ALGORITHMS FOR APPROXIMATION OF J-FIXED POINTS OF NONEXPANSIVE - TYPE MAPS, ZEROS OF MONOTONE MAPS, SOLUTIONS OF FEASIBILITY AND VARIATIONAL INEQUALITY PROBLEMS

Dissertation Presented in Partial Fulfilment
of the
Requirements for the Degree of
Doctor of Philosophy
in Mathematics
of the
African University of Science and Technology
www.aust.edu.ng
P.M.B 681, Garki, Abuja F.C.T
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CERTIFICATE OF APPROVAL

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Ph.D. THESIS

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This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for award of the degree of Doctor of Philosophy degree in Mathematics.

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Dedication

This thesis with every sense of humility is dedicated to my lovely parents Mr and Mrs Nnakwe E. Ogudu.

Acknowledgement

First and foremost, I sincerely owe God a big thank you for the life I live today and making this program a huge success. For all He has done, to Him alone be glorified.

I would like to express my sincere gratitude to my supervisor, the Acting president, African University of Science and Technology, Abuja and HOD institute of Pure and Applied Mathematics Professor C. E. Chidume for his leadership in the university and for the intense/intensive knowledge he has impacted in me. Your constructive criticisms, guidance, patience, support and encouragement through out my PhD program has moulded me into what I am today. Indeed, is a rare opportunity and honour working with a distinguished professor of your capacity.

I would like to thank the members of my PhD defence committee, Professor Eric Ofoedu, Professor Bashir Ali, Professor M.k. Osilike and Dr. Jeremiah Ezeroa for their remarks and suggestions.

My gratitude goes to Dr. A.U. Bello, Dr. Ma'aruf Shehu Minjibir and Dr. Okpala Maduabuchi for their constant support when the need arises.

To all my lecturers; Professor Khalil Ezzinbi, Professor Ngalla Djitte, Professor Gane Samb Lo, Professor V. Berinde, Professor Geraldo and Professor A.U Osisiogu; *discussing with you was very inspiring.*

My special thanks goes to my colleagues; *Dr. Ezea Chinedu, Dr. M.O. Romanus, Dr. Nnyaba Victoria U., Mr Otubo E. Ezzaka, Mr A. Adamu, Miss Lois Okereke and Miss Aisha Aminu for their reassuring presence and always devoting their time to help when the need arises.*

A special thanks to the staff of AUST, especially Miss Udigwe Amaka for her special care and administrative supports. I call her **Sis**. To all my friends: Ken Nwigbo, Uba Markjoe, Uzorchukwu Mark and Aroh Innocent, thank you for all your encouragements.

A special thank you to *Mr and Mrs Oyebamiji Tajudeen Aderemi, Mr and Mrs Kenneth Uwah; and Mr and Mrs Ukwueze* for their sincere humility and assistance during this adventure.

To these special ones, *Ifebude Barnabas, Ojewale Mubarack, Abiola Saka, Mr Michael, Miss Nnaji Comfort, Mr Godfregy and Mr James, Pastor Peter, Pastor Igwe and Pastor Tolu*, you people are wonderful.

To all my seniors and friends: Izuchukwu Chinedu, Eribo, Gbodo, N. Elshadai, N. Sunday, Jagaras, Ayokunle, Okpala Ifunanya, $\cdots$ God bless you.

My sincere gratitude goes to my fiancee, Miss Akpa Grace Chinenye for her consistent encouragements and patience during the course of this program. Baby, I love you!

This special thank you goes to my lovely aunt, Big Sis (Okorchukwu I. Roseline). She is a woman with a heart of Gold. God bless you ma.

A special thank you to my lovely parents, Mr and Mrs Nnakwe Ogudu, my siblings and the rest of my family members for standing by me in all this times. I will forever remain thankful for all your cares.

Special gratitude to my mentor and friend his blessed memory, Late Mr Okorchukwu Martins who taught me how to burn my candle at nights, *"I wish the dead could see". Daddy I love you.*

Thanks to all and sundry that I fail to mention here.

Nnakwe Monday Ogudu.

Abstract

It is well known that many physically significant problems in different areas of research can be transformed at equilibrium state into an inclusion problem of the form

$$0 \in Au,$$

where A is either a multi-valued *accretive* map from a real Banach space into itself or a multi-valued *monotone* map from a real Banach space into its dual space.

In several applications, the solutions of the inclusion problem, when the map A is monotone, corresponds to minimizers of some convex functions.

It is known that the sub-differential of any convex function, say g, and denoted by ∂g is monotone, and for any vector, say v, in the domain of g, $0 \in \partial g(v)$ if and only if v is a minimizer of g.

Setting $\partial g \equiv A$, solving the inclusion problem, is equivalent to finding minimizers of g.

The method of approximation of solutions of the inclusion problem $0 \in Au$, when the map A is monotone in real Banach spaces, was not known until in 2016 when Chidume and Idu [52] introduced J-fixed points technique. They proved that the J-fixed points correspond to zeros of monotone maps which are minimizers of some convex functions.

In general, finding closed form solutions of the inclusion problem, where A is monotone is extremely difficult or impossible. Consequently, solutions are sought through the construction of iterative algorithms for approximating J-fixed points of nonlinear maps.

In chapter three, four and seven of the thesis, we present a convergence result for approximating zeros of the inclusion problem $0 \in Au$.

Let H_1 and H_2 be real Hilbert spaces and $K_1,\ K_2,\ \cdots,\ K_N$, and $Q_1,\ Q_2,\ \cdots,\ Q_P$, be nonempty, closed and convex subsets of H_1 and H_2, respectively, with nonempty intersections K and Q, respectively, that is,

$$K = K_1 \cap K_2 \cap \cdots \cap K_N \neq \emptyset \text{ and } Q = Q_1 \cap Q_2 \cap \cdots \cap Q_P \neq \emptyset.$$

Let $B : H_1 \to H_2$ be a bounded linear map, $G_i : H_1 \to H_1$, $i = 1, \cdots, N$ and $A_j : H_2 \to H_2$, $j = 1, \cdots, P$ be given maps. The *common split variational inequality* problem introduced by

Censor *et al.* [32] in 2005, and denoted by (CSVIP), is the problem of finding an element $u^* \in K$ for which

$$\begin{cases} \langle u - u^*, G_i(u^*)\rangle \geq 0, \quad \forall \ \ u \in K_i, \ i = 1, 2, \ \cdots, N, \quad \text{such that} \\ v^* = Bu^* \in Q \text{ solves} \quad \langle v - v^*, A_j(v^*)\rangle \geq 0, \quad \forall \ \ v \in Q_j, \ j = 1, 2, \cdots, P. \end{cases}$$

The motivation for studying this class of problems with $N > 1$ stems from a simple observation that if we choose $G_i \equiv 0$, the problem reduces to finding $u^* \in \cap_{i=1}^N K_i$, which is the known *convex feasibility problem* (CFP) such that $Bu^* \in \cap_{j=1}^P VI(Q_j, A_j)$. If the sets K_i are the fixed point sets of maps $S_i : H_1 \to H_1$, then, the *convex feasibility problems* (CFP) is the *common fixed points problem*(CFPP) whose image under B is a *common solution to variational inequality problems* (CSVIP).

If we choose $G_i \equiv 0$ and $A_j \equiv 0$, the problem reduces to finding $u^* \in \cap_{i=1}^N K_i$ such that the point $Bu^* \in \cap_{j=1}^P Q_j$ which is the well known *multiple-sets split feasibility problem* or *common split feasibility problem* which serves as a model for many *inverse problems* where the constraints are imposed on the solutions in the domain of a linear operator as well as in the range of the operator.

A lot of research interest is now devoted to split variational inequality problem and its generalizations.

In chapter five and six of the thesis, we present convergence theorems for approximating solutions of variational inequalities and a convex feasibility problem; and solutions of split variational inequalities and generalized split feasibility problems.

Publications arising from the thesis

[A] Papers Accepted/Published from the Thesis

1. C. E. Chidume and M. O. Nnakwe, A new Halpern-type algorithm for a generalized mixed equilibrium problem and a countable family of generalized J-nonexpansive maps, with applications. Carpathian J. of Math. 34(2018), N0.2, 191 - 198.

2. C. E. Chidume and M. O. Nnakwe, Convergence theorems of subgradient extragradient algorithm for solving variational inequalities and a convex feasibility problem, Fixed Point Theory and Applications (FPTA), http://doi.org/10.1186/s13663-018-06414.

3. C. E. Chidume and M. O. Nnakwe, Iterative algorithms for split Variational Inequalities and generalized split feasibility problems, with applications, J. Nonlinear Variational Analysis Vol. 3, Issue 2, pages 127-140 (2019).

4. O. M. Romanus, U. V. Nnyaba and M. O. Nnakwe, *Relaxed iterative algorithms for a system of generalized mixed equilibrium problems and a countable family of some nonlinear multi-valued nonexpansive-type maps,* Acta Mathematica Vol. 38, Issue 6, Nov. 2018, pages 1805-1820, https://doi:10.1016/S0252-9602(18)308488.

5. C. E. Chidume, M. O. Nnakwe and A. Adamu, A strong convergence theorem for Generalized Φ-strongly monotone maps, with applications Fixed Point Theory and Applications 2019:11 https://doi.org/10.1186/s13663-019-0660-9.

6. C. E. Chidume, M. O. Nnakwe and E. E. Otubo, A new iterative algorithm for a generalized mixed equilibrium problem and a countable family of nonexpansive-type maps, with applications. Fixed Point Theory. 21(1) (2020) 109-124 https://doi.org/10.24193/fpt-ro.2020.1.08.

7. C. E. Chidume and M. O. Nnakwe, A strong convergence theorem for an inertial algorithm for a countable family of generalized nonexpansive maps. Fixed Point Theory. 21(2) (2020) 441-452 https://doi.org/10.24193/fpt-ro.2020.2.31.

Contents

CHAPTER 1

Introduction

The contents of this thesis fall within the general area of Nonlinear Operator Theory, an area of mathematics that has applications in numerous fields of science. The contributions of this thesis are in the following important topics in the area, namely:

- The theory of J-fixed points.
- Inertial algorithm for approximating fixed points of generalized J-nonexpansive maps.
- Approximation of solutions of generalized split feasibility problems and variational inequality problems.

1.0.1 The Theory of J-Fixed Points

Let X be a real normed space. Fixed point theory is concerned with solutions of the equation:

$$Tv = v, \tag{1.1}$$

where $T : D(T) \subset X \to X$ is a map and domain of T is denoted by $D(T)$. The set of solutions of the equation (1.1) is denoted by $F(T) := \{v \in X : Tv = v\}$, and it is called the *fixed points set* of T.

The origin of fixed point theory lies in the method of successive approximations used for proving existence of solutions of differential equations introduced by Liouville [143] in 1837, and was developed systematically by Picard [132] in 1890. The theory started formally in the twentieth century with investigations into the existence properties of solutions to certain boundary value problems arising in ordinary and partial differential equations, using the techniques devised by Picard [132], which involved the iteration of an integral operator equation to obtain solutions to such problems.

In 1912, Brouwer [19] proved the following well known fixed point theorem for a continuous self map in a finite dimensional space.

Theorem 1.1 (Brouwer, [19])**.** *Let B be the closed unit ball in $\mathbb{R}^n$ and $T : B \to B$ be a continuous map. Then, T has a fixed point.*

Brouwer's fixed point theorem has many applications in analysis and differential equations. However, the weakness of the theorem is that it is not applicable in infinite dimensional spaces.

Consider, the following example given by Kakutani [94]:

Let $l_2 := \{u = (u_1, u_2, u_3, \cdots),\ u_i \in \mathbb{R} : \sum_{i=1}^{\infty} |u_i|^2 < \infty\}$. Let B be a nonempty, closed, bounded and convex subset of l_2 and $T : B \to B$ be a map defined by

$$Tu = (\sqrt{1 - ||u||^2}, u_1, u_2, u_3, \cdots),\ \forall\ u \in l_2.$$

Clearly, T is a continuous map, and has no fixed point.

In 1922, Banach [8] and Caccioppoli [31] proved the following fixed point theorem which was precisely, an abstract formulation of Picard's techniques in what is now called the *Banach Contraction Mapping Principle.*

Let (X, ρ) be a metric space. A map $T : (X, \rho) \to (X, \rho)$ is called a *contraction* if there exists a constant $k \in [0, 1)$ such that for any $u, v \in X$, the following inequality holds:

$$\rho(Tu, Tv) \leq k\rho(u, v).$$

Theorem 1.2 (Banach contraction mapping principle, [8])**.** *Let X be a complete metric space and $T : X \to X$ be a contraction map. Then, T has a unique fixed point. Furthermore, for arbitrary $x_0 \in X$, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = T(x_n),\ n \geq 0,$$

converges to the unique fixed point of T.

The Banach contraction mapping principle which is involved in many of the existence and uniqueness proofs of differential equations is, perhaps, the most useful fixed point theorem.

Brouwer's fixed point theorem has an advantage over Theorem 1.2 in that it applies to a larger class of maps. However, the theorem is not applicable in infinite dimensional spaces.

Attempts to extend Brouwer's fixed point theorem to infinite dimensional spaces resulted to the following very important theorem of Schauder [141].

Theorem 1.3 (Schauder-Tychonov, [141])**.** *Let B be a nonempty, compact and convex subset of a Banach space. Let $T : B \to B$ be a continuous function. Then, T has a fixed point.*

The Schauder fixed point theorem has numerous applications in approximation theory, game theory, engineering, economics and optimization theory (see e.g., Zeidler [169], Dugundji and Granas [79], and the references therein). However, the limitation of this theorem is the compactness condition imposed on the domain of the function. A modification of the theorem which has been proved without any compactness condition on the domain of the function is the following theorem.

Theorem 1.4 (Schauder-Tychonov, [141])**.** *Let B be a nonempty, closed, bounded and convex subset of a Banach space. Let $T : B \to B$ be a continuous function such that $T(B)$ is compact. Then, T has a fixed point.*

A class of maps closely related to the class of contraction maps is the class of so called *nonexpansive maps.* A map $T : (X, \rho) \to (X, \rho)$ is called *nonexpansive* if for all $u, v \in X$,

$$\rho(Tu, Tv) \leq \rho(u, v).$$

This class of maps is an important generalization of the class of contraction maps as has been observed by Bruck [27], mainly for the following two reasons:

(*i*) *Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960's and constitute one of the first classes of nonlinear maps for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.*

(*ii*) *Nonexpansive maps appear in applications as transition operators for initial value problems of differential equations.*

However, if $X = \mathbb{R}$ and $T : \mathbb{R} \to \mathbb{R}$ is a map defined by $Tu = u + 2$, then, T is nonexpansive and has no fixed point. In general, if K is a nonempty, closed, bounded and convex subset of a Banach space and $T : K \to K$ is nonexpansive, it is known that T may not have a fixed point (unlike the case when T is a strict contraction), and even when it has, the Picard's sequence, $x_0 \in K$, $x_{n+1} = Tx_n,\ \ n \geq 0$, may fail to converge to such a fixed point. This can be observed by considering an anti-clockwise rotation of the unit disc of $\mathbb{R}^2$ about the origin through an angle of $\frac{\Pi}{4}$ (see e.g., Chidume [45]). This map is nonexpansive with the origin as the unique fixed point, but the Picard's sequence, $x_0 \in K$, $x_{n+1} = Tx_n,\ n \geq 0$, fails to converge with any starting point $x_0 \neq 0$.

In 1965, Kirk [100] proved the following fixed point theorem for nonexpansive maps in a reflexive Banach space.

Theorem 1.5 (Kirk, [100])**.** *Let X be a reflexive Banach space and K be a nonempty, closed, bounded and convex subset of X with normal structure. Let $T : K \to K$ be a nonexpansive map. Then, T has a fixed point.*

Browder [20] and Gȯhde [84], also in 1965, proved independently a fixed point theorem for nonexpansive maps in uniformly convex Banach spaces. We remark that every uniformly convex Banach space has normal structure (see, e.g., Browder [20], Gȯhde [84] and Chidume [44], pp. 178).

We observed that Picard's sequence may fail to converge to a fixed point of T if T is nonexpansive. To overcome this difficulty, the following iterative algorithms have been developed for approximating fixed points of nonexpansive maps:

1. the *Krasnosel'skii algorithm* given by Schaefer [140]:

$$x_0 \in K, \quad x_{n+1} = (1-\lambda)x_n + \lambda Tx_n,\ n \geq 0, \quad \lambda \in (0,1), \tag{1.2}$$

2. the *Halpern algorithm* [86]:

$$u \in K, \quad x_{n+1} = \lambda_n u + (1-\lambda_n)Tx_n,\ n \geq 0, \tag{1.3}$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

$$(i)\ \lim_{n\to\infty} \lambda_n = 0 \quad \text{and} \quad (ii)\ \sum_{n=0}^{\infty} \lambda_n = \infty,$$

3. the *Mann sequence* [114]:

$$x_0 \in K, \quad x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n, \quad n \geq 0, \tag{1.4}$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

$$(i) \lim_{n\to\infty} \lambda_n = 0 \quad \text{and} \quad (ii) \sum_{n=0}^{\infty} \lambda_n = \infty.$$

The recursion formula (1.4), is consequently called the *Krasnosel'skii-Mann* (KM) formula for finding fixed points of nonexpansive (ne) maps. The KM recursion formula is used nowadays in several applications.

> *"Many well known algorithms in signal processing and image reconstruction are iterative in nature. A wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the KM iteration procedure, for particular choices of the ne operator $\cdots$." (Charles Byrne, [29]).*

For the past 50 years or so, the study of the Krasnosel'skii-Mann iterative sequence for the approximation of fixed points of nonexpansive maps and fixed points of some of their generalizations, and approximation of zeros of accretive maps has been a flourishing area of research for many mathematicians.

Let H be a real Hilbert space. A map $A : D(A) \subset H \rightrightarrows H$ is called *monotone* if for each $u, v \in D(A)$, the following inequality holds:

$$\langle \eta - \zeta, u - v \rangle \geq 0, \ \forall \, \eta \in Au, \ \zeta \in Av. \tag{1.5}$$

The map A is called *maximal monotone* if, in addition, the graph of A is not properly contained in the graph of any other monotone map. Also, A is maximal monotone if and only if it is monotone and $R(I + tA) = H$, for all $t > 0$.

The map A is called *strongly monotone* if there exists $k \in (0,1)$ such that for all $u, v \in D(A)$, the following inequality holds:

$$\langle \eta - \zeta, u - v \rangle \geq k||u - v||^2, \ \forall \, \eta \in Au, \ \zeta \in Av. \tag{1.6}$$

The map A is called *ϕ-strongly monotone* if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for all $u, v \in D(A)$, the following inequality holds:

$$\langle \eta - \zeta, u - v \rangle \geq \phi(||u - v||)||u - v||, \ \forall \, \eta \in Au, \ \zeta \in Av. \tag{1.7}$$

The map A is called *generalized-Φ-strongly monotone* if there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for all $u, v \in D(A)$, the following inequality holds:

$$\langle \eta - \zeta, u - v \rangle \geq \Phi(||u - v||), \ \forall \, \eta \in Au, \ \zeta \in Av. \tag{1.8}$$

The class of generalized-Φ-strongly monotone maps contains the class of ϕ-strongly monotone maps and the class of strongly monotone maps. The class of generalized-Φ-strongly monotone maps is the largest class of monotone maps for which, if a solution of the inclusion $0 \in Au$ exists, it is always unique.

Monotone maps were studied in Hilbert spaces first by Zarantonello [165], Minty [119] and a host of other authors. Interest in such maps stems from their usefulness in numerous applications. Consider, for example, the following:

Let $g : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *sub-differential of* g at $u \in H$ is defined by

$$\partial g(u) := \{u^* \in H : \langle u^*, v - u\rangle \leq g(v) - g(u), \ \forall \ v \in H\}.$$

It is easy to see that ∂g is a monotone operator, and $0 \in \partial g(u)$ if and only if u is a minimizer of g. Setting $\partial g \equiv A$, *it follows that solving the inclusion* $0 \in Au$, *in this case, is solving for a minimizer of* g. In the case where the map A is single valued, $0 \in Au$ reduces to $Au = 0$.

Let X be a real normed space with dual space X^*. A map $A : D(A) \subset X \to 2^X$ is called *accretive* if for each $u, v \in D(A)$, there exists $j(u - v) \in J(u - v)$ such that the following inequality holds:

$$\langle \eta - \zeta, j(u - v)\rangle \geq 0. \tag{1.9}$$

where $J : X \to 2^{X^*}$ is the normalized duality map given by

$$J(v) = \{v^* \in X^* : \langle v, v^*\rangle = ||v||^2, \ ||v|| = ||v^*||\}. \tag{1.10}$$

In a real Hilbert space H, J is the identity map on H and inequality (1.9) reduces to inequality (1.5). Thus, in a real Hilbert space, accretive maps and monotone maps coincide.

Accretive maps were introduced independently in 1967 by Browder [23] and Kato [98]. Interest in this class of maps stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in Banach spaces. It is known (see e.g., Zeidler [170]) that many physically significant problems can be modelled as an evolution problem of the form:

$$0 \in \frac{du}{dt} + Au, \quad u(0) = u_0, \tag{1.11}$$

where A is a multi-valued accretive map defined on an appropriate Banach space X. If u is independent of time t, i.e., at equilibrium, then, inclusion problem (1.11) reduces to:

$$0 \in Au. \tag{1.12}$$

Solutions of inclusion problem (1.12), in many cases, where the map A is accretive, represent equilibrium states of some dynamical systems.

To solve the inclusion problem (1.12), where A is an accretive map, Browder [23] in 1967, introduced a map $T : X \to 2^X$ defined by $T := I - A$, where I is the identity map on X and A is an accretive map. He called the map T a *pseudocontraction*. It is clear that fixed points of T correspond to solutions of the inclusion problem (1.12). This connection is the key motivation for the huge interest in the fixed points for pseudocontractive maps. Replacing the map A in inequality (1.9) by $I - T$, where T is a pseudocontraction, we have that T is a psuedocontraction if for each $u, v \in X$, there exists $j(u - v) \in J(u - v)$ such that for all $\beta \in Tu, \gamma \in Tv$, the following inequality holds:

$$\langle \beta - \gamma, j(u - v)\rangle \leq ||u - v||^2. \tag{1.13}$$

This class of maps is an important generalization of the class of nonexpansive maps. Hence, solutions of inclusion problem (1.12), in this case, correspond to fixed points of T. Consequently, approximating zeros of accretive maps is equivalent to approximating fixed points of pseudocontractions, assuming existence. For more on approximation of fixed points of pseudocontractions, see e.g., Browder and Petryshyn [24], Reich [135], Bruck [28], Takahashi and Ueda [153], Schu [142], Kirk [101], Chidume and Mutangadura [67], Berinde [12], Ofoedu *et al.*, [126], Ofoedu [125], Chidume [[47], [46]], Chidume and Chidume [50], Chidume *et al.* [49] and the references contained in them.

Let X be a real normed space with dual space X^*. A map $A : D(A) \subset X \to 2^{X^*}$, where $D(A)$ denotes the domain of A, is called *monotone* if for all $u, v \in D(A)$, and for all $\eta \in Au$, $\zeta \in Av$, the following inequality holds:

$$\langle \eta - \zeta, u - v \rangle \geq 0. \tag{1.14}$$

> *The extension of the monotonicity definition to maps from Banach space into its dual has been the starting point for the development of nonlinear functional analysis. The maps constitute the most manageable classes, because of the very simple structure of the monotonicity condition. The monotone maps appear in a rather wide variety of contexts, since they can be found in many functional equations. Many of them appear also in calculus of variations as sub-differential of convex functions (Pascali and Sburian, [130]).*

In many cases, it has been observed that, in a Hilbert space, solutions of inclusion (1.12), where A is a monotone map, represent minimizers of some convex functions.

Now, consider, for example, the following optimization problem:

$$\text{find } v^* \in X : g(v^*) = \inf_{v \in X} g(v),$$

where $g : X \to \mathbb{R} \cup \{\infty\}$ is a map and X is a real Banach space.

It is known that if $v^* \in X$ exists and the function g is Fréchet differentiable at v^*, then, $g'(v^*) = 0$ (see e.g., Diop *et al.*, [78]). This gives a method for obtaining a minimizer of g explicitly. However, maps in general are not differentiable in the usual sense. Take for instance, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(v) = |v|$, $\forall\, v \in \mathbb{R}$. This function is not differentiable at zero. However, the function g has a *sub-differential* at zero given by $[-1, 1]$.

Let X be a real normed space. Let $g : X \to \mathbb{R} \cup \{\infty\}$ be a convex and proper function. The *sub-differential* of g at $v \in X$; $\partial g : X \to 2^{X^*}$, is given by

$$\partial g(v) := \{v^* \in X^* : \langle u - v, v^* \rangle \leq g(u) - g(v),\ \forall\, u \in X\}. \tag{1.15}$$

It is known that $\partial g(v)$ is monotone and $0 \in \partial g(v)$ if and only if v is a minimizer of g (see e.g., Minty [120]). Setting $\partial g \equiv A$, it follows that solving equation (1.12), where $A : X \to 2^{X^*}$ is a monotone map, is equivalent to solving for a minimizer of g.

It is clear that the fixed point technique introduced by Browder [23] in 1967, for solving the inclusion (1.12), where $A : X \to 2^X$ is an accretive map, for obvious reasons, is not applicable in this case where $A : X \to 2^{X^*}$ is a monotone map. Hence, there is the need to introduce and develop techniques for approximating solutions of the inclusion (1.12), when $A : X \to 2^{X^*}$ is a monotone map.

To solve the inclusion problem (1.12), when A is monotone, Chidume and Idu [52] in 2016, introduced a map $T : X \to 2^{X^*}$ defined by $T := J - A$, where J is the normalized duality map on X and $A : X \to 2^{X^*}$ is a monotone map. They called the map T, a *J-pseudocontraction*.

Interest in J-pseudocontractions stems mainly from their firm connection with the important class of nonlinear monotone maps. Replacing the map A in inequality (1.14) by $J - T$, where

$T : X \to 2^{X^*}$ is a J-pseudocontraction, then, we have that for each $u, v \in X$, the following inequality holds:

$$\langle \beta - \gamma, u - v \rangle \leq \langle \alpha - \delta, u - v \rangle, \; \forall \; \beta \in Tu, \; \gamma \in Tv, \; \alpha \in Ju, \; \delta \in Jv. \tag{1.16}$$

where J is the normalized duality map on X. This class of maps is an important generalization of a new class of maps called *J-nonexpansive maps* (see e.g., Chidume *et al.,* [52] and the references therein).

The notion of J-fixed points was first introduced by Zegeye [166] in 2008, which he called *semi-fixed points*. This was later called *duality fixed points* by Liu [111] and also Su and Xu [151] in 2012. In 2016, Chidume and Idu [52] studied this notion of fixed points and called it *J-fixed points.*

Let X be a real normed space with dual space X^*. Let $T : X \to 2^{X^*}$ be a multivalued map.

An element $v \in X$ is called a *J-fixed point* of T if and only if there exists $\beta \in Tv$ such that $\beta \in Jv$, where J is the normalized duality map on X.

If T is a single-valued map, then, $v \in X$ is a *J-fixed point* of T if $\quad Tv = Jv$.

We observe that if $X \equiv H$, a real Hilbert space, where J is the identity map on X, then, the notion of J-fixed points coincides with the usual notion of fixed points. Hence, setting $\partial g \equiv A$, where A is a monotone map, it follows that solving the inclusion problem (1.12) or equivalently, finding J-fixed points of T, assuming existence, is equivalent to solving for a minimizer of g.

In general, finding closed form solutions of the inclusion problem (1.12), when A is nonlinear and monotone is extremely difficult or impossible. Consequently, solutions are sought through the construction of iterative algorithms for approximating J-fixed points of nonlinear maps.

In 2016, Chidume and Idu [52] proved the following theorem for approximating J-fixed points of *J-pseudocontractive* and *bounded maps* in uniformly convex and uniformly smooth real Banach spaces.

