MODELING OF THE ORIGIN AND INTERACTION OF MULTISOLITON SOLUTIONS OF THE (2+4)KdV EQUATION

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APPROVAL

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ABSTRACT

Most of the relevant research work has addressed the properties of internal solitons in a greatly simplified environment, usually in the framework of different versions of the two layer fluid. The simplest equation of this class is the well-known Korteweg-de Vries (kdV) equation that describes the motion of weakly nonlinear internal waves in the long-wave limit. However, in many areas of the world's ocean, the vertical stratification has a clearly pronounced three-layer structure, with well-defined seasonal thermocline at a depth of about 100m or higher. Hence , the need for a redefinition of the famous KdV equation to tackle such scenarios and clearly accounts for nonlinearity in such environments. In this work, we first derived an analytical solution for the (2+4) KdV-like equation which mimics such situations and numerically solved it using the pseudospectral methods due to its robustness. After numerical simulations, we observed that the multisoliton solution interactions, particularly the three soliton solution interaction showed similar properties with the two soliton solution interaction.

Keywords: solitons, KdV equation, interaction, elastic.

DEDICATION

I will like to dedicate this piece of work to my father, Wasu Ningang Joseph of blessed memory .

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CHAPTER 1

LITERATURE REVIEW ON SOLITONS

1.1 **Introduction**

Natural occurrences are predominated by nonlinearity. Fluid and plasma mechanics, gas dynamics, elasticity, relativity, chemical reactions, combustion, ecology and many more other physical phenomena are all governed inherently by nonlinear equations. These nonlinear equations could be ordinary differential equations (ODEs) or partial differential equations (PDEs). Linear systems are governed by linear equations. Majority of natural systems are nonlinear and are therefore modelled by nonlinear systems of equations. Linear systems are systems whose solutions satisfy the superposition principle. That is, a linear combination of two or more solutions to the equation gives rise to another solution to that same equation. The superposition principle is however, not true for nonlinear systems. This superposition principle allows the solution to a linear problem to be broken into pieces, which are then solved independently by, for example, the Laplace transform, and then added back to form a solution to the original problem. The last five decades has seen a tremendous progress in solving nonlinear systems, guided by advances in experiments, great success in the computer simulation of nonlinear systems. Nature abounds with examples of nonlinear waves.

1.2 **Soliton**

This is a particular type of Internal waves which is a solution of a nonlinear PDE which represents a solitary travelling wave, which:

- Has a permanent form.
- Is localised within a region.
- Does not disperse.
- Does not obey the superposition principle.

1.3 **History of Soliton**

The initial observation of a solitary wave in shallow water was made by John Scott Russell, shown in Figure 1.1 . Russell was a Scottish engineer and naval architect who was conducting experiments for the Union Canal Company to design a more efficient canal boat [1].

Figure 1.1: John Scott Russell. Source: G. S. Emmerson (1977). Courtesy of John Murray Publishers.

Russell built a water tank to replicate the phenomenon and research the properties of the solitary wave he had observed [2] and in 1995, scientists gathered at Heriot-Watt University for a conference and successfully recreated a solitary wave but of smaller dimensions than the one observed by Russell 161 years earlier (see Figure 1.2).

Figure1.2: Recreation of a solitary wave on the Scott Russell Aqueduct on the Union Canal. Photograph courtesy of Heriot-Watt University.

In 1965, Zabusky and Kruskal introduced the concept of a soliton for the Korteweg-de Vries(KdV) equation [3]. Two years later, by using the Inverse Scattering method on the Schrödinger equation, Gardner *et al.* (GGKM) solved the KdV equation for exact *N*-soliton solutions [4], which can be used to model the interaction of unidirectional solitary waves, on water. Their discovery establishes the mathematical foundation of the unidirectional water wave interaction. The KdV equation is the leading-order approximation of the Euler equation from a perturbation scheme under the assumption that the wave height is relatively small and the wavelength is relatively long compared with the water depth. It also assumes that the wave propagates in one direction, which is not a good assumption to model the reflection of water waves on a vertical wall. For reflection of water waves, we need a model that allows the bidirectional wave interactions, including head-on and overtaking collisions. A solid mathematical foundation of the bidirectional water wave interaction has been well-established [5].

For over forty years now, optical solitons have been shown to form and propagate inside a nonlinear Kerr medium. The concept of soliton describes various physical phenomena ranging from solitary waves on a water surface to ultrashort optical pulses from a laser. The study of optical solitons is interesting for its important applications. The generation of a train of soliton pulses from continuous wave light in optical fibers was first suggested by Hasegawa and Tappert [6,7] and first realized experimentally in single-mode fibers for the case of negative group velocity dispersion by Mollenauer et al. [8] and in single-mode optical fibers with large positive group velocity by Nakatsuka et al. [9] . The problem of soliton instabilities, leading to the collapse, and which depend on the number of space dimensions and strength of nonlinearity has been reported in two principal directions. The first direction in the study of the collapse stabilization is the use of a weaker nonlinearity, such as saturable [10], cubic-quintic [11–13], quadratic ($\chi^{(2)}$) [14–16], or that induced by the self-induced transparency [17]. Moreover, in a recent experiment, it has been established that the optical susceptibility of CdS_xSe_{1-x}-doped glass processes a considerable level of fifth-order susceptibility $\chi^{(5)}$. In semiconductor doubled optical fibers [18], the doping silica fibers with two appropriate semiconductor particles may lead to an increased value of third-order susceptibility $\chi^{(3)}$and a decreased value of $\chi^{(5)}$. Thus, in order to investigate pulse propagation in such materials, it is necessary to consider higher-order nonlinearities in place of the usual Kerr nonlinearity. However, when the saturation is very strong, a self-focusing $\chi^{(7)}$ is also needed. Quite recently, an experiment has been reported in material such as chalcogenide glass which exhibits not only third-order nonlinearities but even seventh-order nonlinearities [19,20]. In other words, chalcogenide glass can be classified as a cubic-quintic-septic nonlinear material. In the past few years, the higher-order nonlinear Schrödinger equation with cubic-quintic-septic nonlinearity were used, modeling the propagation of ultrashort femtosecond optical pulse [19,20].

The second direction is the dynamics of optical solitons in the presence of higher-order dispersions [21]. It is well known that for 1D case, the dispersion effect broadens the pulse in the longitudinal direction, which is compensated by the self- focusing effect of the nonlinearity to generate a stable soliton propagation. Contrary, for the 2D and 3D cases, the dispersion as well as the diffraction effects broaden the pulse in the longitudinal and the transversal directions, respectively, which must be compensated by the stronger self-focusing effect than for the 1D case. Picosecond pulses are well described by the nonlinear Schrödinger (NLS) equation which accounts for the second-order dispersion and self-phase modulation (SPM). It is known that the NLS equation does not give correct prediction for pulse width smaller than one picosecond. For example, in solid state, solitary wave lasers, where pulses as short as 10 femtoseconds are generated, the approximation breaks down. Thus, quasi-monochromaticity is no longer valid and so, higher-order dispersion terms creep in, such as the third-order dispersion and self-steepening. The third-order dispersion is significant because it qualitatively changes the linear dispersion relation. Its effect on the NLS soliton is to generate continuous wave radiation and causes the soliton decay [21]. On the other hand, it was shown by Fewo and Kofane [22] that, taking in account the third-orderdispersion term, different numerical simulations lead to some changes of propagation properties, where the main observation is the shifting of the temporal position of the pulse under several values of the third-order coefficient. This quality may be detrimental for transmission systems in the sense that the transmitted pulse will arrive practically out of its bit slot at the receiver system, and this may cause a possible loss of information by generating a timing jitter [23]. It has been found that the fourth-order dispersion term in NLS-type equation stabilizes instabilities [24–26]. Sometimes, an additional fourth-order term destabilizes the soliton and the second derivative order term can work as a stabilizer for the soliton [27]. If the group velocity dispersion is close to zero, one needs to consider the third and higher-order dispersion for performance enhancement along trans-oceanic and trans-continental distances. Also, for short pulse widths where group velocity dispersion changes within the spectral bandwidth of the signal cannot be neglected, one needs to take into account the fourth and sixth-order dispersion terms in addition to the third-order dispersion term [28,29].

