C_o−SEMIGROUPS OF CONTRACTION ON BANACH SPACES AND APPLICATIONS

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CERTIFICATION

This is to certify that **ISEDOWO JOSHUA WALE** of Matric Number 40943 compiled this thesis based on his eighteen months programme of Masters of Science in Pure and Applied Mathematics at African University of Science and Technology Abuja.

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ABSTRACT

Let X be a Banach space and $A : D(A) \subset X \longrightarrow X$ be an unbounded linear operator on X. We study the concept of C_0-semigroup of contraction on arbitrary Banach space X and give the two characterizations of A called infinitesimal generator of C_0-semigroup on X namely, Hille-Yosida and Lumer Phillips characterizations. In the later part, we apply the approach of C_0-semigroups to some partial differential equations with boundary conditions.

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DEDICATION

This thesis is dedicated to the Holy Spirit, my family (present and future), commencement of a future career in Mathematics and world of Mathematics at large.

Contents

Chapter 1

INTRODUCTION

Semigroup theory is an important concept in the study of evolution equations which are of the form

$$\begin{aligned} \frac{du(t)}{dt} + Au(t) &= 0, t \geq 0 \\ & u(0) = u_0 \in X \end{aligned} \tag{1.0.1}$$

where X is a Banach space with A an unbounded linear operator on X and $D(A)$ (domain of A) dense in X. In solving 1.0.1, we consider a family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X called semigroup with the following properties

1. $T(0) = I$ (I :Identity operator on X)
2. $T(t+s) = T(t)T(s) \ \forall \ t, s \geq 0,$
3. $\lim_{t\to 0^+} T(t)x = x$ for all $x \in X$.

Under proper assumptions on A, we write the solution $u(t) = T(t)x_0$, $x_0 \in X$. Since A is a linear operator, we define

$$D(A) = \{x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \ \ exists \ \ in \ \ X\},$$

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}|_{t=0} \ \ for \ \ all \ \ x \in D(A).$$

A is called the infinitesimal generator of $(T(t))_{t\geq 0}$.

In the first chapter, C_0-semigroups are introduced with some properties and examples. The next two chapters discuss two characterizations of infinitesimal generators of C_0-semigroups; Hille-Yosida first and then Lumer-Phillips. The fourth chapter contains the application of C_0-semigroups to some partial differential equations on suitable defined spaces with boundary conditions.

Chapter 2

C_0-SEMIGROUPS

In this chapter we start with the introduction of strongly continuous semigroup called C_0-semigroup on a Banach space. We state some definitions and properties with their respective proofs.

Definition 2.0.1. A semigroup $T(t)$, $0 \leq t \leq \infty$ on X is called a strongly continuous semigroup if $\lim\limits_{t \to 0^+} T(t)x = x$ for all $x \in X$. It is called a semigroup of class C_0. It is hereby necessary to give some theorems about this class of semigroup.

Example is $T(t) = e^{tA}, A \in \mathcal{L}(X)$.

Claim: $(T(t))_{t \geq 0}$ is a $C_0 - semigroup$, we give the following lemma

Lemma 2.0.2. *Let B , $C \in \mathcal{L}(X)$ such that $BC = CB$, $\Rightarrow e^{B+C} = e^B \cdot e^C$*

- $T(0) = I_d$ $(Identity)$
- By **lemma 2.0.2** , $T(t+s) = T(t) \circ T(s)$ for all $s, t \geq 0$
- For the continuity, $e^{tA} = \sum\limits_{n \geq 0} \frac{t^n A^n}{n!}$, the series $\sum\limits_{n \geq 0} \frac{t^n A^n}{n!}$ is uniformly convergent on $\mathbb{R}$,

$$\begin{aligned} \Rightarrow \lim_{t \to 0} e^{tA} &= \sum_{n \geq 0} \lim_{t \to 0} \frac{t^n A^n}{n!} = I_d \\ \Rightarrow \|e^{tA} - I_d\| &\underset{n \to \infty}{\longrightarrow} 0. \square \end{aligned}$$

Theorem 2.0.3. *Let $(T(t))_{t \geq 0}$ be a C_o-semigroup then , there exist constants $\alpha \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\alpha t} \quad for \quad all \quad t \quad \in \mathbb{R}^+.$$

Proof:

We show that there is $\eta \in \;]0, 1]$ such that

$$\underset{t \in [0, \eta]}{Sup} ||T(t)|| < \infty.$$

Assume by contradiction, that for all $\eta = \frac{1}{n} \in]0,1]$ *with* $n \in \mathbb{N}$*, there exist* $t_n \in]0, \frac{1}{n}]$ *such that* $||T(t_n)|| = +\infty$. *By uniform boundedness principle, there exist* $x \in X$ *such that* $\underset{n\geq 0}{Sup}||T(t_n)x|| = \infty$. *And* $t \rightarrow T(t)x$ *is continuous at* 0*, then there exist* $(t_{n_k})_{k\geq 1}$ *such that* $\|T(t_{n_k})x\| \rightarrow 0$ *as* $t_{n_k} \rightarrow 0$. *Hence we have contradiction. Thus there exist* $\eta \in]0,1]$ *such that* $\underset{t\in[0,\eta]}{Sup}||T(t)|| < \infty$.

Let $M = \underset{t\in[0,\eta]}{Sup}||T(t)||$*, since* $T(0) = I$, $\|T(0)\| = 1$, $M \geq 1$. *Let* $\alpha = \eta^{-1} logM \geq 0$. $t \geq 0$, *we have* $t = n\eta + \delta$ *where* $0 \leq \delta < \eta$.

Then,

$$\| T(t) \| = \| T(n\eta + \delta) \| = \| T(\delta)T(\eta)^n \| \leq \| T(\delta) \| \| T(\eta) \|^n$$

$$\Rightarrow \| T(t) \| \leq M \cdot M^{n+1}.$$

Since $n = \frac{t-\delta}{\eta} \leq \frac{t}{\eta}$

$$\Rightarrow \| T(t) \| \leq M \cdot M^n \leq M \cdot M^{\frac{t}{\eta}} = Me^{\alpha t}.$$

Corollary 2.0.4. *If* $(T(t))_{t\geq 0}$ *is a* C_0 *semigroup, then for all* $x \in X$, $t \rightarrow T(t)x$ *is a continuous function on* $\mathbb{R}^+$.

