ITERATIVE METHODS FOR APPROXIMATION OF FIXED POINTS OF CERTAIN MULTIVALUED MAPPINGS IN HADAMARD SPACES

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Certification

This is to certify that the thesis titled "ITERATIVE METHODS FOR APPROXIMATION OF FIXED POINTS OF CERTAIN MULTIVALUED MAPPINGS IN HADAMARD SPACES" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research work carried out by Sani Salisu in the Department of Pure and Applied Mathematics.

Approval

ITERATIVE METHODS FOR APPROXIMATION OF FIXED POINTS OF CERTAIN MULTIVALUED MAPPINGS IN HADAMARD SPACES

By

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A THESIS PRESENTED TO THE DEPARTMENT OF PURE AND APPLIED MATHEMATICS

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ABSTRACT

Let (X,d) be a Hadamard space and let D be its closed convex nonempty set. We studied countable family of multivalued demicontractive mappings $\{T_i\}$ from D to $\mathscr{CB}(D)$ with constants $\{k_i\} \subset (0,1)$ and developed an iterative scheme for it, where $\mathscr{CB}(D)$ denotes the family of nonempty closed bounded subsets of D. Furthermore, with the assumption that the family has at least one common fixed point, we showed that a sequence generated by the proposed algorithm **delta** and **strongly** converges to a common fixed point of $T_i's$. Our results generalize and improve many result in the literature.

Keywords and phrases

Multivalued demicontractive mapping, fixed point, Hadamard space, Hausdorff distance, delta convergence.

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Dedication

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Contents

CHAPTER 1

GENERAL INTRODUCTION

The contents of this thesis fall within the general area of nonlinear functional analysis and applications, precisely, nonlinear operator theory. We are interested in approximating fixed points of a countable family of multivalued demicontractive mappings in complete CAT(0) spaces (Hadamard spaces). In this chapter, we give a general introduction on iterative methods for approximating fixed points of certain multivalued mappings in Hadamard spaces. We also give basic definitions and some tools used in the thesis.

1.1 Introduction

Let A and D be nonempty sets and let 2^D denote the power set of D. A map that sends a point to a set, $T : A \to 2^D$, is called a multivalued mapping of A into D. The notation $D(T)$ (called the effective domain of T) means the subset of the domain such that $Tx \neq \emptyset$, i.e., $D(T) = \{x \in A \ : \ Tx \neq \emptyset\}$. An example of a multivalued map is the subdifferential of a convex functional: let f be a proper and convex map from a normed space E to $R \cup \{\infty\}$. The subdifferential of f, $\partial f : E \to 2^{E^*}$, is defined by $\partial f(x_0) := \{x^* \in E^* \ : \ \langle x^*, y - x_0\rangle \leq f(y) - f(x_0), \forall y \in E\}$, where E^* is the dual of E. For instance, let $\mathbb{H}$ be a Hilbert space and $f : \mathbb{H} \to \mathbb{R}$ be defined by $f(x) = \|x\|$. Then $\partial f(x) = \begin{cases} \frac{x}{\|x\|}, & x \neq 0 \\ \{x \in \mathbb{H} \ : \ \|x\| \leq 1\}, & x = 0. \end{cases}$

Remark 1.1.1 *Images of a singlevalued map can be collected as singleton sets instead of* *points. In that sense, one can easily include the collection of singlevalued mappings in to that of multivalued maps.*

Let T be a map (singlevalued or multivalued). The notion of a fixed point of T makes sense if the intersection of its domain and codomain is not empty. For such a map T (singlevalued), a point p in the domain of T is called a fixed point of T if $Tp = p$. When T is multivalued, $p \in D(T)$ is a fixed point of T if $p \in Tp$. In both cases, we denote by $F(T)$ the fixed points set of the map T. Concerning the study of fixed point(s) of certain maps (multivalued or singlevalued), researchers pay much attention to the question of whether the fixed point set of the map is not empty. If yes, how to get a member of the set. This second problem is addressed by iterative methods for approximating fixed points of maps (singlevalued or multivalued) and this is the concern of this research work.

For many years, the study of fixed point theory of multivalued maps has received the attention of many researchers. Interest in such study stems, perhaps, due to its real-world applications for instance, in the following:

- Nonsmooth Differential Equations
- Game Theory, Cryptography, Automata Theory, e.t.c.

Nonsmooth Differential Equations: Modelling countless problems from science and engineering results in differential equations with discontinuous right-hand sides. For instance, consider the following initial value problem.

$$\begin{cases} \frac{dv}{dt} = g(t,v) \; a.e. \; t \in I := [-b,b]; \\ v(0) = v_0, \quad b, v_0 \; fixed \; in \; \mathbb{R}. \end{cases} \tag{1.1.1}$$

Problem (1.1.1) may have no solution in classical sense. However, one may talk of its solutions in the sense of Fillipov (see [Filippov, 1964]) which are solutions of the differential inclusion

$$\begin{cases} \frac{dv}{dt} \in F(t,v), a.e. \; t \in I; \\ v(0) = v_0, \quad v_0 \; fixed \; in \; \mathbb{R}, \end{cases} \tag{1.1.2}$$

where

$$F(t,x) = \left[\liminf_{y \to x} f(t,y), \limsup_{y \to x} f(t,y)\right].$$

Let $H := L^2(I)$ and $N_F : H \to 2^H$ be the multivalued map defined by

$$N_F(v) := \{u \in H \ : \ u(t) \in F(t, v(t)) \ a.e. \ t \in I\}.$$

Now, Let $T : H \to 2^H$ be the multivalued map defined by $T := N_F o L^{-1}$, where L^{-1} is the inverse of the derivative operator $Lv = v'$ given by

$$L^{-1}u(t) := v_0 + \int_0^t u(s)ds.$$

One can see that (1.1.2) reduces to the fixed point problem: $v \in Tv$. For details see [Chidume et al., 2013].

Game theory: It has been established that the existence of equilibrium of certain games can be shown by the application of some multivalued fixed point theorem(s). For example, Brouwer fixed point theorem is used to show the existence of equilibria of noncooperative static game (see e.g., [Chidume et al., 2013]). This made Nash a recipient of Nobel Prize in Economic Sciences in 1994. Consequently, many theorems have been proved on the existence (and uniqueness in some cases) of fixed point(s). However, most of these theorems do not indicate how to find such a fixed point(s). For example, they do not construct a scheme that start at any point in the domain of the map(s) and converges to the fixed point. Thus the need for iterative schemes for approximating fixed point(s) of map(s).

1.2 Basic Definitions

In this section, we give definitions and examples of some related terms.

1.2.1 Demicontractive Mappings

The class of demicontractive mappings on Hilbert spaces was introduced by John Hicks and Lorraine Kubicek in [Hicks and Kubicek, 1977] as a super class of the class of strictly pseudocontractive mappings defined by [Browder and Petryshyn, 1967]. The class of strictly pseudocontractive map is itself a superclass of the class of nonexpansive maps. In addition, the class of demicontractive maps is clearly a superclass of the class of quasi-nonexpansive maps. We recall the definitions.

Definition 1.2.1 *Let (X,d) and (Y,d) be metric spaces. A singlevalued map $T: X \to Y$ is called*

(i) contractive if there exists $k \in [0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y),\ \forall\, x,y \in X.$$

(ii) nonexpansive if

$$d(Tx,Ty) \leq d(x,y),\ \forall\, x,y \in X.$$

(iii) quasinonexpansive if $X = Y$ and for any $p \in F(T)$

$$d(Tx,p) \leq kd(x,p),\ \forall\, x \in X.$$

(iv) strictly pseudocontractive if there exists $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2,\ \forall\, x,y \in C.$$

(v) demicontractive if for any $p \in F(T)$ there exists $k \in [0,1)$ such that

$$\|Tx - p\|^2 \leq \|x-p\|^2 + k\|x - Tx\|^2,\ \forall\, x \in C.$$

Definition 1.2.2 *Let H be a Hilbert space. A singlevalued map $T: H \to H$ is called*

(i) *strictly pseudocontractive if there exists* $k \in [0,1)$ *such that*

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \ \forall\, x,y \in C.$$

(ii) *demicontractive if there exists* $k \in [0,1)$ *such that for any* $p \in F(T)$,

$$\|Tx - p\|^2 \leq \|x-p\|^2 + k\|x - Tx\|^2, \ \forall\, x \in C.$$

The inclusions for the classes of these maps can easily be seen from the definitions. However, the converses are not always true. To see this, we give examples as follows:

Example 1.2.3 ([Browder and Petryshyn, 1967]) *Consider the metric space* $(\mathbb{R}, |\cdot|)$. *Let* $T : [-1,1] \to [-1,1]$ *be a real valued function defined by*

$$Tx = \begin{cases} \frac{2}{3}x\sin\frac{1}{x}, & if x \neq 0 \\ 0, & if x = 0. \end{cases}$$

Then T *is demicontractive but not pseudocontractive and hence not strictly pseudocontractive.*

Proof Clearly $T0 = 0$ and 0 is the only fixed point of T. Indeed, since for all $x \in \mathbb{R}$, $\sin x \neq \frac{3}{2}$, we have $\frac{2}{3}x\sin(1/x) \neq x$ for $x \neq 0$, $x \in [-1,1]$. Now for $x \in [-1,1]$ we have

$$|Tx - 0|^2 = |Tx|^2 \leq \left|\frac{2}{3}x\sin\frac{1}{x}\right|^2 \leq \left|\frac{2}{3}x\right| \leq |x|^2 \leq |x-0|^2 + k|x - Tx|^2, \ for\ any\ k < 1.$$

Thus T is demicontractive. We now show that T is not pseudocontractive. To show this, we take $x = \frac{2}{3\pi} \in [-1,1]$ and $y = \frac{2}{\pi} \in [-1,1]$. Then

$$\begin{aligned} |Tx - Ty|^2 &= \left|\frac{2}{3}\frac{2}{3\pi}\sin\frac{3\pi}{2} - \frac{2}{3}\frac{2}{\pi}\sin\frac{\pi}{2}\right|^2 \\ &= \left|-\frac{4}{9\pi} - \frac{4}{3\pi}\right|^2 \\ &= \frac{256}{81\pi^2} \\ &> \frac{160}{81\pi^2} \end{aligned}$$

$$\begin{aligned}|Tx-Ty|^2 &> \frac{160}{81\pi^2}\\ &= \frac{144+16}{81\pi^2}\\ &= \frac{16}{9\pi^2}+\frac{16}{81\pi^2}\\ &= \left|\frac{2}{3\pi}-\frac{6}{\pi}\right|^2+\left|\left(\frac{2}{3\pi}-\frac{4}{9\pi}\right)-\left(\frac{2}{\pi}-\frac{4}{3\pi}\right)\right|^2\\ &= |x-y|^2+|(x-Tx)-(y-Ty)|^2.\end{aligned}$$

Thus, T is not pseudocontractive and hence not strictly pseudocontractive. ■

Example 1.2.4 *Let $\mathbb{R}$ be endowed with the usual metric and $A=[-\pi,\pi]$. Define $T:A\to A$ by $Tx = xcosx$ for all $x\in A$. T is not nonexpansive but quasinonexpansive.*

Proof Clearly $F(T)=\{0\}$. We also have

$$\big|Tx-0\big| = |x||\cos x| \le |x| = |x-0|.$$

Thus, T is quasinonexpansive. We now show that T is not nonexpansive. Take $x=\pi$ and $y=\frac{\pi}{2}$. Then

$$\big|Tx-Ty\big| = \left|\pi\cos\pi-\frac{\pi}{2}\cos(\pi/2)\right| = \pi > \frac{\pi}{2} = \left|\pi-\frac{\pi}{2}\right| = |x-y|.$$

■

Example 1.2.5 *Let $\mathbb{R}$ be endowed with the usual metric and $T:\mathbb{R}\to\mathbb{R}$ be defined by $Tx=-3x$. Then T is strictly pseudocontractive but not nonexpansive.*

Proof Let $x,y\in\mathbb{R}$. Then

$$\begin{aligned}\big|Tx-Ty\big|^2 &= |-3x+3y|^2\\ &= |3x-3y|^2\\ &= 9|x-y|^2\end{aligned}$$

$$\begin{aligned}
\left|Tx - Ty\right|^2 &= 9|x-y|^2 \\
&= |x-y|^2 + \frac{16}{2}|x-y|^2 \\
&= |x-y|^2 + \frac{1}{2}|4x-4y|^2 \\
&= |x-y|^2 + \frac{1}{2}|(x-(-3x)) - (y-(-3y))|^2 \\
&= |x-y|^2 + \frac{1}{2}\left|(x-Tx) - (y-Ty)\right|^2.
\end{aligned}$$

Thus, T is strictly pseudocontractive map. However, for $x \neq y$ we have

$$\left|Tx - Ty\right| = 3|x-y| > |x-y|.$$

Hence T is not nonexpansive map. ■

Example 1.2.6 *Consider $\mathbb{R}$ with the usual metric. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = -\frac{3}{2}x$. Then T is not quasinonexpansive but demicontractive.*

Proof Clearly, $F(T) = \{0\}$. For $x \neq 0$,

$$\left|Tx - 0\right| = \frac{3}{2}|x| > |x|.$$

Thus T is not quasinonexpansive. We now show that T is demicontractive as follows:

$$\left|Tx - 0\right|^2 = \left|\frac{3}{2}x\right| = \frac{9}{4}|x|^2 = |x|^2 + \frac{5}{4}|x|^2 = |x|^2 + \frac{5}{25}\left(\frac{25}{4}|x|^2\right) = |x|^2 + \frac{1}{5}|x - Tx|^2.$$

■

1.2.2 Hausdorff Metric

Let A be a nonempty subset of a metric space (X, d). We denote the family of nonempty closed bounded subsets of A by $\mathscr{CB}(A)$ and $dist(b, A) := \inf_{a \in A} d(b, a)$ for any $b \in X$.

Definition 1.2.7 (Hausdorff Metric) *The map* $d_H : \mathscr{CB}(A) \times \mathscr{CB}(A) \to \mathbb{R}$ *defined by*

$$d_H(B,D) := max\left\{ \sup_{b \in B} dist(b,D), \sup_{d \in D} dist(d,B) \right\}, \ \forall\, B,D \in \mathscr{CB}(A)$$

is called Hausdorff metric.

