OPERATOR THEORY AND ANALYTIC FUNCTIONS

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Certification/Approval

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Abstract

The theory of analytic functions plays a central role in operator theory. It has been a source of methods, examples and problems, and has led to numerous important results. *Weighted shifts* (which we shall see in the sequel) have been studied with analytic function theory approach.

In this thesis, inspired by the work A. L. Shields, we give excellent exposition of an interplay between weighted shift operators and analytic functions. Essential ingredients of the considerations therein were viewing a weighted shift operator as "multiplication by z" on a Hilbert space consisting of **formal power/Laurent series** and showing that the structure of this space is in fact analytic. This enabled using *multiplication operators and bounded point evaluations*, tools known to be very powerful in variety of problems in operator theory.

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CHAPTER 1

Preliminaries

In this chapter, we will give definition of some terms and results of interest used in the thesis.

1.1 Definition of terms

Along this text, we denote by $\mathcal{H}$ a Hilbert space and by $\mathcal{L}(\mathcal{H})$, the algebra of all bounded linear operators defined in $\mathcal{H}$. For an operator $T \in \mathcal{L}(\mathcal{H})$, we denote by $R(T)$, $N(T)$, $D(T)$ and T^* the range of T, the kernel of T, the domain of T and the adjoint operator of T respectively. Let $M \neq \emptyset$ be a subset of $\mathcal{H}$. we denote the orthogonal complement of M by

$$M^{\perp} = \{y \in \mathcal{H} \text{ such that } \langle x, y\rangle = 0 \text{ for every } x \in M\}.$$

Given a subset A in $\mathcal{H}$, we set $span(A)$ for the smallest vector subspace of H containing A and $vect - T(A)$ for the span of the set $\{T^n x, n \geq 0 \text{ and } x \in A\}$.

Definition 1.1.1 (Normed algebra, Banach algebra) *A normed algebra $\mathcal{N}$ is a normed space which is an algebra such that for all $f, g \in \mathcal{N}$,*

$$\|fg\| \leq \|f\|\|g\|$$

and if $\mathcal{N}$ has an identity e,

$$\|e\| = 1.$$

*$\mathcal{N}$ is **commutative** if the multiplication is commutative, that is, if for all $f, g \in \mathcal{N}$, $fg = gf$. A Banach algebra is a normed algebra which is complete, considered as a normed space.*

Example 1.1.2 *The Banach space of all bounded linear operators on a complex Banach space $X \neq 0$ is a Banach algebra with identity I and multiplication defined by composition of operators.*

Definition 1.1.3 (Spectrum, Resolvent, Spectral radius of an operator) *Let $A \in \mathcal{H}$. The spectrum, denoted by $\Lambda(A)$ is a nonempty and compact subset of $\mathbb{C}$ defined by*

$$\Lambda(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is bijective from } \mathcal{H} \text{ to } \mathcal{H}\}$$

and the resolvent set denoted by $\rho(A)$, is the complement of the spectrum, i.e.,

$$\rho(A) = \mathbb{C} \backslash \Lambda(A)$$

The spectral radius of A is defined by

$$r(A) = \sup\{|\lambda| : \lambda \in \Lambda(A)\}.$$

If $A \in \mathcal{B}$ is invertible then

$$\Lambda(A^{-1}) = \left\{ \frac{1}{\lambda} : \lambda \in \Lambda(A) \right\} = [\Lambda(A)]^{-1}$$

and so

$$(r(A^{-1}))^{-1} = \inf\{|\lambda| : \lambda \in \Lambda(A)\}$$

In this case, $\Lambda(A)$ lies in the annulus $[r(A^{-1})]^{-1} \leq |z| \leq r(A)$.

We now define the different parts the of spectrum of an operator. We proceed with the following definition and give some characterizations.

Definition 1.1.4 (Bounded Below, Lower Bound) *Let A be an operator on the Hilbert space $\mathcal{H}$, then A is bounded below if there exists $\varepsilon > 0$ such that*

$$\|Af\| \geq \varepsilon \|f\| \text{ for all } f \in \mathcal{H}.$$

The lower bound of A denoted by $m(A)$ is defined

$$m(A) = \inf\{\|Af\| : \|f\| = 1\}.$$

Remark 1.1.5 *(i) W $\|Af\| \geq m(A)\|f\|$, for all $f \in \mathcal{H}$*

(ii) It is easy to see that if $|\lambda| < m(A)$, then $A - \lambda$ is bounded below.

(iii) If A is bounded below, then $m(A) > 0$.

(iv) If A is invertible and A^{-1} is bounded, then $m(A) = \frac{1}{\|A^{-1}\|}$. Indeed,

$$\begin{aligned}
\|A^{-1}\| &= \sup\left\{\|A^{-1}g\| : g \in R(A) \text{ and } \|g\| = 1\right\} \\
&= \sup\left\{\frac{\|A^{-1}g\|}{\|g\|} : x \in R(A) \setminus \{0\}\right\} \\
&= \sup\left\{\frac{\|A^{-1}Af\|}{\|Af\|} : f \in D(A) \setminus \{0\}\right\} \\
&= \sup\left\{\frac{\|f\|}{\|Af\|} : f \in D(A) \setminus \{0\}\right\} \\
&= \sup\left\{\frac{1}{\|Af\|} : f \in D(A) \text{ and } \|f\| = 1\right\} \\
&= \frac{1}{\inf\{\|Af\| : f \in D(A) \text{ and } \|f\| = 1\}} = \frac{1}{m(A)}.
\end{aligned}$$

(v) If A and B are two bounded below operators on $\mathcal{H}$, then

$$m(AB) \geq m(A)m(B).$$

Proposition 1.1.6 *Let A be an operator on $\mathcal{H}$. Then the sequence $\{(m(A^n))^{\frac{1}{n}}\}_n$ converges with limit $\sup_n (m(A^n))^{\frac{1}{n}}$.*

Proof: Fix $k \in \mathbb{N}$. For all n, $n = kq_n + r_n := kq + r$ where $0 \leq r < k$. By Remark (1.1.5),

$$m(A^n) \geq m(A^{kq})m(A^r) \geq m(A^k)^q m(A^r) = m(A^k)^{\frac{n-r}{k}} m(A^r).$$

Taking the $n-$th root of both sides and then lim inf of both sides as $n \to \infty$, we obtain

$$\liminf_{n\to\infty} m(A^n)^{\frac{1}{n}} \geq m(A^k)^{\frac{1}{k}}.$$

Since k is abitrary, this holds for all k. Thus $m(A^k)^{\frac{1}{k}}$ is bounded above and has a supremum and hence

$$\liminf_{n\to\infty} m(A^n)^{\frac{1}{n}} \geq \sup_k m(A^k)^{\frac{1}{k}}. \tag{1.1.1}$$

Again, from Remark (1.1.5), for all n

$$m(A^{2n})^{\frac{1}{2n}} \geq m(A^n)^{\frac{1}{n}}.$$

Taking supremum of both sides over n and then lim sup of both sides over n, we obtain

$$\sup_n m(A^n)^{\frac{1}{n}} \geq \limsup_n m(A^n)^{\frac{1}{n}}. \tag{1.1.2}$$

Combining (1.1.1) and (1.1.2), we obtain our result. ■

Using similar approach, by noting that $r(AB) \leq r(A)r(B)$, one can show that

$$r_{\mathcal{B}}(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$$

We denote $\lim_{n\to\infty}(m(A^n))^{\frac{1}{n}}$ by $r_1(A)$.