Theorem 1.6 (Chidume and Idu, [52])**.** *Let X be a uniformly convex and uniformly smooth real Banach space. Let $T : X \to 2^{X^*}$ be a J-pseudocontractive and bounded map with $F_J(T) \neq \emptyset$. For arbitrary $x_1, u \in X$, define a sequence $\{x_n\}$ iteratively by*

$$x_{n+1} = J^{-1}[(1 - \lambda_n)Jx_n + \lambda_n \eta_n - \lambda_n \theta_n (Jx_n - Ju)], \; \eta_n \in Tx_n, \; n \geq 1,$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:
(i) $\sum \lambda_n = \infty$,
(ii) $\sum \lambda_n \theta_n = \infty$,
(iii) $\lambda_n \leq \gamma_0 \theta_n; \; \delta_X^{-1}(\lambda_n M_0^*) \infty \leq \gamma_0 \theta_n$,
(iv) $\frac{\delta_X^{-1}(\frac{\theta_{n-1} - \theta_n}{\theta_n} K)}{\lambda_n \theta_n} \to 0, \; \frac{\delta_{X^*}^{-1}(\frac{\theta_{n-1} - \theta_n}{\theta_n} K)}{\lambda_n \theta_n} \to 0$ *as $n \to \infty$, for some $M_0 > 0$, $M_0^* > 0$, $K > 0$ and $\gamma_0 > 0$, where $\delta_X :]0, \infty[\to]0, \infty[$ is the modulus of convex of X. Then, the sequence $\{x_n\}$ convergences strongly to a J-fixed point of T.*

The prototypes for Theorem 1.6 are the following:

$$\lambda_n = \frac{1}{(n+1)^a} \text{ and } \theta_n = \frac{1}{(n+1)^b},$$

where $p > 1$, $q > 1$, $0 < b < \frac{1}{r}.a$, $a + b < \frac{1}{r}$, and $r := \max\{p, q\}$.

They also applied this theorem to prove strong convergence theorems for approximating a zero of an *m-accretive operator*; and solutions of *Hammerstein integral equations.* In fact, they obtained the following theorem.

Theorem 1.7 (Chidume and Idu, [52])**.** *Let X be a uniformly convex and uniformly smooth real Banach space. Let $A : X \to 2^{X^*}$ be a multi-valued maximal monotone and bounded map with $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1, u \in X$, define a sequence $\{x_n\}$ iteratively by*

$$x_{n+1} = J^{-1}[(1 - \lambda_n)Jx_n + \lambda_n\mu_n - \lambda_n\theta_n(Jx_n - Ju)], \quad n \geq 1, \; \mu_n \in Ax_n,$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions (i)-(iv). Then, the sequence $\{x_n\}$ convergences strongly to a zero of A.

Theorem 1.8 (Chidume and Idu, [52])**.** *Let X be a uniformly smooth and uniformly convex real Banach space and $F : X \to X^*$, $K : X^* \to X$ be maximal monotone and bounded maps, respectively. For (x_1, y_1), $(u_1, v_1) \in X \times X^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in X and X^* respectively, by*

$$\begin{aligned} u_{n+1} &= J^{-1}[Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Jx_1)], n \geq 1 \\ v_{n+1} &= J_*^{-1}[Jv_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(J_*v_n - J_*y_1)], \quad n \geq 1. \end{aligned}$$

Assume that the equation $u + KFu = 0$ has a solution. Then, $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

Also, in 2016, Chidume *et al.* [65] studied an iterative algorithm of *Mann-type* to approximate the zero of a *generalized Φ-strongly monotone* and *bounded* map in a uniformly convex and uniformly smooth real Banach space. They proved the following theorem.

Theorem 1.9 (Chidume *et al.*, [65])**.** *Let X be a uniformly convex and uniformly smooth real Banach space and let X^* be its dual space. Let $A : X \to X^*$ be a generalized Φ-strongly monotone and bounded map with $A^{-1}(0) \neq \emptyset$. For arbitrary $u_1 \in X$, let $\{u_n\}$ be the sequence defined iteratively by*

$$u_{n+1} = J^{-1}(Ju_n - \alpha_n Au_n), \quad n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ is a sequence satisfying the following conditions:
(i) $\sum \alpha_n = \infty$, (ii) $\sum 2\alpha_n\omega(\alpha_n M)M < \infty$, (iii) $\omega(\alpha_n M) \leq \gamma_0$; and $\omega : (0, \infty) \to (0, \infty)$ is the modulus of continuity of J^{-1} on the bounded subsets of X^*. Then, the sequence $\{u_n\}$ converges strongly to the solution of the equation $Au = 0$.

In 2012, Klin-earn *et al.* [102] studied the following CQ-algorithm:

$$\begin{cases} x_1 = x \in C, \; C_0 = Q_0 = C, \; x_{n+1} = R_{C_n \cap Q_n}x, \; n \geq 0, \\ u_n = \alpha_n x_n + (1 - \alpha_n)T_n x_n, \\ C_n = \big\{v \in C_{n-1} \cap Q_{n-1} : \phi(u_n, v) \leq \phi(x_n, v)\big\}, \\ Q_n = \big\{v \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Jz\rangle \geq 0\big\}, \end{cases} \tag{1.17}$$

for a family of generalized nonexpansive maps in a uniformly smooth and uniformly convex real Banach space. They proved that the sequence generated by algorithm (1.17), under some mind conditions on $\{\alpha_n\} \subset (0, 1)$, *converges strongly* to a fixed point of T.

Motivated by the result of Klin-earn *et al.* [102], Chidume *et al.* [55] in 2016, proposed the following CQ-algorithm:

$$\begin{cases} x_1 = x \in C, \; C_0 = Q_0 = C, \; x_{n+1} = R_{C_n \cap Q_n} x, \; n \geq 0, \\ u_n = J^{-1}\big(\alpha_n x_n + (1-\alpha_n) J(J_* o T_n) x_n\big), \\ C_n = \big\{v \in C_{n-1} \cap Q_{n-1} : \phi(u_n, v) \leq \phi(x_n, v)\big\}, \\ Q_n = \big\{v \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Jz\rangle \geq 0\big\}. \end{cases} \tag{1.18}$$

for an infinite family of generalized J-nonexpansive maps in a uniformly smooth and uniformly convex real Banach space. They proved that the sequence generated by algorithm (1.18), under some mind conditions on $\{\alpha_n\} \subset (0,1)$, *converges strongly* to a fixed point of T.

These theorems and definitions have motivated great interest on the theory of J-fixed points in various real Banach spaces. Hence, new definitions are introduced and new convergence theorems on J-fixed points have been proved in various directions, leading to flourishing areas of research in recent times, see e.g., Zegeye [166], Liu [111], Su and Xu [151], Cheng *et al.,* [43], Chidume *et al.*, [53], Chidume and Nnakwe [60, 62], Shahzad and Zegeye [139].

In chapter 3 of this thesis, a new Halpern-type algorithm for a generalized mixed equilibrium problem and a countable family of generalized nonexpansive-type maps is presented. This is achieved by using the idea of J-fixed points to define analogues of some general class of nonexpansive-type maps and establish some new lemmas which constitute the basis for the formulation of the theorem.

The theorem proved is a complementary analogue of a theorem of Klin-earn *et al.* [102], and also, extends and improves results of Klin-earn *et al.* [102], Martinez-Yanes and Xu [115], Nakajo and Takahashi [121], Pen and Yao [131], Qin and Su [134], and Tada and Takahashi [152].

Also, In chapter 7 of this thesis, we present a Mann-type iterative algorithm that approximates the zero of a generalized Φ-strongly monotone map. A strong convergence theorem of a sequence generated by the algorithm is proved. Furthermore, the theorem is applied to approximate the solution of a convex optimization problem, a Hammerstein integral equation and a variational inequality problems. This theorem generalizes, improves and complements results of Diop *et al.* [78], Chidume and Bello [51], and Chidume *et al.*, [54, 65]. Finally, examples of generalized Φ-strongly monotone maps are constructed and numerical experiments are presented which illustrate the convergence of the sequence generated by the algorithm.

1.0.2 Inertial algorithm for approximating J-fixed points of generalized J-nonexpansive maps

Let K be a nonempty, closed and convex subset of a real Hilbert space, H. Let $T : K \to K$ be a nonexpansive map. One of the most used algorithms for approximating fixed points of T is the *Mann algorithm* given by

$$x_0 \in K, \quad x_{n+1} = \lambda_n x_n + (1-\lambda_n) T x_n, \; n \geq 0. \tag{1.19}$$

The sequence $\{x_n\}$ generated by equation (1.19), converges *weakly* to a fixed point of T provided $\{\lambda_n\} \subset (0,1)$ is such that $\sum\limits_{n=0}^{\infty} \lambda_n(1-\lambda_n) = \infty$ (see e.g., Dong *et al.* [72] and the references therein). In general, the convergence rate of a sequence generated by the Mann algorithm for nonexpansive maps is *slow* as observed by Sakuria and Liduka [138]. So, interest in the study of fast algorithms for approximating fixed points of nonexpansive maps and their generalizations has greatly increased.

One method to improve the convergence rate of a sequence generated by the Mann algorithm or Mann-type algorithm is to incorporate the *inertial extrapolation term* with the algorithm.

The inertial extrapolation algorithm was first introduced by Polyak [133] in 1964, from the heavy ball experiment of two order time dynamical system, given by:

$$u^{''}(t) + \gamma u^{'}(t) + \nabla\psi(u(t)) = 0, \tag{1.20}$$

where $\gamma > 0$ and $\psi : H \to \mathbb{R}$ is a differentiable functional. The dynamical system (1.20) is discretized such that, given x_n and x_{n-1}, the next term x_{n+1}, can be determined using

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} + \gamma\frac{x_n - x_{n-1}}{h} + \nabla\psi(u(t)) = 0, \tag{1.21}$$

where h is the step size. Equation (1.21) yields the following iterative algorithm:

$$x_{n+1} = x_n + \beta(x_n - x_{n-1}) - \alpha\nabla\psi(x_n), \quad n \geq 0, \tag{1.22}$$

where $\beta = 1 - \gamma h$, $\alpha = h^2$ and $\beta(x_n - x_{n-1})$ is called the *inertial extrapolation term*, which is intended to speed up the convergence of the sequence generated by equation (1.22).

In 1964, Polyak used equation (1.22) to solve a minimization problem given by:

$$\min \psi(x),$$

where $x \in H$, a real Hilbert space, and $\psi : H \to \mathbb{R}$ is a Fréchet differentiable functional.

Following the idea of Polyak [133], we have the following definition:

An *inertial-type algorithm* is a two-step iterative process in which the next iterate is defined by making use of the previous *two* iterates.

A lot of research interest in nonlinear operator theory is now devoted to inertial-type algorithms.

In 2016, Dong *et al.* [77] introduced the following *inertial extragradient algorithm* in a real Hilbert space, by incorporating an inertial extrapolation term with the extragradient algorithm:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \quad y_n = P_C(w_n - \tau f(w_n)), \\ x_{n+1} = (1 - \lambda_n)w_n + \lambda_n P_C(w_n - \tau f(y_n)). \end{cases} \tag{1.23}$$

They proved that the sequence generated by algorithm (1.23), under some mild conditions on $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, 1)$, *converges weakly* to a solution of variational inequality problems.

In 2017, Dong *et al.* [73] introduced the following *inertial projection and contraction algorithm*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \quad y_n = P_C(w_n - \tau f(w_n)), \\ d(w_n, y_n) = (w_n - y_n) - \tau(f(w_n) - f(y_n)), \\ x_{n+1} = (1 - \lambda_n)w_n + \lambda_n P_C(w_n - \tau f(y_n)), \end{cases} \tag{1.24}$$

for solving variational inequality problems in a real Hilbert space. They proved that the sequence generated by algorithm (1.24), under some mild conditions on $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, 1)$, *converges weakly* to a solution of variational inequality problems.

As observed by Amir and Marc [6], and also Dong and Yuan [76], *"the convergence rate of a sequence generated by CQ-algorithm is very slow"*. This is partly because the algorithm requires at *each step of the iteration process*, the computation of two subsets C_n and Q_n of C; their intersection $C_n \cap Q_n$ and the projection of the initial vector onto this intersection. However, the sequence generated by CQ-algorithm, in many cases, *converges strongly* to an element in the solution set under consideration.

Therefore, there is the need to improve on the rate of convergence of the sequence generated by the algorithm. For more on approximation of fixed points of nonexpansive maps and their generalizations using inertial-type algorithms, see e.g., Alvarez and Attouch [5], Mainge [113], Moudafi and Oliny [118], He [88], Bot and Csetnek [18], Lorenz and Pock [112], Dong *et al.* [75], Chan *et al.* [41], Chidume and Nnakwe [62], Dong *et al.* [72, 75].

In Chapter 4 of this thesis, a strong convergence theorem for an *inertial CQ-algorithm* for a countable family of generalized nonexpansive maps is presented. The theorem presented extends and improves results of Klin-earn *et al.* [102] and Dong *et al.* [75] from a uniformly smooth and *uniformly convex* real Banach space, and real Hilbert space, respectively, to a uniformly smooth and *strictly convex* real Banach space. Furthermore, the theorem is applied to prove a strong convergence theorem for an *inertial CQ-algorithm* for a countable family of generalized J-nonexpansive maps, which itself is an improvement of a result of *Chidume et al.* [55]. Finally, a numerical example is presented to illustrate the efficiency of the sequence generated by our algorithm over the sequence generated by the algorithms of Klin-earn *et al.* [102], Dong *et al.* [75] and *Chidume et al.* [55].

1.0.3 Approximation of solutions of generalized split feasibility problems, variational inequality and split variational inequality problems

Fixed point theory for common fixed point problems started with the investigation into the existence of common fixed point for a collection of maps defined on certain "well-behaved" subset of locally convex topological vector space. The earliest theorem on common fixed point problems is the following theorem of Markov-Kakutani.

Theorem 1.10. *(Markov-Kakutani, [95]) Let X be a locally convex topological vector space and let K be a compact convex subset of X. Suppose Γ is a set of maps of K to itself satisfying the following conditions:*

(a) The set Γ is a family of continuous affine transformations, i.e., for any $\phi \in \Gamma$, $x, y \in K$ and α, $\beta > 0$ such that $\alpha + \beta = 1$. Then,

$$\phi(\alpha x + \beta y) = \alpha\phi(x) + \beta\phi(y).$$

(b) The set Γ is abelian, i.e., for every ϕ, $\psi \in \Gamma$ and every $x \in K$,

$$\phi(\psi(x)) = \psi(\phi(x)).$$

Then, there exists a point $x_0 \in K$ such that $\phi(x_0) = x_0$, for all $\phi \in \Gamma$.

This theorem was originally proved by Markov [96] in 1936, with an alternative proof provided in 1938 by Kakutani [95]. The Markov-Kakutani's theorem is an important theorem in that it determines a common fixed point for a family of maps. For more on common fixed point problems, see e.g., Bauschke [9], Browder [22], Chidume and Ofoedu [63], Jung and Kim [93], O'Hara *et al.* [124], Shimizu and Takahashi [146], Chidume and Okpala [64] and Chidume *et al.* [66].

Let H be a real Hilbert space and $K_1, K_2, K_3, \cdots, K_N$ be closed and convex subsets of H, with nonempty intersection K: that is,

$$K = K_1 \cap K_2 \cap K_3 \cap \cdots \cap K_N \neq \emptyset.$$

The *convex feasibility problem* denoted by (CFP) is a problem of finding an element $u^* \in H$ such that

$$u^* \in K. \tag{1.25}$$

This problem has largely been studied due to its numerous applications in the field of science, such as in image restoration, computer tomography and in radiation therapy treatment planning (see e.g., Eremin [80], Censor and Lent [33], Deutsch [70], Herman [89, 90], Censor [34] and the references therein).

A special case of the convex feasibility problem is the following *split feasibility problem* (SFP):

$$\text{Find} \quad u^* \in K \quad \text{such that} \quad Au^* \in Q,$$

where K and Q are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear map.

In 1994, Censor and Elfving [35] introduced the *split feasibility problem* (SFP) for modelling inverse problems which arise from phase retrievals and in image reconstruction. To approximate a solution of the problem (SFP), Censor and Elfving [35] studied the following algorithm:

$$x_{n+1} = A^{-1} P_Q\Big(P_{A(K)}(Ax_n)\Big), \quad \forall \;\; n \geq 0,$$

where $K, Q \subset \mathbb{R}^N$ are closed and convex sets, P_K is the metric projection from $\mathbb{R}^N$ onto K, A is a full rank $N \times N$ matrix, and

$$A(K) = \{y \in \mathbb{R}^N : y = Ax, \; x \in K\}.$$

In 2005, Censor *et al.* [32] formulated a *generalized split feasibility problem* denoted by ($GSFP$), as the problem of finding an element $u^* \in H_1$ for which

$$u^* \in \bigcap_{i=1}^{\infty} K_i \;\; \text{such that} \;\; Au^* \in \bigcap_{j=1}^{\infty} Q_j, \tag{1.26}$$

where $K_i, \; i = 1, 2, \cdots$, and $Q_j, \; j = 1, 2, \cdots$, are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear map.

If the sets $K_i, \; i = 1, \cdots$, and $Q_j, \; j = 1, \cdots$, are the sets of fixed points of maps $S_i : H_1 \to H_1$, $i = 1, \cdots$, and $T_j : H_2 \to H_2, \; j = 1, \cdots$, respectively, then, the *generalized feasibility problem* ($GSFP$) becomes the *generalized split fixed point problem* denoted by ($GSFPP$), which is the problem of finding

$$u^* \in \bigcap_{i=1}^{\infty} F(S_i) \;\; \text{such that} \;\; Au^* \in \bigcap_{j=1}^{\infty} F(T_j), \tag{1.27}$$

where $A : H_1 \to H_2$ is a bounded linear map. For more on convex feasibility problems and its generalizations, see e.g., Censor and Elfving [35], Byrne [23, 30], Censor *et al.* [36], Chang *et al.* [42], Xu *et al.* [161], Stark [149], Combettes and Hawkes [69], Bauschke and Borwein [10].

Let H be a real Hilbert space and K be a nonempty closed and convex subset of H. Let $A : K \to H$ be a map. A *variational inequality problem* is the problem of finding an element $u^* \in K$ such that the following inequality holds:

$$\langle Au^*, u - u^* \rangle \geq 0, \quad \forall\, u \in K. \tag{1.28}$$

The set of solutions of inequality (1.28) is denoted by $VI(A, K)$.

The first problem involving a variational inequality was developed to solve *equilibrium problems*, precisely, the *Signorini problem* posed by Antonio [148] in 1959 and was solved by Fichera [82] in 1963. In 1964, Stampacchia [150] proved the *Stampacchia* theorem to study the regularity problem for partial differential equations. Consequently, he coined the name *"variational inequality"* for all the problems involving inequality of the form (1.28), which he further applied to study the existence of solutions to such problems.

Variational inequality problems have received great attention due to their numerous applications in problems arising in economics, optimization, operations research and engineering (see e.g., Kinderlehrer and Stampacchia [99], Todd [160], Alber [1], Fang and Petersen [81], Shi [144], Noor [123, 122] Zegeye and Shahzad [168], Allen [4] and the references therein).

Using the projection technique, it is well known that solutions of a variational inequality problem are equivalent to fixed points problem.

The simplest algorithm for approximating solutions of variational inequality problem in a real Hilbert space is the following *projection method* given by

$$\Big\{ x_1 \in H, \;\; x_{n+1} = P_K\big(x_n - \tau f(x_n)\big), \;\; \forall\, n \geq 1, \tag{1.29}$$

where f is Lipschitz and η-strongly monotone with $\tau \in \big(0, \frac{2\eta}{L^2}\big)$.

In 2012, Yao *et al.,* [164] showed that the projection gradient method (1.29) may not converge if the strong monotonicity assumption is relaxed to monotonicity. To overcome this difficulty, Korpelevich [105] proposed the following *extragradient method* in a real Hilbert space:

$$\Big\{ x_1 \in H, \;\; y_n = P_K\big(x_n - \tau f(x_n)\big), \;\; x_{n+1} = P_K\big(x_n - \tau f(y_n)\big), \tag{1.30}$$

for each $n \geq 1$, which converges if f is monotone and Lipschitz. However, the weakness of this extragradient method is that one needs to calculate two projections onto K in each iteration process. It is known that if K is a general closed and convex set, this iteration process might require a huge amount of computation time, see e.g., Censor *et al.* [40] and the references therein.

To overcome this difficulty, Censor *et al.* [40] introduced the *subgradient extragradient method* in a real Hilbert space:

$$\begin{cases} x_0 \in H, \;\; y_n = P_K\big(x_n - \tau f(x_n)\big), \\ T_n = \{w \in H : \langle x_n - \tau f(x_n) - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S P_{T_n}\big(x_n - \tau f(y_n)\big), \;\; \forall\, n \geq 0, \end{cases} \tag{1.31}$$

and proved a weak convergence theorem for approximating a common element of solution set of a variational inequality problem and fixed points of a nonexpansive map. The algorithm 1.31 is a modification of the extragradient method presented by Korpelevich [105] by replacement of one of the projections onto K with a projection onto a specific *constructible subgradient half-space of* T_n. This projection method has an advantage in computing over the extragradient method proposed by Korpelevich [105] as demonstrated by Censor *et al.* [40].

Developing algorithms for solving variational inequality problems has continued to attract the interest of numerous researchers in nonlinear operator theory. For earlier and more recent works on variational inequality problem, see e.g., Stampacchia [150], Browder [25, 26], Hartman and Stampacchia [87], Lions and Stampacchia [110], Brezis [14], Brezis and Stampacchia [17], Lewy [107], Lewy and Stampacchia [108], Yao *et al.* [164], Censor *et al.* [40], Gang C. *et al.* [83], Anh and Hieu [7], Chidume and Nnakwe [61], Dong *et al.* [74].

In Chapter 5 of this thesis, we present a weak convergent theorem of subgradient extragradient algorithm for solving variational inequalities and convex feasibility problems. This theorem is an improvement of a result of Censor *et al.* [40] in the following sense:

- Theorem 5.2 which approximates a common solution of a variational inequality problem and a common fixed point of a countable family of relatively nonexpansive maps extends Theorem 7.1 of Censor *et al.* [40] from a *Hilbert space* to a *uniformly smooth* and 2-*uniformly convex* real Banach space with weakly sequentially continuous duality map, and from a single *nonexpansive map* to a countable family of *relatively nonexpansive maps*.

- The control parameters in Algorithm **1** of Theorem 5.2 are two arbitrarily fixed constants $\beta \in (0,1)$ and $\tau \in (0,1)$ which are to be computed once and then used at each step of the iteration process, while the parameters in equation (1.31) studied by Censor *et al.* [40] are $\alpha_k \in (0,1)$ and $\tau \in (0,1)$, and α_k is to be computed at each step of the iteration process. Consequently, the sequence of Algorithm **1** is of *Krasnoselskii-type* and the sequence defined by equation (1.31) is of *Mann-type*. It is well known that a Krasnoselskii-type sequence converges as fast as a geometric progression which is slightly better than the convergence rate obtained from any Mann-type sequence.

Let H_1 and H_2 be real Hilbert spaces and $K_1,\ K_2,\ \cdots,\ K_N$, and $Q_1,\ Q_2,\ \cdots,\ Q_P$, be nonempty closed and convex subsets of H_1 and H_2, respectively, with nonempty intersections K and Q, respectively: that is,

$$K = K_1 \cap K_2 \cap \cdots \cap K_N \neq \emptyset \text{ and } Q = Q_1 \cap Q_2 \cap \cdots \cap Q_P \neq \emptyset.$$

Let $B : H_1 \to H_2$ be a bounded linear map, $G_i : H_1 \to H_1$, $i = 1, \cdots, N$ and $A_j : H_2 \to H_2$, $j = 1, \cdots, P$ be given maps. The *common split variational inequality* problem introduced by Censor *et al.* [32] in 2005, and denoted by (CSVIP), is the problem of finding an element $u^* \in K$ for which

$$\begin{aligned} &\langle u - u^*, G_i(u^*)\rangle \geq 0, \quad \forall \ \ u \in K_i,\ i = 1, 2, \ \cdots, N, \quad \text{such that} \\ &v^* = Bu^* \in Q \text{ solves } \quad \langle v - v^*, A_j(v^*)\rangle \geq 0, \quad \forall \ \ v \in Q_j,\ j = 1, 2, \cdots, P. \end{aligned} \tag{1.32}$$

The set of solutions of the (*CSVIP*) is given by:

$$CSVIP(K, G_i, Q, A_j) = \{u^* \in \cap_{i=1}^N VI(K_i, G_i) : \quad Bu^* \in \cap_{j=1}^P VI(Q_j, A_j)\}.$$

We observe that $u^* \in$ (*CSVIP*) if and only if $u^* = P_{K_i}(I - \mu G_i)u^*$, for each $i = 1, \cdots, N$, such that $Bu^* = P_{Q_j}(I - \gamma A_j)Bu^*$, for each $j = 1, \cdots, P$, where P_{K_i} , P_{Q_j} are the *metric projections* of K_i on H_1 and Q_i on H_2, respectively, and $\mu > 0,\ \gamma > 0$.

Obviously, if $N = 1$, then, problem (1.32) reduces to the well-known *split variational inequality problem (SVIP)* studied by Censor *et al.* [37].

The motivation for studying problem (1.32) with $N > 1$ stems from a simple observation that if we choose $G_i \equiv 0$, the problem reduces to finding $u^* \in \cap_{i=1}^N K_i$, which is the known *convex feasibility problem* (*CFP*) such that $Bu^* \in \cap_{j=1}^P VI(Q_j, A_j)$. If the sets K_i are the fixed point sets of maps $S_i : H_1 \to H_1$, then, the *convex feasibility problems* is the *common fixed points problem*(*CFPP*) whose image under B is a *common solution to variational inequality problems.*

If we choose $G_i \equiv 0$ and $A_j \equiv 0$, problem (1.32) reduces to finding $u^* \in \cap_{i=1}^N K_i$ such that the point $Bu^* \in \cap_{j=1}^P Q_j$ which is the well known *multiple-sets split feasibility problem* or *common split feasibility problem* which serves as a model for many *inverse problems* where constraints are imposed on the solutions in the domain of a linear operator as well as in the range of the operator.

A lot of research interest is now devoted to split variational inequality problem and its generalizations.

In 2010, Censor *et al.* [38] studied the following split variational inequality problem in a real Hilbert space, given by the algorithm:

$$\Big\{x_1 = x \in C, \quad x_{n+1} = P_C(I - \lambda f)\big(x_n - \gamma A^*(P_Q(I - \lambda_n g) - I)Ax_n\big), \quad \forall \;\; n \geq 1, \tag{1.33}$$

where C and Q are nonempty closed and convex subsets of the real Hilbert spaces H_1 and H_2, respectively, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are inverse strongly monotone maps and $A : H_1 \to H_2$ is a bounded linear map. They proved the following theorem.

Theorem 1.11. *Let $A : H_1 \to H_2$ be a bounded linear map, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be respectively α_1 and α_2 inverse strongly monotone maps and set $\alpha := \min\{\alpha_1, \alpha_2\}$. Assume that* (*SVIP*) *is consistent, $\gamma \in (0, \frac{1}{L})$ with L being the spectral radius of the map A^*A, $\lambda \in (0, 2\alpha)$ and suppose that for all x^* solving* (*SVIP*),

$$\langle f(x), P_C(I - \lambda f)(x) - x^* \rangle \geq 0, \forall \in H_1.$$

Then, the sequence $\{x_n\}$ generated by algorithm (1.33) converges weakly to a solution of (***SVIP***).

Recently, Tian and Jiang [159] studied the following algorithm:

$$\begin{cases} x_1 = x \in C, \quad y_n = P_C\big(x_n - \gamma_n A^*(I - T)Ax_n\big), \\ t_n = P_C\big(y_n - \lambda_n f(y_n)\big) \quad x_{n+1} = P_C\big(y_n - \lambda_n f(t_n)\big), \quad \forall \;\; n \geq 1, \end{cases} \tag{1.34}$$

where $\{\gamma_n\} \subset [a, b]$ for some $a, \;\; b \in (0, \frac{1}{||A||^2})$, $\{\lambda_n\} \subset [c, d]$ for some $c, \;\; d \in (0, \frac{1}{K})$, for approximating a solution of the split feasibility problem (***SFP***) in a real Hilbert space. They proved the following theorem.

