A STRONG CONVERGENCE FOR THE SUM OF THREE MONOTONE OPERATORS IN A REAL BANACH SPACE

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By

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Certification

This is to certify that the thesis titled "A STRONG CONVERGENCE FOR THE SUM OF THREE MONOTONE OPERATORS IN A REAL BANACH SPACE" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research carried out by Auta, Jonathan Timothy in the Department of Pure and Applied Mathematics.

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A THESIS APPROVED BY THE DEPARTMENT OF PURE AND APPLIED MATHEMATICS

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Abstract

Let E be a real 2-uniformly convex Banach space with topological dual E^*. We established strong convergence for the class of variational inclusion for the sum of three monotone operators. Moreover, we give a variant of this algorithm in which the stepsizes which are diminishing and non-summable. More precisely, we provide the following theorem:

Theorem. *Let E be a real 2-uniformly convex Banach space. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $B : E \to E^*$ be $\gamma-$ strongly monotone and L-Lipschitz and $C : E \to E$ be monotone and c-cocoercive . For $x_{-1},\, x_0 \in E$ define the sequence $\{x_k\}$ iteratively by*

$$x_{k+1} = J^A_{\alpha_k} \circ J^{-1}(Jx_k - \alpha_k Bx_k - \alpha_{k-1}(Bx_k - Bx_{k-1}) - \alpha_k Cx_k,$$

where $\alpha_n \subseteq (0,\infty)$ and $\sum\limits_{n=0}^{\infty} \alpha_n = +\infty, \lim\limits_{n\to\infty} \alpha_n = 0$ converges strongly to x^* an element of $(A + B + C)^{-}(0)$.

Finally, few applications were also provided to illustrate the relevance of our proposed scheme. Our results extend and complement several existing results in the literature.

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Dedication

Dedicated to the Almighty God who made it all possible.

Contents

Chapter 1

General Introduction and Literature Review

1.1 Motivation of Research

The contents of this thesis falls within the general area of nonlinear functional analysis, an area which has attracted the attention of prominent mathematicians due to its diverse application in numerous fields of sciences. In this thesis, we focus on variational inclusion problems involving monotone operators

The interest in the study of Monotone variational inclusion problems stems from the very fact that many mathematical problems can be cased as monotone inclusion problems; Monotone operators serve as an extension of functions with positive derivative and the fact that they are easier to handle when compared with their counterparts operators such as compact operators and completely continuous operators. The inception of nonlinear functional analysis was due to the extension of monotonicity definition to maps from arbitrary real Banach space into its dual. Monotone operators have found applicability in diverse areas such as control theory, optimization, variational inequality, equilibrium problems, differential equations and fixed point theory.

Let E be a real Banach space and E^* its topological dual. In this research work, we propose an algorithm for approximating the zeros of the sum of three monotone operators in a real Banach space E. More precisely, we consider the monotone inclusion problem of the form:

$$\text{find } x \in E \text{ such that } \; 0 \in (A+B+C)(x) \tag{1.1.1}$$

With $(A+B+C)^{-1}(0) \neq \emptyset$, where $A : E \to 2^{E^*}$ is a maximal monotone operator, $B : E \to E^*$ a $\gamma-$strongly monotone operator and $C : E \to E$ a monotone and c-cocoercive operator. Monotone inclusions problem of the form in (1.1.1) arises from several applications in nonlinear analysis such as image recovery, signal processing, machine learning and several others. These problems are modeled in form of a nonlinear operator equation and this operator is decomposed as a sum

of three simpler nonlinear operators.
Many problems can be reformulated as a problem of form (1.1.1). This thesis introduces a new three operator-splitting algorithm for solving a variety of problems that are reduced to a monotone inclusion of three operators, one of which is cocoercive. Our new splitting algorithm leads to the solution of new and simple algorithms for a variety of other related monotone inclusion problems, such as: The 3-set split feasibility problems, 3-objective minimization problems, and multiple regularization problems to mention a few.
Our splitting algorithm establishes simple numerical solutions to a large number of problems that appear in signal processing, machine learning, and statistics. We now give some physical problems that can be cast as monotone inclusion problem of the form (1.1.1). We begin with the following problems:

1.1.1 The 3-set (split) feasibility problem

The split feasibility problem introduced by Censor and Elving (13) is to

$$\text{find } x^* \in C \text{ and } Ax^* \in Q \tag{1.1.2}$$

where $A : H_1 \to H_2$ is a linear operator, C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Split feasibility problems have been extremely powerful tool for the treatment of a wide range of inverse problems, such as medical image reconstruction and intensity-modulated radiation therapy treatments. The 3-set feasibility problem is a problem of finding

$$x \in C_1 \cap C_2 \cap C_3, \tag{1.1.3}$$

where C_1, C_2, C_3 are three nonempty convex sets and the projection to each set can be computed numerically.
The more general 3-set split feasibility problem is to find

$$x \in C_1 \cap C_2, \text{such that } \; Lx \in C_3 \tag{1.1.4}$$

where L is a linear mapping. We can reformulate the problem as

$$\underset{x}{\text{minimize}} \frac{1}{2} d^2(Lx, C_3) \text{ subject to, } \; x \in C_1 \cap C_2, \tag{1.1.5}$$

where $d(Lx, C_3) := ||Lx - P_{C_3}(Lx)||$ and P_{C_3} denotes the projection onto C_3. By setting

$$Ax = N_{C_1}(x) \;\; Bx = N_{C_2}(x); \quad Cx = \nabla_x \frac{1}{2} d^2(Lx, C_3) = L^*(Lx - P_{C_3}(Lx)),$$

where $N_C(x)$ is the normal cone of x at C. Then (1.1.4) can be reformulated as (1.1.1)

1.2 Approximating zeros of monotone operators

One of the most successful and effective methods for solving optimization problem in Hilbert spaces is the fixed point method. As a result of this, and the numerous applications of problem