Remark 1.1.7 *Following Remark (1.1.5), if A is invertible, then $r(A^{-1}) = \frac{1}{r_1(A)}$.*

Lemma 1.1.8 *Let A be an operator on $\mathcal{H}$ then A is invertible if and only if A is bounded below and has a dense range.*

We may now classifying the parts of the spectrum of an operator using Lemma (1.1.8). Consequently, if $\lambda \in \Lambda(A)$, let $\sqcap(A)$ denote the set of complex numbers λ such that $A - \lambda I$ is not bounded from below, and $\Gamma(A)$ the set of complex numbers λ such that $Cl(\mathcal{R}(A - \lambda I)) \subsetneq \mathcal{H}$, then

$$\Lambda(A) = \sqcap(A) \cup \Gamma(A).$$

The set $\sqcap(A)$ is called the **approximate point spectrum** of A. An important subset of the approximate point spectrum is the **point spectrum/eigenvalues** $\sqcap_0(A)$; a number λ belongs to $\sqcap_0(A)$ if and only if there exists a nonzero vector f such that $Af = \lambda f$. Such a vector$f$ is then called an eigen vector associated with the eigen values λ.

The set $\Gamma(A)$ is called the **compression spectrum** of A.

Proposition 1.1.9 *The following are equivalent:*

(i) $\lambda \in \sqcap(A)$.

(ii) There exists a sequence $\{f_n\}_n$ *of unit vectors such that* $\|(A-\lambda)f_n\| \to 0$.

(iii) $m(A - \lambda I) = 0$

Proof: $(i) \implies (ii)$. $\lambda \in \sqcap(A)$ means $A - \lambda I$ is not bounded below. Thus for all $n \geq 0$, there exists $\{g_n\}_n$ such that

$$\|(A-\lambda)g_n\| < \frac{1}{n}\|g_n\|.$$

Dividing through by $\|g_n\|$ and letting $n \to \infty$ we obtain (ii)
$(ii) \implies (iii)$. Consider the set

$$D := \{\alpha : \|(A-\lambda I)f\| \geq \alpha\|f\| \text{ for all } f\} \subset \mathbb{R}.$$

Clearly 0 is a lower bound of D. Combining this result with (ii), we obtain (iii).
$(iii) \implies (i)$. We proceed by contrapositive. Suppose $\lambda \notin \sqcap(A)$, then $A - \lambda I$ is bounded below. Thus $\exists c > 0$ such that

$$\|(A-\lambda I)f\| \geq c\|f\| \quad \text{for all } f.$$

Taking infimum over all f with unit norm, we obtain $m(A - \lambda I) \geq c > 0$. ■

Definition 1.1.10 (Weighted Shift Operator) *An operator T on the (complex) separable Hilbert space $\mathcal{H}$ is said to be a weighted shift operator if there is some orthogonal basis $\{e_n\}_n$ and weight sequence $\{w_n\}_n$ such that*

$$Te_n = w_n e_{n+1}, \text{ for all } n.$$

T is unilateral if n runs over $\mathbb{N}$ and bilateral if n runs over $\mathbb{Z}$.

Remark 1.1.11 *(i) We require $\mathcal{H}$ to be separable to guarantee the existence of a countable orthonormal basis e_n for $\mathcal{H}$.*

(ii) It is easy to see that $\|T\| = \sup_n |w_n|$. In particular, T is bounded if and only if $\{w_n\}_n$ is bounded.

(iii) T is injective if and only if none of the weights is zero. Indeed, $x \in ker(T) = 0$ if and only if

$$Tx = T(\sum \alpha_n e_n) = \sum \alpha_n T(e_n) = \sum \alpha_n w_n e_{n+1} = 0,$$

if and only if

$$\alpha_n w_n = 0, \text{ for all } n$$

It follows that

$$N(T) = span\{e_n \text{ such that } w_n = 0\}$$

An operator A on an n-dimensional Hilbert space $n < \infty$ is called **finite-dimensional weighted shift** if there are numbers $\{\beta_1, \beta_2, ..., \beta_{n-1}\}$ and orthonormal basis $\{e_1, e_2, ..., e_n\}$ such that

$$Ae_i = \beta_i e_{i+1} \quad (i < n)$$
$$Ae_n = 0.$$

Such an operator is nilpotent with degree n if none of the $\beta_i' s$ is zero. If T is unilateral and finitely many weights are zero then T is a direct sum of a finite-dimensional weighted shifts and an infinite-dimensional injective weighted shift. If infinitely many weights are zero then T is the direct sum of an infinite family of finite-dimensional weighted shifts.

If T is a bilateral shift with one zero weight (say w_0), we ill see below that T is is a direct sum of a unilateral shift (on the space spanned by $\{e_k\}_k$) and the adjoint of a unilateral weighted shift (on the orthogonal complement). If additional weights are zero, there is a further direct sum decomposition as the unilateral case.

Definition 1.1.12 (Diagonal Operator) *Suppose that $\mathcal{H}$ is a separable Hilbert space, and that $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$, an operator D is called **diagonal** if $De_n = a_n e_n$ for some family of scalar numbers a_n. The family $\{\alpha_n\}$ may be called the diagonal of D.*

Remark 1.1.13 *A weighted shift operator can be seen as the product of a standard shift (one-sided or two, with $w_n = 1$) and a compatible diagonal operator. That is, $W = SD$ where S is a shift ($Se_n = e_{n+1}$).*

Definition 1.1.14 (Unitary operator) *An operator U on $\mathcal{H}$ is said to be unitary if U is bijective and $U^* = U^{-1}$.*

It is easy to see that a diagonal operator D is a unitary operator on $\mathcal{H}$ if and only if $|\alpha_n| = 1$. That is,

$$D^* e_n = D^{-1} e_n = \bar{a}_n.$$

Definition 1.1.15 (Unitary equivalence) *Let A and B be operators on $\mathcal{H}$. A and B are said to be **unitarily equivalent** if there exists a unitary operator U on $\mathcal{H}$ such that*

$$A = U^* BU \text{ and } B = UAU^*$$

Thus we say the operators A and B are "abstractly identical". That is, there is never any loss of generality in restricting attention to either of the operators in place of the other.

Remark 1.1.16 *Two unitarily equivalent operators A and B on $\mathcal{H}$ have the same spectrum. Indeed, for every $\lambda \in \mathbb{C}$*

$$A - \lambda I = U^* BU - \lambda I = U^*(B - \lambda I)U$$

So $B - \lambda I$ is invertible iff $A - \lambda I$ is invertible, which leaves $\Lambda(S) = \Lambda(T)$.

Definition 1.1.17 (Commutant) *Let $\mathcal{U} \subset \mathcal{L}(\mathcal{H})$ then the commutant of $\mathcal{U}$ denoted by $\mathcal{U}'$ is the set of all operators in $\mathcal{L}(\mathcal{H})$ which commutes with every operator in $\mathcal{U}$.*

1.2 Elementary properties of Weighted Shift Operators

Proposition 1.2.1 *Suppose that $\{w_i\}_i$ is bounded, then $\|T^n\|$ exists and*

$$\|T^n\| = \sup_i |w_i w_{i+1} ... w_{i+n-1}|, n = 1, 2, ...$$

Proof: From definition of T we get

$$T^n e_i = (w_i w_{i+1} ... w_{i+n-1}) e_{i+n} \text{ for all } i \tag{1.2.1}$$

and thus the result follows. ■

Proposition 1.2.2 *Let T be a bilateral shift then*

$$m(T^n) = \inf_i |w_i ... w_{i+n-1}| \quad n = 1, 2, ...$$

Proof: This also follows from Equation (1.2.1). ■

Proposition 1.2.3 *If T is a bilateral weighted shift then*

$$T^* e_n = \overline{w}_{n-1} e_{n-1} \textit{ for all } n$$

If T is a unilateral weighted shift then

$$\begin{aligned} &T^* e_n = \overline{w}_{n-1} e_{n-1} \textit{ for all } n \geq 1 \\ &T^* e_0 = 0 \end{aligned}$$

