ITERATIVE ALGORITHMS AND EXISTENCE THEOREMS FOR SOLUTIONS OF NONLINEAR EQUATIONS IN BANACH SPACES

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DISSERTATION

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By

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CERTIFICATE OF APPROVAL

Ph.D. Thesis

This is to certify that the Ph.D. thesis of

Ma'aruf Shehu Minjibir

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics, this day, 28 December, 2013

Thesis Committee: ______________________

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DEDICATION

To my parents, wife and daughter.

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All praise be to God the Almighty, the Lord of the worlds and the Master of the day of judgement. Peace and blessings of God be unto His favourite servant and prophet, his companions, his family members and those who follow their righteous way till the day of judgement.

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ABSTRACT

- Let E be a q-uniformly smooth real Banach space and D be a nonempty, open and convex subset of E. Assume that $T : \overline{D} \to CB(\overline{D})$ is a multi-valued continuous (with respect to the Hausdorff metric), bounded and pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_1 \in \overline{D}$ by

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n u_n - \lambda_n\theta_n(x_n - x_1),\ u_n \in Tx_n,$$

  where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions: (i) $\{\theta_n\}$ decreases to 0; (ii) $\lambda_n(1+\theta_n) < 1, \sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$, $\lambda_n^{q-1} = o(\theta_n)$; (iii) $\limsup_{n\to\infty} \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} \leq 0$, $\sum_{n=1}^{\infty}\lambda_n^q < \infty$. Then, there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n^{q-1} < \gamma_0\theta_n$ for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

- Let K be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T : K \to CB(K)$ is a multi-valued k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n,$$

  where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

- Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Suppose that T :

$K \to CB(K)$ is a multi-valued k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$ and such that $Tp = \{p\}$ for all $p \in F(T)$. For arbitrary $x_1 \in K$ and $\lambda \in (0, \mu)$ with $\mu := \min\left\{1, \left(\frac{qk^{q-1}}{d_q}\right)^{\frac{1}{q-1}}\right\}$, let $\{x_n\}$ be a sequence defined iteratively by

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n,$$

where $y_n \in Tx_n$. Then, $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$.

- Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E. Suppose that $T : K \to CB(K)$ is a multi-valued quasi-nonexpansive mapping such that $Tp = \{p\}$ for some $p \in F(T)$. Then for any fixed $x_0 \in K$ and arbitrary $\lambda \in (0, 1)$, define a sequence $\{x_n\}$ by

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \quad n \geq 0,$$

where $y_n \in Tx_n$. Then, $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$.

- Let H be a real Hilbert space, $K : D(K) \subset H \to H$, $F : D(F) \subset H \to H$ be two Lipschitz monotone mappings such that $D(F)$ and $D(K)$ are closed, convex, bounded and $R(F) \subset D(K)$. Let $E^H := H \times H$ and let $A : D(A) \subset E^H \to E^H$ be a mapping such that $D(F) \times D(K) = D(A)$ and $A[u, v] = [Fu - v, Kv + u]$, $[u, v] \in D(A)$. Suppose that

$$\lim_{\lambda\to 0^+} \frac{\text{dist}(w - \lambda Aw, D(A))}{\lambda} = 0 \text{ for} \quad \text{all } w \in D(A).$$

Then, the Hammerstein equation $u + KFu = 0$ has a solution.

- Let $K : D(K) \subset L^p \to L^p$, $F : D(F) \subset L^p \to L^p$ be two Lipschitz mappings satisfying the following conditions: (a) there exists $\alpha > 0$ such that for each $u_1, u_2 \in D(F)$,

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \alpha||u_1 - u_2||^2;$$

(b) there exists $\beta > 0$ such that for each $u_1, u_2 \in D(K)$,

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \beta||u_1 - u_2||^2.$$

Let $D(F)$ and $D(K)$ be closed, convex, bounded such that $R(F) \subset D(K)$. Let $E := L^p \times L^p$ and let $A : D(A) \subset E \to E$ be a mapping such that $D(F) \times D(K) = D(A)$ and $A[u,v] = [Fu - v, Kv + u], \quad \forall [u,v] \in D(A)$. Suppose that

$$\lim_{\lambda \to 0^+} \frac{\text{dist}(w - \lambda Aw, D(A))}{\lambda} = 0 \text{ for} \quad \text{all } w \in D(A).$$

Let $\gamma := \min\{\alpha, \beta\}$. If $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$ or $1 + \sqrt{1 - \gamma} < p \leq 2$, then the Hammerstein equation $u + KFu = 0$ has a solution.

- Let $K : D(K) \subset L^p \to L^p,\ F : D(F) \subset L^p \to L^p$ be two continuous accretive mappings satisfying the following conditions: (a) there exists $\alpha > 0$ such that for each $u_1, u_2 \in D(F)$,

$$\langle Fu_1 - Fu_2, j(u_1 - u_2) \rangle \geq \alpha ||u_1 - u_2||^2;$$

(b) there exists $\beta > 0$ such that for each $u_1, u_2 \in D(K)$,

$$\langle Ku_1 - Ku_2, j(u_1 - u_2) \rangle \geq \beta ||u_1 - u_2||^2.$$

Let $D(F)$ and $D(K)$ be closed, convex, such that $R(F) \subset D(K)$. Let $E := L^p \times L^p$ and let $A : D(A) \subset E \to E$ be a mapping such that $D(F) \times D(K) = D(A)$ and $A[u,v] = [Fu - v, Kv + u]$ for $[u,v] \in D(A)$. Suppose that $\langle Aw, J(w) \rangle \geq 0$ for all $w \in D(A)$ with $\|w\| \geq R$ or $\lim \|Aw\| = \infty$ as $\|w\| \to \infty$ and suppose

$$\lim_{\lambda \to 0^+} \frac{\text{dist}(w - \lambda Aw, D(A))}{\lambda} = 0 \text{ for} \quad \text{all } w \in D(A).$$

Let $\gamma := \min\{\alpha, \beta\}$. If $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$ or $1 + \sqrt{1 - \gamma} < p \leq 2$, then the Hammerstein equation $u + KFu = 0$ has a solution.

- Let H be a real Hilbert space, $K : H \to H,\ F : H \to H$ be two Lipschitz strongly monotone mappings with constants α, β, respectively. Let $A : E^H \to E^H$ be a mapping defined by $A[u,v] = [Fu - v, Kv + u]$. Then, the Hammerstein equation $u + KFu = 0$ has a solution.

- Let $K : L^p \to L^p,\ F : L^p \to L^p$ be two Lipschitz mappings satisfying mappings satisfying the following conditions: (a)there exists $\alpha > 0$ such that for each $u_1, u_2 \in L^p$,

$$\langle Fu_1 - Fu_2, j(u_1 - u_2) \rangle \geq \alpha ||u_1 - u_2||^2;$$

there exists $\beta > 0$ such that for each $u_1, u_2 \in L^p$,

$$\langle Ku_1 - Ku_2, j(u_1 - u_2) \rangle \geq \beta ||u_1 - u_2||^2.$$

Let $E := L^p \times L^p$ and let $A : E \to E$ be a mapping defined by $A[u,v] = [Fu - v, Kv + u]$. Let $\gamma := \min\{\alpha, \beta\}$. If $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$ or $1 + \sqrt{1 - \gamma} < p \leq 2$, then, the Hammerstein equation $u + KFu = 0$ has a solution.

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- C. E. Chidume, C. O. Chidume, N. Djitté and M. S. Minjibir; *Iterative algorithm for fixed points of multi-valued pseudo-contractive mappings in Banach spaces*, (Accepted for publication, June 2013), to appear in: **Journal of Nonlinear and Convex Analysis**.
- C. E. Chidume and M. S. Minjibir; *Krasnoselskii algorithm for fixed points of multi-valued quasi-nonexpansive mappings in certain Banach Spaces*, (Accepted for publication), to appear in: **Fixed Point Theory**.
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Contents

CHAPTER 1

General Introduction

1.1 Introduction

The contributions of this thesis fall within the general area of nonlinear functional analysis. Within this area, our attention is focused on *two* important topics. They are:

1. Approximation of fixed points of pseudo-contractive-type and quasi-nonexpansive-type multi-valued mappings.
2. Existence of solutions of nonlinear Hammerstein equations.

1.1.1 Motivation for study of multi-valued mappings

Let (X, d) be a metric space. Let $K \subseteq X$, and let $T : K \to 2^X$ be a multi-valued mapping. An element $x \in K$ is called a *fixed point of* T if $x \in Tx$. For a single-valued mapping T, this reduces to $Tx = x$.

For several years, the study of fixed point theory for *multi-valued nonlinear mappings* has attracted, and continues to attract, the interest of several well known mathematicians (see, for example, Brouwer [8], Kakutani [47], Nash [64, 63], Geanakoplos [39], Nadler [62], Downing and Kirk [35]).

Interest in such studies stems, perhaps, mainly from the usefulness of such fixed point theory in real-world applications. We describe briefly the connection of fixed point theory for multi-valued mappings with some of these real-world applications, namely *Game Theory* and *Non-Smooth Differential Equations*.

Application to Game Theory

In game theory, the existence of equilibrium was uniformly obtained by the application of a fixed point theorem. In fact, Nash [64, 63] showed the existence of equilibria for non-cooperative static games as a direct consequence of Brouwer [8] or Kakutani [47] fixed point theorem. More precisely, under some regularity conditions, given a game, there always exists a *multi-valued mapping* whose fixed points coincide with the equilibrium points of the game. A model example of such an application is the *Nash equilibrium theorem* (see, *e.g.,* [64]).

Consider a game $G = (u_n, K_n)$ with N players denoted by n, $n = 1, \cdots, N$, where $K_n \subset \mathbb{R}^{m_n}$ is the set of possible strategies of the n'th player and is assumed to be nonempty, compact and convex and $u_n : K := K_1 \times K_2 \cdots \times K_N \to \mathbb{R}$ is the payoff (or gain function) of the player n and is assumed to be continuous. The player n can take *individual actions*, represented by a vector $\sigma_n \in K_n$. All players together can take a *collective action*, which is a combined vector $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$. For each n, $\sigma \in K$ and $z_n \in K_n$, we will use the following standard notations:

$$K_{-n} := K_1 \times \cdots \times K_{n-1} \times K_{n+1} \times \cdots \times K_N,$$

$$\sigma_{-n} := (\sigma_1, \cdots, \sigma_{n-1}, \sigma_{n+1}, \cdots, \sigma_N),$$

$$(z_n, \sigma_{-n}) := (\sigma_1, \cdots, \sigma_{n-1}, z_n, \sigma_{n+1}, \cdots, \sigma_N).$$

A strategy $\bar{\sigma}_n \in K_n$ permits the n'th player to maximize his gain *under the condition* that the *remaining players* have chosen their strategies σ_{-n} if and only if

$$u_n(\bar{\sigma}_n, \sigma_{-n}) = \max_{z_n \in K_n} u_n(z_n, \sigma_{-n}).$$

Now, let $T_n : K_{-n} \to 2^{K_n}$ be the multi-valued mapping defined by

$$T_n(\sigma_{-n}) := \operatorname*{Arg\,max}_{z_n \in K_n} u_n(z_n, \sigma_{-n}) \;\forall\, \sigma_{-n} \in K_{-n}.$$

Definition 1.1.1 *A collective action $\bar{\sigma} = (\bar{\sigma}_1, \cdots, \bar{\sigma}_N) \in K$ is called a Nash equilibrium point if, for each n, $\bar{\sigma}_n$ is the best response for the n'th player to the action $\bar{\sigma}_{-n}$ made by the remaining players. That is, for each n,*

$$u_n(\bar{\sigma}) = \max_{z_n \in K_n} u_n(z_n, \bar{\sigma}_{-n}) \tag{1.1.1}$$

or equivalently,

$$\bar{\sigma}_n \in T_n(\bar{\sigma}_{-n}). \tag{1.1.2}$$

This is equivalent to $\bar{\sigma}$ *is a fixed point of the multi-valued mapping* $T : K \to 2^K$ *defined by*

$$T(\sigma) := T_1(\sigma_{-1}) \times T_2(\sigma_{-2}) \times \cdots \times T_N(\sigma_{-N}).$$

From the point of view of social recognition, game theory is perhaps the most successful area of application of fixed point theory of multi-valued mappings. In fact, one of the ideas for which John N. Nash was given a Nobel Prize in Economic Sciences (shared with John C. Harsanyi and Reinhard Selten) in the year 1994, was the application of a *multi-valued fixed-point theorem* to establish an existence of equilibrium of a game. However, it has been remarked that the applications of this theory to equilibrium are mostly *static*: they enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a non-equilibrium point and convergent to equilibrium solution. This is part of the problem that is being addressed by *iterative methods for fixed point of multi-valued mappings*.

Application to Non-smooth Differential Equations

The mainstream of applications of fixed point theory for multi-valued mappings has been initially motivated by the problem of differential equations (DEs) with discontinuous right-hand sides which gave birth to the existence theory of differential inclusion (DIs). Here is a simple model for this type of application.

For a start, consider the following differential equation.

$$\begin{cases} \dfrac{dy}{dt} & = & f(t), \\ y(0) & = & t_0, \end{cases} \tag{1.1.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) = \sin([x]),$$

$[\cdot]$ is the greatest integer function. Clearly f does not satisfy the Darboux property of the derivatives on any interval containing an integer. So the differential equation cannot have a solution in the classical sense.

∘ Consider the initial value problem

$$\frac{du}{dt} = f(t,u), \ a.e.\, t \in I := [-a,a], \ u(0) = u_0, \tag{1.1.4}$$

a, u_0 fixed in $\mathbb{R}$. If $f : I \times \mathbb{R} \to \mathbb{R}$ is discontinuous with bounded jumps, measurable in t, one looks for *solutions* in the sense of Filippov [38] which are solutions of the differential inclusion

$$\frac{du}{dt} \in F(t,u), \ a.e.\, t \in I, \ u(0) = u_0, \tag{1.1.5}$$

where

$$F(t,x) = [\liminf_{y \to x} f(t,y), \ \limsup_{y \to x} f(t,y)]. \tag{1.1.6}$$

Now set $H := L^2(I)$ and let $N_F : H \to 2^H$ be the *multi-valued Nemystkii operator* defined by

$$N_F(u) := \{v \in H \ : \ v(t) \in F(t, u(t)) \ a.e.\, t \in I\}.$$

Finally, let $T : H \to 2^H$ be the multi-value mapping defined by $T := L^{-1} o N_F$, where L^{-1} is the inverse of the derivative operator $Lu = u'$ given by

$$L^{-1}v(t) := u_0 + \int_0^t v(s)ds.$$

One can see that problem (1.1.5) reduces to the fixed point problem: $u \in Tu$.

∘ Consider the following differential inclusion: Let $\Omega = (0, \pi)$. Consider the multi-valued initial value problem:

$$\begin{cases} -\dfrac{d^2u}{dt^2} & \in & u - \dfrac{1}{4} - \dfrac{1}{4}\text{sign}(u-1) \text{ on } \Omega; \\ u(0) & = & 0; \\ u(\pi) & = & 0, \end{cases} \tag{1.1.7}$$

where for $x \in \mathbb{R}$,

$$\text{sgn}(x) := \begin{cases} 1, & \text{if } \ x > 0; \\ [-1, 1], & \text{if } \ x = 0; \\ -1, & \text{if } \ x < 0. \end{cases}$$

Let $H := H_0^1(\Omega)$ and let $\langle \cdot, \cdot \rangle_H$ and denote the gradient inner product on $H^1(\Omega)$, i.e., for each $u, v \in H^1(\Omega)$,

$$\langle u, v \rangle_H = \int_\Omega u'v'dt.$$

Then, by Riesz Representation Theorem, there exists a unique vector $Au \in H$ such that

$$\langle Au, v\rangle_H = \int_\Omega u'v'dt \quad \text{for all } v \in H.$$

Let $u \in H$ and set $f(u) = u - \frac{1}{4} - \frac{1}{4}\text{sign}(u-1)$. For a $w \in f(u)$, let

$$L_u^w : H \to \mathbb{R}$$

be the linear functional defined by

$$L_u^w(v) = \int_\Omega wvdt, \ \ v \in H.$$

One can easily see that L_u^w is a bounded linear functional from H to $\mathbb{R}$. So, by Riesz Representation Theorem, again, there exists a unique vector $b_u^w \in H$ such that

$$\langle b_u^w, v\rangle_H = L_u^w(v) = \int_\Omega wvdt, \ \ v \in H.$$

Define a multi-valued map $B : H \to 2^H$ by

$$Bu = \{b_u^w : w \in f(u)\}.$$

Then, defining a multi-valued map

$$T : H \to 2^H$$

by

$$Tu = u - Bu + Au,$$

one can see that

$$u \in Tu \Leftrightarrow u \text{ solves the inclusion (1.1.7)}.$$

• Monotone Operators

Given a real Banach space E, a mapping $A : D(A) \subseteq E \to 2^E$ is called *accretive* if for all $x, y \in D(A)$ and for all $u \in Ax, v \in Ay$ the following inequality holds for some $j(x-y) \in J(x-y)$:

$$\langle u - v, j(x-y)\rangle \geq 0. \tag{1.1.8}$$

In a real Hilbert space H, a mapping $A : D(A) \subseteq H \to 2^H$ is called *monotone* if for all $x, y \in D(A)$ and for all $u \in Ax, v \in Ay$ the following inequality holds:

$$\langle u - v, x - y\rangle \geq 0. \tag{1.1.9}$$

Consider the differential inclusion

$$0 \in \frac{du}{dt} + Au(t). \tag{1.1.10}$$

When A is single-valued, the inclusion (1.1.10) reduces to equation

$$\frac{du}{dt} + Au(t) = 0. \tag{1.1.11}$$

The inclusion (1.1.10), with A being monotone (or accretive), describes the evolution of many physical phenomena which generate energy over time. At equilibrium state, $\frac{du}{dt} = 0$ and therefore $0 \in Au$ describes the equilibrium state of the inclusion (1.1.10). Such equilibrium points are very desirable in many applications in, for example, ecology, economics, physics, and so on.

Interest in monotone (accretive) mappings stems mainly from this firm connection they have with evolution equations. The class of monotone mappings was first introduced by Minty [59] while the class of accretive operators is due to Browder [9] and Kato [48]. An operator $A : D(A) \subseteq E \to 2^E$ is also called accretive if

$$||x - y|| \;\; \leq \;\; ||x - y + \lambda(u - v)||, \tag{1.1.12}$$

for all $u \in Ax,\ v \in Ay$ (see Kato [48]). A result of Kato [48] shows that (1.1.8) is equivalent to (1.1.12). The inequality (1.1.12) implies that $I + \lambda A$ is one-to-one for each $\lambda > 0$, where a multi-valued map $M : X \to 2^Y$ is one-to-one if $x \neq y$ implies that $Mx \cap My = \emptyset$. Therefore, $(I + \lambda A)^{-1}$ is single-valued on $\mathcal{R}(I + \lambda A)$, the range of the mapping $I + \lambda A$. If the range of $I + A$ is the whole E, then A is called *m-accretive* (or *m-monotone*). Hence, for m-accretive mappings, $(I + A)^{-1}$ is defined on the whole E. It can be shown that the range of $I + A$ is the whole E if and only if the range of $I + \lambda A$ is the whole E for all $\lambda > 0$ (see, for example, [48]). The class of m-accretive mappings is one of the most important classes of multi-valued mappings due to its usefulness in the theory of existence of solutions of nonlinear differential equations.

A standard technique in solving $0 \in Au$ (or $Au = 0$), is to define another operator T as $T := I - A$, where I is an identity mapping and seek fixed points of T. It is evident that the set of fixed points of T is the same as the set of zeros of A. Such a T is called *pseudo-contractive*. A single-valued mapping $T : D(T) \subseteq E \to E$ is called nonexpansive if

$$||Tx - Ty|| \;\; \leq \;\; ||x - y|| \;\; \forall x, y \in D(A). \tag{1.1.13}$$

From the definition of accretive mappings, one can easily see that a mapping $A : D(A) \subseteq E \to E$ is accretive if and only if for all $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that $u \in Ax, v \in Ay$ implies

$$\langle (x-u) - (y-v), j(x-y) \rangle \le ||x-y||^2. \tag{1.1.14}$$

If A is single-valued, inequality (1.1.14) is nothing but

$$\langle (I-A)x - (I-A)y, j(x-y) \rangle \le ||x-y||^2. \tag{1.1.15}$$

The inequality (1.1.15) is clearly satisfied by $A := I - T$ where T is any nonexpansive (single-valued) map for any $x, y \in D(T)$ and for any $j(x-y) \in J(x-y)$. Thus, for any nonexpansive mapping T, $I - T$ is accretive. This implies that every nonexpansive mapping is pseudo-contractive. The converse is not true. Clearly every nonexpansive mapping is continuous, in fact uniformly continuous. But there are pseudo-contractive mappings which are not continuous. For instance, the mapping

$$T : \mathbb{R} \to \mathbb{R}$$

$$Tx = x - [x],$$

where $[\cdot]$ denotes the greatest integer function, is pseudo-contractive but not continuous. So T cannot be nonexpansive. Therefore, the class of pseudo-contractive mappings is not only connected with accretive mappings, but is also a proper superclass of the important class of noexpansive mappings. For the nonexpansive mappings, apart from being an obvious generalization of contraction mappings, they are important as has been observed by Bruck [17], mainly for the following reasons:

- Nonexpansive mappings are intimately connected with the monotonicity methods developed since the early 1960's and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.