Remark 1.2.8 *The Hausdorff metric is not defined on arbitrary family of subsets but on a family of closed and bounded subsets of a given set. We give examples below to illustrate this requirement.*

Example 1.2.9 *Consider* $\mathbb{R}$ *with the usual metric and* $\alpha, \beta \in \mathbb{R}$ *fixed such that* $\alpha < \beta$. *Take* $B = [\alpha, \beta]$ *and* $D = [\alpha, \beta)$. *Then* $B \neq D$ *and* $d_H(B,D) = 0$. *We note that D is bounded but not closed in* $\mathbb{R}$.

Example 1.2.10 *Take* $\mathbb{R}$ *with the usual metric. Consider* $B = \{\alpha\}$, $D = (-\infty, \beta]$ *with* α, β *fixed in* $\mathbb{R}$. *Then* $d_H(B,D) = +\infty$. *We observe that D closed but not bounded in* $\mathbb{R}$.

In respect to the definition of Hausdorff metric, we now give definitions of contractive, nonexpansive, quasinonexpansive and demicontractive multivalued maps as stated in the literature.

Definition 1.2.11 *Let* T *be a multivalued map defined on a nonempty subset of* (X,d) *with closed and bounded images.* T *is said to be*

- *contractive if there exists* $k \in [0,1)$ *such that*

$$d_H(Tx,Ty) \leq kd(x,y), \ \forall\, x,y \in D(T).$$

- *Lipschitz if there exists* $L \in [0,+\infty)$ *such that*

$$d_H(Tx,Ty) \leq Ld(x,y), \ \forall\, x,y \in D(T).$$

- *nonexpansive if*

$$d_H(Tx,Ty) \leq d(x,y), \ \forall\, x,y \in D(T).$$

- *quasinonexpansive if for any* $p \in Tp$,

$$d_H(Tx,Tp) \leq d(x,p), \ \forall\, x \in D(T).$$

- *demicontractive if there exists* $k \in [0,1)$ *such that for any* $p \in Tp$,

$$d_H(Tx,Tp)^2 \leq d(x,p)^2 + kdist(x,Tx)^2, \ \forall\, x \in D(T).$$

- *hemicontractive if for any* $p \in Tp$,

$$d_H(Tx,Tp)^2 \leq d(x,p)^2 + dist(x,Tx)^2, \ \forall\, x \in D(T).$$

1.2.3 Geodesic Space

A metric space (X,d) in which every pair $x,y \in X$ can be joined by an arc isometric to a compact interval of the real line is of considerable interest.

Definition 1.2.12 (Geodesic Path) *Let* (X,d) *be a metric space. A geodesic path joining* $x \in X$ *to* $y \in X$ *(or, more briefly, a geodesic from* x *to* y*) is a map* $\gamma : [0,l] \subset \mathbb{R} \to X$, *for some* $l > 0$, *such that*

(i) $\gamma(0) = x$, $\gamma(l) = y$,

(ii) $d(\gamma(t),\gamma(s)) = |t-s|, \ \forall\, t,s \in [0,l]$.

In particular γ *is an isometry and* $d(x,y) = l$.

Definition 1.2.13 (Geodesic Segment) *Let* (X,d) *be a metric space and* $x,y \in X$. *Let* γ *be a geodesic path from* x *to* y. *The image of* γ *is called a geodesic segment joining* x *and* y. *When* γ *is unique, this segment is denoted by* $[x,y]$.

Definition 1.2.14 (Geodesic Space) *Let* (X,d) *be a metric space. The space* (X,d) *is said to be a geodesic space if for every two distinct points of* X *there exists a geodesic segment in* X *that connects them.*

Definition 1.2.15 (Uniquely Geodesic Space) *Let (X,d) be a geodesic space. The space (X,d) is said to be uniquely geodesic if for any two distinct points of X there is exactly one geodesic segment joining them.*

Definition 1.2.16 (r-Geodesic Space) *Let (X,d) be a metric space and $r \in (0,+\infty]$. The space (X,d) is $r-$geodesic if for any two distinct points $x,y \in X$ with $d(x,y) < r$ there exists a geodesic path connecting them.*

Remark 1.2.17 *For $r = +\infty$, the above definition of r-geodesic space is equivalent to that of geodesic space.*

Definition 1.2.18 (Geodesic Triangle) *Let (X,d) be a geodesic space. A geodesic triangle $\triangle(p,q,r)$ consists of three points $p,q,r \in X$ (the vertices of $\triangle$) and a geodesic segment connecting each pair of vertices (the edges of $\triangle$). The triangle is simply defined by* $\triangle(p,q,r) := [p,q] \cup [q,r] \cup [r,p]$.

Definition 1.2.19 (Parameter of a Geodesic Triangle) *Let (X,d) be a geodesic space and $p,q,r \in X$. The sum $d(p,q)+d(q,r)+d(r,p)$ is called the parameter of the geodesic triangle $\triangle(p,q,r)$.*

Remark 1.2.20 *If the space is not uniquely geodesic, a geodesic triangle is not uniquely determined by its vertices. We will however denote it by $\triangle(p,q,r)$, keeping in mind that the definition implicitly presumes a choice of geodesics segments $[p,q]$, $[q,r]$, $[r,p]$.*

Definition 1.2.21 (Convex Set) *Let (X,d) be a geodesic space. A subset $Y \subseteq X$ is convex if any two distinct points in Y are connected by a geodesic path whose segment is in Y.*

1.2.4 Comparison Spaces

The concept of curvature for metric spaces, or more precisely a notion of upper bounds for the curvature of a metric space is mainly introduced via three spaces. These are: Euclidean n-space $\mathbb{E}^n$, the n-sphere $\mathbb{S}^n$ and the hyperbolic n-space $\mathbb{H}^n$.

Definition 1.2.22 (Euclidean $n-$ space $\mathbb{E}^n$) *Let $n \in \mathbb{N}$. We call the map*

$$d_E : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$
$$(x,y) \mapsto d_E(x,y) := \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

the Euclidean metric on $\mathbb{R}^n$, where $x = (x_1, x_1, \cdots, x_n)$ and $y = (y_1, y_1, \cdots, y_n)$. The euclidean n-space is defined by $E^n := (\mathbb{R}^n, d_E)$. We shall use the notation $d_E(x,y) = \|x - y\|, \forall x, y \in \mathbb{R}^n$.

Definition 1.2.23 (Euclidean Segment) *For each $x, y \in \mathbb{E}^n$ with $x \neq y$, the straight line segment between x and y is denoted $[x,y]$, i.e., $[x,y] = \{(1-t)\cdot x + t\cdot y \ : \ t \in [0,1] \subset \mathbb{R}\}$.*

Definition 1.2.24 (The Sphere $\mathbb{S}^n$) *Let $n \in \mathbb{N}$ and $S^n := \{x \in \mathbb{R}^{n+1} \ : \ \|x\| = 1\}$. The spherical metric is a map $d_S : S^n \times S^n \to \mathbb{R}$ defined such that $\cos d_S(x,y) = \langle x,y \rangle$, where $\langle x,y \rangle = \sum_1^n x_i y_i$. Then the spherical space endowed with this metric is defined by $\mathbb{S}^n := (S^n, d_S)$.*

Definition 1.2.25 (Minimal Great Arc) *Let $x, y \in \mathbb{S}^n$ and v be a vector in the direction $y - \langle x,y \rangle \cdot x$ such that $\langle x,v \rangle = 0$. The minimal great arc from x to y is defined by $[x,y] := \{\cos t \cdot x + \sin t \cdot v \ : \ t \in [0, d_s(x,y)] \}$.*

Definition 1.2.26 (The Hyperbolic Space $\mathbb{H}^n$) *Let $n \in \mathbb{N}$ and $H^n := \{x \in \mathbb{R}^{n+1} \ : \ \langle\langle x,x \rangle\rangle = -1, \quad x_{n+1} > 0 \}$, where $\langle\langle \cdot,\cdot \rangle\rangle$ is a map $\langle\langle \cdot,\cdot \rangle\rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $\langle\langle x,y \rangle\rangle := \sum_{i=1}^n x_i y_i - x_{n+1} y_{n+1}$. The hyperbolic n-space is defined by $\mathbb{H}^n := (H^n, \rho_h)$, where ρ_h is a map $\rho_h : H^n \times H^n \to \mathbb{R}$ called hyperbolic metric, define such that $\cosh \rho_h(x,y) := -\langle\langle x,y \rangle\rangle$.*

Definition 1.2.27 (Hyperbolic Segment) *For $x, y \in \mathbb{H}^n$ with $x \neq y$, let $u \in \mathbb{R}^{n+1}$ be the unit vector (i.e. $\langle\langle u,u \rangle\rangle = 1$) in the direction of $y + \langle\langle x,y \rangle\rangle \cdot x$. The hyperbolic segment from x to y is the set $[x,y] := \{(\cosh t)\cdot x + (\sinh t)\cdot u \ : \ t \in [0, \rho_h(x,y)] \}$.*

The three aforementioned spaces $\mathbb{E}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$ are the basic model spaces of constant curvature 0, 1 and -1 respectively. Now we give the general comparison spaces as follows.

Definition 1.2.28 (Model Space (M_k^n) and Comparison Space) *For $n \in \mathbb{N}$ and $k \in \mathbb{R}$, the Model $n-$space of constant curvature k is define by*

$$M_k^n := \begin{cases} \mathbb{E}^n, & if k = 0, \\ (S^n, \frac{1}{\sqrt{k}} d_S), & if k > 0; \\ (H^n, \frac{1}{\sqrt{-k}} \rho_h), & if k < 0. \end{cases}$$

*For $n = 2$, the above definition become the **comparison space** M_k^2. i.e.,*

$$M_k^2 := \begin{cases} \mathbb{E}^2, & if k = 0, \\ (S^2, \frac{1}{\sqrt{k}} d_S), & if k > 0. \\ (H^2, \frac{1}{\sqrt{-k}} \rho_h), & if k < 0. \end{cases}$$

Let k be fixed in $\mathbb{R}$. We denote the diameter of the comparison space M_k^2 by D_k, and this is equal to $+\infty$ for $k \leq 0$ and $\frac{\pi}{\sqrt{k}}$ for $k > 0$.

Remark 1.2.29 *It is known (see, e.g., [Bridson and Haefliger, 1999]) that in a D_k-geodesic space, a comparison triangle of any geodesic triangle with parameter less than $2D_k$ exists.*

Definition 1.2.30 (Comparison Triangle) *Let (X,d) be a geodesic space and $\triangle(p,q,r)$ be a geodesic triangle in X. A geodesic triangle $\bar{\triangle}(\bar{p},\bar{q},\bar{r})$ in comparison space M_k^2 is called comparison triangle of $\triangle(p,q,r)$ if $d(p,q) = \|\bar{p}-\bar{q}\|$, $d(q,r) = \|\bar{q}-\bar{r}\|$ and $d(r,p) = \|\bar{r}-\bar{p}\|$.*

Definition 1.2.31 (Comparison Point) *Let (X,d) be a geodesic space and $\triangle(p,q,r)$ be a geodesic triangle with a comparison triangle $\bar{\triangle}(\bar{p},\bar{q},\bar{r})$. For $w \in [p,q] \subset \triangle(p,q,r)$ the comparison point in $\bar{\triangle}(\bar{p},\bar{q},\bar{r})$ is the point $\bar{w} \in [\bar{p},\bar{q}]$ with $d(\bar{p},\bar{w}) = d(p,w)$. Comparison points in $[p,r]$ and $[r,q]$ are defined in a similar way.*

Definition 1.2.32 (Oplus Notation) *Let (X,d) be a uniquely geodesic space and $x,y \in X$. A point $z \in [x,y]$ is denoted by $(1-t)x \oplus ty$ if $d(x,z) = td(x,y)$ and $d(y,z) = (1-t)d(x,y)$ for some fixed $t \in [0,1]$.*

Remark 1.2.33 *For a fixed $t \in [0,1]$, such a unique point z always exists in uniquely geodesic spaces. To see this, let (X,d) be a uniquely geodesic space and let $x,y \in X$. The* *result holds trivially for $x = y$. Now suppose $x \neq y$. Then set $l = d(x,y)$ and let $\gamma : [0,l] \to X$ be the geodesic path connecting x to y. For every $t \in [0,1]$ we have $tl \in [0,l]$ and*

$$d(x,\gamma(tl)) = d(\gamma(0),\gamma(tl)) = |tl - 0| = tl = td(x,y).$$

Also,

$$d(y,\gamma(tl)) = d(\gamma(l),\gamma(tl)) = |l - tl| = (1-t)l = (1-t)d(x,y).$$

We then take $z = \gamma(tl)$. For uniqueness, suppose there exists $w \in [x,y]$ such that $d(x,w) = td(x,y)$ and $d(y,w) = (1-t)d(x,y)$. Then, $w \in [x,y]$ implies, there exists $t_0 \in [0,1]$ such that $\gamma(t_0) = w$. So, $t_0 = d(\gamma(0),\gamma(t_0)) = d(x,w) = td(x,y) = d(x,\gamma(tl)) = d(\gamma(0),\gamma(tl)) = tl$. Thus $w = \gamma(t_0) = \gamma(tl) = z$.

1.2.5 CAT(k) Space

The acronym CAT was coined in 1987 by an American-French-Russian mathematician, Mikhail Gromov, in recognition to the pioneering work of Elie Joseph **Cartan** (1869 - 1951), Alexander Danilovich **Alexandrov** (1912 - 1999) and Victor Andreevich **Toponogov** (1930 - 2004).