Proof: For all n and m,

$$\begin{aligned} \langle T^* e_n, e_m \rangle &= \langle e_n, T e_m \rangle \\ &= \langle e_n, w_m e_{m+1} \rangle \\ &= \overline{w}_m \langle e_n, e_{m+1} \rangle \end{aligned}$$

and the result follows. ■

Now, we consider weighted shift operators S and T on $\mathcal{H}$ with respect to the same orthonormal basis $\{e_n\}$ and weight sequences $\{v_n\}$ and $\{w_n\}$ respectively. Let A be an operator on $\mathcal{H}$, we define the matrix associated to A by $[a_{i,j}] = [\langle Ae_j, e_i \rangle]$. We give the following proposition:

Proposition 1.2.4 *(a) If S and T are unilateral shifts, then $AS = TA$ if and only if*

$$\begin{aligned} v_j a_{i+1,j+1} &= w_i a_{i,j} \textit{ for all } i, j \geq 0 \\ v_j a_{0,j+1} &= 0 \textit{ for all } j \geq 0 \end{aligned}$$

(b) If S and T are bilateral shifts, then $AS = TA$ if and only if

$$v_j a_{i+1,j+1} = w_i a_{i,j} \textit{ for all } i, j \geq 0$$

Proof: We prove (a). For all i, j,

$$\langle ASe_j, e_{i+1}\rangle = v_j\langle Ae_{j+1}, e_{i+1}\rangle = v_j a_{i+1,j+1}$$

$$\langle TAe_j, e_{i+1}\rangle = \langle Ae_j, T^*e_{i+1}\rangle = \begin{cases} w_i a_{i,j}, & \text{if } i \geq 0 \\ 0, & \text{if } i+1 = 0 \end{cases}$$

and the result follows. ∎

The following theorem gives a characterization for two injective weighted shifts S and T to be similar (the existence of invertible operator A such that $AS = TA$).

Theorem 1.2.5 *(a) Let S, T be injective bilateral weighted shifts with weight sequences $\{v_n\}$, $\{w_n\}$ Then S and T are similar if and only if there exists an integer k and positive constants C_1 and C_2 such that*

$$0 < C_1 \leq |\frac{w_{k+m}...w_{k+n}}{v_m...v_n}| \leq C_2 \text{ for all } m \leq n.$$

(b) Let S, T be injective unilateral weighted shifts with weight sequences $\{v_n\}$, $\{w_n\}$ Then S and T are similar if and only if positive constants C_1 and C_2 such that

$$0 < C_1 \leq |\frac{w_0...w_n}{v_0...v_n}| \leq C_2 \text{ for all } n.$$

In both of these cases, the operator A that implements the similarity can be chosen so that

$$\max(\|A\|, \|A^{-1}\|) < \max(C_2, \frac{1}{C_1}).$$

Recall that T is a contraction if $\|T\| \leq 1$ and is power bounded if there exists a $c > 0$ such that $\|T^n\| \leq c,\ n = 1, 2, ...$ It is clear that every operator that is similar to a contraction is power bounded. The converse has motivated several authors in the last century and has been negatively resolved after numerous investigation. From the previous theorem, we have the next positive answer for weighted shifts. The proof can be seen in page 55, [Shields, 1974].

Corollary 1.2.6 *If T is a weighted shift operator and power bounded, then T is similar to a contraction.*

Proposition 1.2.7 *(a) If S, T are two unilateral weighted shifts with weight sequences $\{v_n\}$, $\{w_n\}$, and if*

$$|v_n| = |w_n|, \text{ for all}$$

then S and T are unitarily equivalent. The converse is true if S and T are injective

(b) If S, T are two bilateral weighted shifts with weight sequences $\{v_n\}$, $\{w_n\}$, and if there exists an integer k such that

$$|v_n| = |w_{n+k}|, \forall n$$

then S and T are unitarily equivalent. The converse is true if S and T are injective.

Proof: We prove (a). We take our required unitary operator to be the diagonal operator $D = \{\lambda_n\}_n$ which we shall find constructively.

Now, assume that $TD = DS$. Applying e_n to both sides we have,

$$w_n\lambda_n = v_n\lambda_{n+1}, \text{ for all } n.$$

Set $\lambda_0 = 1$. If $v_n \neq 0$, put $\lambda_{n+1} = (w_n/v_n)\lambda_n$. If $v_n = 0$, then $w_n = 0$ since $|v_n| = |w_n|$. Thus, put $\lambda_{n+1} = 1$. The result is a sequence λ of complex numbers of modulus 1. The steps leading to this sequence are reversible. Given the sequence $\{\lambda_n\}$, let it induce a diagonal operator D; we note that since $|\lambda_n| = 1$ for all n, the operator D is unitary; and, finally, note that since $TDe_n = DSe_n$ for all n, the operator D transforms S onto T. ∎

Corollary 1.2.8 *T is unitarily equivalent to the weighted shift operator with weight sequence $\{|w_n|\}$.*

Corollary 1.2.9 *If $|c| = 1$, then T and cT are unitarily equivalent.*

From now, on we shall assume that T has no zero weight that is T is injective. Due to the previous corollary, it suffices to say T is a weighted shift operator with real weight sequence $\{w_n\}$, $w_n > 0$ for all n.

Remark 1.2.10 *With the restriction that $w_n > 0$ for all n, the unilateral shift operator is never invertible but the bilateral shift can be. To see this, we use the following.*

Lemma 1.2.11 *For any subset $M \neq \emptyset$ of a Hilbert space $\mathcal{H}$,*

1. *$M^\perp$ is a closed invariant subspace*
2. *$(M^\perp)^\perp = \overline{span(M)}$*
3. *the span of M is dense in $\mathcal{H}$ if and only if $M^\perp = \{0\}$.*

Now, let T be a unilateral weighted shift. We observe that $e_0 \in M^\perp(R(T^*))$. Thus, $R(T^*))$ is a proper subset of $\mathcal{H}$ and hence T^* is not inverible (infact not surjective). This implies that T is cannot be invertible.

Example 1.2.12 *The bilateral weighted shift with weight sequence $\left\{\frac{|n|+2}{|n|+1}\right\}_{n\in\mathbb{Z}}$ is invertible with inverse having weight sequence $\left\{\frac{|n|+1}{|n|+2}\right\}_{n\in\mathbb{Z}}$.*

We shall see that a bilateral weighted shift is invertible if and only if $\left\{\frac{1}{w_n}\right\}_{n\in\mathbb{Z}}$ is bounded.

1.3 Weighted sequence spaces

Definition 1.3.1 *Let $\{f_n\}_n$ be an orthogonal basis for $\mathcal{H}$, an operator T is said to shift this orthogonal basis if*

$$Tf_n = f_{n+1}.$$

We now give the following useful characterization which allows us represent weighted shift operators as ordinary shift.

Proposition 1.3.2 *T is a weighted sift operator if and only if it shifts some orthogonal basis $\{f_n\}$.*

Proof: Given T, unilateral with weight sequence $\{w_n\}$ and orthonormal basis $\{e_n\}$, define

$$f_n = w_0 ... w_{n-1} e_n, \text{ for n} \geq 1$$
$$f_0 = e_0$$

For the bilateral case for $n < 0$, define

$$f_n = \frac{e_n}{w_n w_{n+1} ... w_{-1}}$$

then T shifts $\{f_n\}$.

If T shifts some orthogonal basis $\{f_n\}$, define for all n

$$e_n = \frac{f_n}{\|f_n\|}, \qquad w_n = \frac{\|f_{n+1}\|}{\|f_n\|}$$

then T is weighted shift operator. ■

Let f be an analytic function about zero. We denote $f = \{\hat{f}_n\}$. Let $\{\beta_n\}_n$ be a sequence of positive numbers such that $\beta_0 = 1$. We define the following Hilbert spaces of formal power series and formal Laurent series by

$$H^2(\beta) = \left\{ \{\hat{f}_n\}_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}} |\hat{f}_n|^2 \beta_n^2 < \infty \right\}$$

and

$$L^2(\beta) = \left\{ \{\hat{f}_n\}_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2 \beta_n^2 < \infty \right\} \text{ respectively,}$$

endowed with the inner product

$$\langle f, g \rangle = \sum \hat{f}_n \overline{\hat{g}}_n \beta_n^2$$

and norm

$$\|f\| = \sqrt{\sum |\hat{f}_n|^2 \beta_n^2}.$$

Whenever we do not wish to distinguish either of the spaces, we denote H.
This heuristic expression of members of H as formal power (Laurent) series suggests a multiplicative and analytic structure which have significant relationship to the weighted shift operator T.