- Nonexpansive mappings appear in applications as the transition operators for initial value problems of differential inclusions of the form $0 \in \frac{du}{dt} + T(t)u$, where $\{T(t)\}$ are, in general, set-valued and are accretive or *dissipative* or *minimally continuous*.

We shall say more on the importance of nonexpansive mappings later in this chapter.

- **An important example of a monotone operator**

A convex and continuous functional defined on a real Hilbert space, $f : H \to \mathbb{R}$, need not have a differential (e.g. $x \mapsto |x|, \quad x \in \mathbb{R}$). However such f has what is called a *subdifferential*. The subdifferential of f is a multi-valued mapping defined as

$$\partial f : H \to 2^H$$

$$\partial f(x) := \{u \in H : \langle u, y - x\rangle \leq f(y) - f(x), \quad \forall y \in H\}.$$

It is clear from this definition that $0 \in \partial f(x)$ if and only if x is a global minimizer of f. Therefore, finding zeros of the multi-valued mapping ∂f corresponds to finding minimizers of f. This among other things makes the subdifferential very important and from the definition of the subdifferential, one easily sees that it is a multi-valued monotone mapping.

- **Iterative algorithms for fixed points of nonexpansive mappings**

Let (X, d) and (Y, ρ) be two metric spaces. A mapping $T : X \to Y$ is called *Lipschitz* mapping if there exists $L > 0$ such that for all $x, y \in X$,

$$\rho(Tx, Ty) \leq Ld(x, y). \tag{1.1.16}$$

T is called a *contraction* if $L < 1$, and it is called *nonexpansive* if $L = 1$.

For a contraction mapping $T : X \to X$, X a complete metric space, given arbitrary $x_0 \in X$, the sequence of *Picard iterates* $\{x_n\}$ defined by $x_{n+1} = Tx_n, \ n \geq 0$, converges to the unique fixed point of T. Unlike contraction mappings, trivial examples show that for a nonexpansive mapping $T : X \to X$ having a fixed point, the Picard sequence $x_{n+1} = Tx_n, \ x_0 \in X, \ n \geq 0$, may fail to converge even when T has a unique fixed point. It suffices, for example, to take T to be the anti-clockwise rotation of the closed unit ball in $\mathbb{R}^2$ around the origin of coordinates through an acute angle. Krasnoselskii [53], however, showed that in this example, $\{x_n\}$ converges to the unique fixed point of T where, starting with any fixed element of the ball, $\{x_n\}$ is generated by $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}Tx_n, \ n \geq 0$. Schaefer [73], gave a generalization of this scheme which proved successful in the approximation of fixed points of nonexpansive mappings $T : K \to K$ (when they exist). The recurrence relation of Schaefer is the following: $x_0 \in K$,

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \ n = 0, 1, 2, \cdots , \tag{1.1.17}$$

where K is a closed convex nonempty subset of a normed linear space and $\lambda \in (0, 1)$ is fixed. The sequence $\{x_n\}$ generated by the scheme (1.1.17) is called *Krasnoselskii sequence*. However, the most general iterative scheme now studied for approximating fixed points of nonexpansive mappings is the following: $x_0 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \; n = 0, 1, 2, \cdots , \quad (1.1.18)$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$ is a real sequence satisfying appropriate conditions (see, Chidume [28], for more on this). Under the conditions

$(i) \lim_{n\to\infty} \alpha_n = 0$, and $(ii) \sum_{n=0}^{\infty} \alpha_n = \infty$, the sequence $\{x_n\}$ generated by (1.1.18) is generally referred to as the *Mann* sequence in the light of Mann [57]. The recursion formula (1.1.17) is consequently called (by many) the *Krasnoselskii-Mann* (KM) formula for finding fixed points of nonexpansive mappings.

The following quotation further shows the importance of iterative methods for approximating fixed points of nonexpansive mappings.

> " Many well-known algorithms in signal processing and image reconstruction are iterative in nature $\cdots$. A wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the KM iteration procedure, for particular choices of the operator $\cdots$." (Byrne, [19]).

We have seen, early on, that the class of pseudo-contractive mappings is a proper superclass of the class of nonexpansive mappings. It is not difficult to see, in fact, that the class of pseudo-contractive mappings which are Lipschitz is a proper superclass of the class of nonexpansive mappings. It suffices to consider the mapping $T := \gamma I, \; \gamma < -1$ defined on, say $\mathbb{R}$, where I is the identity map. Existence theorems for fixed points of Lipschitz pseudo-contractions or, more generally, continuous pseudo-contractions have been proved (see, for example, Caristi [20] and Kirk and Schöneberg [52]). Efforts to approximate fixed points of Lipschitz pseudo-contractive mappings, when they exist, using the Mann sequence proved abortive. In 1974, Ishikawa [44], introduced the following iteration scheme for a Lipschitz pseudo-contractive mapping $T : K \to K$, where K is a nonempty compact convex subset of a Hilbert space: $x_0 \in K$ and for $n \geq 0$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n; \quad y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad (1.1.19)$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions

$$(i)\ 0 \leq \alpha_n \leq \beta_n < 1;\ (ii) \lim_{n\to\infty} \beta_n = 0;\ (iii) \sum_{n\geq 0} \alpha_n\beta_n = \infty.$$

He proved that the sequence $\{x_n\}$ generated by this scheme converges strongly to a fixed point of T. The recursion formula (1.1.19) with the conditions $(i), (ii)$ and (iii) is generally referred to as the *Ishikawa iteration process.* The theorem of Ishikawa is given below:

Theorem 1.1.2 (Ishikawa, [44]) *If K is a compact convex subset of a Hilbert space H and $T : K \to K$ is a Lipchitzian pseudo-contactive mapping and x_0 is any point of K, then then the sequence $\{x_n\}$ generated by* (1.1.19) *with $\{\alpha_n\}, \{\beta_n\}$ sequences of positive numbers satisfying the conditions $(i), (ii)$ and (iii) above converges strongly to a fixed point of T.*

Since its publication in 1974, it had remained an open question (see, e.g., Borwein and Borwein [4], Chidume and Moore [23], Hicks and Kubicek [43]) whether or not the Mann recursion formula defined by (1.1.18), which is certainly simpler than the Ishikawa recursion formula (1.1.19) and which worked for nonexpansive mappings, converges under the setting of Ishikawa's result (above) to a fixed point of T if the operator T is pseudo-contractive and Lipschitz. Hicks and Kubicek [43], gave an example of a discontinuous pseudo-contraction with a unique fixed point for which the Mann iteration process does not always converge. Borwein and Borwein [4], gave an example of a Lipschitz map (which is not pseudo-contractive) with a unique fixed point for which the Mann sequence fails to converge. The problem for *Lipschitz* pseudo-contractive mappings still remained open. This was eventually resolved in the negative by Chidume and Mutangadura in 2001, [24]. They produced an example of a *Lipschitz* pseudo-contractive mapping defined from a nonempty compact convex subset of $\mathbb{R}^2$ into itself having a unique fixed point for which no Mann sequence converges. For details see the monograph of Chidume [28].

Two other iteration processes have been introduced and have been successfully used to approximate fixed points of Lipschitz pseudo-contractive mappings in certain Banach spaces *more general than Hilbert spaces.* One of them, which seems to be much simpler than the Ishikawa iteration process is the following:

Let E be a real normed linear space, K a nonempty closed convex subset of E, $T : K \to K$ a Lipschitz pseudo-contractive mapping. For arbitrary $x_1 \in K$, *let the sequence* $\{x_n\}$ *be defined iteratively by*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n\theta_n(x_n - x_1), n \geq 1, \quad (1.1.20)$$

where $\{\lambda_n\}$, $\{\theta_n\}$ *are sequences in* $(0,1)$ *satisfying the conditions*

$$(i)\ \lim \theta_n = 0;\ (ii)\ \lambda_n(1+\theta_n) < 1, \sum \lambda_n\theta_n = \infty, \lim \frac{\lambda_n}{\theta_n} = 0;$$

and $\quad (iii)\ \lim \dfrac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} = 0.$

This iteration process was given by Chidume and Zegeye, [26]. They proved that if K is a noempty closed convex subset of a reflexive real Banach space E which has a uniformly Gâteaux differential norm; and $T : K \to K$ is a Lipschitz pseudo-contractive mapping having a fixed point, then the sequence $\{x_n\}$ generated by the formula (1.1.20) is an approximate fixed point sequence for T (where a sequence in the domain of a mapping T is called an approximate fixed point sequence of T if $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$). Convergence to a fixed point of T is proved with minor compactness-type conditions on K.

- **Iterative algorithms for fixed points of multi-valued mappings**

Let (X,d) be a metric space. For a nonempty subset D of X, let $CB(D)$, K(D), $P(D)$ and $KC(D)$ denote the families of, closed and bounded, compact, proximinal, and compact convex subsets of D, respectively. The *Hausdorff metric* on $CB(X)$ is defined by:

$$H(A,B) = \max\left\{\sup_{a\in A} \operatorname{dist}(a,B),\ \sup_{b\in B} \operatorname{dist}(b,A)\right\}$$

for all $A, B \in CB(X)$, where for $x \in X$ and $A \subseteq X$ nonempty, $\operatorname{dist}(x,A) := \inf_{a\in A} d(x,a)$. A mapping $T : X \to CB(X)$ is said to be *Lipschitz* if there exists $L > 0$ such that

$$H(Tx,Ty) \leq Ld(x,y), \quad \forall x,y \in X. \quad (1.1.21)$$

If $L = 1$, then T is called nonexpansive and when $L < 1$, T is called a contraction.

In 1969, Nadler [62] obtained an existence theorem for fixed point of multi-valued contraction mapping. This is one of the earliest results obtained in the fixed point theory for multi-valued mappings via iteration process. He proved the following theorem.

Theorem 1.1.3 *Let (X, d) be a complete metric space. If $F : X \to CB(X)$ is a multi-valued contraction mapping, then F has a fixed point. Moreover, if k is the constant of contraction, then starting from any $x_0 \in X$, the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_{n+1} \in F(x_n), \\ d(x_{n+1}, x_n) \leq H(F(x_{n+1}), F(x_n)) + k^n, \ n \geq 0, \end{cases} \tag{1.1.22}$$

converges to a fixed point of F.

Since then, several results have been obtained in the field of approximation of fixed-points of multi-valued mappings (see, e.g., [54, 65, 66, 71, 72, 74, 75], etc.).

Let D be nonempty closed and convex subset of a real Banach space E and $T : D \to K(D)$ be a mapping. Let $u \in D$. For each $t \in (0, 1)$, let $T_t : D \to K(D)$ be a mapping defined by

$$T_t x = tTx + (1 - t)u, \quad x \in D. \tag{1.1.23}$$

Since $K(D) \subseteq CB(D)$ and

$$H(tTx, tTy) = tH(Tx, Ty) \ \forall x, y \in X,$$

then for T nonexpansive, we have T_t is a contraction for each $t \in (0, 1)$, and therefore by Theorem 1.1.3, it admits a fixed point $x_t \in D$. That is, there exists $x_t \in D$ such that

$$x_t \in tTx_t + (1 - t)u. \tag{1.1.24}$$

If T is single-valued, then the inclusion (1.1.24) reduces to

$$x_t = tTx_t + (1 - t)u. \tag{1.1.25}$$

The strong convergence of the net $\{x_t\}$ for a self or non-self nonexpansive single-valued mapping T has been investigated by many mathematicians (see, for example, Browder [11], Halpern [41], Reich [70], Xu [79], Morales and Jung [61], etc.). In 1967, Browder [11] established the following strong convergence theorem.

Theorem 1.1.4 (Browder, [11]) *Let D be a bounded closed convex subset of a real Hilbert space H and let $T : D \to D$ be nonexpansive. For a fixed $u \in D$ define a net $\{x_\lambda\}$ in D by*

$$x_\lambda = \lambda T x_\lambda + (1 - \lambda)u, \quad \lambda \in (0, 1).$$

Then $\{x_\lambda\}$ converges strongly to the element of $F(T)$ nearest to u as $\lambda \to 1$.

Reich [70] extended Theorem 1.1.4 to uniformly smooth real Banach spaces as follows.

Theorem 1.1.5 (Reich, [70]) *Let D be a bounded closed convex subset of a uniformly smooth real Banach space E and let $T : D \to D$ be nonexpansive. For a fixed $u \in D$, define a net $\{x_t\}$ in D by*

$$x_t = tTx_t + (1-t)u, \quad t \in (0,1).$$

Then $\{x_t\}$ converges strongly as $t \to 1$ to the element of $F(T)$ nearest to u.

Theorems 1.1.4 and 1.1.5 were extended by Morales and Jung [61] to more general reflexive real Banach spaces and to the class of continuous pseudo-contractive mappings. They proved the following theorem.

Theorem 1.1.6 (Morales and Jung, [61]) *Let E be a real Banach space and let D be a nonempty closed convex subset of E. Let $T : D \to E$ be a continuous pseudo-contractive mapping satisfying the weakly inward condition. Then for a fixed $z \in D$, there exists a unique path $t \mapsto x_t$ in $D, t \in [0,1)$, satisfying*

$$x_t = tTx_t + (1-t)z. \tag{1.1.26}$$

Moreover, if in addition E is assumed to have a uniformly Gâteaux differentiable norm and is such that every closed convex and bounded subset of K has the fixed point property for nonexpansive self-mappings, then as $t \to 1$, the path converges strongly to a fixed point of T.

Pietramala [69] gave an example which shows that Browder's theorem, Theorem 1.1.4, cannot be extended to the multi-valued case without adding extra assumption.

Acedo and Xu [2] established the strong convergence of $\{x_t\}$ in a Hilbert space for a subclass of nonexpansive multi-valued mappings having unique fixed points. Later on, Kim and Jung [51] extended the result of Acedo and Xu to a Banach space having a weakly sequentially continuous duality mapping. Sahu [71] studied this problem in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Recently, Jung [46] proved the strong convergence of $\{x_t\}$ defined by $x_t \in tTx_t + (1-t)u,\ u \in D$, D being the domain of T, for nonexpansive non-self multi-valued mapping T

in a uniformly convex real Banach space having a uniformly Gâteaux differentiable norm. More recently, Shahzad and Zegeye [74] extended the results of Jung [46] where they proved strong convergence of $\{x_t\}$ as in [46] but for multi-valued mappings which are not necessarily nonexpansive but which are associated with some nonexpansive multi-valued mappings.

In 2010, Ofoedu and Zegeye proved the following lemma.

Lemma 1.1.7 (Ofoedu and Zegeye, [65]) *Let D be an open convex subset of a real Banach space E. Let $T : \overline{D} \to CB(E)$ be continuous (relative to Hausdorff metric) pseudo-contractive mapping satisfying weakly inward condition and let $u \in D$ be fixed. Then for each $t \in (0, 1)$ there exists $x_t \in \overline{D}$ such that $x_t \in (1 - t)u + tTx_t$.*

Using this lemma, Ofoedu and Zegeye [65] introduced an iteration scheme for approximating a fixed point of a *multi-valued* Lipschitz pseudo-contractive mapping. They proved the following theorem.

Theorem 1.1.8 (Ofoedu and Zegeye, [65]) *Let E be a reflexive real Banach space having a uniformly Gâteaux differentiable norm, D be a nonempty open convex subset of E such that every closed convex bounded nonempty subset of $\overline{D}$ has the fixed point property for nonexpansive self-mappings. Let $T : \overline{D} \to K(\overline{D})$ be a pseudo-contractive Lipschitzian mapping with constant $L > 0$ and let $u \in \overline{D}$ be fixed. Let $\{x_n\}$ be generated from arbitrary $x_0 \in \overline{D}$, $w_0 \in Tx_0$ by*

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n w_n - \lambda_n\theta_n(x_n - u),\ w_n \in Tx_n, \tag{1.1.27}$$

where $\{\lambda_n\}, \{\theta_n\}$ are as in (1.1.20) Suppose that $\|w_n - w_{n-1}\| = \text{dist}(w_{n-1}, Tx_n)$, $n \geq 1$. If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to some fixed point of T.

It was remarked by Nadler [62] that requiring a *multi-valued mapping* to be *Lipschitz* is placing a *strong continuity restriction* on the mapping.

We prove in Chapter 2 of this Thesis that this condition can be weakened in the setting of q-uniformly smooth real Banach spaces. In fact, the Lipschitz condition of T in Theorem 1.1.8 is weakened to *continuity* and *boundedness* for T. Moreover, in many applications, the real Banach space E is either an L_p-space, a $W^{m,p}$-space, $1 < p < \infty$, $m \geq 1$, or a real Hilbert space. All these real Banach spaces are q-uniformly smooth and reflexive.

We use the recursion formula (1.1.27), dispensing with the restriction that w_n is chosen in Tx_n satisfying $\|w_n - w_{n-1}\| = \text{dist}(w_{n-1}, Tx_n)\ \ \forall n \geq 1$. Our iteration process, in the setting of q-uniformly smooth real Banach spaces, is direct, much easier to use in application than the process in Theorem 1.1.8 since one takes any w_n in Tx_n in our process without having to create a sub-programme to first compute w_n at each step of the iteration process in Theorem 1.1.8.

- **Strictly pseudo-contractive multi-valued mappings**

An important subclass of the class of Lipscitz pseudo-contractive mappings was introduced by Browder and Petryshyn, [12].

Definition 1.1.9 *A mapping $T : K \to E$ is said to be* strictly pseudo-contractive in the sense of Browder and Petryshyn *if*

$$\langle Tx - Ty, j(x-y)\rangle \leq ||x-y||^2 - \lambda||x-y-(Tx-Ty)||^2 \qquad (1.1.28)$$

holds for any $x, y \in E$ and for some $\lambda > 0$.

It is easy to see that such mappings are Lipschitz with Lipschitz constant $L = \dfrac{1+\lambda}{\lambda}$. Browder and Petryshyn actually defined this mapping in a Hilbert as follows:

Definition 1.1.10 *Let K be a nonempty subset of a real Hilbert space. A mapping $T : K \to K$ is said to be strictly pseudo-contractive if*

$$||Tx - Ty||^2 \leq ||x-y||^2 + k||x-y-(Tx-Ty)||^2 \qquad (1.1.29)$$

for some $k \in (0,1)$ and for all $x, y \in K$.

Clearly, nonexpansive mappings satisfy inequality (1.1.29) and it is also easy to see that the inequalities (1.1.28) and (1.1.29) are equivalent in real Hilbert spaces. It is worthy of mention here that, while the example of Chidume and Mutangadura [24], shows that the Mann sequence will not always converge to a fixed point of a Lipschitz pseudo-contractive mapping, the Mann sequence can be used to approximate a fixed point of this important subclass of the class Lipschitz pseudo-contractive mappings consisting of *mappings which are strictly pseudo-contractive in the sense of Browder and Petryshyn.*

This result was obtained by Chidume *et al.* [27].

Sastry and Babu [72] introduced the following iterative schemes. Let H be a real Hilbert space and let $T : H \to P(H)$ be a multi-valued nonexpansive mapping and let x^* be a fixed point of T. Define iteratively the sequence $\{x_n\}_{n\in\mathbb{N}}$ from $x_0 \in E$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \;\; y_n \in Tx_n, \;\; \|y_n - x^*\| = \text{dist}(Tx_n, x^*), \quad (1.1.30)$$

where $\{\alpha_n\}$ is a real sequence in (0,1) satisfying the following conditions: $(i)\ \sum_{n=1}^{\infty} \alpha_n = \infty$ and $(ii)\ \lim \alpha_n = 0$.

They also introduced the following scheme:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n z_n,\ z_n \in Tx_n,\ \|z_n - x^*\| = \text{dist}(x^*, Tx_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n,\ u_n \in Ty_n,\ \|u_n - x^*\| = \text{dist}(Ty_n, x^*), \end{cases} \quad (1.1.31)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying the following conditions:

$(i)\ \ 0 \leq \alpha_n, \beta_n < 1$, $(ii)\ \lim_{n\to\infty} \beta_n = 0$ and $(iii)\ \sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.

Sastry and Babu called a process defined by (1.1.30) a Mann iteration process and a process defined by (1.1.31) where the iteration parameters α_n and β_n satisfy conditions (i), (ii) and (iii) an Ishikawa iteration process. They proved in [72] that the Mann and Ishikawa iteration schemes for a multi-valued mapping T with nonempty fixed point set converge to a fixed point of T under certain conditions. More precisely, they proved the following result for a multi-valued nonexpansive mapping with compact domain.

Theorem 1.1.11 (Sastry and Babu, [72]) *Let H be real Hilbert space, K be a nonempty, compact and convex subset of H, and $T : K \to CB(K)$ be a multi-valued nonexpansive mapping with a fixed point p. Assume that $(i)\, 0 \leq \alpha_n, \beta_n < 1$; $(ii)\, \beta_n \to 0$ and $(iii) \sum \alpha_n\beta_n = \infty$. Then, the sequence $\{x_n\}$ defined by* (1.1.31) *converges strongly to a fixed point of T.*

Panyanak [66] extended Theorem 1.1.11 to uniformly convex real Banach spaces. He proved the following result.

Theorem 1.1.12 (Panyanak, [66]) *Let E be a uniformly convex real Banach space, K be a nonempty, compact and convex subset of E, and $T : K \to CB(K)$ a multi-valued nonexpansive mapping with a fixed point p. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n \beta_n = \infty$. Then, the sequence $\{x_n\}$ defined by (1.1.31) converges strongly to a fixed point of T.*

Panyanak [66] also modified the iteration schemes of Sastry and Barbu [72]. Let K be a nonempty, closed and convex subset of a real Banach space and let $T : K \to P(K)$ be a multi-valued mapping such that $F(T)$ is a nonempty proximinal subset of K.