Proof:

Let t , $h \geq 0$,

$$\begin{aligned} \| T(t+h)x - T(t)x \| &\leq \| T(t) \| \| T(h)x - x \| \\ &\leq Me^{\alpha t} \| T(h)x - x \| \rightarrow 0 \ \ as \ \ h \rightarrow 0 \end{aligned}$$

and also,

$$\begin{aligned} \| T(t-h)x - T(t)x \| &\leq \| T(t-h) \| \| x - T(h)x \| \\ &\leq Me^{\alpha t} \| x - T(h)x \| \rightarrow 0 \ \ as \ \ h \rightarrow 0 \end{aligned}$$

this shows that $t \rightarrow T(t)x$ *is continuous.*

Theorem 2.0.5. *Let* $(T(t))_{t\geq 0}$ *be a* C_0 *semigroup and* A *its infintesimal generator. Then*

i.

$$for \ \ all \ \ x \in X, \lim_{h\rightarrow 0} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x.$$

ii.

$$for \ \ all \ \ x \in X, \int_0^t T(s)xds \in D(A), A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

iii. $for \ \ all \ \ x \in D(A), T(t)x \in D(A)$ *and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

iv. *for all* $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(r)Axdr = \int_s^t AT(r)xdr.$$

Proof:

i. This is trivial from the continuity of $t \to T(t)x$ for all $x \in X$.

ii. Let $x \in X$, $h > 0$ and $\int_0^t T(s)xds \in D(A)$,

$$\begin{aligned}
\frac{T(h) - I}{h} \int_0^t T(s)xds &= \frac{1}{h} \int_0^t (T(h)T(s)x - T(s)x)\, ds \;\; (limit \;\; withheld) \\
&= \frac{1}{h} \int_0^t (T(s+h)x - T(s)x)\, ds \\
&= \frac{1}{h} \int_h^{t+h} T(s)xds - \frac{1}{h} \int_0^t T(s)xds \\
&= \frac{1}{h} \int_t^{t+h} T(s)xds - \frac{1}{h} \int_0^h T(s)xds \underset{h \to 0}{\longrightarrow} T(t)x - x.
\end{aligned}$$

But $\lim\limits_{h\to 0} \frac{T(h)-I}{h} \int_0^t T(s)xds = A\left(\int_0^t T(s)xds\right)$, by uniqueness of limits,

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

iii. Let $x \in D(A)$, $h > 0$,

$$\frac{T(h) - I}{h} T(t)x = T(x) \left(\frac{T(h) - I}{h}\right) x \underset{h \to 0}{\longrightarrow} T(t)Ax.$$

Therefore, $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$

$$\Rightarrow \frac{d^+}{dt} T(t)x = AT(t)x = T(t)Ax \quad (Right \quad Derivative).$$

Since $t \to T(t)x$ is continuous, then $t \to T(t)x$ for all $x \in X$ is a C^1 function on $\mathbb{R}^+$and $\frac{d}{dt}T(t)x = AT(t)x$.

iv. Integrating from s to t for all $x \in D(A)$,

$$\int_s^t T(r)Axdr = \int_s^t \frac{d}{dr} T(r)xdr = [T(r)x]_{r=s}^{r=t} = T(t)x - T(s)x.$$

Corollary 2.0.6. *If A is the infinitesimal generator of a C_0 semigroup $T(t)$, then $D(A)$ is dense in X and A is a closed linear operator.*

Proof:

Let $x \in X$, set $x_n = n\int_0^{\frac{1}{n}} T(s)xds$ from Theorem 2.0.4 , $x_n \in \overline{D(A)}\ \forall\ n > 0$. Also let $x_n \underset{n\to\infty}{\longrightarrow} x \in X$, therefore $\overline{D(A)} = X$,

Let $(x, y) \in \overline{G(A)}$ where $G(A)$ is the graph of A, then there exist $(x_n)_{n\geq 1} \subset D(A)$ such that $(x_n, Ax_n) \to (x, y)$.

By Theorem 2.0.4, we have

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n ds.$$

<u>Claim:</u> $\int_0^t T(s)Ax_n ds \underset{n\to\infty}{\longrightarrow} \int_0^t T(s)yds$ uniformly on bounded interval .

$$\begin{aligned}
\|\int_0^t T(s)Ax_n ds - \int_0^t T(s)yds\| &\leq \int_0^t \|T(s)(Ax_n - y)\|ds \\
&\leq \int_0^t Ne^{\alpha s}\|(Ax_n - y)\|ds \\
&\leq Ne^{\alpha t}t\|Ax_n - y)\|.
\end{aligned}$$

Since $Ax_n \to y$, then

$$\int_0^t T(s)Ax_n ds \underset{n\to\infty}{\longrightarrow} \int_0^t T(s)yds.$$

Our claim is true. But $T(t)x_n - x_n \underset{n\to\infty}{\longrightarrow} T(t)x - x$, from uniqueness of limits,

$$T(t)x - x = \int_0^t T(s)yds. \tag{2.0.1}$$

Dividing 2.0.1 by $t > 0$ and letting $t \to 0$, $\Rightarrow x \in D(A)$ and $Ax = y$ from Theorem 2.0.4 □

Theorem 2.0.7. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be C_o semigroups of bounded linear operators with infinitesimal generators A and B respectively. If $A = B$, then $T(t)x = S(t)x$ for all $t > 0$.*

Proof:

$(T(t))_{t\geq 0}$ is a semigroup and A its infinitesimal generator

$$for\ \ all\ \ x \in D(A) = D(B), Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}.$$

$(S(t))_{t\geq 0}$ is a semigroup and B its infinitesimal generator

$$for\ \ all\ \ x \in D(B) = D(A), Bx = \lim_{t\to 0^+} \frac{S(t)x - x}{t} = \frac{d^+S(t)x}{dt}.$$

If $Ax = Bx$ for $x \in D(A) = D(B)$, then

$$T(t)x - x = \int_0^t T(s)Axds = \int_0^t T(s)Bxds = S(t)x - x.$$

$$\Rightarrow \quad \frac{d^+T(t)x}{dt} = \frac{d^+S(t)x}{dt} \quad for\ all\ x \in D(A) = D(B).$$

Since $\overline{D(A)} = X$ *from Corollary 2.0.5 and due to the continuity of* $(T(t))_{t\geq 0}$ *and* $(S(t))_{t\geq 0}$*,we have* $T(t)x = S(t)x$*.*

Let A *be the infinitesimal generator of the* C_0 *semigroup* $(T(t))_{t\geq 0}$*. Then* $\bigcap_{n\geq 1} D(A^n)$*is dense in* X*.*

Proof:

Let $\mathfrak{D}$ *be the set of all infintely differentiable compact supported functions on* $\mathbb{R}^+$*.*