Theorem 1.12. *Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear map such that $A \neq 0$, $f : C \to H_1$ be a monotone and K-Lipschitz continuous mapping and $T : H_2 \to H_2$ be a nonexpansive mapping. Setting $\Gamma = \{z \in VI(C, f) : Az \in F(T)\}$, assume that $\Gamma \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by algorithm (1.34). Then, the sequence $\{x_n\}$ converges weakly to a point $z \in \Gamma$, where $z = \lim P_\Gamma x_n$.*

Furthermore, they proved a *weak convergence* theorem for approximating a solution of a *split variational inequality problem* in a real Hilbert with the following algorithm:

$$\begin{cases} x_1 = x \in C, \quad y_n = P_C\big(x_n - \gamma_n A^*(I - P_Q(I-\mu g))Ax_n\big), \\ t_n = P_C\big(y_n - \lambda_n f(y_n)\big) \quad x_{n+1} = P_C\big(y_n - \lambda_n f(t_n)\big), \quad \forall \quad n \geq 1, \end{cases} \tag{1.35}$$

where $A : H_1 \to H_2$, is a bounded linear map, $f : C \to H_1$ is a monotone and K-Lipschitz map, $g : H_2 \to H_2$ is an α-inverse strongly map, $\{\gamma_n\} \subset [a, b]$ for some $a,\ b \in (0, \frac{1}{||A||^2})$, $\{\lambda_n\} \subset [c, d]$ for some $c,\ d \in (0, \frac{1}{K})$ and $\mu \in (0, 2\alpha)$.

For more on split variational inequality problem and its generalizations, see e.g., Byrne [23, 30], Censor and Elfving [35], Censor *et al.* [38, 39], Bauschke and Borwein [10], Tiang and Jiang [159].

In Chapter 6 of this thesis, an iterative algorithms for split variational inequalities and generalized split feasibility problems, with applications is presented. The theorem proved complements and improves recent results of Censor *et al.* [39], and also Tian and Jiang [159] in the following sense:

• If $T_i \equiv P_{Q_i} = (I - \mu\mathcal{F}_i) \equiv 0$, $\nabla \equiv I$ and $\lambda \equiv \delta$, then, the *(CSSVIP)* reduces to the *(CSVIP)* and equation (6.1) in Theorem 6.2 reduces to the theorem of Censor *et al.* [39] for solving *(CSVIP)*.

• Theorem 6.1 yields a *strong convergence* of the sequence generated by equation (6.1) for a *finite family of maps* while a *weak convergence* result is proved in Tian and Jiang [159] for a single operator.

• Finally, in Theorem 6.2, a *strong convergence* theorem for approximating a common solution for a *finite family of split variational inequality problems (CSSVIP)* is proved while in the theorem of Tian and Jiang [159], a *weak convergence* theorem for approximating a *split variational inequality problem is proved.*

CHAPTER 2

Preliminaries

In this chapter, we give some fundamental definitions, lemmas and results which constitute the basis for the formulation of our theorems and for effective reading of the subsequent chapters.

2.0.1 Some geometric properties of Banach spaces

Definition 2.1. A real normed space X is called *uniformly convex* if for all $\epsilon \in (0, 2]$, there exists $\delta := \delta(\epsilon) > 0$ such that for each $u, v \in X$ with $||u|| \leq 1,\ ||v|| \leq 1$, and $||u - v|| \geq \epsilon$, we have that $||\frac{1}{2}(u + v)|| \leq 1 - \delta$.

Definition 2.2. A real normed space X is called *strictly convex* if for all $u, v \in X$, $u \neq v$ and $||u|| = ||v|| = 1$, then, the following inequality holds:

$$||\alpha u - (1 - \alpha)v|| < 1 \text{ for all } \alpha \in (0, 1).$$

Remark 2.3. Every uniformly convex space is strictly convex. However, the converse is not generally true (see e.g., Chidume [45]). Moreover, it is well known that every uniformly convex space is a reflexive space. L_p spaces, $1 < p < \infty$, and l_p spaces, $1 < p < \infty$, are uniformly convex spaces.

Definition 2.4. Let X be a real normed space with dual space, X^* and let $U = \{u \in X : ||u|| = 1\}$ be a unit sphere of X. Then, the space X is called *smooth* if:

$$\lim_{t \to 0} \frac{||u + tv|| - ||u||}{t} \text{ exists, for each } u, v \in X.$$

Let the dimension of X, $dim(X) \geq 2$. The *modulus of smoothness* of X denoted by ρ_X, is given by

$$\rho_X(\tau) := \left\{ \frac{||u + v|| + ||u - v||}{2} - 1 : ||u|| = 1, ||v|| = \tau \right\},\ \ \tau > 0.$$

A normed space X is called *uniformly smooth* if $\lim\limits_{\tau \to 0} \frac{\rho_X(\tau)}{\tau} = 0$. It is well known that ρ_X is nondecreasing (see, e.g., Chidume [45], and also Lindenstrauss and Tzafriri [109]). A normed space X is called a *q-uniformly smooth space* if there exists a constant $c > 0$ and a real number

$q > 1$ such that $\rho_X(\tau) \leq c\tau^q$. Typical examples of such spaces include the L_p, l_p and W_p^m spaces, $1 < p < \infty$, where

$$L_p \ (or \ l_p) \ or \ W_p^m = \begin{cases} 2 - \text{uniformly smooth} & \text{if} \ \ 1 < p < 2, \\ p - \text{uniformly smooth} & \text{if} \ \ p \geq 2. \end{cases}$$

Definition 2.5. Let X be a real normed space with dual space, X^* and $p > 1$. The generalized duality map $J_p : X \to 2^{X^*}$ is given by

$$J_p(u) := \{u^* \in X^* : \langle u, u^* \rangle = ||u||||u^*||, \quad ||u^*|| = ||u||^{p-1}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between elements of X and X^*. For $p = 2$, it follows that

$$J_2(u) := \{u^* \in X^* : \langle u, u^* \rangle = ||u||||u^*||, \quad ||u^*|| = ||u||\},$$

where J_2 is called the normalized duality map on X denoted by J.

We make the following remarks (see, e.g., Cioranescu [68]).

1. The normalized duality map exists in any Banach space and its domain is the whole space.
2. In Hilbert spaces, the normalized duality map is precisely the identity map, while in L_p spaces, $1 < p < \infty$, the duality map is given by
$$J(f) = |f|^{p-1} \cdot sign \frac{f}{||f||^{p-1}}.$$
3. The value of the duality map in spaces higher than L_p spaces, $1 < p < \infty$ are not known hitherto.
4. If X is an arbitrary Banach space, then, J is monotone,
5. If X is strictly convex, then, J is strictly monotone,
6. If X is smooth, then, J is single-valued and semi-continuous,
7. If X is uniformly smooth, then, J is uniformly continuous on bounded subset of X,
8. If X is smooth, strictly convex and reflexive, then, the normalized duality map J is single-valued, one-to-one and onto,
9. If X is a reflexive, strictly convex and smooth Banach space and J is the duality map from X into X^*, then, J^{-1} is also single-valued, bijective and is also the duality map from X^* into X and thus, $JJ^{-1} = I_{X^*}$ and $J^{-1}J = I_X$,
10. X is uniformly smooth if and only if X^* is uniformly convex,
11. If X is a reflexive and strictly convex Banach space, then, J^{-1} is norm-weak*-continuous.

Definition 2.6. Let $F : X \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. Then, the sub-differential operator $\partial F : D(F) \subset X \to 2^{X^*}$ is given by

$$\partial F(u) = \{u^* \in X^* : \langle v - u, u^* \rangle \leq F(v) - F(u), \ \forall \ v \in X\}.$$

We remark that the sub-differential ∂F is monotone, and if $u \notin D(F)$, then, $\partial F(u) = \emptyset$.

Lemma 2.7. (see e.g., Bello, [11]) Let X be a real normed space and $F : X \to \mathbb{R} \cup \{\infty\}$ be a function given by

$$F(u) = \frac{1}{2}||u||^2, \ \forall \ u \in X.$$

Then, for any $u \in X$, $\partial F(u) = J(u)$, where J is the normalized duality map on X.

2.0.2 Some nonlinear functions

Let X be a smooth real Banach space with dual space X^*. Consider a map $\phi : X \times X \to \mathbb{R}$ defined by

$$\phi(u, v) = ||u||^2 - 2\langle u, Jv\rangle + ||v||^2, \text{ for all } u, v \ \in X. \tag{2.1}$$

This map which was introduced by Alber [1], and has been studied by Alber and Guerre-Delabriere [2], Kamimura and Takahashi [97], and a host of other authors. For any $u, v, z \in X$, we have the following properties:

(P_1) $(||u|| - ||v||)^2 \leq \phi(u, v) \leq (||u|| + ||v||)^2$,

(P_2) $\phi(u, z) = \phi(u, v) + \phi(v, z) + 2\langle v - u, Jz - Jv\rangle$,

(P_3) $\phi(u, v) \leq ||u||||Ju - Jv|| + ||v||||u - v||$.

Let $V : X \times X^* \to \mathbb{R}$ be a map defined for all $(u, u^*) \in X \times X^*$ by

$$V(u, u^*) = \|u\|^2 - 2\langle u, u^*\rangle + \|u^*\|^2, \tag{2.2}$$

Observe that for all $u \in X, \ u^* \in X^*$, $V(u, u^*) = \phi(u, J^{-1}(u^*))$. The following lemmas will be needed in the sequel.

Lemma 2.8 (Alber, [1])**.** Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Then,

$$V(u, u^*) + 2\langle J^{-1}u^* - u, v^*\rangle \leq V(u, u^* + v^*), \text{ for all } u \in X \text{ and } u^*, v^* \in X^*.$$

Lemma 2.9 (Chidume, [48])**.** Let X be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{u \in X : ||u|| \leq r\}$. Then, for arbitrary $u, v \in B_r(0)$, the following inequality holds:

$$\psi(u, v) \leq ||u - v||^2 + ||u||^2.$$

Lemma 2.10 (Tan and Xu, [158])**.** Let $\{a_n\}$ and $\{\sigma\}$ be sequences of nonnegative real numbers. For some $N_o \in \mathbb{N}$, the following relation hold:

$$a_{n+1} \leq a_n + \sigma_n, \quad n \geq 0.$$

(a) If $\sum \sigma_n < \infty$, then, $\lim a_n$ exists. (b) If in addition, the sequence $\{a_n\}$ has a subsequence that converges to 0, then, $\{a_n\}$ converges to 0.

Lemma 2.11 (Alber and Ryazantseva,[3])**.** Let X be a uniformly convex Banach space with dual space X^*. Then, for any $R > 0$ and for any $u, v \in X^*$ such that $||u|| \leq R$, $||v|| \leq R$, the following inequality holds:

$$||J^{-1}u - J^{-1}v|| \leq c_2\delta_X^{-1}(4RL||u - v||),$$

where $c_2 = 2\max\{1, R\}$ and $1 < L < 1.7$ is the Fiégel constant.

Lemma 2.12 (Alber and Ryazantseva, [3]). Let X be a uniformly convex Banach space with dual space X^*. Then, for any $R > 0$ and for any $u, v \in X$ such that $||u|| \leq R$, $||v|| \leq R$, the following inequality holds:

$$||Ju - Jv|| \leq c_2 \delta_{X^*}^{-1}(4RL||u - v||),$$

where $c_2 = 2\max\{1, R\}$ and $1 < L < 1.7$ is the Fiégel constant..

Definition 2.13 (Takahashi and Yao, [157]). Let K be a nonempty closed convex subset of a real Banach space X.

(a) A map R from X onto K is called a *retraction* if $R^2 = R$.

(b) A map R is called *sunny* if $R(Rx + t(x - Rx)) = Rx$, for all $x \in X$ and $t \geq 0$.

(c) A nonempty closed subset K of a smooth Banach space X is called a *sunny generalized nonexpansive retract* (respectively, *generalized nonexpansive retract*) of X if there exists a sunny generalized nonexpansive retraction (respectively, *generalized nonexpansive retraction*) R from X onto K. If X is a smooth real Banach space, then, the sunny nonexpansive retraction is unique if it exists.

(d) A map T is called *closed* if for any $\{x_n\} \subset X$ such that $x_n \to x^*$ and $Tx_n \to y$, then, $y = Tx^*$.

(e) A point x^* is called an *asymptotic fixed point of* T if there exists a sequence $\{x_n\} \subset K$ such that $x_n \rightharpoonup x^*$ and $||Tx_n - x_n|| \to 0$, as $n \to \infty$. We shall denote the set of asymptotic fixed points of T by $\widehat{F}(T)$.

(f) T is called *relatively nonexpansive* if the fixed point set of T; $F(T) = \widehat{F}(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$, for all $x \in K,\ p \in F(T)$.

Definition 2.14 (Ibaraki and Takahashi, [91]). Let K be a nonempty subset of a smooth Banach space X. A map $T : K \to K$ is called *generalized nonexpansive* if

$$F(T) \neq \emptyset \text{ and } \phi(Tu, v) \leq \phi(u, v), \text{ for all } u \in K,\ v \in F(T).$$

Definition 2.15 (Chidume and Idu, [52]). Let K be a nonempty subset of a real normed space X and $T : K \to X^*$ be a map. A point $u^* \in K$ is called a *J-fixed point of* T if and only if

$$Tu^* = Ju^*.$$

Lemma 2.16 (Kamimura and Takahashi, [97]). Let X be a uniformly convex and uniformly smooth real Banach space and $\{u_n\}$, $\{v_n\}$ be sequences in X such that either $\{u_n\}$ *or* $\{v_n\}$ is bounded. If $\lim\limits_{n\to\infty} \phi(u_n, v_n) = 0$, then, $\lim\limits_{n\to\infty} ||u_n - v_n|| = 0$.

Remark 2.17. Using (P_3), it is easy to see that the converse of Lemma 2.16 is also true whenever $\{u_n\}$ and $\{v_n\}$ bounded.

Lemma 2.18 (Xu, [162]). Let X be a uniformly convex real Banach space. Let $r > 0$. Then, there exists a strictly increasing continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that g(0)=0 and the following inequality holds: for all $x, y \in B_r(0)$, for all $\lambda \in [0, 1]$,

$$||\lambda x + (1 - \lambda)y||^2 \leq \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)g(||x - y||),$$

where $B_r(0) := \{v \in X : ||v|| \leq r\}$.

Lemma 2.19 (Xu, [162]). Let X be a 2-uniformly convex real Banach space. Then, there exists a constant $c_2 > 0$ such that for every $x, y \in X$,

$$c_2||x-y||^2 \leq \langle x-y, jx-jy\rangle, \ \forall \ \ jx \in Jx, \ jy \in Jy.$$

Lemma 2.20 (Xu, [162]). Let X be a 2-uniformly convex and smooth real Banach space. Then, there exists $\alpha > 0$ such that for any $x, y \in X$,

$$\alpha||x-y||^2 \leq \phi(x,y).$$

Lemma 2.21 (Kohsaka and Takahashi, [104]). Let K be a closed convex subset of a uniformly convex and uniformly smooth Banach space X. Let $T_i : K \to X, \quad i = 1, 2, \cdots$ be a countable sequence of relatively nonexpansive maps such that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\} \subset (0,1)$ and $\{\beta_i\}_{i=1}^{\infty} \subset (0,1)$ are sequences such that $\sum\limits_{i=1}^{\infty} \alpha_i = 1$. Let $U : C \to X$ be a map defined by

$$Ux := J^{-1}\left(\sum_{i=1}^{\infty} \alpha_i\big(\beta_i Jx + (1-\beta_i)JT_i x\big)\right), \text{ for each x} \in K.$$

Then, U is relatively nonexpansive and $F(U) = \cap_{i=1}^{\infty} F(T_i)$.

Lemma 2.22 (Ibaraki and Takahashi, [92]). Let K be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space X such that there exists a sunny generalized nonexpansive retraction R from X onto K. Then, the following hold:
(i) $z = Rx$ iff $\langle y - z, Jz - Jx\rangle \geq 0$, for all $y \in K$,
(ii) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$, for all $z \in K$.

Lemma 2.23 (Kohsaka and Takahashi, [103]). Let K be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space X. Then, the following are equivalent:

(i) K is a sunny generalized nonexpansive retract of X,

(ii) K is a generalized nonexpansive retract of X and (iii) JK is closed and convex.

Lemma 2.24 (Chang *et al.*, [97]). Let X be a uniformly smooth and strictly convex real Banach space with Kadec-Klee property and K be a closed convex subset of X. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \to p$ and $\lim\limits_{n\to\infty} \phi(x_n, y_n) = 0$. Then, $y_n \to p$.

Remark 2.25. Using (P_3), it is easy to see that the converse of Lemma 2.24 is also true whenever $\{x_n\}$ and $\{y_n\}$ converge to the same limit point.

Lemma 2.26. A real normed space X is said to have the *Kadec-Klee property*, if for any sequence $\{x_n\} \subset X$ such that $x_n \rightharpoonup x \in X$ and $||x_n|| \to ||x||$, then, $||x_n - x|| \to 0$.

Lemma 2.27 (Ibaraki and Takahashi, [92]). Let K be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then, the sunny generalized nonexpansive retraction from X onto K is uniquely determined.

Lemma 2.28 (Klin-earn *et al.* [102]). Let X be a smooth, strictly convex and reflexive Banach space and let K be a closed subset of X such that JK is closed and convex. Let T be a generalized nonexpansive mapping from K into X. Then, $F(T)$ is closed and $J\big(F(T)\big)$ is closed and convex.

Lemma 2.29 (Klin-earn *et al.* [102]). Let X be a smooth, strictly convex and reflexive Banach space and let K be a closed subset of X such that JK is closed and convex. Let T be a generalized nonexpansive map from K into X. Then, $F(T)$ is a sunny generalized nonexpansive retract of X.

NST-Condition. Let $\{T_n\}$ and Γ be two families of generalized nonexpansive maps from K into X such that $\cap_{n=1}^{\infty} F(T_n) = F(\Gamma) \neq \emptyset$, where $F(T_n)$ is the set of fixed points of T_n and $F(\Gamma)$ is the set of fixed points of Γ. Then, $\{T_n\}$ is said to satisfy the *NST-condition with* Γ if for each bounded sequence $\{x_n\} \subset K$, $\lim_{n\to\infty} ||x_n - T_n x_n|| = 0 \implies \lim_{n\to\infty} ||x_n - Tx_n|| = 0$,$\forall\, T \in \Gamma$, (see, e.g., Klin-earn *et al.* [102]).

Lemma 2.30 (Goebel and Reich, [85]). Let M be a nonempty closed and convex subset of a real Hilbert space H. Then, the following hold:
(1) $||v - P_M u||^2 + ||P_M u - u||^2 \leq ||v - u||^2, \quad \forall\, v \in M,\ u \in H,$
(2) $z = P_M u$ iff $\langle z - v, u - z\rangle \geq 0,\ \forall\, v \in M$, where P_M is the metric projection of H onto M.

Lemma 2.31. (Matsushita and Takahashi, [116]). Let X be a smooth, strictly convex and reflexive Banach space and K be a nonempty closed convex subset of X. Then, the following hold
(1) $\phi(x, \Pi_K y) + \phi(\Pi_K y, y) \leq \phi(x, y), \quad \forall\, x \in K,\ y \in X.$
(2) $z = \Pi_K x \iff \langle z - y, Jx - Jz\rangle \geq 0,\ \forall\, y \in K.$

Definition 2.32. Let $T : H \to H$ be a map.

(1) T is called α-*averaged* if $T = (1-\alpha)I + \alpha S$, where $\alpha \in (0, 1)$, $S : H \to H$ is a *nonexpansive map* and I is the identity map.

(2) $(I - T)$ is *demi-closed* at 0 if T is nonexpansive.

Lemma 2.33 (Xu, [163]). We denote inverse strongly monotone maps by *ism*.
(1) T is nonexpansive if and only if $(I - T)$ is $\frac{1}{2}$-*ism*.
(2) If T is v-*ism* and $\gamma > 0$, then, γT is $\frac{v}{\gamma}$-*ism*.
(3) T is averaged if and only if $(I - T)$ is v-*ism* for some $v > \frac{1}{2}$. Indeed, for $\eta \in (0, 1)$, T is η-averaged if and only if $(I - T)$ is $\frac{1}{2\eta}$-*ism*.
(4) If T_1 is η_1-averaged and T_2 is η_2-averaged, where $\eta_1,\ \eta_2 \in (0, 1)$, then, $T_1 o T_2$ is η-averaged, where $\eta = \eta_1 + \eta_2 - \eta_1\eta_2$.
(5) If T_1 and T_2 are averaged and have a common fixed point, then, $F(T_1 o T_2) = F(T_1) \cap F(T_2)$.

Lemma 2.34 (Takahashi *at el.*, [156]). Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Then, $A^*(I - T)A$ is $\frac{1}{2||A||^2}$-*ism*.

Lemma 2.35 (Tiang and Jiang, [159]). Let H_1 and H_2 be real Hilbert spaces, and M be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $\gamma > 0$ and $z \in H_1$. Suppose $M \cap A^{-1}F(T) \neq \emptyset$, then, the following are equivalent:
(1) $z = P_M\big(I - \gamma A^*(I - T)A\big)z$, (2) $0 \in A^*(I - T)A)z + N_M z$ and (3) $z \in M \cap A^{-1}F(T)$.

Lemma 2.36 (Kraikaew and Saejung, [106]). Let $f : H \to H$ be a monotone and L-Lipschitz map on M. Let $U := P_M(I - \tau f)$, where $\tau > 0$. If $\{x_n\}$ is a sequence in M such that $x_n \rightharpoonup x^*$ and $x_n - U(x_n) \to 0$, then, $x^* \in VI(M, f) = F(U)$.

Lemma 2.37 (Tiang and Jiang, [159]). Let M be a nonempty closed convex subset of a real Hilbert space H. Let Θ be a convex function of H into $\mathbb{R}$. If Θ is differentiable, then, z is a solutions of constrained convex minimization problem if and only if $z \in VI(M, \Theta')$.

Definition 2.38 (Rockafellar, [136]). The normal cone of C at $v \in C$ denoted by $N_C(v)$ is given by $N_K(v) := \{w \in X^* : \langle y - v, w \rangle \leq 0, \quad \forall\, y \in K\}$.

Definition 2.39. A map $T : X \to 2^{X^*}$ is called *monotone* if $\langle \eta - \eta, x - y \rangle \geq 0, \quad \forall\, x, y \in X$ and $\eta \in Tx,\ \eta \in Tx$. Furthermore, T is *maximal monotone* if its monotone and the graph $G(T) := \{(x, y) \in X \times X^* : y \in T(x)\}$ is not properly contained in the graph of any other monotone operator.

Definition 2.40. Let f be a function from X to X^* with domain $D = D(f) \subset X$. The map f is *hemicontinuous* if $u \in D$, $v \in X$ and $u + t_n v \in D$, where t_n is a sequence of positive numbers such that $t_n \to 0$, imply $f(u + t_n v) \rightharpoonup fu$.

Lemma 2.41. (Rockafellar, [136]) Let K be a nonempty closed and convex subset of a reflexive Banach space X. Let $f : K \to X^*$ be a monotone and hemicontinous map and $T \subset X \times X^*$ be a map defined by

$$Tv = \begin{cases} f(v) + N_K(v), & if\ v \in K, \\ \emptyset, & if\ \ v \notin K. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(f, K)$.

Remark 2.42. It has been observed that a monotone map T is maximal if given $(x, y) \in X \times X^*$ and if $\langle x - u, y - v \rangle \geq 0,\ \forall\, (u, v) \in G(T)$, then, $y \in Tx$ (Rockafellar, [136]).

Definition 2.43. Let M be a nonempty closed convex subset of a real Hilbert space H, and $\mathcal{G} : M \times M \to \mathbb{R}$ be a map. The *equilibrium problem* is to find $x^* \in M$ such that

$$\mathcal{G}(x^*.y) \geq 0, \quad \forall\, y \in M. \tag{2.3}$$

We shall denote the set of solutions of the equilibrium problem by $EP(\mathcal{G})$.

Remark 2.44. Let M be a nonempty closed and convex subset of a real Hilbert space H. For solving equilibrium problem, we assume that the bifunctional $\mathcal{G} : M \times M \to \mathbb{R}$ satisfies the following conditions:
(P_1) $\mathcal{G}(x, x) = 0,\ for\ all\ x \in M,$
(P_2) $\mathcal{G}$ is monotone, i.e. $\mathcal{G}(x, y) + \mathcal{G}(y, x) \leq 0,\ for\ all\ x, y \in M,$
(P_3) $\limsup_{t \downarrow 0} \mathcal{G}(x + t(z - x), y) \leq \mathcal{G}(x, y),\ for\ all\ x, y, z \in M,$
(P_4) For all $x \in M$, $\mathcal{G}(x, \cdot)$ is convex and lower semi-continuous.

Definition 2.45. Let M be a nonempty closed convex subset of a real Hilbert space H. The *convex minimization problem* is to find $x^* \in M$ such that

$$\Theta(x^*) = \min_{x \in M} \Theta(x). \tag{2.4}$$

We shall denote the set of solutions of convex minimization problem by $Argmin(M, \Theta)$.

Now, we turn over to the next chapter.

CHAPTER 3

A new Halpern-type algorithm for a generalized mixed equilibrium problem and a countable family of generalized nonexpansive-type maps

Introduction

In this chapter, we present a strong convergence theorem for obtaining a common element in the set of solutions of a generalized mixed equilibrium problem and set of common fixed points of a countable family of generalized-J-nonexpansive maps in a uniformly smooth and uniformly convex real Banach space. The theorem presented is an analogue of the result of Klin-earn *et al.*, [102]. Also, in the special case of a real Hilbert space, the theorem presented complements, extends and improves the results of Martinez-Yanes and Xu [115], Nakajo and Takahashi [121], Pen and Yao [131], Qin and Su [134], Tada and Takahashi [152], and a host of other recent results. Finally, we give numerical experiments to illustrate the convergence of the sequence generated by algorithm (3.11).

3.0.1 Generalized mixed equilibrium problem

Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let K be a nonempty closed and convex subset of X such that JK is closed and convex, where $J : X \to X^*$ is the normalized duality map. Let $\chi : JK \to \mathbb{R}$ be a map, $\Theta : JK \times JK \to \mathbb{R}$ be a bifunction and $B : K \to X^*$ be a map. The *generalized mixed equilibrium problem* is the problem of finding an element $u \in K$ such that

$$\Theta(Ju, Jz) + \chi(Jz) - \chi(Ju) + \langle Bu, z - u\rangle \geq 0, \ \forall \ \ z \in K. \tag{3.1}$$

The set of solutions of the generalized mixed equilibrium problem is given by

$$GMEP(\Theta, B, \chi) = \{u \in K : \Theta(Ju, Jz) + \chi(Jz) - \chi(Ju) + \langle Bu, z - u\rangle \geq 0, \ \forall \ z \in K\}.$$

It has been observed that the class of generalized mixed equilibrium problems contains, as special cases, numerous important classes of nonlinear problems such as equilibrium problems, optimization problems, variational inequality problems, and so on, which themselves, have diverse applications in a large variety of problems arising in Economics, Operation research, Physics, Engineering (see e.g., Blum and Otelli [13], Chang *et al.* [145], Takahashi and Zembayashi [154, 155], Browder [26], Onjai-Uea and Kumam [129] and the references contained in them).

We make the following basic Assumptions for proving the theorem of this chapter.

Let K be a nonempty closed subset of a uniformly smooth and uniformly convex real Banach space X with dual space X^* such that JK is closed and convex. Let $\chi : JK \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $B : K \to X^*$ be a continuous and monotone map. For solving the generalized mixed equilibrium problems, (1.1), we assume that the bifunctional $\Theta : JK \times JK \to \mathbb{R}$ satisfies the following conditons:

(B_1) $\Theta(u^*, u^*) = 0,$ for all $u^* \in JK,$

(B_2) Θ is monotone, i.e. $\Theta(u^*, v^*) + \Theta(v^*, u^*) \leq 0,$ for all $u^*, v^* \in JK,$

(B_3) $\limsup_{\lambda \downarrow 0} \Theta(u^* + \lambda(z^* - u^*), v^*) \leq \Theta(u^*, v^*),$ for all $u^*, v^*, z^* \in JK,$

(B_4) $\Theta(u^*, \cdot)$ is convex and lower semi-continuous, for all $u^* \in JK$.

