| &\leq \|x_{n_k} - Sy_{n_k}\| + \|Sy_{n_k} - Sx_{n_k}\| \\ &\leq \|x_{n_k} - Sy_{n_k}\| + \|y_{n_k} - x_{n_k}\| \\ &\Rightarrow \lim_{k\to\infty} \|x_{n_k} - Sx_{n_k}\| = 0. \end{aligned} \tag{7.2.8}$$

Since $0 < \lambda \leq \lambda_n$, for all $n \geq 1$, by Remark 2.3.9 (ii), we have that

$$\|W^{A,B}_{\lambda} x_{n_k} - x_{n_k}\| \leq 2\|W^{A,B}_{\lambda_{n_k}} x_{n_k} - x_{n_k}\|, \quad \text{which implies that}$$

$$\lim_{k\to\infty} \|W^{A,B}_{\lambda} x_{n_k} - x_{n_k}\| = 0. \tag{7.2.9}$$

Let $z_t = tf(z_t) + (1-t)W^{A,B}_{\lambda} z_t, \ \ \forall t \in (0,1)$. Then, by Lemma 2.3.14 $\{z_t\}$ converges strongly to a point $z \in F(W^{A,B}_{\lambda})$.

Now, using Lemma 2.3.2 and the fact that $W^{A,B}_{\lambda}$ is nonexpansive, we have

$$\begin{aligned} \|z_t - x_{n_k}\|^q &= \|t(f(z_t) - x_{n_k}) + (1-t)(W^{A,B}_{\lambda} z_t - x_{n_k})\|^q \\ &\leq (1-t)^q \|W^{A,B}_{\lambda} z_t - x_{n_k}\|^q + qt\langle f(z_t) - x_{n_k}, j_q(z_t - x_{n_k})\rangle \\ &= (1-t)^q \|W^{A,B}_{\lambda} z_t - x_{n_k}\|^q + qt\langle f(z_t) - z_t, j_q(z_t - x_{n_k})\rangle \\ &\quad + qt\langle z_t - x_{n_k}, j_q(z_t - x_{n_k})\rangle \\ &\leq (1-t)^q \big(\|W^{A,B}_{\lambda} z_t - W^{A,B}_{\lambda} x_{n_k}\| + \|W^{A,B}_{\lambda} x_{n_k} - x_{n_k}\|\big)^q \\ &\quad + qt\langle f(z_t) - z_t, j_q(z_t - x_{n_k})\rangle + qt\|z_t - x_{n_k}\|^q \\ &\leq (1-t)^q \big(\|z_t - x_{n_k}\| + \|W^{A,B}_{\lambda} x_{n_k} - x_{n_k}\|\big)^q \\ &\quad + qt\langle f(z_t) - z_t, j_q(z_t - x_{n_k})\rangle + qt\|z_t - x_{n_k}\|^q \end{aligned}$$

which implies that

$$\begin{aligned} \langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle &\leq \frac{(1-t)^q}{qt}\Big(\|z_t - x_{n_k}\| + \|W^{A,B}_{\lambda} x_{n_k} - x_{n_k}\|\Big)^q \\ &\quad + \frac{(qt-1)}{qt}\|z_t - x_{n_k}\|^q. \end{aligned} \tag{7.2.10}$$

This implies, using (7.2.9)

$$\begin{aligned} \limsup_{k\to\infty} \langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle &\leq \frac{(1-t)^q}{qt} M + \frac{(qt-1)}{qt} M \\ &= \Big(\frac{(1-t)^q + qt - 1}{qt}\Big) M, \end{aligned} \tag{7.2.11}$$

where $M = \limsup_{k\to\infty} \|z_t - x_{n_k}\|^q$. It is easy to see that $\frac{(1-t)^q+qt-1}{qt} \to 0$, as $t \to 0$. By the uniform continuity of j_q on bounded subsets of E and the fact that $z_t \to z$, we have that

$$\|j_q(x_{n_k} - z_t) - j_q(x_{n_k} - z)\| \to 0, \quad as \quad t \to 0.$$

Hence,

$$\begin{aligned}
&|\langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle - \langle f(z) - z, j_q(x_{n_k} - z)\rangle| \\
&\quad = |\langle f(z_t) - f(z), j_q(x_{n_k} - z_t)\rangle + \langle f(z) - z, j_q(x_{n_k} - z_t)\rangle \\
&\qquad + \langle z - z_t, j_q(x_{n_k} - z_t)\rangle - \langle f(z) - z, j_q(x_{n_k} - z)\rangle| \\
&\quad \leq |\langle f(z) - z, j_q(x_{n_k} - z_t) - j_q(x_{n_k} - z)\rangle| + |\langle f(z_t) - f(z), j_q(x_{n_k} - z_t)\rangle| \\
&\qquad + |\langle z - z_t, j_q(x_{n_k} - z_t)\rangle| \\
&\quad \leq \|f(z) - z\|\|j_q(x_{n_k} - z_t) - j_q(x_{n_k} - z)\| + (1+k)\|z_t - z\|\|x_{n_k} - z_t\|^{q-1}.
\end{aligned}$$

Thus, as $t \to 0$, we have

$$\langle f(z_t) - z_t, j_q(x_{n_k} - z_t)\rangle \to \langle f(z) - z, j_q(x_{n_k} - z)\rangle.$$

It follows from (7.2.11), as $t \to 0$, that

$$\limsup_{k\to\infty} \langle f(z) - z, j_q(x_{n_k} - z)\rangle \leq 0. \tag{7.2.12}$$

Furthermore,

$$\begin{aligned}
\|x_{n_k+1} - x_{n_k}\| &= \|\tau_{n_k} f(x_{n_k}) + \sigma_{n_k} x_{n_k} + \mu_{n_k} S y_{n_k} - x_{n_k}\| \\
&\leq \tau_{n_k}\|f(x_{n_k}) - x_{n_k}\| + \mu_{n_k}\|S y_{n_k} - x_{n_k}\|
\end{aligned}$$

implies $\big($using the fact that $\{x_n\}$ is bounded, $\lim_{n\to} \tau_n = 0$ and inequality (7.2.7)$\big)$ that

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{7.2.13}$$

It follows from (7.2.12) and (7.2.13) that

$$\limsup_{k\to\infty} \langle f(z) - z, j_q(x_{n_k+1} - z)\rangle \leq 0.$$

This implies that $\limsup\limits_{k\to\infty} \theta_{n_k} \leq 0$. Hence, by Lemma 2.3.12, $\lim\limits_{n\to\infty} d_n = 0$, i.e.,

$$\lim_{n\to\infty} x_n = z \in F\big(W_\lambda^{A,B}\big) = (A+B)^{-1}0.$$

Furthermore, by (7.2.8) and Lemma 2.3.15, $z \in F(S)$. Thus, $z \in \Omega$. This completes the proof.

■

7.3 Applications

In this section, we shall utilize the generalized viscosity implicit rules presented in the previous section to study convex minimization problem and convexly constrained linear inverse problem.

7.3.1 Application to convex minimization problem

Let $h : H \to \mathbb{R}$ be a convex smooth function and $g : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex and lower-semicontinuous function. We consider the following convex minimization problem of finding $x^* \in H$ such that

$$h(x^*) + g(x^*) = \min_{x \in H} \Big\{ h(x) + g(x) \Big\}. \tag{7.3.1}$$

Problem (7.3.1) is equivalent, by Fermat's rule, to the problem of finding $x^* \in H$ such that

$$0 \in \nabla h(x^*) + \partial g(x^*), \tag{7.3.2}$$

where ∇h is the gradient of h and ∂g is the subdifferential of g. Set $A = \nabla h$ and $B = \partial g$ in Algorithm 7.2.3. It is well-known that if ∇h is $(1/L)$-Lipschitz continuous, then it is L-inverse strongly monotone and ∂g is maximal monotone. Hence from Algorithm 7.2.3 we have the following result.

Algorithm 7.3.1 *Hybrid viscosity-type forward-backward splitting algorithm.*

Step 0. *(Initialization) choose an arbitrary point $x_1 \in H$, and set $n = 1$,*

Step 1. *Compute*

$$z_n = \gamma_n x_n + (1 - \gamma_n) J_{\lambda_n}^{\partial g}(I - \lambda_n \nabla h) x_n,$$

Step 2. *Compute*

$$y_n = s_n x_n + (1 - s_n) J_{\lambda_n}^{\partial g}(I - \lambda_n \nabla h) z_n,$$

Step 3. *Compute*

$$x_{n+1} = \tau_n f(x_n) + \sigma_n x_n + \mu_n S y_n,$$

where $\{\tau_n\}, \{\sigma_n\}$ and $\{\mu_n\}$ are sequences in $(0, 1)$ such that $\tau_n + \sigma_n + \mu_n = 1$, $\{\gamma_n\}, \{s_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ with $0 < \lambda \leq \lambda_n < 2\alpha$.

Step 4. *Update $n = n + 1$ and go to Step 1.*

7.3.2 Application to image restoration problems

In this subsection, we apply our method to image deblurring. General image recovery problem can be formulated by the inversion of the following observation model:

$$b = Ax + v, \tag{7.3.3}$$

where $x \in \mathbb{R}^n$, x, v and b are unknown original image, unknown additive random noise and known degraded observation, respectively, and A is a linear operator that depends on the concerned image recovery problem.

This model (7.3.3), is approximately equivalent to several different formulations available for optimization problems. In the literature, there is a growing interest in using l_1 norm for solving optimization problems. The l_1 regularization problem is given by

$$\min_{x} \left\{ \frac{1}{2} \|Ax - b\|_2^2 + \lambda_n \|x\|_1 \right\}, \tag{7.3.4}$$

where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^k$, A is a $k \times n$ matrix and λ_n is a nonnegative parameter. Next, we use our algorithm to approximate the solution of the following convex minimization problem:

$$\text{Find } x \in \text{Argmin}_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \|Ax - b\|_2^2 + \lambda_n \|x\|_1 \right\}, \tag{7.3.5}$$

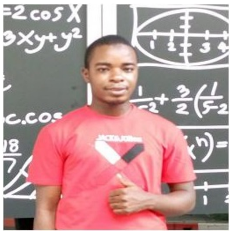
where b is the degraded image, and A an operator representing the mask. Therefore, we use our algorithm 7.3.1 to solve (7.3.5). We set $g(x) = \|x\|_1$, $h(x) = \frac{1}{2}\|Ax - b\|_2^2$, $\lambda_n = 0.023$, $\gamma_n = \frac{1}{n}, s_n = \frac{1}{n+1}, \tau_n = \frac{1}{n+1}, \sigma_n = \frac{1}{n+1}, \mu_n = \frac{n-1}{n+1}$ and $S(x) = x$. The gradient ∇h and subdifferential ∂g are:

$$\partial g = \begin{cases} \frac{x}{\|x\|}, & x \neq 0 \\ B(0,1), & x = 0 \end{cases} \qquad \text{and} \qquad \nabla h(x) = A^*(Ax - b),$$

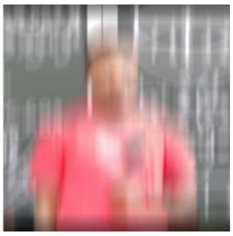
where $B(0,1) := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. The image went through a random blur and random noise. Finally, we use the structural similarity index method (SSIM), which is used to measure the quality of the restored images to analyse the images restored with our algorithm. The SSIM is defined as follows:

$$\text{SSIM} = \frac{(2u_x u_{x_r} + c_1)(2\sigma_{xx_r} + c_2)}{(u_x^2 + u_{x_r}^2 + c_1)(\sigma_x^2 + \sigma_{x_r}^2 + c_2)},$$

where x is the original image, x_r is the restored image, u_x and u_{x_r} are the mean values of the original image x and restored image x_r, respectively, σ_x^2 and $\sigma_{x_r}^2$ are the variances, $\sigma_{xx_r}^2$ is the covariance of two images, $c_1 = (K_1 L)^2$ and $c_2 = (K_2 L)^2$ with $K_1 = 0.01$ and $K_2 = 0.03$, and L is the dynamic range of pixel values. SSIM ranges from 0 to 1 , and 1 means perfect recovery.