(1.1.1) so many research efforts have been devoted in developing different fixed point iterative algorithms for approximating the solution of (1.1.1), a large class of algorithms (14; 27; 35) and their generalization (9; 11; 10; 12; 19; 20; 21; 22; 31; 41) are the applications of one of the following operator-splitting schemes: forward-backward-forward splitting (FBFS) (40), forward-backward splitting (FBS) (34), and Douglas-Rachford splitting (DRS) (32), which all split the sum of two operators. The recently introduced forward-Douglas-Rachford splitting (FDRS) turns out to be a special case of FBS applied to a suitable monotone inclusion (24). In general, finding a solution of an optimization problem in Hilbert space is equivalent to finding a fixed point of a suitable nonlinear mapping, for instance, a solution of a minimization problem is a fixed point of the resolvent of the convex function associated with the minimization problem, also, a solution of a monotone inclusion problem is a fixed point of the resolvent of the monotone operator with the monotone inclusion problem. Thus, the fixed point method for solving optimization problems is concerned with developing different iterative algorithms for finding the fixed points of resolvent of mappings associated with these problems.
A special case of interest for three-operator splitted problem is the following two-operator sum problem in a Hilbert space H i.e. when we substitute $C = 0$ in problem (1.1.1);

$$\text{Find } x \in \text{H such that } 0 \in (A + B)(x) \tag{1.2.1}$$

A notable approach in solving (1.2.1) is the forward–backward method in (32; 34), the Douglas–Rachford method (32), the forward–backward–forward splitting method (40) to mention but a few. Recently, this method has been widely studied in various ways with different assumptions, see, e.g., (5; 26; 28; 36; 38).

In a recent work, Malitsky and Tam (33), introduced the *forward-reflected-backward* splitting method generated by iteratively by

$$x_{n+1} = J^A_{\lambda_n}(x_n - \lambda_n B(x_n) - \lambda_{n-1}(B(x_n) - B(x_{n-1}) \quad \forall n \in \mathbb{N},$$

with $\lambda_n \in (\epsilon, \frac{1-2\epsilon}{2L})$ and $\epsilon > 0$. The forward-reflected-backward splitting method requires only one evaluation of the operator B per iteration and the sequence $\{x_n\}$ converges weakly to some solution of (1.1.1)
Davis and Yin (25) have studied the problem (1.1.1) and introduced the following three-operator splitting scheme:

$$\begin{cases} z^0 \in H \\ x^n_B = J_{\gamma B}(z^n) \\ x^n_A = J_{\gamma A}(2x^n_B - z^n - \gamma(Cx^n_B + e^n_C)) + e^n_A, \\ z^{n+1} = z^n + \lambda_n(x^n_A - x^n_B), \end{cases} \tag{1.2.2}$$

where $\gamma \in (0, 2\beta(2-\epsilon))$ with $\epsilon \in (0,1)$, and $\{\lambda_n\} \subset (0, 2-\epsilon)$ and $\sum_{n=0}^{\infty} \lambda_n(2-\epsilon-\lambda_n) = +\infty$ and $\{e^n_A\}, \{e^n_B\}, \{e^n_c\}$ are the sequences in H which are absolutely summable. The authors in (42) proved that the sequences $\{x^n_A\}$ and $\{x^n_B\}$ converge weakly to some solution of the

problem (1.1.1). Under several additional conditions imposed on the operators A, B, C, the strong convergence of these sequences has been obtained.
Recently, Dang and Pham (29) considered a three-operator splitting algorithm for a class of variational inclusion problem, which is of the form:

$$\begin{cases} \text{Assume } \ x_{n-1}, x_n \in H \ \text{ are known} \\ \textbf{Step 1}. \text{ compute } \ x_{n+1} = J_{\lambda_n}(x_n - \lambda_n B(x_n) - B(x_{n-1})) - \lambda_n C(x_n) \\ \textbf{Step 2}. \text{ if } \ max\{||x_{n+1} - x_n||, ||x_n - x_{n-1}||\} \leq \epsilon, \text{then STOP} \\ \text{Else, set } \ n = n + 1 \text{and go to } \textbf{Step 1} \end{cases} \tag{1.2.3}$$

The major achievement of their proposed algorithm is that it can be easily implemented without the prior knowledge of Lipschitz constant, strongly monotone constant and cocoercive constant of each of the operators simply because the algorithm uses a sequence of stepsizes which is diminishing and non-summable. The strong convergence of the algorithm is established.

1.3 Our Contribution

In this work, we extend the result of Dang and Pham (29) to 2-uniformly convex Banach spaces. Moreover, we give some applications to illustrate how our results can be applied. Our results extends and complements several results in the literatures.

Chapter 2

Preliminaries

In this chapter, we give some basic definitions and Lemmas that would be used in the proof of our main results in the next chapter. Some of these results, which can be easily found in functional analysis textbooks are given without proofs or with a sketch of the proof only.

2.1 Geometry of Banach Spaces

Throughout this section, E denotes a real norm space and E^* denotes its corresponding dual. We shall denote by the pairing $\langle x, x^* \rangle$ the value of the function $x^* \in E^*$ at $x \in E$. The norm in E is denoted by $||.||$, while the norm in E^* is denoted by $||.||_*$. If there is no danger of confusion we omit the asterisk from the notation $||.||_*$ and denote both norm in E and E^* by the symbol $||.||$.

As usual We shall use the symbol $\rightarrow$ and $\rightharpoonup$ to indicate strong and weak convergence in E and E^* respectively. We shall also use w^*-lim to indicate the weak-star convergence in E^*. The space E^* endowed with the weak-star topology is denoted by E_w

Let H be a real Hilbert space. Then for all $x, y \in H$ the following identities holds:

$$||x + y||^2 = ||x||^2 + 2\langle x, y \rangle + ||y||^2 \tag{2.1.1}$$

$$||\lambda x + (1 - \lambda)y||^2 = \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)||x - y||^2 \tag{2.1.2}$$

Consequently, we shall soon study the duality map J, which provides analogues of identities (2.1.1) and (2.1.2) in general Banach spaces.