The sequence of Mann iterates is defined by $x_0 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n, \ \ \alpha_n \in [a, b], \, 0 < a < b < 1, \qquad (1.1.32)$$

where $y_n \in Tx_n$ is such that $\|y_n - u_n\| = \text{dist}(u_n, Tx_n)$ and $u_n \in F(T)$ is such that $\|x_n - u_n\| = \text{dist}(x_n, F(T))$.

The sequence of Ishikawa iterates is defined by $x_0 \in K$,

$$y_n = (1 - \beta_n)x_n + \beta_n z_n, \ \ \beta_n \in [a, b], \, 0 < a < b < 1, \qquad (1.1.33)$$

where $z_n \in Tx_n$ is such that $\|z_n - u_n\| = \text{dist}(u_n, Tx_n)$ and $u_n \in F(T)$ is such that $\|x_n - u_n\| = \text{dist}(x_n, F(T))$. The sequence defined iteratively by the following:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z'_n, \ \ \alpha_n \in [a, b], \, 0 < a < b < 1, \qquad (1.1.34)$$

where $z'_n \in Ty_n$ is such that $\|z'_n - v_n\| = \text{dist}(v_n, Ty_n)$ and $v_n \in F(T)$ is such that $\|y_n - v_n\| = \text{dist}(y_n, F(T))$.

Theorem 1.1.13 (Panyanak, [66]) *Let E be a uniformly convex real Banach space, K be a nonempty, closed, bounded and convex subset of E, and $T : K \to P(K)$ be a multi-valued nonexpansive mapping that satisfies condition (I) (see Section 1.2, p.32 for definition). Assume that (i) $0 \leq \alpha_n < 1$ and (ii) $\sum \alpha_n = \infty$. Suppose that $F(T)$ is a nonempty proximinal subset of K. Then, the sequence $\{x_n\}$ defined by (1.1.32) converges strongly to a fixed point of T.*

Panyanak [66] then asked the following question.

Question (P). Is Theorem 1.1.13 true for the Ishikawa iteration defined by (1.1.33) and (1.1.34)?

For multi-valued mappings, the following lemma is a consequence of the definition of Hausdorff metric, as remarked by Nadler [62].

Lemma 1.1.14 *Let (X, d) be metric space and let $A, B \in CB(X)$, and $a \in A$. For every $\gamma > 0$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \gamma. \tag{1.1.35}$$

Recently, Song and Wang [76] modified the iteration process due to Panyanak [66] and improved the results therein. They gave their iteration scheme as follows:

Let K be a nonempty, closed and convex subset of a real Banach space and let $T : K \to CB(K)$ be a multi-valued mapping. Let $\alpha_n, \beta_n \in [0, 1]$ and $\gamma_n \in (0, \infty)$ be such that $\lim_{n\to\infty} \gamma_n = 0$. Choose $x_0 \in K$,

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n, \end{cases} \tag{1.1.36}$$

where $z_n \in Tx_n$ and $u_n \in Ty_n$ are such that

$$\|z_n - u_n\| \leq H(Tx_n, Ty_n) + \gamma_n$$

and

$$\|z_{n+1} - u_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n.$$

They then proved the following result.

Theorem 1.1.15 (Song and Wang, [76]) *Let K be a nonempty, compact and convex subset of a uniformly convex real Banach space E. Let $T : K \to CB(K)$ be a multi-valued nonexpansive mapping with $F(T) \neq \emptyset$ satisfying $T(p) = \{p\}$ for all $p \in F(T)$. Assume that $(i)\, 0 \leq \alpha_n, \beta_n < 1$; $(ii)\, \beta_n \to 0$ and $(iii) \sum \alpha_n\beta_n = \infty$. Then, the sequence defined by (1.1.36) converges strongly to a fixed point of T.*

More recently, Shahzad and Zegeye [75] extended and improved the results of Sastry and Babu [72], Panyanak [66] and Son and Wang [76] to *multi-valued quasi-nonexpansive mappings*. Also, in an attempt to remove the restriction $Tp = \{p\}$ for all $p \in F(T)$ in Theorem 1.1.15, they introduced a new iteration scheme as follows:

Let K be a nonempty, closed and convex subset of a real Banach space, $T : K \to P(K)$ be a multi-valued mapping and $P_T x := \{y \in Tx : \|x - y\| = \text{dist}(x, Tx)\}$. Let $\alpha_n, \beta_n \in [0, 1]$. Choose $x_0 \in K$, and define $\{x_n\}$ as follows:

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n z_n, \\ \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \end{cases} \tag{1.1.37}$$

where $z_n \in P_T x_n$ and $u_n \in P_T y_n$. They then proved the following result.

Theorem 1.1.16 (Shahzad and Zegeye, [75]) *Let X be a uniformly convex real Banach space, K be a nonempty, closed and convex subset of X, and $T : K \to P(K)$ be a multi-valued mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the Ishikawa iterates defined by* (1.1.37)*. Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then, $\{x_n\}$ converges strongly to a fixed point of T.*

We make the following remarks regarding Theorems 1.1.11, 1.1.12, 1.1.13 and 1.1.16.

Remark 1.1.17 *In the recursion formula* (1.1.30)*, the authors take $y_n \in T(x_n)$ such that $\|y_n - x^*\| = \text{dist}(x^*, Tx_n)$. The existence of y_n satisfying this condition is guaranteed by the assumption that Tx_n is proximinal. In general such a y_n is not easy to pick. If Tx_n is proximinal, then it is closed. If, in addition, it is a convex subset of a real Hilbert space, then y_n is unique and is characterized by*

$$\langle x^* - y_n, y_n - u_n \rangle \geq 0 \ \forall \, u_n \in Tx_n.$$

One can see from this inequality that it is not easy to pick $y_n \in Tx_n$ satisfying

$$\|y_n - x^*\| = \text{dist}(x^*, Tx_n)$$

at every step of the iteration process. So, the recursion formula (1.1.30) *is not convenient to use in application. Also, the recursion formula defined in* (1.1.36) *is not convenient to use in application. The sequences $\{u_n\}$ and*

$\{z_n\}$ are not known precisely. Only their **existence** *is guaranteed by Lemma 1.1.14. Unlike in the case of formula* (1.1.30)*, characterizations of $\{u_n\}$ and $\{z_n\}$ guaranteed by Lemma 1.1.14 are not even known. So, the recursion formulas* (1.1.36) *are not really easy to use.*

We have seen that while pseudo-contractive mappings are generally not continuous, the strictly pseudo-contractive mappings (see definitions (1.1.10) and (1.1.9)) inherit *Lipschitz property* from their definitions. The study of fixed point theory for strictly pseudo-contractive mappings may help in the study of fixed point theory for nonexpansive mappings and for Lipschitz pseudo-contractive mappings. Consequently, the study by several authors of iterative methods for fixed points of *multi-valued nonexpansive mappings* (see, for example [72, 66, 50, 76, 49, 1] and the references therein) and their generalizations (see *e.g.*, [40, 30]), has motivated the study of iterative methods for approximating fixed points of the more general strictly pseudo-contractive mappings.

In chapter three of this thesis, therefore, we introduce the important class of *multi-valued strictly pseudo-contractive mappings* which is more general than the class of *multi-valued nonexpansive mappings*. Then, we prove strong convergence theorems for this class of mappings. The recursion formula used in our more general setting is of the *Krasnoselskii-type* [53]. This is part of the novelty of this work. We achieve these results by means of an incisive result similar to the result of Nadler [62] which we prove below (Lemma 1.2.13).

- **Strictly pseudo-contractive multi-valued mappings in Banach spaces**

In chapter four of this thesis, we extend Definition (1.1.9), that is the definition of single-valued strictly pseudo-contractive mappings in a general Banach space to multi-valued mappings. Then, we prove strong convergence theorems for this class of mappings. The recursion formula used here is again of the *Krasnoselskii-type* [53].

- **Quasi-nonexpansive multi-valued mappings in Banach space**

Another generalization of single-valued nonexpansive mappings is the notion of *quasi-nonexpansive* mappings.

Definition 1.1.18 *Let K be a nonempty subset of E. A mapping $T : K \to K$ is called* **quasi-nonexpansive** *if (i) $F(T) \neq \emptyset$ and (ii) $\|Tx - x^*\| \leq \|x - x^*\|$ for all $x \in K,\ x^* \in F(T)$.*

It is clear from this definition that every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive. The following example shows that the class of quasi-nonepansive mappings contains *properly* the class of nonexpansive mappings with nonempty fixed point sets.

Example 1.1.19 (Dotson [34], see also Chidume [22]) *Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = \frac{x}{2}\sin\left(\frac{1}{x}\right), x \neq 0,$ and $T0 = 0$.*

Krasnoselskii [53] proved that if E is a uniformly convex real Banach space, then for any $x_0 \in K$ fixed, the sequence $\{x_n\}$ generated by $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}Tx_n,\ n \geq 0$ converges strongly to a fixed point of T for T nonexpansive. More generally, for any fixed element $x_0 \in K$, Schaefer [73] extended the result of Krasnoselskii [53] by considering the sequence $\{x_n\}$ generated by $x_{n+1} = (1-\lambda)x_n + \lambda Tx_n,\ n \geq 0,\ \ \lambda \in (0, 1)$. Edelstein [37] observed that the result of Krasnoselskii [53] holds even in a strictly convex real Banach space. The natural question of whether this result holds in any Banach space more general than strictly convex real Banach space remained open for many years. This question was answered in the affirmative by Edelstein and O'Brien [36] who showed that the sequence $\{\|x_n - Tx_n\|\}$ converges to 0 uniformly in any normed linear space provided K is bounded.

Dotson [34] proved that $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$, where $\{x_n\}$ is a Krasnoselskii sequence and T is a single-valued quasi-nonexpansive mapping in the setting of a uniformly convex Banach space. Unlike in the case of nonexpansive mappings, an example given by Chidume [22] shows that $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ does not hold for quasi-nonexpansive mappings in arbitrary Banach space even when K is bounded, where $\{x_n\}$ is a Krasnoselskii sequence.

Kuhfittig [54] proved strong convergence theorem for a multi-valued mapping T which is nonexpansive around a known fixed point. He generated a Krasnoselskii sequence *using the known fixed point* and obtained strong convergence to additional fixed point. The following definition gives a class of multi-valued mappings which exists in the literature.

Definition 1.1.20 ($*$-nonexpansive mappings, see, e.g., [78]) *Let (X, ρ) be a metric space. A mapping $T : X \to K(X)$ is said to be $*$-nonexpansive if* *for all $x, y \in X$ and $u_x \in Tx$ with $\rho(x, u_x) = dist(x, Tx)$, there exists $u_y \in Ty$ with $\rho(y, u_y) = dist(y, Ty)$ such that*

$$\rho(u_x, u_y) \leq \rho(x, y).$$

It is obvious that this notion reduces to the notion of non-expansiveness for single-valued mappings. However, for multi-valued mappings, $*$-nonexpansive mappings are not comparable to nonexpansive mappings in general (see, e.g., [78]). We recall that associated with a mapping $T : X \to K(X)$ is the mapping $P_T : X \to K(X)$ defined by $P_T(x) := \{u \in Tx : \rho(u, x) = \text{dist}(x, Tx)\}$. It is clear that from the definition of P_T, $P_T(x^*) = \{x^*\}$ for every fixed point x^* of P_T. It is also known that T is $*$-nonexpansive if and only if P_T is nonexpansive [2].

Recently, Shahzad and Zegeye [75] proved strong convergence of the sequence of *Ishikawa-type iterates* to a fixed point of a quasi-nonexpansive mapping on uniformly convex Banach space. They extended and improved the results of Sastry and Babu [72], Panyanak [66] and Son and Wang [76]. Furthermore, they proved the convergence of Ishikawa-type sequence: $x_0 \in K$ fixed arbitrarily, $\alpha_n, \beta_n \in [0, 1]$,

$$y_n = (1 - \beta_n)x_n + \beta_n z_n, \;\; n \geq 0, \;\; z_n \in Tx_n,$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z_n', \;\; n \geq 0, \;\; z_n' \in Ty_n,$$

to a fixed point of $*$-nonexpansive mapping.

In chapter five of this thesis we prove strong convergence theorems for multi-valued quasi-nonexpansive mappings extending some of the results known only for single-valued quasi-nonexpansive mappings. Our algorithm does not require knowing any fixed point a priori and the sequence generated is of the *Krasnoselskii-type*.

1.1.2 Hammerstein Equations

A nonlinear integral equation of Hammerstein type (see, e.g., Hammerstein [42]) has the form

$$u(x) + \int_{\Omega} k(x, y) f(y, u(y)) dy = h(x), \tag{1.1.38}$$

where dy is a σ- finite measure on Ω; the kernel k is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear and h is a function on Ω. If we now define an operator

$$K : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$$

by

$$Kv(x) := \int_{\Omega} \kappa(x, y)v(y)dy; \; x \in \Omega,$$

and the so-called *superposition* or *Nemytskii* operator (with respect to the function f),

$$F : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$$

by

$$Fu(y) := f(y, u(y)),$$

where $\mathcal{F}(\Omega, \mathbb{R})$ denotes a space of functions from Ω into $\mathbb{R}$, then, the integral equation (1.1.38) can be put in operator theoretic form as follows:

$$u + KFu = 0, \tag{1.1.39}$$

where, without loss of generality, we have taken $h \equiv 0$.

For $h \neq 0$, we have for any $u \in \mathcal{F}(\Omega, \mathbb{R})$,

$$\begin{aligned} u + KFu = h \;\; &\Leftrightarrow \;\; u - h + KFu = 0 \\ &\Leftrightarrow \;\; w + KF(w + h) = 0, \quad w = u - h. \end{aligned}$$

Thus

$$u + KFu = h$$

has a solution in if and only if

$$u + K\overline{F}u = 0$$

has a solution, where $\overline{F}w = F(w + h)$.

Interest in (1.1.39) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's function can, as a rule, be transformed into the form (1.1.39). Among these is the problem of the forced oscillations of finite amplitude of a pendulum (see e.g. Pascali and Sburlan [67], Chapter IV),

Example. We consider the problem of the pendulum.

$$\begin{cases} \frac{d^2v(t)}{dt^2} + a^2 \sin v(t) = z(t), \quad t \in [0,1], \\ \\ v(0) = v(1) = 0, \end{cases} \tag{1.1.40}$$

where the driving force z is odd. The constant $a \neq 0$ depends on the length of the pendulum and gravity. Since the Green's function of the problem

$$v''(t) = 0, v(0) = v(1) = 0$$

is the function

$$k(t,s) := \begin{cases} t(1-s), \quad 0 \leq t \leq s \leq 1, \\ \\ s(1-t), \quad 0 \leq s \leq t \leq 1, \end{cases}$$

problem (1.1.40) is equivalent to the nonlinear integral equation

$$v(t) = -\int_0^1 k(t,s)[z(s) - a^2 \sin v(s)]\mathrm{d}s, \quad t \in [0,1]. \tag{1.1.41}$$

If

$$g(t) := \int_0^1 k(t,s)z(s)\mathrm{ds}, \quad u(t) := v(t) + g(t), \quad t \in [0,1],$$

then

$$v = u - g$$

and (1.1.41) can be written as

$$u(t) + \int_0^1 k(t,s)a^2 \sin\Big(g(s) - u(s)\Big)\mathrm{d}s = 0$$

which is in Hammerstein equation form:

$$u(t) + \int_0^1 k(t,s)f(s,u(s))\mathrm{d}s = 0,$$

where $f(t,s) = a^2 \sin\Big(g(t) - s\Big), \quad t,s \in [0,1].$

Equations of Hammerstein type play crucial role in the theory of optimal control systems, in automation and in network theory (see, e.g., Dolezale

[33]).

Several existence theorems for the solution of equation (1.1.39) have been proved by a host of authors using various technique (see, for instance, Brézis and Browder ([5, 6, 7]), Browder [16], Browder, De Figueiredo and Gupta [15], Browder and Gupta [13], Chepanovich [21]).

We highlight here the techniques used by Browder and Gupta [13] and Petryshyn and Fitzpatrick [68]. In [13] Browder and Gupta proved the following theorem (see [13] for definition of terms).

Theorem 1.1.21 (Browder-Gupta, [13]) *Let X be a real Banach space, X^* its dual space. Let K be a monotone angle-bounded continuous linear mapping of X into X^* with constant of angle-boundedness $c \geq 0$. Let F be a hemicontinuous (possibly nonlinear) mapping of X^* into X such that for a given constant $k \geq 0$,*

$$\langle v_1 - v_2, Fv_1 - Fv_2 \rangle \geq -k\|v_1 - v_2\|^2_{X^*} \tag{1.1.42}$$

for all v_1 and v_2 in X^*. Suppose finally that there exists a constant R with $k(1+c^2)R < 1$ such that for u in X

$$\langle Ku, u \rangle \leq R\|u\|^2_X. \tag{1.1.43}$$

Then there exists exactly one solution w in X^* of the nonlinear equation

$$w + KFw = 0. \tag{1.1.44}$$

The main tool used by the authors in proving the above theorem is that of splitting the linear operator K via a Hilbert space. Precisely, they proved that if X is a real Banach space, X^* its dual space, and K is a bounded *linear* mapping of X into X^* which is monotone and angle-bounded, then there exist a Hilbert space H, a continuous linear mapping S of X into H with adjoint S^* injective, and a bounded skew-symmetric linear mapping B of H into H such that (see the figure below)

$$K = S^*(I + B)S.$$

This factorization enabled the authors to transform the problem into another problem in a Hilbert such that the Hammerstein equation (1.1.39) has a solution if and only if the new problem has a solution in H. They set $f = (I+B)^{-1} + KFK^*$, $D := B(0,1)$, the closed unit ball in H, and showed

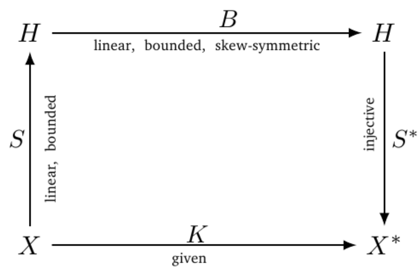

Figure 1.1: Factorization of operator, K

that f is hemicontinuous, monotone, and satisfies $\langle u, f(u)\rangle \geq 0\ \forall u \in D$. With these facts, they then used the following result of Minty [60] to prove theorem 1.1.21 (see [60] for definitions of terms).

Theorem 1.1.22 (Minty, [60]) *Let $D \subset X$ be bounded and surrounds 0; let $C \subset X$ contain $\overline{co}(D)$ and surround every point of $\overline{co}(D)$ densely. Let*

$$f : C \to X^*$$

be monotone and hemicontinuous at every point of $\overline{co}(D)$, and suppose

$$u \in D \text{ implies } \langle u, f(u)\rangle \geq 0. \tag{1.1.45}$$

Then there exists $u \in \overline{co}(D)$ such that $f(u) = 0$.

Petryshyn and Fitzpatrick employed variational techniques in proving existence Theorem for solution of equation (1.1.39). They proved the following theorems.

Theorem 1.1.23 (Petryshyn-Fitzpatrick, [68]) *Let X be a real reflexive Banach space and let K be a linear, monotone and symmetric mapping of X into X^*. Suppose f is a weakly (sequential) lower semi-continuous functional on X^* such that*

$$f(u) \geq -\frac{1}{2}a_1\|u\|^2 - a_2\|u\|^\delta - a_3 \tag{1.1.46}$$

where $a_1\|K\| < 1, a_2 > 0, a_3 > 0$ and $0 < \delta < 2$. Suppose also that $F : X^* \to X$ is such that $grad(f) = F$. Then the equation

$$w + KFw = 0 \tag{1.1.47}$$

has a solution in X^*.

Theorem 1.1.24 (Petryshyn-Fitzpatrick, [68]) *Let X be a reflexive Banach space with $K : X \to X^*$ linear, monotone and symmetric. Let $F : X^* \to X$ be potential and have a Gâteaux derivative which satisfies the inequality*

$$DF(u, v, v) \geq -a\|v\|^2 \qquad (v, u \in X^*)$$

and DN(tu, v, v) is continuous in $t \in [0, 1]$ for u and v fixed, where $a\|K\| < 1$. Then the equation $w + KFw = 0$ has a solution in X^*.

Let X be a real Banach space and let F and K be mappings defined from X to X such that the composition $KF : X \to X$ makes sense. Chidume and Zegeye [25] defined a mapping $A : E \to E$ with $E = X \times X$, as follows:

$$A[u, v] = [Fu - v, Kv + u], \quad \text{for all } u, v \in X$$

and observed that $A[u, v] = 0$ if and only if

$$\begin{cases} Fu - v &= 0 \\ Kv + u &= 0, \end{cases} \tag{1.1.48}$$

so that u solves equation (1.1.39). The system (1.1.48) is equivalent to the following fixed point problem:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -K \\ F & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

In chapter six of this thesis, *we introduce a new method*, perhaps simpler than methods used so far in the literature, *of proving existence* of solutions of Hammerstein equation in certain cases. To achieve this, we use the above technique recently introduced by Chidume and Zegeye [25], some existence results of Deimling [31] for zeros of accretive mappings, and some surjectivity results of Browder [14] for Lipschitz strongly accretive mappings. No linearity assumption is imposed on any of our mappings.