For long-distance communication systems, compensation of attenuation of pulse is an important issue. One approach is the use of periodically spaced amplifiers. In the second approach, the losses can be compensated by the erbium-doped amplifiers. When frequency and intensity dependent gain and loss have to be taken into account, the governing equation is the cubic complex Ginzburg–Landau (CGL) equation [30]. In multidimensional settings, where various important applications of all-optical devices and switches are expected for very high-speed digital communication [31], direct time-independent techniques provides straight and accurate solutions. So far, the finite-difference time-domain (FDTD) method [32] has been applied to model the nonlinear optical pulse propagation in one dimension (1D) [33], two dimensions (2D) [34–36], and in three dimensions (3D) [37]. However, it is well known that (2 + 1)-Dimensional [(2 + 1)D] (two transverse plus one longitudinal dimensions) solitons in self-focusing Kerr medium are inherently unstable [38]. On the other hand, it was predicted [39] that a 2D spatial cylindrical soliton can be quite effectively stabilized in a bulk *layered* medium, with opposite signs of the Kerr coefficient in adjacent layers, corresponding to self-focusing and self-defocusing chromatic regimes, respectively [40]. The models which include loss, diffraction in one transverse direction, and a combination of diffusion and dispersion in the other one have been used to find stable localized pulses [41–43] and collisions between two and three stable dissipative solitons have been reported in driven optical cavities [44]. Regions of existence of two-dimensional solitons have been studied extensively, either numerically [45,46] or with the semi analytical method of moments [47]. Stable (2 + 1)D necklace-ring solitons carrying zero, integer, and even fractional angular momentum have also been investigated [48,49]. Stable quasi 2D spatiotemporal soliton in $\chi^{(2)}$ crystals has been obtained experimentally [50], and predicted for 2D solitons in Bose-Einstein condensates [51–53]. It has been numerically demonstrated the existence of stable light bullets using the three-dimensional CQLE model in both regimes of chromatic dispersion [54–57]. Akhmediev et al. [57,58] have performed numerical simulations which reveal the existence of stationary bell-shaped formation of double soliton complexes. Kamagate et al. [59] have used a collective variable approach to map domains of existence for (3 + 1)D spatiotemporal stationary and pulsating dissipative light bullets of a cubic-quintic Ginzburg–Landau equation. Investigation of spatiotemporal optical solitons [i.e., (3 + 1)D] (two spatial, one temporal and plus one longitudinal dimensions), the so-called light bullets localized in all spatial and time dimensions are new fundamental entities which have the potential applications in all optical processing devices, like all optical switching in bulk media or the digital logic gates and eventually in all optical computation and communication systems [60,61] , and also on the possibility of violet collapse in a self-focusing medium in higher dimensions [62]. By selecting proper parameters in the (3 + 1)D cubic-quintic complex Ginzburg–Landau equation, He et al. [63] have proven that the spatiotemporal necklace-ring solitons carrying zero or nonzero angular momentum can be self-trapped for a very large distance. Some of the (3 + 1)D soliton solutions admit spherical symmetry between temporal and all spatial variables. Liu et al. [64] have also intensified the investigation on the study of the (3D) complex Ginzburg–Landau (CGL) equation by introducing the external annularly periodic potentials in the basic model of the CGL equation. To enlarge the information capacity, it is necessary to transmit ultrashort optical solitons at high bit rate in the picosecond and femtosecond regimes. However, several new effects such as sixth-order dispersion term, self-steepening (Kerr dispersion), self-frequency shift arising from stimulated Raman scattering and cubic, quintic, septic nonlinearities of dispersive and dissipative types greatly influence their propagation properties.

1.4 **Soliton Occurrences**

1.4.1 **Tsunami**

A tsunami is a type of tidal wave caused by sudden movement of water (displacement). They are one of the world's most dangerous natural disasters, having calamitous effects on coastal communities. They can reach speeds of up to 800km/h, and grow to over 30m in height and travel right across the Pacific [65]. Tsunamis belong to the same family as common sea waves that we enjoy at the beach; however, tsunamis are distinct in their mode of generation and in their characteristic period, wavelength and velocity. Unlike common sea waves that evolve from persistent surface winds, most tsunamis spring from sudden shifts of the ocean floor. These sudden shifts can originate from undersea landslides and volcanoes, but mostly, submarine earthquakes parent tsunamis. Tsunamis are often called seismic waves. Compared with wind- driven waves, seismic waves have periods, wavelengths and velocities ten or a hundred times lager. Tsunamis thus have profoundly different propagation characteristics and sholine consequences than do their common cousins. (See Figure 1.3). Tsunamis can be prevented through a number of ways among which are: buildings can be built on reinforced concrete 'stilts' to raise them out of flood waters after a tsunami, tall platforms can be constructed along coastlines for people to reach high ground quickly and planting trees along coastlines help as the trees help to break apart the wave before it reaches homes.

Figure 1.3: Tsunami wave.

1.4.2 **Atmospheric stratification**

The Earth is surrounded by five gaseous layers which is constrained by Earth's gravitational pull. The Earth's atmosphere is divided into several layers namely; Troposphere, Stratosphere, Mesosphere and upper atmospheres. Basically, the atmospheres are classified on the basis of variation of temperature with respect to height [66]. The temperature decreases with increase in height until the Troposphere. But in Stratosphere, the temperature increases with height. Due to the temperature difference between any two layer, a thermocline exists, as such any disturbance in the atmosphere will result in the generation and propagation of an internal wave through the thermocline. (see Figure 1.4)

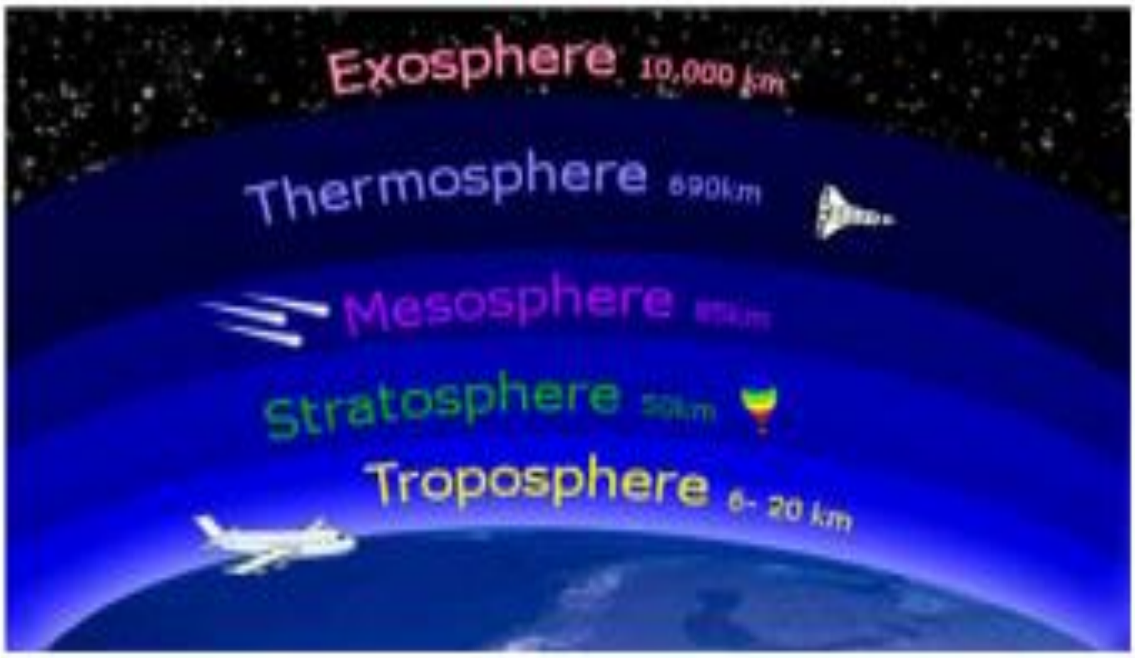

Figure 1.4: Layers of Earth Atmosphere.