2.1.1 Uniformly Convex Spaces

Definition 2.1.1. *Let X be a normed linear space. Then X is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exist a $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in E$ with $||x|| \leq 1$, $||y|| \leq 1$, and $||x - y|| \geq \epsilon$, we have $||\frac{1}{2}(x + y)|| \leq 1 - \delta$*

Remark 2.1.2. *All Hilbert spaces are uniformly convex. Also, for $1 < p < \infty$ L_p and l_p are uniformly convex.*

Proposition 2.1.3. *Let X be a uniformly convex space, let $\alpha \in (0,1)$ and $\epsilon > 0$, then for any $d > 0$, $x, y \in E$ such that $||x|| \leq d$, $||y|| \leq d$, and $||x-y|| \geq \epsilon$ there exist $\delta(\epsilon) > 0$ independent of x and y such that*

$$||\alpha x + (1-\alpha)y|| \leq [1 - 2\delta(\epsilon)\min\{\alpha, 1-\alpha\}]d$$

Definition 2.1.4. *Let $p > 1$ be a real number. Then a normed space E is said to be p-uniformly convex if there is a constant $c > 0$ such that*

$$\delta_E(\epsilon) > c\epsilon^p$$

Example 2.1.5. *If $E = L_p$ (or l_p), $1 < p < \infty$, then*

- $\delta_E(\epsilon) \geq \frac{1}{2^{p+1}}$, *if* $1 < p < 2$
- $\delta_E(\epsilon) \geq \epsilon^p$, *if* $2 \leq p < \infty$

2.1.2 Strictly Convex Spaces

Definition 2.1.6. *A normed linear space E is said to be strictly convex if for all $x, y \in E$, $x \neq y$, $||x|| = ||y|| = 1$ we have*

$$||\alpha x + (1-\alpha)y|| < 1.$$

Proposition 2.1.7. *Every uniformly convex space is strictly convex*

Example 2.1.8. *$\mathbb{R}^n$ with $||.||_1$ is not strictly convex. To see this we choose the canonical bases e_1, e_2 in $\mathbb{R}^n$ and take $\frac{1}{2}$. Clearly $||e_1|| = ||e_2|| = 1$, $e_1 \neq e_2$ and*

$$||\frac{1}{2}e_1 - \frac{1}{2}e_2|| = \frac{1}{2}||e_1 - e_2|| = 1$$

Thus we have $\mathbb{R}^n$ with $||.||_1$ is not strictly convex.

Definition 2.1.9. *Let E be a normed space with $dimE \geq 2$. The modulus of convexity of E is a function $\delta_X : (0,2] \to [0,1]$ defined by*

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \ x, y \in B_E, \ \|x-y\| \geq \epsilon\right\} \tag{2.1.3}$$

In the particular case of an inner product space H, we have

$$\delta_H(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}$$

Proposition 2.1.10. *For every normed linear space E, the function δ_E is convex and continuous. Moreover $\frac{\delta_X(\epsilon)}{\epsilon}$ is non-decreasing on $(0,2]$.*

Proposition 2.1.11. *A normed space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0,2]$*

2.1.3 Duality Maps

In this session, we present the notion of duality mappings which will provide us with a pairing between elements of a normed space E and elements of its dual space E^*, which we shall also denote by $\langle\ ,\ \rangle$ and will serve as a suitable analogue of the inner product in Hilbert spaces.

Definition 2.1.12. *A continuous and strictly increasing function $\phi : [0, +\infty) \to [0, +\infty)$ such that $\phi(0) = 0$ and $\lim\limits_{t\to\infty} \phi(t) = \infty$ is called the guage function.*

Definition 2.1.13. *Given a gauge function ϕ, the mapping $J_\phi : E \to 2^{E^*}$ by*

$$J_\phi x := \{x^* \in E^* : \langle x, x^*\rangle = ||x||||x^*||;\ ||x^*|| = \phi(||x||)\}$$

It is easy to see that $J_\phi(x) \neq \emptyset\ \ \forall x \in E$. Indeed, this is as a result of the Hahn-Banach theorem.

Remark 2.1.14. *If $\phi(x) = x$ then J_ϕ is the normalized duality mapping and is denoted by J*

Definition 2.1.15. *For each $p > 1$, let $\phi(t) = t^{p-1}$ be a guage function. Following Definition 2.1.13, we define the* ***generalized duality map*** *$J_p : X \to 2^{X^*}$ by*

$$J_{\phi(t)} := J_p(x) = \{x^* \in X^* : \langle x, x^*\rangle = ||x||.||x^*||$$

$$||x^*|| = \phi(||x||) = ||x||^{p-1}.$$

We observe that for $p = 2$, we write $J_p = J_2 = J$ which is the normalized duality map on X and is a particular case of $J_\phi(t) = t\ \ \forall t \in \mathbb{R}^+$

Definition 2.1.16. *Let E be a real normed space. The normalized duality mapping of E into E^* is defined by*

$$Jx := \{x^* \in E^* : \langle x^*, x\rangle = \|x^*\|^2 = \|x\|^2\}$$

for all $x \in E$.

The normalized duality mapping J has the following properties (see, e.g., (39)):

- if E is reflexive and strictly convex with the strictly convex dual space E^*, then J is single valued, one-to-one and onto mapping. In this case, we can define the single-valued mapping $J^{-1} : E^* \to E$ and we have $J^{-1} = J^*$, where J^* is the normalized duality mapping on E^*;
- if E is uniformly smooth, then J is uniformly norm-to-norm continues on each bounded subset of E.

Proposition 2.1.17. *Let H be a real Hilbert space and identify H^* with H, then*

$$Jx = \{x\}\ for\ \ all\ \ x \in H$$

i.e The duality map J in Hilbert spaces is the identity map

Proposition 2.1.18. *Let E be a real Banach and J be the normalized duality mapping on E, then*

$$J(\lambda x) = \lambda J(x) \ \forall \ \lambda \in \mathbb{R} \ \forall x \in E$$

Proof

Let $y^* \in J(x)$ and $\lambda \in \mathbb{R}$. For $\lambda = 0$ the result follows trivially. suppose $\lambda \neq 0$, then we have

$$\langle \lambda x, \lambda y^* \rangle = \lambda^2 \langle x, y^* \rangle = ||\lambda x|| ||\lambda y^*||,$$

we also have

$$||\lambda x|| = ||\lambda y^*||$$

. Thus we have $\lambda y^* \in J(\lambda x)$, which implies $\lambda J(x) \subset J(\lambda x)$. From the preceding inclusion we also obtained that $\frac{1}{\lambda} J(\lambda x) \subset J(x)$ which implies $J(\lambda x) \subset \lambda J(x)$. Therefore

$$J(\lambda x) = \lambda J(x) \ \forall \lambda \mathbb{R}, \ \forall x \in E$$

.

