THE AUMANN INTEGRAL OF SET-VALUED MAPS

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Master of Science

By
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The Aumann Integral of Set-Valued Maps

Certification

This is to certify that the thesis titled "**The Aumann Integral of Set-Valued Maps**" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of Masters degree is a record of original research carried out by Chinedu Anthony Eleh in the Department of Pure and Applied Mathematics.

Approval

THE AUMANN INTEGRAL OF SET-VALUED MAPS

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Abstract

This thesis focuses on the Aumann integral of set-valued random variables and its properties. We started off by studying the space in which this integral lies: *hyperspace* endowed with the Hausdorff metric. We considered convergence on a hyperspace with respect to the Hausdorff metric and reviewed the works of Kuratowski, Mosco in trying to abstract topologically, the Hausdorff convergence; this led to a comparison between *weak, Wijsmann, Kuratowski-Mosco* convergences to Hausdorff convergence. We proceeded to see the conditions under which a set-valued random variable is measurable, integrable and integrably bounded. Finally, we defined the class of integrable selections of an integrable set-valued random variable and used it to define the Aumann integral, and went further to prove and outline sufficient conditions for the Aumann integral to be convex and closed-valued respectively.

Dedication

In memory of my class representative, Abdulfatah Suleman. May you rest in peace.

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Contents

CHAPTER 1

Hyperspaces and Hausdorff Metric

1 Introduction to Hyperspaces

As technology advances, scientific research prospect is on the increase too. Prior to the advent of **high performance computing (HPC)**, experimental designs were limited to parameters that have 'measurable' real number values. For instance

- height of each member of a population;
- number of hydrogen and oxygen required to produce a molecule of water;
- amount of interest that accrues in a time given a capital, etc.

All these are modeled with real-valued functions.

Recently, interests have deviated a bit. Scholars now think of how to model abstract things like:

- how can we keep track of the face of a lady even when she is wearing a makeup?
- how can we keep tract of body cells and be able to say-this is a cancerous cell with a particular property?

These are interesting questions researchers working on *image processing* try to answer.

Experimental designs of this nature are done on **hyperspaces**.

Let (E, d) be a metric space. We seek to study the collection of all nonempty closed subsets of E called Hyperspace and maps that are

hyperspace-valued. Of particular interest are those set-valued maps that are measurable, called *set-valued random variables or random sets.* Generally, E will denote a metric space (or a normed linear space);
$\mathcal{P}(E)$, the power set of E;
$\mathbf{P}(E)$, family of all nonempty closed subsets of E;
$c,\ k,\ b$ will denote convex, compact and bounded respectively so that $\mathbf{P}_{bc}(E)$ is the family of all nonempty closed bounded convex subsets of E;
$\mathbf{P}_{kc}(E)$, family of all nonempty compact convex subsets of E, etc.
In this wrok, most theorems, lemmas, definitions etc are adapted from Li *et al.* (2002).

Definition 1. Let E be a normed linear space.

(1). For $A, B \in \mathbf{P}(E)$ the sum of A and B denoted by $A \oplus B$ is defined as

$$A \oplus B = cl\{a + b : a \in A, b \in B\},$$

where cl is the closure of the set $A + B = \{a + b : a \in A, b \in B\}$ taken in E;

(2). For $\lambda \in \mathbb{R}$,

$$\lambda A = \{\lambda a : a \in A\}.$$

We remark that for $A, B \in \mathbf{P}(X)$ and $\lambda \in \mathbb{R}$,

$$A \oplus B \in \mathbf{P}(X) \quad \text{and} \quad \lambda A \in \mathbf{P}(X).$$

Theorem 1. *If $A, B \in \mathbf{P}_{kc}(E)$, then*

(i). $A \oplus B = A + B$*;*

(ii). $A + B$ *is compact;*

(iii). $A + B$ *is convex.*

Proof Let $A, B \in \mathbf{P}_{kc}(E)$.

i). By definition, $A + B \subset A \oplus B$.

Let $x \in A \oplus B$. Then we can find $\{x_n\} \subset A + B,\ \{a_n\} \subset A$, $\{b_n\} \subset B,\ n \geq 1$ such that $a_n + b_n = x_n \longrightarrow x$ as $n \longrightarrow \infty$.