CHAPTER 2

Multiplication Operator and The Commutant

2.1 Multiplication Operator

A weighted shift operator could be viewed as "multiplication by z" on a Hilbert space of formal power series or formal Laurent series. This point of view was taken by R. Gellar in [Gellar, 1968], [Gellar, 1969b] and [Gellar, 1969a] where he considered much more general spaces. Further exposition was done by N.K. Nikol'skii [Nikol'skii, 1968], and S.Grabiner [Gellar and Herrero, 1974].

We define the linear multiplication operator M_z on H as follows

$$M_z f(z) = z \times \sum \hat{f}_n z^n = \sum \hat{f}_n z^{n+1}.$$

For the formal Laurent series case, $(\hat{M_z f})_n = \hat{f}_{n-1}$ for all n, while for the formal power series case, $(\hat{M_z f})_n = \hat{f}_{n-1}$ for $n \geq 1$, $(\hat{M_z f})_0 = 0$.
It is easy to see that $\{f_n(z) = z^n\}_n$ is an orthonormal basis for H with $\|f_n\| = \beta_n$. Furthermore, M_z shifts the orthogonal basis $\{z^n\}_n$; for each n, $M_z f_n = z^{n+1} = f_{n+1}$

Remark 2.1.1 *M_z need not be bounded. Consider $\beta_n = \sqrt{n!}$, then $\|M_z f_n\| = (n+1)\|f_n\|$*

Following Proposition 1.3.2, we have that M_z is unitarily equivalent to an injective weighted shift operator with weight sequence.

$$w_n = \frac{\|f_{n+1}\|}{\|f_n\|} = \frac{\beta_{n+1}}{\beta_n} \text{ for all } n \tag{2.1.1}$$

Conversely, every weighted shift operator is unitarily equivalent to M_z acting on H with β_n given by

$$\beta_n = \begin{cases} w_0 \ldots w_{n-1} & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ \frac{1}{w_{-1} \ldots w_{-n}} & \text{if } n < 0 \end{cases}$$

Proposition 2.1.2 *M_z is bounded if and only if $\frac{\beta_{j+1}}{\beta_j}$ is bounded and*

$$\|M_z^n\| = \sup_j \left[\frac{\beta_{j+n}}{\beta_j}\right], \quad n = 0, 1, 2...$$

Proof: Since M_z is unitarily equivalent to T with weight sequence $\{w_n\}_n$ as given above, this follows from Proposition 1.2.1. ■

Proposition 2.1.3 *Let T be a bilateral weighted shift (represented as M_z on $L^2(\beta)$). T is invertible if and only if $\{\frac{1}{w_n} = \frac{\beta_n}{\beta_{n+1}}\}_n$ is bounded. Furthermore, T^{-1} is a bilateral weighted shift. In this case,*

$$\|T^{-n}\| = \sup_j \frac{\beta_j}{\beta_{n+j}} = \left[\inf_j \frac{\beta_{n+j}}{\beta_j}\right]^{-1}, \quad n = 0, 1, 2, ... \tag{2.1.2}$$

Proof: Suppose $\left\{\frac{1}{w_n}\right\}_n$ is bounded. Then the operator S defined by $Se_n = \frac{1}{w_{n-1}}e_{n-1}$ for all $n \in \mathbb{Z}$ is well defined. We see that $TS = ST = I$. This implies T^{-1} exists and equals S. Moreover, S is a bilateral shift operator: $Sv_n = s_n v_{n+1}$, where $v_n = e_{-n}$ and $s_n = \frac{1}{w_{-(n+1)}}$. Thus T^{-1} is a bilateral weighted shift operator. By Proposition 1.2.1, and equation 2.1.1, we obtain equation 2.1.2.

Conversely, suppose that T is invertible. Then M_z is invertible. Since $M_z f_n = f_{n+1}$, we have $M_z^{-1} f_n = f_{n-1}$ for all n. Let $f_n^{'} = f_{-n}$. We see that M_z^{-1} shifts the orthogonal basis $\{f_n^{'}\}$. Thus we may represent T^{-1} as a multiplication by z on $L^2(\beta')$ where $\beta_n' = \|f_n^{'}\| = \|f_{-n}\| = \beta_{-n}$ and thus ■

2.2 Classical examples

Let $\mathbb{D}$ denote the open unit disc

Example 2.2.1 (Unweighted Hardy shift)

$$w_n = 1 \Rightarrow \beta_n = 1 \Rightarrow H_\beta = H^2(\mathbb{D}) = \left\{ f = \{\hat{f}_n\} : \sum_{n \geq 0} |\hat{f}_n|^2 < \infty \right\}.$$

Example 2.2.2 (Weighted Hardy shift)

$$w_n = \left(\frac{n+1}{n}\right)^\alpha \Rightarrow \beta_n = n^\alpha \Rightarrow H_\beta = \left\{ f = \{\hat{f}_n\} : \sum_{n \geq 1} |\hat{f}_n|^2 n^\alpha < \infty \right\}.$$

In the examples above, we define the Hardy shift on M_z on H_β by the multiplication operator

$$M_z f(z) = z f(z).$$

Example 2.2.3 (Bergmann shift, [Giselsson, 2012]) *Let $\mathcal{E}$ be a Hilbert space and let $n \in \mathbb{N}$. We denote by*

$$A_n(\mathcal{E}) = \left\{ f(z) = \sum_{k \geq 0} \hat{f}_k z^k, \ z \in \mathbb{D} : \sum_{k \geq 0} \|\hat{f}_k\|^2 \mu_{n,k} < \infty \ \textit{where} \ \mu_{n,k} = \frac{1}{\binom{k+n-1}{k}} \right\}$$

The Bergman shift operator is the operator S_n on $A_n(\mathcal{E})$ is given by multiplication by the complex coordinate:

$$S_n f(z) = zf(z), z \in \mathbb{D}.$$

Example 2.2.4 (Dirichlet shift, [Richter and Sundberg, 1994]) *The Dirichlet space D is defined as the space of all analytic functions $f = \{f_n\}$ on $\mathbb{D}$ which have a finite Dirichlet integral*

$$D(f) = \iint_D |f'(z)|^2 dA(z).$$

Here $dA(z) = \frac{1}{\pi} rdrdt$ denotes the normalized area measure on $\mathbb{D}$.
The Dirichlet shift $(M_{z,\mathbb{D}})$ is the operator of multiplication by z on D.

2.3 The commutant

The commutant of a weighted shift operator was first described by R. L. Kelley [Kelley, 1966, p. 5]. The more useful description of the commutant (of a bilateral shift) as a space of formal Laurent series occurs in Gellar [Gellar, 1969b].
We adopt the following notations as suggested by Gellar [Gellar, 1974], and A. L. Shields [Shields, 1974].

$$H^\infty(\beta) = \{\phi = \{\hat{\phi}_n\}_{n\in\mathbb{N}} : \phi H^2(\beta) \subset H^2(\beta)\}$$

and

$$L^\infty(\beta) = \{\phi = \{\hat{\phi}_n\}_{n\in\mathbb{Z}} : \phi L^2(\beta) \subset L^2(\beta)\}$$

where ϕH means the multiplication of formal power (resp. Laurent) series ϕf for all $f \in H$.

With $\{f_n(z) = z^n\}_n$ as an orthonormal basis for H, we give the following remarks.