(a) original image



(b) degradation image

(c) ssim degradation image with ssim value is 0.5743

(d) our algorithm image with $k = \frac{1}{2}$

(e) our algorithm image with $k = \frac{89}{99}$

(f) our algorithm image with $k = \frac{98}{99}$

(g) ssim our algorithm image with ssim value is 0.7485

(h) ssim our algorithm image with ssim value is 0.7629

(i) ssim our algorithm image with ssim value is 0.7671

Figure 7.1: Figure (a) shows the original image size 258×258, figure (b) shows the degradation image, figure (c) shows ssim the degradation image, figure (d), figure (e) and figure (f) show the restoration our algorithm images with $k = \frac{1}{2}, k = \frac{89}{99}$ and $k = \frac{98}{99}$ respectively and figure (g), figure (h) and figure (i) show the ssim the our algorithm images of figure (d), figure (e) and figure (f) respectively.

The original, degradation and restoration images are given in Figure 7.1. All codes were written in Matlab 2017b and run on Samsung i-3 Core laptop.

7.4 Numerical Illustration

In this section, we present a numerical example to compare the convergence of the sequence generated by our algorithm 7.2.3 and that of [Takahashi et al., 2010].

Example 7.4.1

In Theorems 1.3.8 and 7.2.5, set $E = L_3([0,1])$, and let $A : E \to E$, $B : E \to E$, $S : E \to E$, $f : E \to E$ be defined as

$$Ax(t) := 2x(t) + t^2 + \sin t, \quad Bx(t) := 7x(t), \quad Sx(t) := x(t), \quad fx(t) := \frac{x(t)}{8}.$$

It is easy to see that A is $\frac{1}{2}$-isa, B is m-accretive, S is nonexpansive and f is a contraction. Furthermore, the solution set $\Omega = F(S) \cap (A+B)^{-1}0 = \left\{\frac{-(t^2+\sin t)}{9}\right\}$. In Theorem 1.3.8, we take $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n+1}{2n}$ and $\lambda_n = \frac{1}{4}$ and in Theorem 7.2.5, we take $\tau_n = \frac{1}{n}$, $\sigma_n = \mu_n = \frac{n-1}{2n}$, $\gamma_n = \frac{1}{(n+1)^8}$, $s_n = \frac{1}{(n+1)^9}$, $\lambda_n = \frac{1}{4}$, for all $n \in \mathbb{N}$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorems 1.3.8 and 7.2.5, respectively. Observe that

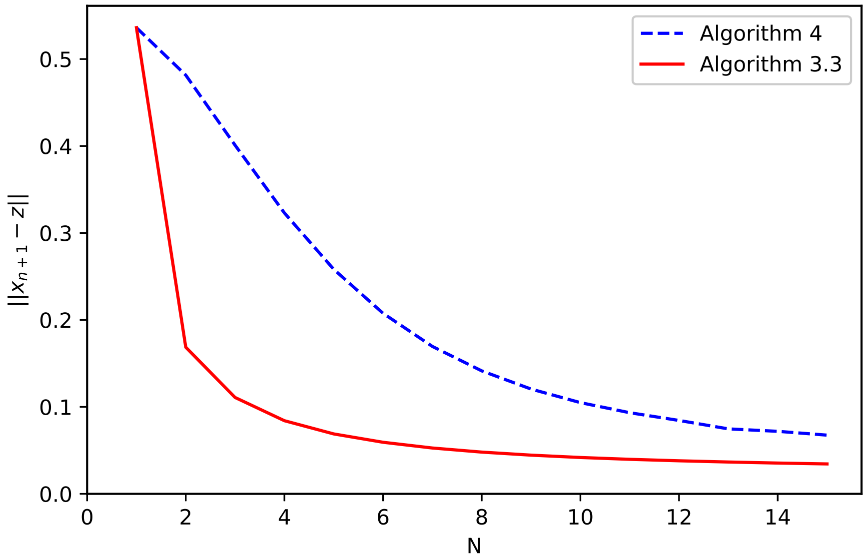
$$J_\lambda^B(I - \lambda A)x(t) = \frac{1-2\lambda}{1+7\lambda}x(t) - \frac{\lambda}{1+7\lambda}(t^2 + \sin t).$$

Finally, we using a tolerance of 10^{-3}, setting maximum number of iterations $n = 15$ and choosing $x_1(t) = t^4$, we obtain the following results (see Table 7.1 and Figure 7.2).

Table 7.1: Numerical results for Example 7.4.1

N	Algorithm (1.3.4) $\|x_{n+1} - z\|$	Algorithm (7.2.3) $\|x_{n+1} - z\|$
1	0.5361	0.5361
2	0.4815	0.1686
3	0.4009	0.1108
5	0.2582	0.0689
7	0.1695	0.0527
9	0.1205	0.0445
11	0.0933	0.0397
13	0.0747	0.0366
14	0.0719	0.0354
15	0.0675	0.0344

In the graph sketched below, the y-axis represents the values of $\|x_{n+1} - z\|$ while the x-axis represents the number of iterations N.



(a) Graph of the first 15 iterates of algorithms (1.3.4)and (7.2.3) choosing $x_1(t) = t^4$

Figure 7.2: Graphical illustration of the data in Table 7.1

The results obtained in this chapter have been **Accepted** for publication in the journal of **Numerical Functional Analysis and Optimization**.

CHAPTER 8

AN ACCELERATED HALPERN-TYPE FORWARD-BACKWARD SPLITTING METHOD WITH APPLICATION TO IMAGE PROCESSING

8.1 Introduction

In this chapter, an inertial Halpern-type forward-backward iterative algorithm for approximating a zero of sum of two accretive operators is studied. Strong convergence theorem is established in a uniformly convex and q-uniformly smooth real Banach space. The convergence result obtained is applied to convex minimization and image restoration problems. Furthermore, numerical experiments are carried out on some classical test images and personal images degraded with motion blur and random noise. Finally, numerical illustrations are presented to compare the convergence of the sequence of the proposed algorithm and that of some recent important algorithms.

8.2 Main result

The following assumptions are central in the Theorem 8.2.6.

Assumption 8.2.1 *The real Banach space E is uniformly convex and q-uniformly smooth, $A : E \to E$ is an α-isa operator of order q, $B : E \to 2^E$ is a set-valued m-accretive operator and the solution set $\Omega := (A + B)^{-1}0 \neq \emptyset$.*

Assumption 8.2.2 *Choose sequences $\{\beta_n\}$, $\{\gamma_n\} \subset (0, 1)$ and $\{\epsilon_n\}, \{\lambda_n\} \subset (0, \infty)$ such that*

(A1) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,

(A2) $\lim_{n\to\infty} \epsilon_n = 0$,

(A3) $0 < \lambda \leq \kappa_q \lambda_n^{q-1} < \alpha q$.

Based on Assumptions 8.2.1 and 8.2.2, we now give our algorithm.

Algorithm 8.2.3 *Inertial Halpern-type forward-backward splitting algorithm.*

Step 0. *(Initialization) choose arbitrary points $x_0, x_1 \in E$, $a \in (0,1)$ and set $n = 1$.*

Step 1. *Choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha_n}$, where*

$$\bar{\alpha_n} = \begin{cases} \min\left\{a, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}, \\ a, & \textit{otherwise.} \end{cases}$$

Step 2. *Compute*

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) \\ v_n = \beta_n u + (1-\beta_n) J_{\lambda_n}^B (y_n - \lambda_n A y_n) \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n) v_n. \end{cases}$$

Step 3. *Update $n = n + 1$ and go to Step 1.*

Remark 8.2.4 *Observe that Assumption 8.2.2 and Step 1 imply $\lim\limits_{n\to\infty} \alpha_n \|x_n - x_{n-1}\| = 0$. To see this, using the fact that $0 \leq \alpha_n \bar{\alpha_n}$ and $\lim\limits_{n\to\infty} \epsilon_n = 0$, we compute as follows: If $x_n = x_{n-1}$, the conclusion follows trivially. For $x_n \neq x_{n-1}$,*

$$\begin{aligned} 0 \leq & \alpha_n \|x_n - x_{n-1}\| \\ & \leq \bar{\alpha_n} \|x_n - x_{n-1}\| \leq \epsilon_n. \end{aligned}$$

This implies that $\lim\limits_{n\to\infty} \alpha_n \|x_n - x_{n-1}\| = 0$. Thus, there exists $M^* > 0$ such that

$$\alpha_n \|x_n - x_{n-1}\| \leq M^*, \quad \forall\ n \geq 1. \tag{8.2.1}$$

Lemma 8.2.5 *Let $\{x_n\}$ be the sequence generated by Algorithm 8.2.3, then $\{x_n\}$ is bounded.*

Proof. Let $W_n = J_{\lambda_n}^B (I - \lambda_n A)$ and $z = Q(u)$, where Q is the sunny nonexpansive retraction of E onto Ω. Using the fact that the resolvent operator is nonexpansive and Lemma 2.3.13, we obtain

$$\begin{aligned} \|W_n x - W_n y\|^q &= \|J_{\lambda_n}^B (I - \lambda_n A)x - J_{\lambda_n}^B (I - \lambda_n A) y\|^q \\ &\leq \|(I - \lambda_n A)x - (I - \lambda_n A)y\|^q \\ &\leq \|x - y\|^q - \lambda_n(\alpha q - \kappa_q \lambda_n^{q-1})\|Ax - Ay\|^q \\ &\leq \|x - y\|^q. \end{aligned}$$

Thus, W_n is nonexpansive.