1.2 Preliminaries

Let E be a real normed space with dual E^* and let $S := \{x \in E : \|x\| = 1\}$. The space E is said to have *Gâteaux differentiable norm* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S$; E is said to have *uniformly Gâteaux differentiable norm* if for each $y \in S$, the limit is attained uniformly for $x \in S$.

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E , ρ_E, is defined by:

$$\rho_E(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}; \quad \tau > 0.$$

A normed linear space E is called *uniformly smooth* if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$. It is well known (see, *e.g.* [28] p.16, [56]) that ρ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or \ l_p) \ or \ W_p^m \text{ is } \begin{cases} 2 - \text{uniformly smooth} & \text{if} \quad 2 \leq p < \infty; \\ p - \text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{cases}$$

In fact, we have the following result.

Lemma 1.2.1 (Lindenstrauss and Tzafriri, [56]) *For* $1 < p < \infty$,

$$\rho_{L_p}(\tau) = \rho_{l_p}(\tau) = \rho_{W_p^m}(\tau) = \begin{cases} (1 + \tau^p)^{\frac{1}{p}} - 1 < \frac{1}{p}\tau^p, & 1 < p \leq 2, \\ \frac{(p-1)}{2}\tau^2 + o(\tau^2) < \frac{p-1}{2}\tau^2, & p \geq 2. \end{cases}$$

Every uniformly smooth real normed space has uniformly Gâteaux differentiable norm (see, *e.g.*, [28], p.17).

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{ f \in E^* : \langle x, f \rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \right\}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. J_2 is called the *normalized duality mapping* and is denoted by J. It is well known that if E is smooth, J_q is single-valued.

Remark 1.2.2 *It is known that (see, e.g., [79])* $J_p(x) = ||x||^{p-2}J(x)$ *for* $x \neq 0$.

Definition 1.2.3 (see, *e.g.*, [66, 72, 76]) *Let* K *be a nonempty subset of a normed space* E. *The set* K *is called proximinal if for each* $x \in E$, *there exists* $u \in K$ *such that*

$$\|x - u\| = \inf\{\|x - y\| : y \in K\} =: \operatorname{dist}(x, K).$$

Definition 1.2.4 *Let* E *be a real Banach space. A mapping* $T : D(T) \subseteq E \to CB(E)$ *is called quasi-nonexpansive if (i) the fixed point set* $F(T)$ *is not empty and (ii)* $H(Tx, Tx^*) \leq d(x, x^*)$ *for all* $x \in D(T),\ x^* \in F(T)$.

Lemma 1.2.5 *Let* H *be a real Hilbert space. Then*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$$

for all $x, y \in H$ *and for all* $t \in [0, 1]$.

Lemma 1.2.6 *Let* E *be a real normed linear space and* $q > 1$. *Then, the following inequality holds:*

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x+y)\rangle \ \ \forall\, j_q(x+y) \in J_q(x+y),\ \forall\, x, y \in E. \quad (1.2.1)$$

Proof Let $q > 1$ and $x, y \in E$. By the definition of J_q, the following holds

$$\begin{aligned} ||x + y||^q &= \langle x + y, j_q(x + y)\rangle \\ &= \langle x, j_q(x + y)\rangle + \langle y, j_q(x + y)\rangle \end{aligned}$$

for every $j_q(x + y) \in J_q(x + y)$. Using Schwartz and Young inequalities and the fact that $\|j_q(z)\| = \|z\|^{q-1}$ for all $z \in E$, we have

$$\begin{aligned} ||x + y||^q &\leq \|x\|\|x + y\|^{q-1} + \langle y, j_q(x + y)\rangle \\ &\leq \frac{1}{q}\|x\|^q + \frac{1}{p}\|x + y\|^q + \langle y, j_q(x + y)\rangle, \end{aligned}$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Observing that $1 - \frac{1}{p} = \frac{1}{q}$, then the results follows. ∎

Lemma 1.2.7 (Xu, [79]) *Let* $q > 1$ *and* E *be a smooth real Banach space. Then the following are equivalent.*

(i) E is q-uniformly smooth.

(ii) There exists a constant $d_q > 0$ such that for all $x, y \in E$,

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q. \tag{1.2.2}$$

(iii) There exists a constant $c_q > 0$ such that for all $x, y \in E$ and $\lambda \in [0,1]$

$$\|(1-\lambda)x + \lambda y\|^q \geq (1-\lambda)\|x\|^q + \lambda\|y\|^q - w_q(\lambda)c_q\|x-y\|^q, \tag{1.2.3}$$

where $w_q(\lambda) := \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$.

A real Banach space E is called *uniformly convex* if for any $\epsilon \in (0,2]$ there exists $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $||x|| = ||y|| = 1$ and $||x-y|| \geq \epsilon$, then $\big|\big|\frac{1}{2}(x+y)\big|\big| \leq 1-\delta$. E is called strictly convex if for any $x, y \in E, x \neq y, ||x|| = ||y|| = 1$, we have $||\lambda x + (1-\lambda)y|| < 1$ for all $\lambda \in (0,1)$. It can easily be shown that every uniformly convex space is strictly convex.

Lemma 1.2.8 (Xu, [79]) *Let E be a uniformly convex real Banach space and $R > 0$. Then there exists a continuous, convex, strictly increasing function*

$$g : [0,\infty) \to [0,\infty), \quad g(0) = 0,$$

such that for all $x, y \in B(0,R) := \{u \in E : \|u\| < R\}$ and $\lambda \in (0,1)$,

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|).$$

Lemma 1.2.9 (Xu, [80]) *Let $\{\rho_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$\rho_{n+1} \leq (1-\alpha_n)\rho_n + \alpha_n\sigma_n + \gamma_n, \;\; n \geq 0,$$

where,
(i) $\{\alpha_n\} \subset (0,1)$, $\sum \alpha_n = \infty$; (ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;
(ii) $\gamma_n \geq 0$, $\sum \gamma_n < \infty$. Then, $\rho_n \to 0$ as $n \to \infty$.

Lemma 1.2.10 (Kato, [48]) *Let E be real Banach space and let J be the normalized duality map. Then for any $x, y \in E$, the following are equivalent:*

(i) $||x|| \leq ||x + \lambda y||$ $\forall \lambda > 0$
(ii) there exists $u^* \in Jx$ such that $\langle y, u^*\rangle \geq 0$.

Some of the definitions below have already been given but we restate them here for convenience.

A mapping $A : D(A) \subset E \to E$ is called *accretive* if for all $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that the following inequality holds:

$$\langle Ax - Ay, j(x-y) \rangle \geq 0. \tag{1.2.4}$$

If E is a real Hilbert space, the mapping A called *monotone*. In this case, A satisfies the following condition:

$$\langle Ax - Ay, x - y \rangle \;\; \geq \;\; 0. \tag{1.2.5}$$

The mapping A is called *strongly accretive* if there exists $c > 0$ such that, for all $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$, such that

$$\langle Ax - Ay, j(x-y) \rangle \geq c\|x - y\|^2.$$

Equivalently, using Lemma 1.2.10, A is accretive if for all $\lambda > 0$,

$$||x - y|| \leq ||x - y + \lambda(Ax - Ay)||$$

for all $x, y \in D(A)$. If A is multi-valued, i.e., $A : D(A) \subset E \to 2^E$, then A is *accretive* if for all $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that for all $u \in Ax, v \in Ay$, the following inequality holds:

$$\langle u - v, j(x-y) \rangle \geq 0.$$

Equivalently, A is *accretive* if for all $\lambda > 0$,

$$||x - y|| \leq ||x - y + \lambda(u - v)||$$

for all $x, y \in D(A)$ and for all $u \in Ax, v \in Ay$.

A mapping $T : D(T) \subset E \to E$ is called *pseudo-contractive* if for all $x, y \in D(T)$ and for all $\lambda > 0$, the following inequality holds:

$$||x - y|| \leq ||(1 + \lambda)(x - y) - \lambda(Tx - Ty)||.$$

Equivalently (using Lemma 1.2.10), T is *pseudo-contractive* if for all $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq ||x - y||^2.$$

A multi-valued mapping $T : D(T) \subset E \to 2^E$ is called *pseudo-contractive* if for all $x, y \in D(T)$, and for all $\lambda > 0$, the following inequality holds:

$$||x - y|| \leq ||(1 + \lambda)(x - y) - \lambda(u - v)||$$

for all $u \in Tx, v \in Ty$. Equivalently, T is *pseudo-contractive* if for all $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that for all $u \in Tx, v \in Ty$, the following inequality holds:

$$\langle u - v, j(x - y)\rangle \leq ||x - y||^2.$$

Remark 1.2.11 *A mapping $T : D(T) \subset E \to 2^E$ is pseudo-contractive if and only if $(I - T)$ is accretive, where I denotes the identity mapping.*

Let E be a real normed linear space and let K be a convex subset of E. For $x \in E$, the *inward set*, $I_k(x)$, of x relative to K is defined as follows:

$$I_K(x) = \{x + c(u - x) : c \geq 1, u \in K\}.$$

A mapping $T : K \to E$ is said to be *inward* if $Tx \in I_K(x)$ for each $x \in K$, and *weakly inward* if Tx belongs to the closure of $I_K(x)$ for each $x \in K$. For a multi-valued mapping $T : K \to 2^E$, it is called *inward* if $Tx \subset I_K(x)$ for each $x \in K$ and *weakly inward* if Tx is a subset of the closure of $I_K(x)$ for each $x \in K$.

Let E be a real Banach space and let $T : K \to CB(K)$ be a multi-valued mapping. The mapping $(I - T)$ is said to be *strongly demiclosed* at 0 (see *e.g.*, [40]) if for any sequence $\{x_n\} \subseteq D(T)$ such that $\{x_n\}$ converges strongly to x^* and $\{\text{dist}(x_n, Tx_n)\}$ converges strongly to 0, then $d(x^*, Tx^*) = 0$. T is called *hemicompact* (see, e.g., [3]) if, for any sequence $\{x_n\}$ in K such that $\text{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges. We say that T satisfies *Condition (I)* if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\text{dist}(x, T(x)) \geq f(\text{dist}(x, F(T)) \; \forall x \in K.$$

Remark 1.2.12 *We note that if K is compact, then every multi-valued mapping $T : K \to CB(K)$ is hemicompact.*

We conclude the preliminaries with the following lemma which will play crucial role in chapters 3 and 4.

Lemma 1.2.13 *Let E be a reflexive real Banach space and let $A, B \in CB(X)$. Assume that B is weakly closed. Then, for every $a \in A$, there exists $b \in B$ such that*

$$\|a - b\| \leq H(A, B). \tag{1.2.6}$$

Proof Let $a \in A$ and let $\{\lambda_n\}$ be a sequence of positive real numbers such that $\lambda_n \to 0$ as $n \to \infty$. From Lemma 1.1.14, for each $n \geq 1$, there exists $b_n \in B$ such that

$$\|a - b_n\| \leq H(A, B) + \lambda_n. \tag{1.2.7}$$

It then follows that the sequence $\{b_n\}$ is bounded. Since E is reflexive and B is weakly closed, there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ that converges weakly to some $b \in B$. Now, using inequality (1.2.7), the fact that $\{a - b_{n_k}\}$ converges weakly to $a - b$ and $\lambda_{n_k} \to 0$, as $k \to \infty$, it follows that

$$\|a - b\| \leq \liminf \|a - b_{n_k}\| \leq H(A, B).$$

This proves the lemma. ■

CHAPTER 2

Iterative algorithm for fixed point of multi-valued pseudo-contractive mappings in Banach spaces

2.1 Introduction

Let $q > 1$ and let E be a q-uniformly smooth real Banach space and K be a nonempty, closed and convex subset of E. Let $CB(K)$ be the collection of all closed and bounded subsets of K. Suppose that $T : K \to CB(K)$ is a multi-valued bounded continuous pseudo-contractive mapping with a nonempty fixed point set. In this chapter, a new iterative algorithm is constructed and the corresponding sequence $\{x_n\}$ is proved to converge strongly to a fixed point of T under appropriate conditions on the iteration parameters.

Our theorems are improvements on Theorem 1.1.8 which we restate here for convenience.

Theorem 2.1.1 (Ofoedu and Zegeye, [65]) . *Let E be a reflexive real Banach space having uniformly Gâteaux differentiable norm, D be a nonempty open convex subset of E, such that every closed convex bounded nonempty subset of $\overline{D}$ has the fixed point property for nonexpansive self-mappings. Let $T : \overline{D} \to K(\overline{D})$ be a pseudo-contractive Lipschitzian mapping with constant $L > 0$ and let $u \in \overline{D}$ be fixed. Let $\{x_n\}$ be generated from arbitrary $x_0 \in \overline{D}$, $w_0 \in Tx_0$ by*

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n w_n - \lambda_n \theta_n (x_n - u), \ w_n \in Tx_n, \tag{2.1.1}$$

where $\{\lambda_n\}, \{\theta_n\}$ *are as in* (1.1.20)*. Suppose that* $\|w_n - w_{n-1}\| = \mathrm{dist}(w_{n-1}, Tx_n)$*,* $n \geq 1$*. If* $F(T) \neq \emptyset$*, then* $\{x_n\}$ *converges strongly to some fixed point of* T*.*

Remark 2.1.2 *To establish Theorem 2.1.1, the authors assume that* $\|w_n - w_{n-1}\| = \mathrm{dist}(w_{n-1}, Tx_n)$ *for all* $n \geq 1$*. A sufficient condition to guarantee this is to assume that for each* x*, the set* Tx *is proximinal. In this case,* Tx *is closed. If, in addition* Tx *is convex and* E *is, for example, a real Hilbert space, such* w_n *is characterized as follows:*

$$\langle w_{n-1} - w_n, w_n - u_n \rangle \geq 0 \qquad \forall\ u_n \in Tx_n.$$

Consequently, this condition requires that a sub-programme be constructed to first compute w_n ***at each step of the iteration process****.*

Remark 2.1.3 *Nadler [62] remarked that requiring a* **multi-valued mapping** *to be Lipschitz is placing a* strong continuity condition *on the mapping. We shall therefore weaken this condition imposed in Theorem OZ in our theorems to* ***boundedness and continuity****.*

With Remarks 2.1.2 and 2.1.3 in mind, it is our purpose in this chapter to prove strong convergence theorems for fixed points of *multi-valued bounded continuous pseudo-contractive mappings* defined on q-uniformly smooth real Banach spaces. We use the recursion formula (2.1.1), dispensing with the restriction that $\|w_n - w_{n-1}\| = \mathrm{dist}(w_{n-1}, Tx_n)$ $\forall n \geq 1$.

In the sequel we shall need the following results.

Lemma 2.1.4 (Ofoedu and Zegeye, [65]) *Let* D *be a nonempty open convex subset of a real Banach space* E *and* $T : \overline{D} \to CB(E)$ *be continuous (with respect to the Hausdorff metric) pseudo-contractive mapping satisfying weakly inward condition and* $u \in \overline{D}$ *be fixed. Then, for* $t \in (0,1)$ *there exists* $y_t \in \overline{D}$ *satisfying* $y_t \in tTy_t + (1-t)u$*. If, in addition,* E *is reflexive and has uniformly Gâteaux differentiable norm, and is such that every closed convex bounded subset of* $\overline{D}$ *has the fixed point property for nonexpansive self-mappings, then* T *has a fixed point if and only if* $\{y_t\}$ *remains bounded as* $t \to 1$*; moreover, in this case,* $\{y_t\}$ *converges strongly to a fixed point of* T *as* $t \to 1$*.*

Remark 2.1.5 *If* E *is* q*-uniformly smooth real Banach space, then* E *is a reflexive real Banach space with uniformly Gâteaux differentiable norm and every nonempty, closed, convex and bounded subset of* E *has the fixed point property for nonexpansive self-mappings.*

Remark 2.1.6 *We note that in Lemma 2.1.4, in the case that $F(T) \neq \emptyset$, the path $\{y_t\}$ is bounded. Furthermore, if E is assumed to have a uniformly Gâteaux differentiable norm and is such that every closed convex and bounded subset of $\overline{D}$ has the fixed point property for nonexpansive self-mappings, then as $t \to 1$, the path $\{y_t\}$ converges strongly to a fixed point of T.*

For the rest of this chapter, $q > 1$ is a real number, d_q will denote the constant appearing in Lamma 1.2.7 and $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions:

(i) $\{\theta_n\}$ deceases to 0;

(ii) $\lambda_n(1+\theta_n) < 1, \sum_{n=1}^{\infty} \lambda_n\theta_n = \infty, \quad \lambda_n^{q-1} = o(\theta_n);$

(iii) $\limsup_{n\to\infty} \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} \leq 0, \quad \sum_{n=1}^{\infty} \lambda_n^q < \infty.$

2.2 Main results

In what follows, $\{y_n\}$ denotes the sequence defined by $y_n := y_{t_n} = t_n z_n + (1-t_n)x_1$, where $x_1 \in \overline{D}$ and $t_n = (1+\theta_n)^{-1}$ $\forall\ n \geq 1$, guaranteed by Lemma 2.1.4 and Remark 2.1.6. We observe that with this t_n, the sequence $\{y_n\}$ satisfies the following conditions:

$$\theta_n(y_n - x_1) + (y_n - z_n) = 0, \ n \geq 1; \tag{2.2.1}$$

$$y_n \to y^* \text{ with } y^* \in F(T). \tag{2.2.2}$$

for some $z_n \in Ty_n$.

We now prove the following theorem.

Theorem 2.2.1 *Let E be a q-uniformly smooth real Banach space and D be a nonempty, open and convex subset of E. Assume that $T : \overline{D} \to CB(\overline{D})$ is a multi-valued continuous (with respect to the Hausdorff metric), bounded and pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_1 \in \overline{D}$ by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n u_n - \lambda_n\theta_n(x_n - x_1), \ u_n \in Tx_n. \tag{2.2.3}$$

Then, there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n^{q-1} < \gamma_0\theta_n$, for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof Let $x^* \in F(T)$. There exists $r > 0$ sufficiently large such that $x_1 \in B(x^*, r/2)$. Define $B := \overline{B(x^*, r)} \cap \overline{D}$. Since T is bounded, it follows that $(I - T)(B)$ is bounded. So,

$$M_0 := \sup\Big\{\|x - u + \theta(x - x_1)\|^q : \ x \in B,\, u \in Tx,\, 0 < \theta \leq 1\Big\} + 1 < \infty.$$

Set

$$M := d_q M_0; \ \ \gamma_0 := \Big(\frac{2^{q-1} - 1}{2^q M}\Big) r^q.$$

Step1. *We prove that $\{x_n\}$ is bounded.* Indeed, it suffices to show that x_n is in B for all $n \geq 1$. The proof is by induction. By construction, $x_1 \in B$. Suppose that $x_n \in B$ for some $n \geq 1$. We prove that $x_{n+1} \in B$.

Using the recursion formula (2.2.3) and inequality (1.2.2) of Lemma 1.2.7, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^q &= \|x_n - x^* - \lambda_n(x_n - u_n + \theta_n(x_n - x_1))\|^q \\ &\leq \|x_n - x^*\|^q - q\lambda_n\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - x^*)\rangle \\ &\quad + d_q\lambda_n^q\|x_n - u_n + \theta_n(x_n - x_1)\|^q \\ &\leq \|x_n - x^*\|^q - q\lambda_n\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - x^*)\rangle \\ &\quad + M\lambda_n^q. \end{aligned} \tag{2.2.4}$$

Using the fact that T is pseudo-contractive, we obtain

$$\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - x^*)\rangle \geq \theta_n\|x_n - x^*\|^q + \theta_n\langle x^* - x_1, j_q(x_n - x^*)\rangle.$$

Therefore, using inequality (2.2.4) and Schwartz inequality, we have the following estimates:

$$\begin{aligned} \|x_{n+1} - x^*\|^q &\leq (1 - q\lambda_n\theta_n)\|x_n - x^*\|^q - q\lambda_n\theta_n\langle x^* - x_1, j_q(x_n - x^*)\rangle + M\lambda_n^q \\ &\leq (1 - q\lambda_n\theta_n)\|x_n - x^*\|^q + q\lambda_n\theta_n\|x^* - x_1\|\|x_n - x^*\|^{q-1} + M\lambda_n^q \\ &\leq (1 - q\lambda_n\theta_n)\|x_n - x^*\|^q + q\lambda_n\theta_n\Big(\frac{1}{q}\|x^* - x_1\|^q + \frac{1}{p}\|x_n - x^*\|^q\Big) \\ &\quad + M\lambda_n^q, \end{aligned}$$

with $1/p + 1/q = 1$. Thus,

$$\|x_{n+1} - x^*\|^q \ \leq \ (1 - \lambda_n\theta_n)\|x_n - x^*\|^q + \lambda_n\theta_n\|x^* - x_1\|^q + M\lambda_n^q.$$

So, using the induction assumption, the fact that $x_1 \in B(x^*, r/2)$ and the condition $\lambda_n^{q-1} \leq \gamma_0\theta_n$, we obtain:

$$\|x_{n+1} - x^*\|^q \leq \Big(1 - \frac{1}{2}\lambda_n\theta_n\Big) r^q \leq r^q.$$

Therefore $x_{n+1} \in B$. Thus by induction, $\{x_n\}$ is bounded.

Step 2. *We prove that $\{x_n\}$ converges to a fixed point of T.* Let $\{y_n\}$ be the sequence obtained from Lemma 2.1.4 and satisfying equation (2.2.1) and condition (2.2.2).