Remark 2.3.1 *(i) Since $\phi f_0 = \phi$ for all formal power (resp. Laurent) series, $L^\infty(\beta) \subset L^2(\beta)$ and $H^\infty(\beta) \subset H^2(\beta)$*

(ii) For all $\phi \in H$ and n, m,

(2.3.1)
$$\langle f_m\phi, f_n\rangle = \hat{\phi}_{n-m}\beta_n^2$$

Let $\phi \in H$, we denote the operator of multiplication by ϕ on H by M_ϕ. We give the following propositions.

Proposition 2.3.2 *M_ϕ is a bounded linear transformation.*

Proof: It is clear that M_ϕ is linear. Let the matrix associated with M_ϕ be A. Then given the orthogonal basis $\{f_n\}_n$ and from equation 2.3.1, for all i, j,

(2.3.2)
$$a_{i,j} = \frac{\langle M_\phi f_j, f_i\rangle}{\|f_i\|^2} = \frac{\hat{\phi}_{i-j}\beta_i^2}{\beta_i^2 =}.$$

For the formal power series case, $a_{i,j} = 0$ for $i < j$. Using the fact that an everywhere defined matrix transformation is bounded [Cohen and Dunford, 1937], we conclude that M_ϕ is bounded. ∎

Proposition 2.3.3 *Both $H^\infty(\beta)$ and $L^\infty(\beta)$ are commutative algebras.*

Proof: Let $\phi, \psi \in L^\infty(\beta)$. Denote the matrix of transformation of M_ϕ and M_ψ by $[a_{i,j}]$ and $[b_{i,j}]$ respectively. The product operator $M_\phi M_\psi$ is thus given by the product of the matrices $[a_{i,j}][b_{i,j}]$ with entries

$$c_{i,j} = \sum_k a_{i,k} b_{k,j}.$$

This sum converges due to the boundedness of M_ϕ and M_ψ and **the result from Cohen and Dunford** (since the product of bounded maps is bounded).
But

$$\sum_k a_{i,k} b_{k,j} = \sum_k \hat{\phi}_{i-k} \hat{\psi}_{k-j} = \sum_l \hat{\phi}_l \hat{\psi}_{i-j-l} = (\hat{\phi\psi})(i-j)$$

which by equation 2.3.2 is precisely the $i,j-$entry of the matrix associated with $M_{\phi\psi}$. Thus

$$M_{\phi\psi} H = M_\phi M_\psi H = M_\phi (M_\psi H) \subset M_\phi H \subset H.$$

This gives $\phi\psi \in H$. Hence H is a commutative algebra.
Following the same process, we also have that the product of the matrices $[b_{i,j}][a_{i,j}]$ with entries $d_{i,j} = (\hat{\phi\psi})(i-j)$. ■

From Proposition 2.3.2, we have that $\|M_\phi\|$ exists. We shall use the following notation for norm of M_ϕ.

$$\|M_\phi\| = \|\phi\|_\infty \tag{2.3.3}$$

We now give a characterization of the commutant of M_z on H

Theorem 2.3.4 *(a) Let A be an operator on $H^2(\beta)$ that commutes with M_z, then $A = M_z$, for some $\phi \in H^2(\beta)$.*

(b) Let A be an operator on $L^2(\beta)$ that commutes with M_z, then $A = M_z$, for some $\phi \in L^2(\beta)$.

Corollary 2.3.5 *Both $H^\infty(\beta)$ and $L^\infty(\beta)$ are complete in the norm given by 2.3.3.*

CHAPTER 3

The Spectrum of Weighted shift operator

Here we investigate the parts of the spectrum of the weighted shift mostly due to R. L. Kelley [Kelley, 1966]. However, the exposition on the approximate point spectrum was done by W. Ridge [Ridge, 1970]. Independently, N. K. Nikolskii [Nikol'skii, 1968] identified the eigenvalues of an injective bilateral shift.

From now on we assume that T is a bounded weighed shift operator. From corollary 1.2.9 and the fact that unitarily equivalent have the same spectrum, we get that the spectrum of a weighted shift operator has circular symmetry about the origin: If λ is in the spectrum of T and if $|c| = 1$ then $c\lambda$ is in the spectrum of cT and hence in the spectrum of T.
We now investigate on the spectrum of T. First off, the unilateral case.

Theorem 3.0.1 *Let T be a unilateral weighted shift. Then the spectrum of T is the disc $|z| \leq r(T)$.*

Proof: It is known that

$$\Lambda(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r(T)\}.$$

To prove the converse, it suffices to show that the resolvent set of T is the punctured disc $|z| > r(T)$.
T can be represented by M_z on a corresponding $H^2(\beta)$. Let $\lambda \in \rho(T)$. Then $(T - I\lambda)^{-1}$ exists.
Now,

$$\begin{aligned} T(T - I\lambda)^{-1} &= ((T - I\lambda)T^{-1})^{-1} \\ &= (TT^{-1} - I\lambda T^{-1})^{-1} \\ &= (T^{-1}T - T^{-1}I\lambda)^{-1} \\ &= (T^{-1}(T - I\lambda))^{-1} \\ &= (T - I\lambda)^{-1}T. \end{aligned}$$

Thus, $(T - I\lambda)^{-1}$ is commutative with T and hence can be represented by an operator M_ϕ on $H^2(\beta)$ for some formal power series $\phi \in H^\infty(\beta)$. That is, $M_\phi = (M_z - I\lambda)^{-1}$.
Applying f_0 to both sides,

$$M_\phi f_0 = (M_z - I\lambda)^{-1} f_0.$$

Thus we obtain,

$$(z-\lambda)\phi(z) = 1$$

From which by computation (comparison of coefficients), we find that $\hat{\phi}_n = (-1)/\lambda^{n+1}$ for all $n \geq 0$. Let $k, m \in \mathbb{N}$. From equation 2.3.1 we have

$$|\hat{\phi}_k \beta_{k+m}^2| = |\langle M_\phi f_m, f_{m+k}\rangle| \leq \|M_\phi\| \beta_m \beta_{k+m}.$$

Hence,

$$\frac{\beta_{k+m}}{\beta_m} \leq \frac{\|M_\phi\|}{|\hat{\phi}_k|} = \|M_\phi\||\lambda^{k+1}|$$

So,

$$\|M_z^k\| = \sup_m \left[\frac{\beta_{k+m}}{\beta_m}\right] \leq \frac{\|M_\phi\|}{|\hat{\phi}_k|} = \|M_\phi\||\lambda^{k+1}| = \|\lambda M_\phi\||\lambda^k|$$

By taking $k-$th root and letting $k \to \infty$ yields $r(T) \leq |\lambda|$ or $|\lambda| \leq r(T)$. In fact, strict inequality must holds since, by circular symmetry, the entire circle $|\lambda| = r(T)$ is in the spectrum. Thus we obtain our result. ■

We now consider the bilateral case.

Theorem 3.0.2 *(a) If T is an invertible bilateral weighted shift, then the spectrum of T is the annulus $[r(T^{-1})]^{-1} \leq |z| \leq r(T)$.*

(b) If T is a non-invertible weighted shift, then the spectrum is the disc $|z| \leq r(A)$.