For all $x \in X$ *,* $\psi \in \mathfrak{D}$*,*

$$y = x(\psi) = \int_0^\infty \psi(s)T(s)xds$$

if $h > 0$*, then*

$$\begin{aligned}
\frac{T(h)-I}{h}y &= \frac{T(h)-I}{h}\left(\int_0^\infty \psi(s)T(s)xds\right) \\
&= \int_0^\infty \psi(s)\left(\frac{T(s+h)x - T(s)x}{h}\right) ds \\
\lim_{h\to 0}\frac{T(h)-I}{h}y &= \int_0^\infty \psi(s)\lim_{h\to 0}\left(\frac{T(s+h)x - T(s)x}{h}\right) ds \\
&= \int_0^\infty \psi(s)T'(s)xds \\
&= \langle T'(t)x, \psi(t)\rangle \\
&= -\langle T(t)x, \psi'(t)\rangle \\
&= \int_0^\infty \psi'(s)T(s)xds
\end{aligned}$$

Therefore

$$Ay = \int_0^\infty \psi'(s)T(s)xds.$$

$\psi \in \mathfrak{D} \Rightarrow \psi^{(\mathrm{n})} \in \mathfrak{D}$,

$$\Rightarrow A^n y = (-1)^n \int_0^\infty \psi^{(n)}(s)T(s)xds \forall n \geq 1$$

$$\Rightarrow y \in \bigcap_{n\geq 1} D(A^n).$$

Let $Y = \{x(\psi) : x \in X, \psi \in \mathfrak{D}\}$ *, then* $Y \subseteq \bigcap_{n\geq 1} D(A^n)$*.*

We need to prove that Y is dense in X. Assume by contradiction that Y is not densed in X, by Hann-Banach's theorem, $\exists$ a function $x^* \in X^*$, $x^* \neq 0$ such that $x^*(y) = 0\ \forall\ y \in Y$ and then

$$\int_0^\infty \psi(s)x^*T(s)xds = x^*\left(\int_0^\infty \psi(s)T(s)xds\right) = 0\ \ \forall\ \ x \in X, \psi \in \mathfrak{D}$$

$\Rightarrow s \to x^*\left(T(s)x\right)$ vanish identically on $[0, \infty[$. Thus, for $s = 0, x^*(x) = 0$. It holds for every $x \in X$ and $x^* = 0$ (Contradition).$\square$

2.1 More Examples of C_0-semigroups and their generators

1. .Let $X = C_1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}$ continuous such that $\forall\ M > 0, \exists\ c > 0 : |f(x)| \leq M\ \ \forall\ x \in \mathbb{R} \setminus [-c, c],\}$ endowed with the supremum norm

   let $V : \mathbb{R} \to \mathbb{R}$ continuous where V is endowed with the supremum norm.

   The operator $M_v f = V \cdot f$ with domain $D(M_v)$ where $D(M_v) = \{f \in C_1(\mathbb{R}) : V \cdot f \in C_1(R)\}$ and $(T_v(t))_{t\geq 0}$ is a semigroup where $T_v(t)f = e^{vt}f$.

   Claim : $(T_v(t))$ is a $C_0 - semigroup$

   - $T_v(0)f = e^0 f = f \Rightarrow T_v(0) = I_d$
   - for all $s, t \geq 0$,

   $$T_v(s+t)f = e^{v(s+t)}f = e^{vs}e^{vt}f = T_v(s)T_v(t)f.$$

   Also, $\lim\limits_{t\to 0^+} T_v(t)f = \lim\limits_{t\to 0^+} e^{vt}f = If = f \Rightarrow \lim\limits_{t\to 0^+} T_v(t)f = f$ for all $f \in C_1(\mathbb{R})$.$\square$

2. X is a Banach space of bounded uniformly continuous function on $\mathbb{R}^+$ with supremum norm. For all $f \in X$ define

   $$(T(t))f(s) = f(t+s)\ \ s, t \in \mathbb{R}^+$$

   Then $(T(t))_{t\geq 0}$ is a C_0-semigroup with

   $$Af = \frac{df}{dt}\ \ and\ \ D(A) = \{f \in X : \frac{df}{dt} \in X\}.$$

Proof: $X = BUC(\mathbb{R}^+, \mathbb{R})$ provided with the supremum norm

$$\begin{aligned} Let\ \ T(t):& X \to X \\ & f \to T(t)f \\ & where\ \ (T(t)f)(s) = f(t+s)\ \ t,s \geq 0. \end{aligned}$$

<u>Claim 1</u>: $(T(t))_{t\geq 0}$ is a $C_0-semigroup$ on X

i. $T(0) = Id$

ii. $(T(t+s)f)(\tau) = f(t+s+\tau)$

$$\begin{aligned} T(t)(T(s)f)(\tau) &= (T(s)f)(t+\tau) \\ &= f(t+s+\tau) \\ & \quad Hence\ \ T(t+s) = T(t)\circ T(s)\ \ for\ \ all\ \ t,s \geq 0. \end{aligned}$$

iii. For the continuity,

$$|T(t)f - f| = \underset{s\geq 0}{Sup}|f(t+s) - f(s)|.$$

Since f is uniformly continuous, then for all $\epsilon > 0\ \exists\ \delta > 0$ such that $|t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| < \epsilon$.

Take $t_1 = t+s$ and $t_2 = s$, then if

$$\begin{aligned} 0 < t < \delta &\Rightarrow for\ \ all\ \ s \geq 0, |f(t+s) - f(s)| < \epsilon \\ &\Rightarrow \underset{s\geq 0}{Sup}|f(t+s) - f(s)| \leq \epsilon \\ &\Rightarrow |T(t)f(s) - f(s)| \underset{t\to 0}{\longrightarrow} 0. \end{aligned}$$

It is worthnoting that not every $(T(t))_{t\geq 0}$ is given by an exponential function.

<u>Claim 2</u>: There does not exist $A \in \mathcal{L}(X)$ such that $T(t) = e^{tA}$. Assume that there exist $A \in \mathcal{L}(X)$ such that $T(t) = e^{tA} \Rightarrow \frac{d}{dt}T(t) = \frac{d}{dt}e^{tA} = Ae^{tA}$. To prove this claim, we give a lemma that will be used in the process of proving it.

Lemma: Let $\rho :]a,b[\longrightarrow X$ be continuous such that $D^+\rho$ (right derivative) exists and is continuous on $]a,b[$, then ρ is a $C^1 function$ on $]a,b[$.

Let $f \in X$ such that

$$\frac{T(t)f - f}{t} \underset{t \to 0^+}{\longrightarrow} Af \ \ on \ \ X.$$

Then for all $s \geq 0$,

$$\begin{aligned}
\frac{(T(t)f)(s) - f(s)}{t} &\underset{t \to 0^+}{\longrightarrow} (Af)(s) \\
\Rightarrow \frac{f(t+s) - f(s)}{t} &\underset{t \to 0^+}{\longrightarrow} (Af)(s).
\end{aligned}$$

But

$$\frac{f(t+s) - f(s)}{t} \underset{t \to 0^+}{\longrightarrow} D^+ f(s).$$

Hence, $(Af)(s) = (D^+ f)(s)$ (Uniqueness of limit). By Lemma, we get that f is a $C^1 function$ and $Af = f'$, then $D(A) \subseteq \{f \in C^1 \cap X : f' \in X\}$.