Definition 1.2.34 (CAT(k) Inequality) *Let k be fixed in $\mathbb{R}$. A geodesic triangle satisfies the CAT(k) inequality if for every pair of points u and v on it we have $d(u,v) \leq d(\bar{u},\bar{v})$ where $\bar{u}$ and $\bar{v}$ are the respective corresponding points of a comparison triangle in M_k^2.*

Definition 1.2.35 (CAT(k) Space) *For $k \in \mathbb{R}$, a D_k-geodesic space (X,d) is called CAT(k) space if all geodesic triangles in X of perimeter less than $2D_k$ satisfy the CAT(k) inequality.*

Remark 1.2.36 *CAT(0) space is simply a geodesic space in which every triangle is thinner then its comparison triangle in the Euclidean space $\mathbb{E}^2$.*

Definition 1.2.37 (Hadamard Space) *A complete CAT(0) space is called Hadamard space, named after a French mathematician Jacques Salomon Hadamard (1865-1963).*

Remark 1.2.38 *The following facts are true (see, e.g., [Kirk and Shahzad, 2014]):*

- *If X is a $CAT(k_1)$ space, then it is a $CAT(k_2)$ space for every $k_2 \geq k_1$.*
- *If X is a $CAT(k_1)$ space for every $k_1 > k_2$, then it is a $CAT(k_2)$ space.*
- *X is $CAT(k)$ space for all $k \in \mathbb{R}$ if and only if X is an $R-tree$, where a metric space (X,d) is called R-tree if*

  (i) for $x,y \in X$, there is a unique geodesic segment $[x,y]$, and

  (ii) $[x,y] \cap [y,z] = \{y\} \implies [x,z] = [x,y] \cup [y,z]$.
- *If X is a $CAT(0)$ space, then the distance function $d : X \times X \to \mathbb{R}$ is convex.*

Definition 1.2.39 (Curvature) *A metric space (X,d) is said to be of curvature $\leq k$ if it is locally a CAT(k) space, i.e., for every $x \in X$ there exists $r_x > 0$ such that the ball $B(x,r_x)$, endowed with the induced metric is a CAT(k) space. In particular, if X is of curvature ≤ 0, then we say that it is non-positively curved.*

Definition 1.2.40 (Asymptotic Centre and Radius) *Let $\{x_n\}$ be a bounded sequence in a metric space (X,d). We set $r(x,\{x_n\}) := \limsup_n d(x,x_n)$. Then*

(i) the asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is defined by

$$r(\{x_n\}) := inf\{r(x,\{x_n\}) \ : \ x \in X\}.$$

(ii) the asymptotic centre $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) := \{x \in X \ : \ r(x,\{x_n\}) = r(\{x_n\})\}.$$

Remark 1.2.41 *It is known (see, e.g., [Dhompongsa et al., 2006]) that in a $CAT(0)$ space, $A(\{x_n\})$ is singleton.*

Definition 1.2.42 (Strong and Delta Convergences) *Let (X,d) be a metric space. A sequence $\{x_n\} \subset X$ is said to*

- *strongly converges to a point* $x \in X$ *if* $\lim_n d(x_n, x) = 0$. *In this case we write* $\lim_n x_n = x$ *and call* x *the limit of* $\{x_n\}$.

- *delta converges to a point* $x \in X$ *if the* $\limsup_k d(x_{n_k}, x) \leq \limsup_k d(x_{n_k}, y)$ *for every* $\{x_{n_k}\}$ *subsequence of* $\{x_n\}$ *and for every* $y \in X$. *In other words, if* x *is the unique asymptotic centre for every* $\{x_{n_k}\}$ *subsequence of* $\{x_n\}$. *In this case we write* $\triangle - \lim_n x_n = x$ *and call* x *the* $\triangle$*-limit of* $\{x_n\}$.

Remark 1.2.43 *Strong convergence implies delta convergence. To see this, let* $\{x_n\}$ *be a sequence in* X *such that* $\lim_n x_n = x$. *Let* $y \in X$. *Then for any subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *we have*

$$\limsup_k d(x_{n_k}, x) = \lim_k d(x_{n_k}, x) = \lim_n d(x_n, x) = 0 \leq \limsup_k d(x_{n_k}, y).$$

However, the converse is not always true. We give a counterexample below.

Example 1.2.44 *Let* $X = l^2$ *be endowed with* $\|\cdot\|_2$, *where* l^2 *is the space of real sequences* $\{x_k\}$ *such that* $\sum_{k=1}^{+\infty} |x_k|^2$ *converges. Consider a sequence* $\{x_n\}$ *such that* $x_n = (0, 0, \cdots, 0, 1, 0, \cdots)$ *with* 1 *in the* n^{th}*-place. Then* $\{x_n\}$ *delta converges to* 0 *but not strongly converges to* 0.

Proof Let $y = \{y_n\} \in X$. Then for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ we have

$$\begin{aligned}
\limsup_k d^2(x_{n_k}, y) &\geq \liminf_k d^2(x_{n_k}, y) \\
&= \liminf_k \left\|x_{n_k} - y\right\|_2^2 \\
&= \liminf_k \left(\sum_{i=1}^{n_k - 1} |y_i|^2 + \left|1 - y_{n_k}\right|^2 + \sum_{i=n_k+1}^{+\infty} |y_i|^2 \right) \\
&\geq \liminf_k \left(\sum_{i=1}^{n_k - 1} |y_i|^2 + \left|1 - y_{n_k}\right|^2 + \sum_{i=n_k+1}^{+\infty} |y_i|^2 \right) \\
&\geq \liminf_k \sum_{i=1}^{n_k - 1} |y_i|^2 + \liminf_k \left|1 - y_{n_k}\right|^2 + \liminf_k \sum_{i=n_k+1}^{+\infty} |y_i|^2 \\
&= \lim_k \sum_{i=1}^{n_k - 1} |y_i|^2 + \lim_k \left|1 - y_{n_k}\right|^2 + \lim_k \sum_{i=n_k+1}^{+\infty} |y_i|^2
\end{aligned}$$

$$\begin{aligned}\limsup_k d^2(x_{n_k},y) &\geq \lim_k \sum_{i=1}^{n_k-1}|y_i|^2 + \lim_k \left|1-y_{n_k}\right|^2 + \lim_k \sum_{i=n_k+1}^{+\infty}|y_i|^2 \\ &\geq \lim_k \left|1-y_{n_k}\right|^2 \\ &= 1 \\ &= \limsup_k \|x_{n_k}\|_2^2 \\ &= \limsup_k d^2(x_{n_k},0).\end{aligned}$$

This implies $\limsup_k d^2(x_{n_k},y) \geq \limsup_k d^2(x_{n_k},0)$. Thus, $\{x_n\}$ delta converges to 0. However, $\{x_n\}$ does not converge strongly to 0 since $\|x_n\| = 1$ for all n. ∎

Remark 1.2.45 *The concept of delta convergence in $CAT(0)$ spaces give us a condition that is known in Banach space theory as the opial property. For instance, given $\{x_n\} \subset X$ such that $\{x_n\}$ $\triangle-$converges to x and given $y \in X$ with $y \neq x$, $\limsup_n d(x_n,x) < \limsup_n d(x_n,y)$. In similar regard, $CAT(0)$ also satisfies what is known as Kadec-klee property. Furthermore, every bounded closed convex set in a Hadamard space is $\triangle-$compact (see, e.g., [Kirk and Shahzad, 2014]).*

Definition 1.2.46 ($\triangle$-demiclosed) *Let (X,d) be a Hadamard space. A self-mapping T is said to be demiclosed at zero if the conditions $\{x_n\}$ $\triangle$-converges to x and $d(x_n,Tx_n) \to 0$, imply $x \in F(T)$.*

CAT(0) attracts the interest of many scholars, perhaps, because not only does it contain Hilbert spaces but also inherits some geometric properties of Euclidean spaces, Hilbert spaces and uniformly convex Banach spaces. For example, closed convex set are uniquely proximinal, descending sequence of nonempty closed convex sets have nonempty intersections, nearest point projections onto closed convex sets are nonexpansive and asymptotic centre techniques apply (see, e.g., [Kirk and Shahzad, 2014]). The following inequality (which is known as the CN inequality of Bruhat and Tits (see [Bruhat and Tits, 1972])) holds in CAT(0) spaces and is crucial in obtaining some of these properties.

$$d(v,\frac{1}{2}w_1 \oplus \frac{1}{2}w_2)^2 \leq \frac{1}{2}d(v,w_1)^2 + \frac{1}{2}d(v,w_2)^2 - \frac{1}{4}d(w_1,w_2)^2, \tag{1.2.1}$$

where v, w_1, w_2 are arbitrary points in a geodesic space (X,d). In fact one of the characterization of $CAT(0)$ spaces is that a geodesic space (X,d) is a $CAT(0)$ space if and only if it satisfies (1.2.1) (see e.g., [Bridson and Haefliger, 1999]).

Example 1.2.47 *Using (1.2.1) above, one can easily see that finite product of CAT(0) spaces is also a CAT(0) space. For instance, if* (X_1,d_1) *and* (X_2,d_2) *are CAT(0) spaces then* (X,d) *is a CAT(0) space, where* $X := X_1 \times X_2$ *and* $d(x,y)^2 = d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2$ *for* $x,y \in X$.

Remark 1.2.48 *CAT(0) spaces are uniformly convex spaces in the following sense:*

Let (X,d) be a $CAT(0)$ space, $v \in X$ and $\varepsilon \in (0,2]$. Let $w_1, w_2 \in \bar{B}_1(v)$ such that $d(w_1,w_2) > \varepsilon$. Then (1.2.1) gives

$$\begin{aligned} d(v, \frac{1}{2}w_1 \oplus \frac{1}{2}w_2)^2 &\leq \frac{1}{2}d(v,w_1)^2 + \frac{1}{2}d(v,w_2)^2 - \frac{1}{4}d(w_1,w_2)^2 \\ &\leq \frac{1}{2}(1) + \frac{1}{2}(1) - \frac{1}{4}\varepsilon^2 \\ &= 1 - \frac{1}{4}\varepsilon^2, \end{aligned}$$

and so

$$d(v, \frac{1}{2}w_1 \oplus \frac{1}{2}w_2) \leq 1 - \delta, \text{ where } \delta = 1 - \sqrt{1 - \frac{1}{4}\varepsilon^2}. \tag{1.2.2}$$

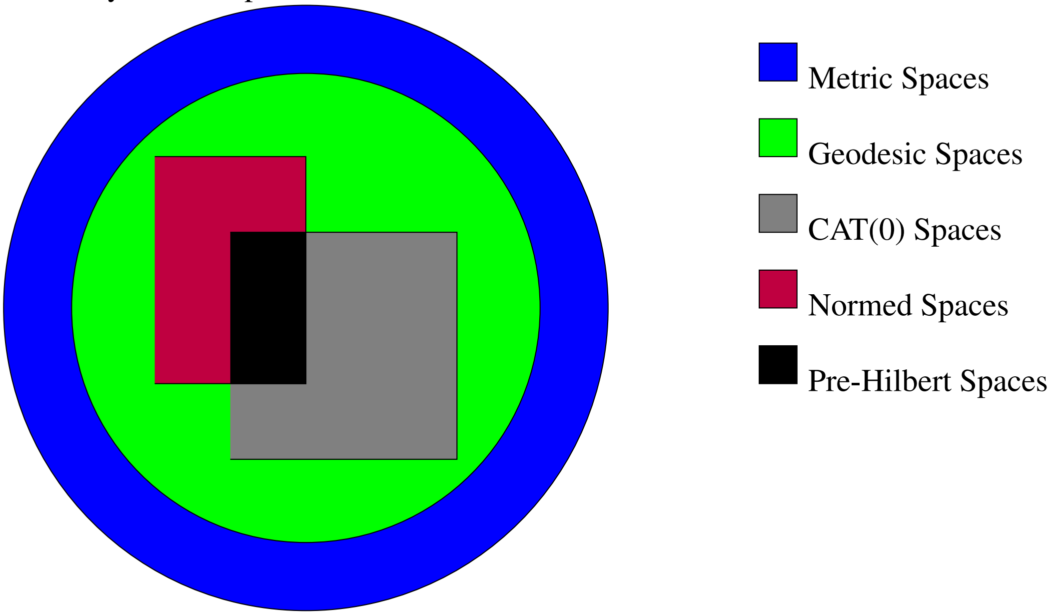
Thus (1.2.2) gives analog to uniformly convex spaces.

Remark 1.2.49 *Every pre-Hilbert (inner product) space is a CAT(0) space. The converse is not always true. To see this, let X be pre-Hilbert space and* $v, w_1, w_2 \in X$. *Then*

$$\begin{aligned} d\left(v, \frac{1}{2}w_1 + \frac{1}{2}w_2\right)^2 &= \left| \left(\frac{1}{2}w_1 + \frac{1}{2}w_2\right) - v \right|^2 \\ &= \left| \frac{1}{2}(w_1 - v) + \frac{1}{2}(w_2 - v) \right|^2 \\ &= \frac{1}{2}\|w_1 - v\|^2 + \frac{1}{2}\|w_2 - v\|^2 - \frac{1}{4}\|w_1 - w_2\|^2 \\ &= \frac{1}{2}d(v,w_1)^2 + \frac{1}{2}d(v,w_2)^2 - \frac{1}{4}d(w_1,w_2)^2. \end{aligned}$$

Thus (1.2.1) is actually equality in pre-Hilbert space. Therefore, every pre-Hilbert space is a CAT(0) space. For the converse, take R-tree (which is CAT(k) space for all $k \in \mathbb{R}$).

An important fact is that a normed space is a CAT(0) space if and only if it is a pre-Hilbert space (see, e.g., [Bridson and Haefliger, 1999]). The following diagram represents the hierarchy of such spaces.



CHAPTER 2

LITERATURE REVIEW

The study of fixed point theory of singlevalued maps started long ago. Many theorems have been proved on the existence (and uniqueness in some cases) of fixed point(s). These theorems include the Banach contraction mapping principle, Brouwer fixed point theorem, Schauder fixed point theorem, e.t.c. (see, e.g., [Banach, 1922]). Most of the results obtained from the singlevalued mappings were later extended to the setting of multivalued maps (see, e.g., [Nadler, 1969], [Markin, 1973], [Sastry and Babu, 2005], [Chidume et al., 2013]). However, there is always a question of how to get at least one of the fixed point(s) when they exist. Iterative methods of approximating fixed point is one way of finding such a fixed point. Consequently, many iterative schemes were proposed and studied by many scholars. These schemes include the Picard iterative scheme, Mann scheme, Ishikawa scheme and Agarwal iterative scheme (see, e.g.,[Picard, 1890], [Mann, 1953], [Ishikawa, 1974], [Khan and Abbas, 2011]). Most of these constructed sch-emes strongly converge or weakly converge (or delta converge) to a fixed point of certain map in some spaces. It is the aim of this chapter to review some of these related literature that involves demicontractive maps (from singlevalued to multivalued).