Claim: $\|x_{n+1} - y_n\| \to 0$ as $n \to 0$. Since $\{x_n\}$ and $\{y_n\}$ are bounded and T is bounded, there exists some positive constant M such that:

$$\begin{aligned}\|x_{n+1} - y_n\|^q &= \|x_n - y_n - \lambda_n(x_n - u_n + \theta_n(x_n - x_1))\|^q \\ &\leq \|x_n - y_n\|^q - q\lambda_n\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - y_n)\rangle \\ &\quad + d_q\lambda_n^q\|x_n - u_n + \theta_n(x_n - x_1)\|^q \\ &\leq \|x_n - y_n\|^q - q\lambda_n\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - y_n)\rangle + M\lambda_n^q.\end{aligned}$$

Using equation (2.2.1) and the fact that T is pseudo-contractive, we have

$$\begin{aligned}\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - y_n)\rangle &= \langle (x_n - u_n) - (y_n - z_n), j_q(x_n - y_n)\rangle \\ &\quad +\theta_n\|x_n - y_n\|^q \\ &\quad +\langle y_n - z_n + \theta_n(y_n - x_1), j_q(x_n - y_n)\rangle \\ &\geq \frac{\theta_n}{q}\Big\|x_n - y_n\Big\|^q.\end{aligned}$$

Therefore,

$$\|x_{n+1} - y_n\|^q \leq (1 - \lambda_n\theta_n)\|x_n - y_n\|^q + M\lambda_n^q. \tag{2.2.5}$$

Using again the fact that T is pseudo-contractive, we obtain:

$$\|y_{n-1} - y_n\| \leq \left\|y_{n-1} - y_n + \frac{1}{\theta_n}\Big[(y_{n-1} - z_{n-1}) - (y_n - z_n)\Big]\right\|. \tag{2.2.6}$$

Observing from equation (2.2.1) that

$$y_{n-1} - y_n + \frac{1}{\theta_n}\Big[(y_{n-1} - z_{n-1}) - (y_n - z_n)\Big] = \frac{\theta_n - \theta_{n-1}}{\theta_n}(y_{n-1} - x_1),$$

it follows from inequality (2.2.6) that

$$\|y_{n-1} - y_n\| \leq \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)\|y_{n-1} - x_1\|. \tag{2.2.7}$$

By Lemma 1.2.6, we have

$$\begin{aligned}\|x_n - y_n\|^q &= \|(x_n - y_{n-1}) + (y_{n-1} - y_n)\|^q \\ &\leq \|x_n - y_{n-1}\|^q + q\langle y_{n-1} - y_n, j_q(x_n - y_n)\rangle.\end{aligned}$$

Using Schwartz's inequality, we obtain:

$$\|x_n - y_n\|^q \leq \|x_n - y_{n-1}\|^q + q\|y_{n-1} - y_n\|\|x_n - y_n\|^{q-1}. \tag{2.2.8}$$

Using inequalities (2.2.5), (2.2.7), (2.2.8) and the fact that $\{x_n\}$ and $\{y_n\}$ are bounded, we have,

$$\begin{aligned} \|x_{n+1} - y_n\|^q &\leq (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^q + C\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big) + M\lambda_n^q \\ &= (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^q + (\lambda_n\theta_n)\sigma_n + \gamma_n \end{aligned}$$

for some positive constant C where

$$\sigma_n := \frac{C\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}\Big)}{\lambda_n\theta_n} = C\left(\frac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n\theta_n}\right), \;\; \gamma_n := M\lambda_n^q.$$

Thus, using Lemma 1.2.9, the conditions $\limsup\limits_{n\to\infty} \dfrac{\Big(\frac{\theta_{n-1}}{\theta_n} - 1\Big)}{\lambda_n\theta_n} \leq 0$ and $\sum\limits_{n=1}^{\infty} \lambda_n^q < \infty$, it follows that $x_{n+1} - y_n \to 0$. From condition (2.2.2), we have that $x_n \to y^*$ and $y^* \in F(T)$. This completes the proof. ■

Corollary 2.2.2 *Let E be a q-uniformly smooth real Banach space, $q > 1$ and D be a nonempty, open and convex subset of E. Assume that $T : \overline{D} \to CB(\overline{D})$ is a multi-valued Lipschitz pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_1 \in \overline{D}$ by*

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n u_n - \lambda_n\theta_n(x_n - x_1), \;\; u_n \in Tx_n. \tag{2.2.9}$$

Then, there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n^{q-1} < \gamma_0\theta_n$, for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof We need only to show that T is bounded. For this, let B be a nonempty bounded subset of $\overline{D}$. We show that $T(B) := \bigcup\limits_{x\in B} Tx$ is bounded. Let $y_0 \in T(B)$ be fixed. Then, there exists $x_0 \in B$ such that $y_0 \in Tx_0$. Set $r_1 :=$ diameter(B) and $r_2 :=$ diameter(Tx_0). We note that r_1 and r_2 are finite. Let $y_1, y_2 \in T(B)$. Then there exist $x_i \in B$ such that $y_i \in Tx_i$, $i = 1, 2$. Using Lemma 1.1.14 and the fact that T is Lipschitz, it follows that there exist $z_1, z_2 \in Tx_0$ such that

$$\begin{aligned} \|y_1 - y_2\| &\leq \|y_1 - z_1\| + \|z_1 - z_2\| + \|z_2 - y_2\| \\ &\leq 2(Lr_1 + 1) + r_2, \end{aligned}$$

which implies that diameter$(T(B)) < \infty$. Therefore, $T(B)$ is bounded. ■

Corollary 2.2.3 *Let $E = L_p$, $1 < p < \infty$ and $q := \min\{2, p\}$. Let D be a nonempty, open and convex subset of E and $T : \overline{D} \to CB(\overline{D})$ be a multi-valued continuous (with respect to the Hausdorff metric), bounded and pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_1 \in \overline{D}$ by*

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n u_n - \lambda_n \theta_n (x_n - x_1),\ u_n \in Tx_n. \tag{2.2.10}$$

Then, there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n^{q-1} < \gamma_0 \theta_n$, for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof Since L_p spaces, $1 < p < \infty$ are q-uniformly smooth spaces with $q := \min\{2, p\}$, the proof follows from Theorem 2.2.1. ■

Corollary 2.2.4 *Let H be a real Hilbert space and D be a nonempty open convex subset of H. Assume that $T : \overline{D} \to CB(\overline{D})$ is a multi-valued continuous (with respect to the Hausdorff metric), bounded and pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from arbitrary $x_1 \in \overline{D}$ by*

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n u_n - \lambda_n \theta_n (x_n - x_1),\ u_n \in Tx_n, \tag{2.2.11}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0, 1)$ satisfying the following conditions:

(i) $\{\theta_n\}$ *decreases to 0;*

(ii) $\lambda_n(1 + \theta_n) < 1, \sum_{n=1}^{\infty} \lambda_n \theta_n = \infty, \quad \lambda_n = o(\theta_n);$

(iii) $\limsup_{n\to\infty} \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n \theta_n} \leq 0, \quad \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$

Then, there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n < \gamma_0 \theta_n$, for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof Since Hilbert spaces are 2-uniformly smooth spaces, the proof follows from Theorem 2.2.1. ■

Remark 2.2.5 *Our theorems are improvements on recent results of Ofoedu and Zegeye [65] in the sense that the class of mappings in our theorems is a superclass of the class of Lipschitz pseudo-contractions studied by Ofoedu and Zegeye [65]. In q-uniformly smooth real Banach spaces, our theorems extend Theorem 2.1.1* from multi-valued Lipschitz *pseudo-contractive mappings to the much more general class of* multi-valued continuous, bounded *and pseudo-contractive mappings.*

Furthermore, our iteration process, in the setting of q-uniformly smooth real Banach spaces, is direct, much more applicable than the process in Theorem 2.1.1 since it does not require the creation of a sub-programme to first compute w_n *at each step of the iteration process.*

CHAPTER 3

Convergence theorems for fixed points of multi-valued strictly pseudo-contractive mappings in Hilbert Spaces

3.1 Introduction

In this chapter, we extend the notion of single-valued strictly pseudo-contra ctive mappings defined by Browder and Petryshyn [12] on Hilbert spaces to the multi-valued case. This class of mappings is a superclass of multi-valued nonexpansive mappings. For such a mapping $T : K \to 2^K$ where K be a nonempty, closed and convex subset of a real Hilbert space H and $F(T) \neq \emptyset$, we construct a *Krasnoselskii-type iteration sequence* $\{x_n\}$ which we prove to be an approximate fixed point sequence of T, i.e., $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$ holds. We then prove convergence theorems under appropriate additional conditions.

Definition 3.1.1 *Let H be a real Hilbert space. A multi-valued mapping $T : D(T) \subseteq H \to CB(H)$ is said to be k-strictly pseudo-contractive if there exist $k \in (0,1)$ such that for all $x, y \in D(T)$ and for all $u \in Tx$, $v \in Ty$ we have:*

$$\Big(H(Tx, Ty)\Big)^2 \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2. \tag{3.1.1}$$

Proposition 3.1.2 *Let K be a nonempty subset of a real Hilbert space H and let $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive* *mapping. Assume that for every $x \in K$, the set Tx is weakly closed. Then, T is Lipschitzian.*

Proof Let $x, y \in D(T)$ and $u \in Tx$. From Lemma 1.2.13, there exists $v \in Ty$ such that

$$\|u - v\| \leq H(Tx, Ty). \tag{3.1.2}$$

Using the fact that T is k-strictly pseudo-contractive, and inequality (4.1.2), we obtain the following estimates

$$\begin{aligned} \Big(H(Tx, Ty)\Big)^2 &\leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2 \\ &\leq \Big(\|x - y\| + \sqrt{k}\|x - u - (y - v)\|\Big)^2, \end{aligned}$$

so that,

$$\begin{aligned} H(Tx, Ty) &\leq \|x - y\| + \sqrt{k}\Big(\|x - y\| + \|u - v\|\Big) \\ &\leq \|x - y\| + \sqrt{k}\Big(\|x - y\| + H(Tx, Ty)\Big) \end{aligned}$$

Hence,

$$H(Tx, Ty) \leq \left(\frac{1 + \sqrt{k}}{1 - \sqrt{k}}\right) \Big\|x - y\Big\|.$$

Therefore, T is L-Lipschitzian with $L := \dfrac{1 + \sqrt{k}}{1 - \sqrt{k}}$. ∎

Remark 3.1.3 *We note that for a single-valued mapping T, for each $x \in D(T)$, the set Tx is always weakly closed.*

We now prove the following lemma which will also be crucial in what follows.

Lemma 3.1.4 *Let K be a nonempty and closed subset of a real Hilbert space H and let $T : K \to P(K)$ be a k-strictly pseudo-contractive mapping. Assume that for every $x \in K$, the set Tx is weakly closed. Then, $(I - T)$ is strongly demiclosed at zero .*

Proof Let $\{x_n\} \subseteq K$ be such that $x_n \to x$ and $\text{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$. Since K is closed, we have that $x \in K$. Since, for every n, Tx_n is proximinal, let $y_n \in Tx_n$ such that $\|x_n - y_n\| = \text{dist}(x_n, Tx_n)$. Using Lemma 1.2.13, for each n, there exists $z_n \in Tx$ such that

$$\|y_n - z_n\| \leq H(Tx_n, Tx).$$

We then have

$$\begin{aligned}\|x - z_n\| &\leq \|x - x_n\| + \|x_n - y_n\| + \|y_n - z_n\| \\ &\leq \|x - x_n\| + \|x_n - y_n\| + H(Tx_n, Tx) \\ &\leq \|x - x_n\| + \|x_n - y_n\| + \frac{(1+\sqrt{k})}{(1-\sqrt{k})}\Big\|x_n - x\Big\|.\end{aligned}$$

Observing that $\text{dist}(x, Tx) \leq \|x - z_n\|$, it then follows that

$$\text{dist}(x, Tx) \leq \|x - x_n\| + \|x_n - y_n\| + \left(\frac{1+\sqrt{k}}{1-\sqrt{k}}\right)\Big\|x_n - x\Big\|.$$

Taking the limit as $n \to \infty$, we have that $\text{dist}(x, Tx) = 0$. Therefore, $x \in Tx$, completing the proof. ∎

3.2 Main results

We prove the following theorem.

Theorem 3.2.1 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T : K \to CB(K)$ is a multi-valued k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \tag{3.2.1}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$.

Proof Using Lemma 1.2.5, inequality (3.1.1) and the assumption that $Tp = \{p\}$ for all $p \in F(T)$, we obtain the following estimates

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\lambda)(x_n - p) + \lambda(y_n - p)\|^2 \\ &= (1-\lambda)\|x_n - p\|^2 + \lambda\|y_n - p\|^2 - \lambda(1-\lambda)\|x_n - y_n\|^2 \\ &\leq (1-\lambda)\|x_n - p\|^2 + \lambda\Big(H(Tx_n, Tp)\Big)^2 - \lambda(1-\lambda)\|x_n - y_n\|^2 \\ &\leq (1-\lambda)\|x_n - p\|^2 + \lambda\Big(\|x_n - p\|^2 + k\|x_n - y_n\|^2\Big) \\ &\quad -\lambda(1-\lambda)\|x_n - y_n\|^2 \\ &= \|x_n - p\|^2 + \lambda k\|x_n - y_n\|^2 - \lambda(1-\lambda)\|x_n - y_n\|^2 \\ &= \|x_n - p\|^2 - \lambda(1-k-\lambda)\|x_n - y_n\|^2. \end{aligned} \tag{3.2.2}$$

It then follows that

$$\lambda(1-k-\lambda)\sum_{n=1}^{\infty}\|x_n-y_n\|^2 \leq \|x_0-p\|^2$$

which implies that

$$\sum_{n=1}^{\infty}\|x_n-y_n\|^2 < \infty.$$

Hence, $\lim_{n\to\infty}\|x_n-y_n\|=0$. Since $y_n\in Tx_n$, we have that $\lim_{n\to\infty}\text{dist}(x_n,Tx_n)=0$. ■

We now prove the following corollaries of Theorem 3.2.1.

Corollary 3.2.2 *Let K be a nonempty, closed and convex subset of a real Hilbert space H and let $T: K\to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T)\neq\emptyset$ such that $Tp=\{p\}$ for all $p\in F(T)$. Suppose that T is hemicompact and continuous. Let $\{x_n\}$ be a sequence defined by $x_0\in K$,*

$$x_{n+1}=(1-\lambda)x_n+\lambda y_n, \tag{3.2.3}$$

where $y_n\in Tx_n$ and $\lambda\in(0,1-k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof From Theorem 3.2.1, we have that $\lim_{n\to\infty}\text{dist}(x_n,Tx_n)=0$. Since T is hemicompact (see section 1.2, p.32 for definition), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to q$ as $k\to\infty$ for some $q\in K$. Since T is continuous, we also have $\text{dist}(x_{n_k},Tx_{n_k})\to\text{dist}(q,Tq)$ as $k\to\infty$. Therefore, $\text{dist}(q,Tq)=0$ and so $q\in F(T)$. Setting $p=q$ in the proof of Theorem 3.2.1, it follows from inequality (3.2.2) that $\lim_{n\to\infty}\|x_n-q\|$ exists. So, $\{x_n\}$ converges strongly to q. This completes the proof. ■

Corollary 3.2.3 *Let K be a nonempty, compact and convex subset of a real Hilbert space H and $T: K\to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T)\neq\emptyset$ such that $Tp=\{p\}$ for all $p\in F(T)$. Suppose that T is continuous. Let $\{x_n\}$ be a sequence defined by $x_0\in K$,*

$$x_{n+1}=(1-\lambda)x_n+\lambda y_n, \tag{3.2.4}$$

where $y_n\in Tx_n$ and $\lambda\in(0,1-k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof Observing that if K is compact, every mapping $T : K \to CB(K)$ is hemicompact, the proof follows from Corollary 3.2.2. ∎

Corollary 3.2.4 *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T : K \to CB(K)$ be a multi-valued nonexpansive mapping such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is hemicompact. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \tag{3.2.5}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof Since T is nonexpansive and hemicompact then it is strictly pseudo-contractive, hemicompact and continuous. So, the proof follows from Corollary 3.2.2. ∎

Remark 3.2.5 *In Corollary 3.2.2, the continuity assumption on T can be dispensed with if we assume that for every $x \in K$, Tx is proximinal and weakly closed. In fact, we have the following result.*

Corollary 3.2.6 *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T : K \to P(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ such that for every $x \in K$, Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is hemicompact. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \tag{3.2.6}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof Following the same arguments as in the proof of Corollary 3.2.2, we have $x_{n_k} \to q$ as $k \to \infty$ and $\lim_{k\to\infty} \text{dist}(x_{n_k}, Tx_{n_k}) = 0$. Furthermore, from Lemma 3.1.4, $(I - T)$ is strongly demiclosed at zero (see section 1.2, p.32 for definition). It then follows that $q \in Tq$. Setting $p = q$ and following the same computations as in the proof of Theorem 3.2.1, we have from inequality (3.2.2) that $\lim \|x_n - q\|$ exists. Since $\{x_{n_k}\}$ converges strongly to q, it follows that $\{x_n\}$ converges strongly to $q \in F(T)$, completing the proof. ∎

Corollary 3.2.7 *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T : K \to P(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ such that for every $x \in K$, Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T satisfies condition (I). Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \tag{3.2.7}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof From Theorem 3.2.1, we have that $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$. Using the fact that T satisfies condition (I) (see section 1.2, p.32 for definition), it follows that $\lim_{n\to\infty} f(\text{dist}(x_n, F(T))) = 0$. Thus, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\} \subset F(T)$ such that

$$\|x_{n_k} - p_k\| < \frac{1}{2^k} \ \forall\, k.$$

By setting $p = p_k$ and following the same arguments as in the proof of Theorem 3.2.1, we obtain from inequality (3.2.2) that

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| < \frac{1}{2^k}.$$

We now show that $\{p_k\}$ is a Cauchy sequence in K. Notice that

$$\begin{aligned} \|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence in K and thus converges strongly to some $q \in K$. Using the fact that T is L-Lipschitzian and $p_k \to q$, we have

$$\begin{aligned} \text{dist}(p_k, Tq) &\leq H(Tp_k, Tq) \\ &\leq L\|p_k - q\|, \end{aligned}$$

so that $\text{dist}(q, Tq) = 0$ and thus $q \in Tq$. Therefore, $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Setting $p = q$ in the proof of Theorem 3.2.1, it follows from inequality (3.2.2) that $\lim_{n\to\infty} \|x_n - q\|$ exists. So, $\{x_n\}$ converges strongly to q. This completes the proof. ■

Corollary 3.2.8 *Let K be a nonempty compact convex subset of a real Hilbert space H and $T : K \to P(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ such that for every $x \in K$, Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \tag{3.2.8}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof From Theorem 3.2.1, we have that $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$. Since $\{x_n\} \subseteq K$ and K is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges strongly to some $q \in K$. Furthermore, from Lemma 3.1.4, $(I-T)$ is strongly demiclosed at zero. It then follows that $q \in Tq$. Setting $p = q$ and following the same arguments as in the proof of Theorem 3.2.1, we have from inequality (3.2.2) that $\lim \|x_n - q\|$ exists. Since $\{x_{n_k}\}$ converges strongly to q, it follows that $\{x_n\}$ converges strongly to $q \in F(T)$. This completes the proof. ∎

Corollary 3.2.9 *Let K be a nonempty, compact and convex subset of a real Hilbert space E and $T : K \to P(K)$ be a multi-valued nonexpansive mapping. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \tag{3.2.9}$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.2.10 *The recursion formula* (3.2.1) *of Theorem 3.2.1 is the Krasnoselkii type (see e.g., [53]) and seems to be superior to the recursion formula the Mann algorithm (see e.g., Mann [57]) in the sense that the recursion formula* (3.2.1) *requires less computation time than the Mann algorithm because the parameter λ in formula* (3.2.1) *is fixed in* $(0, 1-k)$ *whereas in the algorithm of Mann, λ is replaced by a sequence $\{c_n\}$ in* $(0, 1)$ *satisfying the following conditions: $\sum_{n=1}^{\infty} c_n = \infty$ and $\lim c_n = 0$. The c_n must be computed at each step of the iteration process.*

Remark 3.2.11 *Our theorem and corollaries improve convergence theorems for multi-valued nonexpansive mappings in [72, 66, 76, 49, 1, 30] in the sense that:*
(i) in our algorithm, $y_n \in Tx_n$ is arbitrary and does not have to satisfy the

very restrictive condition $\|y_n - x^*\| = \mathrm{dist}(x^*, Tx_n)$ *in the recursion formula* (1.1.30)*, and similar restrictions in the recursion formula* (1.1.31)*. These restrictions on* y_n *depend on* x^**, a fixed point that is being approximated.*
(ii) *the algorithms used in our theorem and corollaries which are proved for the much larger class of multi-valued strict pseudo-contractions are of the Krasnoselskii type.*

Let K be a nonempty, closed and convex subset of a real Hilbert space, $T : K \to P(K)$ be a multi-valued mapping. We recall that $P_T : K \to CB(K)$ be defined by

$$P_T(x) := \{y \in Tx \,:\, \|x - y\| = \mathrm{dist}(x, Tx)\}.$$

Remark 3.2.12 *In [75], the authors replace the condition* $Tp = \{p\}$ *for all* $p \in F(T)$ *with the following two restrictions:* (i) *on the sequence* $\{y_n\}$*:* $y_n \in P_T x_n$ *e.g.,* $y_n \in Tx_n$ *and* $\|y_n - x_n\| = \mathrm{dist}(x_n, Tx_n)$*. We observe that if* Tx_n *is a closed convex subset of a real Hilbert space, then* y_n *is unique and is characterized by*

$$\langle x_n - y_n, y_n - u_n\rangle \geq 0 \;\forall\, u_n \in Tx_n;$$

(ii) *on* P_T*: the authors demand that* P_T *be nonexpansive. So, the first restriction makes the recursion formula difficult to use in any possible application, while the second restriction reduces the class of mappings to which the results are applicable. This is the price to pay for removing the condition* $Tp = \{p\}$ *for all* $p \in F(T)$*.*

Remark 3.2.13 *Corollary 3.2.2 is an extension of Theorem 12 of Browder and Petryshyn [12] from single-valued to multi-valued strictly pseudo-contractive mappings.*

Remark 3.2.14 *The addition of bounded error terms to the recursion formula* (3.2.1) *leads to no generalization.*

We take the following examples where for each $x \in K$, Tx is proximinal and weakly closed.