2.1.4 Uniformly Smooth Spaces

Definition 2.1.19. *A normed space X is called smooth if for every $x \in X$, $||x|| = 1$, there exists a unique x^* in X^* such that $||x^*|| = 1$ and $\langle x, x^* \rangle = ||x||$.*

Definition 2.1.20. *Let E be a real normed linear space. Let S_E and B_E denote the unit sphere and the closed unit ball of E, respectively. The modulus of smoothness of E $\rho_E : [0, \infty) \to [0, \infty)$ is defined by*

$$\rho_E(t) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x \in S_E, \|y\| = t \right\}.$$

Proposition 2.1.21. *For every normed linear space E, the modulus of smoothness, ρ_E, is a convex and continuous function. Moreover, $\frac{\rho_E(t)}{t}$ is nondecreasing.*

Definition 2.1.22. *A normed space E is said to be uniformly smooth if*

$$\lim_{t \to 0^+} \frac{\rho_E(t)}{t} = 0$$

Proposition 2.1.23. *Every uniformly smooth normed space E is smooth.*

The following result gives the duality between uniformly convex and uniformly smooth spaces.

Theorem 2.1.24. *Let X be a Banach space.*

(a) X is uniformly smooth if and only if X^* is uniformly convex.

(b) X is uniformly convex if and only if X^* is uniformly smooth.

Definition 2.1.25. *For, $q > 1$, a Banach space X is said to be $q-$uniformly smooth if there exists a constant $c > 0$ such that*

$$\rho_X(t) \leq c\epsilon^q \quad t > 0$$

Definition 2.1.26. *A real Banach space E is said to be 2-uniformly smooth (i.e., substituting $q = 2$ in Definition 2.1.25 above), if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^2$. The space E is said to be smooth if*

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1.4}$$

exists for all $x, y \in S_E$. The space E is also said to be uniformly smooth if (2.1.4) converges uniformly in $x, y \in S_E$.

2.1.5 Some Distance Functions and Related Inequalities

Definition 2.1.27. *Let E be a smooth real Banach space with dual E^*. The functional, $\phi : E \times E \to \mathbb{R}$, defined by*

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \text{ for } x,y \in E, \tag{2.1.5}$$

where J is the normalized duality mapping on E will play a central role in the sequel. It was introduced by Alber and has been studied by Alber (1), Alber and Guerre-Delabriere (2), Kamimura and Takahashi (30), Reich (37), Chidume et al. (17; 18) and a host of other authors.

Lemma 2.1.28. *(1; 4) Let E be a real uniformly convex smooth Banach space. Then, the following identities hold:*

(i) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E.$

(ii) $\phi(x, y) + \phi(y, x) = 2\langle x - y, Jx - Jy\rangle, \quad \forall x, y \in E.$

Lemma 2.1.29. *(4) Let E be a real 2-uniformly convex Banach space. Then, there exists $\mu \geq 1$ such that*

$$\frac{1}{\mu}\|x - y\|^2 \leq \phi(x, y) \quad \forall x, y \in X.$$

Lemma 2.1.30. *(6) Let $A : E \to 2^{E^*}$ be a maximal monotone mapping and $B : E \to E^*$ be a Lipschitz continuous and monotone mapping. Then the mapping $A + B$ is a maximal monotone.*

Lemma 2.1.31. *(30) Let E be a uniformly convex and smooth Banach space, $\{x_n\}$ and $\{y_n\}$ be two sequence of E. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.1.32. *(3) Let E be a uniformly convex Banach space. Then*

$$\phi(x, y) \leq 8||x - y||^2 + c_1\rho_E(||x - y||)$$

where $c_1 = 8\max(L, R)$, $1 < L < 1.7$ is the Figiel constant and $\rho_E(\tau)$ modulus of smoothness of a Banach space E

2.2 Monotone Operators

Definition 2.2.1. *Let E be a real normed space. A map $A : E \to 2^{E^*}$ is called monotone if for each $x, y \in E$,*

$$\left\langle \eta - \nu, x - y \right\rangle \geq 0, \quad \forall \qquad \eta \in Ax, \qquad \nu \in Ay. \tag{2.2.1}$$

If A is single-valued, the map $A : E \to E$ is called monotone if

$$\left\langle Ax - Ay, x - y \right\rangle \geq 0, \quad \forall \quad x, y \in E. \tag{2.2.2}$$

The mapping $A : E \to E^*$ is called

- *γ-strongly monotone if there exists $\gamma \in (0, 1)$ such that*

$$\langle Ax - Ay, x - y\rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in E.$$

  In this case, we say that A is γ-strongly monotone;

- *coercive if there exists $\beta > 0$ such that if*

$$\langle Ax - Ay, x - y\rangle \geq \beta \|Ax - Ay\|^2 \quad \forall x, y \in X;$$

- *Lipschitz continuous if there exists a constant $L > 0$ such that*

$$\|Ax - Ay\| \leq L\|x - y\| \quad \forall \quad x, y \in E.$$

Lemma 2.2.2. *Let $A : E \to E^*$ be cocoercive operator. Then, A is $\frac{1}{\beta}$-Lipschitz continuous.*

Proof. Let $x, y \in E$. Since A is β-cocoercive, by Cauchy-Schwarz inequality we have

$$\begin{aligned} \beta||Ax - Ay||^2 &\leq \langle Ax - Ay, x - y\rangle \\ &\leq ||Ax - Ay||||x - y|| \end{aligned}$$

Hence,

$$||Ax - Ay|| \leq \frac{1}{\beta}||x - y||$$

Which expresses that A is $\frac{1}{\beta}$-Lipschitz continuous

Lemma 2.2.3. *(3) Let E be a uniformly convex Banach space. Then, the normalized duality mapping, J, is uniformly monotone on every bounded set. That is, for every $R > 0$ and arbitrary $x, y \in E$ with $\|x\| \leq R$ and $\|y\| \leq R$ there exists a real non-negative and continuous function $\psi_R : [0, \infty) \to [0, \infty)$ such that $\psi_R(t) > 0$ for $t > 0$, $\psi_R(0) = 0$ and*

$$\langle Jx - Jy, x - y\rangle \geq \psi_R(\|x - y\|).$$

Example 2.2.4. *Every nondecreasing function on $\mathbb{R}$ is monotone.*

To see this, let $f : \mathbb{R} \to \mathbb{R}$ be a non decreasing function. Then for arbitrary $x, y \in \mathbb{R}$ with $x \leq y$, we have $f(x) \leq f(y)$. Thus, we see that

$$\langle y - x, f(y) - f(x) \rangle \geq 0$$

for all $x, y \in \mathbb{R}$ which shows the monotonicity of f.