2.1 Singlevalued Maps

2.1.1 Contraction Maps

One of the most important theorems in the study of fixed points is the Banach contraction mapping principle. This theorem was first proved in setting of complete normed spaces by Stefan Banach in 1922 (see [Banach, 1922]). Then later extended to setting of complete metric spaces. The theorem is stated as follows:

Theorem 2.1.1 (Banach contraction mapping principle) *Let (X,d) be a complete metric space and let $T : X \to X$ be a contraction mapping, then T has a unique fixed point. Furthermore, for arbitrary $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, $n = 0,1,2,\cdots$ converges strongly to the unique fixed point.*

The iterative process of obtaining the sequence $\{x_n\}$ above is called Picard scheme named after a French mathematician Charles-Émile Picard, and it is essential in most of the proofs of Banach contraction mapping principle. The beauty of this theorem is that it does not only give us existence of fixed point but also gives uniqueness, convergence rate and method of approximating the fixed point. However, this theorem does not hold for nonexpansive maps. For instance, a nonexpansive map may not have a fixed point. To see this, we give the example as follows.

Example 2.1.2 *Take $X = \mathbb{R}$ with the usual metric and fix $\alpha \in \mathbb{R}/\{0\}$. Consider a map $T : X \to X$ defined by $Tx := x + \alpha$. Then T is nonexpansive and it has no fixed point.*

Proof Clearly, $|Tx - Ty| = |x - y|, \forall\, x,y \in \mathbb{R}$. Now suppose T has a fixed point (say x_0). Then $x_0 = x_0 + \alpha$. Thus $\alpha = 0$ which is a contradiction. Therefore T has no fixed point.

■

Furthermore, a nonexpansive map may have a fixed point but the so-called Picard scheme may fail to approximate such a fixed point. Too see this, consider the following example:

Example 2.1.3 *Take $X = \mathbb{R}$ with the usual metric and fix $\alpha \in \mathbb{R}/\{0\}$. Consider a map $T : X \to X$ defined by $Tx := -(x + \alpha)$. Then T is nonexpansive and it has a unique fixed point that cannot be achieved using Picard scheme.*

Proof Clearly for any $x, y \in \mathbb{R}$, $|Tx - Ty| = |x - y|$ and $F(T) = \{-\frac{\alpha}{2}\}$. Now if we take $x_0 \in \mathbb{R}/\{-\frac{\alpha}{2}\}$, we have the sequence $\{x_n\}$ generated by the Picard scheme as $\{x_0, -(x_0 + \alpha), x_0, -(x_0 + \alpha), x_0, -(x_0 + \alpha), \cdots\}$. Thus $\{x_n\}$ is oscillating and therefore does not converge to the fixed point. ∎

Thus, the Picard scheme used in proving Banach contraction mapping principle is not the suitable method of approximating fixed point(s) of nonexpansive maps.

2.1.2 Nonexpansive Maps

Due to the limitation of Picard scheme and the fact that Banach contraction mapping principle does not works for bigger classes of maps (e.g. class of nonexpansive maps), researchers began to study different iterative schemes that may work for classes of maps that are bigger than the class of contraction maps. For example, in 1953, a Polish mathematician, Robert Mann studied class of continuous maps in normed space (see [Mann, 1953]) and introduced the following iterative scheme:

$$\begin{cases} x_0 \in D; \\ v_n = \sum_{k=1}^{n} b_{nk} x_k; \\ x_{n+1} = T(u_n), n \geq 1, \end{cases} \tag{2.1.1}$$

where $b_{nk},\ k, n \in \mathbb{N}$ are entries of a matrix of the form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ b_{21} & b_{22} & 0 & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ b_{n1} & b_{n2} & . & . & . & b_{nn} & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{bmatrix}.$$

The entries of B above are selected such that $\sum_{j=1}^{i} b_{ij} = 1,\ b_{ij} \geq 0,\ \forall\, i, j$, and $b_{ij} = 0, \forall\, i, j$ with $j > i$. One of the theorems Robert Mann proved uses scheme (2.1.1) above and this theorem is as follows:

Theorem 2.1.4 *If either of the sequences* $\{x_n\}$ *or* $\{u_n\}$ *converges, then the other also converges to the same point, and their common limit is a fixed point of* T.

He also consider a special case of matrix B for which $b_{ij} = 0$ when $i < j$ and $b_{ij} = \frac{1}{i}$ for all $i \geq j$, and used it in proving the following:

Theorem 2.1.5 *If* $T : [a,b] \to [a,b]$ *has a unique fixed point* $p \in [a,b]$, *then the sequence* $\{u_n\}$ *converges to* p *for all choice of* $x_0 \in [a,b]$.

In an attempt to obtain analog of Banach contraction mapping principle in a bigger classes of maps, a Russian mathematician in [Krasnoselskii, 1955] introduced the following iterative scheme:

$$\begin{cases} x_0 \in D; \\ x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}T(x_n), n \geq 1. \end{cases} \tag{2.1.2}$$

He used this scheme and proved the following theorem:

Theorem 2.1.6 *Suppose* E *is uniformly convex,* D *closed and convex, and* T *is nonexpansive with* $T(D)$ *contained in a compact subset of* D. *Then* $\{x_n\}$ *given by* (2.1.2) *converges strongly to a fixed point of* T.

In 1957, a German mathematician H. Schaefer in [Schaefer, 1957] discovered that similar result can be obtained using a sequence generated by the following iterative scheme:

$$\begin{cases} x_0 \in D; \\ x_{n+1} = (1-\lambda)x_n + \lambda T(x_n), n \geq 1, \lambda \in (0,1). \end{cases} \tag{2.1.3}$$

Now, the advanced (or the most general) method for approximation of fixed points of nonexpansive mappings in the light of [Mann, 1953] is as follows:

$$\begin{cases} x_1 = x \in D \\ x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n,\ n \in \mathbb{N}, \end{cases} \tag{2.1.4}$$

where $\{\lambda_n\} \subset (0,1)$ such that $\lambda_n \to 0$ and $\sum \lambda_n = \infty$. This method is called the *Mann* *iteration formula* and the scheme in (2.1.3) is called the *Krasnoselskii-Mann (KM) formula* (see, e.g., [Chidume, 2009]).

2.1.3 Strictly Pseudocontractive Maps

[Browder and Petryshyn, 1967] introduced the class of strictly pseudocontractive mappings and studied their fixed points properties in the setting of Hilbert spaces. This class of maps does not only generalised the class of nonexpansive maps but also inherits Lipschitzian property (see e.g., [Chidume et al., 2013]). In addition to showing some nice properties of this class of maps, Browder and Petryshyn proved the following theorem in 1967.

Theorem 2.1.7 *Let $\mathbb{H}$ be a real Hilbert space and C a bounded closed convex subset of $\mathbb{H}$. Let $T : C \to C$ be a k-strictly pseudocontractive map. Then for any fixed $\alpha \in (1-k,1)$, the sequence generated from an arbitrary $x_0 \in C$ by*

$$x_{n+1} = (1-\alpha)x_n + \alpha T x_n$$

converges weakly to a fixed point of T.

[Ishikawa, 1974] introduced a new iterative method for approximating fixed points of pseudocontractive mappings in Hilbert spaces. This method is called Ishikawa scheme and for a pseudocontractive self-map (i.e. with domain equals to codomain) T on a compact subset of a Hilbert space. He generated a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1 = x \in C \\ y_n = (1-a_n)x_n + a_n T x_n \\ x_{n+1} = (1-b_n)x_n + b_n T y_n \quad a_n, b_n \in (0,1), \ n \in \mathbb{N}, \end{cases} \tag{2.1.5}$$

It is known (see e.g., [Rhoades, 1976]) that the Mann scheme defined in (2.1.4) ia a special case of the Ishikawa scheme defined in (2.1.5) above.

2.1.4 Demicontractive Maps

Insighted by the class of strictly pseudocontractive maps as introduced by Browder and Petryshyn in 1967, researchers have been making efforts in studying iterative methods of fixed point(s) of such class of maps and their generalizations. For example, Charles Chidume and Ştefan Măruşter in 2010, studied iterative methods for computation of fixed points and survey some convergence properties of Mann-type scheme for demicontractive map. They proved the following theorem:

Theorem 2.1.8 *Let $T : C \subset \mathbb{H} \to \mathbb{H}$, where $\mathbb{H}$ is a real Hilbert space. Suppose that T is demicontractive with a constant k, such that $0 < t_n < 1-k$ and that $\{x_n\}$ given by the Mann iteration for some x_0 belongs to C. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that*

$$\lim_j d(x_{n_j}, Fix(T)) = 0.$$

[Igbokwe and Udofia, 2013] studied an implicit iteration method for approximation of common fixed points of finite family of demicontractive maps and proved some theorems including the following:

Theorem 2.1.9 *Let $\mathbb{H}$ be a real Hilbert space and let K be a nonempty closed convex subset of $\mathbb{H}$. Let $\{T_i\}_{i=1}^N$ be N demicontractive self-maps of K such that $\mathscr{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, where $F(T_i) = \{x \in K : T_i x = x\}$. Let $\{a_n\}_{n=1}^\infty$ be a sequence in $(0,1)$ such that $\sum_{n=1}^\infty (1 - a_n) = \infty$. Let $x_1 \in K$ and let $\{x_n\}$ be defined by*

$$x_n = a_n x_{n-1} + (1-a_n) T_n x_n, n \geq 1,$$

where $T_n = T_{n\, modN}$. Then,

(i) $\lim_n \|x_n - p\|$ *exists for all* $p \in \mathscr{F}$. *(ii)* $\liminf_n \|x_n - T_n x_n\| = 0$.

(iii) $\{x_n\}$ *converges strongly to a common fixed point p of the mappings $\{T_i\}_{i=1}^N$ if there is subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to p.*

[Lim, 1976] introduced a notion of convergence on arbitrary metric space which he called delta convergence. Kirk was among the first to study fixed point theorems in Hadamard spaces (see, e.g, [Kirk, 2003], [Kirk, 2004]). He showed the existence of fixed point for nonexpansive mappings on a nonempty closed convex subset of CAT(0) space. Since then, the study of existence and approximation of fixed points has attracted the attention of many researchers (see, e.g, [Dhompongsa et al., 2005], [Dhompongsa et al., 2006], [Dhompongsa et al., 2009]). In 2008, [Kirk and Panyanak, 2008] studied the delta convergent concept of Lim in CAT(0) spaces and showed that many results which involve weak convergence (e.g., opial property) have analogues in this setting. Using the result of [Kirk and Panyanak, 2008], [Dhompongsa and Panyanak, 2008] Obtained delta convergence theorem for the Picard, Mann and Ishikawa Iterations involving one mapping in the CAT(0) space setting.

2.2 Multivalued Mappings

The study of fixed point theory of multivalued mappings started long ago. It has attracted the attention of prominent scholars and continues to do so (see, e.g., [Brouwer, 1912], [Kakutani, 1941], [Nash, 1951], [Nadler, 1969], [Downing and Kirk, 1977]). This is, perhaps, as a result of its application in convex optimizations, differential equations, control theory, economics and many other fields.

[Nadler, 1969] introduced an iterative method for multivalued mappings which is similar to that of Picard scheme and used it to develop a theorem. This theorem is consider as an analogue of Banach contraction mapping principle for multivalued mappings and is stated as follows:

Theorem 2.2.1 *Let (X,d) ba a complete metric space and $T : X \to CB(X)$ be a multivalued contraction mapping. Then T has a fixed point. Furthermore, the following scheme is used in approximating such a fixed point:*

$$\begin{cases} x_0 \in X; \\ x_{n+1} \in Tx_n : d(x_n, x_{n+1}) \leq d_H(Tx_n, Tx_{n+1}) + k^n, \ k \in [0,1), \ n \geq 1. \end{cases} \tag{2.2.1}$$

The theorem above guaranteed existence of fixed point(s) with a scheme of approximating it. However it does not give uniqueness of such a point.

In 1973, [Markin, 1973] used Hausdorff metric to introduce the study of fixed points for the class of nonexpansive multivalued mappings. Since then, researches developed much interest in the study of fixed point theory of multivalued maps (see e.g., [Abbas et al., 2011], [Chidume et al., 2013], [Chidume et al., 2014])

With regard to iterative methods of approximating fixed points of multivalued mappings, [Sastry and Babu, 2005] introduced the analogue of Mann and Ishikawa schemes. For a multivalued map $T : X \to P(X)$ with a fixed point p, where $P(X)$ denotes the proximinal set of X, they defined the analogue of

- the Mann Scheme by:

$$\begin{cases} x_1 = x \in C, \ z_n \in Tx_n : \|z_n - p\| = d(p, Tx_n); \\ x_{n+1} = (1 - a_n)x_n + a_n z_n, \ a_n \in (0,1] : \sum a_n = \infty, \ n \in \mathbb{N}. \end{cases} \tag{2.2.2}$$

- the Ishikawa Scheme by:

$$\begin{cases} x_1 = x \in C, \ z_n \in Tx_n : \|z_n - p\| = d(p, Tx_n); \\ y_n = (1 - a_n)x_n + a_n z_n, \quad w_n \in Ty_n : \|w_n - p\| = d(p, Ty_n); \\ x_{n+1} = (1 - b_n)x_n + b_n w_n \ \ a_n, b_n \in (0,1] : b_n \to 0, \sum a_n b_n = \infty, \ n \in \mathbb{N}. \end{cases} \tag{2.2.3}$$

They used these schemes in proving some convergence theorems for nonexpansive mappings on a compact convex subset of a Hilbert space. Two of these theorems are stated below.