3.0.2 Main result

Definition 3.1. A map $T : K \to X^*$ is called *generalized J-nonexpansive* if

$$F_J(T) \neq \emptyset \text{ and } \phi((J^{-1}oT)x, p) \leq \phi(x, p), \text{ for all } x \in K, \; p \in F_J(T),$$

where ϕ denotes the Alber's functional and $F_J(T)$ denotes the J-fixed points of T.

Lemma 3.2. Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let K be a closed subset of X such that JK is closed and convex. Let T be a generalized J-nonexpansive map on K with $F_J(T) \neq \emptyset$. Then, $F_J(T)$ is closed and $JF_J(T)$ is closed and convex.

Proof. Obviously $F_J(T)$ is closed. Let $\{v_n^*\} \subset JF_J(T)$ be such that $v_n^* \to v^*$, for some $v^* \in X^*$. Since JK is closed, we have that $v^* \in JK$. Hence, there exist $v \in K$ and $\{v_n\} \subset F_J(T)$ such that $v^* = Jv$ and $v_n^* = Jv_n$. Utilizing the definition of T, we have that

$$\begin{aligned} \phi((J^{-1}oT)v, v) &= \lim_{n\to\infty} \phi((J^{-1}oT)v, v_n) \leq \lim_{n\to\infty} (||v||^2 - 2\langle v, Jv_n \rangle + ||v_n||^2) \\ &= \lim_{n\to\infty} (||v||^2 - 2\langle v, v_n^* \rangle + ||v_n^*||^2) = \phi(v, v) = 0. \end{aligned} \tag{3.2}$$

Utilizing the strictly convex of X and inequality (3.2), we have that $J^{-1}v^* \in F_J(T)$. Hence, $JF_J(T)$ is closed.

Next: let $u^*, v^* \in JF_J(T)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta = 1$. Then, we compute as follows:

$$\begin{aligned} & \phi((J^{-1}oT)J^{-1}(\alpha u^* + \beta v^*), J^{-1}(\alpha u^* + \beta v^*)) \\ = \;& ||(J^{-1}oT)J^{-1}(\alpha u^* + \beta v^*)||^2 - 2\langle (J^{-1}oT)J^{-1}(\alpha u^* + \beta v^*), \alpha u^* + \beta v^* \rangle + ||\alpha u^* + \beta v^*||^2 \\ & + \; \alpha||u||^2 + \beta||v||^2 - (\alpha||u||^2 + \beta||v||^2) \\ = \;& \alpha(||(J^{-1}oT)J^{-1}(\alpha u^* + \beta v^*)||^2 - 2\langle (J^{-1}oT)J^{-1}(\alpha u^* + \beta v^*), Ju \rangle + ||u||^2) \\ & + \; \beta(||(J^{-1}oT)J^{-1}(\alpha u^* + \beta v^*)||^2 - 2\langle (J^{-1}oT)J^{-1}(\alpha u^* + \beta v^*), Jv \rangle + ||v||^2) \\ & + \; ||\alpha u^* + \beta v^*||^2 - (\alpha||u||^2 + \beta||v||^2) \\ = \;& \alpha\phi((J^{-1}oT)J^{-1}(\alpha u^* + \beta v^*), u) + \beta\phi((J^{-1}oT)J^{-1}(\alpha u^* + \beta v^*), v) \\ & + \; ||\alpha u^* + \beta v^*||^2 - (\alpha||u||^2 + \beta||v||^2) \end{aligned}$$

$$
\begin{aligned}
&\phi((J^{-1}oT)J^{-1}(\alpha u^* + \beta v^*), J^{-1}(\alpha u^* + \beta v^*)) \\
\le\ & \alpha\phi\big((J^{-1}(\alpha u^* + \beta v^*), u\big) + \beta\phi\big((J^{-1}(\alpha u^* + \beta v^*), v\big) + ||\alpha u^* + \beta v^*||^2 \\
&- \ (\alpha||u||^2 + \beta||v||^2) \\
=\ & \alpha(||\alpha u^* + \beta v^*||^2 - 2\big\langle(J^{-1}(\alpha u^* + \beta v^*), Ju\big\rangle + ||u||^2) + \ \beta(||\alpha u^* + \beta v^*||^2 \\
&- \ 2\big\langle(J^{-1}(\alpha u^* + \beta v^*), Jv\big\rangle + ||v||^2) + ||\alpha u^* + \beta v^*||^2 - (\alpha||u||^2 + \beta||v||^2) \\
=\ & 2||\alpha u^* + \beta v^*||^2 - 2\big\langle(J^{-1}(\alpha u^* + \beta v^*), \alpha u^* + \beta v^*\big\rangle = 0.
\end{aligned}
$$

Since X strictly convex and $\phi((J^{-1}oT)J^{-1}(\alpha u^* + \beta v^*), J^{-1}(\alpha u^* + \beta v^*)) = 0$, we have that $J^{-1}(\alpha u^* + \beta v^*) \in F_J(T)$. Hence, $\alpha u^* + \beta v^* \in JF_J(T)$. □

NST-condition. Let $\{S_n\}$ and Υ be two families of generalized J-nonexpansive maps from K into X^* such that $\cap_{n=1}^{\infty} F_J(S_n) = F_J(\Upsilon) \neq \emptyset$, where $F_J(S_n)$ is the set of J-fixed points of S_n and $F_J(\Upsilon)$ is the set of J-fixed points of Υ.

The sequence $\{S_n\}$ from K to X^* is said to satisfy the *NST-condition with* Υ if for each bounded sequence $\{x_n\} \subset K$, $\lim_{n\to\infty} ||Jx_n - S_n x_n|| = 0 \implies \lim_{n\to\infty} ||Jx_n - Sx_n|| = 0, \ \forall\ S \in \Upsilon$.

Remark 3.3. In particular, if $\Upsilon = \{S\}$, then, $\{S_n\}$ is said to satisfy the *NST-condition* with S. It is obvious that $\{S_n\}$ with $S_n = S$, for all $n \in \mathbb{N}$, satisfies NST-condition with $\Upsilon = \{S\}$.

Lemma 3.4. Let K be a closed subset of a uniformly convex and uniformly smooth real Banach space X with dual space X^*. Let S be a generalized J-nonexpansive map from K into X^* with $F_J(S) \neq \emptyset$. Let $\{\beta_n\} \subset (0,1)$. For each $n \in \mathbb{N}$, define a map $S_n : K \to X^*$ by

$$S_n u = J(\beta_n u + (1-\beta_n)J^{-1}oSu), \ \text{ for all } \ u \in K.$$

Then, $\{S_n\}$ is a countable family of generalized J-nonexpansive maps satisfying the *NST*-condition with S.

Proof. Clearly, $F_J(S_n) = F_J(S), \ \forall\ n \in \mathbb{N}$. Hence, $\bigcap_{n\ge 1} F_J(S_n) = F_J(S)$. For $u \in K, \ v \in F_J(S_n)$,

$$
\begin{aligned}
\phi(J^{-1}oS_n u, v) &= \phi(\beta_n u + (1-\beta_n)J^{-1}oSu, v) \\
&\le \beta_n\phi(u,v) + (1-\beta_n)\phi(J^{-1}oSu, v) \\
&\le \beta_n\phi(u,v) + (1-\beta_n)\phi(u,v) = \phi(u,v).
\end{aligned}
$$

Hence, $\{S_n\}$ is generalized J-nonexpansive, where the map ϕ is the Alber's functional.

Let $\{u_n\}$ be a bounded sequence in K such that $\lim ||Ju_n - S_n u_n|| = 0$. Since $\{u_n\}$ is bounded, then, $\{J^{-1}oSu_n\}$ is bounded. Using the definition of S_n, we have that

$$||u_n - J^{-1}oSu_n|| = \frac{1}{1-\beta_n}||u_n - J^{-1}oS_n u_n|| \le 2||u_n - J^{-1}oS_n u_n||.$$

Since $\lim ||Ju_n - S_n u_n|| = 0$ and the fact that J^{-1} and J are uniformly continuous on bounded subsets of X^* and X, respectively, we have that $\lim ||Ju_n - Su_n|| = 0$. □

Lemma 3.5. Let K be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space X, with dual space X^*, such that JK is closed and convex. Let $\chi : JK \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $B : K \to X^*$ be continuous and monotone, and $\Theta : JK \times JK \to \mathbb{R}$ be a bifunction. Let $r > 0$ and $x \in X$ be any point. Define a map $T_r : X \to K$ by

$$T_r(x) = \{u \in K : \Theta(Ju, Jz) + \chi(Jz) - \chi(Ju) + \langle Bu, z - u\rangle + \frac{1}{r}\langle u - x, Jz - Ju\rangle \geq 0, \ \forall \ \ z \in K\}.$$

Then, the following conclusions hold:

(a) T_r is single-valued,

(b) T_r is a firmly nonexpansive-type map, i.e.,

$$\forall \ x, y \ \in X, \ \ \big\langle T_r x - T_r y, JT_r x - JT_r y\big\rangle \leq \big\langle x - y, JT_r x - JT_r y\big\rangle,$$

(c) $F(T_r) = GMEP(\Theta, \chi, B)$,

(d) $GMEP(\Theta, \chi, B)$ is closed and $J(GMEP(\Theta, \chi, B))$ is closed and convex,

(e) $\phi(x, T_r x) + \phi(T_r x, q) \leq \phi(x, q), \forall \ q \in F(T_r), \ \ x \ \ \in X$, where the map ϕ is the Alber's functional.

Proof. (a) Let $x \in X$ and $r > 0$. Let $u_1, u_2 \in T_r(x)$. Then, we have that

$$\Theta(Ju_1, Ju_2) + \chi(Ju_2) - \chi(Ju_1) + \langle Bu_1, u_2 - u_1\rangle + \frac{1}{r}\langle u_1 - x, Ju_2 - Ju_1\rangle \geq 0, \quad (3.3)$$

$$\Theta(Ju_2, Ju_1) + \chi(Ju_1) - \chi(Ju_2) + \langle Bu_2, u_1 - u_2\rangle + \frac{1}{r}\langle u_2 - x, Ju_1 - Ju_2\rangle \geq 0. \quad (3.4)$$

From inequalities (3.3) and (3.4), condition (B_2) and the monotonicity of B, we have that

$$\frac{1}{r}\langle u_1 - u_2, Ju_2 - Ju_1\rangle \geq 0. \quad (3.5)$$

From monotonicity of J and strict convexity of X, we have that $u_1 = u_2$, which implies that T_r is single-valued.

(b) For any $x, y \in K$, we have that

$$\Theta(JT_r x, JT_r y) + \chi(JT_r y) - \chi(JT_r x) + \langle BT_r x, T_r y - T_r x\rangle + \frac{1}{r}\langle T_r x - x, JT_r y - JT_r x\rangle \geq 0,$$

$$\Theta(JT_r y, JT_r x) + \chi(JT_r x) - \chi(JT_r y) + \langle BT_r y, T_r x - T_r y\rangle + \frac{1}{r}\langle T_r y - y, JT_r x - JT_r y\rangle \geq 0.$$

From the above inequalities, condition (B_2) and the monotonicity of B, we conclude that

$$\big\langle T_r x - T_r y, JT_r x - JT_r y\big\rangle \leq \big\langle x - y, JT_r x - JT_r y\big\rangle.$$

(c) Claim. $F(T_r) = GMEP(\Theta, \chi, B)$.

$$\begin{aligned} u \in F(T_r) \quad &\iff \quad u = T_r u \\ &\iff \quad \Theta(Ju, Jz) + \chi(Jz) - \chi(Ju) + \langle Bu, z - u\rangle + \frac{1}{r}\langle u - u, Jz - Ju\rangle \geq 0, z \in K \\ &\iff \quad \Theta(Ju, Jz) + \chi(Jz) - \chi(Ju) + \langle Bu, z - u\rangle \geq 0, \ \forall \ z \in K \\ &\iff \quad u \in GMEP(\Theta, \chi, B). \end{aligned}$$

(d) Claim. $GMEP(\Theta, \chi, B)$ is closed, and $J(GMEP(\Theta, \chi, B))$ is closed and convex. Clearly, $GMEP(\Theta, \chi, B)$ is closed. Let $\{u_n^*\} \subset J(GMEP(\Theta, \chi, B))$ such that $u_n^* \to u^*$, for some $u^* \in X^*$. Since JK is closed, we have that $u^* \in JK$. Hence, there exist $u \in K$ and $\{u_n\} \subset (GMEP(\Theta, \chi, B))$ such that $u^* = Ju$ and $u_n^* = Ju_n$, $\forall\, n \in \mathbb{N}$. Utilizing the definitions of Θ, B, χ and the fact that J^{-1} is uniformly continuous on bounded subset of X^*, we have:

$$\begin{aligned}
\chi(u^*) \leq \liminf \chi(u_n^*) &\leq \liminf \left[\Theta(u_n^*, Jy) + \chi(Jy) + \langle BJ^{-1}u_n^*, y - J^{-1}u_n^*\rangle\right] \\
&\leq \limsup \left[\Theta(u_n^*, Jy) + \chi(Jy) + \langle BJ^{-1}u_n^*, y - J^{-1}u_n^*\rangle\right] \\
&\leq \Theta(u^*, Jy) + \chi(Jy) + \langle BJ^{-1}u^*, y - J^{-1}u^*\rangle.
\end{aligned}$$

Hence, $J(GMEP(\Theta, \chi, B))$ is closed.

Let $u_1^*, u_2^* \in J(GMEP(\Theta, \chi, B))$. Then, $u_1^* = Ju_1$, $u_2^* = Ju_2$, for some u_1, $u_2 \in K$. For $\lambda, t \in (0, 1]$, let $u_\lambda^* = \lambda u_1^* + (1-\lambda)u_2^* \in JK$. For any $y \in K$, set $z_t^* = tJy + (1-t)u_\lambda^*$. By conditions (B_1) to (B_4), we have that

$$\begin{aligned}
0 &= \Theta(z_t^*, z_t^*) + \chi(z_t^*) - \chi(z_t^*) + \langle B(J^{-1}z_t^*), y - J^{-1}z_t^*\rangle - \langle B(J^{-1}z_t^*), y - J^{-1}z_t^*\rangle \\
&\leq \Theta(z_t^*, Jy) + \chi(Jy) - \chi(z_t^*) + \langle B(J^{-1}z_t^*), y - J^{-1}z_t^*\rangle \\
&= \Theta(u_\lambda^* + t(Jy - u_\lambda^*), Jy) + \chi(Jy) - \chi(u_\lambda^* + t(Jy - u_\lambda^*)) \\
&\quad + \left\langle BJ^{-1}(u_\lambda^* + t(Jy - u_\lambda^*)), y - J^{-1}(u_\lambda^* + t(Jy - u_\lambda^*))\right\rangle.
\end{aligned}$$

Applying condition (B_3) we conclude that

$$\Theta(u_\lambda^*, Jy) + \chi(Jy) - \chi(u_\lambda^*) + \langle B(J^{-1}u_\lambda^*), y - J^{-1}u_\lambda^*\rangle \geq 0.$$

Hence, $u_\lambda^* \in J(GMEP(\Theta, \chi, B))$. Therefore, $J(GMEP(\Theta, \chi, B))$ is convex.

(e) Claim. $\phi(x, T_rx) + \phi(T_rx, q) \leq \phi(x, q)$, $\forall\, q \in F(T_r)$, $x \in X$. Let $x, y \in K$. Then, we have:

$$\phi(T_rx, T_ry) + \phi(T_ry, T_rx) = 2\langle T_rx - T_ry, JT_rx - JT_ry\rangle \tag{3.6}$$

$$\phi(x, T_ry) + \phi(y, T_rx) - \phi(x, T_rx) - \phi(y, T_ry) = 2\langle x - y, JT_rx - JT_ry\rangle. \tag{3.7}$$

Applying Lemma 3.5(b), equations (3.6) and (3.7), we have that

$$\phi(T_rx, T_ry) + \phi(T_ry, T_rx) \leq \phi(x, T_ry) + \phi(y, T_rx) - \phi(x, T_rx) - \phi(y, T_ry),\ \forall\, x, y \in K. \tag{3.8}$$

For $y = u \in F(T_r)$, we have that

$$\phi(T_rx, u) + \phi(u, T_rx) \leq \phi(x, u) + \phi(u, T_rx) - \phi(x, T_rx) - \phi(u, u),\ \forall\, x \in K. \tag{3.9}$$

Hence, we conclude that

$$\phi(x, T_rx) + \phi(T_rx, u) \leq \phi(x, u),\ \forall\, x, \in K,\ u \in F(T_r). \tag{3.10}$$

This proof is complete. □

Theorem 3.6. *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let K be a nonempty closed and convex subset of X such that JK is closed and convex. Let $\chi : JK \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $B : K \to X^*$ be a continuous and monotone map. Let $\Theta : JK \times JK \to \mathbb{R}$ be a bifunction satisfying conditions $(B_1) - (B_4)$. Let $S_n : K \to X^*, n = 1, 2, \cdots$ be a countable family of generalized J-nonexpansive maps and Υ be a family of closed and generalized J-nonexpansive maps from K to X^* such that*

$\cap_{n=1}^{\infty} F_J(S_n) \cap GMEP(\Theta, B, \chi) = F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \neq \emptyset$, $\beta_n \in (0,1)$ *with* $\lim \beta_n = 0$ *and* $\{r_n\} \subset [a, \infty)$, *for some* $a > 0$. *Let* $\{x_n\}$ *be a sequence generated iteratively by:*

$$\begin{cases} x_1 \in K, \ K_1 = K, \\ z_n = \beta_n x_1 + (1-\beta_n)(J^{-1} o S_n) x_n, \\ u_n = T_{r_n} z_n, \quad x_{n+1} = R_{K_{n+1}} x_1, \forall\ n \geq 1, \\ K_{n+1} = \big\{ v \in K_n : \phi(u_n, v) \leq \beta_n \phi(x_1, v) + (1-\beta_n)\phi(x_n, v) \big\}. \end{cases} \tag{3.11}$$

Assume that $\{S_n\}$ *satisfies the NST-condition with* Υ. *Then,* $\{x_n\}$ *converges strongly to* $R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)} x$, *where* $R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)}$ *is the sunny generalized* J*-nonexpansive retraction of* X *onto* $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)$.

Proof. The proof is divided into 5 steps. Here, the map ϕ denotes the Alber's functional.

Step 1: The sequence $\{x_n\}$ is well defined and $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \subset K_n$.
First, we show that JK_n is closed and convex. Clearly $JK_1 = JK$ is closed and convex. Assume that JK_n is closed and convex, for some $n \geq 1$, applying the definition of K_{n+1}, it is clear that $K_{n+1} = \{v \in K_n : 2\langle \beta_n x_1 + (1-\beta_n)x_n - u_n, Jv\rangle \leq \beta_n ||x_1||^2 + (1-\beta_n)||x_n||^2 - ||u_n||^2\}$. Thus, JK_{n+1} is closed and convex. Hence, JK_n is closed and convex. By Lemma 2.23, K_n is a sunny generalized-J-nonexpansive retract of X. Hence, $\{x_n\}$ is well defined.

Next, we show that $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \subset K_n, \ \forall\ n \geq 1$. Clearly, $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)$ is a subset of K_1. Assume that $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \subset K_n$, for some $n \geq 1$. Let $q \in F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)$. Applying Lemma 3.5 and definition of S_n, we have that

$$\begin{aligned} \phi(u_n, q) &= \phi(T_{r_n} z_n, q) \leq \phi(z_n, q) \\ &= ||\beta_n x_1 + (1-\beta_n)(J^{-1} o S_n)x_n||^2 - 2\big\langle \beta_n x_1 + (1-\beta_n)(J^{-1} o S_n)x_n, Jq \big\rangle + ||q||^2 \\ &\leq \beta_n \phi(x_1, q) + (1-\beta_n)\phi((J^{-1} o S_n)x_n, q) \leq \beta_n \phi(x_1, q) + (1-\beta_n)\phi(x_n, q). \end{aligned} \tag{3.12}$$

This implies that $q \in K_{n+1}$. Hence, $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \subset K_n$.

Step 2: We show that $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ converge to a solution of $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)$.
First, we show that $\{x_n\}$ is bounded. From the definition of $\{x_n\}$ and Lemma 2.22, we have: $\phi(x_1, x_n) = \phi(x_1, R_{K_n} x_1) \leq \phi(x_1, q) - \phi(R_{K_n} x_1, q) \leq \phi(x_1, q)$, $\forall\ q$ in $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)$ $\subset K_n$. This implies that $\{\phi(x_1, x_n)\}$ is bounded. It follows from the definition of ϕ that $\{x_n\}$ is bounded. Since $x_{n+1} = R_{K_{n+1}} x_1 \in K_{n+1} \subset K_n$ and $x_n = R_{K_n} x_1$, we have that $\phi(x_1, x_n) \leq \phi(x_1, x_{n+1})$ and this implies that $\{\phi(x_1, x_n)\}$ is nondecreasing. Hence, $\lim_{n\to\infty} \phi(x_1, x_n)$ exists. Also, for $m > n$, from Lemma 2.22 and $x_n = R_{K_n} x_1$, we have that

$$\begin{aligned} \phi(x_n, x_m) = \phi(R_{K_n} x_1, R_{K_m} x_1) &\leq \phi(x_1, R_{K_m} x_1) - \phi(x_1, R_{K_n} x_1) \\ &= \phi(x_1, x_m) - \phi(x_1, x_n) \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

Hence, $\lim_{n\to\infty} \phi(x_n, x_m) = 0$. It follows from Lemma 2.16 that $\lim_{n\to\infty} ||x_n - x_m|| = 0$. Hence, $\{x_n\}$ is a Cauchy sequence in K. Thus, there exists $x^* \in K$ such that $\lim_{n\to\infty} x_n = x^*$. From inequality (3.12) and using the fact that $\lim_{n\to\infty} \beta_n = 0$ by assumption, it follows that
$\phi(u_n, x_m) \leq \beta_n \phi(x_1, x_m) + (1-\beta_n)\phi(x_n, x_m) \to 0$ as $n \to \infty$. By Lemma 2.16, we have that

$$\lim_{n\to\infty} ||u_n - x_m|| = 0. \ \text{ Hence, } \lim_{n\to\infty} ||u_n - x_n|| = 0. \text{ This implies that } \lim_{n\to\infty} u_n = x^*. \tag{3.13}$$

From inequality (3.12), Lemma 3.5 and equation (3.13), we get that

$$\begin{aligned}\phi(z_n, u_n) = \phi(z_n, T_{r_n} z_n) &\leq \phi(z_n, q) - \phi(u_n, q)\\ &\leq \beta_n \phi(x_1, q) + (1-\beta_n)\phi(x_n, q) - \phi(u_n, q)\\ &\leq \beta_n \phi(x_1, q) + \phi(x_n, q) - \phi(u_n, q) \to 0.\end{aligned}$$

By Lemma 2.16, it follows that $\lim_{n\to\infty} ||u_n - z_n|| = 0$. Thus, $\lim_{n\to\infty} z_n = x^*$. Using this and equation (3.13), we conclude that $\lim_{n\to\infty} x_n = x^*$, $\lim_{n\to\infty} u_n = x^*$ and $\lim_{n\to\infty} z_n = x^*$.

Step 3: $\lim_{n\to\infty} ||Jx_n - Sx_n|| = 0, \ \forall\ S \in \Upsilon$.
From equation (3.1), we obtain that

$$(1-\beta_n)||x_n - (J^{-1} o S_n)x_n|| \leq ||x_n - z_n|| + \beta_n ||x_1 - x_n||. \tag{3.14}$$

First, we observe that $\{(J^{-1} o S_n)x_n\}$ is bounded in X. Using step 2 and the fact that $\lim_{n\to\infty} \beta_n = 0$ by assumption in inequality (3.14), we obtain that $\lim_{n\to\infty} ||x_n - (J^{-1} o S_n)x_n|| = 0$. By uniform continuity of J on bounded subset of X, we get that $\lim_{n\to\infty} ||Jx_n - S_n x_n|| = 0$. Since $\{S_n\}$ satisfies the NST-condition with Υ, we conclude that $\lim_{n\to\infty} ||Jx_n - Sx_n|| = 0, \ \forall\ S \in \Upsilon$.

Step 4: $x^* \in F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)$.
From step 3, we have that $\lim_{n\to\infty} ||Jx_n - Sx_n|| = 0, \ \forall\ S \in \Upsilon$. We also proved that $x_n \to x^* \in K$. Since S is closed, we conclude that $x^* \in F_J(\Upsilon)$. Furthermore, from step 2, we have that $\lim_{n\to\infty} ||z_n - u_n|| = 0$. Since $\{r_n\} \subset [a, \infty)$ by assumption, we obtain that $\lim_{n\to\infty} \frac{||z_n - u_n||}{r_n} = 0$. Since $u_n = T_{r_n} z_n$ in equation (4.9) and by Lemma 3.5, we have that

$$F(Ju_n, Jz) + \frac{1}{r_n}\Big\langle u_n - z_n, Jz - Ju_n \Big\rangle \geq 0, \quad \forall\ z \in K. \tag{3.15}$$

By B_2, we have that $\frac{1}{r_n}\Big\langle u_n - z_n, Jz - Ju_n\Big\rangle \geq F(Jz, Ju_n)$. Since $z \mapsto F(Ju, Jz)$ is convex and lower semi-continuous, we obtain from the above inequality that $0 \geq F(Jz, Jx^*), \forall\ z \in K$. For $\lambda \in (0, 1]$ and $z \in K$, letting $z^*_\lambda = \lambda Jz + (1-\lambda)Jx^*$, then, $z^*_\lambda \in JK$ since JK is closed and convex. Hence, $0 \geq F(z^*_\lambda, Jx^*)$. By B_1, we have that

$$0 = F(z^*_\lambda, z^*_\lambda) \ \leq \ \lambda F(z^*_\lambda, Jz) + (1-\lambda)F(z^*_\lambda, Jx^*) \leq F(Jx^* + \lambda(Jz - Jx^*), Jz).$$

Letting $\lambda \downarrow 0$, by B_3, we obtain that $F(Jx^*, Jz) \geq 0$. Hence, $x^* \in GMEP(\Theta, B, \chi)$. Using this and the fact that $x^* \in F_J(\Upsilon)$, we conclude $x^* \in F_J(\Gamma) \cap GMEP(\Theta, B, \chi)$.

Step 5: $\lim_{n\to\infty} x_n = R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)} x_1$. From Lemma 2.22, we obtain that

$$\phi(x, R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)} x_1) \leq \phi(x_1, x^*). \tag{3.16}$$

Also, for $x^* \in F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) \subset K_{n+1}$, $x_{n+1} = R_{K_{n+1}} x_1$, and by Lemma 2.22, we have that $\phi(x_1, x_{n+1}) \leq \phi(x_1, R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)} x_1)$. Since $\lim_{n\to\infty} x_n = x^*$, and by inequality 3.16, we get that $\phi(x_1, R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)} x_1) \leq \phi(x_1, x^*) \leq \phi(x_1, R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)} x_1)$. By uniqueness of $R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)} x_1$, we conclude that $x^* = R_{F_J(\Upsilon)\cap GMEP(\Theta,B,\chi)} x_1$.
This proof is complete. □

Corollary 3.7. Let H be a real Hilbert space. Let K be a nonempty closed and convex subset of H. Let $\chi : K \to \mathbb{R}$ be a lower semi-continuous and convex function. *Let* $B : K \to H$ be a continuous and monotone map. Let $\Theta : K \times K \to \mathbb{R}$ be a bifunction satisfying conditions $(B_1) - (B_4)$. Let $S_n : K \to H,\ n = 1, 2, \cdots$ be a countable family of generalized nonexpansive maps and Υ be a family of closed and generalized nonexpansive maps from K to H such that $\cap_{n=1}^{\infty} F(S_n) \cap GMEP(\Theta, B, \chi) = F(\Upsilon) \cap GMEP(\Theta, B, \chi) \neq \emptyset$, $\beta_n \in (0,1)$ with $\lim_{n\to\infty} \beta_n = 0$ and $\{r_n\} \subset [a, \infty)$, for some $a > 0$. Let $\{x_n\}$ be a sequence generated iteratively by

$$\begin{cases} x_1 \in K,\ \ K_1 = K, \\ z_n = \beta_n x + (1-\beta_n) S_n x_n, \\ u_n = T_{r_n} z_n,\ \ x_{n+1} = P_{K_{n+1}} x_1,\ \ \forall\, n \geq 1, \\ K_{n+1} = \big\{ v \in K_n : ||u_n - v||^2 \leq \beta_n ||x_1 - v||^2 + (1-\beta_n)||x_n - v||^2 \big\}. \end{cases} \tag{3.17}$$

Assume that $\{S_n\}$ satisfies the NST-condition with Υ. Then, $\{x_n\}$ converges strongly to $P_{F(\Upsilon)\cap GMEP(\Theta,B,\chi)} x_1$.