Example 2.2.5. *Let H be a real Hilbert space, $T : H \to H$ be a non-expansive map, i.e.,*

$$||Tx - Ty|| \leq ||x - y||$$

for all $x, y \in H$. Then the operator $I - T$ is monotone.

In fact, for $x, y \in H$, we have

$$\begin{aligned}
\langle x - y, (I - T)x - (I - T)y \rangle &= \langle x - y, x - Tx - y + Ty \rangle \\
&= \langle x - y, x - y + Ty - Tx \rangle \\
&= ||x - y||^2 - \langle x - y, Tx - Ty \rangle \\
&\geq ||x - y||^2 - ||x - y|| ||Tx - Ty|| \\
&= ||x - y||^2 - ||x - y||^2 \\
&= 0
\end{aligned}$$

Here, we have used Cauchy-Schwarz inequality and the fact that T is non-expansive. Thus, we have that $I - T$ that is monotone on H

2.2.1 Maximal Monotone Operator

Definition 2.2.6. *A multivalued monotone operator $A : E \to E^*$ is said to be maximal monotone if $A = B$ whenever $B : E \to 2^{E^*}$ is monotone and $G(A) \subset G(S)$, where $G(A) = \{(x, x^*) : x^* \in Ax\}$ is the graph of A.*

2.2.2 Resolvent Operators

Definition 2.2.7. *Let E be a real reflexive, strictly convex and smooth Banach space and let $A : E \to 2^{E^*}$ be a maximal monotone operator. Then for each $r > 0$ the resolvent of A, $J_r^A : E \to E$ is defined by*

$$J_r^A(x) = (J + rA)^{-1} Jx,$$

where J is the normalized duality mapping on E.

It is easy to show that $A^{-1}0 = F(J_r^B)$ for all $r > 0$, where $F(J_r^A)$ denotes the set of fixed points of J_r^A.

2.2.3 Subdifferential Operators

Definition 2.2.8. *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper and convex function. let $x \in D(f)$, then the subdifferential $\partial f(x)$ of f at x is the set*

$$\partial f(x) = \{x^* \in E^* : \langle y - x, x^* \rangle \leq f(y) - f(x) \quad \forall y \in E\}$$

We remarked here that if x is not in $D(f)$ then $\partial f(x) = \emptyset$

Chapter 3

Main Results

In this section, we propose an algorithm for approximating zeros of the sum of three monotone operators in 2-uniformly convex Banach spaces. We begin with the following results:

Lemma 3.0.1. *Let J be the normalized duality map from a Banach space E into its dual E^*, we define the mapping $T_\alpha : E \to E^*$ by*

$$T_\alpha = J_\alpha^A \circ J^{-1}(Jx - \alpha B(x) - \alpha Cx) \quad \forall x \in E$$

where $A : E \to 2^{E^*}$ is a maximal monotone operator and $B : E \to E^*$ be $\gamma-$ strongly monotone and L-Lipschitz and $C : E \to E$ be monotone and c-cocoercive. Then $x \in F(T_\alpha)$ if and only if $x \in (A + B + C)^{-}(0)$ i.e.

$$\begin{aligned} x \in \quad & Fix(T_\alpha) \\ \iff \quad & x = T_\alpha(x) \\ \iff \quad & Jx - \alpha B(x) - \alpha C(x) \in (J + \alpha A)(x) \\ \iff \quad & 0 \in (A + B + C)(x) \\ \iff \quad & x \in (A + B + C)^{-1}(0) \end{aligned}$$

Algorithm 3.0.2. ***(Three-Operator Splitting)***

Initialization: Choose two points x_{-1}, $x_0 \in E$, a real number $\epsilon > 0$ and we define a sequence of stepsizes $\{\alpha_k\} \subset (0, +\infty)$, such that:

$$\lim_{k\to\infty} \alpha_k = 0, \text{and} \sum_{k=1}^{\infty} \alpha_k = +\infty$$

Iterative Step: Assume that $x_{k-1}, x_k \in E$ are known, calculate x_{k+1} as follows:
Step 1. Compute
$x_{k+1} = J_{\alpha_k}^A \circ J^{-1}(Jx_k - \alpha_k Bx_k - \alpha_{k-1}(Bx_k - Bx_{k-1}) - \alpha_k Cx_k$,
where $A : E \to 2^{E^*}$ is a maximal monotone operator and $B : E \to E^*$ be $\gamma-$ strongly monotone

and L-Lipschitz and $C : E \to E$ be monotone and c-cocoercive.
Step 2: If $\max\{||x_{k+1} - x_k||, ||x_k - x_{k-1}||\} \leq \epsilon$
Else, set $k = k + 1$ and then go to **Step 1**

It is easily seen that the implementation of Algorithm (3.0.2) is not difficult, since the stepsizes were previously chosen only with the fact that $\lim_{k\to\infty} \alpha_k = 0$, and $\sum_{k=1}^{\infty} \alpha_k = +\infty$ are satisfied . Whereas, the results obtained in (25; 33; 42), the condition imposed on the stepsizes is independent of the characteristic constants of operator as the Lipschitz constant, the strongly monotone constant, and the cocoercive constant. Hence, these constants are not the input parameters of Algorithm (3.0.2). It is particularly interesting, since those constants are often unknown, or even for nonlinear problems, they are difficult to estimate. Therefore, for the implementation of Algorithm (3.0.2), we first remark that, if for some $k \geq 1$, the following equalities are satisfied

$$x_{k+1} = x_k = x_{k-1} \tag{3.0.1}$$

then x_{n+1} is a solution of problem (1.1.1) and the following holds

$$x_p = x_{k+1}, \quad \forall p \geq k + 2. \tag{3.0.2}$$

Accordingly, from relation (3.0.1) and the definition of x_{k+1}, we obtain

$$x_{k+1} = J^A_{\alpha_k} \circ J^{-1}(Jx_{k+1} - \alpha_k(B(x_{k+1} + Cx_{k+1})))$$