Now, using Remark 2.3.9 (i) and the nonexpansivity of W_n, we have

$$\begin{aligned}\|v_n - z\| &= \|\beta_n u + (1-\beta_n)J^B_{\lambda_n}(y_n - \lambda_n A y_n) - z\| \\ &\le \beta_n\|u - z\| + (1-\beta_n)\|W_n y_n - W_n z\| \\ &\le \beta_n\|u - z\| + (1-\beta_n)\|y_n - z\|. \end{aligned} \tag{8.2.2}$$

Thus, using inequalities (8.2.2) and (8.2.1), we obtain

$$\begin{aligned}\|x_{n+1} - z\| &= \|\gamma_n y_n + (1-\gamma_n)v_n - z\| \\ &\le \gamma_n\|y_n - z\| + (1-\gamma_n)\|v_n - z\| \\ &\le \gamma_n\|y_n - z\| + (1-\gamma_n)\big(\beta_n\|u - z\| + (1-\beta_n)\|y_n - z\|\big) \\ &= \gamma_n\|y_n - z\| + (1-\gamma_n)\beta_n\|u - z\| + (1-\gamma_n)(1-\beta_n)\|y_n - z\| \\ &= (1-\gamma_n)\beta_n\|u - z\| + (1-\beta_n(1-\gamma_n))\|y_n - z\| \\ &\le (1-\gamma_n)\beta_n\|u - z\| + (1-\beta_n(1-\gamma_n))\|x_n - z\| \\ &\quad + (1-\beta_n(1-\gamma_n))\alpha_n\|x_n - x_{n-1}\| \\ &\le (1-\beta_n(1-\gamma_n))\|x_n - z\| + (1-\beta_n(1-\gamma_n))M^* + (1-\gamma_n)\beta_n\|u - z\| \\ &\le \max\{\|x_n - z\| + M^*, \|u - z\|\}. \end{aligned}$$

This implies (by induction) that $\{x_n\}$ is bounded. Thus, $\{v_n\}$ and $\{y_n\}$ are also bounded. ∎

Theorem 8.2.6 *Let $\{x_n\}$ be the sequence generated by Algorithm 8.2.3. Then $\{x_n\}$ converges strongly to $z \in \Omega$.*

Proof. Let $z \in \Omega$. Using Remark 2.3.9 (i), Lemmas 2.3.2 and 2.3.10, we have

$$\begin{aligned}\|v_n - z\|^q &= \|\beta_n u + (1-\beta_n)W_n y_n - z\|^q \\ &\le (1-\beta_n)^q\|W_n y_n - W_n z\|^q + q\beta_n\langle u - z, j_q(v_n - z)\rangle \\ &\le (1-\beta_n)^q\Big(\|y_n - z\|^q - \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\ &\quad - (1-\beta_n)^q\varphi(\|y_n - \lambda_n(Ay_n - Az) - W_n y_n\|)\Big) \\ &\quad + q\beta_n\langle u - z, j_q(v_n - z)\rangle \\ &= (1-\beta_n)^q\|y_n - z\|^q - \lambda_n(1-\beta_n)^q(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\ &\quad - (1-\beta_n)^q\varphi(\|y_n - \lambda_n(Ay_n - Az) - W_n y_n\|) \\ &\quad + q\beta_n\langle u - z, j_q(v_n - z)\rangle. \end{aligned} \tag{8.2.3}$$

Next, using Lemma 2.3.4(i), inequality (8.2.3) and Lemma 2.3.2, we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^q &= \|\gamma_n y_n + (1-\gamma_n)v_n - z\|^q \\
&\le \gamma_n \|y_n - z\|^q + (1-\gamma_n)\|v_n - z\|^q \\
&\le \gamma_n \|y_n - z\|^q + (1-\gamma_n)\Big((1-\beta_n)^q \|y_n - z\|^q \\
&\quad - \lambda_n (1-\beta_n)^q (\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\beta_n)^q \varphi(\|y_n - \lambda_n(Ay_n - Az) - W_n y_n\|) \\
&\quad + q\beta_n \langle u - z, j_q(v_n - z)\rangle\Big) \\
&= \gamma_n \|y_n - z\|^q + (1-\gamma_n)(1-\beta_n)^q \|y_n - z\|^q \\
&\quad - \lambda_n (1-\gamma_n)(1-\beta_n)^q (\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\gamma_n)(1-\beta_n)^q \varphi(\|y_n - \lambda_n(Ay_n - Az) - W_n y_n\|) \\
&\quad + q\beta_n (1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle \\
&\le (1 - (1-\gamma_n)\beta_n)\|y_n - z\|^q \\
&\quad - \lambda_n (1-\gamma_n)(1-\beta_n)^q (\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\gamma_n)(1-\beta_n)^q \varphi(\|y_n - \lambda_n(Ay_n - Az) - W_n y_n\|) \\
&\quad + q\beta_n (1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle \\
&\le (1 - (1-\gamma_n)\beta_n)\|x_n - z\|^q \\
&\quad + q(1 - (1-\gamma_n)\beta_n)\alpha_n \langle x_n - x_{n-1}, j_q(y_n - z)\rangle \\
&\quad - \lambda_n (1-\gamma_n)(1-\beta_n)^q (\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\gamma_n)(1-\beta_n)^q \varphi(\|y_n - \lambda_n(Ay_n - Az) - W_n y_n\|) \\
&\quad + q\beta_n (1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle. \qquad (8.2.4)
\end{aligned}$$

Thus, from inequality (8.2.4), we deduce that

$$\begin{aligned}
\|x_{n+1} - z\|^q &\le (1 - (1-\gamma_n)\beta_n)\|x_n - z\|^q \\
&\quad + q(1 - (1-\gamma_n)\beta_n)\alpha_n \langle x_n - x_{n-1}, j_q(y_n - z)\rangle \\
&\quad + q\beta_n (1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - z\|^q &\le \|x_n - z\|^q - \lambda_n (1-\gamma_n)(1-\beta_n)^q (\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\gamma_n)(1-\beta_n)^q \varphi(\|y_n - \lambda_n(Ay_n - Az) - W_n y_n\|) \\
&\quad + q(1 - (1-\gamma_n)\beta_n)\alpha_n \langle x_n - x_{n-1}, j_q(y_n - z)\rangle \\
&\quad + q\beta_n (1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle,
\end{aligned}$$

for each $n \ge 1$.

$$\text{Set} \quad d_n = \|x_n - z\|^q, \qquad \theta_n = \beta_n(1-\gamma_n)$$

$$\tau_n = \frac{q(1 - (1-\gamma_n)\beta_n)\alpha_n}{\beta_n(1-\gamma_n)}\langle x_n - x_{n-1}, j_q(y_n - z)\rangle + q\langle u - z, j_q(v_n - z)\rangle$$

$$\begin{aligned}
\eta_n &= \lambda_n (1-\gamma_n)(1-\beta_n)^q (\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad + (1-\gamma_n)(1-\beta_n)^q \varphi(\|y_n - \lambda_n(Ay_n - Az) - W_n y_n\|)
\end{aligned}$$

$$\rho_n = q(1-(1-\gamma_n)\beta_n)\alpha_n\langle x_n - x_{n-1}, j_q(y_n - z)\rangle + q\beta_n(1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle$$

$$d_{n+1} \le (1-\theta_n)d_n + \theta_n\tau_n \quad \text{and} \quad d_{n+1} \le d_n - \eta_n + \rho_n.$$

Observe that $\sum_{n=1}^{\infty}\beta_n = \infty$ implies $\sum_{n=1}^{\infty}\theta_n = \infty$. By the boundedness of $\{y_n\}$ and $\{v_n\}$, and the fact that $\lim_{n\to\infty}\beta_n = 0 = \lim_{n\to\infty}\alpha_n\|x_n - x_{n-1}\|$, we obtain that $\lim_{n\to\infty}\rho_n = 0$.

Next, by Lemma 2.3.12, it remains to show $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\tau_{n_k} \le 0$, for any subsequence $\{n_k\} \subset \{n\}$. Let $\{\eta_{n_k}\}$ be a subsequence of $\{\eta_n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$. Then, by the property of φ, we have

$$\lim_{k\to\infty}\|Ay_{n_k} - Az\| = \lim_{k\to\infty}\|y_{n_k} - \lambda_{n_k}(Ay_{n_k} - Az) - W_{n_k}y_{n_k}\| = 0.$$

Thus, by the triangle inequality,

$$\lim_{k\to\infty}\|W_{n_k}y_{n_k} - y_{n_k}\| = 0.$$

Furthermore,

$$\begin{aligned}\|W_{n_k}y_{n_k} - x_{n_k}\| &\le \|W_{n_k}y_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \\ &\le \|W_{n_k}y_{n_k} - y_{n_k}\| + \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\|\end{aligned}$$

implies

$$\lim_{k\to\infty}\|W_{n_k}y_{n_k} - x_{n_k}\| = 0.$$

Also,

$$\begin{aligned}\|y_{n_k} - v_{n_k}\| &\le \|x_{n_k} - v_{n_k}\| + \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\| \\ &\le \beta_{n_k}\|x_{n_k} - u\| + (1-\beta_{n_k})\|x_{n_k} - W_{n_k}y_{n_k}\| + \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\|\end{aligned}$$

implies

$$\lim_{k\to\infty}\|y_{n_k} - v_{n_k}\| = 0.$$

By Assumption 8.2.2 there exists $\lambda > 0$ such that $\lambda_n \ge \lambda$, for all $n \ge 1$. Hence, using Remark 2.3.9 (ii), we have

$$\|W_\lambda y_{n_k} - y_{n_k}\| \le 2\|W_{n_k}y_{n_k} - y_{n_k}\|.$$

This implies that

$$\limsup_{k\to\infty}\|W_\lambda y_{n_k} - y_{n_k}\| \le 2\limsup_{k\to\infty}\|W_{n_k}y_{n_k} - y_{n_k}\| = 0.$$

$$\text{So,} \quad \limsup_{k\to\infty}\|W_\lambda y_{n_k} - y_{n_k}\| = 0. \quad \text{Thus,} \quad \lim_{k\to\infty}\|W_\lambda y_{n_k} - y_{n_k}\| = 0.$$

Observe that

$$\|W_\lambda y_{n_k} - v_{n_k}\| \le \|W_\lambda y_{n_k} - y_{n_k}\| + \|y_{n_k} - v_{n_k}\|$$

implies

$$\lim_{k\to\infty} \|W_\lambda y_{n_k} - v_{n_k}\| = 0.$$

Also,

$$\begin{aligned}\|W_\lambda v_{n_k} - v_{n_k}\| &\le \|W_\lambda v_{n_k} - W_\lambda y_{n_k}\| + \|W_\lambda y_{n_k} - v_{n_k}\| \\ &\le \|v_{n_k} - y_{n_k}\| + \|W_\lambda y_{n_k} - v_{n_k}\|\end{aligned}$$

implies $\lim\limits_{k\to\infty} \|W_\lambda v_{n_k} - v_{n_k}\| = 0$.

Now, let $z_t = tu + (1-t)W_\lambda z,\ \ t \in (0,1)$. By a well-known theorem of Reich (see [Reich, 1980]), z_t converges strongly to the unique fixed point $z = Q(u) \in F(W_\lambda) = (A+B)^{-1}0$.