Proof. In a real Hilbert space, J is the identity map and $\phi(y,z) = ||y-z||^2,\ \ \forall\, y, z \in H$. The result follows from Theorem 3.6. □

Example 3.8. Let $X = l_p,\ 1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$, and $K = \overline{B_{l_p}}(0,1) = \{u \in l_p : ||u||_{l_p} \leq 1\}$. Consider the following maps:

$\chi : JK \to \mathbb{R}$ defined by $\chi(u^*) = ||u^*||,\ \forall\, u^* \in JK$;

$\Theta : JK \times JK \to \mathbb{R}$ defined by $\Theta(u^*, v^*) = \langle J^{-1}u^*, v^* - u^* \rangle,\ \ \forall\, v^* \in JK$;

$B : K \to l_q$ defined by $Bu = J(u_1, u_2, u_3, \cdots),\ \forall\, u = (u_1, u_2, u_3, \cdots) \in K$;

$S : K \to l_q$ defined by $Su = J(0, u_1, u_2, u_3, \cdots),\ \forall\, u = (u_1, u_2, u_3, \cdots) \in K$;

$S_n : K \to l_q$ defined by $S_n u = J(\alpha_n u + (1-\alpha_n) J^{-1} o S u),\ \forall\, n \geq 1,\ u \in K,\ \ \alpha_n \in (0,1)$.

Let $\{\beta_n\} := \{\frac{1}{n+1}\},\ \forall\, n \geq 1,\ \{r_n\} \subset [1, \infty),\ \forall\, n \geq 1$ and $\Upsilon = S$. Then, by Theorem 3.6, the sequence $\{x_n\}$ generated by algorithm (3.11) converges strongly to an element of $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi)$.

Proof. (a) We show that $JK = \overline{B_{l_q}}(0,1)$.

Let $u^* \in JK$. Then, there exists $u \in K$ such that $u^* = Ju$. Clearly, $u^* \in l_q$. Then, $||u^*|| = ||u|| \leq 1$. Hence, $JK \subseteq \overline{B_{l_q}}(0,1)$. Conversely, let $u \in \overline{B_{l_q}}(0,1)$. This implies that $J^{-1}u \in l_p$ and $||J^{-1}u|| = ||u|| \leq 1$. Hence, $J^{-1}u \in K$, which implies that $u \in JK$. Thus, $\overline{B_{l_q}}(0,1) \subseteq JK$. Hence, $JK = \overline{B_{l_q}}(0,1)$.

(b) We show that $\chi : JK \to \mathbb{R}$ defined by $\chi(u^*) = ||u^*||,\ \forall\, u^* \in JK$, is lower semi-continuous and convex. Obviously, χ is lower semi-continuous and convex.

(c) We show that $\Theta : JK \times JK \to \mathbb{R}$ defined by $\Theta(u^*, v^*) = \langle J^{-1}u^*, v^* - u^* \rangle,\ \ \forall\, v^* \in JK$ satisfies conditions B_1 to B_4 in the following sense.

(B_1) $\Theta(u^*, u^*) = \langle J^{-1}u^*, u^* - u^* \rangle = 0,\ \forall\, u^* \in JK$.

(B_2) $\Theta(u^*, v^*) + \Theta(v^*, u^*) = \langle J^{-1}u^* - J^{-1}v^*, v^* - u^* \rangle \leq 0,\ \forall\, u^*, v^* \in JK$.

$$\begin{aligned} (B_3)\ \ \limsup_{\lambda \downarrow 0} \Theta(u^* + \lambda(z^* - u^*), v^*) &= \limsup_{\lambda \downarrow 0} \langle J^{-1}(u^* + \lambda(z^* - u^*)), v^* - u^* + \lambda(u^* - z^*) \rangle \\ &\leq \langle J^{-1}u^*, v^* - u^* \rangle = \Theta(u^*, v^*),\ \forall\, u^*, v^*, z^* \in JK. \end{aligned}$$

(B_4) Let $x^*, v^* \in JK$ and $\alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$. Then,

$$\begin{aligned}\Theta(u^*, \alpha x^* + \beta v^*) &= \langle J^{-1}u^*, \alpha x^* + \beta v^* - u^* \rangle = \alpha \langle J^{-1}u^*, x^* - u^* \rangle + \beta \langle J^{-1}u^*, v^* - u^* \rangle \\ &= \alpha \Theta(u^*, x^*) + \beta \Theta(u^*, v^*), \quad \forall\, u^* \in JK.\end{aligned}$$

Let $v_n^* \rightharpoonup v^*$ as $n \to \infty$. Then,
$\liminf \Theta(u^*, v_n^*) = \liminf \langle J^{-1}u^*, v_n^* - u^* \rangle \geq \langle J^{-1}u^*, v^* - u^* \rangle = \Theta(u^*, v^*), \ \forall\, u^* \in JK$. Hence, Θ is convex and lower semi-continuous.

(d) We show that $B : K \to l_q$ defined by $Bu = J(u_1, u_2, u_3, \cdots), \ \forall\, u = (u_1, u_2, u_3, \cdots) \in K$ is continuous and monotone. Clearly, B is continuous and monotone since J is continuous and monotone. Observe that $0 \in GMEP(\Theta, B, \chi)$.

(e) Let $S : K \to l_q$ be a map defined by $Su = J(0, u_1, u_2, u_3, \cdots), \ \forall\, u = (u_1, u_2, u_3, \cdots) \in K$; $S_n : K \to l_q$ be a map defined by $S_n u = J(\alpha_n u + (1-\alpha_n)J^{-1}oSu), \ \forall\, n \geq 1, \ u \in K, \alpha_n \in (0,1)$. Clearly, $Su = Ju$ if and only if $u = \bar{0}$, and $S_n u = Ju$ if and only if $Su = Ju$. Hence, $F_J(S) = F_J(\Upsilon) = F_J(S_n) = \{0\}, \ \ \forall\, n \geq 1$. Hence, $F_J(\Upsilon) \cap GMEP(\Theta, B, \chi) = \{0\}$.

Next, we show that $\{S_n\}$ is generalized-J-nonexpansive, for each n, and satisfies the NST-condition with Υ. The proof follows from Lemma 3.4. □

3.0.3 Discussion

1. Theorem 3.6 is a complementary analogue of a result of Klin-earn [102] in the sense that, in the result of Klin-earn [102], the family $\{T_n\}$ maps from a subset $C \subset X$ to the space X while in Theorem 3.6, the family $\{S_n\}$ maps from a subset $K \subset X$ to the dual X^*. Furthermore, in Hilbert spaces, both theorems virtually agree and yield the same conclusion. Finally, in Theorem 3.6, generalized mixed equilibrium problem is also studied which is not the case in the result of Klin-earn [102].

2. Theorem 3.6 extends and improves the theorem of Martinez-Yanes and Xu [115], Nakajo and Takahashi [121], in the sense that these theorems are special cases of Theorem 3.6 in which X is a real Hilbert space. Furthermore, in the theorem of Martinez-Yanes and Xu [115], T is a single self-map on $C \subset X$ while in Theorem 3.6, $\{S_n\}$ is a family of maps from a subset $C \subset X$ *to the dual space* X^*. Finally, in Theorem 3.6, generalized mixed equilibrium problem is also studied which is not the case in the theorem of Martinez-Yanes and Xu [115], also Nakajo and Takahashi [121].

3. In Corollary 3.7, the set of generalized mixed equilibrium problem is studied which is not considered in Pen and Yao [131], Qin and Su [134], Tada and Takahashi [152].

4. Corollary 3.7 extends the result in Pen and Yao [131], Qin and Su [134], Tada and Takahashi [152] from a nonexpansive self-map to a countable family of generalized nonexpansive non self-maps.

5. The iteration process of Corollary 3.7 is more efficient than that considered in Pen and Yao [131] which requires more arithmetic at each stage to implement because of the extra y_n and z_n terms involved in the iteration process.

6. Finally, the sequence of *Halpern-type* algorithm considered in Theorem 3.7 requires less computation time at each step of the iteration process than the sequence of *Mann-type* algorithm studied in Pen and Yao [131], Qin and Su [134], Tada and Takahashi [152], thereby reducing computational cost.

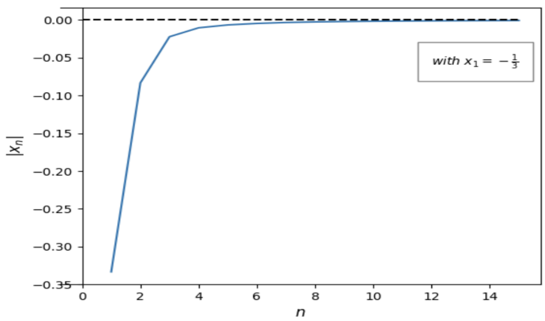
3.0.4 Numerical experiment

Here, we present a numerical example to illustrate the convergence of the sequence $\{x_n\}$ in Theorem 3.6.

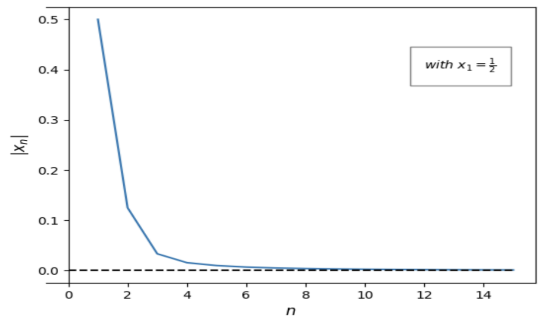
Example 3.9. Let $X = \mathbb{R}$, $K = [\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$. Clearly, $x \in \mathbb{R}$,

$$R_K x := \begin{cases} \alpha, & if \quad x < \alpha, \\ x, & if \quad x \in [\alpha, \beta], \\ \beta, & if \quad x > \beta. \end{cases} \tag{3.18}$$

Now, set $K = [-1, 3]$ and $Sx = sin(x)$ in Theorem 3.6. Clearly, S is generalized-J-nonexpansive with 0 as its unique fixed point. With $x_1 = \frac{-1}{3}$ and $x_1 = \frac{1}{2}$ in K respectively, by Theorem 3.6, the sequence generated by algorithm (3.10) converges strongly to zero. The numerical result is sketched in the figures below with initial points $x_1 = \frac{-1}{3}$ and $x_1 = \frac{1}{2}$, respectively, where the y-$axis$ represents the value of $|x_n - 0|$ while the x-$axis$ represents the number of iterations (n).



Numerical example with $x_1 = \frac{-1}{3}$



Numerical example with $x_1 = \frac{1}{2}$

All computations and graphs were implemented in python 3.6 using some abstractions developed at $AUST$ and other open source python library such as numpy and matplotlib on Zinox with intel core $i7$ processor.

The result of this chapter is published in Carpathian J. Math. 34(2018), No.2, 191 - 198.

CHAPTER 4

Inertial algorithm for a countable family of generalized J-nonexpansive maps

Introduction

In this chapter, we present a strong convergence theorem for an inertial algorithm for a countable family of generalized nonexpansive maps. This theorem presented is applied to prove a strong convergence theorem for a countable family of generalized-J-nonexpansive maps. The theorem presented is an improvement in the results of Klin earn *et al.*, [102]. Chidume *et al.*, [55], and Dong *et al.*, [75]. Finally, we give a numerical experiment to illustrate the efficiency and advantage of the *inertial algorithm* over algorithm *without inertial term*.

4.0.1 Main result

Theorem 4.1. *Let X be a uniformly smooth and strictly convex real Banach space with Kadec-Klee-property and dual space X^*. Let K be a nonempty closed and convex subset of X such that JK is closed and convex. Let $T_k : X \to X,\ k = 1, 2, \cdots$ be a countable family of generalized-nonexpansive maps and Γ be a family of closed and generalized-nonexpansive maps from X to X such that $\cap_{k=1}^{\infty} F(T_k) = F(\Gamma) \neq \emptyset$. Let $\{v_k\}$ be generated by:*

$$\begin{cases} v_0, v_1 \in X,\ K_1 = X, \\ w_k = v_k + \beta_k(v_k - v_{k-1}), \\ y_k = \alpha w_k + (1-\alpha)T_k w_k,\ \ v_{k+1} = R_{K_{k+1}} v_0, \quad \forall\ \ k \geq 1, \\ K_{k+1} = \big\{ v \in K_k : \phi(y_k, v) \leq \phi(w_k, v) \big\}. \end{cases} \tag{4.1}$$

Assume that $\{T_k\}$ satisfies the NST-condition with Γ and $\beta_k,\ \alpha \in (0,1)$. Then, $\{v_k\}$ converges strongly to $R_{F(\Gamma)}v_0$, where $R_{F(\Gamma)}$ is the sunny generalized-nonexpansive retraction of X onto $F(\Gamma)$.

Proof. We divide our proof into five steps. Here, the map ϕ denotes the Alber's functional.

Step 1: The sequence $\{v_k\}$ is well defined and $F(\Gamma) \subset K_k,\ \forall\ k \geq 1$.
First, we show that JK_k is closed and convex. Clearly, $JK_1 = JX$ is closed and convex. Assume that JK_k is closed and convex, for some $k \geq 1$. Utilizing the definition of K_{k+1}, it

is clear that $K_{k+1} = \{2\langle w_k - y_k, Jv\rangle \leq ||w_k||^2 - ||y_k||^2\}$. Thus, JK_{k+1} is closed and convex. Hence, we conclude that JK_k is closed and convex. Furthermore, by Lemma 2.23, K_k is a sunny generalized-nonexpansive retract of X. Hence, $\{v_k\}$ is well defined.

Next, we prove that $F(\Gamma) \subset K_k$. Clearly, $F(\Gamma) \subset K_1$. Assume that $F(\Gamma) \subset K_k$, for some $k \geq 1$. Let $p \in F(\Gamma)$. Then, we have that

$$\begin{aligned} \phi(y_k, p) &= ||\alpha w_k + (1-\alpha)T_k w_k||^2 - 2\langle \alpha w_k + (1-\alpha)T_k w_k, Jp\rangle + ||p||^2 \\ &\leq \alpha\phi(w_k, p) + (1-\alpha)\phi(T_k w_k, p) \\ &\leq \alpha\phi(w_k, p) + (1-\alpha)\phi(w_k, p) = \phi(w_k, p), \end{aligned} \tag{4.2}$$

which implies that $p \in K_{k+1}$. Hence, $F(\Gamma) \subset K_k$.

Step 2: $\lim_{k\to\infty} \phi(v_k, v_0)$ exists and the sequence $\{v_k\}$ is convergent.

First, we prove that $\{v_k\}$ is bounded. From the definition of $\{v_k\}$ and Lemma 2.22, we have that $\phi(v_k, v_0) \leq \phi(p, v_0)$, $\forall\ p \in F(\Gamma) \subset K_k$. This implies that $\{\phi(v_k, v_0)\}$ is bounded. Furthermore, $\{v_k\}$ is bounded. Since $v_{k+1} \in K_k$ and $v_k = R_{K_k} v_0$, we have that $\phi(v_k, v_0) \leq \phi(v_{k+1}, v_0)$, and this implies that $\{\phi(v_k, v_0)\}$ is nondecreasing. Hence, $\lim_{n\to\infty} \phi(v_k, v_0)$ exists.

Since X is reflexive and $\{v_k\}$ is bounded; there exists a subsequence $\{v_{k_j}\}$ of $\{v_k\}$ such that $v_{k_j} \rightharpoonup x^* \in K_{K_j}$. In view of $v_{k_j} = R_{K_j} v_0$ and Lemma 2.22, we get that $\phi(v_{k_j}, v_0) \leq \phi(p, v_0)$, $\forall\ j \geq 1$. Applying the weak lower semi-continuity of norm $||\cdot||$, we obtain that

$$\phi(x^*, v_0) \leq \liminf_{j\to\infty} \phi(v_{k_j}, v_0) \leq \limsup_{j\to\infty} \phi(v_{k_j}, v_0) \leq \phi(x^*, v_0), \tag{4.3}$$

which implies that $\lim_{j\to\infty} \phi(v_{k_j}, v_0) = \phi(x^*, v_0)$. Furthermore, $\lim_{j\to\infty} ||v_{k_j}|| = ||x^*||$. By Lemma 2.26, we obtain that $\lim_{j\to\infty} v_{k_j} = x^*$. Since $\lim_{k\to\infty} \phi(v_k, v_0)$ exists and $\lim_{j\to\infty} \phi(v_{k_j}, v_0) = \phi(x^*, v_0)$, then, $\lim_{k\to\infty} \phi(v_k, v_0) = \phi(x^*, v_0)$.

Next we show that $\lim_{k\to\infty} v_k = x^*$. Suppose for contraction that there exists a subsequence $\{v_{k_i}\}$ of $\{v_k\}$ such that $\lim_{i\to\infty} v_{k_i} = x$ with $x^* \neq x$, then, by Lemma 2.22, we have that

$$\begin{aligned} \phi(x^*, x) &= \lim_{i,j\to\infty} \phi(v_{k_j}, v_{k_i}) = \lim_{i,j\to\infty} \phi(v_{k_j}, R_{K_{k_i}} v_0) \\ &\leq \lim_{i,j\to\infty} \Big(\phi(v_{k_j}, v_0) - \phi(v_{k_i}, v_0)\Big) \\ &= \phi(x^*, v_0) - \phi(x^*, v_0) = 0. \end{aligned}$$

This implies that $x^* = x$, which is a contradiction. Hence, $\lim_{k\to\infty} v_k = x^*$.

Step 3: $\lim_{k\to\infty} ||w_k - v_k|| = \lim_{k\to\infty} ||v_k - y_k|| = \lim_{k\to\infty} ||w_k - y_k|| = 0$.

Using the definition of w_k in equation (4.1) and convergence of $\{v_k\}$ in step **2**, we obtain that $\lim_{k\to\infty} ||w_k - v_k|| = 0$. By Lemma 2.24 and Remark 2.17, we obtain that $\lim_{k\to\infty} \phi(w_k, v_k) = 0$. Since $v_k \in K_k$, and by inequality (4.2), we have that $\phi(y_k, v_k) \leq \phi(w_k, v_k)$, and this implies that $\lim_{k\to\infty} \phi(y_k, v_k) = 0$. Hence, by Lemma 2.24, we obtain that $\lim_{k\to\infty} ||y_k - v_k|| = 0$. By triangle inequality, we obtain that $||y_k - w_k|| \leq ||y_k - v_k|| + ||v_k - w_k|| \to 0$ as $k \to \infty$. This completes the proof of Step **3**.

Step 4: $\lim_{k\to\infty} ||w_k - Tw_k|| = 0$ and $x^* \in F(T)$.

From equation (4.1) and step **3**, we obtain that

$$\begin{aligned}||v_k - y_k|| &= ||v_k - \alpha w_k - (1-\alpha)T_k w_k|| \\ &= ||(1-\alpha)(v_k - T_k w_k) - \alpha(w_k - v_k)|| \\ &\geq (1-\alpha)||v_k - T_k w_k|| - \alpha||w_k - v_k||,\end{aligned}$$

which implies that

$$||v_k - T_k w_k|| \leq \frac{1}{1-\alpha}\Big(||v_k - y_k|| + \alpha||w_k - v_k||\Big) \to 0 \text{ as } k \to \infty. \tag{4.4}$$

From inequality (4.4) and step **3**, we obtain that

$$||w_k - T_k w_k|| \leq ||w_k - v_k|| + ||v_k - T_k w_k|| \to 0 \text{ as } k \to \infty. \tag{4.5}$$

Since T_k satisfies the NST-condition with Γ, we obtain from inequality (4.5) that

$$\lim_{k\to\infty} ||w_k - T w_k|| = 0, \quad \forall\, T \in \Gamma.$$

Furthermore, since $\lim\limits_{k\to\infty} w_k = x^*$ in step **3** and T is closed by assumption, then, $x^* \in F(T)$.

Step 5: $\lim\limits_{k\to\infty} v_k = R_{F(\Gamma)}v_0$.

From Lemma 2.22, we obtain that

$$\phi(R_{F(\Gamma)}v_0, v_0) \leq \phi(x^*, v_0). \tag{4.6}$$

For $x^* \in F(\Gamma) \subset K_k$, $v_k = R_{K_k}v_0$ and by Lemma 2.22, we have that

$\phi(v_k, v_0) \leq \phi(R_{F(\Gamma)}v_0, v_0)$. Taking limit of both sides of this inequality, we obtain that

$$\phi(x^*, v_0) \leq \phi(R_{F(\Gamma)}v_0, v_0). \tag{4.7}$$

Combining inequality (4.6) and (4.7), we obtain that $\phi(x^*, v_0) = \phi(R_{F(\Gamma)v_0}, v_0)$.
By uniqueness of $R_{F(\Gamma)}v_0$, we conclude that $x^* = R_{F(\Gamma)}v_0$. □

Example 4.2. Let $X = l_p,\ 1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$ and $K = \overline{B_{l_p}}(0,1)$.
Let $T : l_p \to l_p$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, \cdots)$. Let $T_k : l_p \to l_p$ be defined by $T_k x = \alpha_k x + (1-\alpha_k)Tx,\ \forall\, k \geq 1,\ x \in l_p$ and $\alpha_k \in (0,1)$. Let $\Gamma = T$.
Clearly, $X,\ K,\ T_k$ and Γ satisfy all the conditions of Theorem 3.6. Hence, the sequence $\{v_k\}$ generated by equation (4.1) converges to $\bar{0}$, the unique element of $\cap_{k=1}^{\infty}(T_k) = F(\Gamma)$.

Corollary 4.3. Let K be a nonempty closed and convex subset of L_p, l_p *or* $W_p^m(\Omega)$ spaces, $1 < p < \infty$, such that JK is closed and convex. Let $T_k : X \to X,\ k = 1, 2, \cdots$ be a countable family of generalized nonexpansive maps and Γ be a family of closed and generalized nonexpansive maps from X to X such that $\cap_{k=1}^{\infty} F(T_k) = F(\Gamma) \neq \emptyset$. Let $\{v_k\}$ be generated by:

$$\begin{cases} v_0, v_1 = v \in X,\ K_1 = X, \\ w_k = v_k + \beta_k(v_k - v_{k-1}),\ y_k = \alpha w_k + (1-\alpha)T_k w_k, \\ K_{k+1} = \big\{v \in K_k : \phi(y_k, v) \leq \phi(w_k, v)\big\}, \\ v_{k+1} = R_{K_{k+1}}v_0, \forall\, k \geq 1. \end{cases} \tag{4.8}$$

Assume that $\{T_k\}$ satisfies the NST-condition with Γ and $\beta_k,\ \alpha \in (0,1)$. Then, $\{v_k\}$ converges strongly to $R_{F(\Gamma)}v_0$, where $R_{F(\Gamma)}$ is the sunny generalized-nonexpansive retraction of X onto $F(\Gamma)$.

Proof. Since these spaces are uniformly smooth and strictly convex, the result follows from Theorem 4.1. □

4.0.2 Applications

Consider a countable family of maps from a space X to its dual space X^*. In this case, the usual notion of fixed points for maps from the space X into itself, obviously does not make sense. However, a new notion of fixed points called *J-fixed points* has been defined for maps from a normed space X to its dual X^*, (see e.g., Zegeye [166], and Chidume and Idu, [52]).

Theorem 4.4. *Let X be a uniformly smooth and strictly convex real Banach space with Kadec-Klee-property and dual space X^*. Let K be a nonempty closed and convex subset of X such that JK is closed and convex. Let $S_k : X \to X^*$, $k = 1, 2, \cdots$ be a countable family of generalized-J-nonexpansive maps and Γ be a family of closed and generalized-J-nonexpansive maps from X to X^* such that $\cap_{k=1}^{\infty} F_J(S_k) = F_J(\Gamma) \neq \emptyset$. Let $\{v_k\}$ be generated by:*

$$\begin{cases} v_0, v_1 \in X, \ K_1 = X, \\ w_k = v_k + \beta_k(v_k - v_{k-1}), \\ y_n = \alpha w_k + (1-\alpha)(J_* o S_k) w_k, \ \ v_{k+1} = R_{K_{k+1}} v_0, \ \forall \ k \geq 1, \\ K_{k+1} = \big\{ v \in K_k : \phi(y_k, v) \leq \phi(w_k, v) \big\}. \end{cases} \tag{4.9}$$

Assume that $\{S_k\}$ satisfies the NST-condition with Γ, and β_k, $\alpha \in (0,1)$. Then, $\{v_k\}$ converges strongly to $R_{F_J(\Gamma)} v_0$, where $R_{F_J(\Gamma)}$ is the sunny generalized-J-nonexpansive retraction of X onto $F_J(\Gamma)$.

Proof. Define $T_k := J_* o S_k$. Then, $T_k : X \to X$, $k = 1, 2, \cdots$. Furthermore, T_k is a generalised nonexpansive map and $\cap_{k=1}^{\infty} F(T_k) = \cap_{k=1}^{\infty} F_J(S_k) = F(\Gamma)$. Hence, by Theorem 3.6, $\{v_k\}$ converges strongly to some $x^* \in R_{F_J(\Gamma)}$. □

4.0.3 Discussion

1. Theorem 4.1 extends the result of Klin-earn *et al.* [102] from a uniformly smooth and *uniformly convex* real Banach space to a uniformly smooth and *strictly convex* real Banach space. Furthermore, an *inertial term* is incorporated in the algorithm of Theorem 4.1, whereas the algorithm of Klin-earn *et al.* [102] does not involve this term. Moreover, the computation at each iteration process of two subsets C_n *and* Q_n *of* C*, their intersection* $C_n \cap Q_n$ and the retraction of the initial vector onto the intersection which is required in the theorem of Klin-earn *et al.* [102] has been dispensed with and replaced with a single retraction onto the subset K_{k+1} *of* X.

2. In the theorem of Dong *et al.* [75], the authors proved a strong convergence theorem in a real *Hilbert space* for a *nonexpansive map.* Our Theorem 4.1 extends this result to a *uniformly smooth and strictly convex* real Banach space and to a *countable family of generalized nonexpansive maps* that satisfy the *NST-condition.* We observe that the *NST-condition* imposed in our Theorem 4.1 is trivially satisfied for a single operator T as in the theorem of Dong *et al.* [75]. Furthermore, the control parameter in our algorithm is one arbitrarily fixed constant $\alpha \in (0,1)$ which is to be computed once and then used at each step of the iteration process, whereas the parameter in the algorithm studied by Dong *et al.* [75] is $\beta_n \in (0,1)$ which is to be computed at each step of the iteration

process. Consequently, the sequence of equation (4.1) is of *Krasnoselskii-type* and the sequence defined by equations (1.17) and (1.18) are of *Mann-type*.

3. Theorem 4.4 extends a theorem of Chidume *et al.* [55] from a uniformly smooth and *uniformly convex* real Banach space to a uniformly smooth and *strictly convex* real Banach space. Furthermore, an *inertial term* is incorporated in the algorithm of Theorem 4.4, whereas the algorithm of the theorem of Chidume *et al.* [55] does not involve this term. Moreover, the computation at each iteration process of two subsets C_n *and* Q_n *of* C, *their intersection* $C_n \cap Q_n$ and the retraction of the initial vector onto the intersection in the theorem of Chidume *et al.* [55] has been dispensed with and replaced with a single retraction onto the subset K_{k+1} *of* X. In addition, the condition that T be J_*-closed in the theorem of Chidume *et al.* [55] has also been dispensed with in our Theorem 4.4.