Thus, $x_{k+1} \in Fix(T_{\alpha_k})$, which follows from Remark 2.1 that $x_{k+1} \in (A + B + C)^{-1}(0)$. Now, we prove conclusion (3.0.2). From definition of x_{k+2} and the fact $x_k = x_{k+1}$, we see that

$$\begin{aligned} x_{k+1} &= J^A_{\alpha_k} \circ J^{-1}(Jx_{k+1} - \alpha_{k+1}B(x_{k+1}) - \alpha_k(B(x_{k+1}) - B(x_k)) - \alpha_{k+1}) \\ &= J^A_{\alpha_k} \circ J^{-1}(Jx_{k+1} - \alpha_{k+1}(B(xk + 1) + Cx_{k+1})) \end{aligned} \tag{3.0.3}$$

Since $x_{k+1} = x_k = x_{k-1}$,

$$x_{k+1} = J^A_{\alpha_k} \circ J^{-1}(Jx_{k+1} - \alpha_{k+1}(B(x_{k+1}) + Cx_{k+1})). \tag{3.0.4}$$

Combining relations (3.0.3) and (3.0.4), we have

$$x_{k+2} = x_{k+1}.$$

Thus, by induction, we conclude that $x_p = x_{k+1}$ for all $p \geq k + 2$.
Furthermore, we assume that condition (3.0.1) does not hold. This means that the sequence x_k generated by Algorithm (3.0.2) is infinite. In this case, we have the following main result.

Theorem 3.0.3. *Let E be a real 2-uniformly convex Banach space. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $B : E \to E^*$ be $\gamma-$ strongly monotone and L-Lipschitz and $C : E \to E$ be monotone and c-cocoercive . Then, the sequence x_n generated by Algorithm* (3.0.2) *converges strongly to the unique solution of x^* of problem* (1.1.1)

Proof

Let $x^* \in (A+B+C)^{-1}(0)$. then from Algorithm (3.0.2) we have that

$$z_k = (Jx_k - \alpha_k Jx_{k+1} - \alpha_k Bx_k - \lambda_{k-1}(Bx_k - Bx_{k-1}) - Jx_{k+1}) \in \alpha_k Ax_{k+1} \quad \text{and} \tag{3.0.5}$$

$$z^* = -\alpha_k B(x^*) - \alpha_k C(x^*) \in \alpha_k A(x^*) \tag{3.0.6}$$

Now, it follows from monotonicity of A, (3.0.5) and (3.0.6) that

$$\langle z_k - z^*, x_{k+1} - x^* \rangle \tag{3.0.7}$$

i.e.,

$$\begin{aligned} \langle Jx_k - Jx_{k+1}, x_{k+1} - x\rangle &- \alpha_k\langle B(x_k) - B(x^*), x_{k+1} - x^*\rangle - \alpha_{k-1}(Bx_k) - Bx_{k-1}, x_{k+1} - x^*\rangle \\ &- \alpha_k\langle Cx_k - Cx^*, x_{k+1} - x^*\rangle \geq 0 \end{aligned} \tag{3.0.8}$$

Thus, using $\gamma-$strongly monotonicity of B, we obtain,

$$\begin{aligned} 0 &\leq 2\langle Jx_k - Jx_{k+1}, x_{k+1} - x^*\rangle - 2\alpha_k\langle B(x_k) - B(x^*), x_{k+1} - x^*\rangle \\ &+ 2\alpha_{k-1}(Bx_{k-1}) - Bx_k, x_{k+1} - x^*\rangle - 2\alpha_k\langle Cx_{k+1} - Cx^*, x_{k+1} - x^*\rangle \\ &= 2\langle Jx_k - Jx_{k+1}, x_{k+1} - x^*\rangle - 2\alpha_k\langle Bx_k - Bx_{k+1}, x_{k+1} - x^*\rangle \\ &- 2\alpha_k\langle Bx_{k+1} - Bx^*, x_{k+1} - x^*\rangle + 2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_{k+1} - x_k\rangle \\ &+ 2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_k - x^*\rangle - 2\alpha_k\langle Cx_k - Cx^*, x_{k+1} - x^* > \\ 0 &\leq 2\langle Jx_k - Jx_{k+1}, x_{k+1} - x^*\rangle - 2\alpha_k\langle Bx_k - Bx_{k+1}, x_{k+1} - x^*\rangle \\ &- 2\gamma\alpha_k||x_{k+1} - x^*||^2 + 2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_k - x^*\rangle + 2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_{k+1} - x_k\rangle \\ &- 2\alpha_k\langle Cx_k - Cx^*, x_{k+1} - x^*\rangle \end{aligned} \tag{3.0.9}$$

Now, using the fact that $\langle x - z, Jz - Jy\rangle = \dfrac{1}{2}(\phi(x,y) - \phi(x,z) - \phi(z,y))$, and the $L-$ Lipschitz property of B, we have

$$\begin{aligned} 2\gamma\alpha_k||x_{k+1} - x^*||^2 &\leq \phi(x_k, x^*) - \phi(x_{k+1}, x_k) - \phi(x^*, x_{k+1}) - 2\alpha_k\langle Bx_k - Bx_{k+1}, x_{k+1} - x^*\rangle \\ &+ 2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_k - x^*\rangle + 2\alpha_{k-1}L||x_{k-1} - x_k||||x_{k+1} - x_k|| \\ &- 2\alpha_k\langle Cx_k - Cx^*, x_{k+1} - x^*\rangle \end{aligned} \tag{3.0.10}$$

Now, using the c-cocoercitivity of C, we obtain

$$\begin{aligned}
2\langle Cx_k - Cx^*, x_{k+1} - x^*\rangle &= 2\langle Cx_k - Cx^*, x_{k+1} - x_k\rangle + 2\langle Cx_n - Cx^*, x_k - x^*\rangle \\
&\geq -2||Cx_k - Cx^*||||x_k - x^*|| + 2c||Cx_k - Cx^*||^2 \\
&\geq -2c||Cx_k - Cx^*||^2 - \frac{1}{2c}||x_{k+1} - x_k||^2 + 2c||Cx_k - Cx^2||^2 \\
&= -\frac{1}{2c}||x_{k+1} - x_k||^2 \qquad (3.0.11)
\end{aligned}$$