1.5 Conclusion

In this chapter, we presented the soliton definition, the history of the soliton discovery, high-lighted the importance of the KdV equation in the development of the soliton theory. A presentation of the different fields of application of soliton was equally made. We also saw that by a numerical study of the KdV equation for the time, some interesting properties of the soliton such as collision was revealed.

CHAPTER 2

METHODOLOGY

2.1 Introduction

In this chapter, we will present the analytic and numerical methods for solving the KdV equation, we will derive the standard KdV equation, state some properties of the KdV equation and state some limitations of the KdV and mKdV equations.

2.2 Derivation of the KdV equation

We shall start this section by introducing some terminology. Let the vector $u(x,t) = (u(x,y,t), v(x,y,t))^T$ denote the (two-dimensional) flow velocity at an arbitrary point x and time t. If we assume an incompressible and irrotational flow, mathematically speaking $divu = 0$ and $rotu = 0$, we may rewrite the components u and v of the velocity using the velocity potential$\emptyset$ and the stream function φ:

$$u = \frac{\partial\emptyset}{\partial\emptyset} = \frac{\partial\emptyset}{\partial\emptyset} \ and \ v = \frac{\partial\emptyset}{\partial y} = -\frac{\partial\emptyset}{\partial x}$$

Which are related to the flow velocity$u$by grad$\emptyset = u$and$u = rot(\partial\varphi\tilde{z})$.Inorder to demonstrate how Korteweg and de Vries derived their equation, we will start from Eulers equation of fluid dynamics and formulate two physical boundary conditions: the free surface condition and the kinematic boundary condition. Secondly, we will perform a Taylor expansion of the velocity components which will be subsequently inserted into the boundary conditions leading to the KdV equation.

Surface Condition

Eulers equation for an inviscid fluid reads

$$\frac{\partial u}{\partial t} + (u.\nabla)u = -\nabla\left(\frac{p}{\varrho} + X\right). \tag{2.2.1}$$

Where p is the density of the fluid, p the pressure and g = grad X. We write Eq. (2.2.1) using the identity $(u.\nabla)\,u = (rotu)\,\Lambda u + \nabla(\frac{1}{2}u^2)$ and the velocity potential$\emptyset$:

$$\frac{\partial \emptyset}{\partial t}+\frac{1}{2}u^2+X=C(t)+\frac{p_0-p}{\varrho}\,. \qquad (2.2.2)$$

Where we integrated once, thus obtaining an integration constant p_o which we identify with the atmospheric pressure. C(t) is an arbitrary function which only depends on time. As already pointed out Korteweg and de Vries included the effect of surface tension which causes a net upward force per unit area of surface of $T\frac{d^2\eta}{dx^2}\Delta x$. This upward force has to correspond to the difference of pressures, in other words $p_0 — p = T\frac{d^2\eta}{dx^2}$. Therefore, rewrite Eq. (2.2.2):

$$\frac{\partial \emptyset}{\partial t}+\frac{1}{2}u^2+g\eta-C(t)=\frac{T}{\varrho}\frac{\partial^2\eta}{\partial x^2} \quad if \ y= \eta(x,t). \qquad (2.2.3)$$

This is known as the free surface condition (with surface tension).

Kinematic Condition

Let $\eta(x,t)$ describe the shape of a one-dimensional wave. We define a function $F(x,y,t) = \eta(x,t)-y$ which vanishes as long as a particle is on the surface. Taking the total derivative with respect to time we obtain

$$\frac{dF}{dt}=\frac{\partial F}{\partial t}+(u.\nabla)F=\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial t}-v=0 \ if \ y= \eta(x,t). \qquad (2.2.4)$$

Equation (2.2.4) is referred to as the kinematic boundary condition.

The Expansion

Following Korteweg and de Vries expanded the velocity components using Taylor series under the assumption of shallow water $h << 1$. By doing so they obtained

$$u(x,t)= f(x,t) — \frac{y^2}{2!}f''(x,t)+\frac{y^4}{4}f^4 — \ldots and v(x,t)= -yf'(x,t) + \frac{y^3}{3!}f^3(x,t) — \ldots \qquad (2.2.5)$$

Under the assumption of shallow water $h << 1$. In addition, they made the ansatz $f(x,t)=q_o-\frac{g}{q_0}\,(\eta(x,t)\ +\ a\ +\ \gamma(x,t))$,

where q_0 is an unknown constant velocity, α is a small constant describing the uniform motion of the liquid and γ is small compared to η. By inserting Eq. (2.2.5) into the kinematic condition (2.2.4) and into the derivative with respect to time of the free surface condition (2.2.3), we obtain two equations (for h and γ). Combining these we may eliminate $\gamma(x,t)$ and eventually obtain the Korteweg de Vries equation as it was originally presented in the dissertation of de Vries:

$$\frac{\partial\eta}{\partial t} = \frac{3}{2}\frac{g}{q_0}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2}\right)\ with\ \sigma = \frac{1}{3}h^3 - \frac{Th}{\varrho g}. \qquad (2.2.6)$$

Equation (2.2.6) can be rewritten in a moving frame $\xi := x - \left(\sqrt{gh} - \sqrt{\frac{g}{h}\alpha}\right)t\ with\ qo = -\sqrt{gh}\ and\ t = \tau.$ In this way we are left with

$$\frac{\partial\eta}{\partial\tau} + \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial\xi}\left(\frac{1}{2}\eta^2 + \frac{1}{2}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2\eta}{\partial\xi^2}\right) = 0. \qquad (2.2.7)$$

where we neglected the added (constant) velocity . Furthermore, Eq.(2.2.6) becomes dimensionless by introducing the variables $t := \frac{1}{2}\sqrt{\frac{g}{h}\tau}$, $x := \sigma^{\frac{-1}{3}}$ ξ, $u \coloneqq \sigma^{\frac{-1}{3}}\left(\frac{1}{2}\eta + \frac{1}{3}\alpha\right)$ and Eq.(2.2,7) simplifies to

$$u_t(x,t) — 6u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0\,. \qquad (2.2.8)$$

The subscripts in Eq. (2.2.8) denote partial differentiations. This (simplified) form is how the KdV equation usually appears in literatures.

2.3 Properties of KdV equation

The KdV equation (2.2.8) obeys the conservation laws. A conservation law is an equation of the form $\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x}$ = 0.

The laws are as follows:

- **Conservation of mass:** $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u_{xx} - 3u^2) = 0.$

This implies $\int_{-\infty}^{\infty} u dx = M.\ where\ M\ \ is\ a\ constant.$

- **Conservation of momentum:** $u_t - 6uu_x + u_{xxx} = 0.$

This implies that $\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + \frac{\partial}{\partial x}\left(uu_{xx} - 2u^3 - \frac{u^2{}_x}{2}\right) = 0.$

Hence, we have that $\int_{-\infty}^{\infty}\frac{u^2}{2}dx = p\ .\ where\ p\ is\ constant.$

- **Conservation of energy:** $3u^2xKdV + u_x x\frac{\partial}{\partial x}KdV.$

This implies $\int_{-\infty}^{\infty}\left(u^3 + \frac{u_x^2}{2}\right)\,dx\ = constant = E.$

2.4 Analytical methods for solving the KdV equation

Over the past five decades, the construction of exact solutions for a broad class of nonlinear equations including the KdV equation has been an extremely active domain of research. Much of the literature of the theory of nonlinear equations uses the soliton solution model of the KdV equation as example to introduce nonlinear theory. In order to obtain the multiple solutions of the KdV equation, thus showing its richness, numerous analytical methods leading to the exact solutions of the KdV equation have been studied [67]. In the following, we will present some analytical methods for obtaining exact solutions of the soliton-type, for different forms of the KdV equation. We will limit ourselves to methods able of finding one-soliton solution, and also multisoliton solutions.