By Lemma 2.3.2 and the fact that W_λ is nonexpansive, we obtain

$$\begin{aligned}\|z_t - v_{n_k}\|^q &= \|tu + (1-t)W_\lambda z_t - v_{n_k}\|^q \\ &\le (1-t)^q \|W_\lambda z_t - v_{n_k}\|^q + qt\langle u - v_{n_k}, j_q(z_t - v_{n_k})\rangle \\ &\le (1-t)^q \big(\|W_\lambda z_t - W_\lambda v_{n_k}\| + \|W_\lambda v_{n_k} - v_{n_k}\|^q\big)^q \\ &\quad + qt\langle u - v_{n_k}, j_q(z_t - v_{n_k})\rangle \\ &\le (1-t)^q \big(\|z_t - v_{n_k}\| + \|W_\lambda v_{n_k} - v_{n_k}\|^q\big)^q + qt\langle u - v_{n_k}, j_q(z_t - v_{n_k})\rangle \\ &\le (1-t)^q \big(\|z_t - v_{n_k}\| + \|W_\lambda v_{n_k} - v_{n_k}\|^q\big)^q + qt\langle u - z_t, j_q(z_t - v_{n_k})\rangle \\ &\quad + qt\langle z_t - v_{n_k}, j_q(z_t - v_{n_k}\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\langle z_t - u, j_q(z_t - v_{n_k})\rangle \le \frac{(1-t)^q}{qt}\big(\|z_t - v_{n_k}\| + \|W_\lambda v_{n_k} - v_{n_k}\|^q\big)^q \\ + \frac{(qt-1)}{qt}\|z_t - v_{n_k}\|^q.\end{aligned}$$

Hence,

$$\begin{aligned}\limsup_{k\to\infty}\langle z_t - u, j_q(z_t - v_{n_k}\rangle &\le \frac{(1-t)^q}{qt}C^q + \frac{(qt-1)}{qt}C^q \\ &= \Big(\frac{(1-t)^q + qt - 1}{qt}\Big)C^q, \qquad (8.2.5)\end{aligned}$$

where $C = \limsup\limits_{k\to\infty}\|z_t - v_{n_k}\|$. Observe that $\lim\limits_{t\to 0}\dfrac{(1-t)^q + qt - 1}{qt} = 0$. By the uniform continuity of j_q on bounded sets and the fact that $z_t \to z$, as $t \to 0$, we have

$$\lim_{t\to 0}\|j_q(z_t - v_{n_k}) - j_q(z - v_{n_k})\| = 0.$$

Thus,

$$\begin{aligned}&|\langle z_t - u, j_q(z_t - v_{n_k})\rangle - \langle z - u, j_q(z - v_{n_k})\rangle| \\ &\quad = |\langle (z_t - z) + (z - u), j_q(z_t - v_{n_k})\rangle - \langle z - u, j_q(z - v_{n_k})\rangle| \\ &\quad \le |\langle z_t - z, j_q(z_t - v_{n_k})\rangle| + |\langle z - u, j_q(z_t - v_{n_k}) - j_q(z - v_{n_k})\rangle| \\ &\quad \le \|z_t - z\|\|z_t - v_{n_k}\|^{q-1} + \|z - u\|\|j_q(z_t - v_{n_k}) - j_q(z - v_{n_k})\|.\end{aligned}$$

Hence, $\lim_{t\to 0}\langle z_t - u, j_q(z_t - v_{n_k})\rangle = \langle z - u, j_q(z - v_{n_k})\rangle$. From inequality (8.2.5), we deduce that

$$\limsup_{k\to\infty}\langle z - u, j_q(z - v_{n_k})\rangle \leq 0.$$

Furthermore, since

$$\zeta_k\langle x_{n_k} - x_{n_k-1}, j_q(y_{n_k} - z)\rangle \leq \zeta_k\|x_{n_k} - x_{n_k-1}\|\|y_{n_k} - z\|^{q-1},$$

$$\limsup_{k\to\infty}\zeta_k\langle x_{n_k} - x_{n-1}, j_q(y_{n_k} - z)\rangle \leq 0,$$

where $\zeta_k = \frac{(1-(1-\gamma_{n_k})\beta_{n_k})\alpha_{n_k}q}{(1-\gamma_{n_k})\beta_{n_k}}$.

Hence, obtain that $\limsup_{k\to\infty}\tau_{n_k} \leq 0$. Hence, by Lemma 2.3.12, $\lim_{n\to\infty} d_n = 0$, i.e.,

$$\lim_{n\to\infty} x_n = z \in (A+B)^{-1}0.$$

This completes the proof.

■

Next, we give a corollary of our main Theorem 8.2.6 in L_q, $2 < q < \infty$ spaces.

Assumption 8.2.7 *The space $E = L_q$, $2 < q < \infty$, $A : E \to E$ is an α-isa operator of order q, $B : E \to 2^E$ is a set-valued m-accretive operator and the solution set $\Omega := (A+B)^{-1}0 \neq \emptyset$.*

Assumption 8.2.8 *Choose sequences $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ and $\{\epsilon_n\}, \{\lambda_n\} \subset (0,\infty)$ such that*

(A4) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=1}^{\infty}\beta_n = \infty$,

(A5) $\lim_{n\to\infty}\epsilon_n = 0$,

(A6) $0 < \lambda \leq (q-1)\lambda_n^{q-1} < \alpha q$.

Based on Assumptions 8.2.7 and 8.2.8, we now give the following algorithm.

Algorithm 8.2.9 *Inertial Halpern-type forward-backward splitting algorithm.*

Step 0. *(Initialization) choose arbitrary points $x_0, x_1 \in E$, and set $n = 1$.*

Step 1. *Choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha_n}$, where*

$$\bar{\alpha_n} = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}, \\ \alpha, & \textit{otherwise}. \end{cases}$$

Step 2. *Compute*

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) \\ v_n = \beta_n u + (1-\beta_n)J^B_{\lambda_n}(y_n - \lambda_n A y_n) \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n)v_n. \end{cases}$$

Step 3. *Update $n = n+1$ and go to Step 1.*

Corollary 8.2.10 *Let $\{x_n\}$ be the sequence generated by Algorithm 8.2.9. Then $\{x_n\}$ converges strongly to $z \in \Omega$.*

Proof. Since L_q, $2 < q < \infty$ spaces are uniformly convex and q-uniformly smooth spaces, the proof follows from Theorem 8.2.6. ∎

8.3 Applications

In this section, we shall apply Theorem 8.2.6 to convex minimization problem and convexly constrained linear inverse problem.

8.3.1 Application to convex minimization problem

Let H be a real Hilbert space and let $h : H \to \mathbb{R}$ be a convex smooth function and $g : H \to \mathbb{R} \cup \{\infty\}$ be a proper lower-semicontinuous and convex function. We consider the following convex minimization problem:

$$\text{Find} \quad x^* \in H \quad \text{such that} \quad h(x^*) + g(x^*) = \min_{x \in H}\Big\{h(x) + g(x)\Big\}. \tag{8.3.1}$$

Problem (8.3.1) is equivalent, by Fermat's rule, to the problem of finding $x^* \in H$ such that

$$0 \in \nabla h(x^*) + \partial g(x^*), \tag{8.3.2}$$

where ∇h is the gradient of h and ∂g is the subdifferential of g. Set $A = \nabla h$ and $B = \partial g$ in Algorithm 8.2.3. It is well-known that if ∇h is $(1/\alpha)$-Lipschitz continuous, then it is α-inverse strongly monotone and ∂g is maximal monotone. Hence from Algorithm 8.2.3 we have the following algorithm:

Algorithm 8.3.1 *Inertial Halpern-type forward-backward splitting algorithm.*

Step 0. *(Initialization) choose arbitrary points $x_0, x_1 \in H$, and set $n = 1$.*

Step 1. *Choose α_n such that $0 \le \alpha_n \le \bar{\alpha_n}$, where*

$$\bar{\alpha_n} = \begin{cases} \min\Big\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\Big\}, & x_n \neq x_{n-1}, \\ \alpha, & \textit{otherwise}. \end{cases}$$

Step 2. *Compute*

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) \\ v_n = \beta_n u + (1-\beta_n)J_{\lambda_n}^{\partial g}(I - \lambda_n \nabla h)y_n \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n)v_n. \end{cases}$$

Step 3. *Update $n = n+1$ and go to Step 1.*

Theorem 8.3.2 *Let $\{x_n\}$ be the sequence generated by Algorithm 8.3.1. Then $\{x_n\}$ converges strongly to $z \in \Omega$.*

Proof. Since Hilbert spaces are uniformly convex and q-uniformly smooth spaces, the proof follows from Theorem 8.2.6. ∎

8.3.2 Application to image restoration problems

General image restoration problem can be formulated by the inversion of the following observation model:

$$b = Lx + y,$$

where b is the observed image, x is the unknown image, y is the noise and L is a linear operator that depends on the concerned image recovery problem. It is well-known that regularization methods are used in image restoration problems. The l_1-regularization is a powerful tool in image denoising. The restoration process is given by:

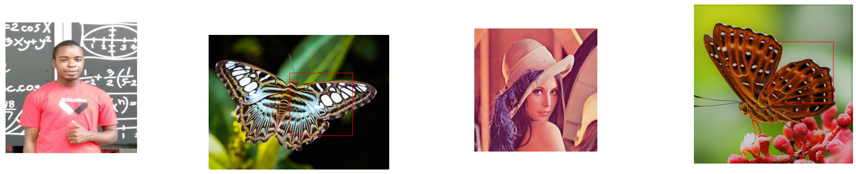
$$\min_x \frac{1}{2}\|Lx - b\|^2 + \lambda\|x\|_1, \tag{8.3.3}$$

where $\|\cdot\|$ denotes the Euclidean norm, λ is a positive regularization parameter and $\|\cdot\|_1$ is the l_1-regularization term.

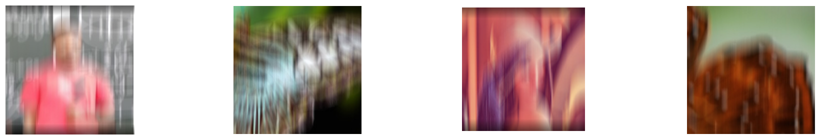
Now, we use algorithms (1.3.4), (1.3.6) and (8.3.1) to approximate the solution of the following convex minimization problem:

$$\text{Find} \quad x \in Argmin_{x\in\mathbb{R}^n}\Big\{\frac{1}{2}\|Lx - b\|^2 + \lambda_n\|x\|_1\Big\}.$$

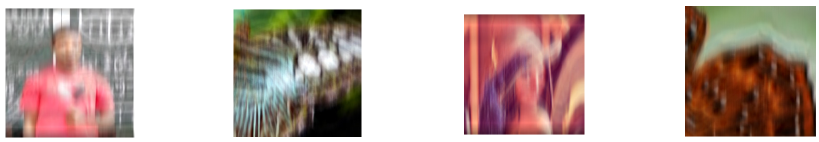
In algorithm (1.3.4), we set $\alpha_n = \frac{1}{1000n}$, $\beta_n = \frac{n}{2n+1}$, $\lambda_n = 0.001$, in algorithm (1.3.6), we take $\beta_n = \frac{1}{1000n}$, $\delta_n = \frac{1}{200n}$, $\lambda_n = 0.001$, $f(u) = \frac{u}{3}$, and in algorithm (8.3.1), we take $\alpha = 0.5$, $\alpha_n = \bar{\alpha_n}$, $\beta_n = \frac{1}{1000n}$, $\epsilon_n = \frac{1}{(n+1)^6}$, $\gamma_n = \frac{1}{(n+1)^8}$, $\lambda_n = 0.001$ as our parameters and in all these algorithms, we set $A = \nabla g$ and $B = \partial h$, where $g(x) = \frac{1}{2}\|Lx - b\|^2$, $h(x) = \lambda_n\|x\|_1$. We consider the blur function in MATLAB "fspecial ('motion', 30, 60)" and add random noise. The test images are Abubakar, Barbra and butterfly (see Figure 8.1) and the stopping criterion of the algorithms is $\frac{\|x_{n+1}-x_n\|}{\|x_{n+1}\|} < 10^{-4}$. As we can see from Figure 8.1 and Table 8.1, our proposed algorithm is competitive and promising.