For the converse, let $f \in C^1 \cap X$ such that $f' \in X$,

$$\begin{aligned}
|\frac{T(t)f - f}{t} - f'| &= |\frac{f(t+s) - f(s)}{t} - f'(s)| \\
&= |\frac{f(t+s) - f(s) - tf'(s)}{t}|.
\end{aligned}$$

Let $\rho(t) = f(t+s) - tf'(s)$, by Mean-Value theorem,

$$|\rho(t) - \rho(0)| \leq t \underset{s \in [0,t]}{Sup} |\rho'(s)|.$$

And,

$$\begin{aligned}
|\rho(t) - \rho(0)| &= |f(t+s) - tf'(s) - f(s)| \\
&\leq t \underset{s \in [0,t]}{Sup} |f'(t+s) - f'(s)|
\end{aligned}$$

$$\underset{s \geq 0}{Sup} |\frac{f(t+s) - f(s) - tf'(s)}{t}| \leq \underset{s \geq 0}{Sup} \underset{s \in [0,t]}{Sup} |f'(t+s) - f'(s)| \underset{t \to 0}{\longrightarrow} 0 \ \ since \ \ f \ \ is \ \ uniformly \ \ continuous.$$

$\Rightarrow D(A) \subseteq \{f \in C^1 \cap X : f' \in X\}$ and the equality follows. This cannot be possible because $Af = f'$ is not bounded on X. Therefore there does not exist $A \in \mathcal{L}(X)$ such that $T(t) = e^{tA}$.

Claim 3: Let A be defined by $D(A) \subseteq \{f \in C^1 \cap X : f' \in X\}, Af = f'$ then, A is the infinitesimal generator of $(T(t))_{t\geq 0}$. Infact, let $f \in D(A)$, there exist $g \in X$ such that

$$\lim_{t\to 0^+} \frac{T(t)f - f}{t} = Af = g$$

$$\Rightarrow \frac{(T(t)f)(s) - f(s)}{t} \underset{t\to 0^+}{\longrightarrow} (Af)(s) = g(s).$$

But, $\frac{f(t+s)-f(s)}{t} \underset{t\to 0^+}{\longrightarrow} \frac{d^+f(s)}{ds}$, hence $\frac{d^+f(s)}{ds} = g(s)$ for all $s \geq 0$, $\Rightarrow f$ is a $C^1 function$ and $f' = g$.$\square$

Chapter 3

HILLE-YOSIDA THEOREM

The next natural question is to see if an unbounded operator on X is the generator of a C_0-semigroup. The following theorem is central in the semigroup theory hereby providing a clear answer to this question. This characterizes the infinitesimal generator of a C_0 semigroup.

Definition 3.0.1. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup, since $||T(t)|| \leq Me^{\alpha t}$, $M \geq 1$, $\alpha \geq 0$

i. if $\alpha = 0$, $T(t)$ is uniformly bounded.

ii. if $\alpha = 0$ and $M = 1$, $T(t)$ is called C_0-semigroup of contractions.

If A is a linear (not necessarily bounded) operator in X, the resolvent set of A, denoted by $\rho(A)$ defined as $\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \ \ is \ \ invertible\}$. $R(\lambda : A) = (\lambda I - A)^{-1}, \lambda \in \rho(A)$ is called the **resolvent of** A**.**

Theorem 3.0.2. *A linear (unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geq 0}$, if and only if*

i. *A is closed and $\overline{D(A)} = X$.*

ii. *$\rho(A) \supseteq]0, \infty[$ and $\forall \ \lambda > 0, \|R(\lambda : A)\| \leq \frac{1}{\lambda}$.*

Proof:

Assume A is the infinitesimal generator of a C_0 semigroup ,by Corollary 2.0.4, it is closed and $\overline{D(A)} = X$. For $\lambda > 0$ and $x \in X$, let $R(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)xdt$.

Since $t \to T(t)x$ is continuous and uniformly bounded, $R(\lambda)$ defines a bounded linear operator. That is ,

$$\begin{aligned} \|R(\lambda)x\| &\leq \int_0^\infty e^{-\lambda t}\|T(t)x\|dt \\ &\leq \|x\|\int_0^\infty e^{-\lambda t}dt \ \ since \ \ (\|T(t)x\| \leq \|x\|) \\ &= \frac{1}{\lambda}\|x\|. \end{aligned}$$

For any $h > 0$,

$$\begin{aligned}
\frac{T(h)-I}{h}R(\lambda)x &= \frac{T(h)-I}{h}\int_0^\infty e^{-\lambda t}T(t)xdt \\
&= \frac{1}{h}\int_0^\infty e^{-\lambda t}\left(T(t+h)x - T(t)x\right)dt \\
&= \frac{1}{h}\int_0^\infty e^{-\lambda t}T(t+h)xdt - \frac{1}{h}\int_0^\infty e^{-\lambda t}T(t)xdt \\
&= \frac{1}{h}\int_h^\infty e^{-\lambda(t-h)}T(t)xdt - \frac{1}{h}\int_0^\infty e^{-\lambda t}T(t)xdt \\
&= \frac{1}{h}\int_0^\infty e^{-\lambda(t-h)}T(t)xdt - \frac{1}{h}\int_0^h e^{-\lambda(t-h)}T(t)xdt - \frac{1}{h}\int_0^\infty e^{-\lambda t}T(t)xdt \\
&= \frac{1}{h}\int_0^\infty \left(e^{-\lambda(t-h)} - e^{-\lambda t}\right)T(t)xdt - \frac{1}{h}\int_0^h e^{-\lambda(t-h)}T(t)xdt \\
&= \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}T(t)xdt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}T(t)xdt.
\end{aligned}$$

Taking limit $h \downarrow 0$, R.H.S $= \lambda R(\lambda)x - T(0)x = \lambda R(\lambda)x - x$,

$\Rightarrow$ for all $x \in X$ and $\lambda > 0$, $R(\lambda)x \in D(A)$ and

$$AR(\lambda) = \lambda R(\lambda) - I$$

$$\Rightarrow \lambda R(\lambda) - AR(\lambda) = I$$

$$\Rightarrow (\lambda - AI)R(\lambda) = I.$$

Also , for all $x \in D(A)$, we have

$$R(\lambda)Ax = \int_0^\infty e^{-\lambda t}T(t)Axdt = \int_0^\infty e^{-\lambda t}AT(t)xdt$$

$$= A\left(\int_0^\infty e^{-\lambda t}T(t)xdt\right) = AR(\lambda)x.$$

Therefore , since A is closed , $R(\lambda)(\lambda - AI)x = x$ for all $x \in D(A)$.