2.4.1 Inverse Scattering method

The Inverse Scattering Transform is a method to solve the KdV equation for certain initial values. The Inverse Scattering Transform uses quantum mechanical principles to solve an equation. The Inverse Scattering Method (ISM) played a very important role in the development of soliton theory. Indeed, the ISM uses linear techniques to solve the problem of value of a large number of nonlinear wave equations and provides N-soliton solutions (N=1,2,3, ...) or multisoliton solutions. ISM was discovered and developed by Kruskal, Greene, Gardner and Miura in 1967 [68], and it was first applied to find soliton solutions to the KdV equation. A general formulation of the ISM was quickly followed by Peter Lax who assumed that it is possible to find two linear operators $L(u)\ and\ B(u)$, which depend on the solution $u(x,t)$ of the KdV equation, and which can satisfy the following operator equation :

$$iL_t = [B, L] = BL - LB. \tag{2.4.1}$$

The form of the KdV equation for the operators $L(u)$ and $B(u)$ are:

$$L = \frac{\partial^2}{\partial x^2} + u(x,t), \quad B = -4i\frac{\partial^3}{\partial x^3} + 3i\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right). \tag{2.4.2}$$

Let us note that L_t is equivalent to u_t . The operator L is a function of time by its dependence on $u(x,t)$, so, we will denote it by $L(t)$. Then, the eigenvalue problem can be formed by:

$$L\psi = \lambda\psi. \tag{2.4.3}$$

If B is self-adjoint ($B = B^T$), Eq.(2.4.1) implies that the eigenvalue λ in previous expression is independent of time. In addition, the proper function X evolves over time according to the following equation $i\psi_t = B\psi$. (2.4.4)

The idea of the ISM is therefore based on three phases: direct scattering, the temporal evolution of scattering data and the inverse scattering. It does this in the following way: Firstly, the initial wave shape $u(x, 0)$ is seen as a potential in the time-independent Schrödinger equation. Then, the scattering data of that potential is calculated. This scattering data can be transformed in a way corresponding to the KdV -equation, and the new scattering data can be used to calculate the corresponding potential $u(x, t)$ in certain situations.

Direct Scattering

The initial solution $u(x, 0)$ solves Eq. (2.4.3) at $t = 0$, for the scattering data at $|x| = \infty$. In the case of KdV equation, the eigenvalue at t = 0 satisfies the following equation:

$$L(0)\psi = \frac{\partial^2}{\partial x^2}\psi + u(x, 0)\psi = \lambda\psi , \quad (2.4.5)$$

which is the Schrödinger equation in quantum mechanics.

Temporal evolution of Scattering data

Using Eq. (2.1.4), with asymptotic form of operator for $|x| = \infty$, scattering data temporal evolution can be calculated for $t > 0$.

Inverse Scattering method

From the knowledge of the scattering data for $t > 0$, the solution $u(x, t > O)$ can be built. This is accomplished by solving an integral equation derived from equation (2.4.3). The name of the method comes from this last step. In addition to the KdV equation, the Inverse Scattering Method can be applied to all nonlinear partial differential equations provided that it is integrable.

2.4.2 Hirota bilinear method

In most dynamical systems, integrability implies existence of several conserved quantities. This important feature allows us to make some predictions over time and hence, plays a key role in obtaining analytical solutions. In the case of solitons, it means conservation of waves in the medium in question such as fiber optics. Most dispersive wave systems generate solitary waves that would scatter inelastically, meaning loss of energy and hence loss of information. However, dynamics of dispersive wave systems governed by integrable nonlinear Schrödinger equation generate nonlinear waves called solitons. Indeed, these nonlinear travelling waves collide, on the contrary, elastically, meaning, after a nonlinear interaction phase, the waves recover their shape and retain their energy and information.

In 1971, Ryogo Hirota published an article giving a new method called the Hirota direct method to find the exact solution of the KdV equation for multiple collisions of solitons. In his successive articles, he dealt also with many other nonlinear evolution equations such as the modified Korteweg-de Vries (mKdV), sine-Gordon (sG), nonlinear Schrödinger (NLS) and Toda lattice equations. The Hirota direct method has taken an important role in the study of integrable systems. Most equations (even non-integrable ones) having Hirota bilinear form possess automatically one- and two-soliton solutions. When we come to the three solitons, we come across a very restrictive condition. Actually, this condition is not sufficient to search the integrability of an equation, but it can be used as a powerful tool for this purpose. This condition was also used to produce new integrable equations by Hietarinta.

Hirota D operator and bilinear transformation

Definition 2.4.1 Let $S:\ C^2 \rightarrow\ C$ be a space of differentiable functions. Then, the Hirota D-operator $D : S\ x\ S\ \rightarrow S$ is defined as

$[D_x^{m1} D_t^{m2} ..]\{f(x,t).g(x,t)\} = [(\partial_x - \partial_{x\prime})^{m1}(\partial_t - \partial_{t\prime})^{m2} \ldots]\, f(x,t) x g(x',t')|x = x', t = t' \ldots|$, (2.4.6)

where $m_i, i =\ 1,2 \ldots$ are positive integers and x; t; ... are independent variables. For example:

$Dt\ f.g = f_t g - f g_t$. (2.4.7)

$D_{tt} f.g\ =\ f_{tt} g -\ 2 f_t g_t +\ f g_{tt}$.(2.4.8)

$D_{ttt} f.g = f_{ttt} g - 3 f_{tt} g_t + 3 f_t g_{tt-} f\ g_{ttt}$. (2.4.9)

$D_t D_x^2\{f.g\} = f_{xxt} g_t - 2 f_{xt} g_x + 2 f_x g_{xt} + f_x g_{xx} - f g_{xxt}$.(2.4.10)

If $f(x,t,\dots) =\ g(x,t,\dots)$, *we get that*

$D_t f.g\ =\ D_t f.f\ =\ f_t f -\ f f_t$. (2.4.11)

However, we have that:

$D_{tt} f.f =\ f_{tt} f - 2 f_t f_t + f f_{tt}$. (2.4.12)

Hirota D is a linear operator and hence:

$(\propto_1 D_T + \propto_2 D_x) f.g = \propto_1 D_t f.g + \propto_2 D_x f.g,$ (2.4.13) where

$(\propto_1 D_t + \propto_2 D_x) f.g = P(D)\{f.g\}.$ (2.4.14)

Proposition 2.4.2 Let P(D) be an Hirota D-operator, g and f be two differentiable functions, then

$P(D)\{f.g\} = P(-D)\{f.g\}.$ (2.4.15)

Proof. We can simply take $P(D) = D_x^m$. The other combinations of D-operators follow in same manner. We can write

$P(D) = D_x^m.\{f.g\}.$ (2.4.16)

Then, we have that

$P(D)\{f, g) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f_{(m-k)x} g_{kx}$. (2.4.17)

That again gives

$P(D)\{f.g\} = f_{mx}\, g - m f_{(m-1)xgx} + \cdots . + (-1)^m f.g_{mx},$ (2.4.18)

where the subscripts of the functions f and g define the order of the partial derivatives with respect to x. Indeed,

$$P(D)\{f.g\} = (-1)^m \left[f_{mx} g - m f_{xg(m-1)x} + \cdots + (-1)^{m-1} m f_{(m-1)xgx} + (-1)^m f_{mxg} \right],$$ (2.4.19)

which is equal to $(-D)\{f.g)$. Hence, $P(D)\{f.g\}$. Note that if m is a positive even integer, interchanging the functions does not change the value of the Hirota bilinear equation.