Theorem 2.2.2 *Let D be a compact convex subset of a Hilbert space X. Suppose that a nonexpansive map $T : D \to P(D)$ has a fixed point. Assume that*

(i) $0 \leq a_n, b_n < 1$,

(ii) $b_n \to 0$ *and*

(iii) $\sum a_n b_n = \infty$. *Then the sequence of Ishikawa scheme in (2.2.3) converges to a fixed point of T.*

Theorem 2.2.3 *Let D be a compact convex subset of a Hilbert space X. Suppose that a nonexpansive map $T : D \to P(D)$ has a fixed point p. Assume that*

(i) $0 \leq a_n < 1$

(ii) $\sum a_n = \infty$. *Then the sequence of Mann scheme in (2.2.2) converges to a fixed point q of T.*

In 2007, [Panyanak, 2007] generalized the results of [Sastry and Babu, 2005] to uniformly convex Banach spaces and considered the iteration processes in a new sense, which he used to prove a convergence theorem for a mapping defined on a noncompact domain.

In 2011, [Abbas et al., 2011] studied two multivalued nonexpansive mappings in the setting of uniformly convex Banach spaces. They introduced a new one-step iterative process for approximating common fixed point of the mappings. Under some regularity conditions, they proved weak and strong convergence theorems for the proposed scheme.

Following the paper [Browder and Petryshyn, 1967] of Browder and Petryshyn,[Chidume et al., 20 introduced the class of multivalued strictly pseudocontractive mappings in Hilbert spaces as follows:

Definition 2.2.4 *Let $\mathbb{H}$ be a Hilbert space. A multivalued mapping $T : D(T) \subseteq \mathbb{H} \to \mathscr{C}\mathscr{B}(\mathbb{H})$ is said to be $k-strictly\ pseudocontractive$ if there exists $k \in (0,1)$ such that for all $x, y \in D(T)$ one has $(d_H(Tx,Ty))^2 \leq \|x-y\|^2 + k\|(x-u)-(y-v)\|^2, \forall u \in Tx, v \in Ty$. The map T above is said to be pseudocontractive if $k = 1$.*

They constructed a Krasnoselskii-type algorithm for a sequence $\{x_n\}$ such that

$$\begin{cases} x_0 \in K; \\ x_{n+1} = (1-\lambda)x_n + \lambda y_n; \\ y_n \in Tx_n, \quad \lambda \in (0,1), \end{cases} \tag{2.2.4}$$

and proved the following theorem for a sequence generated by such scheme.

Theorem 2.2.5 *Let K be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Suppose that $T : K \to CB(K)$ is a multivalued strictly pseudocontractive mapping such that $F(T) \neq \emptyset$. Assume that $T(p) = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be the sequence defined in* (2.2.4). *Then $\lim_n d(x_n, Tx_n) = 0$.*

[Chidume and Ezeora, 2014] extended the result of [Chidume et al., 2013] to a finite family of multivalued strictly pseudocontractive mappings in real Hilbert spaces and proved the theorem below.

Theorem 2.2.6 *Let K be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$, and let $T_i : K \to CB(K)$ be a finite family of multivalued k_i-strictly pseudocontractive mappings, $k_i \in (0,1)$, $i = 1, \cdots, m$, such that $F(T) \neq \emptyset$. Assume that $T(p) = p$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_0 \in K; \\ x_{n+1} = \lambda_0 x_n + \lambda_1 y_n^1 + \lambda_2 y_n^2 +, \cdots, \lambda_m y_n^m; \quad n \geq 1, \\ y_n^i \in T_i x_n,\ \lambda_i \in (k,1), k = max\{k_i,\ i = 1,2,\cdots,m\},\ \sum_{i=0}^m \lambda_i = 1. \end{cases} \tag{2.2.5}$$

Then $\lim_n d(x_n, T_i x_n) = 0,\ \forall i = 1,2,\ldots,m.$

[Isiogugu and Osilike, 2014] studied convergence of a sequence with Mann-type scheme for multivalued demicontractive mappings in real Hilbert spaces and proved the following theorem.

Theorem 2.2.7 *Let K be a nonempty closed convex subset of a real Hilbert space H. Suppose that $T : K \to P(K)$ is a demicontractive mapping from K into the family of all prox-*

iminal subsets of K with $k \in (0,1)$ and $T(p) = \{p\}$ for all $p \in F(T)$. Suppose $(I-T)$ is weakly demiclosed at zero. Then, the Mann type sequence defined by

$$\begin{cases} x_0 \in K; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n; \\ y_n \in Tx_n, \quad \{\alpha_n\} \subset (0,1) : \alpha_n \to \alpha \in (0, 1-k), \end{cases} \tag{2.2.6}$$

converges weakly to $q \in F(T)$

[Chidume et al., 2014] proved strong and $\triangle$-convergence theorems for a Krasnoselskii-type sequence to a common fixed point of a finite family of demicontractive mappings in the setting of CAT(0) spaces. Below is their main theorem.

Theorem 2.2.8 *Let K be a nonempty closed and convex subset of a complete $CAT(0)$space. Let $T_i : K \to CB(K)$, $i = 1,2,\cdots,m$, be a family of demicontractive mappings with constants $k_i \in (0,1)$, $i = 1,\cdots m$, such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that T_i is $\triangle$-demiclosed at 0 for all $i = 1,\cdots,m$, and $T_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^{m} F(T_i)$. Then a sequence $\{x_n\}$ define by*

$$\begin{cases} x_1 \in K; \\ x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_m y_n^m; \quad n \geq 1, \\ y_n^i \in T_i x_n, \ \alpha_0 \in (k,1), \ \alpha_i \in (0,1), \ k = max\{k_i, \ i = 1,2,\cdots,m\}, \ \sum_{i=0}^{m} \alpha_i = 1, \end{cases} \tag{2.2.7}$$

$\triangle$-converges to a point $p \in \bigcap_{i=1}^{m}$.

The notation $\oplus$ used in Theorem 2.2.8 above is an ordered operation defined inductively by

$$\bigoplus_{i=1}^{n} \alpha_i x_i := \alpha_1 x_1 \oplus (1-\alpha_1)\left(\frac{\alpha_2}{1-\alpha_1}x_2 \oplus \frac{\alpha_3}{1-\alpha_1}x_3 \oplus \cdots \oplus \frac{\alpha_n}{1-\alpha_1}x_n\right).$$

For better understanding we illustrate how this operation work for $n = 3$. If we fix $\alpha_1, \alpha_2, \alpha_3 \in$

$(0,1)$ such that $\alpha_1+\alpha_2+\alpha_3=1$ and $x_1,x_2,x_3 \in X$, we have

$$\begin{aligned}\bigoplus_{i=1}^{3}\alpha_i x_i &:= \alpha_1 x_1 \oplus (1-\alpha_1)\left(\frac{\alpha_2}{1-\alpha_1}x_2 \oplus \frac{\alpha_3}{1-\alpha_1}x_3\right)\\ &= \alpha_1 x_1 \oplus (\alpha_2+\alpha_3)\left(\frac{\alpha_2}{\alpha_2+\alpha_3}x_2 \oplus \frac{\alpha_3}{\alpha_2+\alpha_3}x_3\right)\\ &= \alpha_1 x_1 \oplus \alpha\prime x\prime,\end{aligned}$$

where $\alpha\prime=\alpha_2+\alpha_3$ and $x\prime=\frac{\alpha_2}{\alpha_2+\alpha_3}x_2 \oplus \frac{\alpha_3}{\alpha_2+\alpha_3}x_3$. If we set $w=\bigoplus_{i=1}^{3}\alpha_i x_i$ and recall Definition 1.2.32, then one can easily see that $x\prime$ is the unique point in $[x_2,x_3]$ such that $d(x_2,x\prime)=\frac{\alpha_3}{\alpha_2+\alpha_3}d(x_2,x_3)$ and $d(x_3,x\prime)=\frac{\alpha_2}{\alpha_2+\alpha_3}d(x_2,x_3)$. Then the notation $\bigoplus_{i=1}^{3}\alpha_i x_i$ which we set as w, becomes the unique point in $[x_1,x\prime]$ satisfying $d(x_1,w)=\alpha\prime d(x_1,x\prime)$ and $d(x\prime,w)=\alpha_1 d(x_1,x\prime)$. In this regard, the operation defined by $\bigoplus_{i=1}^{n}$ take a similar pattern for any fixed $n\geq 2$. However, this is not the only method for this notation. Another method for this notation (see, e.g., [Dhompongsa et al., 2012]) is defined in reverse order as follows:

$$\bigoplus_{i=1}^{n}\alpha_i x_i := (1-\alpha_n)\left(\frac{\alpha_1}{1-\alpha_n}x_1 \oplus \frac{\alpha_2}{1-\alpha_n}x_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n}x_{n-1}\right) \oplus \alpha_n x_n.$$

For conveniences we give some illustration for $n=3$ as follows. For fixed $\alpha_1,\alpha_2,\alpha_3 \in (0,1)$ such that $\alpha_1+\alpha_2+\alpha_3=1$ and $x_1,x_2,x_3 \in X$, this operation gives

$$\begin{aligned}\bigoplus_{i=1}^{3}\alpha_i x_i &:= (1-\alpha_3)\left(\frac{\alpha_1}{1-\alpha_3}x_1 \oplus \frac{\alpha_2}{1-\alpha_3}x_2\right) \oplus \alpha_3 x_3\\ &= (\alpha_1+\alpha_2)\left(\frac{\alpha_1}{\alpha_1+\alpha_2}x_1 \oplus \frac{\alpha_2}{\alpha_1+\alpha_2}x_2\right) \oplus \alpha_3 x_3\\ &= \alpha_0 x_0 \oplus \alpha_3 x_3,\end{aligned}$$

where $\alpha_0=\alpha_1+\alpha_2$ and $x_0=\frac{\alpha_1}{\alpha_1+\alpha_2}x_1 \oplus \frac{\alpha_2}{\alpha_1+\alpha_2}x_2$. Now setting $z=\bigoplus_{i=1}^{3}\alpha_i x_i$ and using Definition 1.2.32, one see that x_0 is the unique point in $[x_1,x_2]$ such that $d(x_1,x_0)=\frac{\alpha_2}{\alpha_1+\alpha_2}d(x_1,x_2)$ and $d(x_2,x_0)=\frac{\alpha_1}{\alpha_1+\alpha_2}d(x_1,x_2)$. Thus z becomes the unique point in $[x_0,x_3]$ satisfying $d(x_0,z)=\alpha_3 d(x_0,x_3)$ and $d(x_3,z)=\alpha_0 d(x_0,x_3)$. It follows similar argument for any fixed $n\geq 2$.

Remark 2.2.9 *We observe that the two definitions for the $\bigoplus$ notation involving finite elements given above have different meanings. However, for $n = 2$, one can easily see that these methods coincide with that of the Definitions 1.2.32.*

Several iterative schemes have been constructed for approximations of fixed point(s) of certain maps (singlevalued and multivalued) in Hilbert spaces. Some of these algorithms were later extended to CAT(0) spaces. However, only few algorithms are developed for multivalued demicontractive maps in CAT(0) spaces. These algorithms are actually for finite family (see, e.g., [Chidume et al., 2014]). The question is can we have similar results for countable family of multivalued demicontractive maps. It is our purpose in this thesis to introduce new iterative schemes, and prove delta & strong convergence theorems for a sequence (generated by the proposed schemes) to a common fixed point of a countable family of multivalued demicontractive mappings in Hadamard spaces.

CHAPTER 3

METHODOLOGY

In this chapter, we give some of the important results in special metric spaces (e.g. CAT(0) spaces). The proofs of most of these results are in the literature. However, for sake of understanding we also give a shorter and simplified proofs to some.

Lemma 3.0.1 *[Dhompongsa et al., 2007] Let D be a nonempty closed convex subset of a Hadamard space* (X,d) *and* $\{x_n\}$ *be a bounded sequence in D. Then the asymptotic centre* $A(\{x_n\})$ *of* $\{x_n\}$ *is in D.*

Lemma 3.0.2 *[Dhompongsa and Panyanak, 2008] Let* (X,d) *be a* $CAT(0)$ *space. Then*

(i) (X,d) *is uniquely geodesic.*

(ii) for all $x,y \in X$ *with* $x \neq y$ *and* $z,w \in [x,y]$ *such that* $d(x,z) = d(x,w)$*, we have* $z = w$.

(iii) for each $x,y \in X$ *and* $t \in [0,1]$*, there exists a unique point* $z \in [x,y]$ *such that* $d(x,z) = td(x,y)$ *and* $d(y,z) = (1-t)d(x,y)$*. From now on, we use the notation* $(1-t)x \oplus ty$ *for such a point z.*

(iv) for all $x,y \in X$ *with* $x \neq y$ *we have* $[x,y] = \{(1-t)x \oplus ty \mid t \in [0,1]\}$.

(v) for all $x,y,z \in X, \quad d(x,z) + d(z,y) = d(x,y)$ *if and only if* $z \in [x,y]$.

(vi) *for all* $p,x,y \in X$, $m_1 \in [p,x]$, $m_2 \in [p,y]$, $t \in [0,1]$, *satisfying* $d(p,m_1) = td(p,x)$ *and* $d(p,m_2) = td(p,y)$, *we have* $d(m_1,m_2) \leq td(x,y)$.

(vii) *the mapping* $\gamma : [0,1] \to [x,y]$ *defined by* $\gamma(t) := (1-t)x \oplus ty$ *is continuous and bijective.*

Lemma 3.0.3 *Let* $\bar{\triangle}(\bar{p},\bar{q},\bar{r})$ *be a comparison triangle of a geodesic triangle* $\triangle(p,q,r)$. *Let* $z \in \triangle(p,q,r)$ *such that* $z = (1-t)p \oplus tq$ *for some* $t \in [0,1]$. *Then its comparison point* $\bar{z} = (1-t)\bar{p} + t\bar{q}$.