Thus, $R(\lambda)$ is the inverse of $\lambda I - A$ $\forall \lambda > 0$, hence conditions **i.** and **ii.** are sufficient for A to be the infinitesimal generator.

Lemma 3.0.3. *Let A satisfy conditions i. and ii. and $R(\lambda : A) = (\lambda I - A)^{-1}$. Then*

$$\lim_{\lambda\to\infty} \lambda R(\lambda : A)x = x \;\; for \;\; all \;\; x \in X.$$

Proof:

Let $x \in D(A)$,

$$\begin{aligned} \|\lambda R(\lambda : A)x - x\| &= \|AR(\lambda : A)x\| \\ &= \|R(\lambda : A)Ax\| \\ &\leq \|R(\lambda : A)\|\|Ax\| \\ &\leq \frac{1}{\lambda}\|Ax\| \underset{\lambda\to\infty}{\longrightarrow} 0. \end{aligned}$$

We have that $\lambda R(\lambda : A)x \longrightarrow x$ *for* $x \in D(A)$,

Let $\epsilon > 0, y \in X$, *by density of* $D(A)$, $\exists$ $x \in D(A)$ *such that* $\|x - y\| < \epsilon$.

Now,

$$\begin{aligned} \|\lambda R(\lambda : A)y - y\| &\leq \|\lambda R(\lambda : A)y - \lambda R(\lambda : A)x\| + \|\lambda R(\lambda : A)x - x\| + \|x - y\| \\ &\leq \|y - x\| + \|\lambda R(\lambda : A)x - x\| + \|x - y\| \\ &\leq 2\epsilon + \|\lambda R(\lambda : A)x - x\|. \end{aligned}$$

But $\|\lambda R(\lambda : A)x - x\| < \epsilon$ *for* $\lambda > \lambda_0$ $(x \in D(A))$, *then we have*

$$\|\lambda R(\lambda : A)y - y\| \leq 3\epsilon \ \ for \ \ \lambda > \lambda_0$$

which means that $\lambda R(\lambda : A)y \underset{\lambda\to\infty}{\longrightarrow} y.\square$

3.1 Yosida Approximation

Now, define **Yosida Approximation** of A by

$$A_\lambda = \lambda AR(\lambda : A) = \lambda^2 R(\lambda : A) - \lambda I \ \ for \ \ all \ \ \lambda > 0.$$

Also,

$$\lim_{\lambda\to\infty} A_\lambda x = \lim_{\lambda\to\infty} \lambda AR(\lambda : A)x = \lim_{\lambda\to\infty} \lambda R(\lambda : A)Ax = Ax \ \ for \ \ all \ \ x \in D(A).$$

Lemma 3.1.1. *If* A_λ *is the Yosida Approximation of* A, *then* A_λ *is the infinitesimal generator of a uniformly continuous semigroup of contractions* e^{tA_λ}. *And, for all* $x \in X$ $\lambda, \mu > 0$ *, we have*

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leq t\|A_\lambda x - A_\mu x\|.$$

Proof:

A_λ is obviously a bounded linear operator and thus is the infinitesimal generator of $(e^{tA_\lambda})_{t\geq 0}$

$$Therefore \;\; \|e^{tA_\lambda}\| = \|e^{t(\lambda^2 R(\lambda:A)-\lambda I)}\| = \|e^{-\lambda t}e^{t(\lambda^2 R(\lambda:A))}\|$$

$$= e^{-\lambda t}\|e^{t(\lambda^2 R(\lambda:A))}\| \leq e^{-\lambda t}e^{t\lambda^2\|R(\lambda:A)\|}$$

$$\leq e^{-\lambda t}e^{t\lambda^2(\frac{1}{\lambda})} = e^{-\lambda t}e^{\lambda t} = e^0 = I.$$

Therefore e^{tA_λ} is a semigroup of contractions.

Also, it is clear that $e^{tA_\lambda}, e^{tA_\mu}, A_\lambda$ and A_μ commute with each other. Then,

$$\begin{aligned}
\|e^{tA_\lambda}x - e^{tA_\mu}x\| &= \|\int_0^1 \frac{d}{ds}\left(e^{tsA_\lambda}e^{(1-s)A_\mu}x\right)ds\| \\
&\leq \int_0^1 t\|\left(A_\lambda x - A_\mu x\right)e^{tsA_\lambda}e^{t(1-s)A_\mu}\|ds \\
&\leq t\|A_\lambda x - A_\mu x\|\int_0^1 \|e^{tsA_\lambda}e^{t(1-s)A_\mu}\|ds \\
&\leq t\|A_\lambda x - A_\mu x\|\int_0^1 1ds = t\|A_\lambda x - A_\mu x\|.\square
\end{aligned}$$

Continuation of Theorem 3.1 (Sufficiency):

Let $x \in D(A)$,

$$\begin{aligned}
\|e^{tA_\lambda}x - e^{tA_\mu}x\| &\leq t\|A_\lambda x - A_\mu x\| \qquad (3.1.1)\\
&\leq t\|A_\lambda x - Ax\| + t\|Ax - A_\mu x\|.
\end{aligned}$$

__**Claim1**__: For $x \in X$, $\lim_{\lambda\to\infty} e^{tA_\lambda}x = T(t)x$ for all $t \geq .0$

Proof: Let $x \in D(A)$, by 3.1.1 and Yosida Approximation, we have $(e^{tA_\lambda}x)_{\lambda>0}$ as a Cauchy sequence for all $x \in D(A)$.

Thus, given $\epsilon > 0, \exists \;\; \delta(\epsilon) > 0$ such that

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| < \frac{\epsilon}{3}, \mu > \delta, for \;\; all \;\; x \in D(A).$$

Let $\lim_{\lambda\to\infty} e^{tA_\lambda}x = T(t)x$,

Since $D(A)$ is dense in X, for $x \in X, \epsilon > 0 \;\; \exists y \in D(A)$ such that $\|x - y\| < \frac{\epsilon}{3}$.