2.5 Numerical solutions to the KdV equation

Since the discovery of the soliton and the derivation of nonlinear partial differential equations, including families of KdV equations, many studies have been carried out in order to find numerical solutions for these equations. To this end, several numerical methods have been proposed for the numerical processing of families of KdV equations according to the initial and boundary conditions. These are the Spectral and Pseudospectral methods, the Fourier spectral method, the Finite difference method (explicit, implicit and exponential), the Finite element method and many others. Semi-analytical methods such as the domian decomposition method, the Variational iteration method and the Homotopy analysis method were also used [67]. The numerical study of the KdV equation is essential, because of the unavailability of solitonic solutions for KdV family equations at determined boundary conditions. In what follows, we will briefly present some of these methods.

2.5.1 Fourier Spectral methods for the KdV equation

In this section, a general spectral method will be presented.

- **Step 1:** Select a computational domain ($[\pi L, \pi L]$ or $[0, 2\pi L]$) and scale the partial derivative equation (PDE) accordingly.
- **Step 2:** Take the Fourier transform of the scaled equation.
- **Step 3:** Use an integrating factor or an exponential integrator to solve the linear term exactly (if applicable).
- **Step 4:** Use the method of lines to rewrite the PDE as a system of ODEs (ordinary derivative equation) in time. The exponential integrator alleviates the often stiff system of ODEs.
- **Step 5:** Apply a time stepping technique to solve the resulting system of ODEs. The set of steps above give a general idea of the algorithm. Other advanced steps can be included, for instance, preconditioning, interring, dealiasing, smoothing, and other post-processing techniques. To start, we state the general abstract problem. '.We would like to solve the KdV equation given an initial condition and enforcing periodic boundary conditions:

$$u_t + 6uu_x + u_{xxx} = 0, \;\; x \in [p, p], \;\; t > 0 \, , \tag{2.5.1}$$

$$u(x, 0) = f(x) \, , \tag{2.5.2}$$

$$u(-p, t) = u(p, t), \quad t > 0 \, . \tag{2.5.3}$$

(i) Step 1:

We re-write the KdV equation in the form

$$u_t + 3(u^2)_{\,x} + u_{xxx} = 0, \tag{2.5.4}$$

where the initial condition and boundary conditions are as given above. The first step is to use change of variables to scale the domain of $[p, p]$ to a computational domain of $[0, 2\pi]$, where $P = \pi L$. This is done because, eventually, we will discretize the Fourier transform and employ the Fast Fourier Transform (FFT). To do this transformation, we make a change of variables:

$$x = \frac{\pi X}{p} + \pi \, . \tag{2.5.5}$$

We have changed the solution interval from $[p, p]$ to $[0, 2\pi]$. Thus, the above equation becomes

$$u_t + \frac{3\pi}{p}(u^2)_{\,x} + \frac{\pi^3}{p^3} u_{xxx} = 0, \qquad x \in [0, 2\pi]. \tag{2.5.6}$$

(ii) Step 2:

Let the continuous univariable Fourier transform be denoted by the symbol F(.)

Definition 2.5.1 Given a function $f \in C^0$, the Fourier transform of f is

$$F(f(t)) = \tilde{f}(k) = \int_{-\infty}^{+\infty} f(x) \exp(-ikx)\, dx. \tag{2.5.7}$$

The above definition requests that f is continuous. However, this condition can be weakened. In order for spectral accuracy f will have to be smoother than just continuous. Also, the reader may be aware that there are many different (but equivalent) Fourier transform definitions. The integral form in above definition was adopted for simplicity. For more information, see the comprehensive Fourier analysis book [69].

There are two properties of the Fourier transform that we are mainly interested in for the spectral method. The first property is linearity.

Fourier Transform property 1

Given functions $f, g \in C^0$ and scalars$a, b \in \mathbb{R}$, we have

$$F[af + bf] = aFf + bFg \tag{2.5.8}$$

The second property deals with differentiation identities:

Fourier Transform property 2

If $f \in C^m$, then

$$F(f^{(n)}(t) = (ik)^n F\big(f(t)\big), \tag{2.5.9}$$

where $f^{(n)}$ is the n^{th} derivative of f and i is the imaginary unit. Taking the Fourier transform on both sides ofEq. (2.5.6) yields

$$\frac{du_k(t)}{dt} + \frac{3ik\pi}{p}\big(\widehat{u^2}\big)k - \frac{ik^3\pi^3}{p^3}u_K = 0, \tag{2.5.10}$$

where $\tilde{u}k = F(u)$

(iii) Step 3:

In Eq. (2.5.7), we isolate linear and nonlinear terms and obtain

$$\frac{du\tilde{k}(t)}{dt} - \frac{ik^3\pi^3}{p^3}u_K = \frac{3ik\pi}{p}(\tilde{u}^2)k\,. \tag{2.5.11}$$

To reduce numerical stiffness, we use an integrating factor in order to exactly solve the linear portion of Eq. (2.5.11). Since integrating factors are a standard technique in the theory of linear differential equation, they will be introduced first.

(iv) Step 4: The method of integrating factors seeks a function *I(t)* (called an integrating factor) such that the linear portion of a differential equation multiplied by *I(t)* can be written as the derivative of a product. In our case, the integrating factor takes the form

$$\mathrm{I(t)} = \exp\left(-ik^3\pi^3\frac{t}{p^3}\right). \qquad (2.5.12)$$

Multiplying equation (2.5.11) by (2.5.12), we have

$$I(t)\frac{d\tilde{u}k(t)}{dt} - I(t)\frac{ik^3\pi^3}{p^3}uk = -I(t)\frac{3ik\pi}{p}(\widetilde{u^2})k. \qquad (2.5.13)$$

To complete the integrating factor method, we make a change of variables in Eq. (2.5.13) ,namely: $\widetilde{w}_k = I(t)\tilde{u}_k$. We carry time derivative of $\widetilde{w}_k$ using the product rule. We have

$$\frac{d\widetilde{w}_k}{dt} = -\frac{ik^3\pi^3}{p^3}I(t)\tilde{u}k + \frac{d\tilde{u}k}{dt}I(t). \qquad (2.5.14)$$

Making substitution of Eq. (2.5.14) into Eq. (2.5.13) and cancelling out common terms, we obtain the transformed equation, where the linear term is gone, and the problem is no longer stiff.

$$\frac{d\widetilde{w}_k}{dt} = \frac{3ik\pi}{p}\exp\left(-ik^3\pi^3\frac{t}{p^3}\right)\left[(F^{-1}(\widetilde{w}\exp\left(ik^3\pi^3\frac{t}{p^3}\right)\right]^2. \qquad (2.5.15)$$

Equation (2.5.15) has the distinct advantage of recurring the linear portion of the differential equation exactly when one applies an ODE solver. In practice, one replaces t with ∇t in the function *I(t)* part.