Combining (3.0.10) and (3.0.11), we obtain,

$$\begin{aligned}
2\gamma\alpha_n||x_{k+1} - x^*||^2 &\leq \phi(x_k, x^*) - \phi(x_{k+1}, x_k) - \phi(x^*, x_{k+1}) - 2\alpha_k\langle Bx_k - Bx_{k+1}, x_{k+1} - x^*\rangle \\
&+ 2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_k - x^*\rangle + 2\alpha_{k-1}L||x_{k-1} - x_k||||x_{k+1} - x_k|| \\
&+ \frac{\alpha_k}{2c}||x_{k+1} - x_k||^2
\end{aligned}$$

$$\implies 2\gamma\alpha_k||x_{k+1} - x^*||^2 \leq \phi(x_k, x^*) - \phi(x_{k+1}, x_k) - \phi(x^*, x_{k+1}) - 2\alpha_k\langle Bx_k - Bx_{k+1}, x_{k+1} - x^*\rangle$$
$$+2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_k - x^*\rangle + \alpha_{k-1}L||x_{k-1} - x_k||^2 + \alpha_{k-1}L||x_{k+1} - x_k||^2 + \frac{\alpha_k}{2c}||x_{k+1} - x_k||^2$$

Using Lemma 2.1.29 , we have

$$\begin{aligned}
2\gamma\alpha_k||x_{k+1} - x^*||^2 &\leq \phi(x_k, x^*) - \phi(x_{k+1}, x_k) - \phi(x^*, x_{k+1}) - 2\alpha_k\langle Bx_k - Bx_{k+1}, x_{k+1} - x^*\rangle \\
&+ 2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_k - x^*\rangle + \langle_{k-1}\mu L\phi(x_k, x_{k-1}) + \alpha_{k-1}L\mu\phi(x_{k+1} - x_k) \\
&+ \frac{\alpha_k}{2c}\phi(x_{k+1} - x_k)
\end{aligned}$$

Thus we get,

$$\begin{aligned}
2\gamma\alpha_k||x_{k+1} - x^*||^2 &+ \phi(x_{k+1}, x^*) + 2\alpha_k\langle Bx_k - Bx_{k+1}, x_{k+1} - x^*\rangle \\
&+ (1 - \alpha_{k-1}\mu L - \frac{\alpha_k\mu}{2c})\phi(x_{k+1}, x_k) \\
&\leq \phi(x_k, x^*) \\
&+ 2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_k - x^*\rangle + \alpha_{k-1}\mu L\phi(x_k, x_{k-1}) \quad (3.0.12)
\end{aligned}$$

Now, Let,

$$\mathit{\Gamma}_k = \phi(x^*, x_k) + 2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_k - x^*\rangle + \alpha_{k-1}\mu L\phi(x_k, x_{k-1})$$

Then, (3.0.12) becomes:
$2\gamma\alpha_k||x_{k+1} - x^*||^2 + (1 - \alpha_{k-1}\mu L - \frac{\alpha_k\mu}{2c} - \alpha_k L)\phi(x_{k+1}, x_k) \leq \mathit{\Gamma}_k - \mathit{\Gamma}_{k+1}$
Now, using Lipschitz continuity of B, we have

$$\begin{aligned}
\mathit{\Gamma}_k &= \phi(x^*, x_k) + 2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_k - x^*\rangle + \lambda_{k-1}\mu L\phi(x_k, x_{k-1}) \\
&\geq \phi(x^*, x_k) + \alpha_{k-1}L\mu\phi(x_k, x_{k-1}) - 2\alpha_{k-1}\langle Bx_{k-1} - Bx_k, x_k - x^*\rangle \\
&\geq \phi(x^*, x_k) + \alpha_{k-1}L\mu\phi(x_k, x_{k-1}) - \alpha_{k-1}||x_{k-1} - x_k||^2 - \alpha_{k-1}||x_k - x^*||^2 \\
&\geq \phi(x^*, x_k) + \alpha_{k-1}L\mu\phi(x_k, x_{k-1}) - \alpha_{k-1}L\mu\phi(x_{k-1}, x_k) - \alpha_{k-1}L\mu\phi(x^*, x_k) \\
&= \phi(x^*, x_k) - \alpha_{k-1}L\mu\phi(x^*, x_k) \\
&= (1 - \alpha_{k-1}L\mu)\phi(x^*, x_k) \qquad (3.0.13)
\end{aligned}$$

Let $\delta \in (0,1)$ be fixed, since $\alpha_k \to 0$ as $k \to \infty$, we find that

$$\lim_{k\to\infty}(1-\alpha_{k-1}L\mu - \frac{\alpha_k\mu}{2c} - \alpha_k L) = 1 > \delta$$

Thus, there exists $n_0 \geq 1$, such that

$$(1-\alpha_{k-1}L\mu - \frac{\alpha_k\mu}{2c} - \alpha_k L) \geq \delta \tag{3.0.14}$$

In addition, we have,

$$1-\alpha_{k-1}L\mu \geq \delta, \quad \forall n \geq k_0 \geq k_0 \tag{3.0.15}$$

So, it follows from (3.0.13) and (3.0.15) that the sequence $\{\Gamma_n\}_{n\geq n_o}$ is non-negative. And the relation (3.0.12) and (3.0.14) ensure that

$$2\alpha_k\gamma||x_{k+1}-x^*||^2 + \delta\phi(x_{k+1},x_k) \leq \mathit{\Gamma}_k - \mathit{\Gamma}_{k+1} \quad \forall n \geq n_0 \tag{3.0.16}$$

Thus, the sequence $\{\mathit{\Gamma}_k\}_{k\geq k_0}$ is non-decreasing. Consequently, the limit of $\{\mathit{\Gamma}_k\}$ exists. Thus, from (3.0.13), we conclude that the sequence x_k is bounded. For $K \geq k_0$ we obtain from (3.0.16) that

$$\sum_{k=k_0}^{\infty}[2\alpha_k\gamma||x_{k+1}-x^*||^2 + \delta\phi(x_{k+1},x_k)] \leq \mathit{\Gamma}_{k_0} - \mathit{\Gamma}_{K+1} \leq \mathit{\Gamma}_{k_0} \tag{3.0.17}$$