Then, we have

$$\begin{aligned}
\|e^{tA_\lambda}x - e^{tA_\mu}x\| &\leq \|e^{tA_\lambda}x - e^{tA_\mu}y\| + \|e^{tA_\lambda}y - e^{tA_\mu}y\| + \|e^{tA_\lambda}y - e^{tA_\mu}x\| \ \ for\ \ all\ \ \lambda, \mu > \delta \\
&\leq \|x - y\| + \frac{\epsilon}{3} + \|y - x\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \ \ for\ \ all\ \ \lambda, \mu > \delta.
\end{aligned}$$

Hence, $(e^{tA_\lambda}x)_{\lambda>0}$ is a Cauchy sequence for all $x \in X$, since $x \in X$ is arbitrary chosen. Also,$\|e^{tA_\lambda}\| \leq 1$ for all $t > 0$, we then get $\lim_{\lambda\to\infty} e^{tA_\lambda}x = T(t)x \ \ for\ \ all\ \ x \in X$. Obviously, $\Rightarrow e^{tA_\lambda}x$ converges as $\lambda \to \infty$ and is uniform on bounded intervals.

Since $D(A)$ is dense in X and $\|e^{tA_\lambda}\| \leq 1$, then

$$\lim_{\lambda\to\infty} e^{tA_\lambda}x = T(t)x \ \ \forall \ \ \in X.$$

Obviously , $(T(t))_{t\geq 0}$ satisfies the semigroup property and $\|T(t)\| \leq 1$. Also, $t \to T(t)x$ is continuous $\forall\ t \geq 0$ as the uniform limit of the continuous functions $t \to e^{tA_\lambda}x$.

Therefore, $(T(t))_{t\geq 0}$ is a semigroup of contractions on X.

Claim: A is the infinitesimal generator of $(T(t))_{t\geq 0}$.

$\forall\ x \in D(A)$,

$$T(t)x - x = \lim_{\lambda\to\infty} \left(e^{tA_\lambda}x - x\right) = \lim_{\lambda\to\infty} \int_0^t e^{tA_\lambda}xA_\lambda xds = \int_0^t T(s)Axds.$$

Therefore,

$$\lim_{t\to 0^+} \frac{T(t)x - x}{t} = \lim_{t\to 0^+} \frac{1}{t} \int_0^t T(s)Axds \ \ \forall\ \ x \in D(A).$$

Hence, A is the infintesimal generator of $(T(t))_{t\geq 0}.\square$

From the preceding proof, if A is an infinitesimal generator of C_0 semigroup of contractions $T(t)$ and A_λ is the Yosida Approximation of A, then

$$T(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x \ \ \forall\ \ x \in X.$$

Also, the resolvent set of A , $\rho(A) \supseteq \{\lambda : \lambda > 0\}$ and for such λ

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}.$$

Claim: A is unique.

Let C be the infinitesimal generator of $(T(t))_{t\geq 0}$. We claim that $C = A$. Let $x \in D(A)$,

$$e^{tA_\lambda}x - x \ \ = \ \ \int_0^t e^{sA_\lambda}A_\lambda xds.$$

As $\lambda \to \infty$,

$$\begin{aligned}
\Rightarrow \frac{T(t)x - x}{t} &= \frac{1}{t}\int_0^t T(s)Axds \\
\Rightarrow Cx &= Ax.
\end{aligned}$$

Hence, $D(A) \subseteq D(C)$. Now. we show that $D(C) \subseteq D(A)$, since C is the generator of $(T(t))_{t\geq 0}$, then $1 \in \rho(C)$.

$$(I - C)D(A) = (I - C)D(C) = X$$

$$(I - C)^{-1}X = D(A) = D(C)$$

then $A \equiv C$.

Chapter 4

LUMER-PHILLIPS THEOREM

In the previous chapter, we show the Hille-Yosida characterization of the infinitesimal generator of a C_0 semigroup of contractions. Now in this chapter, we discuss another characterization of infinitesimal generators. hence, the need to define some preliminaries.

4.1 Dissipativeness

We state and prove an important theorem that is crucial in this chapter.

Definition 4.1.1. A linear operator A is dissipative if $\|(\lambda I - A)x\| \geq \lambda\|x\|$ for all $x \in D(A)$.

Examples of Dissipative operator

1. $X = l^2 = \{x = (x_n)_{n\geq 0} \subset \mathbb{R} : \sum\limits_{n=0}^{\infty} |x_n|^2 < \infty\}$. Let $(a_n)_{n\geq 0} \subset \mathbb{R}^-$ and define

$$A : D(A) \subset l^2 \longrightarrow l^2$$

$$let\ \ x = (x_n)_n, Ax = (a_n x_n)_n\ \ and\ \ D(A) = \{(x_n)_n \in l^2 : (a_n x_n)_n \in l^2\}.$$

Then, A is dissipative.

Proof

Let $\lambda > 0, then$

$$\|(\lambda I - A)x\|^2 \;\; = \;\; \lambda^2\|x\|^2 - 2\langle \lambda x, Ax\rangle + \|Ax\|^2.$$

Since $\langle \lambda x, Ax\rangle = \sum an\lambda_n x_n^2 \leq 0$ if $a_n \leq 0$. Therefore $\|(\lambda I - A)x\|^2 \geq \lambda^2\|x\|^2\ \ which\ \ implies\ \ \|(\lambda I - A)x\| \geq \lambda\|x\|$ for all $x \in D(A)$.

2. Ω is smooth and open in $\mathbb{R}$, $m : \Omega \longrightarrow \mathbb{R}^-$ a measurable function. $L^2(\Omega) = \{f : \Omega \to \mathbb{R}^n\ \ measurable : \int_\Omega f^2 < \infty\}$. $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ be defined by $Af = mf$ and $D(A) = \{f \in L^2(\Omega) : mf \in L^2(\Omega)\}$ then A is dissipative.

Proof

For $\lambda > 0, then$

$$\|(\lambda I - A)f\|^2 \;=\; \lambda^2\|f\|^2 - 2\langle \lambda f, Af\rangle + \|Af\|^2 \geq \lambda^2\|f\|^2.$$

Hence $\|(\lambda I - A)f\|^2 \geq \lambda^2\|f\|^2 \Longrightarrow \|(\lambda I - A)f\| \geq \lambda\|f\|.\Box$

Theorem 4.1.2. ***(Lumer-Phillips):*** *Let A be a linear operator with dense domain $D(A)$, if A is dissipative and $\exists\ \lambda_0 > 0$ such that the range $R(\lambda_0 I - A)$ of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.*

Proof:

i. *Let $\lambda > 0$,by* ***Definition 4.1.1 ,*** $\|(\lambda I - A)x\| \geq \lambda\|x\|\ \forall\ x \in D(A)$. *Since $R(\lambda_0 I - A) = X$, it follows that with $\lambda = \lambda_0$, $(\lambda_0 I - A)^{-1}$ is a bounded linear operator and hence closed. Also, $\lambda_0 I - A$ is closed and therefore A is closed. If $R(\lambda I - A) = X\ \forall\ \lambda > 0$, then $\rho(A) \supseteq \mathbb{R}^+$and $\|R(\lambda : A)\| \leq \frac{1}{\lambda}$. Consequently, A is the infinitesimal generator of a C_0 semigroup of contractions on X by Hille-Yosida Theorem.*