(v) Step 5:

In order to increase the flexibility within the model, we discretize the spatial dimension in Eq. (2.5.15). This results in a system of coupled ODEs that we can approximate using an ODE solver. Truncating to finite domain, we divide the interval $[O, 2\pi]$ into N evenly spaced grid points defined by

$$x_j = \frac{2\pi}{N}, \;\; j = 0,1 \ldots\ldots, N-1\,. \qquad (2.5.16)$$

We truncate the Fourier transform of $\widetilde{w}(k,t)$. Let

$$\widetilde{w}(k,t) = F(w) = \frac{1}{N}\sum_{j=0}^{N-1} w(x_j,t)\exp(-ikx_j), \quad -\frac{N}{2} < k < \frac{N}{2} - 1. \qquad (2.5.17)$$

In similar manner, the inverse Fourier transform is truncated to

$$\widetilde{w}(k,t) = F^{-1}(w) = \frac{1}{N}\sum_{k=\frac{-N}{2}}^{\frac{N}{2}-1} w(x_j,t)\,exp(-ikx_j), 0 < j < N-1. \qquad (2.5.18)$$

With this discretization, a system of ordinary differential equations arises from Eq. (2.5.15):

$$\frac{\widetilde{w}(k,t)}{dt} = -\frac{3ik\pi}{p}\exp\left(-ik^3\pi^3\frac{t}{p^3}\right)F\left[F^{-1}(\widetilde{w}\exp\left(ik^3\pi^3\frac{t}{p^3}\right))^2\right]. \qquad (2.5.19)$$

Let

$$W = [w(x_0, t), w(x_1, t), w(x_2, t), \ldots\ldots w(x_{N-1}, t)]. \qquad (2.5.20)$$

The Eq. (2.5.19) can be written in the vector form

$$\frac{dW}{dt} = F(W), \qquad (2.5.21)$$

where F denotes the right hand side of Eq. (2.5.19). To complete the approximation of the KdV equation, we need to apply an ODE solver to the system of ODEs found in Eq. (2.5.21) and take the inverse discrete Fourier transform to the resulting solution. As mentioned above, the Fourier transform has many variations of its definition, and this is also true for the discrete Fourier transform.

2.6 Limitation of KdV and mKdV equations

The KdV and mKdV equations could only explain the two layer model which only conditionally represents the vertical structures of seas and oceans. It direct extensions, three layer stratification has proven to be a proper approximation of the sea water density profile in some basins in the wall oceans with specific hydrological conditions.

2.7 Conclusion

In this chapter, we saw the analytic and numerical methods for solving the KdV equation. We equally derived the standard KdV equation and stated some properties of the KdV equations. We concluded the chapter with limitation of the KdV and mKdV equations.

CHAPTER 3

RESULTS AND DISCUSSION

Introduction

In this chapter, we will present the (2 + 4) KdV model equation, derive its analytical solution and analyse the multisoliton interactions of its solutions. We then end the chapter with a general conclusion.

3.1 2D Nonlinear evolution equation in the shallow water for interfacial waves in a symmetric three-layer fluid

We present, in this section, the schematic representation of the symmetric three layer fluid from which our model is stated and the relevant evolution equation derived.

3.1.1 Formulation of the problem

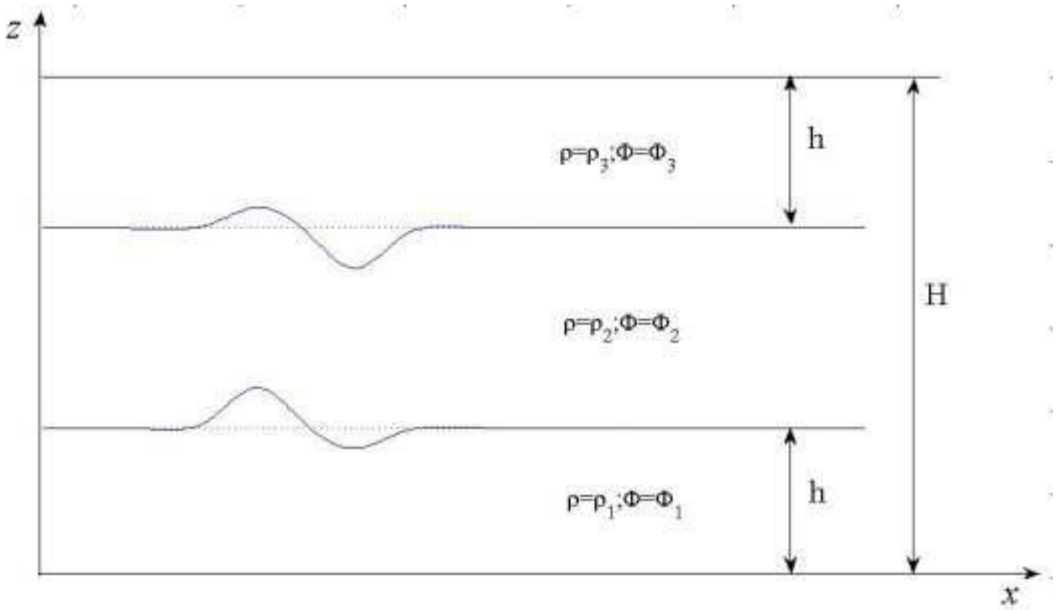

Figure 3.1: Schematic representation of symmetric three layer fluid.

Most of the relevant research work has addressed the properties of internal solitons in a greatly simplified environment, usually in the framework of different versions of the two layer fluid. The simplest equation of this class is the well-known Korteweg-de Vries equation that describes the motion of weakly nonlinear internal waves in the long-wave limit. However, in many areas of the world ocean, the vertical stratification has a clearly pronounced three-layer structure, with well-defined seasonal thermocline at a depth of about 100m or higher. Several Basins such as the Baltic sea host more or less continuously three-layer model so, the need for an extended KdV equation.

3.1.2 Derivation of an approximate equation

In his paper titled ''Higher-order (2+4)Korteweg-de Vries like equation for interfacial waves in a symmetric three-layer fluid", Kurkina et al. [70] went through a series of asymptotic expansions to derive the (2+4) KdV model equation for interfacial waves in a symmetric three-layer fluid. This equation in its original form incorporates the cubic and quintic nonlinearities, and a dispersive term and has the following form:

$$u_t + \alpha_1 u^2 u_x + \alpha_3 u^4 u_x + \beta u_{xxx} = 0, \qquad (3.1.1)$$

where α_1 and α_3 are nonlinear coefficients and β is a dispersive coefficient. For full derivation of the Eq. (3.11), we make reference to Kurkina et al. [70].

3.2 Analytic solution of the (2+4) Korteweg-de Vries-like equation

We shall consider the (2 + 4) KdV equation given by

$$u_t + \alpha_1 u^2 u_x + \alpha_3 u^4 u_x + \beta u_{xxx} = 0. \qquad (3.2.1)$$

We assume a travelling wave solution of the form $u(x,t) = f(z)$ where $z = x - ct$ to Eq. (3.2.1). Now, we have that

$$u_t = -c\frac{df}{dz}. \qquad (3.2.2)$$

$$u_x = \frac{df}{dz} \ . \qquad (3.2.3)$$

Consequently, it becomes

$$u_{xx} = \frac{d^2 f}{dz^2} \ . \qquad (3.2.4)$$

$$u_{xxx} = \frac{d^3f}{dz^3} . \tag{3.2.5}$$

Substituting Equations (3.2.2), (3.2.3) and (3.2.5) into (3.2.1), we obtain

$$-c\frac{df}{dz} + \alpha_1 f^2 \frac{df}{dz} + \alpha_3 f^4 \frac{df}{dz} + \beta \frac{d^3f}{dz^3} = 0. \tag{3.2.6}$$

Integrating (3.2.6) with respect to z, we get

$$-cf + \alpha_1 \frac{f^3}{3} + \alpha_3 \frac{f^5}{5} + \beta \frac{d^2f}{dz^2} = A, \tag{3.2.7}$$

Where A is a constant.

We multiply Eq. (3.2.7) by $\frac{df}{dz}$ and then integrate to obtain

$$-c\frac{f^2}{2} + \alpha_1 \frac{f^4}{12} + \alpha_3 \frac{f^6}{30} + \frac{\beta}{2}\left(\frac{df}{dz}\right)^2 = Af + B , \tag{3.2.8}$$

where A and B are constants.