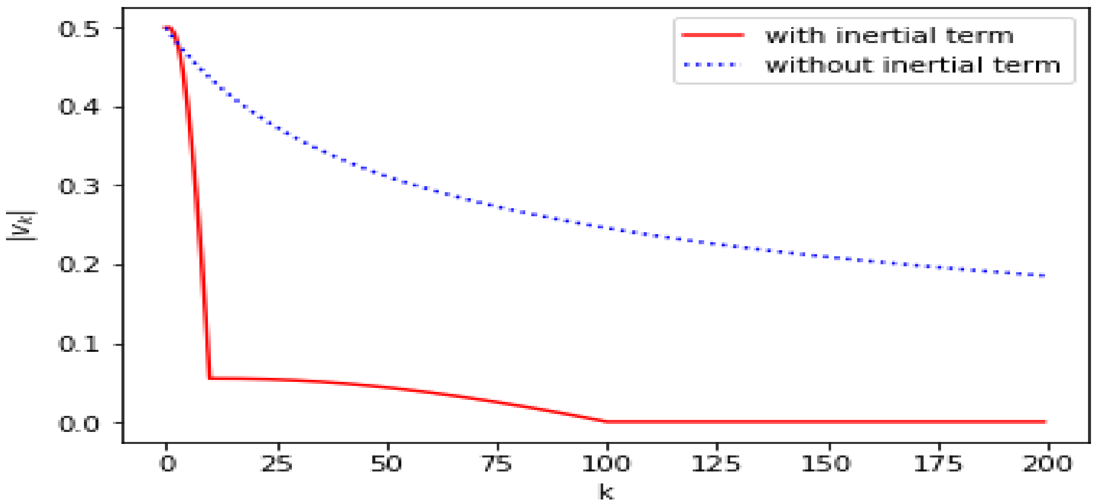
4.0.4 Numerical experiment

Here, we present a numerical example to compare the speed of convergence of inertial algorithm and the algorithm without inertial term.

Example 4.5. Let $E = \mathbb{R}$, $K = [\gamma, \eta]$, $\gamma, \eta \in \mathbb{R}$.

$$R_K v = \begin{cases} \gamma, & if \quad v < \gamma, \\ v, & if \quad v \in K, \\ \eta, & if \quad v > \eta. \end{cases} \tag{4.10}$$

Now, set $Tv = \sin v$, $K = [-1, 1]$ in Theorem 4.1. Clearly, S is generalized-J-nonexpansive with 0 as its unique fixed point. Set $\beta_k = \frac{k}{k+\zeta-1}$, $\zeta = 5$, $v_0 = v_1 = \frac{1}{2}$. Then, by Theorem 4.1, the sequence generated by algorithm (4.1) converges to zero. The numerical results are sketched in the figure below, where the *y-axis* represents the value of $|v_k - 0|$ while the *x-axis* represents the number of iterations (k).



All computations and graph were done using spyder 3.2.6 on Hp Intel CORE DUO 2gb Ram. We observe from the figure above that the algorithm with inertial term converges much faster than algorithm without inertial term.

The result of this chapter is published in Fixed Point Theory 21(2) (2020), 441 - 452.

CHAPTER 5

Convergence theorem of subgradient extragradient algorithm for solving variational inequalities and a convex feasibility problem

Introduction

In this chapter, we present a *Krasnoselskii-type subgradient extragradient algorithm* and prove a *weak convergence* theorem for approximating a common solution of variational inequality problems and common fixed points for a countable family of relatively-nonexpansive maps in a uniformly smooth and 2-uniformly convex real Banach space. The theorem presented is an improvement of the result of Censor *et al.* [40].

5.0.1 Main result

A Krasnoselskii-type Subgradient Extragradient algorithm.

Let $\{v_k\}_{k=1}^{\infty}$ be a sequence generated iteratively by

$$\begin{cases} v_1 \in X \text{ and } \tau > 0,\ \beta \in (0,1), \\ y_k = \Pi_C J^{-1}(Jv_k - \tau f(v_k)), \\ T_k = \big\{ w \in X : \big\langle w - y_k, (Jv_k - \tau f(v_k)) - Jy_k \big\rangle \le 0 \big\}, \\ v_{k+1} = J^{-1}\Big(\beta J v_k + (1-\beta) J S \Pi_{T_k} J^{-1}(Jv_k - \tau f(y_k)) \Big),\ \forall\ k \ge 1. \end{cases} \tag{5.1}$$

We shall make the following assumption.

C_1. The map f is monotone on X,

C_2. The map f is Lipschitz on X, with constant $K > 0$,

C_3. $\mathcal{G} := VI(f, C) \cap F(S) \neq \emptyset$, $F(S)$ is the set of fixed points of S.

C_4. $\mathcal{V} := \cap_{i=1}^{\infty} F(T_i) \cap VI(f, C) \neq \emptyset$, where $F(T_i) := \{x \in X : T_i\ x = x,\ \forall\ i \ge 1\}$.

Lemma 5.1. Let X be a uniformly smooth and 2-uniformly convex real Banach space with dual space X^*. Let C be a nonempty closed and convex subset of X. Let $S : X \to X$ be a relatively nonexpansive map and $f : X \to X^*$ be a map satisfying conditions C_1 and C_2 with

$\tau \in (0, \frac{\alpha}{K})$. Assume condition C_3 holds and J is weakly sequentially continuous on X. Then, the sequence $\{v_k\}_{k=1}^{\infty}$ generated by Algorithm **1** converges weakly to some $v^* \in \mathcal{G}$.

Proof. Denote $t_k = \Pi_{T_k} J^{-1}(Jv_k - \tau f(y_k)), \ \ \forall\, k \geq 1$, $Jz_k := Jv_k - \tau f(y_k)$ and $\gamma = 1 - \frac{\tau K}{\alpha}$. Since $\mathcal{G} \neq \emptyset$, let $u \in \mathcal{G}$. Then, we have that

$$\begin{aligned}
\phi(u, t_k) &\leq \phi(u, z_k) - \phi(t_k, z_k) \\
&= ||u||^2 - 2\langle u, Jv_k - \tau f(y_k)\rangle - ||t_k||^2 + 2\langle t_k, Jv_k - \tau f(y_k)\rangle \\
&= \phi(u, v_k) - \phi(t_k, v_k) + 2\tau\langle u - t_k, f(y_k)\rangle \\
&= \phi(u, v_k) - \phi(t_k, v_k) + 2\tau\langle u - y_k, f(y_k) - f(u)\rangle + 2\tau\langle y_k - t_k, f(y_k)\rangle \\
&\quad +2\tau\langle u - y_k, f(u)\rangle.
\end{aligned}$$

By C_1, $\langle u - y_k, f(y_k) - f(u)\rangle \leq 0, \ \ \forall\, k \geq 1$. Consequently, $\langle u - y_k, f(u)\rangle \leq 0, \ \ \forall\, k \geq 1$. Thus, from the last line of the above inequality, and the fact that $t_k \in T_k$, we obtain that

$$\begin{aligned}
\phi(u, t_k) &\leq \phi(u, v_k) - \phi(t_k, v_k) + 2\tau\langle y_k - t_k, f(y_k)\rangle \\
&= \phi(u, v_k) - \phi(y_k, v_k) - \phi(t_k, y_k) + 2\big\langle t_k - y_k, Jv_k - \tau f(y_k) - Jy_k\big\rangle \\
&\leq \phi(u, v_k) - \phi(y_k, v_k) - \phi(t_k, y_k) + 2\tau\big\langle t_k - y_k, f(v_k) - f(y_k)\big\rangle. \qquad (5.2)
\end{aligned}$$

By condition C_2 and Lemma 2.20, we have that

$$\begin{aligned}
\phi(u, t_k) &\leq \phi(u, v_k) - \phi(y_k, v_k) - \phi(t_k, y_k) + \frac{\tau K}{\alpha}\Big(\phi(t_k, y_k) + \phi(y_k, v_k)\Big) \\
&= \phi(u, v_k) - \gamma\phi(t_k, y_k) - \gamma\phi(y_k, v_k) \leq \phi(u, v_k). \qquad (5.3)
\end{aligned}$$

Applying Lemma 2.18, inequality (5.3) and relatively nonexpansivity of S, we obtain that

$$\begin{aligned}
\phi(u, v_{k+1}) &= \phi\big(u, J^{-1}\big(\beta Jv_k + (1-\beta)J(St_k)\big)\big) \\
&\leq \beta\phi\big(u, v_k) + (1-\beta)\phi(u, t_k)\big) - \beta(1-\beta)g\big(||Jv_k - J(St_k)||\big) \qquad (5.4) \\
&\leq \beta\phi\big(u, v_k) + (1-\beta)\Big(\phi(u, v_k) - \gamma\phi(t_k, y_k) - \gamma\phi(y_k, v_k)\Big) \leq \phi(u, v_k). \quad (5.5)
\end{aligned}$$

This implies that $\lim\limits_{k\to\infty} \phi\big(u, v_k\big)$ exists. Consequently, $\{v_k\}_{k=1}^{\infty}$ is bounded. From inequality (5.3), $\{t_k\}_{k=1}^{\infty}$ is bounded. Also, from inequality (5.4), we obtain that

$$\phi(y_k, v_k) \leq \frac{1}{\gamma(1-\beta)}\Big(\phi(u, v_k) - \phi(u, v_{k+1})\Big) \ \text{ and } \ \phi(t_k, y_k) \leq \frac{1}{\gamma(1-\beta)}\Big(\phi(u, v_k) - \phi(u, v_{k+1})\Big).$$

From these inequalities, we obtain that

$$\lim_{k\to\infty} \phi(y_k, v_k) = 0 \ \text{ and } \ \lim_{k\to\infty} \phi(t_k, y_k) = 0. \qquad (5.6)$$

By Lemma 2.16, it follows that $\lim ||y_k - v_k|| = 0$ and $\lim ||t_k - y_k|| = 0$. Consequently, we obtain $\lim\limits_{k\to\infty} ||v_k - t_k|| = 0$.

Next, we show that $\Omega_\omega(v_k) \subset \mathcal{G} = F(S) \cap VI(f, C)$, where $\Omega_\omega(v_k)$ is the set of weak subsequential limit of $\{v_k\}$. Let $x^* \in \Omega_\omega(v_k)$ and $\{v_{k_j}\}_{j=1}^{\infty}$ be a subsequence of $\{v_k\}_{k=1}^{\infty}$ such that

$$v_{k_j} \rightharpoonup x^* \ \text{ as } \ j \to \infty. \ \text{Consequently, } \ t_{k_j} \rightharpoonup x^*, \ y_{k_j} \rightharpoonup x^* \ \text{ as } \ j \to \infty \qquad (5.7)$$

. By definition of S, $\{St_k\}_{k=1}^{\infty}$ is bounded. From inequalities (5.4) and (5.5), we have that

$$g\Big(||Jv_k - J(St_k)||\Big) \leq \frac{1}{\beta(1-\beta)}\Big(\phi(u, v_k) - \phi(u, v_{k+1})\Big). \tag{5.8}$$

Applying the property of g, we obtain that $\lim_{k\to\infty} ||Jv_k - J(St_k)|| = 0$.

By the uniform continuity of J^{-1} on bounded subset of X^*, we get that $\lim_{k\to\infty} ||v_k - St_k|| = 0$, so that

$$||St_k - t_k|| \leq ||St_k - v_k|| + ||v_k - t_k|| \to 0 \text{ as } k \to \infty, \tag{5.9}$$

which implies that $Sx^* = x^*$. Hence, $x^* \in F(S)$.

Next, we show that $x^* \in VI(f, C)$. Let $T : X \to X^*$ be a map defined by

$$Tv = \begin{cases} fv + N_C(v), & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases} \tag{5.10}$$

where $N_C(v)$ is the normal cone to C at $v \in C$. Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(f, C)$ (Rockafellar [136]). Let $(v, w) \in G(T)$, where $G(T)$ is the graph of T. Then, $w \in Tv = fv + N_C(v)$. Hence, we get that $w - fv \in N_C(v)$. This implies that $\langle v - t, w - fv\rangle \geq 0, \quad \forall\, t \in C$. In particular,

$$\big\langle v - y_k, w - f(v)\big\rangle \geq 0, \;\; \forall\, k \geq 1. \tag{5.11}$$

Furthermore, $y_k = \Pi_C J^{-1}\big(Jv_k - \tau f(v_k)\big), \;\forall\, k \geq 1$. By characterization of the generalized projection map, we obtain that

$$\Big\langle y_k - v, Jv_k - \tau f(v_k) - Jy_k\Big\rangle \geq 0, \forall\, v \in C. \tag{5.12}$$

This implies that

$$\Big\langle v - y_k, \frac{Jy_k - Jv_k}{\tau} + f(v_k)\Big\rangle \geq 0, \forall\, v \in C. \tag{5.13}$$

Using inequalities (5.11) and (5.13) for some $M_0 > 0$, Cauchy Schwartz inequality and condition C_2, we have that

$$\begin{aligned}
\big\langle v - y_{k_j}, w\big\rangle &\geq \big\langle v - y_{k_j}, f(v)\big\rangle \\
&\geq \big\langle v - y_{k_j}, f(v)\big\rangle - \Big\langle v - y_{k_j}, \frac{Jy_{k_j} - Jv_{k_j}}{\tau} + f(v_{k_j})\Big\rangle \\
&= \big\langle v - y_{k_j}, f(v) - f(y_{k_j})\big\rangle + \big\langle v - y_{k_j}, f(y_{k_j}) - f(v_{k_j})\big\rangle - \Big\langle v - y_{k_j}, \frac{Jy_{k_j} - Jv_{k_j}}{\tau}\Big\rangle \\
&\geq -KM_0||y_{k_j} - v_{k_j}|| - M_0||Jy_{k_j} - Jv_{k_j}||.
\end{aligned} \tag{5.14}$$

Taking limit of both sides of inequality (5.14) and using the fact that J is uniformly continuous on bounded subset of X, we obtain that

$$\big\langle v - x^*, w\big\rangle \geq 0. \tag{5.15}$$

Since T is a maximal monotone operator, it follows that $x^* \in T^{-1}(0) = VI(f, C)$, which implies that $\Omega_\omega(v_k) \subset VI(f, C)$. Hence, $x^* \in \mathcal{G}$.

Now, we show that $v_k \rightharpoonup x^*$ as $k \to \infty$. Define $x_k := \Pi_{VI(f,C)} v_k$. Then, $\{x_k\} \subset VI(f,C)$. Furthermore, from inequality 5.5 and Lemma 2.31, we have that

$$\phi(x_k, v_{k+1}) \leq \phi(x_k, v_k) \text{ and } \phi(x_{k+1}, v_{k+1}) \leq \phi(x_k, v_{k+1}) - \phi(x_k, x_{k+1}), \tag{5.16}$$

which implies that $\{\phi(x_k, v_k)\}$ converges. From inequality (5.16) and for any $m > k$, we have that

$$\phi(x_k, v_m) \leq \phi(x_k, v_k) \text{ and } \phi(x_k, x_m) \leq \phi(x_k, v_m) - \phi(x_m, v_m). \tag{5.17}$$

Furthermore, $\lim_{k\to\infty} \phi(x_k, x_m) = 0$. Hence, by Lemma 2.16, we obtain that $\lim_{k,m\to\infty} ||x_k - x_m|| = 0$, which implies that $\{x_k\}$ is a Cauchy sequence in $VI(f,C)$. Therefore, there exists a $u^* \in VI(f,C)$ such that $\lim_{k\to\infty} x_k = u^*$.

Now, using the definition of $x_k = \Pi_{VI(f,C)} v_k, \ \ \forall\ k \geq\ 0$, it follows from Lemma 2.31 that for any $p \in VI(f,C)$, we have that

$$\langle x_k - p, Jx_k - Jv_k \rangle \geq 0. \tag{5.18}$$

Let $\{v_{k_i}\}$ be any subsequence of $\{v_k\}$. We may assume without loss of generality that $\{v_{k_i}\}$ converges weakly to some $p^* \in VI(f,C)$. By inequality (5.18), weak sequential continuity of J and the fact that $\lim_{k\to\infty} x_k = u^*$, we obtain that

$$\langle u^* - p^*, Jp^* - Ju^* \rangle \geq 0. \tag{5.19}$$

However, from the monotonicity of J, we obtain that

$$\langle u^* - p^*, Ju^* - Jp^* \rangle \geq 0. \tag{5.20}$$

Combining inequalities (5.19) and (5.20), we have that

$$\langle u^* - p^*, Ju^* - Jp^* \rangle = 0. \tag{5.21}$$

By Lemma 2.19, we obtain that

$$||u^* - p^*||^2 \leq \frac{1}{c_2} \langle u^* - p^*, Ju^* - Jp^* \rangle = 0,$$

which implies that $u^* = p^*$. Hence, $v_k \rightharpoonup u^* = \lim_{k\to\infty} x_k$. This completes the proof. □

Theorem 5.2. *Let X be a uniformly smooth and 2-uniformly convex real Banach space with dual space X^*. Let C be a nonempty closed and convex subset of X. Let $T_i : X \to X, i = 1, 2, \cdots$ be a countable family of relatively nonexpansive maps and $f : X \to X^*$ be a map satisfying conditions C_1 and C_2 with $\tau \in (0, \frac{\alpha}{K})$ and let $\beta \in (0,1)$. Assume that condition C_4 holds and J is weakly sequentially continuous on X. Then, the sequence $\{v_k\}_{k=1}^{\infty}$ generated iteratively by Algorithm **1** converges weakly to some $v^* \in \mathcal{V}$, where*

$$Sx = J^{-1}\Bigg(\sum_{i=1}^{\infty} \delta_i \big(\gamma_i Jx + (1-\gamma_i) JT_i x\big) \Bigg), \ \ \sum_{i=1}^{\infty} \delta_i = 1 \ \ \text{and} \ \ \{\gamma_i\}_{i=1}^{\infty} \subset (0,1).$$

Proof. By Lemma 2.21, S is relatively nonexpansive and the $F(S) = \cap_{i=1}^{\infty} F(T_i)$. Also, by Lemma 5.1, the result of Theorem 5.2 follows. □

Corollary 5.3. Let H be a real Hilbert space and let C be a nonempty closed and convex subset of H. Let $T_i : H \to H, \ \ i = 1, 2, \cdots$ be a countable family of nonexpansive maps and $f : H \to H$ be a monotone and K-Lipschitz map. Assume that $C_1, \ C_2$ and C_4 hold with $\tau \in (0, \frac{1}{K})$ and let $\beta \in (0, 1)$. Then, the sequence $\{v_k\}_{k=1}^{\infty}$ generated by Algorithm ***1*** converges weakly to $v^* \in \mathcal{V}$.

Proof. In a Hilbert space, $J, \ J^{-1}$ are identity maps on H, and $\phi(y, z) = ||y - z||^2, \ \ \forall \ y, z \in H$. Thus, the conclusion follows from Theorem 5.2. □

5.0.2 Discussion

1. Theorem 5.2 which approximates a common solution of a variational inequality problem and a common fixed point of a countable family of relatively nonexpansive maps extends Theorem 7.1 of Censor *et al.* [40] from a Hilbert space to a uniformly smooth and 2-uniformly convex real Banach space with weakly sequentially continuous duality map, and from a single *nonexpansive map* to a countable family of *relatively nonexpansive maps*.

2. The control parameters in Algorithm **1** of Theorem 5.2 are two arbitrarily fixed constants $\beta \in (0, 1)$ and $\tau \in (0, 1)$ which are to be computed once and then used at each step of the iteration process, while the parameters in equation (1.31) studied by Censor *et al.* [40] are $\alpha_k \in (0, 1)$ and $\tau \in (0, 1)$, and α_k is to be computed at each step of the iteration process. Consequently, the sequence of Algorithm **1** is of *Krasnoselskii-type* and the sequence defined by equation (1.31) is of *Mann-type*. It is well known that a Krasnoselskii-type sequence converges as fast as a geometric progression which is slightly better than the convergence rate obtained from any Mann-type sequence.

The result of this chapter is published in FPTA, http://doi.org/10.1186/s13663-018-06414.

CHAPTER 6

Iterative algorithms for split Variational Inequalities and generalized split feasibility problems, with applications

Introduction

In this chapter, we present a *strong convergence* theorem for approximating a common solution for a finite family of split variational inequalities and generalized split feasibility problems in a real Hilbert space. The theorem presented improves and extends the results of Censor *et al.* [39], Tian and Jiang [159], which themselves are improvements of important recent results. Furthermore, applications of the theorem presented to equilibrium and optimization problems are given. Finally, a numerical example is presented to illustrate the convergence of the sequence generated by our algorithm.

6.0.1 Main result

A hybrid method for a class of generalized split feasibility problems

Theorem 6.1. *Let H_1 and H_2 be real Hilbert spaces. Let $M_i,\ i=1,\cdots,N$ be nonempty closed and convex subsets of H_1 such that $M=\cap_{i=1}^N M_i\neq\emptyset$. Let $\nabla:H_1\to H_2$ be a bounded linear map such that $\nabla\neq 0$ and ∇^* be the adjoint of ∇. Let $B_i:M_i\to H_1$ $i=1,\cdots,N$ be a finite family of monotone and L-Lipschitz maps and $T_i:H_2\to H_2$ $i=1,\cdots,N$ be a finite family of nonexpansive maps such that $\mathcal{D}=\{z\in\cap_{i=1}^N VI(M_i,B_i):\nabla z\in\cap_{i=1}^N F(T_i)\}\neq\emptyset$. For $v_1=v\in H_1,\ C_1=H_1$, and $W_1=H_1$, let $\{v_n\}$ be a sequence given by:*

$$\begin{cases} y_n^i = P_{M_i}\big(v_n-\lambda\nabla^*(I-T_i)\nabla v_n\big),\ i=1,\cdots,N,\\ u_n^i = P_{M_i}\big(y_n^i-\delta B_i(y_n^i)\big),\ i=1,\cdots,N,\\ t_n^i = P_{M_i}\big(y_n^i-\delta B_i(u_n^i)\big),\ i=1,\cdots,N,\\ C_n^i=\{z\in H:||t_n^i-z||\leq||v_n-z||\},\\ W_n=\{z\in H:\langle z-v_n,v-v_n\rangle\leq 0\},\\ v_{n+1}=P_{C_n\cap W_n}v,\quad\forall\ \ n\geq 1,\end{cases}\tag{6.1}$$

where $C_n=\cap_{i=1}^N C_n^i$, $\lambda\in(0,\frac{1}{||\nabla||^2})$ and $\delta\in(0,\frac{1}{L})$. Then, $\{v_n\}$ converges strongly to $P_{\mathcal{D}}v$.

Proof. **Step 1.** The sequence $\{v_n\}$ is well defined and $\mathcal{D} \subset C_n \cap W_n$.
First, we show that C_n^i and W_n are closed and convex. Clearly, from the definition of C_n^i and W_n, they are either half-spaces or the whole space H. Thus, they are closed and convex. Therefore, $\{v_n\}$ is well defined. Next, we prove that $\mathcal{D} \subset C_n \cap W_n$ for each $n \geq 1$.

Claim 1. $\mathcal{D} \subset C_n^i$ for each $n \geq 1$. Let $u \in \mathcal{D}$. Then, we compute as follows.
From (3), (4) of Lemma 2.33, and Lemma 2.34, we establish that $P_{M_i}\big(I - \lambda\nabla^*(I - T_i)\nabla\big)$ is $\frac{1+\lambda||\nabla||^2}{2}$ - averaged, for each $i = 1, \cdots, N$. Hence, y_n^i can be expressed as

$$y_n^i = (1-\beta)v_n + \beta Q_n^i v_n, \tag{6.2}$$

where $\beta = \frac{1+\lambda||\nabla||^2}{2}$ and Q_n^i is a nonxpansive map, for each $n \geq 1$ and for each $i = 1, \cdots, N$.

$$\begin{aligned} ||y_n^i - u||^2 &= ||(1-\beta)v_n + \beta Q_n^i v_n - u||^2 \\ &= (1-\beta)||v_n - u||^2 + \beta||Q_n^i v_n - u||^2 - \beta(1-\beta)||v_n - Q_n^i v_n||^2 \\ &\leq ||v_n - u||^2 - \beta(1-\beta)||v_n - Q_n^i v_n||^2. \end{aligned} \tag{6.3}$$

Since $u_n^i \in M_i$ and B_i is monotone for each $i = 1, \cdots, N$, we have that

$$\big\langle u_n^i - u, B_i(u_n^i) - B_i(u)\big\rangle \geq 0, \forall \;\; n \geq 1 \text{ and for each } i = 1, \cdots, N.$$

$$\text{With } t_n^i = P_{M_i}\big(y_n^i - \delta B_i(u_n^i)\big), \quad \text{we have that } \; \big\langle u - t_n^i, B_i(u_n^i)\big\rangle \leq \big\langle u_n^i - t_n^i, B_i(u_n^i)\big\rangle. \tag{6.4}$$

Set $z_n^i = y_n^i - \delta B_i(u_n^i)$. Then, for each $i = 1, \cdots, N$, we obtain using Lemma 2.30(1) that

$$\begin{aligned} ||t_n^i - u||^2 &\leq ||z_n^i - u||^2 - ||z_n^i - t_n^i||^2 \\ &= ||(y_n^i - u) - \delta B_i(u_n)||^2 - ||(y_n^i - t_n^i) - \delta B_i(u_n^i)||^2 \\ &= ||y_n^i - u||^2 - ||t_n^i - y_n^i||^2 + 2\delta\big\langle u - t_n^i, B_i(u_n^i)\big\rangle. \end{aligned} \tag{6.5}$$

From inequality (6.4), it follows that

$$\begin{aligned} ||t_n^i - u||^2 &\leq ||y_n^i - u||^2 - ||t_n^i - y_n^i||^2 + 2\delta\big\langle u_n^i - t_n^i, B_i(u_n^i)\big\rangle \\ &= ||y_n^i - u||^2 - ||y_n^i - u_n^i||^2 - ||u_n^i - t_n^i||^2 + 2\big\langle t_n^i - u_n^i, y_n^i - \delta B_i(u_n^i) - u_n^i\big\rangle \\ &= ||y_n^i - u||^2 - ||y_n^i - u_n^i||^2 - ||u_n^i - t_n^i||^2 + 2\big\langle t_n^i - u_n^i, y_n^i - \delta B_i(y_n^i) - u_n^i\big\rangle \\ &\quad + \; 2\delta\big\langle t_n^i - u_n^i, B_i(y_n^i) - B_i(u_n^i)\big\rangle \\ &\leq ||y_n^i - u||^2 - ||y_n^i - u_n^i||^2 - ||u_n^i - t_n^i||^2 + 2\delta\big\langle t_n^i - u_n^i, B_i(y_n^i) - B_i(u_n^i)\big\rangle. \end{aligned} \tag{6.6}$$

Since B_i is L-Lipschitz for each $i = 1, \cdots, N$, it follows from inequality (6.6)

$$\begin{aligned} ||t_n^i - u||^2 &\leq ||y_n^i - u||^2 - ||y_n^i - u_n^i||^2 - ||u_n^i - t_n^i||^2 + 2L\delta||t_n^i - u_n^i||||y_n^i - u_n^i|| \\ &\leq ||y_n^i - u||^2 - ||y_n^i - u_n^i||^2 - ||u_n^i - t_n^i||^2 + L\delta\big(||t_n^i - u_n^i||^2 + ||y_n^i - u_n^i||^2\big) \\ &= ||y_n^i - u||^2 - (1 - L\delta)||y_n^i - u_n^i||^2 - (1 - L\delta)||u_n^i - t_n^i||^2. \end{aligned} \tag{6.7}$$

From inequality (6.3), it follows that

$$||t_n^i - u||^2 \leq ||v_n - u||^2 - (1 - L\delta)||y_n^i - u_n^i||^2 - (1 - L\delta)||u_n^i - t_n^i||^2 \leq ||v_n - u||^2. \tag{6.8}$$

This implies that $\mathcal{D} \subset C_n^i$ for all $n \geq 1$ and for each $i = 1, \cdots, N$. Hence, $\mathcal{D} \subset C_n = \cap_{i=1}^N C_n^i$.