Proof Clearly $\bar{z} \in [\bar{p},\bar{q}]$. Furthermore,

$$d(\bar{p},(1-t)\bar{p}+t\bar{q}) = \|(1-t)\bar{p}+t\bar{q}-\bar{p}\| = \|(t(\bar{q}-\bar{p})\| = t\|\bar{p}-\bar{q}\| = td(p,q) = d(p,z).$$

This completes the proof. ■

Remark 3.0.4 *For any two distinct points* x,y *in a CAT(0) space* (X,d) *and* $s,t \in [0,1]$ *we have* $(1-t)x \oplus ty = (1-s)x \oplus sy$ *if and only if* $s = t$.

We introduced Lemma 3.0.3 above for the purpose of improving proofs of some (known) results. We start by simplifying the proof (given by [Dhompongsa and Panyanak, 2008]) of the following lemma.

Lemma 3.0.5 *[Dhompongsa and Panyanak, 2008] Let* (X,d) *be a CAT(0) space. Let* $x,y,z \in X$ *and* $t \in [0,1]$. *Then*

(i) $d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$,

(ii) $d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2$.

Proof Let (X,d) be a CAT(0) space and $x,y,z \in X$. By definition, $(1-t)x \oplus ty \in [x,y]$. Consider $\bar{\triangle}(\bar{x},\bar{y},\bar{z})$ a comparison triangle of the geodesic triangle $\triangle(x,y,z)$. If we set $w := (1-t)x \oplus ty$, then $w \in \triangle(x,y,z)$ and its comparison point in the triangle $\bar{\triangle}(\bar{x},\bar{y},\bar{z})$

is $\bar{w} = (1-t)\bar{x} + t\bar{y}$ (by Lemma 3.0.3). Using the $CAT(0)$ inequality and the fact that $z, w \in \triangle(x,y,z)$,

$$\begin{aligned} d(w,z) &\leq \|\bar{w} - \bar{z}\| \\ &= \|(1-t)\bar{x} + t\bar{y} - \bar{z}\| \\ &= \|(1-t)\bar{x} + t\bar{y} - t\bar{z} + t\bar{z} - \bar{z}\| \\ &\leq (1-t)\|\bar{x} - \bar{z}\| + t\|\bar{y} - \bar{z}\| \\ &\leq (1-t)d(x,z) + td(y,z). \end{aligned}$$

For (ii) we proceed as follows:

$$\begin{aligned} d(w,z)^2 &\leq \|\bar{w} - \bar{z}\|^2 \\ &= \|(1-t)\bar{x} + t\bar{y} - \bar{z}\|^2 \\ &= \|(1-t)\bar{x} + t\bar{y} - t\bar{z} + t\bar{z} - \bar{z}\|^2 \\ &= (1-t)\|\bar{x} - \bar{z}\|^2 + t\|\bar{y} - \bar{z}\|^2 - t(1-t)\|\bar{x} - \bar{y}\|^2 \\ &= (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2. \end{aligned}$$

For existing proof of this Lemma, we refer the reader to [Dhompongsa and Panyanak, 2008].

■

Lemma 3.0.6 ([Kirk and Panyanak, 2008]) *Let D be a closed convex nonempty subset of a Hadamard space (X,d) and let $T: D \to X$ be a nonexpansive mapping. Then the conditions $\{x_n\}$ $\triangle$-converges to x and $d(x_n, Tx_n) \to 0$, imply $x \in D$ and $Tx = x$.*

Researchers used Lemma 3.0.6 in proving many results (see, e.g., [Khan and Abbas, 2011], [Dhompongsa and Panyanak, 2008]). However, we note that this lemma can be generalised in the following sense.

Lemma 3.0.7 *Let D be a closed convex nonempty subset of a Hadamard space (X,d) and let $T: D \to X$ be a nonexpansive mapping. Then the conditions $x \in \bigcup A(\{u_n\})$ for union taken over subsequences of $\{x_n\}$ and $d(x_n, Tx_n) \to 0$, imply $x \in D$ and $Tx = x$.*

Proof Let $x \in \bigcup A(\{u_n\})$. Then there exists $\{v_n\}$ subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{x\}$. By Lemma 3.0.1, $x \in D$. Furthermore, using the fact that T is nonexpansive and $x \in A(\{v_n\})$ we have

$$\begin{aligned} r(\{v_n\}) &\leq r(Tx, \{v_n\}) \\ &= \limsup_n d(Tx, v_n) \\ &\leq \limsup_n d(Tx, Tv_n) + \limsup_n d(Tv_n, v_n) \\ &\leq \limsup_n d(x, v_n) + 0 \\ &= r(x, \{v_n\}) \\ &= r(\{v_n\}). \end{aligned}$$

Thus $r(\{v_n\}) = r(Tx, \{v_n\})$. So, $Tx \in A(\{v_n\})$. Hence by uniqueness of asymptotic centre in $CAT(0)$ spaces, we have $Tx = x$. ■

Lemma 3.0.8 *[Kirk and Panyanak, 2008] Every bounded sequence in a Hadamard space has a $\triangle$-convergent subsequence.*

Lemma 3.0.9 *Let (X, d) be a geodesic metric space. The following are equivalent:*

(i) X is a $CAT(0)$ space.

(ii) X satisfies the CN inequality: if $x, y \in X$ and $\frac{x \oplus y}{2}$ is the midpoint of x and y, then

$$d(z, \frac{x \oplus y}{2}) \leq \frac{1}{2} d(z, x)^2 + \frac{1}{2} d(z, y)^2 - \frac{1}{2} d(x, y)^2, \qquad \forall z \in X.$$

Lemma 3.0.10 *[Dhompongsa and Panyanak, 2008] If $\{x_n\}$ is a bounded sequence in a Hadamard space (X, d) with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 3.0.11 *Let $\{a_n\}$, $\{b_n\}$ be sequences in $\mathbb{R}$ such that $\lim_n a_n b_n = 0$. If $\liminf_n a_n > 0$, then $\lim_n b_n = 0$.*

Proof Suppose $b_n \nrightarrow 0$, then we have some $\varepsilon_0 > 0$ such that for all $k \in \mathbb{N}$ there exists $N_k \geq k$ such that $|b_{N_k}| \geq \varepsilon_0$. This gives a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ with $|b_{n_k}| \geq \varepsilon_0$, $\forall\, k$. Indeed, for $k = 1$, there exists $N_1 \geq 1$ such that $\left|b_{N_1}\right| \geq \varepsilon_0$. Set $n_1 = N_1$. For $k = N_1 + 1$, there exists $n_2 \geq N_1 + 1$ such that $\left|b_{n_2}\right| \geq \varepsilon_0$. For $k = n_2 + 1$, there exists $n_3 \geq n_2 + 1$ such that $\left|b_{n_3}\right| \geq \varepsilon_0$. Continuing in this manner we obtain such a subsequence $\{b_{n_k}\}$ of $\{b_n\}$. Let $\{a_{n_k}\}$ be the corresponding subsequence of $\{a_n\}$. Then $\liminf_k a_{n_k} \geq \liminf_n a_n > 0$ since $\liminf$ of a sequence is the infimum of the set of all its subsequential limits. Let $\{a_{n_{k_j}}\}$ be a subsequence of $\{a_{n_k}\}$ such that $\lim_j a_{n_{k_j}} = \liminf_k a_{n_k}$. Since $|a_{n_{k_j}} b_{n_{k_j}}| = |a_{n_{k_j}}||b_{n_{k_j}}| \geq |a_{n_{k_j}}|\varepsilon_0 \geq a_{n_{k_j}}\varepsilon_0$, taking limit as $j \to \infty$ we have $\lim_j |a_{n_{k_j}} b_{n_{k_j}}| \geq \varepsilon_0 \liminf_k a_{n_k} > 0$, which is a contradiction. Therefore $\lim_n b_n = 0$. ■

CHAPTER 4

MAIN RESULTS

In this chapter, we propose two iterative schemes; one for finite family and the other for infinite countable family of demicontractive multivalued maps. We start by given a proof of a lemma which will be central to our main theorem.

4.1 Iterative Scheme for Finite Family of Demicontractive Maps

Let (X,d) be a Hadamard space and let D be its closed convex nonempty subset. Let $T_i : D \to \mathscr{CB}(D)$ be multivalued demicontractive mappings with constants $\{k_i\} \subset (0,1)$, $m \in N$, $i = 1, \cdots, m$. A sequence $\{x_n\}$ is defined iteratively as follows:

$$\begin{cases} x_1 \in D; \\ y_n^{(0)} = x_n; \\ y_n^{(i)} = a_{ni} y_n^{(i-1)} \oplus (1-a_{ni}) z_n^{(i-1)}, \qquad i = 1, \cdots, m-1; \\ x_{n+1} = a_{nm} y_n^{(m-1)} \oplus (1-a_{nm}) z_n^{(m-1)}; \\ z_n^{(i-1)} \in T_i y_n^{(i-1)},\ a_{ni} \in [k_i, 1],\ n \in \mathbb{N}, \qquad i = 1, \cdots, m. \end{cases} \tag{4.1.1}$$

Lemma 4.1.1 *Let (X,d) be a Hadamard space and D be its closed convex nonempty subset. Let $T_i : D \to \mathscr{CB}(D)$ be multivalued demicontractive mappings with constants $\{k_i\} \subset (0,1)$, $m \in N$, $i = 1,\cdots,m$ and $\{x_n\}$ be defined by iterative process (4.1.1). Suppose $\mathscr{F} := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $p \in \mathscr{F}$ and for all $i \in \{1,2,\cdots,m\}$. Then, $\lim_n d(x_n, p)$ exists for all $p \in \mathscr{F}$.*

Proof Let $p \in \mathscr{F}$ and $i \in \{1,\cdots,m-1\}$. By Lemma 3.0.5 (ii), the scheme (4.1.1) and the assumptions on $T_i's$ we have

$$
\begin{aligned}
d(y_n^{(i)},p)^2 &\leq a_{ni} d\left(y_n^{(i-1)},p\right)^2 + (1-a_{ni}) d\left(z_n^{(i-1)},p\right)^2 - a_{ni}(1-a_{ni}) d(y_n^{i-1}, z_n^{(i-1)})^2 \\
&\leq a_{ni} d(y_n^{(i-1)},p)^2 + (1-a_{ni}) dist(z_n^{(i-1)}, T_i p)^2 - a_{ni}(1-a_{ni}) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \\
&\leq a_{ni} d(y_n^{(i-1)},p)^2 + (1-a_{ni}) d_H(T_i y_n^{(i-1)}, T_i p)^2 - a_{ni}(1-a_{ni}) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \\
&\leq a_{ni} d(y_n^{(i-1)},p)^2 + (1-a_{ni})[d(y_n^{(i-1)},p)^2 + k_i d(y_n^{(i-1)}, z_n^{(i-1)})^2] \\
&\quad - a_{ni}(1-a_{ni}) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \\
&\leq d(y_n^{(i-1)},p)^2 + (1-a_{ni})(k_i - a_{ni}) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \\
&\leq d(y_n^{(i-1)},p)^2 + (k_i - a_{ni}) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \\
&= d(y_n^{(i-1)},p)^2 - (a_{ni} - k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2, \quad i = 1,\cdots,m-1.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d(x_{n+1},p)^2 &\leq a_{nm} d(y_n^{(m-1)},p)^2 + (1-a_{nm}) d(z_n^{(m-1)},p)^2 - a_{nm}(1-a_{nm}) d(y_n^{m-1}, z_n^{(m-1)})^2 \\
&\leq a_{nm} d(y_n^{(m-1)},p)^2 + (1-a_{nm}) dist(z_n^{(m-1)}, T_m p)^2 - a_{nm}(1-a_{nm}) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&\leq a_{nm} d(y_n^{(m-1)},p)^2 + (1-a_{nm}) d_H(T_m y_n^{(m-1)}, T_m p)^2 - a_{nm}(1-a_{nm}) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&\leq a_{nm} d(y_n^{(m-1)},p)^2 + (1-a_{nm})[d(y_n^{(m-1)},p)^2 + k_m d(y_n^{(m-1)}, z_n^{(m-1)})^2] \\
&\quad - a_{nm}(1-a_{nm}) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&\leq d(y_n^{(m-1)},p)^2 + (1-a_{nm})(k_i - a_{nm}) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&\leq d(y_n^{(m-1)},p)^2 + (k_m - a_{nm}) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&= d(y_n^{(m-1)},p)^2 - (a_{nm} - k_m) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&\leq d(y_n^{(m-2)},p)^2 - (a_{nm-1} - k_{m-1}) d(y_n^{(m-2)}, z_n^{(m-2)})^2 - (a_{nm} - k_m) d(y_n^{(m-1)}, z_n^{(m-1)})^2
\end{aligned}
$$

So,

$$\begin{aligned} d(x_{n+1},p)^2 &\le d(y_n^{(m-2)},p)^2-(a_{nm-1}-k_{m-1})d(y_n^{(m-2)},z_n^{(m-2)})^2-(a_{nm}-k_m)d(y_n^{(m-1)},z_n^{(m-1)})^2\\ &\le d(y_n^{(m-3)},p)^2-\sum_{i=m-2}^{m}(a_{ni}-k_i)d(y_n^{(i-1)},z_n^{(i-1)})^2\\ &\ \vdots\\ &\le d(y_n^{(0)},p)^2-\sum_{i=1}^{m}(a_{ni}-k_i)d(y_n^{(i-1)},z_n^{(i-1)})^2\\ &= d(x_n,p)^2-\sum_{i=1}^{m}(a_{ni}-k_i)d(y_n^{(i-1)},z_n^{(i-1)})^2\\ &\le d(x_n,p)^2. \end{aligned}$$

This implies that $\lim_n d(x_n,p)$ exists, as a monotonic nonincreasing sequence of real numbers (that is bounded below by 0).