We apply the asymptotics conditions, where $f'(z) = f''(z) = f'''(z) = 0$. This implies that $A = B = 0$. Thus, we have that

$$-c\frac{f^2}{2} + \alpha_1 \frac{f^4}{12} + \alpha_3 \frac{f^6}{30} + \frac{\beta}{2}\left(\frac{df}{dz}\right)^2 = 0. \tag{3.2.9}$$

Hence, we have

$$\left(\frac{df}{dz}\right) + \frac{1}{\beta}\left(cf^2 - \frac{\alpha_1 f^4}{6} - \frac{\alpha_3 f^6}{15}\right). \tag{3.2.10}$$

Consequently, we have that

$$d_z = \sqrt{\beta}\left(cf^2 - \frac{\alpha_1 f^4}{6} - \frac{\alpha_3 f^6}{15}\right)^{\frac{-1}{2}} df. \tag{3.2.11}$$

Integrating, we obtain

$$z - z_0 = \sqrt{\beta}\int\left(cf^2 - \frac{\alpha_1 f^4}{6} - \frac{\alpha_3 f^6}{15}\right)^{\frac{-1}{2}} df. \tag{3.2.12}$$

Now, we substitute for $f(z) = u(z)$ into (3.2.10) and set $\beta = \beta_1$, we have

$$\left(\frac{d_u}{dz}\right)^2 = \frac{1}{\beta}\left(cu^2 - \frac{\alpha_1 u^4}{6} - \frac{\propto_3 u^6}{15}\right). \tag{3.2.13}$$

We substitute using the Ansatz

$$u^2 = \frac{\alpha}{1+\lambda cosh\beta z}. \tag{3.2.14}$$

Differentiating (3.2.14) with respect to z, we get

$$2u\frac{du}{dz} = \frac{-\alpha\lambda\beta sinh\beta z}{(1+\lambda cosh\beta z)^2}. \qquad (3.2.15)$$

Hence, we have that

$$\frac{du}{dz} = \frac{-\alpha^{\frac{1}{2}}\lambda\beta sinh\beta z}{2(1+\lambda cosh\beta z)^{\frac{3}{2}}}. \qquad (3.2.16)$$

Thus, we get

$$\left(\frac{du}{dz}\right)^2 = \frac{\alpha\lambda^2\beta^2 sinh^2\beta z}{4(1+\lambda cosh\beta z)^3}. \qquad (3.2.17)$$

Also, we have that

$$u^4 = \frac{\alpha^2}{(1+\lambda cosh\beta z)^3}. \qquad (3.2.18)$$

$$u^6 = \frac{\alpha^3}{(1+\lambda cosh\beta z)^3}. \qquad (3.2.19)$$

We substitute Eq. (3.2.14), (3.2.17), (3.2.18) and (3.2.19) into Eq. (3.2.13), that leads to expressions for α, β and λ , that is

$$\alpha = \frac{12c}{\alpha_1} . \qquad (3.2.20)$$

$$\beta = 2\sqrt{\frac{c}{\beta_1}}. \qquad (3.2.21)$$

$$\lambda = \frac{\sqrt{25\alpha_1^2+240\alpha_3 c}}{5\alpha_1} . \qquad (3.2.22)$$

Consequently, Eq. (3.2.14) becomes

$$u^2 = \frac{60\,c}{5\alpha_1+\sqrt{25\alpha_1^2+240\alpha_3 c}\;cosh2\sqrt{\frac{c}{\beta_1}}} . \qquad (3.2.23)$$

Hence, we have that

$$u(z) = \sqrt{\frac{60\,c}{5\alpha_1+\sqrt{25\alpha_1^2+240\alpha_3 c}\;cosh2\sqrt{\frac{c}{\beta_1}}}} . \qquad (3.2.24)$$

3.3 Numerical method for solving the (2 + 4) Korteweg- de Vries-like equation

Several numerical methods exit that can be used to solve Eq. (3.1.1), among which are the Finite difference methods and the Pseudospectral method. We choose to use the Pseudospectral method in solving the nonlinear PDE (3.1.1) above because of its robustness and good accuracy for large N.

3.3.1 The Pseudospectral method

In the Pseudospectral approach, in a finite difference like manner, the PDEs are solved point wise in physical space (x, t). However, the space derivatives are calculated using orthogonal functions (e.g Fourier integrals, Chebyshev polynomials). They are either evaluated using matrix multiplications, Fast Fourier Transforms (FFT), or convolutions.

Spectral solutions to the time dependent PDE (3.1.1) are formulated in the frequency wave number domain and solutions are obtained in terms of spectra (e.g seismograms). This technique is particularly interesting for geometrics where partial solutions in the $\omega - K$ domain can be obtained analytically (the case of our three-layer symmetric model above). Detail of the procedure is not shown here. The pseudospectral method has some advantages such as it robustness, good accuracy for large N and small time step.

3.3.2 Properties of the (2 + 4) KdV equation

The (2 + 4) KdV Eq. (3.1 .1) has two conservation laws which are as follows:

• Conservation of mass

$$M = \int_{-\infty}^{\infty} u dx. \tag{3.3.1}$$

• Conservation of energy

$$E = \int_{-\infty}^{\infty} u^2 dx \ . \tag{3.3.2}$$

3.4 Interaction of the solitary solution of the (2 + 4) KdV- like equation

An important feature of Eq. (3.1.1) is that it has solitary wave solutions. The decisive parameters for wave motion in media described by Eq. (3,1.1) are the signs of the coefficients at its nonlinear terms. The coefficient α_1 which is the cubic nonlinearity, is sign variable in the vicinity of $\frac{h_{cr}}{H}$. The coefficient α_3 which is the quintic nonlinearity, is negative in this region, but may change its sign at $\frac{h}{H} < 0.1384$.

All our numerical simulations of the initial problem for Eq. (3.1.1) with smooth and localized initial conditions showed a stable wave dynamics, with no evidence of instabilities or collapses even in interactions. Therefore, it seems plausible that the cubic nonlinearity plays a stabilizing role.

Both positive and negative solitary wave solutions are possible for each combination of the signs of the coefficients of Eq. (3.1.1). The wave speeds are directly proportional to the amplitudes. Large amplitude solutions to Eq. (3.1.1) form a table like wide signal. The described table like appearance of solutions for the relatively large amplitudes and rapidly moving disturbances with steep fronts may have substantial consequences in practical applications. Generally speaking, interactions and collisions of solitary solutions to non integrable evolution equations are inelastic. It is therefore not unexpected that solitary solutions to Eq. (3.1.1) interact inelastically with each other and with the back ground wave fields. As a demonstration of the feature, we present here an example of numerically simulated collisions of two and multi solitonic solutions. We shall discuss the two and multi-soliton solution interactions.

3.4.1 Two soliton solution interaction of same polarity

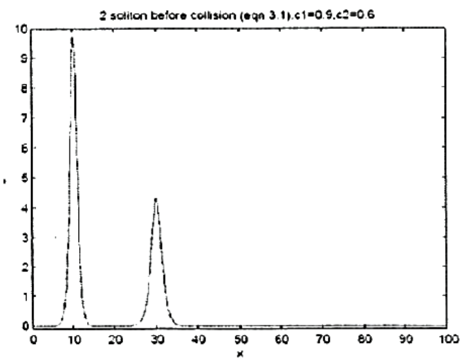

Figure 3.2: Two solitons before collision.

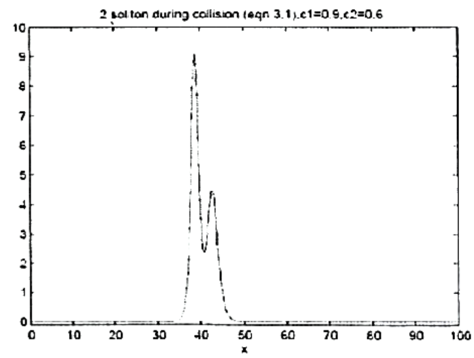

Figure 3.3: Two solitons during collision.