Proof: Since T is invertible, it is known that

$$\Lambda(T) \subset \{\lambda \in \mathbb{C} : [r(T^{-1})]^{-1} \leq |\lambda| \leq r(T)\}.$$

To prove the converse, we follow in similitude the method used in the previous theorem.
Let $\lambda \in \rho(T)$ then $(M_z - I\lambda)^{-1}$ is represented by M_ϕ for some $\phi \in L^\infty(\beta)$ and so we obtain

$$(z-\lambda)\phi(z) = 1 \text{ that is } (z-\lambda)\sum_n \hat{\phi}_n z^n.$$

From which by comparing of coefficients, we get

$$\begin{cases} \hat{\phi}_{-1} - \lambda\hat{\phi}_0 = 1 \\ \lambda^k \hat{\phi}_k = \hat{\phi}_0 \\ \hat{\phi}_{-k-1} = \lambda^k \hat{\phi}_{-1}, \quad k \geq 0 \end{cases} \tag{3.0.1}$$

When $k = \lambda = 0$ we take $\lambda^k = 1$. By equation 2.3.1, and equation 2.1.1,

$$|\hat{\phi}_k|\beta_{m+k}^2 = \langle M_\phi f_m, f_{m+k}\rangle \leq \|M_\phi\|\|f_m\|\|f_{m+k}\| = \|M_\phi\|\beta_m\beta_{m+k} \text{ for all } k, m$$

That is,

$$|\hat{\phi}_k|\beta_{m+k} \leq \|M_\phi\|\beta_m \text{ for all } k, m \tag{3.0.2}$$

We consider the following cases:

Case 1. $\hat{\phi}_0 \neq 0$**.** Multiplying 3.0.2 by $|\lambda|^k$ and applying 3.0.1 we obtain

$$|\hat{\phi}_0|\beta_{m+k} \leq |\lambda|^k \|M_\phi\|\beta_m \text{ for all } m \text{ and } k \geq 0.$$

So that,

$$\beta_{k+m}/\beta_m \leq |\lambda|^k \|M_\phi\|/|\hat{\phi}_0|$$

So,

$$\|M_z^k\| = \sup_m[\beta_{k+m}/\beta_m] \leq |\lambda|^k |M_\phi\|/|\hat{\phi}_k|$$

By taking $k-$th root and letting $k \to \infty$ yields $r(T) \leq |\lambda|$. Equality is excluded since by circular symmetry, the entire circle $|\lambda| = r(T)$ is in the spectrum.

Case 2. $\hat{\phi}_{-1} \neq 0$**.** In equation 3.0.2, let $k = -n$, $n \geq 1$, we have

$$|\hat{\phi}_{-n}|\beta_{m-n} \leq \|M_\phi\|\beta_m \text{ for all } n, m.$$

Applying the last equation in equation 3.0.1, we obtain

$$|\lambda|^{n-1}|\hat{\phi}_{-1}|\beta_{m-n} \leq \|M_\phi\|\beta_m \text{ for all } n, m.$$

Setting $n = 1$, recalling that we take 0^0 to be 1, we obtain that $\frac{\beta_{m-1}}{\beta_m}$ is bounded for all m implying from Proposition 2.1.3 that T is invertible.
Furthermore, for $n \geq 1$, $|\lambda|^{n-1}|\hat{\phi}_{-1}| \leq \|M_\phi\|\beta_m/\beta_{m-n}$. Taking inf over m of both sides and recalling 2.1.2, we obtain

$$|\lambda|^{n-1}|\hat{\phi}_{-1} \leq \|M_\phi\| \inf_m \left[\frac{\beta_m}{\beta_{m-n}}\right] = \|M_\phi\|\|T^{-n}\|^{-1}$$

Taking the n-th root of both sides and letting $n \to \infty$, we obtain $|\lambda| \geq [r(T^{-1})]^{-1}$.
From the first equation in 3.0.1, we see that given λ, at least one of these Cases must occur. If T is invertible then T^{-1} exists and thus the two Cases yield the desired result (a). If T is not invertible then only Case 1 holds and thus (b).

■

3.1 The approximate point spectrum of Weighted shift operator

Proposition 3.1.1 *Let T be a weighted shift represented as M_z (on $H^2(\beta)$ or $L^2(\beta)$) then*

$$m(T^n) = \inf_k \frac{\beta_{k+n}}{\beta_k} \text{ for } n = 1, 2, ...$$

Thus, $r_1(T) = \lim_{n\to\infty} \left[\inf_k \frac{\beta_{k+n}}{\beta_k}\right]^{\frac{1}{n}}$.

Theorem 3.1.2 ([Ridge, 1970], p. 350) *Let T be a unilateral weighted shift. Then $\sqcap(T) = \{\lambda \in \mathbb{C} : r_1(T) \leq |\lambda| \leq r(T)\}$.*

For the bilateral shift case, we shall use the following notations.

$$r_1^+(T) = \lim_{n\to\infty}\left[\inf_{j\geq 0}\frac{\beta_{n+j}}{\beta_j}\right]^{\frac{1}{n}}, \quad r^+(T) = \lim_{n\to\infty}\left[\sup_{j\geq 0}\frac{\beta_{n+j}}{\beta_j}\right]^{\frac{1}{n}}$$
$$r_1^-(T) = \lim_{n\to\infty}\left[\inf_{j<0}\frac{\beta_j}{\beta_{-n+j}}\right]^{\frac{1}{n}}, \quad r^-(T) = \lim_{n\to\infty}\left[\sup_{j<0}\frac{\beta_j}{\beta_{-n+j}}\right]^{\frac{1}{n}}.$$

In the sense that the limits exists following the procedure used in Proposition 1.2.1.

We state the following:

Theorem 3.1.3 ([Ridge, 1970] p.352) *If T is a bilateral shift and if $r^-(T) < r_1^+(T)$, then*

$$\sqcap(T) = \{\lambda \in \mathbb{C} : r_1^- \leq |\lambda| \leq r^-\} \cup \{\lambda \in \mathbb{C} : r_1^+ \leq |\lambda| \leq r^+\}$$

otherwise,

$$\sqcap(T) = \{\lambda \in \mathbb{C} : r_1^- \leq |\lambda| \leq r^-\} \cup \{\lambda \in \mathbb{C} : r_1^+ \leq |\lambda| \leq r^+\}.$$

3.2 The Point Spectrum of Weighted shift operator

If T is a unilateral shift with weight sequence $\{w_n\}$, we define $r_2(T)$ by

$$r_2(T) = \liminf_{n\to\infty}\left[w_0 ... w_{n-1}\right]^{\frac{1}{n}}.$$

Theorem 3.2.1 *Let T be a unilateral weighted shift. Then*

(i) $\sqcap_0(T)$ is empty.

(ii)

$$\{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| < r_2(T)\} \subset \sqcap_0(T^*) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r_2(T)\}.$$

Furthermore, all eigenvalues of T^* are simple that is, of geometric multiplicity one.

Proof:

(i) Let $\lambda \in \mathbb{C}$. It suffices to show that $T - I\lambda$ is one to one. Now, let $f = \sum \alpha_n e_n$ such that $(T - I\lambda)f = 0$, that is, $Tf = \lambda f$. We get

$$\sum_{n\geq 0}\alpha_n w_n e_{n+1} = \sum_{n\geq 0}\lambda\alpha_n e_n$$

From which we get

$$\lambda\alpha_0 = 0, \quad \alpha_n w_n = \lambda\alpha_{n+1} \text{ for all } n \geq 0.$$

If $\lambda = 0$, $Tf = 0$ implies $f = 0$ since T is injective.

If $\lambda \neq 0$, we get that $\alpha_n = 0$ for all $n \geq 0$ implying $f = 0$.