Example 3.2.15 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be an increasing function. Define* $T : \mathbb{R} \to 2^{\mathbb{R}}$ *by:*

$$Tx = [f(x-), f(x+)] \;\forall\, x \in \mathbb{R},$$

where $f(x-) := \lim_{y \to x^-} f(y)$ *and* $f(x+) := \lim_{y \to x^+} f(y)$*. For every* $x \in \mathbb{R}$*,* Tx *is either a singleton or a closed and bounded interval. Therefore,* Tx *is always weakly closed and convex. Hence, for every* $x \in \mathbb{R}$*, the set* Tx *is proximinal and weakly closed.*

Example 3.2.16 *Let H be a real Hilbert space and $f : H \to \mathbb{R}$ be a convex continuous function. Let $T : H \to 2^H$ be the multi-valued mapping defined by:*

$$Tx = \partial f(x)\ \forall\, x \in H,$$

where $\partial f(x)$ is the subdifferential of f at x, i.e.,

$$\partial f(x) = \{z \in H\ :\ \langle z, y - x\rangle \leq f(y) - f(x)\ \forall\, y \in H\}.$$

It is well known that for every $x \in H$, $\partial f(x)$ is nonempty, weakly closed and convex. Therefore, since H is a real Hilbert space, it then follows that for every $x \in H$, the set Tx is proximinal and weakly closed.

The condition $Tp = \{p\}$ for all $p \in F(T)$ which is imposed in all our theorems of this chapter is not crucial. We show how this condition can be replaced with another condition which does not assume that the multi-valued mapping is single-valued on the nonempty fixed point set. This can be found in the paper by Shahzad and Zegeye [75].

We will need the following result.

Lemma 3.2.17 (Song and Cho [77]) *Let K be a nonempty subset of a real Banach space and $T : K \to P(K)$ be a multi-valued mapping. Then, the following are equivalent.*

(i) $x^* \in F(T)$;

(ii) $P_T(x^*) = \{x^*\}$;

(iii) $x^* \in F(P_T)$. *Moreover,* $F(T) = F(P_T)$.

Remark 3.2.18 *We observe from Lemma 3.2.17 that if $T : K \to P(K)$ is any multi-valued mapping with $F(T) \neq \emptyset$, then the corresponding multi-valued mapping P_T satisfies $P_T(p) = \{p\}$ for all $p \in F(P_T)$, condition imposed in all our theorems and corollaries. Consequently, examples of multi-valued mappings $T : K \to CB(K)$ satisfying the condition $Tp = \{p\}$ for all $p \in F(T)$ abound.*

We now prove the following theorem in which the condition $Tp = \{p\}$ for all $p \in F(T)$ is dispensed with.

Theorem 3.2.19 *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T : K \to P(K)$ be a multi-valued mapping such that $F(T) \neq \emptyset$. Assume that P_T is k-strictly pseudo-contractive. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary point $x_1 \in K$ by*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \tag{3.2.10}$$

where $y_n \in P_T(x_n)$ and $\lambda \in (0, 1-k)$. Then, $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$.

Proof Let $p \in F(T)$. Using the recursion formula (3.2.10), Lemma 1.2.5, the fact that P_T is k-strictly pseudo-contractive and Lemma 3.2.17, we obtain the following estimates

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\lambda)(x_n - p) + \lambda(y_n - p)\|^2 \\
&= (1-\lambda)\|x_n - p\|^2 + \lambda\|y_n - p\|^2 - \lambda(1-\lambda)\|x_n - y_n\|^2 \\
&\leq (1-\lambda)\|x_n - p\|^2 + \lambda\Big[H\Big(P_T(x_n), P_T(p)\Big)\Big]^2 \\
&\quad -\lambda(1-\lambda)\|x_n - y_n\|^2 \\
&\leq (1-\lambda)\|x_n - p\|^2 + \lambda\Big(\|x_n - p\|^2 + k\|x_n - y_n\|^2\Big) \\
&\quad -\lambda(1-\lambda)\|x_n - y_n\|^2 \\
&= \|x_n - p\|^2 + \lambda k\|x_n - y_n\|^2 - \lambda(1-\lambda)\|x_n - y_n\|^2 \\
&= \|x_n - p\|^2 - \lambda(1-k-\lambda)\|x_n - y_n\|^2.
\end{aligned}$$

It then follows that

$$\lambda(1-k-\lambda)\sum_{n=1}^{\infty}\|x_n - y_n\|^2 \leq \|x_0 - p\|^2,$$

which implies that

$$\sum_{n=1}^{\infty}\|x_n - y_n\|^2 < \infty.$$

Hence, $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Since $y_n \in P_T(x_n)$ (and hence, $y_n \in Tx_n$), we have that $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$. This completes the proof. ■

We conclude this chapter with examples of multi-valued mappings T_1 and T_2 for which P_{T_1} P_{T_2} are strictly pseudo-contractive, a condition assumed in Theorem 3.2.19.

Example 3.2.20 *Let $H = \mathbb{R}$, with the usual metric and $T : \mathbb{R} \to CB(\mathbb{R})$ be the multi-valued mapping defined by*

$$T_1x = \begin{cases} [0, \frac{x}{2}], & x \in (0, \infty) \\ [\frac{x}{2}, 0], & x \in (-\infty, 0]. \end{cases}$$

Then P_{T_1} is strictly pseudo-contractive. In fact, $P_{T_1}x = \{\frac{x}{2}\}$ for all $x \in \mathbb{R}$.

Example 3.2.21 *It is clear that if T is $*$-nonexpansive (see section 1.2 for definition), then P_T is nonexpansive and hence, strictly pseudo-contractive. We also note that $*$-nonexpansiveness is different from nonexpansiveness for multi-valued mappings. Let $K = [0, +\infty)$ and T_2 be defined by $T_2x = [x, 2x]$ for $x \in K$. Then, $P_{T_2}(x) = \{x\}$ for $x \in K$ and thus it is nonexpansive and hence strictly pseudo-contractive. Note also that T is $*$-nonexpansive but is not nonexpansive (see [78]).*

CHAPTER 4

Krasnoselskii-type algorithm for fixed points of multi-valued strictly pseudo-contractive mappings in Banach spaces

4.1 Introduction

In this chapter, we extend the notion of single-valued strictly pseudo-contractive mapping to the multi-valued case in *arbitrary real normed space* E as follows:

Definition 4.1.1 *A multi-valued mapping* $T : D(T) \subset E \to CB(E)$ *is called* k-strictly pseudo-contractive *if there exists* $k \in (0,1)$ *such that for all* $x, y \in D(T)$,

$$k\big(H(Ax, Ay)\big)^2 \leq \langle u - v, j(x-y)\rangle \quad \forall u \in Ax, v \in Ay. \tag{4.1.1}$$

where $A := I - T$ *and* I *is the identity mapping on* E.

We observe that if T is single-valued, then inequality (4.1.1) reduces to inequality (1.1.28).

Let $T : K \to CB(K)$ be a multi-valued strictly pseudo-contractive mapping with a nonempty fixed point set, where K is a closed convex nonempty subset of a q-uniformly smooth real Banach space, $q > 1$. A Krasnoselskii-type iteration sequence $\{x_n\}$ is constructed and proved to be an approximate fixed point sequence of T, *i.e.*, $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$. This result is then applied to prove strong convergence theorems for fixed point of T under additional appropriate mild conditions.

In the sequel, we need the following results.

Proposition 4.1.2 *Let K be a nonempty subset of a real Banach space E and let $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping. Assume that for every $x \in K$, Tx is weakly closed. Then, T is Lipschitzian.*

Proof We first observe that for any $x \in D(T)$, the set Tx is weakly closed if and only if the set Ax is weakly closed. Now let $x, y \in D(T)$ and $u \in Ax$. From Lemma 1.2.13, there exists $v \in Ay$ such that

$$\|u - v\| \leq H(Ax, Ay). \tag{4.1.2}$$

Using the fact that T is k-strictly pseudo-contractive, and inequality (4.1.2), we have

$$\begin{aligned} k\Big(H(Ax, Ay)\Big)^2 &\leq \langle u - v, j(x - y)\rangle \\ &\leq \|u - v\|\|x - y\| \\ &\leq H(Ax, Ay)\|x - y\|. \end{aligned}$$

So,

$$H(Ax, Ay) \leq \frac{1}{k}\|x - y\| \qquad \forall x, y \in D(T). \tag{4.1.3}$$

From definition of the Hausdorff distance, we have

$$H(Tx, Ty) \leq H(Ax, Ay) + \|x - y\| \quad \forall x, y \in D(T). \tag{4.1.4}$$

Using (4.1.3) and (4.1.4) we obtain

$$H(Tx, Ty) \leq L_k\|x - y\| \qquad \forall x, y \in D(T), \text{where } L_k := \frac{1 + k}{k}.$$

Therefore, T is L_k-Lipschitzian. ∎

Remark 4.1.3 *We note that* for a single-valued map T, *for each $x \in D(T)$, the set Tx is always weakly closed.*

Lemma 4.1.4 *Let $q > 1$, E be a q-uniformly smooth real Banach space, $k \in (0,1)$. Suppose $T : D(T) \subset E \to CB(E)$ is a multi-valued mapping with $F(T) \neq \emptyset$, and for all $x \in D(T)$, $x^* \in F(T)$,*

$$k\left(H(Ax, Ax^*)\right)^2 \leq \langle u - v^*, j(x - x^*)\rangle \qquad \forall u \in Ax, v^* \in Ax^*, \quad (4.1.5)$$

where $A := I - T$, I is the identity mapping on E. If $Tx^* = \{x^*\}$ for all $x^* \in F(T)$. Then, the inequality

$$\langle x - y, j_q(x - x^*)\rangle \geq k^{q-1}\|x - y\|^q, \qquad \forall\, x \in D(T), \forall\, y \in Tx$$

holds.

Proof Let $x \in D(T), u \in Ax, x^* \in F(T)$. Then, from inequality (4.1.5), the definition of Hausdorff metric and the assumption that $Tx^* = \{x^*\}$, we have

$$\begin{aligned} k\left(H(Ax, Ax^*)\right)^2 &\leq \|u\|\|x - x^*\| \\ &\leq H(Ax, Ax^*)\|x - x^*\|. \end{aligned}$$

So

$$kH(Ax, Ax^*) \;\leq\; \|x - x^*\| \qquad \forall x \in D(T), \quad x^* \in F(T). \qquad (4.1.6)$$

Therefore, for all $x \in D(T), y \in Tx, x^* \in F(T)$ such that $x \neq x^*$, using inequalities (4.1.5) and (4.1.6) and the fact that $j_q(x-x^*) = \|x-x^*\|^{q-2}j(x-x^*)$ (see Remark 1.2.2), we obtain

$$\begin{aligned} \langle x - y, j_q(x - x^*)\rangle &= \|x - x^*\|^{q-2}\langle x - y, j(x - x^*)\rangle \\ &\geq k^{q-1}\left(H(Ax, Ax^*)\right)^q \\ &\geq k^{q-1}\|x - y\|^q. \end{aligned}$$

This completes the proof. ■

Lemma 4.1.5 *Let K be a nonempty closed subset of a real Banach space E and let $T : K \to P(K)$ be a k-strictly pseudo-contractive mapping. Assume that for every $x \in K$, Tx is weakly closed. Then, $(I - T)$ is strongly demiclosed at zero.*

Proof Let $\{x_n\} \subseteq K$ be such that $x_n \to x$ and $\text{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$. Since K is closed, we have that $x \in K$. Since, for every n, Tx_n is proximinal, let $y_n \in Tx_n$ such that $\|x_n - y_n\| = \text{dist}(x_n, Tx_n)$. Using Lemma 1.2.13, for each n, there exists $z_n \in Tx$ such that.

$$\|y_n - z_n\| \leq H(Tx_n, Tx).$$

We then have:

$$\begin{aligned}\|x - z_n\| &\leq \|x - x_n\| + \|x_n - y_n\| + \|y_n - z_n\| \\ &\leq \|x - x_n\| + \|x_n - y_n\| + H(Tx_n, Tx) \\ &\leq \|x - x_n\| + \|x_n - y_n\| + L_k\|x_n - x\|.\end{aligned}$$

Observing that $\text{dist}(x, Tx) \leq \|x - z_n\|$, it then follows that

$$\text{dist}(x, Tx) \leq \|x - x_n\| + \|x_n - y_n\| + L_k\|x_n - x\|.$$

Taking limit as $n \to \infty$, we have that $\text{dist}(x, Tx) = 0$. Therefore, $x \in Tx$ and the proof is complete. ■

For the remaining part of this chapter, d_q denotes the constant appearing in Lemma 1.2.7. Let $\mu := \min\left\{1, \left(\frac{qk^{q-1}}{d_q}\right)^{\frac{1}{q-1}}\right\}$.

4.2 Main results

We prove the following theorem.

Theorem 4.2.1 *Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Suppose that $T : K \to CB(K)$ is a multi-valued k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. and such that $Tp = \{p\}$ for all $p \in F(T)$. For arbitrary $x_1 \in K$ and $\lambda \in (0, \mu)$, let $\{x_n\}$ be a sequence defined iteratively by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{4.2.1}$$

where $y_n \in Tx_n$. Then, $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$.

Proof Let $x^* \in F(T)$. Then, using the recursion formula (4.2.1), Lemmas 1.2.7 and 4.1.4, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^q &= \|x_n - x^* - \lambda(x_n - y_n)\|^q \\ &\leq \|x_n - x^*\|^q - \lambda q\langle x_n - y_n, j_q(x_n - x^*)\rangle + \lambda^q d_q\|x_n - y_n\|^q \\ &\leq \|x_n - x^*\|^q - q\lambda k^{q-1}\|x_n - y_n\|^q + \lambda^q d_q\|x_n - y_n\|^q \\ &= \|x_n - x^*\|^q - \lambda\left(qk^{q-1} - d_q\lambda^{q-1}\right)\|x_n - y_n\|^q. \qquad (4.2.2)\end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^q < \infty.$$

Hence, $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Since $y_n \in Tx_n$, we have that $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$. ■

We now prove the following corollaries of Theorem 4.2.1.

Corollary 4.2.2 *Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Let $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ and such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is continuous and hemicompact. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{4.2.3}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof From Theorem 4.2.1, we have $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$. Since T is hemicompact (see section 1.2 for definition), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p$ for some $p \in K$. Since T is continuous, we have $\text{dist}(x_{n_j}, Tx_{n_j}) \to \text{dist}(p, Tp)$. Therefore, $\text{dist}(p, Tp) = 0$ and so $p \in F(T)$. Setting $x^* = p$ in the proof of Theorem 4.2.1, it follows from inequality (4.2.2) that $\lim_{n\to\infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to p. This completes the proof. ■

Corollary 4.2.3 *Let $q > 1$ be a real number and K be a nonempty, compact and convex subset of a q-uniformly smooth real Banach space E. Let $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ and such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is continuous. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{4.2.4}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof Observing that if K is compact, every mapping $T : K \to CB(K)$ is hemicompact, the proof follows from Corollary 4.2.2. ■

Remark 4.2.4 *In Corollary 4.2.2, the continuity assumption on T can be dispensed with if we assume that for every $x \in K$, the set Tx is proximinal and weakly closed. In fact, we have the following result.*

Corollary 4.2.5 *Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Let $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ and such that for every $x \in K$, Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is hemicompact. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{4.2.5}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof Following the same arguments as in the proof of Corollary 4.2.2, we have $x_{n_j} \to p$ and $\lim\limits_{j\to\infty} \text{dist}(x_{n_j}, Tx_{n_j}) = 0$. Furthermore, from Lemma 4.1.5, $(I - T)$ is strongly demiclosed at zero. It then follows that $p \in Tp$. Setting $x^* = p$ and following the same computations as in the proof of Theorem 4.2.1, we have from inequality (4.2.2) that $\lim \|x_n - p\|$ exists. Since $\{x_{n_j}\}$ converges strongly to p as $j \to \infty$, it follows that $\{x_n\}$ converges strongly to $p \in F(T)$, the proof is completed. ■

Convergence theorems have been proved in real Hilbert spaces for multi-valued nonexpansive mappings T under the assumption that T satisfies *condition (I)* defined in section 1.2 (see *e.g.,* [66, 75]). The following corollary extends such theorems to multi-valued strictly pseudo-contractive mappings and to q-uniformly smooth real Banach spaces.

Corollary 4.2.6 *Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Let $T : K \to P(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ and such that for every $x \in K$, Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T satisfies condition (I). Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{4.2.6}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof From Theorem 4.2.1, we have $\lim\limits_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$. Using the fact that T satisfies condition (I), it follows that $\lim\limits_{n\to\infty} f(\text{dist}(x_n, F(T))) = 0$. Thus there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F(T)$ such that

$$\|x_{n_j} - p_j\| < \frac{1}{2^j} \;\forall j \in \mathbb{N}.$$

By setting $x^* = p_j$ and following the same arguments as in the proof of Theorem 4.2.1, we obtain from inequality (4.2.2) that

$$\|x_{n_{j+1}} - p_j\| \leq \|x_{n_j} - p_j\| < \frac{1}{2^j}.$$

We now show that $\{p_j\}$ is a Cauchy sequence in K. Notice that

$$\begin{aligned} \|p_{j+1} - p_j\| &\leq \|p_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - p_j\| \\ &< \frac{1}{2^{j+1}} + \frac{1}{2^j} < \frac{1}{2^{j-1}}. \end{aligned}$$

This shows that $\{p_j\}$ is a Cauchy sequence in K and thus converges strongly to some $p \in K$. Using the fact that T is L-Lipschitzian and $p_j \to p$, we have

$$\begin{aligned} \text{dist}(p_j, Tp) &\leq H(Tp_j, Tp) \\ &\leq L\|p_j - p\|, \end{aligned}$$

so that $\text{dist}(p, Tp) = 0$ and thus $p \in Tp$. Therefore, $p \in F(T)$ and $\{x_{n_j}\}$ converges strongly to p. Setting $x^* = p$ in the proof of Theorem 4.2.1, it follows from inequality (4.2.2) that $\lim_{n\to\infty} \|x_n - p\|$ exists. So, $\{x_n\}$ converges strongly to p. This completes the proof. ■

Corollary 4.2.7 *Let $q > 1$ be a real number and K be a nonempty, compact and convex subset of a q uniformly smooth real Banach space E. Let $T : K \to P(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ and such that for every $x \in K$, the set Tx is weakly closed and $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{4.2.7}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof From Theorem 4.2.1, we have $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$. Since $\{x_n\} \subseteq K$ and K is compact, $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ that converges strongly to some $p \in K$. Furthermore, from Lemma 4.1.5, the mapping $(I - T)$ is strongly demiclosed at zero. It then follows that $p \in Tp$. Setting $x^* = p$ and following the same arguments as in the proof of Theorem 4.2.1, we have from inequality (4.2.2) that $\lim \|x_n - p\|$ exists. Since $\{x_{n_j}\}$ converges strongly to q, it follows that $\{x_n\}$ converges strongly to $p \in F(T)$. This completes the proof. ■

Corollary 4.2.8 *Let $q > 1$ be a real number and K be a nonempty compact convex subset of a q uniformly smooth real Banach space E. Let $T : K \to P(K)$ be a multi-valued nonexpansive mapping. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined iteratively from $x_1 \in K$,*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{4.2.8}$$

where $y_n \in Tx_n$ and $\lambda \in (0, \mu)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Remark 4.2.9 *The recursion formula* (4.2.1) *of Theorem 4.2.1 is of the Krasnoselkii type (see e.g., [53]) and is known to be superior to the recursion formula of the Mann algorithm (see e.g., Mann [57]) in the sense that the recursion formula* (4.2.1) *requires less computation time than the formula of the Mann algorithm because the parameter λ in formula* (4.2.1) *is fixed in* $(0, 1-k)$ *whereas in the algorithm of Mann, λ is replaced by a sequence $\{c_n\}$ in* $(0, 1)$ *satisfying the following conditions:* $\sum_{n=1}^{\infty} c_n = \infty,\ \lim c_n = 0$. *The c_n must be computed at each step of the iteration process.*

Remark 4.2.10 *Our theorems in this chapter are important generalizations of several important recent results in the sense that our theorems extend results proved for multi-valued nonexpansive mappings in real Hilbert spaces (see e.g., [72, 66, 76, 49, 1]) to the much more larger class of multi-valued strictly pseudo-contractive mappings and in the much larger class of q-uniformly smooth real Banach spaces.*

The condition $Tp = \{p\}$ for all $p \in F(T)$ which is imposed in all our theorems of this chapter is not crucial. We show, as in the Hilbert space case in chapter 3, how this condition can be replaced with another condition which does not assume that the multi-valued mapping is single-valued on the nonempty fixed point set. This can be found in the paper by Shahzad and Zegeye [75].