Since A and B are compact, we can find $\{a_{n_k}\} \subset \{a_n\},\ \{b_{n_j}\} \subset \{b_n\},\ a \in A,\ b \in B$ such that

$\|a_{n_k} - a\| \leq \frac{\epsilon}{4}$ and $\|b_{n_j} - b\| \leq \frac{\epsilon}{4},\ \forall\, k \geq N_1,\ j \geq N_2$ for some $N_1,\ N_2$.

Let $N_3 = \max\{N_1,\ N_2\}$. Then

$$\|a_{n_i} - a\| < \frac{\epsilon}{4} \text{ and } \|b_{n_i} - b\| < \frac{\epsilon}{4},\quad \forall\, i \geq N_3.$$

So that

$$\begin{aligned} \|x_n - (a+b)\| &= \|a_n + b_n - a - b\| \\ &\leq \|a_n - a_{n_i}\| + \|b_n - b_{n_i}\| + \|a_{n_i} - a\| + \|b_{n_i} - b\| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

Hence, $A \oplus B = A + B$.

ii). Since E is a metric space, it suffices to prove sequential compactness. Let $\{x_n\} \subset A+B,\ n \geq 1$. Then, we can find $\{a_n\} \subset A,\ \{b_n\} \subset B$ such that $x_n = a_n + b_n,\ n \geq 1$.

By compactness of A and B there exists $\{a_{n_k}\} \subset \{a_n\},\ \{b_{n_j}\} \subset \{b_n\},\ a \in A,\ b \in B$ such that

$\|a_{n_k} - a\| \leq \frac{\epsilon}{2}$ and $\|b_{n_j} - b\| \leq \frac{\epsilon}{2},\ \forall\, k \geq N_1,\ j \geq N_2\ for\ some\ N_1,\ N_2$.

Let $N_3 = \max\{N_1, N_2\}$. Then

$$\|a_{n_i} - a\| < \frac{\epsilon}{2} \text{ and } \|b_{n_i} - b\| < \frac{\epsilon}{2},\quad \forall\, i \geq N_3.$$

Thus, $\forall\ i \geq N_3$,

$$\begin{aligned}\|x_{n_i} - (a+b)\| &= \|a_{n_i} + b_{n_i} - a - b\| \\ &\leq \|a_{n_i} - a\| + \|b_{n_i} - b\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence, $A + B$ is compact.

iii). Let $x = a_1 + b_1 \in A + B$ and $y = a_2 + b_2 \in A + B$.
For $\lambda \in [0, 1]$,

$$\begin{aligned}\lambda x + (1-\lambda)y &= \lambda(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \\ &= \lambda a_1 + (1-\lambda)a_2 + \lambda b_1 + (1-\lambda)b_2 \in A + B\end{aligned}$$

since A and B are convex.

□

It follows from theorem 1 that $A, B \in \mathbf{P}_{kc}(X)$ implies $A + B \in \mathbf{P}(X)$.

2 Hausdorff Metric

Let $A, B \in \mathbf{P}(E)$. Suppose $A \neq B$. i.e., $A \subset B$ and $B \subset A$ do not hold simultaneously. We desire to quantify the discrepancy/mismatch between A and B (that makes them different). A natural place to start is to begin with what is known.
Assume $A = B$. Then for each $a \in A$, we can find $b \in B$ such that

$$a = b \tag{2.1}$$

Equation 2.1 implies $d(a, B) \leq d(a, b) = 0$ for all $a \in A$ where $d(a, B) = \inf_{b\in B} d(a, b)$. Thus,
$\sup_{a\in A} d(a, B) = 0 \;\Rightarrow d(A, B) = 0$ where $d(A, B) = \sup_{a\in A} d(a, B)$.

Following same arguments, we have $d(A,B) = 0$ and $\max\{d(A,B), d(B,A)\} = 0$. Suppose, without loss of generality, there is $a \in A$ such that for all $b \in B$, $a \neq b$. Then A and B are closed implies $d(A,B) > 0$ and $d(B,A) > 0$. So, $d(A,B)$ and $d(B,A)$ give the mismatches between A and B and $\max\{d(A,B), d(B,A)\}$ gives the highest discrepancy between A and B.