(ii) Let $0 \neq \lambda \in \sqcap_0(T^*)$ with $f = \sum_{n\geq 0} \alpha_n e_n$ as a corresponding eigenvector. From $T^* f = \lambda f$ we have

$$\sum_{n\geq 1} \alpha_n w_{n-1} e_{n-1} = \sum_{n\geq 0} \lambda \alpha_n e_n.$$

And so, $\alpha_{n+1} w_n = \lambda \alpha_n$ for all $n \geq 0$. Therefore

$$\alpha_n = \frac{\alpha_0 \lambda^n}{w_0 w_1 ... w_{n-1}} \text{ for all } n \geq 1. \tag{3.2.1}$$

We see that $a_0 \neq 0$ else $a_n = 0$ for all n. Also, $f = \alpha_0 \left[e_0 + \sum_{n\geq 1} \frac{\lambda^n e_n}{w_0 w_1 ... w_{n-1}}\right]$ and

$$\|f\|^2 = |\alpha_0|^2 \left(1 + \sum_{n\geq 1} \frac{\lambda^{2n}}{(w_0 w_1 ... w_{n-1})^2}\right) \tag{3.2.2}$$

In this note, we take $\alpha_0 = 1$. So that by the Cauchy-Hadamard formula, 3.2.2 converges for

$$|\lambda| \leq \liminf_{n\to\infty} [w_0 w_1 ... w_{n-1}]^{\frac{1}{n}} = r_2(T).$$

implying that $\sqcap_0(T^*) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r_2(T)\}$.
Conversely, since $T^* e_0 = 0$, $0 \in \sqcap_0(T^*)$ and if $|\lambda| < r_2$ then $\|f\| < \infty$ and so, f is an eigenvector of T^* corresponding to λ. And thus

$$\{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| < r_2(T)\} \subset \sqcap_0(T^*).$$

Furthermore, from 3.2.1, we see that the scalars $\alpha_n' s$ of the eigenvector f is uniquely determined up to a scalar multiple by the eigenvalue λ. In other words, the geometric multiplicity of λ is 1. Thus, the eigenvalues of T^* are simple. ■

If T is a bilateral shift with weight sequence $\{w_n\}$, we denote

$$r_2^+(T) = \liminf_{n\to\infty} [w_0 ... w_{n-1}]^{\frac{1}{n}}, \quad r_3^+(T) = \limsup_{n\to\infty} [w_0 ... w_{n-1}]^{\frac{1}{n}}$$
$$r_2^-(T) = \liminf_{n\to\infty} [w_{-1} ... w_{-n}]^{\frac{1}{n}}, \quad r_3^-(T) = \limsup_{n\to\infty} [w_{-1} ... w_{-n}]^{\frac{1}{n}}.$$

Then $r_1^- \leq r_2^- \leq r_3^- \leq r^-$ and $r_1^+ \leq r_2^+ \leq r_3^+ \leq r^+$.

Theorem 3.2.2 *Let T be a bilateral weighted shift. Then*

(i) all eigenvalues of T and T^* are simple.

(ii) $\{\lambda \in \mathbb{C} : r_3^+(T) < |\lambda| < r_2^-(T)\} \subset \sqcap_0(T) \subset \{\lambda \in \mathbb{C} : r_3^+(T) \leq |\lambda| \leq r_2^-(T)\}$.

(iii) $\{\lambda \in \mathbb{C} : r_3^-(T) < |\lambda| < r_2^+(T)\} \subset \sqcap_0(T^*) \subset \{\lambda \in \mathbb{C} : r_3^-(T) \leq |\lambda| \leq r_2^+(T)\}$.

(iv) at least one of $\sqcap_0(T)$, $\sqcap_0(T^*)$ is empty.

Remark 3.2.3 *(i) If $r_2^- < r_3^+$, then $\sqcap_0(T) = \emptyset$; if $r_2^+ < r_3^-$ then $\sqcap_0(T^*) = \emptyset$.*

(ii) By circular symmetry, one of the containments in (ii), and one of the containments in (iii) must be equality.

Proof: Let $\lambda \in \sqcap_0(T)$ with its corresponding eigenvector $f = \sum_n \alpha_n \ell_n$. Then we have from $Tf = \lambda f$

$$\sum_{n\in\mathbb{Z}} \alpha_n w_n e_{n+1} = \sum_{n\in\mathbb{Z}} \lambda \alpha_n e_n.$$

From which we obtain $\alpha_{n-1} w_{n-1} = \lambda \alpha_n$ for all n. And so,

$$a_n = \frac{a_0 w_0 ... w_{n-1}}{\lambda^n}, \quad a_{-n} = \frac{a_0 \lambda^n}{w_{-1} ... w_{-n}} \text{ for all } n \geq 1.$$

From this we that the eigenvalues are simple. Further calculations are reversible and so $\lambda \in \sqcap_0(T)$ if and only if the sequence $\{\alpha_n\}$ with $\alpha_0 = 1$ is square summable. This leads to two power series, one in λ and the other in $\frac{1}{\lambda}$, and the result follows from Cauchy-Hadamard formula [Gamelin, 2003] for the radius of convergence.
The case of T^* is also similar.
Finally, let $\lambda \in \sqcap_0(T)$ and $\mu \in \sqcap_0(T^*)$. We wish to show that at least one of these is impossible. By what we have shown above, $r_3^+(T) \leq |\lambda| \leq r_2^-(T)$ and $r_3^-(T) \leq |\mu| \leq r_2^+(T)$. Since $r_3^+ \leq |\lambda| \leq r_2^- \leq r_3^- \leq |\mu| \leq r_2^+ \leq r_3^+$, we have $|\lambda| = |\mu|$. Also, an examination of the series which must converge shows that

$$\sum_{n\geq 1} \frac{|w_0 ... w_{n-1}|^2}{|\lambda|^{2n}} < \infty \quad \text{and} \quad \sum_{n\geq 1} \frac{|\mu|^{2n}}{|w_0 ... w_{n-1}|^2} < \infty,$$

which is impossible since $|\lambda| = |\mu|$. ■

Proposition 3.2.4 *(a) Let T be a unilateral weighted shift with weight sequence $\{w_n\}$. If $w_n \to d$, as $n \to \infty$ then*

$$r_1 = r_2 = r_3 = r = d.$$

(b) Let T be a bilateral weighted shift with weight sequence $\{w_n\}$. If $w_n \to d^+$ as $n \to +\infty$ and $w_n \to d^-$ as $n \to -\infty$ then

$$r_1^- = r_2^- = r_3^- = r^- = d^-, \quad r_1^+ = r_2^+ = r_3^- + r^+ = d^+.$$

CHAPTER 4

Analytic structure

Let $w \in \mathbb{C}$. We denote the the functional of "evaluation at w", defined on polynomials by $\lambda_w(p) = p(w)$.

Definition 4.0.1 (Bounded point evaluation) *w is said to be a bounded point evaluation on $H^2(\beta)$ if there exist $c > 0$ such that*

$$|\lambda_w(p)| \leq c\|p\|_2 \quad \textit{for all polynomials } p,$$

where the norm denotes the norm on $H^2(\beta)$.
We define $\lambda_w(f)$ to be $f(w)$ for all $f \in H^2(\beta)$.

By Riesz representation theorem, there exists $k^w \in H^2(\beta)$ such that

$$f(w) = \lambda_w(f) = \langle f, k^w \rangle = \sum_n \hat{f}_n \overline{\hat{k}^w_n} \beta_n^2, \quad \text{for all } f \in H^2(\beta). \tag{4.0.1}$$

We call k^w the **reproducing kernel** of $H^2(\beta)$ associated with w. Let n be fixed, by taking for f $f_n = z^n = \sum_i \delta_{i,n} z^i$ on $H^2(\beta)$, we obtain

$$w^n = \sum_i \delta_{i,n} \overline{\hat{k}^w_i} \beta_i^2 = \overline{\hat{k}^w_n} \beta_n^2, \quad \text{implying} \quad \hat{k}^w_n = \frac{\overline{w}^n}{\beta_n^2}.$$

Thus, w is a bounded point evaluation if and only if

$$\|k^w\|^2 = \sum_n \frac{|w|^{2n}}{\beta_n^2} < \infty.$$

Note that if p and q are polynomials then $\lambda_w(pq) = \lambda_w(p)\lambda_w(q)$. Holding p fixed, we have, for all $f \in H^2(\beta)$,

$$\lambda_w(pf) = \lambda_w(p)\lambda_w(f). \tag{4.0.2}$$

We state the following theorem:

Theorem 4.0.2 *Let T be a unilateral weighted shift operator represented as M_z on $H^2(\beta)$. Then*

(i) w is a bounded point evaluation if and only if $w \in \sqcap_0(T^*)$.

(ii) If w is a bounded point evaluation and if $f \in H^2(\beta)$, then the power series f converges absolutely at w to the value $\lambda_w(f)$.
Furthermore, the disc $\Delta_2(T) = \{|w| < r_2(T)\}$ is the largest open disc in which all the power series in $H^2(\beta)$ converge.