Claim 2. $\mathcal{D} \subset C_n \cap W_n$, for all $n \in \mathbb{N}$. Clearly, $\mathcal{D} \subset C_1 \cap W_1$. Assume that $\mathcal{D} \subset C_n \cap W_n$, for some $n \geq 1$. From $v_{n+1} = P_{C_n \cap W_n} v$ and Lemma 2.30(2), we have that $\langle z - v_{n+1}, v - v_{n+1} \rangle \leq 0$, for all $z \in C_n \cap W_n$. In particular, $u \in \mathcal{D} \subset C_n \cap W_n$. By definition of W_{n+1}, $\mathcal{D} \subset W_{n+1}$, which implies , $\mathcal{D} \subset C_{n+1} \cap W_{n+1}$. Hence, $\mathcal{D} \subset C_n \cap W_n$, for each $n \geq 1$.

Step 2.

$\lim_{n\to\infty} ||v_n - t_n^i|| = \lim_{n\to\infty} ||y_n^i - u_n^i|| = \lim_{n\to\infty} ||u_n^i - t_n^i|| = \lim_{n\to\infty} ||u_n^i - v_n|| = 0$, for each $i = 1, \cdots, N$.
First, we prove that $\{v_n\}$ is bounded. From the definition of $\{W_n\}$, we have that $v_n = P_{W_n} v$, $\forall \;\; n \geq \; 1$. Hence, by Lemma 2.30(1), we obtain that

$$||v_n - v||^2 = ||P_{W_n} v - v||^2 \leq ||u - v||^2 - ||u - v_n||^2 \leq ||u - v||^2, \; \forall \, u \, \mathcal{D} \subset W_n. \tag{6.9}$$

This implies that $\{||v_n - v||\}$ is bounded. Hence, $\{v_n\}$ is bounded. Consequently, $\{t_n^i\}$, $\{y_n^i\}$, and $\{u_n^i\}$ are bounded for each $i =, 1 \cdots, N$. Since $v_{n+1} = P_{C_n \cap W_n} v \in W_n$ and $v_n = P_{W_n} v$, from the definition of $v_n = P_{W_n} v$, we have that $||v_n - v|| \leq ||v_{n+1} - v||$, $\forall \, n \geq 1$, and this implies that $\{||v_n - v||\}$ is monotone nondecreasing. Hence, $\lim_{n\to\infty} ||v_n - v||$ exists.
From Lemma 2.30(1) and $v_n = P_{W_n} v$, we obtain for arbitrary $m, n \in \mathbb{N}$, with $m > n$, that

$$\begin{aligned} ||v_m - v_n||^2 = ||v_m - P_{W_n} v||^2 &\leq ||v_m - v||^2 - ||P_{W_n} v - v||^2 \\ &= ||v_m - v||^2 - ||v_n - v||^2 \to 0 \text{ as } n \to \infty. \end{aligned} \tag{6.10}$$

Hence, $\{v_n\}$ is Cauchy. Thus, there exists $x^* \in M$ such that $\lim_{n\to\infty} v_n = x^*$. Since $v_m \in C_m \subset C_n = \cap_{i=1}^N C_n^i$, from the definition C_n^i, we obtain that $||t_n^i - v_m|| \leq ||v_n - v_m||$. Using this and inequality (6.10), we have that

$$||v_n - t_n^i|| \leq ||v_n - v_m|| + ||v_m - t_n^i|| \leq 2||v_n - v_m|| \to 0 \text{ (for each } i = 1, \cdots, N). \tag{6.11}$$

From inequality (6.8), set $\gamma = (1 - L\delta)^{-1}$ so that

$$\begin{aligned} ||y_n^i - u_n^i||^2 &\leq \gamma\big(||v_n - u||^2 - ||t_n^i - u||^2\big) \leq \gamma(||v_n - u|| + ||t_n^i - u||)(||v_n - t_n^i||), \text{ and} \\ ||u_n^i - t_n^i||^2 &\leq \gamma\big(||v_n - u||^2 - ||t_n^i - u||^2\big) \leq \gamma(||v_n - u|| + ||t_n^i - u||)(||v_n - t_n^i||). \end{aligned}$$

Hence, we obtain that $\lim_{n\to\infty} ||y_n^i - u_n^i|| = \lim_{n\to\infty} ||u_n^i - t_n^i|| = 0$, for each $i = 1, \cdots, N$.

Furthermore, from inequality (6.3), we obtain using inequality (6.7), that

$$\beta(1 - \beta)||v_n - Q_n^i v_n||^2 \leq ||v_n - u||^2 - ||y_n^i - u||^2 \leq ||v_n - u||^2 - ||t_n^i - u||^2.$$

Hence, from inequality (6.11), we conclude that for each $i = 1, \cdots, N$, $\lim_{n\to\infty} ||v_n - Q_n^i v_n|| = 0$. Also, from equation (6.2), we obtain that $\lim_{n\to\infty} ||y_n^i - v_n|| = 0$. Hence,

$$\lim_{n\to\infty} ||v_n - t_n^i|| = \lim_{n\to\infty} ||y_n^i - u_n^i|| = \lim_{n\to\infty} ||u_n^i - t_n^i|| = \lim_{n\to\infty} ||u_n^i - v_n|| = 0, \quad \text{for each } i = 1, \cdots, N.$$

Step 3. $\Omega_\omega(v_n) \subset \cap_{i=1}^N \big(M_i \cap B_i^{-1} F(T_i)\big)$ and $\Omega_\omega(v_n) \subset \cap_{i=1}^N VI(M_i, B_i)$, where $\Omega_\omega(v_k)$ is the set of weak sub-sequential limits of $\{v_n\}$.
Let $x^* \in \Omega_\omega(v_n)$ and $\{v_{n_j}\}_{j=1}^\infty$ be a subsequence of $\{v_n\}_{n=1}^\infty$ such that

$$v_{n_j} \rightharpoonup x^* \text{ as } j \to \infty. \text{ Consequently, } u_{n_j}^i \rightharpoonup x^* \text{ as } j \to \infty, \text{ for each } i = 1, \cdots, N.$$

From the definition of y_n^i, we have that $\lim_{j\to\infty} ||v_{n_j} - P_{M_i}(v_{n_j} - \lambda\nabla^*(I-T_i)\nabla v_{n_j})|| = 0$, for each $i = 1, \cdots, N$. Since for each $i = 1, \cdots, N$, $\nabla^*(I-T_i)\nabla$ is inverse strongly monotone, then, it is Lipschitz. By Lemma 2.36, we obtain that $x^* \in F\big(P_{M_i}(I - \lambda\nabla^*(I-T_i)\nabla)\big)$, for each $i = 1, \cdots, N$. By Lemma 2.35, we have that $\Omega_\omega(v_n) \subset M_i \cap B^{-1}F(T_i)$, for each $i = 1, \cdots, N$.

Next, we show that $\Omega_\omega(v_n) \subset \cap_{i=1}^N VI(M_i, B_i)$.
Let $x^* \in \Omega_\omega(v_n)$ and $\{u_{n_j}^i\}_{j=1}^\infty$ be a subsequence of $\{u_n^i\}_{n=1}^\infty$ such that $u_{n_j}^i \rightharpoonup x^*$ as $j \to \infty$, for each $i = 1, \cdots, N$.
Applying a result of (Rockafellar [136]), we define maps $R_i : H \to H \quad i = 1, \cdots, N$ by

$$R_i u = \begin{cases} B_i u + N_{M_i}(u), & \text{if } \ u \in M_i, \\ \emptyset, & \text{if } \ u \notin M_i, \end{cases} \tag{6.12}$$

where $N_{M_i}(\cdot)$ is the normal cone of M_i, for each $i = 1, \cdots, N$. Then, R_i is maximal monotone and $R_i^{-1}(0) = VI(M_i, B_i)$, for each $i = 1, \cdots, N$. Let $(u, w) \in G(R_i)$, where $G(R_i)$ is the graph of R_i, for each $i = 1, \cdots, N$. Then, $w \in R_i u = B_i u + N_{M_i}(u)$. Hence, we get that $w - B_i u \in N_{M_i}(u)$. This implies that $\langle u - t, w - B_i u\rangle \geq 0, \ \forall\ t \in M_i$. In particular,

$$\big\langle u - u_n^i, w - B_i u\big\rangle \geq 0. \tag{6.13}$$

But $u_n^i = P_{M_i}\big(y_n^i - \delta B_i(y_n^i)\big), \ \forall\ n \geq 1$ and for each $i = 1, \cdots, N$. By a characterization of the metric projection, we obtain that

$$\Big\langle u_n^i - u, y_n^i - \delta B_i(y_n^i) - u_n \Big\rangle \geq 0, \forall\ u \in M_i. \tag{6.14}$$

This implies that

$$\Big\langle u - u_n^i, \frac{u_n^i - y_n^i}{\delta} + B_i(y_n^i) \Big\rangle \geq 0, \forall\ u \in M_i. \tag{6.15}$$

Using inequalities (6.13) and (6.15) for some $M_0 > 0$, and the fact that B_i is monotone and L-Lipschitz, for each $i = 1, \cdots, N$, we have that

$$\begin{aligned}
\big\langle u - u_{n_j}^i, w\big\rangle &\geq \big\langle u - u_{n_j^i}, B_i u\big\rangle \\
&\geq \big\langle u - u_{n_j}^i, B_i u\big\rangle - \Big\langle u - u_{n_j}^i, \frac{u_{n_j}^i - y_{n_j}^i}{\delta} + B_i(y_{n_j}^i) \Big\rangle \\
&= \big\langle u - u_{n_j}^i, B_i u - B_i(u_{n_j}^i)\big\rangle + \big\langle u - u_{n_j}^i, B(u_{n_j}^i) - B_i(y_{n_j}^i)\big\rangle - \Big\langle u - u_{n_j}^i, \frac{u_{n_j}^i - y_{n_j}^i}{\delta} \Big\rangle \\
&\geq -M_0(1+L)||u_{n_j}^i - y_{n_j}^i||.
\end{aligned} \tag{6.16}$$

Taking limit of both sides of inequality (6.16) as $j \to \infty$, for each $i = 1, \cdots, N$, we obtain that $\big\langle u - x^*, w\big\rangle \geq 0$.
Since R_i is maximal monotone for each $i = 1, \cdots, N$, it follows that $x^* \in R_i^{-1}(0) = VI(M_i, B_i)$, which implies that $\Omega_\omega(v_n) \subset VI(M_i, B_i)$, for each $i = 1, \cdots, N$. Hence, $\Omega_\omega(v_n) \subset \mathcal{D}$.

Step 4. $\lim_{n\to\infty} v_n = P_{\mathcal{D}} v$. From Lemma 2.30(1), we have that

$$||v - P_{\mathcal{D}} v|| \leq ||v - x^*||. \tag{6.17}$$

For $x^* \in \mathcal{D} \subset W_n$, $v_n = P_{W_n} v$ and Lemma 2.30(1), we obtain that $||v - v_n|| \leq ||v - P_{\mathcal{D}} v||$. Since, $\lim_{n\to\infty} v_n = x^*$, we get that $||v - x^*|| \leq ||v - P_{\mathcal{D}} v||$. Using this and inequality (6.17), we obtain that $||v - x^*|| = ||v - P_{\mathcal{D}} v||$. By uniqueness of $P_{\mathcal{D}} v$, we conclude that $x^* = P_{\mathcal{D}} v$. □

Theorem 6.2. *Let H_1 and H_2 be real Hilbert spaces. Let M and Q be nonempty closed convex subsets of H_1 and H_2, respectively, such that $M = \cap_{i=1}^{N} M_i \neq \emptyset$ and $Q = \cap_{i=1}^{N} Q_i \neq \emptyset$. Let $\nabla : H_1 \to H_2$ be a bounded linear map such that $\nabla \neq 0$, and ∇^* be the adjoint of ∇. Let $B_i : M_i \to H_1 \ \ i = 1, \cdots, N$, be monotone and L-Lipschitz maps. Let $\mathcal{F}_i : H_2 \to H_2$ $i = 1, \cdots, N$, be η-inverse strongly monotone map such that $\mathcal{D} = \{z \in \cap_{i=1}^{N} VI(M_i, B_i) : \nabla z \in \cap_{i=1}^{N} VI(Q_i, \mathcal{F}_i)\} \neq \emptyset$. For $v_1 = v \in H_1$, $C_1 = H_1$, and $W_1 = H_1$, with $T_i = P_{Q_i}(I - \mu \mathcal{F}_i)$ and $\mu \in (0, 2\eta)$. Then, the sequence $\{v_n\}$ generated by equation (6.1) converges strongly to $P_{\mathcal{D}} v$.*

Proof. From Lemma 2.30, it is easy to see that $u \in VI(Q_i, \mathcal{F}_i)$ if and only if $u = P_{Q_i}(I - \mu \mathcal{F}_i)u$, for $\mu > 0$ and for each $i = 1, \cdots, N$. Furthermore, with $\mu \in (0, 2\eta)$, $P_{Q_i}(I - \mu \mathcal{F}_i)$ is nonexpansive for each $i = 1, \cdots, N$. Hence, by Theorem 6.1, the result of Theorem 6.2 is immediate. □

6.0.2 Applications

Here, we apply our theorem to solve equilibrium problems and optimization problems.

Theorem 6.3. *Let H_1 and H_2 be real Hilbert spaces. Let M and Q be nonempty closed and convex subsets of H_1 and H_2, respectively, such that $M = \cap_{i=1}^{N} M_i \neq \emptyset$ and $Q = \cap_{i=1}^{N} Q_i \neq \emptyset$. Let $\nabla : H_1 \to H_2$ be a bounded linear map such that $\nabla \neq 0$, and ∇^* be the adjoint of ∇. Let $B_i : M_i \to H_1 \ \ i = 1, \cdots, N$, be monotone and L-Lipschitz maps and $\mathcal{G}_i : Q_i \times Q_i \to \mathbb{R}$ $i = 1, \cdots, N$, be bifunctionals satisfying conditions (P_1) - (P_4) such that $\mathcal{D} = \{z \in \cap_{i=1}^{N} VI(M_i, B_i) : \nabla z \in \cap_{i=1}^{N} EP(\mathcal{G}_i)\} \neq \emptyset$. For $v_1 = v \in H_1$, $C_1 = H_1$, $W_1 = H_1$. Then, $\{v_n\}$ generated by equation (6.1) converges strongly to $P_{\mathcal{D}} v$.*

Proof. Set $T_i = T_r^{\mathcal{G}_i}$, for each $i = 1, \cdots, N$, in Theorem 6.1. By Remark 2.44 and a result of Blum and Oettli [13], Theorem 6.3 is immediate. □

Theorem 6.4. *Let H_1 and H_2 be real Hilbert spaces. Let M and Q be nonempty closed and convex subsets of H_1 and H_2, respectively, such that $M = \cap_{i=1}^{N} M_i \neq \emptyset$ and $Q = \cap_{i=1}^{N} Q_i \neq \emptyset$. Let $\nabla : H_1 \to H_2$ be a bounded linear map such that $\nabla \neq 0$, and ∇^* be the adjoint of ∇. Let $B_i : M_i \to H_1 \ \ i = 1, \cdots, N$, be monotone and L-Lipschitz maps and $\Theta_i : H_2 \to \mathbb{R}$ $i = 1, \cdots, N$, be differentiable convex functions. Suppose Θ_i' is η-inverse strongly monotone such that $\mathcal{D} = \{z \in \cap_{i=1}^{N} VI(M_i, B_i) : \nabla z \in \cap_{i=1}^{N} \underset{y \in Q_i}{Argmin} \Theta_i(y)\} \neq \emptyset$. For $v_1 = v \in H_1$, $C_1 = H_1$, $W_1 = H_1$. Then, the sequence $\{v_n\}$ generated by equation (6.1) converges strongly to $P_{\mathcal{D}} v$.*

Proof. Set $T_i = P_{Q_i}(I - \mu \Theta_i')$, for each $i = 1, \cdots, N$, in Theorem 6.1 with $\mathcal{F}_i = \Theta_i'$, for each $i = 1, \cdots, N$, in Theorem 6.2. By Lemma 2.37, Theorem 6.4 is immediate. □

6.0.3 Discussion

Theorem 6.1 which approximates a common solution of a finite family of *generalized split feasibility problems* and Theorem 6.2 which approximates a common solution of a finite family of *split variational inequalities* in a real Hilbert space, respectively, complement the recent important results of Censor *et al.* [39], Tian and Jiang [159] in the following sense:

We first observe that even for a *single operator*, the algorithms of Theorems 6.1 and 6.2 are slightly different from the algorithms studied by Tian and Jiang [159].

1. If $T_i \equiv P_{Q_i} = (I - \mu\mathcal{F}_i) \equiv 0$, $\nabla \equiv I$ and $\lambda \equiv \delta$, then, the $(CSSVIP)$ reduces to the $(CSVIP)$ and equation (6.1) in Theorem 6.2 reduces to the theorem of Censor *et al.* [39] for solving $(CSVIP)$.

2. Theorem 6.1 yields a *strong convergence* of the sequence generated by equation (6.1) for a *finite family of maps* while a *weak convergence* result is proved in Tian and Jiang [159] for a single operator.

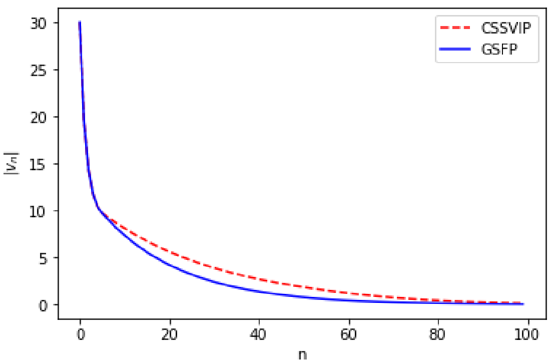
3. Finally, in Theorem 6.2, a *strong convergence* theorem for approximating a common solution for a *finite family of split variational inequality problems* $(CSSVIP)$ is proved while in the theorem of Tian and Jiang [159], a *weak convergence* theorem for approximating a *split variational inequality problem is proved.*

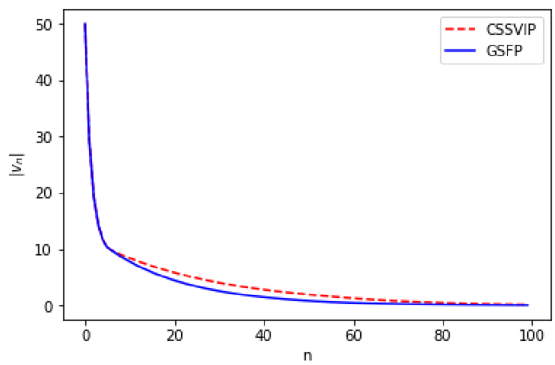
6.0.4 Numerical experiment

Here, we present numerical examples to illustrate the convergence of our sequence $\{v_n\}$ in Theorem 6.1 and Theorem 6.2, respectively. For this example, we take $N = 1$.

Example 6.5. Let $H = \mathbb{R}$, $M = [-20, 10]$, $v_1 = 30$, $\nabla v = 5v$, $Bv = 10v$, $\mathcal{F}v = 3v$, $Tv = \sin v$. Clearly, $v_1 \in \mathbb{R}$, T is nonexpansive and $F(T) = \{0\}$, ∇, B and $\mathcal{F}$ satisfy the conditions of Theorems 6.1 and 6.2, respectively. The parameters $\lambda = 0.04$, $\delta = 0.01, \mu = 0.335$. $\lambda = 0.05$ and $\delta = 0.015$.

The graph of numerical experiments are given below. The *y-axis* represents the values of $|v_n - 0|$, while the *x-axis* represents the number of iterations (n).





All computations and graphs were implemented in python 3.6 using some abstractions developed at $AUST$ and other open source python library such as numpy and matplotlib on Zinox with intel core $i7$ process.

The result of this chapter is published in J. Nonlinear Variational Analysis Vol. 3 (2019), Issue 2, pages 127-140.

CHAPTER 7

Convergence theorem for Generalized Φ-strongly monotone maps, with applications

Introduction

In this chapter, we present a Mann-type iterative algorithm that approximates the zero of a generalized Φ-strongly monotone map. A strong convergence theorem for a sequence generated by the algorithm is proved. Furthermore, the theorem is applied to approximate the solution of a convex optimization problem, a Hammerstein integral equation and a variational inequality problem. This theorem generalizes, improves and complements results of Diop *et al.* [78], Chidume and Bello [51], and Chidume [48], Chidume *et al.*, [54, 65]. Finally, examples of generalized Φ-strongly monotone maps are constructed and numerical experiments are presented which illustrate the convergence of the sequence generated by the algorithm.

Definition 7.1. A map $A : X \to X^*$ is *quasi-bounded* if for every $\mu > 0$, there exists $\gamma > 0$ such that whenever $\langle v, Av\rangle \leq \mu||v||$ and $||v|| \leq \mu$, then, $||Av|| \leq \gamma$.

Lemma 7.2 (Rockafellar, [137], see also Pascali and Sburlin, [130])**.** A monotone map $A : X \to X^*$ is locally bounded at the interior points of its domain.

The following Lemma has been proved. But for completeness, we present the prove here (see, e.g., Pascali and Sburlan [130], chapter III, Lemma 3.6).

Lemma 7.3. Let X be a real normed space with dual space X^*. Every monotone map $A : D(A) \subset X \to X^*$ with $0 \in IntD(A)$ is *quasi-bounded.*

Proof. By Lemma 7.2, A is locally bounded at 0, i.e., there exists $r := r_0 > 0$ such that

$$||Au|| \leq \mu, \quad \forall\ u \in B_r(0), \text{ for some } \mu > 0.$$

Now, using this $\mu > 0$, suppose $\langle v, Av\rangle \leq \mu||v||$ and $||v|| \leq \mu$. Then, by the monotonicity of A, we have that

$$\langle v, Av\rangle \geq \langle u, Av\rangle + \langle v - u, Au\rangle, \quad \forall\ u \in B_r(0).$$

Observe that

$$\langle v - u, Au\rangle \leq ||Au||(||v|| + ||u||) \leq \mu(||v|| + r)$$

Thus,

$$\begin{aligned}\langle u, Av\rangle &\leq \langle v, Av\rangle + \langle u-v, Au\rangle \\ &\leq \mu||v|| + \mu(||v||+r) = \mu(2||v||+r), \quad \forall\ u \in\ B_r(0).\end{aligned}$$

This implies that

$$|\langle u, Av\rangle| \leq \mu(2||v||+r), \quad \forall\ u \in\ B_r(0).$$

Thus,

$$\sup_{||u||\leq r} |\langle u, Av\rangle| \leq \mu(2||v||+r).$$

Therefore,

$$||Av|| \leq \frac{\mu}{r}(2||v||+r).$$

Hence, A is quasi-bounded. □

7.0.1 Main result

In Theorem 7.4 below, the sequence $\{\beta_n\} \subset (0,1)$ is assumed to satisfy the following conditions: (C_1) $\sum \beta_n = \infty$, $\lim \beta_n = 0$; (C_2) $2\sum \delta_X^{-1}(\beta_n M)M < \infty$; (C_3) $2\delta_X^{-1}(\beta_n M) \leq \gamma_0$, for some $M > 0$, $\gamma_0 > 0$, where δ_X is the modulus of convexity (*see e.g., Chidume [45], P.* 5, 6).

Theorem 7.4. *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let $A : D(A) = X \to X^*$ be a generalized-Φ-strongly monotone map, where $D(A)$ is the domain of A and $A^{-1}(0) \neq \emptyset$. For arbitrary $v_1 \in X$, let $\{v_n\}$ be a sequence generated iteratively by*

$$v_{n+1} = J^{-1}(Jv_n - \beta_n Av_n),\ n \geq 1, \tag{7.1}$$

where J is the normalized duality map on X, and the sequence $\{\beta_n\} \subset (0,1)$ satisfies conditions C_1, C_2 and C_3. Then, the sequence $\{v_n\}$ converges strongly to $v^* \in A^{-1}(0)$.

Proof. First, we observe that if the equation $Au = 0$ has a solution, it is necessarily unique. If y^* is a solution of the equation $Au = 0$, then, from inequality (1.8), we have that

$$\langle x - y^*, Ax\rangle \geq \Phi(||x-y^*||),\ \forall\ x \in X. \tag{7.2}$$

Suppose $u^* \neq y^*$ is another solution of the equation $Au = 0$, substituting u^* in inequality (7.2), we have:

$$0 \geq \Phi(||u^* - y^*||),$$

which implies, by the properties of Φ that $u^* = y^*$. This contradiction yields the uniqueness of the solution.

The remainder of the proof is now in two steps.

Step 1. We show that the sequence $\{v_n\}$ is bounded. Here, ψ denotes the Alber's functional. Let $v^* \in A^{-1}(0)$. Let $\mu > 0$ be arbitrary but fixed. Then, there exists $r > 0$ such that

$$r > \max\{4\mu^2 + ||v^*||^2, \psi(v^*, v_1)\}. \tag{7.3}$$

Define $B := \{v \in X : \psi(v^*, v) \leq r\}$. It suffices to show that $\{\psi(v^*, v_n)\}$ is bounded. We proceed by induction. For $n = 1$, by construction, we have that $\psi(v^*, v_1) \leq r$. Assume that

$\psi(v^*, v_n) \le r$, for some $n \ge 1$. Using the property of ψ, we have that $||v_n|| \le ||v^*|| + \sqrt{r}$. Now, we show that $\psi(v^*, v_{n+1}) \le r$. Suppose by contradiction that $\psi(v^*, v_{n+1}) \le r$ does not hold. Then, $\psi(v^*, v_{n+1}) > r$. Since $A : X \to X^*$ is locally bounded at $v \in X$, there exist $r_v > 0$ and $m > 0$ such that

$$\begin{aligned} ||Ax|| &\le m, \ \forall\, x \in B_{r_v}(v). \\ \text{In particular,} \quad ||Av|| &\le m. \\ \text{Therefore,} \quad \langle v, Av\rangle &\le m||v||. \end{aligned}$$

Define $M_0 := \max\{m, ||v^*|| + \sqrt{r}\}$. Then, $\langle v, Av\rangle \le M_0||v||$ and $||v|| \le M_0$. By Lemma 7.3, there exists $M > 0$ such that $||Av|| \le M, \quad \forall\, v \in B$.

Define $\gamma_0 := \min\{1, \frac{\Phi(\mu)}{M}, \frac{\mu}{M}\}$. Using Lemma 2.8, we compute as follows:

$$\begin{aligned} \psi(v^*, v_{n+1}) &= V(v^*, Jv_n - \beta_n Av_n) \\ &\le V(v^*, Jv_n) - 2\beta_n\langle J^{-1}(Jv_n - \beta_n Av_n) - v^*, Av_n - Av^*\rangle \\ &= \psi(v^*, v_n) - 2\beta_n\langle v_n - v^*, Av_n - Av^*\rangle - 2\beta_n\langle v_{n+1} - v_n, Av_n\rangle. \end{aligned} \tag{7.4}$$

Using the fact that A is a generalized Φ-strongly monotone map and Lemma 2.11, it follows from inequality (7.4) that

$$\begin{aligned} \psi(v^*, v_{n+1}) &\le \psi(v^*, v_n) - 2\beta_n\Phi(||v_n - v^*||) + 2\beta_n\delta_X^{-1}(4RL\beta_n||Av_n||)||Av_n|| \\ &\le \psi(v^*, v_n) - 2\beta_n\Phi(||v_n - v^*||) + 2\beta_n\delta_X^{-1}(\beta_n M)M. \end{aligned} \tag{7.5}$$

But from recursion formula (7.1), we have that

$$||Jv_{n+1} - Jv_n|| = \beta_n||Av_n|| \le \beta_n M. \tag{7.6}$$

Applying Lemma 2.11 and inequality (7.6), we have that

$$||v_{n+1} - v_n|| = ||J^{-1}(Jv_{n+1}) - J^{-1}(Jv_n)|| \le 2\delta_X^{-1}(\beta_n M). \tag{7.7}$$

Thus, from inequality (7.7), we obtain that

$$||v_n - v^*|| \ge ||v_{n+1} - v^*|| - 2\delta_X^{-1}(\beta_n M). \tag{7.8}$$

From Lemma 2.9, we have that

$$r < \psi(v^*, v_{n+1}) \le ||v_{n+1} - v^*||^2 + ||v^*||^2. \tag{7.9}$$

Using inequality (7.3), we have that

$$4\mu^2 + ||v^*||^2 - ||v^*||^2 < r - ||v^*||^2 \le ||v_{n+1} - v^*||^2.$$

Hence,

$$2\mu \le ||v_{n+1} - v^*||. \tag{7.10}$$

From inequalities (7.7), (7.8), condition C_3 and definition of γ_0, we have that

$$||v_n - v^*|| \ge 2\mu - 2\delta_X^{-1}(\beta_n M) \ge 2\mu - \mu = \mu. \tag{7.11}$$

Since Φ is strictly increasing, we have that

$$\Phi(||v_n - v^*||) \ge \Phi(\mu). \tag{7.12}$$

From inequality (7.5), condition C_3 and definition of γ_0, we have that

$$r < \psi(v^*, v_{n+1}) \leq \psi(v^*, v_n) - 2\beta_n\Phi(\mu) + 2\beta_n\delta_X^{-1}(\beta_n M)M \tag{7.13}$$
$$\leq r - 2\beta_n\Phi(\mu) + \beta_n\Phi(\mu) < r. \tag{7.14}$$

This is a contradiction. Hence, $\{\psi(v^*, v_n)\}$ is bounded. Consequently, $\{v_n\}$ is bounded.