■

Theorem 4.1.2 *Let X, D, $\{T_i\}$, $\mathscr{F}$, $\{k_i\}$, $\{a_{ni}\}$ and $\{x_n\}$ be as in Lemma 4.1.1. Let $\liminf_n a_{ni} > k_i$ for each $i \in \{1,\cdots,m\}$ and let $T_1,\cdots,T_m$ be Lipschitzian maps. Then $\lim_n dist(x_n,T_ix_n) = 0$ for all $i = 1,\cdots,m$.*

Proof As in the proof of Lemma 4.1.1,

$$\sum_{i=1}^{m}(a_{ni}-k_i)d(y_n^{(i-1)},z_n^{(i-1)})^2 \le d(x_n,p)^2-d(x_{n+1},p)^2$$

and $\lim_n d(x_n,p)$ exists for all $p \in \mathscr{F}$. Thus $\lim_n (a_{ni}-k_i)d(y_n^{(i-1)},z_n^{(i-1)})^2 = 0$ for all $i = 1,\cdots,m$. Since $\liminf_n a_{ni} > k_i$ for each $i \in \{1,\cdots,m\}$, it follows from Lemma 3.0.11 that

$$\lim_n d(y_n^{(i-1)},z_n^{(i-1)}) = 0 \; for\; each\; i = 1,\cdots,m. \tag{4.1.2}$$

Now, let $i \in \{1,\cdots,m\}$. Then,

$$\begin{aligned} d(x_n,z_n^{(i-1)}) &= d(y_n^{(0)},z_n^{(i-1)})\\ &\le d(y_n^{(0)},y_n^{(1)})+d(y_n^{(1)},y_n^{(2)})+,\cdots,+d(y_n^{(i-2)},y_n^{(i-1)})+d(y_n^{(i-1)},z_n^{(i-1)}) \end{aligned}$$

$$\begin{aligned} d(x_n, z_n^{(i-1)}) &\le d(y_n^{(0)}, y_n^{(1)}) + d(y_n^{(1)}, y_n^{(2)}) +, \cdots, + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\ &\le d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, y_n^{(2)}) +, \cdots, + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\ &\le d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, z_n^{(1)}) +, \cdots, + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\ &\vdots \\ &\le d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, z_n^{(1)}) +, \cdots, + d(y_n^{(i-2)}, z_n^{(i-2)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\ &\le \sum_{k=1}^{i} d(y_n^{(k-1)}, z_n^{(k-1)}). \end{aligned}$$

This implies that

$$\lim_n d(x_n, z_n^{(i-1)}) = 0 \; for \; each \; i = 1, \cdots, m. \tag{4.1.3}$$

By triangle inequality we have

$d(x_n, \alpha_n^i) \le d(x_n, z_n^{(i-1)}) + d(z_n^{(i-1)}, \alpha_n^i), \quad \forall\, \alpha_n^i \in T_i x_n$. This implies

$dist(x_n, T_i x_n) \le d(x_n, z_n^{(i-1)}) + d(z_n^{(i-1)}, \alpha_n^i), \quad \forall\, \alpha_n^i \in T_i x_n$. Now taking infimum over $\alpha_n^i \in T_i x_n$ and using the fact that T_i is L_i-Lipschitzian for each $i \in 1, \cdots, m$, we have the following:

$$\begin{aligned} dist(x_n, T_i x_n) &\le d(x_n, z_n^{(i-1)}) + dist(z_n^{(i-1)}, T_i x_n) \\ &\le d(x_n, z_n^{(i-1)}) + d_H(T_i y_n^{(i-1)}, T_i x_n) \\ &\le d(x_n, z_n^{(i-1)}) + L_i d(y_n^{(i-1)}, x_n) \\ &\le d(x_n, z_n^{(i-1)}) + L_i[d(y_n^{(i-1)}, z_n^{i-1}) + d(z_n^{i-1}, x_n)] \\ &\le (1 + L_i) d(x_n, z_n^{(i-1)}) + L_i d(y_n^{(i-1)}, z_n^{i-1}). \end{aligned}$$

Therefore, by (4.1.2) and (4.1.3) we have $\lim_n dist(x_n, T_i x_n) = 0$ for all $i = 1, \cdots, m$.

■

Corollary 4.1.3 *Let X, D, $\{T_i\}$ and $\{x_n\}$ be as in Theorem 4.1.2. Suppose T_i is $\triangle$-demiclosed at 0 for $i \in \{1, \cdots, m\}$. Then $\{x_n\}$ $\triangle$-converges to a common fixed point.*

Proof By Lemma 4.1.1 and Theorem 4.1.2 we have $\lim_n d(x_n, p)$ exists for all $p \in \mathscr{F}$

and $\lim_n d(x_n, T_i x_n) = 0$ for all $i = 1, \cdots, m$. Hence $\{x_n\}$ is bounded. Now, let $u \in \bigcup A(\{w_n\})$ for $\{w_n\}$ subsequence of $\{x_n\}$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 3.0.8 there exists $\{v_n\}$, a subsequence of $\{u_n\}$ such that $\triangle - \lim_n v_n = v$ and by Lemma 3.0.1 we have that $v \in D$. Using Theorem 4.1.2 and the fact that T_i is $\triangle$-demiclosed at 0 for each i, we have $v \in \mathscr{F}$ and hence $\{d(u_n, v)\}$ converges by Lemma 4.1.1. Moreover, Lemma 3.0.10 implies that $u = v \in \mathscr{F}$. Thus $\bigcup A(\{w_n\}) \subseteq \mathscr{F}$. Now we have seen that the collection of asymptotic centres of subsequences of $\{x_n\}$ is in $\mathscr{F}$, it remains to show that $\{x_n\}$ is $\triangle$-convergent. It suffices to show that the collection is actually singleton. To see this, let $A(\{x_n\}) = \{x\}$ and let $\{u_n\}$ be arbitrary subsequence of $\{x_n\}$. Let $A(\{u_n\}) = \{u\}$. We have $u \in \mathscr{F}$, since $\{u\}$ is the asymptotic centre of a $\{u_n\}$ subsequence of $\{x_n\}$. By Lemma 4.1.1 we have that $\{d(x_n, u)\}$ converges and Lemma 3.0.10 implies that $u = x$. This complete the proof. ∎

Corollary 4.1.4 *Let X, D, $\{T_i,\ i = 1, \cdots, m\}$, $\mathscr{F}$ and $\{x_n\}$ be as in Theorem 4.1.2. Suppose D is compact. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i,\ i = 1, \cdots, m\}$.*

Proof We have from Theorem 4.1.2 that $\lim_n d(x_n, T_i x_n) = 0$ for all $i = 1, \cdots, m$. Since D is compact, there exists $\{v_n\}$ a subsequence of $\{x_n\}$ such that $\lim_n d(v_n, w) = 0$ for some $w \in D$. Now using triangular inequality, for $i \in \{1, \cdots, m\}$ we have $d(w, \beta_i) \leq d(w, v_n) + d(v_n, u_n^i) + d(u_n^i, \beta_i)\ \forall \beta_i \in T_i w,\ \forall u_n^i \in T_i v_n$. This implies that
$dist(w, T_i w) \leq d(w, v_n) + d(v_n, u_n^i) + d(u_n^i, \beta_i)\ \forall \beta_i \in T_i w,\ \forall u_n^i \in T_i v_n$. Taking infimum over $\beta_i \in T_i w$ and using the fact that $\{T_i\}$ is lipschitzian we obtain

$$\begin{aligned}
dist(w, T_i w) &\leq d(w, v_n) + d(v_n, u_n^i) + dist(u_n^i, T_i w) \\
&\leq d(w, v_n) + d(v_n, u_n^i) + d_H(T_i v_n, T_i w) \\
&\leq d(w, v_n) + d(v_n, u_n^i) + L_i d(v_n, w) \\
&\leq (1 + L_i) d(w, v_n) + d(v_n, u_n^i), \qquad \forall u_n^i \in T_i v_n.
\end{aligned}$$

Now taking infimum over $u_n^i \in T_i v_n$ we have

$$dist(w, T_i w) \leq (1 + L_i) d(w, v_n) + dist(v_n, T_i v_n).$$

Taking limit of both sides as $n \to +\infty$ we have $dist(w, T_i w) = 0$. Thus $w \in \mathscr{F}$. By Lemma 4.1.1 we have that $\lim_n d(x_n, w)$ exists since $w \in \mathscr{F}$. Thus $\lim_n d(x_n, w) = \lim_n d(v_n, w) = 0$. Hence $\{x_n\}$ converges strongly to w. ■

Theorem 4.1.5 *Let X, D, $\{T_i\}$, $\mathscr{F}$ and $\{x_n\}$ be as in Lemma 4.1.1. Then $\{x_n\}$ converges strongly to a point $p \in \mathscr{F}$ if and only if $\liminf_n dist(x_n, \mathscr{F}) = 0$.*

Proof ($\Longrightarrow$) Suppose $d(x_n, p) \to 0$ *as* $n \to +\infty$ for some $p \in \mathscr{F}$. Since $0 \le dist(x_n, \mathscr{F}) \le d(x_n, p)$,

$$\liminf_n dist(x_n, \mathscr{F}) = \lim_n dist(x_n, \mathscr{F}) = 0.$$

($\Longleftarrow$) Suppose that $\liminf_n dist(x_n, \mathscr{F}) = 0$. It is seen in the proof of Lemma 4.1.1 that $d(x_{n+1}, p) \le d(x_n, p)$ for all $p \in \mathscr{F}$. This implies that $dist(x_{n+1}, \mathscr{F}) \le dist(x_n, \mathscr{F})$. So the $\lim_n dist(x_n, \mathscr{F})$ exists. Using the hypothesis we have $\lim_n dist(x_{n+1}, \mathscr{F}) = 0$. Therefore we can choose $\{x_{n_k}\}$ a subsequence of $\{x_n\}$ and a sequence $\{p_k\}$ in $\mathscr{F}$ such that for all $k \in \mathbb{N}$, $d(x_{n_k}, p_k) < \frac{1}{2^k}$. By Lemma 4.1.1 we have $d(x_{n_{k+1}}, p_k) \le d(x_{n_k}, p_k) < \frac{1}{2^k}$. Hence $d(p_{k+1}, p_k) \le d(x_{n_{k+1}}, p_{k+1}) + d(x_{n_{k+1}}, p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}$. Thus $\{p_k\}$ is a Cauchy sequence in D and therefore converges to some point $q \in D$. For $i \in \{1, \cdots, m\}$, $dist(p_k, T_i q) \le d_H(T_i p_k, T_i q) \le L_i d(p_k, q)$. Taking limit as $k \to +\infty$ we have $dist(q, T_i q) = 0 \; \forall \, i = 1, \cdots, m$. Hence $q \in \mathscr{F}$ and so $\lim_k d(x_{n_k}, q) = 0$. Since $\lim_n d(x_n, q)$ exists, it follows that $\{x_n\}$ converges strongly to q. ■

4.2 Iterative Scheme for Countable Family of Demicontractive Maps

Let (X, d) be a Hadamard space and let D be its closed convex nonempty subset. Let $T_i : D \to \mathscr{C}\mathscr{B}(D)$ be multivalued demicontractive mappings with constants $\{k_i\} \subset (0, 1)$, $i \in N$.

A sequence $\{x_n\}$ is defined iteratively as follows:

$$\begin{cases} x_1 \in D; \\ y_n^{(0)} = x_n; \\ y_n^{(i)} = a_{ni} y_n^{(i-1)} \oplus (1-a_{ni}) z_n^{(i-1)}, \qquad i = 1, \cdots, n-1; \\ x_{n+1} = a_{nn} y_n^{(n-1)} \oplus (1-a_{nn}) z_n^{(n-1)}; \\ z_n^{(i-1)} \in T_i y_n^{(i-1)}, a_{ni} \in [k_i, 1], \ \ n \in \mathbb{N}, \qquad i = 1, \cdots, n. \end{cases} \tag{4.2.1}$$

Lemma 4.2.1 *Let (X,d) be a Hadamard space and let D be its closed convex nonempty subset. Let $T_i : D \to \mathscr{CB}(D)$ be multivalued demicontractive mappings with constants $\{k_i\} \subset (0,1)$, $i \in N$ and $\{x_n\}$ be defined by iterative process in (4.2.1). Suppose $\mathscr{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $p \in \mathscr{F}$. Then, $\lim_n d(x_n, p)$ exists for all $p \in \mathscr{F}$.*

Proof Let $p \in \mathscr{F}$. By lemma 3.0.5 (ii), the scheme (4.2.1) and the assumptions on $T_i's$ we have

$$\begin{aligned} d(x_{n+1},p)^2 &\leq a_{nn} d(y_n^{(n-1)},p)^2 + (1-a_{nn}) d(z_n^{(n-1)},p)^2 - a_{nn}(1-a_{nn}) d(y_n^{n-1}, z_n^{(n-1)})^2 \\ &\leq a_{nn} d(y_n^{(n-1)},p)^2 + (1-a_{nn}) dist(z_n^{(n-1)}, T_n p)^2 - a_{nn}(1-a_{nn}) d(y_n^{(n-1)}, z_n^{(n-1)})^2 \\ &\leq a_{nn} d(y_n^{(n-1)},p)^2 + (1-a_{nn}) d_H(T_n y_n^{(n-1)}, T_n p)^2 - a_{nn}(1-a_{nn}) d(y_n^{(n-1)}, z_n^{(n-1)})^2 \\ &\leq a_{nn} d(y_n^{(n-1)},p)^2 + (1-a_{nn})[d(y_n^{(n-1)},p)^2 + k_n d(y_n^{(n-1)}, z_n^{(n-1)})^2] \\ &\quad - a_{nn}(1-a_{nn}) d(y_n^{(n-1)}, z_n^{(n-1)})^2 \\ &\leq d(y_n^{(n-1)},p)^2 + (1-a_{nn})(k_i - a_{nn}) d(y_n^{(n-1)}, z_n^{(n-1)})^2 \\ &\leq d(y_n^{(n-1)},p)^2 + (k_n - a_{nn}) d(y_n^{(n-1)}, z_n^{(n-1)})^2 \\ &= d(y_n^{(n-1)},p)^2 - (a_{nn} - k_n) d(y_n^{(n-1)}, z_n^{(n-1)})^2. \\ &\leq d(y_n^{(n-2)},p)^2 - (a_{nn-1} - k_{n-1}) d(y_n^{(n-2)}, z_n^{(n-2)})^2 - (a_{nn} - k_n) d(y_n^{(n-1)}, z_n^{(n-1)})^2 \\ &\leq d(y_n^{(n-3)},p)^2 - \sum_{i=n-2}^{n} (a_{ni} - k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \\ &\ \ \vdots \\ &\leq d(y_n^{(0)},p)^2 - \sum_{i=1}^{n} (a_{ni} - k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \end{aligned}$$

$$\begin{aligned} d(x_{n+1},p)^2 &\leq d(x_n,p)^2 - \sum_{i=1}^{n}(a_{ni}-k_i)d(y_n^{(i-1)},z_n^{(i-1)})^2 \\ &\leq d(x_n,p)^2. \end{aligned}$$