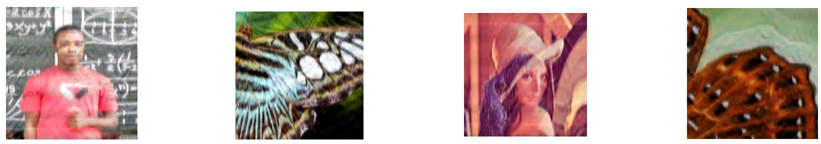
(a) original images



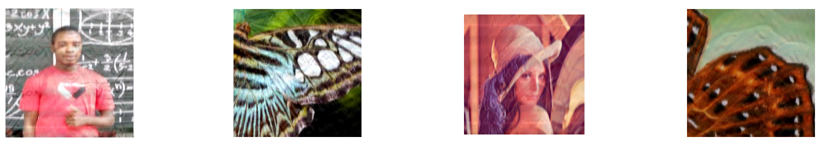
(b) images degraded by motion blur and random noise



(c) restored images with algorithm 1.3.4



(d) restored image with algorithm 1.3.6



(e) restored images with our algorithm 8.3.1

Figure 8.1: Test images and their restorations via algorithms 1.3.4, 1.3.6 and 8.3.1

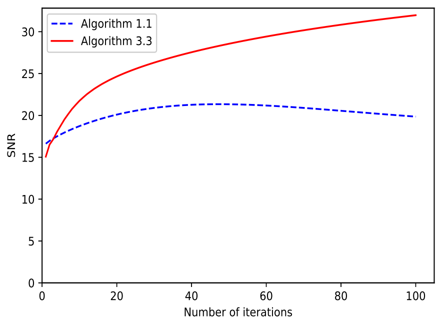
The signal to noise ratio (SNR) is used to measure the quality of the restored images and it is defined as:

$$\mathrm{SNR} := 10 \log \frac{\|x\|^2}{\|x - x_n\|},$$

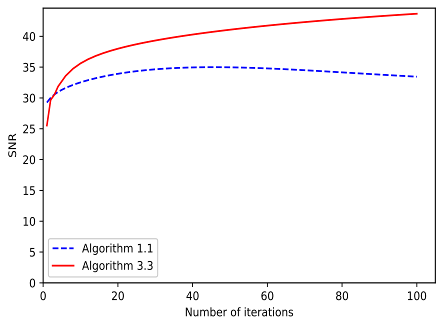
where x and x_n are the original and estimated image at iteration n, respectively. All algorithms were implemented with Ubuntu 64bits and MATLAB 2018b running on a Zinox laptop with Intel(R) Core(TM) i7 CPU and 4 GB of RAM.

Table 8.1: Numerical results of SNR in Figure 8.1

	The Signal to Noise Ratio (SNR)					
	Algorithm 1.3.4		Algorithm 1.3.6		Algorithm 8.3.1	
n	Butterfly Image	Barbra Image	Butterfly Image	Barbra Image	Butterfly Image	Barbra Image
1	16.62	29.25	15.55	26.69	15.07	25.52
10	18.72	32.52	21.25	35.16	21.74	35.61
20	20.09	33.91	23.84	37.3	24.64	37.98
30	20.89	34.64	25.33	38.51	26.31	39.31
40	21.27	34.95	26.38	39.35	27.54	40.28
50	21.33	34.96	27.21	39.99	28.55	41.07
60	21.18	34.79	27.88	40.52	29.41	41.73
70	20.89	34.49	28.44	40.95	30.16	42.31
80	20.55	34.14	28.91	41.32	30.83	42.81
90	20.19	33.79	29.32	41.63	31.42	43.26
100	19.85	33.44	29.66	41.91	31.96	43.66

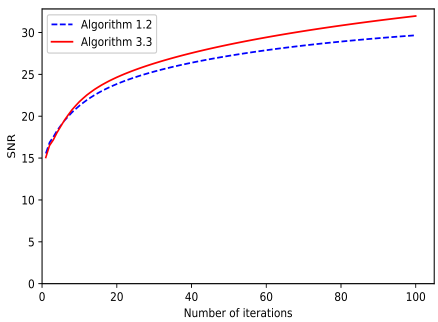


(a) SNR of algorithms 1.3.4 and 8.3.1 shown for colour butterfly image in Table 8.1

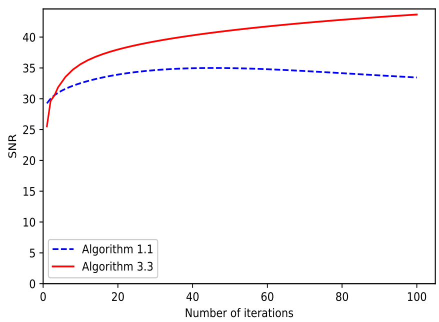


(b) SNR of algorithms 1.3.4 and 8.3.1 shown for Barbra image in Table 8.1

Figure 8.2: Graphical illustration of the data in Table8.1



(a) SNR of algorithms 1.3.6 and 8.3.1 shown for colour butterfly image in Table 8.1



(b) SNR of algorithms 1.3.6 and 8.3.1 shown for Barbra image in Table 8.1

Figure 8.3: Graphical illustration of the data in Table8.1

8.4 Numerical Illustration

In this section, we present a numerical example to compare the convergence of the sequence generated by our algorithm 8.2.3 and that algorithms 1.3.4, 1.3.6 and 1.3.7.

Example 8.4.1

We consider the Banach space $E = L_5([-1,1])$, with norm defined by

$$\|x\|_5 := \Big(\int_{-1}^{1} |x(t)|^5 dt\Big)^{\frac{1}{5}} \quad \forall x, y \in E.$$

Let $A : E \to E, \quad B : E \to E$, be defined as

$$Ax(t) := 5x(t) + t + \cos t, \qquad\qquad Bx(t) := 2x(t).$$

Then, it is easy to see that A is $\frac{1}{5}$-isa of order 2, B is m-accretive. Furthermore, the solution set $\Omega = (A+B)^{-1}0 = \Big\{\frac{-(t+\cos t)}{7}\Big\}$. Observe that

$$J_\lambda^B(I - \lambda A)x(t) = \frac{1-5\lambda}{1+2\lambda}x(t) - \frac{\lambda}{1+2\lambda}(t + \cos t), \quad \forall\, \lambda > 0.$$

We compare the convergence of the sequence of Algorithms 1.3.4, 1.3.6, 1.3.7 and that of Algorithm 8.2.3. In the Algorithm 1.3.4, we take $\alpha_n = \frac{1}{1000n}$, $\beta_n = \frac{n}{2n+1}$, $\lambda_n = \frac{1}{64}$ and $u(t) = \frac{t}{2}$, in Algorithm 1.3.6, we take $\alpha_n = 0.4$, $\beta_n = \frac{1}{1000n}$, $\delta_n = \frac{1}{200n}$, $\lambda_n = \frac{1}{64}$ and $fx(t) = \frac{x(t)}{3}$, in Algorithm 1.3.7, we take $\alpha_n = \frac{1}{1000n}$, $\lambda_n = \frac{1}{64}$ and $\bar{\beta_n} = \beta_n$, $\beta = 0.8$, $a_n(t) = b_n(t) = 0$ and, in Algorithm 8.2.3, we take $\alpha = 0.8$, $\alpha_n = \bar{\alpha_n}$, $\beta_n = \frac{1}{1000n}$, $\epsilon_n = \frac{1}{(n+1)^6}$, $\gamma_n = \frac{1}{(n+1)^8}$, $\lambda_n = \frac{1}{64}$ and $u(t) = \frac{t}{2}$ as our parameters. Clearly, these parameters satisfy the hypothesis of the theorem of [Pholasaa et al., 2016], [Pan and Wang, 2019], [Cholamjiak and Shehu, 2019b] and Theorem 8.2.6, respectively. Finally, we using a tolerance of 10^{-3}, setting maximum number of iterations $n = 15$ and choosing $x_0 = 2t^2 + 1$, $x_1 = -t^3$.

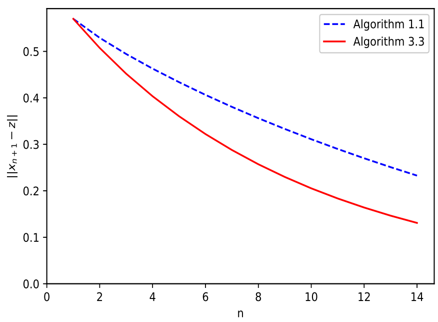
Table 8.2: Numerical results for Example 8.4.1

n	Algorithm 1.3.4 $\|x_n - z\|$	Algorithm 1.3.6 $\|x_n - z\|$	Algorithm 1.3.7 $\|x_n - z\|$	Algorithm 8.2.3 $\|x_n - z\|$
1	0.5701	0.5701	0.5701	0.5701
2	0.5292	1.2778	0.5084	0.5074
3	0.4944	1.4553	0.4535	0.4520
5	0.4338	1.1933	0.3624	0.3605
7	0.3806	0.8370	0.2897	0.2878
9	0.3331	0.5669	0.2317	0.2298
11	0.29	0.3817	0.1853	0.164
13	0.2509	0.2575	0.1482	0.1309
14	0.2327	0.2118	0.1325	0.1309

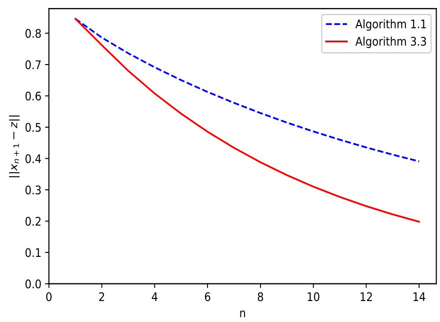
Choosing $x_0 = t - 4$, $x_1 = \sin t$, we obtain the following results

Table 8.3: Numerical results for Example 8.4.1

n	Algorithm 1.3.4 $\|x_n - z\|$	Algorithm 1.3.6 $\|x_n - z\|$	Algorithm 1.3.7 $\|x_n - z\|$	Algorithm 8.2.3 $\|x_n - z\|$
1	0.8463	0.8463	0.8463	0.8463
2	0.7863	2.0722	0.7574	0.7621
3	0.7361	2.3608	0.6725	0.6801
5	0.6504	1.9445	0.5322	0.5432
7	0.5777	1.3661	0.4221	0.4341
9	0.5147	0.9259	0.3561	0.3469
11	0.46	0.6236	0.2684	0.2773
13	0.4123	0.4207	0.2171	0.2216
14	0.3908	0.3461	0.1966	0.1981

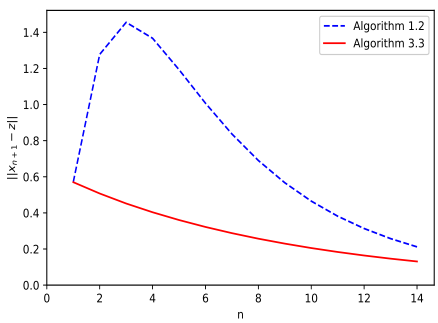


(a) Graph of the first 14 iterates of Algorithms 1.3.4 and 8.2.3 choosing $x_0 = 2t^2 + 1$, $x_1 = -t^3$