Step 2. We show that the sequence $\{v_n\}$ converges strongly to a point $v^* \in A^{-1}(0)$.
Using inequality (7.5), we have that

$$\begin{aligned} \psi(v^*, v_{n+1}) &\leq \psi(v^*, v_n) - 2\beta_n\Phi(||v_n - v^*||) + 2\beta_n\delta_X^{-1}(\beta_n M)M \\ &\leq \psi(v^*, v_n) + 2\beta_n\delta_X^{-1}(\beta_n M)M. \end{aligned} \tag{7.15}$$

By Lemma 2.10, we get that $\{\psi(v^*, v_n)\}$ is convergent. Furthermore, we have that

$$2\beta_n\Phi(||v_n - v^*||) \leq \psi(v^*, v_n) - \psi(v^*, v_{n+1}) + 2\beta_n\delta_X^{-1}(\beta_n M)M. \tag{7.16}$$

Claim. $\liminf \Phi(||v_n - v^*||) = 0$.
Suppose by contradiction that $\liminf \Phi(||v_n - v^*||) = 0$ does not hold. Then,
$\liminf \Phi(||v_n - v^*||) = s > 0$. Hence, there exists $N_1 \in \mathbb{N}$ such that

$$\Phi(||v_n - v^*||) > \frac{s}{2}, \text{ for all } n \geq N_1. \tag{7.17}$$

Using inequality (7.17), conditions C_1 and C_2, we have that

$$s\sum_{n=1}^{\infty}\beta_n \leq \sum_{n=1}^{\infty}\big(\psi(v^*, v_n) - \psi(v^*, v_{n+1})\big) + 2\sum_{n=1}^{\infty}\delta_X^{-1}(\beta_n M)M < \infty. \tag{7.18}$$

This is a contraction. Hence, $\liminf \Phi(||v_n - v^*||) = 0$. Thus, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that

$$\lim_{k\to\infty} \Phi(||v_{n_k} - v^*||) = 0. \tag{7.19}$$

Using the property of Φ, it follows that $\lim\limits_{k\to\infty} ||v_{n_k} - v^*|| = 0$. By Remark 2.17, we have that

$$\lim_{k\to\infty} \psi(v^*, v_{n_k}) = 0. \tag{7.20}$$

Consequently, by Lemma 2.10, we have that $\lim\limits_{n\to\infty} \psi(v^*, v_n) = 0$.

Hence, by Lemma 2.16, we have that $\lim\limits_{n\to\infty} ||v_n - v^*|| = 0$. This completes the proof. □

7.0.2 Application to convex optimization problems

In this section, we apply Theorem 7.4 in solving the problem of finding minimizers of convex functions defined on real Banach spaces. First, we begin with the following known results.

Lemma 7.5 (Xu, [162], see also Chidume [45], p. 43)**.** Let X be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{v \in X : ||v|| \leq r\}$. Then, there exists a continuous strictly increasing convex function $\Phi : [0, \infty) \to [0, \infty)$, $\Phi(0) = 0$, such that for every $u, v \in B_r(0)$, the following inequality holds:

$$\langle u - v, Ju - Jv\rangle \geq \Phi(||u - v||),$$

where J is the single-valued normalized duality map on X.

Lemma 7.6 (Chidume *et al.*, [65]). Let X be a uniformly convex and uniformly smooth real Banach space. Let $g : X \to \mathbb{R}$ be a differentiable convex function. Then, the differential map $dg : X \to X^*$ satisfies the following inequality:

$$\langle u - v, dg(u) - dg(v)\rangle \geq \langle u - v, Ju - Jv\rangle, \quad \forall \ \ u, v \in X,$$

where J is the single-valued normalized duality map on X.

Remark 7.7. If for any $R > 0$ and for any $u, v \in X$ such that $||u|| \leq R, ||v|| \leq R$, then, the map $dg : X \to X^*$ is generalized Φ-strongly monotone. This can easily be seen from Lemmas 7.5 and 7.6.

Theorem 7.8. *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let $g : X \to \mathbb{R} \cup \{\infty\}$ be a differentiable, convex, proper and coercive function such that $(dg)^{-1}(0) \neq \emptyset$. For arbitrary $v_1 \in X$, let the sequence $\{v_n\}$ be generated by*

$$v_{n+1} = J^{-1}\big(Jv_n - \beta_n dg(v_n)\big), \ \ n \geq 1,$$

where J is the normalized duality map on X. Assume $\{\beta_n\} \subset (0, 1)$ satisfies conditions C_1, C_2 and C_3 of Theorem 7.4. Then, g has a unique minimizer $v^* \in X$ and the sequence $\{v_n\}$ converges strongly to v^*.

Proof. Since g is a lower semi-continuous, convex, proper and coercive function, then, g has a minimizer $v^* \in X$. Furthermore, $dg : X \to X^*$ is generalized Φ-strongly monotone. Hence, the conclusion follows from Theorem 7.4. □

7.0.3 Application to Hammerstein integral equation

Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation of Hammerstein-type has the form

$$u(x) + \int_\Omega k(x, y)f(y, u(y))dy = w(x), \tag{7.21}$$

where the unknown function u and inhomogeneous function w lie in a Banach space X of measurable real-valued functions.

If we define a map $F : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ and $K : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ by

$$Fu(y) = f(y, u(y)), \ \ y \in \Omega, \quad \text{and} \quad Kv(x) = \int_\Omega k(x, y)v(y)dy, \ \ x \in \Omega, \tag{7.22}$$

respectively, where $\mathcal{F}(\Omega, \mathbb{R})$ is a space of measurable real-valued functions defined from Ω to $\mathbb{R}$. Then, equation (7.21) can be put in an abstract form

$$u + KFu = w, \tag{7.23}$$

Indeed, if $w \neq 0$, then, $u - w + KFu = 0$. Setting $h = u - w$, we obtain that

$$h + K\overline{F}h = 0,$$

where, $\overline{F}(h) = F(h + w)$.

Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts posses Green's function can, as a rule, be put in the form (7.21) (see e.g., Pascali and Sburian [130], chapter p. 164).

Several existence and uniqueness theorems have been proved for equations of Hammerstein-type (see e.g., Brezis and Browder [15, 16], Chepanovich [147], Browder and Gupta [21], De Figueiredo and Gupta [71], and the references contained in them).

In general, equations of Hammerstein-type are nonlinear and there is no known method to find a close form solutions for them. Consequently, methods for approximating solutions of such equations are of interest. For earlier and more recent works on approximation of solutions of equations of Hammerstein-type, the reader may consult any of the following: Brezis and Browder [15, 16], Chidume and Shehu [57], Chidume and Ofoedu [58], Chidume and Zegeye [59], Chidume and Djitte [56], Ofoedu and Onyi [127], Ofoedu and Malonza [128], Zegeye and Malonza [167], Chidume and Bello [51], Minjibir and Mohammed [117], and the references contained in them.

We now apply Theorem 7.4 to approximate a solution of equation (7.23). The following lemma would be needed in the proof of Theorem 7.11 below.

Lemma 7.9. Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^* and $E = X \times X^*$. Let $F : X \to X^*$ and $K : X^* \to X$ be generalized Φ_1-strongly monotone and generalized Φ_2-strongly monotone maps, respectively. Let $A : E \to E^*$ be defined by $A([u, v]) = [Fu - v, Kv + u]$. Then, A is a generalized Φ-strongly monotone map.

Proof. Let $[u_1, v_1], [u_2, v_2] \in E$. Then,

$$\begin{aligned} & \langle [u_1, v_1] - [u_2, v_2], A([u_1, v_1]) - A([u_2, v_2]) \rangle \\ = & \langle [u_1 - u_2, v_1 - v_2], [Fu_1 - Fu_2 + v_2 - v_1, Kv_1 - Kv_2 + u_1 - u_2] \rangle \\ = & \langle u_1 - u_2, Fu_1 - Fu_2 \rangle + \langle v_1 - v_2, Kv_1 - Kv_2 \rangle \\ \geq & \Phi_1(||u_1 - u_2||) + \Phi_2(||v_1 - v_2||). \end{aligned}$$

□

Remark 7.10. For A defined in Lemma 7.9, $[u^*, v^*]$ is a zero of A if and only if u^* solves (7.23), where $v^* = Fu$.

In Theorem 7.11 below, the sequence $\{\beta_n\} \subset (0, 1)$ is assumed to satisfy the following conditions:

(C_1) $\sum \beta_n = \infty$; $\lim \beta_n = 0$,

(C_2) $2 \sum (\delta_X^{-1}(\beta_n M_1) M_1 + \delta_X^{-1}(\beta_n M_2) M_2) < \infty$,

(C_3) $2 \max\{\delta_X^{-1}(\beta_n M_1) M_1, \delta_{X^*}^{-1}(\beta_n M_2) M_2\} \leq \gamma_0$, for some $M_1 > 0$, M_2, $\gamma_0 > 0$.

(C_4) $\gamma_0 = \min\{1, \frac{\Phi(\mu)}{2M_1}, \frac{\Phi(\mu)}{2M_2}\}$, δ_X is the modulus of convexity (*see e.g., Chidume [45], p.* 5, 6).

We now prove the following theorem.

Theorem 7.11. *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let $F : D(F) = X \to X^*$ and $K : D(K) = X^* \to X$ be generalized Φ_1-strongly monotone and generalized Φ_2-strongly monotone maps, respectively, where $D(F)$ and $D(K)$*

denote the domains of F and K, respectively, and such that the equation (7.23) has a solution. For arbitrary $(u_1, v_1) \in X \times X^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ by

$$u_{n+1} = J^{-1}\Big(Ju_n - \beta_n(Fu_n - v_n)\Big),\ n \geq 1; \quad v_{n+1} = J_*^{-1}\Big(J_*v_n - \beta_n(Kv_n + u_n)\Big),\ n \geq 1.$$

Assume that the sequence $\{\beta_n\} \subset (0,1)$ satisfies conditions C_1, C_2 and C_3 of Theorem 7.4. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of the equation $u + KFu = 0$ and $v^* = Fu^*$.

Proof. Set $E = X \times X^*$ and $A : E \to E^*$ by $A([u,v]) = [Fu - v, Kv + u]$. Then, by Lemma 7.9, A is a generalized Φ-strongly monotone map. Hence, by Theorem 7.4 and Remark 7.10, the result is immediate. □

7.0.4 Application to variational inequality problems

Let X be a real normed space with dual space X^*. Let $A : K \subset X \to X^*$ be a nonlinear map. The *classical variational inequality problem* is the following:

$$\text{find } u \in K \text{ such that } \quad \langle u - v, Au\rangle \geq 0,\ \forall\ v \in K. \tag{7.24}$$

The set of solutions of problem (7.24) is denoted by $VI(A, K)$.

Theorem 7.12. *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*, and C be a nonempty closed and convex subset of X. Let $A : D(A) = X \to X^*$ be a generalized-Φ-strongly monotone map, where $D(A)$ is the domain of A. Let $T_i : C \to X$, $i = 1, 2, \cdots, N$ be a finite family of quasi-ϕ-nonexpansive maps such that $P := \cap_{i=1}^N F(T_i) \neq \emptyset$. For arbitrary $v_1 \in X$, define the sequence $\{v_n\}$ generated by*

$$v_{n+1} = J^{-1}\Big(J(T_{[n]}v_n) - \beta_n A(T_{[n]}v_n)\Big),\ n \geq 1, \quad \textit{where } T_{[n]} := T_n \quad \textit{mod} \quad N. \tag{7.25}$$

Assume that $VI(A, P) \neq \emptyset$, and the sequence $\{\beta_n\} \subset (0,1)$ satisfies conditions C_1, C_2 and C_3 of Theorem 7.4. Then, the sequence $\{v_n\}$ converges strongly to $v^* \in VI(A, P)$.

Proof. The proof is in two steps. Here, ψ denotes the Alber's functional.

Step 1. We show that the sequence $\{v_n\}$ is bounded.

Let $v^* \in G^{-1}(0)$. Let $\mu > 0$ be arbitrary but fixed. Then, there exists $r > 0$ such that

$$r > \max\{4\mu^2 + ||v^*||^2, \psi(v^*, v_1)\}. \tag{7.26}$$

Define $B = \{v \in X : \psi(v^*, v) \leq r\}$. It suffices to show that $\{\psi(v^*, v_n)\}$ is bounded for each $n \in \mathbb{N}$. We proceed by induction. For $n = 1$, by construction, $\psi(v^*, v_1) \leq r$. Assume that $\psi(v^*, v_n) \leq r$ for some $n \geq 1$. Applying the definition of the map ψ, we have that $||v_n|| \leq ||v^*|| + \sqrt{r}$. Now, we show that $\psi(v^*, v_{n+1}) \leq r$. Suppose not, i.e., suppose $\psi(v^*, v_{n+1}) > r$.

By Lemma 7.3, A is quasi-bounded. Thus, there exists $M > 0$ such that $||Av|| \leq M,\ \ \forall\ v \in B$.

Define $\gamma_0 := \min\{1, \frac{\Phi(\mu)}{M}, \frac{\mu}{M}\}$. Using Lemma 2.8, we compute as follows:

$$\begin{aligned}
\psi(v^*, v_{n+1}) &= V(v^*, J(T_{[n]}v_n) - \beta_n A(T_{[n]}v_n)) \\
&\le V(v^*, J(T_{[n]}v_n)) - 2\beta_n\langle J^{-1}(J(T_{[n]}v_n) - \beta_n A(T_{[n]}v_n)) - v^*, A(T_{[n]}v_n)\rangle \\
&= \psi(v^*, T_{[n]}v_n) - 2\beta_n\langle T_{[n]}v_n - v^*, AT_{[n]}v_n\rangle - 2\beta_n\langle v_{n+1} - T_{[n]}v_n, AT_{[n]}v_n\rangle \\
&\le \psi(v^*, v_n) - 2\beta_n\langle T_{[n]}v_n - v^*, AT_{[n]}v_n - Av^*\rangle - 2\beta_n\langle T_{[n]}v_n - v^*, Av^*\rangle \\
&\quad -2\beta_n\langle v_{n+1} - T_{[n]}v_n, A(T_{[n]}v_n)\rangle \\
&\le \psi(v^*, v_n) - 2\beta_n\langle T_{[n]}v_n - v^*, AT_{[n]}v_n - Av^*\rangle - 2\beta_n\langle v_{n+1} - T_{[n]}v_n, A(T_{[n]}v_n)\rangle.
\end{aligned} \tag{7.27}$$

Using the fact that A is a generalized Φ-strongly monotone map and Lemma 2.11, it follows from inequality (7.27) that

$$\begin{aligned}
\psi(v^*, v_{n+1}) &\le \psi(v^*, v_n) - 2\beta_n\Phi(||T_{[n]}v_n - v^*||) + 2\beta_n\delta_X^{-1}(4RL\beta_n||AT_{[n]}v_n||)||AT_{[n]}v_n|| \\
&\le \psi(v^*, v_n) - 2\beta_n\Phi(||v_n - v^*||) + 2\beta_n\delta_X^{-1}(\beta_n M)M. \qquad (7.28)
\end{aligned}$$

But from recursion formula (7.25), we have that

$$||Jv_{n+1} - JT_{[n]}v_n|| = \beta_n||Av_n|| \le \beta_n M. \tag{7.29}$$

Applying Lemma 2.11 and inequality (7.29), we have that

$$||v_{n+1} - T_{[n]}v_n|| = ||J^{-1}(Jv_{n+1}) - J^{-1}(JT_{[n]}v_n)|| \le 2\delta_X^{-1}(\beta_n M). \tag{7.30}$$

Thus, from inequality (7.30), we obtain that

$$||T_{[n]}v_n - v^*|| \ge ||v_{n+1} - v^*|| - 2\delta_X^{-1}(\beta_n M). \tag{7.31}$$

From Lemma 2.9, we have that

$$r < \psi(v^*, v_{n+1}) \le ||v_{n+1} - v^*||^2 + ||v^*||^2. \tag{7.32}$$

Using inequality (7.26) , we have that

$$4\mu^2 + ||v^*||^2 - ||v^*||^2 < r - ||v^*||^2 \le ||v_{n+1} - v^*||^2.$$

Hence,

$$2\mu \le ||v_{n+1} - v^*||. \tag{7.33}$$

From inequalities (7.31), (7.33) and definition of γ_0, we have that

$$||T_{[n]}v_n - v^*|| \ge 2\mu - 2\delta_X^{-1}(\beta_n M) \ge 2\mu - \mu = \mu. \tag{7.34}$$

Since Φ is strictly increasing, we have that

$$\Phi(||T_{[n]}v_n - v^*||) \ge \Phi(\mu). \tag{7.35}$$

From inequality (7.28) and definition of γ_0, we have that

$$r < \psi(v^*, v_{n+1}) \le \psi(v^*, v_n) - 2\beta_n\Phi(\mu) + 2\beta_n\delta_X^{-1}(\beta_n M)M \tag{7.36}$$

$$\le r - 2\beta_n\Phi(\mu) + \beta_n\Phi(\mu) < r. \tag{7.37}$$

This is a contradiction. Hence, $\{\psi(v^*, v_n)\}$ is bounded. Consequently, $\{v_n\}$ is bounded. The remaining part of the proof follows from the proof of Theorem 7.4. □

7.0.5 Examples

Example 7.13. Let $X = l_p,\ 1 < p < 2$, and let $A : l_p \to l_p^*$ be a map defined by

$$Au = Ju,\ \forall\ u \in l_p,\ u = (u_1, u_2, u_3, \cdots),$$

where J is the normalized duality map on X.
Then,

$$\begin{aligned}\langle u - v, Au - Av\rangle &= \langle u - v, Ju - Jv\rangle \\ &\geq (p-1)||u - v||^2,\ \forall\ u, v \in X.\end{aligned}$$

Hence, A is generalized-Φ-strongly monotone map with $\Phi(t) = (p-1)t^2$, (see e.g., Chidume [45], p. 55).

Example 7.14. Let $X = l_p,\ 2 \leq p < \infty$, and let $A : l_p \to l_p^*$ be a map defined by

$$Au = \frac{1}{2}J_p u,\ \forall\ u \in l_p,\ u = (u_1, u_2, u_3, \cdots).$$

Then,

$$\begin{aligned}\langle u - v, Au - Av\rangle &= \frac{1}{2}\langle u - v, J_p u - J_p v\rangle \\ &\geq p^{-1}c_p||u - v||^p,\ \forall\ u, v \in X,\ c_p > 0.\end{aligned}$$

Hence, A is a generalized-Φ-strongly monotone map with $\Phi(t) = p^{-1}c_p t^p$, (see e.g., Chidume [45], p. 54).

7.0.6 Discussion

Our theorem is a significant improvement of the results of Diop *et al.* [78], Chidume and Bello [51], Chidume [48], and Chidume *et al.*, [54, 65] in the following sense:

1. Theorems 7.4 and 7.11 are proved in a more general real Banach space which contains the space of 2-uniformly convex space and L_P spaces, $1 < p < \infty$.

2. The class of *strongly monotone maps* studied in Diop *et al.* [78], Chidume and Bello [51] is extended to the more general class of *generalized-Φ-strongly monotone maps* in Theorems 7.4 and 7.11, respectively.

3. The requirement that the maps, A, K and F be *bounded* which is assumed in the theorems of Diop *et al.* [78], Chidume and Bello [51], respectively; and in the theorem of Chidume *et al.*, [54, 65] and Chidume [48] is dispensed with in our theorems.

7.0.7 Numerical experiment

In this section, we present numerical examples to illustrate the convergence of the sequence generated by our algorithm.

Example 7.15. In Theorem 7.4, set $X = \mathbb{R}^2$ so that $X^* = \mathbb{R}^2$,

$$Av = \begin{pmatrix} 5 & -5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then, it is easy to see that A is a generalized-Φ-strongly monotone map and the vector $v^* = (0,0)$ is the unique solution of the equation $Av = 0$. Take $\beta_n = \frac{1}{n+1}$, $n = 1, 2, \cdots$, as our parameter in Theorem 7.4. With this, we now give the following algorithm which is a specialized version of algorithm 7.1.

Algorithm.
Step 0: Choose any $v_1 \in \mathbb{R}^2$ and set a tolerance $\epsilon_0 > 0$. Let $k = 1$ and set maximum number of iterations, n.
Step 1: If $\|v_k\| \leq \epsilon_0$ or $k > n$, STOP. Otherwise, set $\beta_n = \frac{1}{k+1}$.
Step 2: Compute

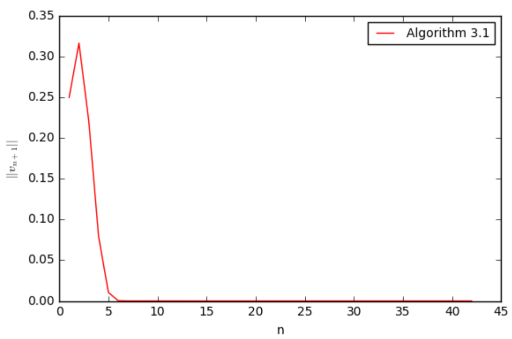
$$v_{k+1} = v_k - \beta_k A v_k$$

Step 3: Set $k = k + 1$ and go to Step 1.

The following table gives our test results using 10^{-6} tolerance.

initial points	Num. of iter	Approx. solution
(1,0)	88	9.6598×10^{-7}
(0,1)	95	9.3690×10^{-7}
(2,1)	103	9.9756×10^{-7}
(1,4)	120	9.5080×10^{-7}
$(\frac{1}{2}, \frac{1}{2})$	86	9.3020×10^{-7}
$(1, \frac{1}{2})$	92	9.6662×10^{-7}

The numerical result for the initial point $(1, \frac{1}{2})$ is sketched below where the y-axis represents the values of $\|v_{n+1} - 0\|$ while the x-axis represents the number of iterations n.



Convergence of the sequence $\{v_n\}$ with initial point $(1, \frac{1}{2})$

Example 7.16. In Theorem 7.11, set $X = \mathbb{R}^2$ so that $X^* = \mathbb{R}^2$,

$$Fu = \begin{pmatrix} 3 & -1 \\ 1 & 8 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad Kv = \begin{pmatrix} 7 & 2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then, it is easy to see that F and K are generalized-Φ-strongly monotone maps and the vector $u^* = (0,0)$ is the unique solution of the equation $u + KFu = 0$. Take $\beta_n = \frac{1}{(n+1)}$, $n = 1, 2, \cdots$, as our parameters in Theorem 7.11. With this, we now give the following algorithm which is a specialized version of algorithm 7.11

Algorithm.

Step 0: Choose any $u_1, v_1 \in \mathbb{R}^2$ and set a tolerance $\epsilon_0 > 0$. Let $k = 1$ and set maximum number of iterations, n.

Step 1: If $\|u_k\| \leq \epsilon_0$ or $k > n$, STOP. Otherwise, set $\beta_k = \frac{1}{(k+1)}$.

Step 2: Compute

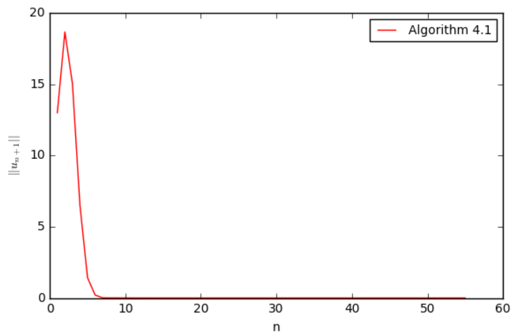
$$\begin{cases} u_{k+1} = u_k - \beta_k(Fu_k - v_k) \\ v_{k+1} = v_k - \beta_k(Kv_k + u_k) \end{cases}$$

Step 3: Set $k = k + 1$ and go to Step 1.

The following table gives our test results using 10^{-6} tolerance.

initial points	Num. of iter	Approx. sol. ($\|u_{n+1}\|$)
(1,0),(0,1)	45	9.7064×10^{-7}
(1,1),(2,3)	49	9.4440×10^{-7}
(2,3),(1,1)	49	9.9188×10^{-7}
$(\frac{1}{2}, \frac{1}{2}),(\frac{1}{2}, \frac{1}{2})$	36	9.6055×10^{-7}
$(\frac{1}{2},1),(\frac{1}{2},2)$	38	9.4539×10^{-7}
(3,5), (2,1)	55	9.7373×10^{-7}

The numerical result for the initial point (3,5), (2,1) is sketched below where the y-axis represents the values of $\|u_{n+1} - 0\|$ while the x-axis represents the number of iterations n.



Convergence of the sequence $\{u_n\}$ with initial point (3,5), (2,1)

Example 7.17. In Theorem 7.12, set $X = \mathbb{R}^2$ so that $X^* = \mathbb{R}^2$,

$$Av = \begin{pmatrix} 5 & -5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad Tv = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then, it is easy to see that A is a generalized-Φ-strongly monotone map, T is quasi-Φ-nonexpansive and the vector $v^* = (0, 0)$ is the common solution. We take $\beta_n = \frac{1}{n+1}$, $n = 1, 2, \cdots$, as our parameter in Theorem 7.12. With this, we now present the following algorithm:

Algorithm.

Step 0: Choose any $v_1 \in \mathbb{R}^2$ and set a tolerance $\epsilon_0 > 0$. Let $k = 1$ and set maximum number of iterations, n.
Step 1: If $||v_k|| \leq \epsilon_0$ or $k > n$, STOP. Otherwise, set $\beta_n = \frac{1}{k+1}$.
Step 2: Compute

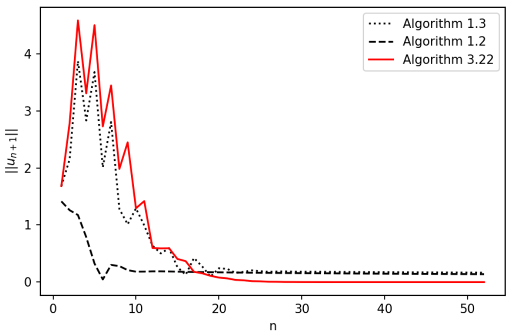
$$v_{k+1} = T_{[k]}v_k - \beta_k A(T_{[k]}v_k)$$

Step 3: Set $k = k + 1$ and go to Step 1.

The following table gives our test results using 10^{-6} tolerance.

initial points	Num. of iter	Approx. solution
(1,0)	24	8.2377×10^{-7}
(1,1)	24	9.6812×10^{-7}
(2,3)	25	9.6103×10^{-7}
(-2,1)	25	9.3095×10^{-7}
$(\frac{1}{2}, \frac{1}{2})$	22	7.1434×10^{-7}
$(-\frac{1}{10}, -1)$	92	9.6662×10^{-7}
(5,8)	27	8.3144×10^{-7}

The numerical result for the initial point (5,8) is sketched below where the y-axis represents the values of $\|v_{n+1} - \mathbf{0}\|$ while the x-axis represents the number of iterations n.



Convergence of the sequence $\{v_n\}$ with initial point (5,8)

The result of this chapter is published in FPTA (2019) https://doi.org/10.1186/s13663-019-0660-9.

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