To complete the proof, it remains to show that $R(\lambda I - A) = X\ \forall\ \lambda > 0$, consider the set $\Theta = \{\lambda : \lambda \in \mathbb{R}^+$ and $R(\lambda I - A) = X\}$. Let $\lambda \in \Theta, \lambda \in \rho(A)$ (obviously). Since $\rho(A)$ is open, $\exists$ a neighbourhood O of λ such that $O \subseteq \rho(A)$ and $O \cap \mathbb{R}^+ \subseteq \Theta$. Hence, Θ is open. Also, let $\lambda_n \in \Theta,\ \lambda_n \to \lambda > 0$. For every $y \in X$ $\exists x_n \in D(A)$ such that $\lambda_n x_n - Ax_n = y$.

$$\begin{aligned}
\lambda_n\|x_n\| &\leq \|(\lambda_n I - A)x_n\| \\
&= \|y\| \\
\Rightarrow \|x_n\| &\leq \lambda_n^{-1}\|y\| \\
&\leq c\ \ (c > 0).
\end{aligned}$$

Now

$$\lambda_m\|x_n - x_m\| \;\leq\; \|\lambda_m(x_n - x_m) - A(x_n - x_m)\|.$$

Since, $\lambda_n x_n - Ax_n = y$ *and* $\lambda_m x_m - Ax_m = y$,

$$\begin{aligned} \Rightarrow \lambda_n x_n - Ax_n &= \lambda_m x_m - Ax_m \\ \Rightarrow \lambda_n x_n - \lambda_m x_m &= A(x_n - x_m) \\ Therefore\ \ \lambda_m \|x_n - x_m\| &\leq \|\lambda_m(x_n - x_m) - (\lambda_n x_n - \lambda_m x_m)\| \\ &= \|\lambda_m x_n - \lambda_n x_n\| \\ &= \|\lambda_n - \lambda_m\| \|x_n\| \\ &\leq c\|\lambda_n - \lambda_m\|. \end{aligned}$$

Hence, (x_n) *is Cauchy. Let* $x_n \longrightarrow x$, *then*

$$Ax_n \longrightarrow \lambda x - y.$$

Since A *is closed,* $x \in D(A)$ *and* $\lambda x - Ax = y$, *then* $R(\lambda I - A) = X$ *and* $\lambda \in \Theta$. Θ *is closed and open in* $]0, \infty[$, $\Theta \neq \phi$ *since* $\lambda \in \Theta$ *and* $]0, \infty[$ *is connected, then* $\Theta =]0, \infty[$.

4.2 Maximal monotone operators on Hilbert spaces

Definition 4.2.1. A linear unbounded operator A on Hilbert space H is said to be **monotone** if for all $x \in D(A), \langle Ax, x\rangle \geq 0$. A linear unbounded operator A on X is said to be **maximal monotone** if it is monotone and $\exists\ \lambda_0 > 0$ such that $R(\lambda_0 I + A) = H$ that is $\forall\ f \in H, \exists\ x \in D(A) : x + Ax = f$.

Theorem 4.2.2. *In Hilbert spaces, a linear operator* $-A$ *is dissipative if and only if* A *is monotone.*

Proof: Let A be an operator on Hilbert space H. If A is monotone then,

$$\begin{aligned} \|(\lambda I + A)x\|^2 &= \lambda^2\|x\|^2 + 2\lambda\langle x, Ax\rangle + \|Ax\|^2 \\ \Rightarrow \|(\lambda I + A)x\|^2 &\geq \lambda^2\|x\|^2\ \ (\langle x, Ax\rangle \geq 0) \\ \Rightarrow \|(\lambda I + A)x\| &\geq \lambda\|x\|\ \ for\ \ all\ \ \lambda > 0\ \ and\ \ x \in D(A). \end{aligned}$$

This implies that $-A$ is dissipative.

Now for the converse, A is maximal monotone $\Rightarrow -A$ is the generator of a C_o semigroup of contractions. Infact, A is maximal $\Rightarrow \overline{D(A)} = H$. By Lumer-Philipps'theorem, we get that $-A$ is the generator of a C_o semigroup of contractions on H. Therefore $-A$ is dissipative. This motivates the next definition.

Before we give examples of maximal monotone operators, it is necessary to state two theorems that will be used in proving the examples.

Theorem 4.2.3. ***(Green's formula):*** *Let Ω be smooth , $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ where $H^2(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \ \ for \ \ \alpha \in \mathbb{N}^n : |\alpha| \leq 2\}$,Then*

$$-\int_\Omega v\Delta u \ dx = \int_\Omega \nabla u\cdot \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial n}\cdot v \ d\sigma,$$

$$\frac{\partial u}{\partial n} = \nabla u \cdot \hat{n} \ \ where \ \hat{n} \ is \ the \ unit \ normal \ vector \ over \ \partial\Omega.$$

Theorem 4.2.4. ***(Lax-Milgram's Theorem):*** *Let H be a real Hilbert space and $a : H \times H \longrightarrow \mathbb{R}$ be bilinear, continuous and coercive. let $f \in H'$(topological dual set of H), Then $\exists! \ u \in H$ such that*

$$a(u, v) = \langle f, v\rangle \ \ for \ \ all \ \ v \in H.$$

Remark: *a is coercive if $\exists\alpha > 0$ such that $a(u, v) \geq \alpha|u|_H^2$ for all $u \in H$.*

4.3 Examples of Maximal Monotone Operators

1. **Laplacian Operator with Dirichlet boundary condition**

   Let Ω be smooth and open in $\mathbb{R}^n$. $-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow L^2(\Omega)$ defined by

$$-\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u\ |_{\partial\Omega}= 0\}$$

   then $-\Delta$ is maximal monotone.

   Proof: $L^2(\Omega) = \{g : \Omega \longrightarrow \mathbb{R}^n \ \ measurable : \int_\Omega g^2 < \infty\}$ and $H^2(\Omega) = \{g \in L^2(\Omega) : \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j} \in L^2(\Omega) \ \ \forall \ \ i, j = 1, ..., n\}$.

Hence $A = -\Delta$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$,

$$\begin{aligned}
\langle u, Au \rangle &= \int_\Omega u(-\Delta u) \\
&= \int_\Omega \nabla u \cdot \nabla u - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot u \; d\sigma \;\; (Green's \;\; formula) \\
&= \int_\Omega \|\nabla u\|^2 \;\; (u \mid_{\partial\Omega} = 0) \\
&\geq 0.
\end{aligned}$$

Therefore A is monotone.