Theorem 2. *Let E be a metric space and for $A, B, C \in \mathbf{P}_b(E)$, let $H : \mathbf{P}_b(E) \times \mathbf{P}_b(E) \to \mathbb{R}$ be given by*

$$H(A,B) = \max\{d(A,B), d(B,A)\}.$$

Then

(i). $H(A,B) \geq 0$,

(ii). $H(A,B) = 0 \iff A = B$,

(iii). $H(A,B) = H(B,A)$,

(iv). $H(A,B) \leq H(A,C) + H(C,B)$.

Proof

i). By definition, $H(A,B) = \max\{d(A,B), d(B,A)\} \geq 0$.

ii). Since A and B are closed,

$$\begin{aligned}
H(A,B)=0 &\Leftrightarrow \max\{d(A,B),d(B,A)\}=0\\
&\Leftrightarrow \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b),\ \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}=0\\
&\Leftrightarrow \sup_{a\in A}\inf_{b\in B} d(a,b)=0 \ \text{ and } \ \sup_{b\in B}\inf_{a\in A} d(a,b)=0\\
&\Leftrightarrow \inf_{b\in B} d(a,b)=0,\ \inf_{a\in A} d(a,b)=0,\quad \forall a\in A,\ b\in B\\
&\Leftrightarrow \forall\ n\geq 1,\ \exists\ b_n\in B,\ a_n\in A: b_n\to a,\ a_n\to b \text{ as } n\to\infty\\
&\Leftrightarrow a\in B \ \text{ and } \ b\in A,\quad \forall\ a\in A,\ b\in B\\
&\Leftrightarrow A=B.
\end{aligned}$$

iii).

$$\begin{aligned}
H(A,B)&=\max\{d(A,B),d(B,A)\}\\
&=\max\{d(B,A),d(A,B)\}\\
&=H(A,B).
\end{aligned}$$

iv). For all $a\in A,\ b\in B$ and $c\in C$,

$$d(a,b)\leq d(a,c)+d(c,b) \tag{2.2}$$

Taking infimum of both sides over $a\in A$,

$$d(b,A)\leq d(c,A)+d(c,b)$$

Taking infimum over $c\in C$ and observing $\inf_{c\in C}\leq d(c,A)$,

$$d(b,A)\leq d(c,A)+d(b,C)$$

We now take supremum over $b\in B$ and $c\in C$ to get

$$\begin{aligned}
d(B,A)&\leq d(C,A)+d(B,C)\\
&\leq \max\{d(C,A),d(A,C)\}+\max\{d(B,C),d(C,B)\}
\end{aligned}$$

So that

$$d(B,A) \leq \max\{d(C,A), d(A,C)\} + \max\{d(B,C), d(C,B)\} \quad (2.3)$$

Following same procedure as in 2.2 beginning with infimum over $b \in B$, we have

$$d(A,B) \leq \max\{d(A,C), d(C,A)\} + \max\{d(C,B), d(B,C)\} \quad (2.4)$$

Hence, by 2.3 and 2.4,

$$\begin{aligned} H(A,B) &= \max\{d(A,B), d(B,A)\} \\ &\leq \max\{d(A,C), d(C,A)\} + \max\{d(C,B), d(B,C)\} \\ &\leq H(A,C) + H(C,B). \end{aligned}$$

□

Remarks 1.

(1). By Theorem 2 we have proved H is a metric on $\mathbf{P}_b(E)$

(2). If E is a normed linear space and we take $B = \{0\}$, then

$$\begin{aligned} H(A,\{0\}) &= \max\{d(A,\{0\}), d(\{0\},A)\} \\ &= \max\left\{\sup_{a\in A} d(a,0), d(0,A)\right\} \\ &= \sup_{a\in A}\|a\| \;\equiv\; \|A\|_{\mathbf{P}} \end{aligned}$$

So, by $\|A\|_{\mathbf{P}}$ we mean the distance between A and $\{0\}$ with respect to the metric H.