We will need the following result which we restate for convenience. We recall that given nonempty, closed and convex subset K of a real Banach space E, associated with a multi-valued mapping $T : K \to P(K)$ is the mapping $P_T : K \to CB(K)$ be defined by

$$P_T(x) := \{y \in Tx \,:\, \|x - y\| = \text{dist}(x, Tx)\}.$$

Lemma 4.2.11 (Song and Cho [77]) *Let K be a nonempty subset of a real Banach space and $T : K \to P(K)$ be a multi-valued map. Then, the following are equivalent.*

(i) $x^* \in F(T)$;

(ii) $P_T(x^*) = \{x^*\}$;

(iii) $x^* \in F(P_T)$. *Moreover,* $F(T) = F(P_T)$.

Remark 4.2.12 *We observe from Lemma 4.2.11 that if $T : K \to P(K)$ is any multi-valued map with $F(T) \neq \emptyset$, then the corresponding multi-valued mapping P_T satisfies $P_T(p) = \{p\}$ for all $p \in F(P_T)$, condition imposed in all our theorems and corollaries. Consequently, examples of multi-valued mappings $T : K \to CB(K)$ satisfying the condition $Tp = \{p\}$ for all $p \in F(T)$ abound.*

Furthermore, we have the following theorem in which the condition $Tp = \{p\}$ for all $p \in F(T)$ is dispensed with.

Theorem 4.2.13 *Let $q > 1$ be a real number and K be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E. Suppose that $T : K \to CB(K)$ is a multi-valued mapping such that $F(T) \neq \emptyset$. Assume that P_T is k-strictly pseudo-contractive. For arbitrary $x_1 \in K$ and $\lambda \in (0, \mu)$, let $\{x_n\}$ be a sequence defined iteratively by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \tag{4.2.9}$$

where $y_n \in P_T(x_n)$. Then, $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$.

CHAPTER 5

Krasnoselskii Algorithm for fixed points of multi-valued quasi-nonexpansive Mappings in Certain Banach Spaces

5.1 Introduction

In this chapter we extend some of the results known for single-valued quasi-nonexpansive mappings to the multi-valued case.

Suppose that $T : K \to CB(K)$ is a multi-valued quasi-nonexpansive mapping. We construct a Krasnoselskii-type iteration sequence $\{x_n\}$ and prove that it is an approximate fixed point sequence of T, that is, $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$ holds. We prove convergence theorems under appropriate additional mild conditions. We need the following in the sequel.

Lemma 5.1.1 *(Acedo and Xu [2]) Let X be a metric space and $T : X \to K(X)$ be a multi-valued mapping. Then T is $*$-nonexpansive if and only if P_T is nonexpansive.*

Lemma 5.1.2 *(Acedo and Xu [2]) Let E be a uniformly convex real Banach space and let D be a closed convex and bounded subset of E. Suppose $T : D \to K(D)$ is $*$-nonexpansive multi-valued mapping . Then T has a fixed point.*

5.2 Main Theorems

We start this section by showing that a result of Edelstein [37] is easily extended to the class of quasi-nonexpansive continuous mappings and for general $\lambda \in (0,1)$. We use the method of Edelstein [37].

Theorem 5.2.1 *Let K be a closed convex nonempty subset of a strictly convex real Banach space E and $T : K \to K$ be a continuous quasi-nonexpansive mapping such that $T(K)$ is contained in a compact subset K_1 of K. Then for each $\lambda \in (0,1)$ and for each $x_0 \in K$, the sequence $\{T_\lambda^n x_0\}$, where $T_\lambda : K \to K$ is defined by $T_\lambda x = ((1-\lambda)I + \lambda T)\,x$, converges strongly to a fixed point of T.*

Proof Clearly, the quasi-nonexpansiveness of T implies that of T_λ. Therefore, for each $x \in K \setminus F(T)$, and $x^* \in F(T)$,

$$\|T_\lambda x - x^*\| = \|(1-\lambda)(x-x^*) + \lambda(Tx - x^*)\| \le \|x - x^*\|.$$

This implies

$$\left\|(1-\lambda)\frac{(x-x^*)}{\|x-x^*\|} + \lambda\frac{(Tx-x^*)}{\|x-x^*\|}\right\| \le 1.$$

Strict convexity of E and quasi-nonexpansiveness of T give

$$\left\|(1-\lambda)\frac{(x-x^*)}{\|x-x^*\|} + \lambda\frac{(Tx-x^*)}{\|x-x^*\|}\right\| < 1.$$

Hence, for all $x \in K \setminus F(T)$, $x^* \in F(T)$ we have,

$$\|T_\lambda x - x^*\| < \|x - x^*\|. \tag{5.2.1}$$

We note that $\{T_\lambda^n x_0\}$ is contained in $\bar{co}\,(K_1 \cup \{x_0\})$ which, by Mazur Theorem [58], is compact. Therefore, $\{T_\lambda^n x_0\}$ has a convergent subsequence $\{T_\lambda^{n_j} x_0\}$ with limit $p \in K$. By the quasi-nonexpansiveness of T_λ, we have that for any $q \in F(T)$ and for any $n \ge 0$,

$$\|T_\lambda^{n+1} x_0 - q\| \le \|T_\lambda^n x_0 - q\|.$$

Hence, $\lim\limits_{n\to\infty} \|T_\lambda^n x_0 - q\|$ exists. Therefore, convergence of a subsequence of $\{T_\lambda^n x_0\}$ to an element of $F(T)$ implies the convergence of the whole sequence to the same element. So, to prove the theorem, it suffices to show that the limit p of $\{T_\lambda^{n_j} x_0\}$ belongs to $F(T)$. We do this by contradiction. Suppose $T(p) \ne p$. Then no term of the sequence $\{T_\lambda^n x_0\}$ is a fixed point of T; for if there exists $N \ge 1$, such that $T_\lambda^N x_0 \in F(T)$, then $T_\lambda^n x_0 = T_\lambda^N x_0, \quad \forall n \ge N$

and so the whole sequence $\{T_\lambda^n x_0\}$ converges to $T_\lambda^N x_0 \in F(T)$ which gives $p \in F(T)$, a contradiction.

Hence, from (5.2.1), for any member $q \in F(T)$ we have

$$\|T_\lambda^{n+1}x - q\| < \|T_\lambda^n x - q\|, \quad \forall n \geq 1. \tag{5.2.2}$$

By continuity of T_λ at p, setting $r := \frac{1}{2}\Big(\|p-q\|-\|T_\lambda p-q\|\Big) > 0$, and $B := \{w \in K : \|w-T_\lambda(p)\| < r\}$, we obtain an open ball B' centered at p such that $T_\lambda(B') \subset B$. Convergence of $\{T_\lambda^{n_j}x_0\}$ to p guarantees the existence $k \geq 1$ such that $T_\lambda^k x_0 \in B'$. So $T_\lambda^{k+1}x_0 \in B$. Using (5.2.2) we have for each $i \geq 1$,

$$\|T_\lambda^{k+i}x_0-q\| < \|T_\lambda^{k+(i-1)}x_0-q\| < \cdots < \|T_\lambda^{k+1}x_0-q\| \leq \|T_\lambda^{k+1}x_0-T_\lambda p\|+\|T_\lambda p-q\|.$$

Therefore,

$$\|T_\lambda^{k+i}x_0 - q\| < r + \|T_\lambda p - q\| = \frac{1}{2}\Big(\|p - q\| + \|T_\lambda p - q\|\Big).$$

This now implies that for each $i \geq 1$,

$$\begin{aligned}
\|T_\lambda^{k+i}x_0 - p\| &= \|T_\lambda^{k+i}x_0 - q + q - p\| \\
&\geq \|q - p\| - \|T_\lambda^{k+i}x_0 - q\| \\
&> \|q - p\| - \frac{1}{2}\Big(\|T_\lambda(p) - q\| + \|p - q\|\Big) \\
&= \frac{1}{2}\Big(\|p - q\| - \|T_\lambda(p) - q\|\Big) = r,
\end{aligned}$$

which contradicts the fact that $\{T_\lambda^{n_j}x_0\}$ converges to p. Thus $p \in F(T)$ and the proof is complete.

■

We now prove the main theorem of this chapter.

Theorem 5.2.2 *Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E. Suppose that $T : K \to CB(K)$ is a multi-valued quasi-nonexpansive mapping such that $Tp = \{p\}$ for some $p \in F(T)$. Then for any fixed $x_0 \in K$ and arbitrary $\lambda \in (0,1)$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \quad n \geq 0, \tag{5.2.3}$$

where $y_n \in Tx_n$. Then, $\lim_{n\to\infty} \mathrm{dist}(x_n, Tx_n) = 0$.

Proof We first note that for any $x, y, z \in K$ such that $Ty = \{z\}$, we have

$$\begin{aligned} H(Tx,Ty) &= \max\left\{\sup_{u\in Tx}\operatorname{dist}(u,Ty), \sup_{u\in Ty}\operatorname{dist}(u,Tx)\right\} \\ &= \max\{\sup_{u\in Tx}\|u-z\|, \operatorname{dist}(z,Tx)\} \\ &= \sup_{u\in Tx}\|u-z\| \\ &\geq \|u-z\| \quad \forall u \in Tx. \end{aligned} \tag{5.2.4}$$

We next show that $\{x_n\}$ is bounded. Let $p \in F(T)$ such that $Tp = \{p\}$. Then using inequality (5.2.4) and the assumption on T we have

$$\begin{aligned} \|x_{n+1}-p\| &= \|(1-\lambda)(x_n-p)+\lambda(y_n-p)\| \\ &\leq (1-\lambda)\|x_n-p\|+\lambda\|y_n-p\| \\ &\leq (1-\lambda)\|x_n-p\|+\lambda H(Tx_n,Ty_n) \\ &\leq (1-\lambda)\|x_n-p\|+\lambda\|x_n-p\| \\ &= \|x_n-p\|, \quad \forall n \geq 0. \end{aligned}$$

This implies that $\{x_n\}$ is bounded and, so $\{y_n\}$ is bounded. We also have that the limit $\lim_{n\to\infty}\|x_n-p\|$ exists.
To prove the assertion of the theorem, let $R > 0$ such that $\{x_n\}, \{y_n\}$ are contained in $B(0,R)$. Then by Lemma 1.2.8, there exists a continuous, convex, and strictly increasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for all $n \geq 0$ we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|(1-\lambda)(x_n-p)+\lambda(y_n-p)\|^2 \\ &\leq (1-\lambda)\|x_n-p\|^2+\lambda\|y_n-p\|^2-\lambda(1-\lambda)g\left(\|x_n-y_n\|\right) \\ &\leq (1-\lambda)\|x_n-p\|^2+\lambda H\left(Tx_n,Tp\right)^2-\lambda(1-\lambda)g\left(\|x_n-y_n\|\right) \\ &\leq (1-\lambda)\|x_n-p\|^2+\lambda\|x_n-p\|^2-\lambda(1-\lambda)g\left(\|x_n-y_n\|\right) \\ &= \|x_n-p\|^2-\lambda(1-\lambda)g\left(\|x_n-y_n\|\right). \end{aligned}$$

It then follows that

$$\lambda(1-\lambda)g\left(\|x_n-y_n\|\right) \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2, \quad \forall n \geq 0.$$

Since $\lim_{n\to\infty}\|x_n-p\|$ exists and $\lambda \in (0,1)$, we have $\lim_{n\to\infty} g\left(\|x_n-y_n\|\right) = 0$. The fact that g is strictly increasing and $g(0) = 0$, imply that $\lim_{n\to\infty}\|x_n-y_n\| = 0$. Since $y_n \in Tx_n$, we have that $\lim_{n\to\infty}\operatorname{dist}(x_n,Tx_n) = 0$. This completes the proof. ∎

The following corollary follows.

Corollary 5.2.3 *Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E. Suppose that $T : K \to K$ is a quasi-nonexpansive mapping. Then for any fixed $x_0 \in K$ and arbitrary $\lambda \in (0,1)$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n. \tag{5.2.5}$$

Then, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Corollary 5.2.4 *Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E and let $T : K \to CB(K)$ be a multi-valued quasi-nonexpansive mapping such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is hemicompact and continuous with respect to the Hausdorff metric. Let $\{x_n\}$ be a sequence generated by* (5.2.3)*. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof From Theorem 5.2.2, we have that $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$. Since $\{x_n\}$ is bounded and T is hemicompact (see section 1.2 for definition), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q$ as $k \to \infty$ for some $q \in K$. Since T is continuous, we also have $\text{dist}(x_{n_k}, Tx_{n_k}) \to \text{dist}(q, Tq)$ as $k \to \infty$. Therefore, $\text{dist}(q, Tq) = 0$ and so, by closedness of Tq, $q \in F(T)$. Setting $p = q$ in the proof of Theorem 5.2.2, it follows from (5.2.5) that $\lim_{n\to\infty} \|x_n - q\|$ exists. So, $\{x_n\}$ converges strongly to q. This completes the proof. ■

Corollary 5.2.5 *Let K be a nonempty, compact and convex subset of uniformly convex real Banach space E and $T : K \to CB(K)$ be a multi-valued mapping such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is continuous with respect to the Hausdorff metric. Let $\{x_n\}$ be a sequence generated by* (5.2.3)*. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof The proof follows from Corollary 5.2.4 and the fact that compactness of K implies hemicompactness of T. ■

Corollary 5.2.6 *Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E and $T : K \to CB(K)$ be a multi-valued nonexpansive mapping with a nonempty fixed point set $F(T)$ such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is hemicompact. Let $\{x_n\}$ be a sequence generated by* (5.2.3)*. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof Since T is nonexpansive with nonempty fixed point set, it is quasi-nonexpansive and continuous with respect to the Hausdorff metric. So, the proof follows from Corollary 5.2.4. ∎

Corollary 5.2.7 *Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E and $T : K \to P(K)$ be a multi-valued quasi-nonexpansive mapping such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T satisfies condition (I). Let $\{x_n\}$ be a sequence generated by* (5.2.3). *Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof From Theorem 5.2.2, we have that $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$. Using the fact that T satisfies condition (I) (see section 1.2 for definition), it follows that $\lim_{n\to\infty} f(\operatorname{dist}(x_n, F(T))) = 0$ which in turn, using the nondecreasing property of f, gives $\lim_{n\to\infty} \operatorname{dist}(x_n, F(T)) = 0$. Thus there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\} \subset F(T)$ such that

$$\|x_{n_k} - p_k\| < \frac{1}{2^k} \ \forall\, k.$$

By setting $p = p_k$ and following the same arguments as in the proof of Theorem 5.2.2, we obtain from inequality (5.2.5) that

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| < \frac{1}{2^k}.$$

We now show that $\{p_k\}$ is a Cauchy sequence in K. Notice that

$$\begin{aligned} \|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence in K and thus converges strongly to some $q \in K$. Using the fact that T is quasi-nonexpansive and $p_k \to q$, we have

$$\begin{aligned} \operatorname{dist}(p_k, Tq) &\leq H(Tp_k, Tq) \\ &\leq \|p_k - q\|, \end{aligned}$$

so that $\operatorname{dist}(q, Tq) = 0$ and thus $q \in Tq$. Therefore, $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Setting $p = q$ in the proof of Theorem 5.2.2, it follows from inequality (5.2.5) that $\lim_{n\to\infty} \|x_n - q\|$ exists. So, $\{x_n\}$ converges strongly to q. This completes the proof. ∎

Corollary 5.2.8 *Let D be a nonempty, closed and convex subset of a uniformly convex real Banach space E and let $T : K \to K(D)$ be a multi-valued $*$-nonexpansive mapping with nonempty fixed point set $F(T)$. Suppose $\lambda \in (0,1)$ and let $\{x_n\}$ be a sequence defined by $x_0 \in D$ fixed and*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \quad n \geq 0, \tag{5.2.6}$$

where $y_n \in P_T x_n$. Then, $\lim_{n\to\infty} \mathrm{dist}(x_n, P_T x_n) = 0$. Moreover, if P_T is hemi-compact or it satisfies condition (I), then $\{x_n\}$ converges to a fixed point of T.

Proof Using the fact that T is $*$-nonexpansive (see section 1.2 for definition), we obtain by virtue of Lemma 5.1.1 that P_T is nonexpansive. Since $F(T) \neq \emptyset$ and $F(T) = F(P_T)$, it follows that P_T is quasi-nonexpansive. The results then follow from Theorem 5.2.2, Corollaries 5.2.4 and 5.2.7. ■

Corollary 5.2.9 *Let D be a nonempty, closed, convex and bounded subset of a uniformly convex real Banach space E and let $T : K \to K(D)$ be a multi-valued $*$-nonexpansive mapping. Suppose $\lambda \in (0,1)$ and let $\{x_n\}$ be a sequence defined by $x_0 \in D$ fixed and*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \quad n \geq 0, \tag{5.2.7}$$

where $y_n \in P_T x_n$. Then, $\lim_{n\to\infty} \mathrm{dist}(x_n, P_T x_n) = 0$. Moreover, if P_T is hemi-compact or it satisfies condition (I), then $\{x_n\}$ converges to a fixed point of T.

Proof Using Lemma 5.1.2, the proof follows from Corollary 5.2.8. ■

Ishikawa [45] proved that the if the Mann sequence $\{x_n\}$ *is bounded*, then it is an approximate fixed point sequence. Observe that in this result of Ishikawa, one can choose $\lambda_n = \lambda \in (0, b], 0 < b < 1, \ \forall n \geq 0$ (which will not be the case in the Mann process where $\lim \lambda_n = 0$ is required). If a nonexpansive mapping has a fixed point, x^* say, it is trivial to see that the sequence $\{\|x_n - x^*\|\}$ is monotone decreasing and so $\{x_n\}$ is bounded. Consequently, to approximate a fixed point of a nonexpansive mapping (using the Mann-type sequence) when existence is known, it can be assumed by the above result of Ishikawa that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Also, Edelstein and O'Brien [36] proved that for a nonexpansive mapping $T : K \to K$, where K is a bounded convex subset of *an arbitrary normed linear space*, the Krasnoselskii sequence always yields $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ *uniformly.* An example of Chidume [22] shows that the above result does not hold if T is a quasi-nonepansive mapping. We have noted that the result holds for single-valued quasi-nonexpansive mappings in uniformly convex real Banach spaces (Dotson [34]). This brings us to the following open question.

Question. Let K be nonempty closed convex subset of a real Banach space E and let $T : K \to K$ be a quasi-nonexpansive continuous mapping. Let $\{x_n\}$ be defined by $x_0 \in K$,

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n \geq 0.$$

Is $\{x_n\}$ an approximate fixed point sequence in any Banach space E more general than uniformly convex Banach spaces?

Remark 5.2.10 *Theorem 5.2.2 extends this result of Dotson to the multi-valued quasi-nonexpansive mappings on uniformly convex real Banach spaces.*

CHAPTER 6

A new method for existence theorems for Hammerstein equations

6.1 Introduction

In this chapter, a *new method* of proving existence theorems for solutions of abstract Hammerstein equations of the form

$$u + KFu = 0 \tag{6.1.1}$$

is presented in the case $K : X \to X, F : X \to X$ are accretive (monotone) mappings. Our method does not involve the complicated method of factorizing a linear mapping via a Hilbert space and then using a deep result of Minty, and does not involve use of deep variational techniques, methods used in the literature to prove existence of solutions for Hammerstein equations. In fact linearity of either K or F is not assumed in our theorems.

We restate the following definitions for convenience.

Definition 6.1.1 (Inward Set) *Let X be a normed linear space and let K be a convex subset of X. For $x \in X$, the* **inward set**, *$I_k(x)$, of x relative to K is defined as follows:*

$$I_K(x) = \{x + c(u - x) : c \geq 1, u \in K\}.$$

Definition 6.1.2 (Inward and weakly inward mappings) *A mapping $T : K \to X$ is said to be* **inward** *if $Tx \in I_K(x)$ for each $x \in K$, and* **weakly inward** *if Tx belongs to the closure of $I_K(x)$ for each $x \in K$.*

A relationship between the weak inward condition and the following condition:

$$\lim_{\lambda \to 0^+} \frac{\text{dist}(x - \lambda Ax, D(A))}{\lambda} = 0 \text{ for } \quad \text{all } x \in D(A),$$

for a mapping $A : D(A) \subset X \to X$ is given in lemma 6.1.6 below.

In the sequel, X is a q-uniformly smooth real Banach space, $q > 1$, and $E := X \times X$ with

$$\|[u, v]\|_E = \Big(\|u\|^q + \|v\|^q\Big)^{\frac{1}{q}} \text{ for } \text{ all } [u, v] \in E.$$

If $X(= H)$ is a real Hilbert space, we shall denote E by $E^H := H \times H$.

In lemmas 6.1.4 and 6.1.5 below, we use the following variant definition of accretive mappings as given by Deimling [31].

Definition 6.1.3 (Accretive mapping in the sense of Deimling, [31]) *Let X be real Banach space. A map $A : D(A) \subset X \to X$ is said to be accretive (in the sense of Deimling) if*

$$\langle A(x - y), x - y \rangle_+ \geq 0, \quad \forall x, y \in D(A), \tag{6.1.2}$$

where

$$\langle x, y \rangle_+ := \sup_{j(y) \in J(y)} \langle x, j(y) \rangle, \quad \textit{for all } x, y \in X.$$

It is evident that in any real Banach space, an accretive mapping is also accretive in the sense of Deimling. The converse is true in any real Banach space whose normalized duality mapping is single-valued. This is certainly the case when X is a q-uniformly smooth real Banach space, $q > 1$.