Now, we show that $R(I - \Delta) = L^2(\Omega)$, that is for all $f \in L^2(\Omega)$ there exists $u \in H^2(\Omega)$ such that $u - \Delta u = f$. Since $u \mid_{\partial\Omega} = 0$, $u \in H_0^1(\Omega)$ where $H_0^1(\Omega) = \{u \in H^1(\Omega) : u \mid_{\partial\Omega} = 0\}$.

Claim: $H_0^1(\Omega)$ is a Hilbert space.

$H_0^1(\Omega)$ is closed since $H_0^1(\Omega) = \overline{D(A)}|_{H^1(\Omega)}$,and $H_o^1(\Omega)$ is provided with a norm defined as

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}}.$$

It is complete.

The inner product is given as

$$\langle u, v \rangle_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v dx.$$

Let $a(\cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be defined as

$$(u, v) \longrightarrow a(u, v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv \;\; for \;\; all \;\; u, v \in H_0^1(\Omega).$$

Then a is continuous, bilinear and coercive (because a is exactly the inner product in $H^1(\Omega)$).

Claim: $L \in (H^1(\Omega))'$ where

$$L : H_0^1(\Omega) \longrightarrow \mathbb{R}$$

$$v \longrightarrow L(v) = \int_\Omega vf$$

$$\int_\Omega vf \leq \|v\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} \leq \|v\|_{H^1(\Omega)}\|f\|_{L^2(\Omega)} \Rightarrow f \in (H^1(\Omega))'.$$

Hence, $L \in (H^1(\Omega))'$, by Lax-Miligram's theorem, $\exists!\ u \in H_0^1(\Omega)$ such that $a(u,v) = \langle f, v\rangle\ \forall\ v \in H_0^1(\Omega),\ f \in L^2(\Omega)$. Therefore, $\forall\ f \in L^2(\Omega)\ \exists!\ u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $u - \Delta u = f$ and then $R(I - \Delta) = L^2(\Omega)$. Therefore A is a maximal monotone operator.$\square$

2. **Laplacian Operator with Neumann boundary condition**

Let Ω be a smooth and open set in $\mathbb{R}^n$. $-\Delta : D(\Delta) \longrightarrow L^2(\Omega)$ defined by

$$\begin{aligned} -\Delta u &= -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \\ D(\Delta) &= \{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0\}. \end{aligned}$$

Then $-\Delta$ is maximal monotone.

Proof:

$$\begin{aligned} \langle u, Au\rangle &= \int_\Omega u(-\Delta u) \\ &= \int_\Omega \nabla u \cdot \nabla u - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot u \ d\sigma \ (Green's\ \ formula) \\ &= \int_\Omega \|\nabla u\|^2 \ \ (\frac{\partial u}{\partial n} = 0 \ \ on \ \ \partial\Omega) \\ &\geq 0. \end{aligned}$$

Therefore A is monotone.

Define $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be defined as

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} uv \ \ for \ \ all \ \ u, v \in H^1(\Omega).$$

$a(u,v)$ is continuous, bilinear and coercive (from the previous example), hence by Lax-Miligram's theorem, $\exists!\ u \in H^2(\Omega)$ such that $a(u,v) = \langle f, v\rangle\ \forall\ v \in H^2(\Omega),\ f \in L^2(\Omega)$. Then, $R(I - \Delta) = L^2(\Omega)$. Therefore A is a maximum monotone operator.

4.4 APPLICATIONS

Now, we discuss how the C_0-semigroups solve some partial differential equations.

Theorem 4.4.1. *Let A be a linear unbounded operator on a real Hilbert space. A generates a C_0-semigroup of contraction if and if only $-A$ is maximal monotone.*

Heat Equation

1. Consider an heat equation on Ω an open bounded set of R^n with smooth boundary and with Dirichlet's condition given as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = 0 \quad on &\quad \Omega \times [0, \infty[\\ u(t,x) \quad = &\quad 0, t \geq 0, x \in \partial\Omega \\ u(x,0) = u_0(x) \quad on &\quad \Omega, u_0 \in L^2(\Omega). \end{aligned} \tag{4.4.1}$$

   Rewriting equation 4.4.1 in evolution form,

$$\begin{aligned} \frac{du(t)}{dt} + Au(t) &= 0, t \geq 0 \\ u(0) = u_0 \in X. \end{aligned}$$

   For $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a solution $u(x,t) = [T(t)u_0](x)$ that satisfies equation 4.4.1

   We have the form $\frac{du}{dt} + Au = 0$ where $A = -\Delta$. The suitable defined space is $X = L^2(\Omega)$ with $\|\cdot\|_{L^2(\Omega)}$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. From example 1, it has been proved that $A = -\Delta$ is maximal monotone, hence by Theorem 4.4.1 , Δ generates a C_0-semigroup of contractions on $L^2(\Omega)$.

2. Consider a heat equation on Ω an open bounded set with smooth boundary and with

Neumann's condition given as

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad on \quad \Omega \times]0, \infty[\tag{4.4.2}$$

$$\frac{\partial}{\partial n} u(t,x) \quad = \quad 0, t \geq 0, x \in \partial\Omega \tag{4.4.3}$$

$$u(x,0) = u_0(x) \quad on \quad \Omega, u_0 \in L^2(\Omega)$$

Also, the suitable defined space is $X = L^2(\Omega)$ with $\|\cdot\|_{L^2(\Omega)}$ and $D(A) = \{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \ on \ \partial\Omega\}$. Then, equation 4.4.2 takes the following form

$$\frac{du}{dt} + Au = 0 \ where \ A = -\Delta.$$

$-\Delta$ is maximal monotone on example 2, hence Δ is the generator of a C_0-semigroup of contractions that satisfies equation 4.4.2. The solution of equation 4.4.2 is given by $u(x,t) = [T(t)u_0](x)$ for $u_0 \in D(\Omega) = \{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \ on \ \partial\Omega\}$.

4.5 CONCLUSION

It has been shown with proofs that continuous semigroups of contractions serve as solutions to some partial differential equations with the necessary conditions satified. Hence, their importances in the theory of partial differential equations cannot be overemphasized.

BIBLIOGRAPHY

[1] Ezzinbi K. ; *Lecture Notes in Functional Analysis and Evolution Equations.* AUST, 2012.

[2] Kaylan B. Sinha, Sachi Srivastara. ; *Theory of Semigroups and Applications.* Springer Nature , Singapore, 2017.

[3] Nwigbo Kenule Tuador ; *Maximal Monotone Operators on Hilbert Spaces and Applications.* AUST, 2016.

[4] Pazy, A. ; *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, New York, 1983.