Therefore,

$$\sum_{k=k_0}^{\infty}[2\alpha_k\gamma||x_{k+1}-x^*||^2 + \delta\phi(x_{k+1},x_k)] \leq \mathit{\Gamma}_{k_0} - \mathit{\Gamma}_{K+1} \leq \mathit{\Gamma}_{k_0} < \infty \tag{3.0.18}$$

and using Lemma 2.1.32 we obtain

$$\sum_{k=k_0}^{\infty}[2\alpha_k\gamma\phi(x^*,x_{k+1}) + \delta\phi(x_{k+1},x_k)] \leq \mathit{\Gamma}_{k_0} - \mathit{\Gamma}_{K+1} \leq \mathit{\Gamma}_{k_0} < \infty \tag{3.0.19}$$

This implies that

$$\lim_{k\to\infty}\phi(x_{k+1},x_k) = 0 \tag{3.0.20}$$

and

$$\sum_{k=k_0}^{\infty}\alpha_k\phi(x^*,x_{k+1}) < \infty \tag{3.0.21}$$

It follows from relation (3.0.21) and the fact that $\sum_{k=k_0}^{\infty}\alpha_k = +\infty$ that

$$\liminf_{k\to\infty}\phi(x^*,x_{k+1}) = 0 \tag{3.0.22}$$

From the relation (3.0.20), the boundedness of $\{x_k\}$, the Lipschitz continuity of B and the fact that $\alpha_k \to 0$, we have

$$\lim_{k\to\infty}[\alpha_k L\mu\phi(x_{k+1},x_k) + 2\alpha_k\langle Bx_k - Bx_{k+1}, x_{k+1-x^*}\rangle] = 0$$

Thus, from the definition of Γ_{k+1}, we obtain that $\lim_{k\to\infty} \Gamma_{k+1} = \lim_{k\to\infty} \phi(x^*, x_{k+1})$. Hence, the limit of $\{\phi(x^*, x_{k+1})\}$ exists. Therefore, from (3.0.22), we can conclude that $\lim\limits_{k\to\infty} \phi(x^*, x_{k+1}) = 0$. This complete the proof.

Corollary 3.0.4. *Let E be a real 2-uniformly convex Banach space. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $C : E \to E$ be monotone and c-cocoercive . Then, the sequence x_k generated by Algorithm* (3.0.2) *converges strongly to the unique solution x^* of problem* (1.1.1).

We now state Theorem 3.0.3 in a real Hilbert space

Corollary 3.0.5. *Let H be a real Hilbert space, $A : H \to 2^H$ be a maximally monotone, let $B : H \to H$ be a monotone and Lipschitzian and $C : H \to H$ be a cocoercive. then the sequence $\{x_k\}$ generated by Algorithm 3.0.2 converges in norm to the unique solution x^* of problem* (1.1.1).

Chapter 4

Application

We now give two applications in the framework of Hilbert spaces and general Banach spaces. Let K be a nonempty closed convex subset of H; $B : H \to H$ be a strongly monotone and Lipschitz continuous operator; and $C : H \to H$ be cocoercive operator.

- **3-objective minimization problem** The most straight-forward example of problem (1.1.1) arises from the optimization problem (3-objective minimization problem) of the form:
$$\underset{x}{\text{minimize}}(f(x) + g(x) + h(x)), \tag{4.0.1}$$
where f, g, h are proper, closed and convex functions and h is Lipschitz differentiable. The first-order optimality condition of (4.0.1) reduces to the sum of three monotone operators as in problem (1.1.1) by setting
$$Ax = \partial f(x), \quad Bx = \partial g(x) \;\; and \quad Cx = \nabla h(x) \tag{4.0.2}$$
where $\partial f, \partial g$ are subdifferentials of f and g respectively, we also note here that C is cocoercive because h is Lipschitz-differentiable. Then the following result can be obtained immediately

  Theorem 4.0.1. *Let E be a real 2-uniformly convex Banach space. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $B : E \to E^*$ be $\gamma-$ strongly monotone and L-Lipschitz and $C : E \to E$ be monotone and c-cocoercive . Define the sequence x_k by the iterative scheme*
$$x_{k+1} = J^A_{\alpha_k} \circ J^{-1}(Jx_k - \alpha_k Bx_k - \alpha_{k-1}(Bx_k - Bx_{k-1}) - \alpha_k Cx_k,$$
where $\alpha_n \subseteq (0, \infty)$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty, \lim_{n\to\infty} \alpha_n = 0$ generated by Algorithm (3.0.2) converges strongly to the minimizer of (4.0.1)

- **Variational inequality problems** A monotone variational inequality problem (VIP) for two-operators sum form is formulated as the problem of finding a point $x \in K$ such that:
$$\langle (B + C)(x), y - x\rangle \geq 0, \forall y \in K \tag{4.0.3}$$

Let N_K be the normal cone of K, then problem (4.0.3) is equivalent to the sum of three monotone inclusion problem as shown below:

$$\text{find } x \in K \text{ such that } \ 0 \in (N_K + B + C)(x)$$

Let $A = N_K$ in Corollary 3.0.5, we find that $J_{\alpha_n} = J^A_{\alpha_n} = (I + \alpha_n N_K)^{-1} = P_K$. Then the following result can be obtained immediately

Theorem 4.0.2. *Let H be a real Hilbert space. Let $B : H \to H$ be strongly monotone and Lipschitz continuous and $C : H \to H$ be cocoercive operator. Define a sequence by the iterative scheme,*

$$x_{n+1} = P_K(x_n - \alpha_n B(x_n) - \alpha_{n-1}(B(x_n) - B(x_{n-1})) - \alpha_n C(x_n), \quad n \geq 0 \tag{4.0.4}$$

where $\alpha_n \subseteq (0, \infty)$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty, \lim_{n\to\infty} \alpha_n = 0$ Then the sequence $\{x_n\}$ generated by (4.0.4) converges strongly to a solution of problem (4.0.3).

Chapter 5

Conclusion

In this research work, we have introduced an explicit three-operator splitting algorithm for solving the sum of three monotone operators in a 2-uniformly convex Banach space. The algorithm uses a sequence of stepsizes which are non-summable and diminishing. The chosen stepsizes are independent with the Lipschitz constant, the modulus of strong monotonicity, and the cocoercive constant of each individual operator. This allows the algorithm to be easily implemented without previously knowing (or estimating) those constants. The strong convergence of the iterative sequence generated by the new algorithm is established and we also provide few applications to demontrate the relevance of our proposed iterative scheme.
We also recommend here that in a further resarch work, some of the assumptions such as the cocoercivity could be drop.

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