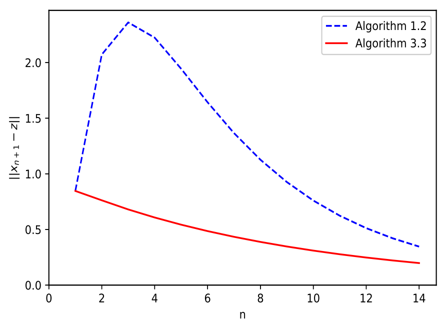


(b) Graph of the first 14 iterates of Algorithms 1.3.4 and 8.2.3 choosing $x_0 = t-4$, $x_1 = \sin t$

Figure 8.4: Graphical illustration of the data in Tables 8.2 and 8.3

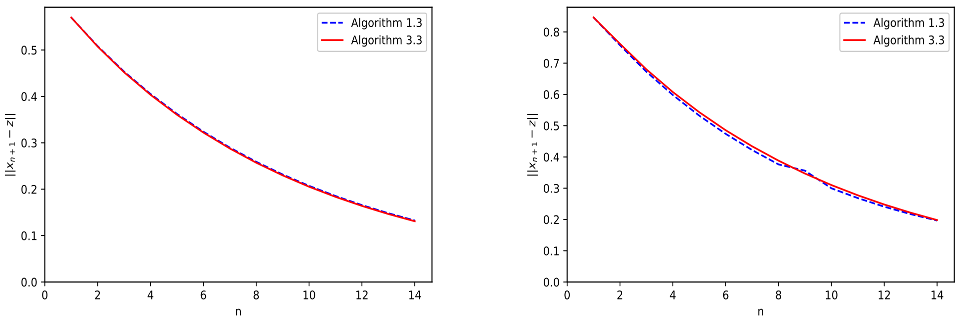


(a) Graph of the first 14 iterates of Algorithms 1.3.7 and 8.2.3 choosing $x_0 = 2t^2 + 1$, $x_1 = -t^3$



(b) Graph of the first 14 iterates of Algorithms 1.3.7 and 8.2.3 choosing $x_0 = t-4$, $x_1 = \sin t$

Figure 8.5: Graphical illustration of the data in Tables 8.2 and 8.3



(a) Graph of the first 14 iterates of Algorithms 1.3.6 and 8.2.3 choosing $x_0 = 2t^2 + 1$, $x_1 = -t^3$

(b) Graph of the first 14 iterates of Algorithms 1.3.6 and 8.2.3 choosing $x_0 = t - 4$, $x_1 = \sin t$

Figure 8.6: Graphical illustration of the data in Tables 8.2 and 8.3

Conclusion. In this Chapter, an inertial version of the algorithm of studied by [Takahashi et al., 2012] is introduced and studied. Strong convergence of the sequence of the proposed algorithm is proved in real Banach spaces that are uniformly convex and q-uniformly smooth. Furthermore, the strong convergence result obtained is applied to convex minimization and image restoration problems. Numerical experiments were carried out on some classical test images and personal images degraded with motion blur and random noise. From the results obtained using these images (see Figure 8.1 and Table 8.1) the proposed algorithm appears to competitive and promising. Finally, a numerical example is presented in section 8.4 to support the main theorem.

The results obtained in this chapter are still in the refereeing process in the **Journal of Computational and Applied Mathematics**.

CHAPTER 9

CONCLUSION

9.1 Key results

This thesis provides a comprehensive study of iterative algorithms for approximating solutions of nonlinear inclusion problems involving monotone and accretive operators in real Banach spaces more general than real Hilbert spaces. The following important to topics are studied in the thesis:

1. Approximation of zeros of m-accretive and maximal monotone operators.
2. Approximation of solutions of Hammerstein equations involving maximal monotone operators.
3. Approximation of zeros of sum of accretive operators with an application to image restoration problems.

The key results presented are outlined below:

- Let E be a smooth and reflexive real Banach space. Any accretive mapping $A : D(A) \subset E \to 2^E$ with $0 \in int\ D(A)$ is quasi-bounded.

- Let E be a uniformly smooth real Banach space and let $A : E \to 2^E$ be a set-valued m-accretive mapping such that the inclusion $0 \in Au$ has a solution. For arbitrary $u_1 \in E$, define inductively a sequence $\{u_n\}$ in E by
$$u_{n+1} = (1 - \alpha_n\beta_n)u_n - \alpha_n\zeta_n,\ \zeta_n \in Au_n,\ n \geq 1. \tag{9.1.1}$$
Then the sequence $\{u_n\}$ converges strongly to a zero of A.

- Let E be a uniformly smooth and uniformly convex real Banach space. Let $A : E \to 2^{E^*}$ be a maximal monotone operator with $D(A) = E$ such that the inclusion $0 \in Au$ has a solution. For arbitrary $u_0, u_1 \in E$, define the sequence $\{u_n\}$ in E by
$$\begin{cases} w_n = u_n + \beta_n(u_n - u_{n-1}), \\ u_{n+1} = J^{-1}\left(Jw_n - \lambda_n\mu_n - \lambda_n\theta_n Jw_n\right),\ \mu_n \in Aw_n,\ n \geq 1. \end{cases} \tag{9.1.2}$$

Then, the sequence $\{u_n\}$ converges strongly to a zero of A.

- Let E be a uniformly convex and uniformly smooth real Banach space. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone mappings. For arbitrary $x \in E$, $y \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively by $u_1 \in E$, $v_1 \in E^*$,

$$\begin{cases} u_{n+1} = J^{-1}\big(Ju_n - \beta_n^2(Fu_n - v_n) - \alpha_n\beta_n(Ju_n - Jx)\big), \\ v_{n+1} = J\big(J^{-1}v_n - \beta_n^2(Kv_n + u_n) - \alpha_n\beta_n(J^{-1}v_n - J^{-1}y)\big). \end{cases} \tag{9.1.3}$$

  Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of $u + KFu = 0$ with $v^* = Fu^*$.

- Let E be a uniformly convex and uniformly smooth real Banach space. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone mappings. For $u_1 \in E$, $v_1 \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively by

$$\begin{aligned} u_{n+1} &= J^{-1}\Big(Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n Ju_n\Big), \\ v_{n+1} &= J\Big(J^{-1}v_n - \alpha_n(Kv_n + u_n) - \alpha_n\theta_n J^{-1}v_n\Big). \end{aligned} \tag{9.1.4}$$

  Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of $u + KFu = 0$ with $v^* = Fu^*$.

- Let E be a uniformly convex and q-uniformly smooth real Banach space. Let $A : E \to E$ be an α-inverse strongly accretive operator of order q, and $B : E \to 2^E$ be a set-valued m-accretive operator, $f : E \to E$ be a contraction with constant $k \in (0,1)$ and $S : E \to E$ be a nonexpansive mapping such that the solution set $\Omega := F(S) \cap (A+B)^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in E$, define the sequence $\{x_n\}$ in E by

$$\begin{aligned} z_n &= \gamma_n x_n + (1-\gamma_n)J_{\lambda_n}^B(I - \lambda_n A)x_n, \\ y_n &= s_n x_n + (1-s_n)J_{\lambda_n}^B(I - \lambda_n A)z_n, \\ x_{n+1} &= \tau_n f(x_n) + \sigma_n x_n + \mu_n S y_n, \end{aligned} \tag{9.1.5}$$

  Then, the sequence $\{u_n\}$ converges strongly to a point in Ω.

- Let E be a uniformly convex and q-uniformly smooth real Banach space. Let $A : E \to E$ be an α-inverse strongly accretive operator of order q, and $B : E \to 2^E$ be a set-valued m-accretive operator such that the inclusion $0 \in (A+B)u$ has a solution. For arbitrary $x_0, x_1 \in E$, define the sequence $\{x_n\}$ in E by

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ v_n = \beta_n u + (1-\beta_n)J_{\lambda_n}^B(y_n - \lambda_n A y_n), \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n)v_n. \end{cases}$$

  Then, the sequence $\{u_n\}$ converges strongly to a zero of $(A+B)$.

9.2 Future work

Areas to consider for future research include:

1. **Equations with d-accretive operators.**

Definition 9.2.1 *Let X be a reflexive, strictly convex and smooth real Banach space and X^* also be strictly convex. An operator $A : X \to 2^X$ with $D(A) \subset X$ is said to be d-accretive if for all $x, y \in D(A)$, for all $u \in Ax$, for all $v \in Ay$,*

$$\langle u - v, Jx - Jy \rangle \geq 0.$$

Remark 9.2.2 *In real Hilbert spaces, this definition coincides with the classical definition of accretive operators.*

The concept of d-accretive operator was introduced by [Alber and Reich, 1994]. Interested readers should see section 1.16 of [Alber and Ryazantseva, 2006] for established results concerning the class of d-accretive operators.

Definition 9.2.3 ([Alber and Ryazantseva, 2006]) *Let X be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of X. The map $\Pi_C : X \to C$ defined by $\tilde{x} = \Pi_C(x) \in C$ such that $\phi(\tilde{x}, x) = \inf_{y \in C} \phi(y, x)$ is called the generalized projection of x onto C.*

Next, we give an interesting example of a d-accretive operator.

Example 9.2.4 (see [Alber and Ryazantseva, 2006] page 109) *The generalized projection Π_C is d-accretive.*

Proof. Let $\hat{x_1} = \Pi_C x_1$ and $\hat{x_2} = \Pi_C x_2$, then the following inequalities hold

$$\phi(\zeta, x_1) \geq \phi(\hat{x_1}, x_1) \quad \text{and} \quad \phi(\eta, x_2) \geq \phi(\hat{x_2}, x_2),$$

for all $\zeta, \eta \in C$ and for all $x_1,\ x_2 \in X$. Assume $\zeta = \hat{x_2}$ and $\eta = \hat{x_1}$. Then

$$\phi(\hat{x_2}, x_1) \geq \phi(\hat{x_1}, x_1) \quad \text{and} \quad \phi(\hat{x_1}, x_2) \geq \phi(\hat{x_2}, x_2).$$

Thus,

$$\begin{aligned} &\|\hat{x_2}\|^2 - 2\langle \hat{x_2}, Jx_1 \rangle + \|x_1\|^2 + \|\hat{x_1}\|^2 - 2\langle \hat{x_1}, Jx_2 \rangle + \|x_2\|^2 \\ &\geq \|\hat{x_1}\|^2 - 2\langle \hat{x_1}, Jx_1 \rangle + \|x_1\|^2 + \|\hat{x_2}\|^2 - 2\langle \hat{x_2}, Jx_2 \rangle + \|x_2\|^2. \end{aligned}$$

Hence,

$$\langle \hat{x_1}, Jx_1 \rangle + \langle \hat{x_2}, Jx_2 \rangle \geq \langle \hat{x_2}, Jx_1 \rangle + \langle \hat{x_1}, Jx_2 \rangle$$

and, thus,

$$\langle \Pi_C x_1 - \Pi_C x_2, Jx_1 - Jx_2 \rangle \geq 0, \quad \forall\ x_1, x_2 \in X.$$

Therefore, Π_C is d-accretive (see, e.g., [Alber and Ryazantseva, 2006] page 108 for another important examples of d-accretive operators). ■

Remark 9.2.5 *It is well known that the generalized projection operator has important applications in the theory of approximation and optimization.*

In the nearest future, we shall explore this concept of d-accretive.