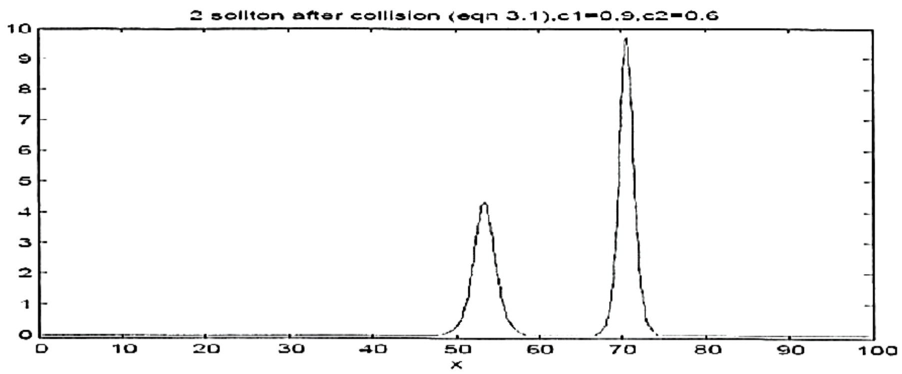

Figure 3.4: Two solitons after collision.

3.4.2 Two soliton solution interaction of different polarity

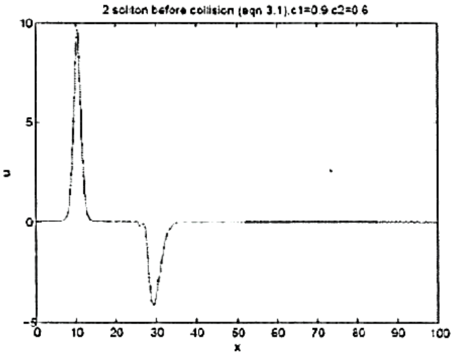

Figure 3.5: Two solitons before collision.

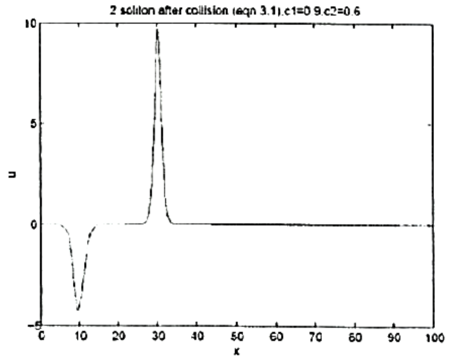

Figure 3.6: Two solitons after collision.

3.5 Multisoliton interaction

Here we limited ourselves to three solitons interaction.

3.5.1 Three soliton solution interaction of same polarity

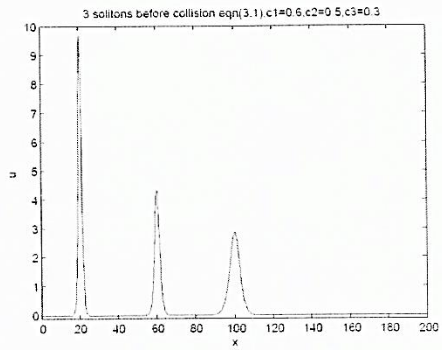

Figure 3.7: Three solitons before collision.

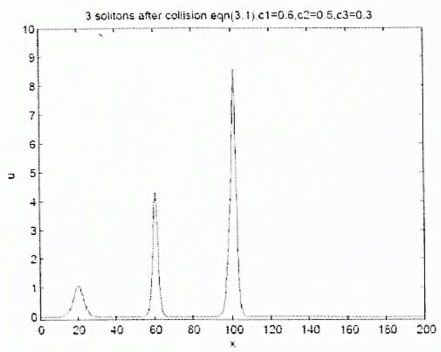

Figure 3.8: Three solitons after collision.

3.5.2 Three soliton solution interaction of different polarity

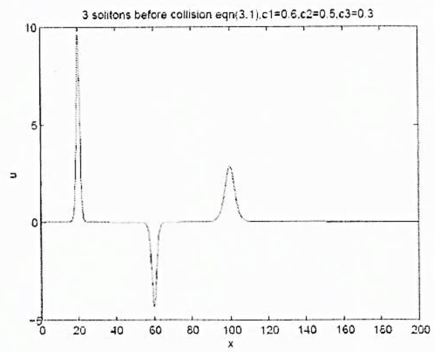

Figure 3.9: Three solitons before collision.

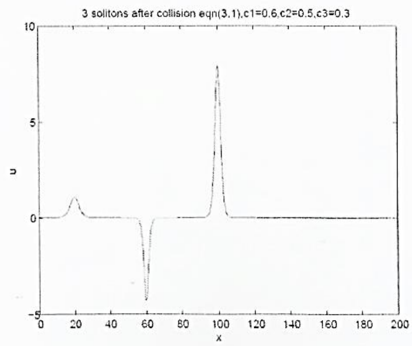

Figure 3.10: Three solitons after collision.

3.6 Results and discussion

The initial states for studies of interaction of these solitary waves were composed simply as a linear superposition of the counterpart. The smaller wave was placed ahead of the larger one. The evolution of solitary wave of elevation resemble the typical scenarios for soliton interaction of similar type in the classical KdV and mKdV frame works in which the taller wave overtakes the smaller one. In such interaction, the counterpart usually loses their identity and merges into a composite structure at a certain instant. After a while, the counterpart emerge again whereas one cannot say whether the counterparts propagate through each other as waves do or collide as particles do.

The interaction process is accompanied as in the case of KdV solitons by a clear decrease of the amplitude of the composite structure during the merging phase and by a substantial phase shift. The entire process was also accompanied by a modest radiation of wave energy from the interaction region. The collision of the solitary wave of different polarities has a similar appearance. Both the counterpart largely survive the collision but the phase shift for the wave of depression is more pronounced (See Fig 3.8) above. Consequently, collisions of solitary wave solutions of Eq. (3.1.1) basically have inelastic nature, although, both the intensity of wave radiation and changes to the amplitude of the solitons are fairly minor. For example, the collision of waves of elevation led to the increase in the amplitude of the taller soliton from 1 to 1.002 and an accompanied decrease in the smaller solutions from 0.5 to 0.477. The collision of wave of different polarity led to much smaller changes. The post collision amplitudes of the waves were 1.001 and - 0.499, respectively. Similar observation was made for the three soliton case as the amplitude of the tallest soliton of elevation slightly increased from 1 to 1.001, while the taller and tall solitons slightly decreased in amplitudes from 0.7 and 0.4 to 0.687 and 0.349, respectively. Also, the post collision amplitudes of the waves were 1.001 , - 0.698 and 0.399, respectively.

CHAPTER 4

GENERAL CONCLUSION

This thesis was devoted to the study of the dynamics of solitary waves described by the model of the (2 +4) KdV equation for a symmetric three layer fluid. The model equation is obtained starting from the Euler equation through a series of asymptotic expressions.

In chapter one, we presented the history of the soliton discovery, the different fields of application of the soliton and concluded the chapter with two examples of nonlinear evolution equations with solitary solution.

In chapter two, we presented the derivation of the KdV equation. Some properties of the KdV equation were examined. Some analytical methods for solving the KdV equation such as the inverse scattering method and some numerical methods such as the Fourier and spectral methods were also presented.

In chapter three, we were concerned with the presentation of different results obtained in our work and their interpretation.

We propose that this work can be extended to an asymmetric three layer fluid. Also, the model in our work can be considered for variable coefficients of nonlinearity and dispersion.

PARAMETERS

In calculations, we used the following parameters of the medium : total depth H= 100m, depth of the uppermost and lowermost layers h=30m. The corresponding values of the linear wave speed and the coefficients of Eqn(3.1.1) are as follows :

$$\alpha_1 = 0.002859\, m^{-1}/s\ .$$

$$\beta = 771.98\, m^3/s\ .$$

$$\alpha_3 = -0.00004924\, m^{-1}/s\ .$$

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