This implies that $\lim_n d(x_n,p)$ exists, as a monotonic nonincreasing sequence of real numbers that is bounded below by 0. ■

Theorem 4.2.2 *Let X, D, $\{T_i\}$, $\mathscr{F}$ and $\{x_n\}$ be as in Lemma 4.2.1. Let $\liminf_n a_{ni} > k_i$ for each $i \in \mathbb{N}$ and let T_i be Lipschitzian maps for all $i \in \mathbb{N}$. Then $\lim_n dist(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$.*

Proof As in the proof of Lemma 4.2.1,

$$\sum_{i=1}^{n}(a_{ni}-k_i)d(y_n^{(i-1)},z_n^{(i-1)})^2 \leq d(x_n,p)^2 - d(x_{n+1},p)^2 \quad for\ all_n \in \mathbb{N}.$$

This implies that

$$\sum_{i=1}^{n}(a_{ni}-k_i)d(y_n^{(i-1)},z_n^{(i-1)})^2 \leq d(x_1,p) \quad for\ all_n \in \mathbb{N}$$

and so

$$\lim_n \sum_{i=1}^{n}(a_{ni}-k_i)d(y_n^{(i-1)},z_n^{(i-1)})^2$$

exists in $\mathbb{R}$. Thus $\lim_n (a_{ni}-k_i)d(y_n^{(i-1)},z_n^{(i-1)})^2 = 0$ for all $i \in \mathbb{N}$. Since $\liminf_n a_{ni} > k_i$ for each $i \in \mathbb{N}$, it follows from Lemma 3.0.11 that

$$\lim_n d(y_n^{(i-1)},z_n^{(i-1)}) = 0 \ for\ each \quad i \in \mathbb{N}. \tag{4.2.2}$$

Now, let $i \in \mathbb{N}$. Then

$$\begin{aligned}
d(x_n, z_n^{(i-1)}) &= d(y_n^{(0)}, z_n^{(i-1)}) \\
&\le d(y_n^{(0)}, y_n^{(1)}) + d(y_n^{(1)}, y_n^{(2)}) + \cdots + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\
&\le d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, y_n^{(2)}) + \cdots + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\
&\le d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, z_n^{(1)}) + \cdots + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\
&\;\vdots \\
&\le d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, z_n^{(1)}) + \cdots + d(y_n^{(i-2)}, z_n^{(i-2)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\
&\le \sum_{k=1}^{i} d(y_n^{(k-1)}, z_n^{(k-1)}).
\end{aligned}$$

This implies that

$$\lim_n d(x_n, z_n^{(i-1)}) = 0 \;\; for\ each\ i \in \mathbb{N}. \tag{4.2.3}$$

By triangle inequality we have $d(x_n, \beta_n^i) \le d(x_n, z_n^{(i-1)}) + d(z_n^{(i-1)}, \alpha_n^i), \quad \forall\, \beta_n^i \in T_i x_n$. This implies $dist(x_n, T_i x_n) \le d(x_n, z_n^{(i-1)}) + d(z_n^{(i-1)}, \beta_n^i), \quad \forall\, \beta_n^i \in T_i x_n$. Now taking infimum over $\beta_n^i \in T_i x_n$ and using the fact that T_i is L_i-Lipschitzian for each $i \in \mathbb{N}$, we have the following

$$\begin{aligned}
dist(x_n, T_i x_n) &\le d(x_n, z_n^{(i-1)}) + dist(z_n^{(i-1)}, T_i x_n) \\
&\le d(x_n, z_n^{(i-1)}) + d_H(T_i y_n^{(i-1)}, T_i x_n) \\
&\le d(x_n, z_n^{(i-1)}) + L_i d(y_n^{(i-1)}, x_n) \\
&\le d(x_n, z_n^{(i-1)}) + L_i [d(y_n^{(i-1)}, z_n^{i-1}) + d(z_n^{i-1}, x_n)] \\
&\le (1 + L_i) d(x_n, z_n^{(i-1)}) + L_i d(y_n^{(i-1)}, z_n^{i-1}).
\end{aligned}$$

Therefore, by (4.2.3) and (4.2.2) we have $\lim_n dist(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$. ∎

Corollary 4.2.3 *Let X, D, $\{T_i\}$ and $\{x_n\}$ be as in Theorem 4.2.2. Suppose T_i is $\triangle$-demiclosed at 0 for each $i \in \mathbb{N}$. Then $\{x_n\}$ $\triangle$-converges to a common fixed point of $\{T_i\}$.*

Proof Using Lemma 4.2.1 in place of Lemma 4.1.1 and Theorem 4.2.2 in place of 4.1.2, the proof follows similar arguments as in the proof of Corollary 4.1.3. ∎

Corollary 4.2.4 *Let X, D, $\{T_i\}$ and $\{x_n\}$ be as in Theorem 4.2.2. Suppose D is compact. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.*

Proof Using Lemma 4.2.1 in place of Lemma 4.1.1 and Theorem 4.2.2 in place of 4.1.2, the proof follows similar arguments as in the proof of Corollary 4.1.4. ■

Theorem 4.2.5 *Let X, D, $\{T_i\}$, $\mathscr{F}$ and $\{x_n\}$ be as in Lemma 4.2.1. Then $\{x_n\}$ converges strongly to a point $p \in \mathscr{F}$ if and only if $\liminf_n dist(x_n, \mathscr{F}) = 0$.*

Proof Using Lemma 4.2.1 in place of Lemmas 4.1.1, the proof follows similar arguments as in the proof of Theorem 4.1.5. ■

Remark 4.2.6 *Our theorems improve and extends many results in the following sense:*

- *we extend some results (see e.g., [Chidume and Ezeora, 2014]) to a more general space than Hilbert spaces (Hadamard spaces).*
- *We proposed new iterative schemes with different methods of proofs for delta and strong convergences.*
- *Our results hold for all complete CAT($k < 0$) spaces, since $CAT(k) \subset CAT(k')$ for $k \leq k'$.*
- *While [Chidume et al., 2014] proved $\triangle$ convergence for a finite family of demicontractive maps, we proved (using different iterative scheme) delta and strong convergence for countable family of demicontractive maps.*

Bibliography

[Abbas et al., 2011] Abbas, M., Khan, S. H., Khan, A. R., and Agarwal, R. P. (2011). Common fixed points of two multivalued nonexpansive mappings by one-step iterative scheme. *Applied Mathematics Letters*, 24(2):97–102.

[Banach, 1922] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. math*, 3(1):133–181.

[Bridson and Haefliger, 1999] Bridson, M. R. and Haefliger, A. (1999). *Metric Spaces of Non-Positive Curvature*. Springer, Berlin, Germany.

[Brouwer, 1912] Brouwer, L. E. J. (1912). Über abbildung von mannigfaltigkeiten. *V Mathematische Annalen*, 71(4):598.

[Browder and Petryshyn, 1967] Browder, F. E. and Petryshyn, W. V. (1967). Construction of fixed points of nonlinear mappings in hilbert space. *Journal of Mathematical Analysis and Applications*, 20:197–228.

[Bruhat and Tits, 1972] Bruhat, F. and Tits, J. (1972). Groupes reductifs sur un corps local. i. donnees radicielles valuees inst hautes etudes sci. *Publ Math*, 41.

[Chidume, 2009] Chidume, C. (2009). *Geometric Properties of Banach Spaces and Nonlinear Iterations*, volume 1965. Springer-Verlag London Limited.

[Chidume et al., 2014] Chidume, C. E., Bello, A. U., and Ndambomve, P. (2014). Strong and $\triangle$-convergence theorems for common fixed points of a finite family of multivalued demicontractive mappings in cat(0) spaces. *Abstract and Applied Analysis*, 2014.

[Chidume et al., 2013] Chidume, C. E., Chidume, C. O., Djitte, N., and Minjibir, M. S. (2013). Convergence theorems for fixed points of multivalued strictly pseudocontractive mappings in hilbert spaces. *Abstract and Applied Analysis*, 2013.

[Chidume and Ezeora, 2014] Chidume, C. E. and Ezeora, J. N. (2014). Krasnoselskii-type algorithm for family of multi-valued strictly pseudo-contractive mappings. *Fixed Point Theory and Applications*, 2014(1):111.

[Dhompongsa et al., 2009] Dhompongsa, S., Fupinwong, W., and Kaewkhao, A. (2009). Common fixed points of a nonexpansive semigroup and a convergence theorem for mann iterations in geodesic metric spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 70(12):4268–4273.

[Dhompongsa et al., 2005] Dhompongsa, S., Kaewkhao, A., and Panyanak, B. (2005). Lim's theorems for multivalued mappings in cat (0) spaces. *Journal of Mathematical Analysis and Applications*, 312(2):478–487.

[Dhompongsa et al., 2012] Dhompongsa, S., Kaewkhao, A., and Panyanak, B. (2012). On kirk's strong convergence theorem for multivalued nonexpansive mappings on cat (0) spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 75(2):459–468.

[Dhompongsa et al., 2007] Dhompongsa, S., Kirk, W. A., and Panyanak, B. (2007). Non-expansive set-valued mappings in metric and banach spaces. *Journal of Nonlinear and Convex Analysis*, 8(1):35.

[Dhompongsa et al., 2006] Dhompongsa, S., Kirk, W. A., and Sims, B. (2006). Fixed points of uniformly lipschitzian mappings. *Nonlinear Analysis: Theory, Methods & Applications*, 65(4):762–772.

[Dhompongsa and Panyanak, 2008] Dhompongsa, S. and Panyanak, B. (2008). On $\triangle$-convergence theorems in cat(0) spaces. *Computers & Mathematics with Applications*, 56:2572—2579.

[Downing and Kirk, 1977] Downing, D. and Kirk, W. A. (1977). Fixed point theorems for set-valued mappings in metric and banach spaces. *Mathematica Japonica*, 22(1):99–112.

[Filippov, 1964] Filippov, A. F. (1964). Diffrential equations with discontinuous right hand side. *Transactions of the American Mathematical Society*, 42:199–232.

[Hicks and Kubicek, 1977] Hicks, T. L. and Kubicek, J. D. (1977). On the mann iteration process in a hilbert space. *Journal of Mathematical Analysis and Applications*, 59(3):498–504.

[Igbokwe and Udofia, 2013] Igbokwe, D. I. and Udofia, U. E. (2013). Approximation of common fixed points of a finite family of ø-demicontractive mappings by an implicit iteration method. *Global Journal of Mathematical Sciences*, 12(1):31–38.

[Ishikawa, 1974] Ishikawa, S. (1974). Fixed points by a new iteration method. *Proceedings of the American Mathematical Society*, 44(1):147–150.

[Isiogugu and Osilike, 2014] Isiogugu, F. O. and Osilike, M. O. (2014). Convergence theorems for new classes of multivalued hemicontractive-type mappings. *Fixed Point Theory and Applications*, 2014(1):93.

[Kakutani, 1941] Kakutani, S. (1941). A generalization of brouwer's fixed point theorem. *Duke mathematical journal*, 8(3):457–459.

[Khan and Abbas, 2011] Khan, S. H. and Abbas, M. (2011). Strong and $\triangle$- convergence of some iterative schemes in cat (0) spaces. *Computers & Mathematics with Applications*, 61(1):109–116.

[Kirk and Shahzad, 2014] Kirk, W. and Shahzad, N. (2014). *Fixed point theory in distance spaces*. Springer.

[Kirk, 2003] Kirk, W. A. (2003). Geodesic geometry and fixed point theory. *Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003)*, 64:195–225.

[Kirk, 2004] Kirk, W. A. (2004). Geodesic geometry and fixed point theory ii. *Fixed Point Theory and Applications*.

[Kirk and Panyanak, 2008] Kirk, W. A. and Panyanak, B. (2008). A concept of convergence in geodesic spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 68(12):3689–3696.

[Krasnoselskii, 1955] Krasnoselskii, M. A. (1955). Two remarks about the method of successive approximations, mat. *Nauk*, 10:123–127.

[Lim, 1976] Lim, T. C. (1976). Remarks on some fixed point theorems. *Proceedings of the American Mathematical Society*, 60(1):179–182.

[Mann, 1953] Mann, W. R. (1953). Mean value methods in iteration. *Proceedings of the American Mathematical Society*, 4(3):506–510.

[Markin, 1973] Markin, J. T. (1973). Continuous dependence of fixed point sets. *Proceedings of the American Mathematical Society*, 38(3):545–547.

[Nadler, 1969] Nadler, S. B. (1969). Multi-valued contraction mappings. *Pacific Journal of Mathematics*, 30(2):475–488.

[Nash, 1951] Nash, J. F. (1951). Non-cooperative games. *Annals of mathematics. Second Series*, 54:286–295.

[Panyanak, 2007] Panyanak, B. (2007). Mann and ishikawa iterative processes for multivalued mappings in banach spaces. *Computers & Mathematics with Applications*, 54(6):872–877.

[Picard, 1890] Picard, E. (1890). Mémoire sur la théorie des équations aux dérivées partielles et la méthode des approximations successives. *Journal de Mathématiques pures et appliquées*, 6:145–210.

[Rhoades, 1976] Rhoades, B. E. (1976). Comments on two fixed point iteration methods. *Journal of Mathematical Analysis and Applications*, 56(3):741–750.

[Sastry and Babu, 2005] Sastry, K. P. R. and Babu, G. V. R. (2005). Convergence of ishikawa iterates for a multi-valued mapping with a fixed point. *Czechoslovak Mathematical Journal*, 55(4):817–826.

[Schaefer, 1957] Schaefer, H. (1957). Über die methode sukzessiver approximationen. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 59:131–140.