2. **Efficient alternative of the FBA.**

   Just like for the case of PPA, the FBA also involves the computation of $\big(I + \lambda B\big)^{-1}$ which is difficult to compute in some applications. As shown numerically in Chapters 3 and 4 of this thesis, the alternatives of the PPA studied in these Chapters converge faster in term of number of iterations and computer time than the PPA and its modifications in all the problems considered. Following this, the following problems are of interest.

   Problem 1. Can an efficient alternative of the FBA be developed which will not involve the computation of $\big(I + \lambda B\big)^{-1}$ at any step of the iteration process and still guarantee strong convergence to a solution of the inclusion $0 \in (A + B)u$?

   Problem 2. One can consider analogues of the theorems proved in Chapters 7 and 8 of this thesis for the class of monotone operators in real Banach spaces. A quick review shows that there are only few results in this direction. Some of the few results in the literature are the following: [Shehu, 2019], [Kimura and Nakajo, 2019], [Cholamjiak et al., 2020]. We have not found any published result in this direction to be the best of our knowledge before 2018.

   Problem 3. One consider application of the FBA and its modifications to signal processing in compressed sensing and machine learning.

3. **Image inpainting.**

   Problem 4. One can consider introducing and studying iterative algorithms for approximating zeros sum of three monotone operators (see [Davis and Yin, 2017] for the setting). This problem has application in image inpainting.

   Problem 5. One can consider improving the existing MATLAB codes for image restoration, signal processing and image inpainting to be able to restore an arbitrary degraded image.

LIST OF PUBLICATIONS ARISING FROM THE THESIS AND OTHER PEER-REVIEW PUBLICATIONS

(A) **Papers Published/Accepted from the Thesis**

1. C.E. Chidume, **A. Adamu**, M.S. Minjibir and U.V. Nnyaba, *On the strong convergence of the proximal point algorithm with an application to Hammerstein equations*, **Journal of Fixed Point Theory and Applications** (2020) 22:61 https://doi.org/10.1007/s11784-020-00793-6.
Q1, ISI indexed

2. C.E. Chidume, **A. Adamu**, M.O. Nnakwe, *Strong convergence of an inertial algorithm for maximal monotone inclusions with applications*, **Fixed Point Theory and Applications** (2020)
https://doi.org/10.1186/s13663-020-00680-2. **Q2, Scopus indexed**

3. C.E. Chidume, **A. Adamu** and L.O. Chinwendu, *Iterative algorithms for solutions of Hammerstein equations in real Banach spaces,* **Fixed Point Theory and Applications** (2020)
https://doi.org/10.1186/s13663-020-0670-7. **Q2, Scopus indexed**

4. C.E. Chidume, **A. Adamu** and L.O. Chinwendu, *Approximation of solutions of Hammerstein equations with monotone mappings in real Banach spaces,* **Carpathian Journal of Mathematics,** 35 (2019), No. 3, 305 - 316. **Q2, ISI indexed**

5. C.E. Chidume, **A. Adamu**, P. Kumam and D. Kitkuan, *Generalized hybrid viscosity-type forward-backward splitting method with application to convex minimization and image restoration problems,* **Numerical Functional Analysis and Optimization (Accepted)**. **Q2, ISI indexed**

6. C.E. Chidume, **A. Adamu**, P. Kumam and D. Kitkuan, *An accelerated Halpern-type forward-backward splitting method with application to image processing,* **Journal of Computational and Applied Mathematics, (to appear)**. **Q2, ISI indexed**

(B) **Other Peer-reviewed Published/Accepted Papers not Included in the Thesis (to Limit its Size)**

7. C.E. Chidume, **A. Adamu** and L.O. Chinwendu, *A Krasnoselskii-type algorithm for approximating solutions of variational inequality problems and convex feasibility problems,* **Journal of Nonlinear Variational Analysis,** Volume 2, Issue 2, Pages 203-218. **Q1, Scopus indexed**

8. C.E. Chidume, **A. Adamu** and L.O. Chinwendu, *Strong convergence theorem for some nonexpansive-type mappings in certain Banach spaces,* **Thai Journal of Mathematics,** Volume 18 Number 3 (2020) Pages 1537-1548. **Q4, ISI indexed**

9. C.E Chidume, P. Kumam and **A. Adamu**, *A hybrid inertial algorithm for approximating solution of convex feasibility problems with applications*, **Fixed Point Theory and Applications** (2020) https://doi.org/10.1186/s13663-020-00678-w. **Q2, Scopus indexed**

10. C.E. Chidume, M. O. Nnakwe and **A. Adamu**, *A strong convergence theorem for generalized Φ-strongly monotone maps, with applications,* **Fixed Point Theory and Applications** (2019) https://doi.org/10.1186/s13663-019-0660-9. **Q2, Scopus indexed**

11. C.E. Chidume, L.O. Chinwendu and **A. Adamu**, *A hybrid algorithm for approximating solutions of a variational inequality problem and a convex feasibility Problem*, **Advances in Nonlinear Variational inequalities** Vol. 21 (2018), No. 1, 46 - 64. **Q4 Scopus indexed**

12. C.E. Chidume, S.I. Ikechukwu and **A. Adamu**, *Inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps,* **Fixed Point Theory and Applications** (2018) https://doi.org/10.1186/s13663-018-0634-3. **Q2, Scopus indexed**

13. C.E. Chidume, G.S De Souza, U.V Nnyaba, O.M Romanus and **A. Adamu**, *Approximation of zeros of m-accretive mappings, with applications to Hammerstein integral equations,* **Carpathian Journal of Mathematics,** 36 (2020), No. 1, 45 - 55. **Q2, ISI indexed**

CHAPTER 10

APPENDICES

10.1 Appendix A. Verification

Verification 10.1.1 **(**∂f **is monotone and** $0 \in \partial f(u) \;\Leftrightarrow\; f(u) = \min_{x \in E} f(x)$**.)**

To see this, let $u, v \in E$ and let $u^* \in \partial f(u)$ and $v^* \in \partial f(v)$. We show that

$$\langle u - v, u^* - v^* \rangle \geq 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between elements of E and E^*. Now, by definition, $u^* \in \partial f(u)$ implies $\langle y - u, u^* \rangle \leq f(y) - f(u)$, for all $y \in E$. In particular, setting $y = v \in E$, we have

$$\langle v - u, u^* \rangle \leq f(v) - f(u). \tag{10.1.1}$$

Similarly, $v^* \in \partial f(v)$ implies

$$\langle u - v, v^* \rangle \leq f(u) - f(v). \tag{10.1.2}$$

Adding inequalities (10.1.1) and (10.1.2), we obtain

$$\langle v - u, u^* \rangle + \langle u - v, v^* \rangle \leq 0 \text{ which implies that } \langle u - v, u^* \rangle - \langle u - v, v^* \rangle \geq 0.$$

Thus, $\langle u - v, u^* - v^* \rangle \geq 0$, establishing the monotonicity of ∂f. Furthermore, $0 \in \partial f(x)$ *if and only if* x *is a minimizer of* f. To see this, by definition,

$$\begin{aligned} 0 \in \partial f(x) &\Leftrightarrow \langle y - x, 0 \rangle \leq f(y) - f(x), \quad \forall y \in E. \\ &\Leftrightarrow f(x) \leq f(y), \quad \forall y \in E. \\ &\Leftrightarrow x \text{ is a minimizer of } f. \end{aligned}$$

Verification 10.1.2 (A consequence of accretivity.)

If $A : E \to 2^E$ is accretive then

$$\|x-y\| \le \|x-y+\lambda(\eta-\zeta)\|, \quad \eta \in Ax,\ \zeta \in Ay,\ \lambda > 0.$$

To see this, let $x, y \in E$, $\eta \in Ax$, $\zeta \in Ay$, and let $\lambda > 0$ be given. Then by accretivity of A, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$0 \le \langle \eta-\zeta, j_q(x-y)\rangle.$$

Now,

$$0 \le \langle \eta-\zeta, j_q(x-y)\rangle \quad \Leftrightarrow 0 \le \langle \lambda(\eta-\zeta), j_q(x-y)\rangle.$$

Thus,

$$\begin{aligned}
0 &\le \langle \lambda(\eta-\zeta), j_q(x-y)\rangle \\
&= -\langle x-y, j_q(x-y)\rangle + \langle x-y+\lambda(\eta-\zeta), j_q(x-y)\rangle \\
&= -\|x-y\|^q + \langle x-y+\lambda(\eta-\zeta), j_q(x-y)\rangle \\
&\le -\|x-y\|^q + \|x-y+\lambda(\eta-\zeta)\|\|j_q(x-y)\| \\
&= -\|x-y\|^q + \|x-y+\lambda(\eta-\zeta)\|\|x-y\|^{q-1}.
\end{aligned}$$

This implies that

$$\|x-y\| \le \|x-y+\lambda(\eta-\zeta)\|.$$

Verification 10.1.3 (J_λ is single value and nonexpansive.)

To see this, let $u \in H$, $\lambda > 0$ be given. Let $\zeta, \eta \in J_\lambda u$. We show that $\zeta = \eta$. Now,

$$\begin{aligned}
\zeta \in J_\lambda u &\Leftrightarrow \zeta \in (I+\lambda A)^{-1}u \\
&\Leftrightarrow (\zeta + \lambda A\zeta) \ni u \\
&\Leftrightarrow \frac{(u-\zeta)}{\lambda} \in A\zeta.
\end{aligned}$$

Similarly, $\eta \in J_\lambda u \Leftrightarrow \frac{(u-\eta)}{\lambda} \in A\eta$. Since A is monotone (accretive), using this $\lambda > 0$ given, we have that

$$\|\zeta-\eta\| \le \Big\|\zeta-\eta+\lambda\Big(\frac{(u-\zeta)}{\lambda} - \frac{(u-\eta)}{\lambda}\Big)\Big\| = 0.$$

Thus, $\zeta = \eta$. Hence, J_λ is single valued. Next, we show that J_λ is nonexpansive, i.e.,

$$\|J_\lambda u - J_\lambda v\| \le \|u-v\|.$$

Let $u, v \in H$ and let $\lambda > 0$ be given. Set $\zeta_u = J_\lambda u$ and $\eta_v = J_\lambda v$. Now, $\zeta_u = J_\lambda u \Leftrightarrow \frac{(u-\zeta_u)}{\lambda} \in A\zeta_u$. Also, $\eta_v = J_\lambda v \Leftrightarrow \frac{(v-\eta_v)}{\lambda} \in A\eta_v$. Using this $\lambda > 0$ and the fact that A is monotone on H, we have that

$$\begin{aligned}
\|\zeta_u - \eta_v\| &\le \Big\|\zeta_u - \eta_v + \lambda\Big(\frac{(u-\zeta_u)}{\lambda} - \frac{(v-\eta_v)}{\lambda}\Big)\Big\| \\
&= \|u-v\|,
\end{aligned}$$

establishing the nonexpansivity of J_λ.