Lemma 6.1.4 (Deimling, [31]) *Let X be a reflexive real Banach space with normal structure and let D be a closed convex bounded subset of X. Let $A : D \to X$ be a Lipschitz and accretive mapping satisfying the condition*

$$\lim_{\lambda \to 0^+} \frac{\textit{dist}(x - \lambda Ax, D)}{\lambda} = 0 \textit{ for } \quad \textit{all } x \in D.$$

Then $0 \in A(D)$.

Lemma 6.1.5 (Deimling, [31]) *Let X be real Banach space and let D be a closed convex subset of X. Let $A : D(A) \subset X \to X$ be an accretive continuous mapping such that $\langle Ax, x\rangle_+ \geq 0$ for all $x \in X$ with $\|x\| \geq R$ or $\lim \|Ax\| = \infty$ as $\|x\| \to \infty$. Suppose A satisfies the condition*

$$\lim_{\lambda \to 0^+} \frac{dist(x - \lambda Ax, D(A))}{\lambda} = 0 \text{ for } \quad \text{all } x \in D(A)$$

and suppose that $A(D)$ is closed. Then $0 \in A(D)$.

Lemma 6.1.6 (Caristi, [20]) *Let D be a convex subset of a normed linear space X and let $A : D \to X$ be a map. Then*

$$\lim_{\lambda \to 0^+} \frac{dist(x - \lambda Ax, D)}{\lambda} = 0 \text{ for } \quad \text{all } x \in D$$

if and only if $I - A$ is weakly inward, I is the identity mapping on D.

Remark 6.1.7 *In view of lemma 6.1.6, if $D(A) = H$ in lemma 6.1.5, then the condition*

$$\lim_{\lambda \to 0^+} \frac{dist(x - \lambda Ax, D(A))}{\lambda} = 0 \text{ for } \quad \text{all } x \in D(A)$$

can be dropped.

From now on, c_q and $\mathrm{d_q}$ denote the constants appearing in lemma 1.2.7.

Lemma 6.1.8 (Chidume, [28], p. 173) *Let X be a q-uniformly smooth real Banach space. Let $F, K : X \to X$ be mappings with $D(K) = F(X) = X$ such that the following conditions hold:*

(i) There exists $\alpha > 0$ such that for each $u_1, u_2 \in D(F)$,

$$\langle Fu_1 - Fu_2, j_q(u_1 - u_2)\rangle \geq \alpha \|u_1 - u_2\|^q; \tag{6.1.3}$$

(ii) there exists $\beta > 0$ such that for each $u_1, u_2 \in D(K)$,

$$\langle Ku_1 - Ku_2, j_q(u_1 - u_2)\rangle \geq \beta \|u_1 - u_2\|^q; \tag{6.1.4}$$

(iii) $(1 + d_q)(1 + c_q) \geq 2^q$, $\min\{\alpha, \beta\} =: \gamma > \frac{(1+d_q)(1+c_q)-2^q}{q(1+c_q)}$.

Let a mapping $T : E \to E$ *be defined by* $Tz := T[u, v] = [Fu - v, u + Kv]$, *for each* $z = [u, v] \in E$, *where* $E := X \times X$. *Then for each* $z_1, z_2 \in E$,

$$\langle Tz_1 - Tz_2, j_q(z_1 - z_2)\rangle \geq \left[\gamma - q^{-1}\left((1 + d_q) - \frac{2^q}{(1 + c_q)}\right)\right] \|z_1 - z_2\|^q. \tag{6.1.5}$$

Lemma 6.1.9 *Let* H *be a real Hilbert space. Let* $K : D(K) \subset H \to H$, $F : D(F) \subset H \to H$ *be two monotone mappings such that* $R(F) \subset D(K)$. *Then the mapping* $A : D(F) \times D(K) \subset E^H \to E^H$ *defined by* $A[u, v] = [Fu - v, Kv + u]$ *is monotone.*

Proof The proof follows from the lines of argument of the proof of lemma 6.1.8 (see Chidume and Zegeye [25]).
■

Lemma 6.1.10 (Chidume, [28], p. 173) *Let* X *be a q-uniformly smooth real Banach space and let* $K : D(K) \subset X \to X$, $F : D(F) \subset X \to X$ *be two Liptschitz mappings such that* $R(F) \subset D(K)$. *Let* $A : D(A) \subset E \to E$ *be a mapping such that* $D(F) \times D(K) = D(A)$ *and define* $A[u, v] = [Fu - v, Kv + u]$. *Then,* A *is Lipschitz.*

Theorem 6.1.11 (Browder, [14]) *Let* X *and* Y *be Banach spaces with* Y^* *uniformly convex and suppose* $f : X \to Y$ *is a strongly* ϕ*-accretive mapping satisfying a Lipschitz condition on each bounded subset of* X. *Then,* $f(X) = Y$

The following corollary follows from theorem 6.1.11.

Corollary 6.1.12 *Let* X *be a real Banach space with uniformly convex dual* X^* *and suppose* $f : X \to X$ *is a strongly accretive Lipschitz mapping. Then,* $f(X) = X$.

6.2 Main results

Let $X := L_p$, $1 < p < 2$ and let $E := X \times X$ with $\|z\|_E^2 := ||[u,v]||_E^2 = ||u||_X^2 + ||v||_E^2$ for arbitrary $z = [u,v] \in E$. For L_p spaces, $1 < p < 2$, the following estimate has been established (see, e.g., Chidume, [28], p. 183):

$$\begin{aligned} A(u_1,u_2,v_1,v_2) &:= [\langle v_1 - v_2, j(u_1-u_2)\rangle + \langle u_1 - u_2, j(\mathbf{v_2} - \mathbf{v_1})\rangle] \quad (6.2.1) \\ &\leq p(2-p)\left(||u_1-u_2||^2 + ||v_1-v_2||^2\right) \quad \forall u_1,u_2,v_1,v_2 \in X. \end{aligned}$$

We begin with a proof of the following Theorem for L_p spaces, $1 < p < 2$, *which is new.*

Theorem 6.2.1 *Let $X = L_p$ $(1 < p < 2)$; $F, K : X \to X$ be mappings such that $D(K) = F(X) = X$ and the following conditions hold:*

(a) there exists $\alpha > 0$ such that for each $u_1, u_2 \in X$,

$$\langle Fu_1 - Fu_2, j(u_1-u_2)\rangle \geq \alpha||u_1-u_2||^2;$$

(b) there exists $\beta > 0$ such that for each $u_1, u_2 \in X$,

$$\langle Ku_1 - Ku_2, j(u_1-u_2)\rangle \geq \beta||u_1-u_2||^2;$$

(c) $\gamma := \min\{\alpha,\beta\}$ with $\gamma > p(2-p)$.

Let $E := X \times X$ and define $T : E \to E$ by

$$Tz := T[u,v] = [Fu - v, u + Kv], \quad \forall [u,v] \in E = X \times X.$$

Then, for arbitrary $z_1, z_2 \in E$, the following inequality holds:

$$\left\langle Tz_1 - Tz_2, j^E(z_1-z_2)\right\rangle \geq [\gamma - p(2-p)]\,||z_1-z_2||^2.$$

Proof We compute as follows;

$$
\begin{aligned}
\left\langle Tz_1 - Tz_2, j^E(z_1 - z_2)\right\rangle &= \langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle - \langle v_1 - v_2, j(u_1 - u_2)\rangle \\
&\quad + \langle Kv_1 - Kv_2, j(v_1 - v_2)\rangle + \langle u_1 - u_2, j(v_1 - v_2)\rangle \\
&\geq \alpha||u_1 - u_2||^2 + \beta||v_1 - v_2||^2 - \langle v_1 - v_2, j(u_1 - u_2)\rangle \\
&\quad + \langle u_1 - u_2, j(v_1 - v_2)\rangle \\
&\geq \gamma\left(||u_1 - u_2||^2 + ||v_1 - v_2||^2\right) - [\langle v_1 - v_2, j(u_1 - u_2)\rangle \\
&\quad - \langle u_1 - u_2, j(v_1 - v_2)\rangle] \\
&\geq \gamma||z_1 - z_2||^2 - A(u_1, u_2, v_1, v_2) \\
&\geq \gamma||z_1 - z_2||^2 - p(2-p)\left(||u_1 - u_2||^2 + ||v_1 - v_2||^2\right) \\
&= (\gamma - p(2-p))||z_1 - z_2||^2 \quad \forall z_1, z_2 \in E,
\end{aligned}
$$

completing proof of the theorem. ■

Remark 6.2.2 *Observe that the condition* $1 + \sqrt{1-\gamma} < p < 2$ *implies* $\gamma > p(2-p)$.

6.2.1 The case of Hilbert spaces

Theorem 6.2.3 *Let* H *be a real Hilbert space,* $K : D(K) \subset H \to H$, $F : D(F) \subset H \to H$ *be two Lipschitz monotone mappings such that* $D(F)$ *and* $D(K)$ *are closed, convex, bounded and* $R(F) \subset D(K)$. *Let* $A : D(A) \subset E^H \to E^H$ *be a mapping such that* $D(F) \times D(K) = D(A)$ *and* $A[u,v] = [Fu - v, Kv + u]$, $[u,v] \in D(A)$. *Suppose that*

$$\lim_{\lambda \to 0^+} \frac{dist(w - \lambda Aw, D(A))}{\lambda} = 0 \textit{ for } \quad \textit{all } w \in D(A).$$

Then, the Hammerstein equation (6.1.1) *has a solution.*

Proof The fact that K and F are Lipschitz and monotone implies A is Lipschitz and monotone (lemmas 6.1.9 and 6.1.10). Since the normalized duality map is the identity mapping in real Hilbert spaces, monotonicity of A is equivalent to accretivity in the sense of Deimling. Also $D(A)$ is closed and convex since $D(F)$ and $D(K)$ are. Therefore, by lemma 6.1.4, $0 \in A(D)$, that is, there exists $[u,v] \in D$ such that $Fu - v = 0$ and $Kv + u = 0$. So u solves equation (6.1.1). This completes the proof.
■

Theorem 6.2.4 *Let H be a real Hilbert space, $K : D(K) \subset H \to H$, $F : D(F) \subset H \to H$ be two continuous monotone mappings such that $D(F)$ and $D(K)$ are closed, convex and $R(F) \subset D(K)$. Let $A : D(A) \subset E^H \to E^H$ be a mapping such that $D(F) \times D(K) = D(A)$ and $A[u, v] = [Fu - v, Kv + u]$, $[u, v] \in D(A)$. Suppose that $\langle Aw, w\rangle \geq 0$ for all $w \in D(A)$ with $\|w\| \geq R$ or $\lim \|Aw\| = \infty$ as $\|w\| \to \infty$ and suppose*

$$\lim_{\lambda \to 0^+} \frac{dist(w - \lambda Aw, D(A))}{\lambda} = 0 \textit{ for } \quad \textit{all } w \in D(A). \tag{6.2.2}$$

Suppose that $A(D(A))$ is closed. Then, the Hammerstein equation (6.1.1) *has a solution.*

Proof The fact that K and F are monotone implies A is monotone (lemma 6.1.9). $D(F)$ and $D(K)$ are closed and convex imply that $D(A)$ is closed and convex. Also since E^H is a real Hilbert space and the normalized duality mapping of any real Hilbert space is the identity map, we have $\langle Aw, w\rangle_+ = \langle Aw, w\rangle$, for all $w \in D(A)$. Therefore, the assumptions on A and $D(A)$ together with lemma 6.1.5 give that $0 \in A(D)$, that is, there exists $[u, v] \in D$ such that $Fu - v = 0$ and $Kv + u = 0$. So u solves equation (6.1.1). This completes the proof.
■

Corollary 6.2.5 *Let H be a real Hilbert space $K, F : H \to H$ be two continuous monotone mappings defined on H. Let $A : E^H \to E^H$ be a mapping defined by $A[u, v] = [Fu - v, Kv + u]$, $[u, v] \in E^H$. Suppose that $\langle Aw, w\rangle \geq 0$ for all $w \in E^H$ with $\|w\| \geq R$ or $\lim \|Aw\| = \infty$ as $\|w\| \to \infty$. Suppose that $A(E^H)$ is closed. Then the Hammerstein equation* (6.1.1) *has a solution.*

Proof Since A is defined on E^H, it satisfies the condition (6.2.2). Therefore, the result follows from theorem 6.2.4.
■

6.2.2 The case for L^p spaces, $1 < p < \infty$.

Theorem 6.2.6 *Let $K : D(K) \subset L^p \to L^p$, $F : D(F) \subset L^p \to L^p$ be two Lipschitz mappings satisfying the following conditions:*

(a) there exists $\alpha > 0$ *such that for each* $u_1, u_2 \in D(F)$,

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \alpha||u_1 - u_2||^2;$$

(b) there exists $\beta > 0$ *such that for each* $u_1, u_2 \in D(K)$,

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \beta||u_1 - u_2||^2.$$

Let $D(F)$ *and* $D(K)$ *be closed, convex, bounded such that* $R(F) \subset D(K)$. *Let* $E := L^p \times L^p$ *and let* $A : D(A) \subset E \to E$ *be a mapping such that* $D(F) \times D(K) = D(A)$ *and* $A[u, v] = [Fu - v, Kv + u], \quad \forall [u, v] \in D(A)$. *Suppose that*

$$\lim_{\lambda \to 0^+} \frac{dist(w - \lambda Aw, D(A))}{\lambda} = 0 \text{ for } \quad \text{all } w \in D(A).$$

Let $\gamma := \min\{\alpha, \beta\}$. *If* $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$ *or* $1 + \sqrt{1 - \gamma} < p \leq 2$, *then the Hammerstein equation* (6.1.1) *has a solution.*

Proof The fact that K and F are Lipschitz implies A is Lipschitz by lemma 6.1.10. Also $D(A)$ is closed and convex since $D(F)$ and $D(K)$ are.

Case 1. $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$; In this case L^p is 2-uniformly smooth space and $c_q = d_q = p - 1$ (see, e.g., [79]). Therefore, $(1 + c_q)(1 + d_q) = p^2 \geq 4 = 2^q$ and

$$\gamma > \frac{1}{2p}(p^2 - 4) = \frac{(1 + d_q)(1 + c_q) - 2^q}{q(1 + c_q)}$$

for $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$. This implies, by lemma 6.1.8, that A is accretive. Therefore, A is accretive in the sense of Deimling. Hence, using lemma 6.1.4, we have that $0 \in A(D)$, that is, there exists $[u, v] \in D$ such that $Fu - v = 0$ and $Kv + u = 0$. So u solves equation (6.1.1).

Case 2. $1 + \sqrt{1 - \gamma} < p \leq 2$; The condition $1 + \sqrt{1 - \gamma} < p \leq 2$ implies that $\gamma > p(2 - p)$. Hence, by theorem 6.2.1, A is accretive. We conclude as in *Case 1.* This completes the proof.

■

Theorem 6.2.7 *Let* $K : D(K) \subset L^p \to L^p$, $F : D(F) \subset L^p \to L^p$ *be two continuous accretive mappings satisfying the following conditions:*

(a) *there exists* $\alpha > 0$ *such that for each* $u_1, u_2 \in D(F)$,

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \alpha||u_1 - u_2||^2;$$

(b) *there exists* $\beta > 0$ *such that for each* $u_1, u_2 \in D(K)$,

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \beta||u_1 - u_2||^2.$$

Let $D(F)$ *and* $D(K)$ *be closed, convex, such that* $R(F) \subset D(K)$. *Let* $E := L^p \times L^p$ *and let* $A : D(A) \subset E \to E$ *be a mapping such that* $D(F) \times D(K) = D(A)$ *and* $A[u, v] = [Fu - v, Kv + u]$ *for* $[u, v] \in D(A)$. *Suppose that* $\langle Aw, w\rangle_+ \geq 0$ *for all* $w \in D(A)$ *with* $\|w\| \geq R$ *or* $\lim \|Aw\| = \infty$ *as* $\|w\| \to \infty$ *and suppose*

$$\lim_{\lambda \to 0^+} \frac{dist(w - \lambda Aw, D(A))}{\lambda} = 0 \text{ for } \quad \text{all } w \in D(A). \tag{6.2.3}$$

Let $\gamma := \min\{\alpha, \beta\}$. *If* $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$ *or* $1 + \sqrt{1 - \gamma} < p \leq 2$, *then the Hammerstein equation* (6.1.1) *has a solution.*

Proof Evidently, continuity of K and F give the continuity of A. Also $D(A)$ is closed and convex since $D(F)$ and $D(K)$ are. The rest follows as in the proof of theorem 6.2.6. This completes the proof.

∎

Corollary 6.2.8 *Let* $K : D(K) \subset L^p \to L^p$, $F : D(F) \subset L^p \to L^p$ *be two continuous accretive mappings satisfying the following conditions:*

(a) *there exists* $\alpha > 0$ *such that for each* $u_1, u_2 \in D(F)$,

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \alpha||u_1 - u_2||^2;$$

(b) *there exists* $\beta > 0$ *such that for each* $u_1, u_2 \in D(K)$,

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \beta||u_1 - u_2||^2.$$

Let $D(F) = L^p = D(K)$. *Let* $E := L^p \times L^p$ *and let* $A : E \to E$ *be a mapping defined by* $A[u, v] = [Fu - v, Kv + u]$, $[u, v] \in D(A)$. *Suppose that* $\langle Aw, w\rangle_+ \geq 0$ *for all* $w \in E$ *with* $\|w\| \geq R$ *or* $\lim \|Aw\| \to \infty$ *as* $\|w\| \to \infty$. *Let* $\gamma := \min\{\alpha, \beta\}$. *If* $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$ *or* $1 + \sqrt{1 - \gamma} < p \leq 2$, *then the Hammerstein equation* (6.1.1) *has a solution.*

Proof Since A is defined on E, it satisfies condition 6.2.3 of theorem (6.2.7). Also $D(A)$ is closed and convex. Therefore, the result follows from theorem 6.2.7.
■

6.2.3 The case of Hilbert spaces with Lipschitz strongly monotone mappings

Theorem 6.2.9 *Let H be a real Hilbert space, $K : H \to H$, $F : H \to H$ be two Lipschitz strongly monotone mappings with constants α, β, respectively. Let $A : E^H \to E^H$ be a mapping defined by $A[u,v] = [Fu - v, Kv + u]$. Then, the Hammerstein equation* (6.1.1) *has a solution.*

Proof Using lemma 6.1.10 we have that A is Lipschitz. Also since every real Hilbert space is q-uniformly smooth with $q = 2, d_q = c_q = 1$, we have that $(1 + c_q)(1 + d_q) = 4 = 2^q$. Also $\min\{\alpha, \beta\} > 0 = \frac{(1+c_q)(1+d_q)-2^q}{q}$. Therefore, A is strongly monotone by lemma 6.1.8. Since E^H is a real Hilbert space and every real Hilbert space is uniformly convex, we invoke corollary 6.1.12 to have $A(E^H) = E^H$. So there exists $[u,v] \in E^H$ such that $A[u,v] = 0$, that is, $Fu - v = 0,\ Kv + u = 0$. Hence u solves equation (6.1.1). This completes the proof.
■

6.2.4 The case of L_p spaces, $1 < p < \infty$, with Lipschitz strongly accretive mappings

Theorem 6.2.10 *Let $K : L^p \to L_p$, $F : L^p \to L_p$ be two Lipschitz mappings satisfying mappings satisfying the following conditions:*

(a) there exists $\alpha > 0$ such that for each $u_1, u_2 \in L^p$,

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \alpha ||u_1 - u_2||^2;$$

(b) *there exists* $\beta > 0$ *such that for each* $u_1, u_2 \in L^p$,

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \beta||u_1 - u_2||^2.$$

Let $E := L^p \times L^p$ *and let* $A : E \to E$ *be a mapping defined by* $A[u,v] = [Fu-v, Kv+u]$. *Let* $\gamma := \min\{\alpha, \beta\}$. *If* $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$ *or* $1 + \sqrt{1-\gamma} < p \leq 2$, *then, the Hammerstein equation* (6.1.1) *has a solution.*

Proof Using lemma 6.1.10 we have that A is Lipschitz.

Case 1. $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$; In this case L^p is 2-uniformly smooth space and $c_q = d_q = p-1$ (see, e.g., [79]). Therefore, $(1+c_q)(1+d_q) = p^2 \geq 4 = 2^q$ and

$$\gamma > \frac{1}{2p}(p^2 - 4) = \frac{(1+d_q)(1+c_q) - 2^q}{q(1+c_q)}$$

for $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$. This implies, by lemma 6.1.8, that A is strongly accretive. Since every L_p spaces, $2 \leq p < \gamma + \sqrt{\gamma^2 + 4}$, is uniformly convex, by corollary 6.1.12, $A(L_p) = L_p$. Therefore there exists $[u,v] \in E$ such that $A[u,v] = 0$, that is, $Fu - v = 0$ and $Kv + u = 0$. So u solves equation (6.1.1).

Case 2. $1 + \sqrt{1-\gamma} < p \leq 2$; The inequality $1 + \sqrt{1-\gamma} < p \leq 2$ implies that $\gamma > p(2-p)$. Hence by theorem 6.2.1, A is strongly accretive. The result now follows as in *Case 1* since every L_p space, $1 + \sqrt{1-\gamma} < p \leq 2$, is uniformly convex. This completes the proof.

■

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