(iii) If $|w| < r(T)$ and if $\phi \in H^\infty(\beta)$, then the power series ϕ converges at w and

$$|\phi(w) \leq \|M_\phi\||.$$

Furthermore, this is the largest open disc in which all the power series in $H^\infty(\beta)$ converge.

(iv) If $\phi \in H^\infty(\beta)$ and $f \in H^2(\beta)$ and w is a bounded point evaluation on $H^2(\beta)0$, then $\lambda_w(\phi f) = \lambda_w(\phi)\lambda_w(f)$.

(v) If $w \in \sqcap_0(T^*)$, then k^w is a common eigenvector for all operators commuting with T^*.

$$M_\phi^* k^w = \overline{\phi(w)} k^w \quad \textit{for } \phi \in \mathcal{H}^\infty(\beta).$$

(vi) If $|w| = \|T\|$ then w is not a bounded point evaluation.

(vii) If the power series ϕ represents a bounded analytic function in the disc $|z| < \|T\|$, then $\phi \in H^\infty(\beta)$ and

$$\|M_\phi\| \leq sup\{|\phi(z)| : |z| < \|T\|\}.$$

Proof: (i) Let w be a bounded point evaluation on $H^2(\beta)$. Then there exists $k^w \in H^2(\beta)$ such that

$$\lambda_w(f) = \langle f, k^w \rangle \quad \text{for all } f \in H^2(\beta).$$

Moreover,

$$\begin{aligned} \langle f, M_z^* k^w \rangle &= \langle M_z f, k_w \rangle \\ &= \lambda_w(zf) \\ &= \lambda_w(z)\lambda_w(f) \quad \{\text{By } (4.0.2)\} \\ &= w\langle f, k^w \rangle \\ &= \langle f, \overline{w} k^w \rangle. \end{aligned}$$

This implies that $M_z^* k^w = \overline{w} k^w$. Thus $\overline{w} \in \sqcap_0(T^*)$. By circular symmetry of the spectrum, $w \in \sqcap_0(T^*)$.
Conversely, suppose that $w \in \sqcap_0(T^*)$, again by circular symmetry, $\overline{w} \in \sqcap_0(T^*)$, thus there exists $k \neq 0$ such that $M_z^* k = \overline{w} k$.

Define a bounded operator on $H^2(\beta)$ $\lambda(f) = \langle f, ck \rangle$ for some $c \neq 0$. Then,

$$\lambda(zf) = \langle M_z f, ck \rangle = \langle f, M_z^* ck \rangle = \langle f, c\overline{w}k \rangle = w\langle f, ck \rangle = w\lambda(f).$$

We observe that if $f_n = z^n$, then $\lambda(f_{n+1}) = w^n\lambda(f_0)$ for all n. $f_0 \neq 0$ else by Riesz representation theorem, $0 = \|\lambda\| = \|ck\|$ implying that $k = 0$. Hence we may choose c so that $\lambda(f_0) = 1$ so that $\lambda(p) = p(w)$ for all polynomials and so w is a bounded point evaluation.

(ii) Let $s_n(z) = \sum_{k=0}^{n} \hat{f}_k z^k$ for $n = 0, 1, ...$, be the partial sums of the power series of f converging to f in the norm of $H^2(\beta)$. Then $\lambda_w(s_n(z)) \to \lambda_w(f)$. Implying $s_n(w) \to \lambda_w(f)$. That is

$$\sum_n \hat{f} z^n = \lambda_w(f).$$

The absolute convergence is a direct consequence of the following:
From (i), $w \in \sqcap_0(T^*)$ and thus by circular symmetry $|w| \in \sqcap_0(T^*)$ implying that that $|w|$ is a bounded point evaluation on $H^2(\beta)$ and so we have convergence at $|w|$ for all $f \in H^2(\beta)$. And, if $f \in H^2(\beta)$, then the power series $\sum |\hat{f}_n| z^n \in H^2(\beta)$.
The second part follows using Cauchy-Hadamard formula by noting that the power series f defined by $\hat{f}_n = \frac{1}{(n+1)\beta_n}$ is in $H^2(\beta)$ and has $r_2(T) = \liminf_{n\to\infty} [\beta_n]^{\frac{1}{n}}$ as its radius of convergence. ■

Let G be an open subset of $\mathbb{C}$. We denote by $H^\infty(G)$ the algebra of all bounded analytic functions in G with the supremum norm. If G is a disc or annulus about the origin, we identify elements of $H^\infty(G)$ with their corresponding power series or Laurent series.
Now, let w be a bounded point evaluation on $H^2(\beta)$. We denote by $H^2_w(\beta)$ the set of functions in $H^2(\beta)$ vanishing at w. That is the kernel of λ_w.

Given a linear space X, a proper linear subspace M is called a linear space of **codimension one** if for a given $x_0 \in X \setminus M$, every $x \in X$ can be represented as in the form

$$x = \alpha x_0 + y$$

where α is a scalar and $y \in M$.

Lemma 4.0.3 *The kernel of a linear functional that is not $\equiv 0$ is a linear subspace of codimension one.*

Proposition 4.0.4 *$H^2_w(\beta)$ is a closed subspace of codimension one in $H^2(\beta)$, and the polynomials in $H^2_w(\beta)$ are dense in $H^2_w(\beta)$.*

Proof: The first part follows directly from the preceding lemma. For the second part, let $f \in H^2(\beta)$ and let $\{p_n\}$ be a sequence of polynomials such that $p_n \to f$. This implies $\lambda_w(p_n) \to 0$. Define a new sequence of polynomials $\{q_n\}$ by $q_n = p_n - \lambda_w(p_n)$. We see that for each n, $q_n \in H^2_w(\beta)$ and $q_n \to f$.
■

Proposition 4.0.5 *If w is a bounded point evaluation, then $(z-w)H^2(\beta)$ is a dense subset of $H^2_w(\beta)$. They are equal if and only if $|w| < r_1(T)$.*

CHAPTER 5

Applications of Analytic Structure

Here, we see some applications of the analytic structure of weighted shift operator. Other notable examples and applications can be found in [Gellar, 1969b], [N. P. Jewell, 1979], [I. B. Jung, 2008], [J. Stochel, 1989] and [Xia, 1983].

5.1 Equivalence of normed spaces

Let $\mathbb{O}$ be an open subset of $\mathbb{C}$. We shall denote $H^\infty(\mathbb{O})$ as the algebra of bounded analytic functions in $\mathbb{O}$ with the supremum norm.

Proposition 5.1.1 *If $r(T) = \|T\|$, then the normed algebras $H^\infty(\beta)$ and $H^\infty(\{z : |z| < \|T\|\})$ are equivalent.*

Proof: This follows directly as stated in (iii) and (vii) of theorem 4.0.2. ■

5.2 No Reducing Subspace

A closed subspace E of $\mathcal{H}$ is invariant under T, if $T(E) \subset E$ and is said to be reducing provided that it is invariant under T and T^*.
Let $\mathcal{H} = E \oplus E^\perp$ and P_E be the orthogonal projection on E. E and $E^\perp$ are invariant under T if and only if $P_E T = T P_E$.

Proposition 5.2.1 *Let T be a weighted shift operator. Then there does not exist any proper invariant subspace E such that $H^2(\beta) = E \oplus E^\perp$.*

Proof: The existence of such subspace would imply that from the analytic structure of a shift, $P_E = M_\phi$ and since $P_E^2 = P_E$ it will follow that $\phi^2 = \phi$ which says $\phi = 1$ or 0. ■

5.3 Non-existence of root in L(H)

Proposition 5.3.1 *If T is unilateral weighted shift, then T has no* nth *root in $L(H)$ for $n > 1$.*

Proof: Suppose $A^n = T$, then A commutes with T. By the analytic structure of T, $A = \phi$. But there is no formal power series ϕ such that $\phi^n = z$ for $n > 1$. ■

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