Verification 10.1.4 (A counter example.)

Composition of two monotone maps need not be monotone. In fact, consider the mappings $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $K : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$Fu = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \text{and} \qquad Kv = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

It is easy to see that K and F are monotone. To see this, since K and F are linear, it is enough to show that $\langle u, Fu \rangle \geq 0$ and $\langle u, Ku \rangle \geq 0$. Now, let $u \in \mathbb{R}^2$. Then,

$$Fu = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (u_1 + u_2, -u_1 + u_2).$$

Thus,

$$\begin{aligned} \langle u, Fu \rangle &= u_1(u_1 + u_2) + u_2(-u_1 + u_2) \\ &= u_1^2 + u_2^2 \geq 0. \end{aligned}$$

Hence, F is monotone. Similarly, K is monotone. However, $KF = \begin{pmatrix} -1 & 3 \\ -3 & 1 \end{pmatrix}$ is *not*. Taking $x_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, we have that $\langle x_0, KFx_0 \rangle = -4$.

Verification 10.1.5 (Is $\mathbb{R}$ uniformly smooth?)

Proof. Let $x, y \in \mathbb{R}$ such that $|x| = |y| = 1$, and let $\tau \geq$ be given. Then,

$$\begin{aligned} \rho(\tau) &= \sup\left\{ \frac{|x + \tau y| + |x - \tau y|}{2} - 1 : |x| = 1; |y| = 1 \right\} \\ &= \begin{cases} 0, & 0 \leq \tau \leq 1; \\ \tau - 1 & 1 < \tau < \infty. \end{cases} \end{aligned} \tag{10.1.3}$$

This implies that $\lim_{\tau \to 0^+} \frac{\rho(\tau)}{\tau} = 0$. ■

Verification 10.1.6 (Prototype of the parameters for Thoerem 3.2.5.)

For L_p spaces, $2 \leq p < \infty$, let $\alpha_n = (n+1)^{-a}$ and $\beta_n = (n+1)^{-b}$, $n \geq 1$ with $0 < b < a$, $\frac{1}{2} < a < 1$ and $a + b < 1$.

Now, we verify conditions (i)-(iii) and $\frac{\rho_E(\alpha_n M_0)}{\alpha_n M_0} \leq \gamma_0 \beta_n$ given in Thoerem 3.2.5.

Clearly, $\lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \frac{1}{(n+1)^b} = 0$ and the sequence β_n is decreasing.

For (ii), using the fact that $a + b < 1$, we have $\sum_{n=1}^{\infty} \alpha_n \beta_n = \sum_{n=1}^{\infty} \frac{1}{(n+1)^{a+b}} = \infty$.

Furthermore, the condition $\frac{1}{2} < a < 1$ implies that

$$\sum_{n=1}^{\infty} \rho_E(\alpha_n M_0) \leq \sum_{n=1}^{\infty} \left(\frac{p-1}{2}\right) \alpha_n^2 M_0^2 \leq M_0^2 \left(\frac{p-1}{2}\right) \sum_{n=1}^{\infty} \frac{1}{(n+1)^{2a}} < \infty.$$

Next, for (iii), using the fact that $(1+x)^s \leq 1 + sx$, for $x > -1$ and $0 < s < 1$, we have

$$\begin{aligned} 0 \leq \frac{\left(\frac{\beta_{n-1}}{\beta_n} - 1\right)}{\alpha_n \beta_n} &= \left[\left(1 + \frac{1}{n}\right)^b - 1\right] \cdot (n+1)^{a+b} \\ &\leq b \cdot \frac{(n+1)^{a+b}}{n} = b \cdot \frac{n+1}{n} \cdot \frac{1}{(n+1)^{1-(a+b)}} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Finally, using the fact that $\rho_E(t) \leq \frac{(p-1)}{2} t^2$, $0 < b < a$ and $\alpha_n = (n+1)^{-a} \leq \beta_n = (n+1)^{-b}$, we obtain:

$$\begin{aligned} \frac{\rho_E(\alpha_n M_0)}{\alpha_n M_0} &\leq \frac{(p-1)}{2\alpha_n} \cdot \alpha_n^2 \\ &= \frac{(p-1)}{2} \alpha_n = \frac{(p-1)}{2} (n+1)^{-a} \\ &\leq \frac{(p-1)}{2} (n+1)^{-b} = \gamma_0 \beta_n, \end{aligned}$$

where $\gamma_0 := \frac{(p-1)}{2}$. This completes the verification.

Similarly, for L_p spaces, $1 < p \leq 2$, let $\alpha_n = (n+1)^{-a}$ and $\beta_n = (n+1)^{-b}$, $n \geq 1$ with $0 < b < (p-1)a$, $\frac{1}{p} < a < 1$ and $a + b < 1$, it can be shown that the conditions (i)-(iii) and $\frac{\rho_E(\alpha_n M_0)}{\alpha_n M_0} \leq \gamma_0 \beta_n$ of Theorem 3.2.5 are satisfied.

Verification 10.1.7 (Prototypes of the parameters for Theorem 4.2.2.)

For $E = L_p$, $1 < p < \infty$, take the following:

For L_p spaces, $2 \leq p < \infty$,

$\lambda_n = (n+1)^{-\frac{1}{2}}$, $\theta_n = (n+1)^{-\frac{1}{4p}}$ and $\beta_n = (n+1)^{-(2+\frac{1}{4p})}$, $n \geq 1$.

For L_p spaces, $1 < p < 2$,

$\lambda_n = (n+1)^{-\frac{1}{4}}$, $\theta_n = (n+1)^{-\frac{1}{16}}$ and $\beta_n = (n+1)^{-\frac{17}{16(p-1)}}$, $n \geq 1$.

With these choices, the conditions (i) - (ix) given in Lemma 4.2.1 and Thoerem 4.2.2 are easily satisfied (see, e.g., [Chidume and Idu, 2016] page 16).

Verification 10.1.8 (Prototype of the parameters for Thoerem 5.2.5.)

We verify conditions (ii)-(iv) of our theorem for L_p spaces, $1 < p < \infty$. For $p > 1$, $q > 1$, $E = L_p$, so that $E^* = L_q$ $(\frac{1}{p} + \frac{1}{q} = 1)$. Then (see, e.g., [Alber and Ryazantseva, 2006], pg 47):

$$\delta_{E^*}(\epsilon) = 1 - \left[1 - \left(\frac{\epsilon}{2}\right)^q\right]^{\frac{1}{q}} \qquad (0 < \epsilon \leq 2).$$

Let $y = \delta_{E^*}(\epsilon)$. Then,

$$\begin{aligned}
y &= 1 - \left[1 - \left(\frac{\epsilon}{2}\right)^q\right]^{\frac{1}{q}} \qquad (0 < \epsilon \leq 2)\\
\left[1 - \left(\frac{\epsilon}{2}\right)^q\right]^{\frac{1}{q}} &= 1 - y\\
1 - \left(\frac{\epsilon}{2}\right)^q &= (1-y)^q\\
\left(\frac{\epsilon}{2}\right)^q &= 1 - (1-y)^q\\
\frac{\epsilon}{2} &= \big(1 - (1-y)^q\big)^q\\
\epsilon &= 2\big(1 - (1-y)^q\big)^q.
\end{aligned}$$

Thus, $\delta_{E^*}^{-1}(\epsilon) = 2\Big(1 - (1-\epsilon)^q\Big)^{\frac{1}{q}} \leq 2q^{\frac{1}{q}}\epsilon^{\frac{1}{q}}$, since $(1-\epsilon)^q > 1 - q\epsilon$, $\;for\; q > 1$. We verify the following:

(ii) $\delta_E^{-1}(\beta_n M_0) \leq \alpha_n \gamma_0$;

(iii) $\delta_E^{-1}(\eta_n) = \delta_E^{-1}\Big(\frac{\alpha_{n-1}\beta_n}{\alpha_n\beta_n} - 1\Big) \to 0$, as $n \to \infty$;

(iv) $\dfrac{\delta_E^{-1}\Big(\frac{\alpha_{n-1}\beta_n}{\alpha_n\beta_n} - 1\Big)}{\alpha_n\beta_n} \to 0$, as $n \to \infty$.

Now, to verify condition (ii), we have

$$\begin{aligned}
\frac{\delta_E^{-1}(\beta_n M_0)}{\alpha_n} &= \frac{2\Big(1 - (1-\beta_n M_0)\Big)^{\frac{1}{p}}}{\alpha_n}\\
&\leq \frac{2p^{\frac{1}{p}}(\beta_n M_0)^{\frac{1}{p}}}{\alpha_n}\\
&= \frac{2(pM_0)^{\frac{1}{p}}(n+1)^{-b}}{(n+1)^{-a}} = 2(pM_0)^{\frac{1}{p}}(n+1)^{a-b} \to 0, \quad as \;\; n \to \infty.
\end{aligned}$$

Next we show that condition (iii) is satisfied

$$
\begin{aligned}
\delta_E^{-1}\Big(\frac{\alpha_{n-1}\beta_n}{\alpha_n\beta_n}-1\Big) &= 2\Big[1-\Big(1-\frac{\alpha_{n-1}\beta_n}{\alpha_n\beta_{n-1}}+1\Big)^p\Big]^{\frac{1}{p}}\\
&= 2\Big[1-\Big(2-\frac{\alpha_{n-1}\beta_n}{\alpha_n\beta_{n-1}}\Big)^p\Big]^{\frac{1}{p}} = 2\Big[1-\Big(2-\frac{n^{-a}(n+1)^{-b}}{(n+1)^{-a}n^{-b}}\Big)^p\Big]^{\frac{1}{p}}\\
&= 2\Big[1-\Big(2-\Big(\frac{n+1}{n}\Big)^a\Big(\frac{n}{n+1}\Big)^b\Big)^p\Big]^{\frac{1}{p}}\\
&\le 2\Big[1-\Big(2-\Big(1+\frac{1}{n}\Big)^a\Big)^p\Big]^{\frac{1}{p}} \le 2\Big[1-\Big(1-\frac{a}{n}\Big)^p\Big]^{\frac{1}{p}}\\
&\le 2p^{\frac{1}{p}}\Big(\frac{a}{n}\Big)^{\frac{1}{p}} \to 0, \quad as \quad n\to\infty.
\end{aligned}
$$

Finally, we verify (iv). Using (iii), we have

$$
\begin{aligned}
\frac{\delta_E^{-1}\left(\frac{\alpha_{n-1}\beta_n}{\alpha_n\beta_n}-1\right)}{\alpha_n\beta_n} &\le \frac{2p^{\frac{1}{p}}\left(\frac{a}{n}\right)^{\frac{1}{p}}}{(n+1)^{-(a+b)}}\\
&= 2p^{\frac{1}{p}}\Big(\frac{a}{n}\Big)^{\frac{1}{p}}(n+1)^{(a+b)}\\
&\le 2^{a+b+1}(pa)^{\frac{1}{p}}n^{a+b-\frac{1}{p}} \to 0, \quad as \quad n\to\infty.
\end{aligned}
$$

10.2 Appendix B. Matlab & python codes

Contact the author (Abubakar Adamu) via email at aadamu@aust.edu.ng for matlab and python codes used for generating the iterates and plots in the numerical sections of this thesis.

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