It easy to verify that $A \subset E$ is bounded if and only if $\|A\|_{\mathbf{P}}$ is finite. Furthermore, for any arbitrary index set I, we say $\{A_\alpha\}_{\alpha\in I}$ is *Uniformly bounded* if $\sup_{\alpha\in I}\|A_\alpha\|_{\mathbf{P}} < \infty$.

(3). H is called the *Hausdorff metric/distance* on $\mathbf{P}_b(E)$.

We now give examples to illustrate how the Hausdorff metric can be computed. For ease of computations, we illustrate in $\mathbb{R}$ and in $\mathbb{R}^2$ with their usual metrics.

Let $X = \mathbb{R}$. Compute the distance between

i). $A = [2, 4],\ B = [7, 10]$

ii) $A = [2, 4],\ B = [-3, 3]$

iii). $A = [-9.5, 1] \cup [1, 20], \quad B = [2, 8] \cup [25, 30]$

iv). $A = \overline{B_1(0)}$ (closed unit ball in $\mathbb{R}^2$), $B = \{(-3, -4), (4, -4), (2, 1)\}$

Solution

i). $A = [2, 4],\ B = [7, 10]$

$$d(A, B) = \sup_{a \in A} d(a, B) \ = \sup_{a \in [2,4]} |a - 7| \ = \ |2 - 7| \ = \ 5$$

$$d(B, A) = \sup_{b \in B} d(b, A) \ = \sup_{b \in [7,10]} |b - 4| \ = \ |10 - 4| \ = \ 6$$

Hence,

$$H(A, B) = \max\{5, 6\} = 6.$$

ii). $A = [2, 4],\ B = [-3, 3]$

$$d(A, B) = \sup_{a \in A} d(a, B) \ = \sup[0, 1] \ = \ 1$$

$$d(B, A) = \sup_{b \in B} d(b, A) \ = \sup[0, 5] \ = \ 5$$

Hence,

$$H(A, B) = \max\{1, 5\} = 5.$$

iii). $A = [-9.5, 1] \cup [1, 20], \quad B = [2, 8] \cup [25, 30]$

$$d(A, B) = \sup_{a \in A} d(a, B) \ = \sup[1, 11.5] \ = \ 11.5$$

$$d(B,A) = \sup_{b \in B} d(b,A) \;\; = \sup[1,10] \;\; = \;\; 10$$

Hence,

$$H(A,B) = \max\{11.5, 10\} = 11.5.$$

iv). $A = \overline{B_1(0)}$, $\quad B = \{(-3,-4),(4,-4),(2,1)\}$

$\|(-3,-4)\| \; = \; \sqrt{(-3)^2+(-4)^2} \; = \; 5; \qquad d((-3,-4),A) = 4$

$\|(4,-4)\| \; = \; \sqrt{4^2+(-4)^2} \; = \; 4\sqrt{2}; \qquad d((4,-4),A) = 4\sqrt{2}-1$

$\|(2,1)\| \; = \; \sqrt{2^2+1^2} \; = \; \sqrt{5}; \qquad d((2,1),A) = \sqrt{5}-1$

$$d(A,B) \; = \; \sup_{a \in A} d(a,B) \; = \; \sup[\sqrt{5}-1, \sqrt{5}+1] \; = \; \sqrt{5}+1$$

$$d(B,A) = \sup_{b \in B} d(b,A) = \max\left\{4,\; 4\sqrt{2}-1,\; \sqrt{5}-1\right\} = 4\sqrt{2}-1$$

Hence,

$$H(A,B) = \max\left\{\sqrt{5}+1,\; 4\sqrt{2}-1\right\} = 4\sqrt{2}-1.$$

Notice in each case, $d(A,B) \neq d(B,A)$ since by definition,

$$\sup_{a \in A} d(a,B) \neq \sup_{b \in B} d(b,A).$$

3 Properties of Hausdorff Metric

Here, we seek to prove, among others, an important property of the Hausdorff metric: $\mathbf{P}_b(E)$ endowed with H is a complete metric space. To this end, we make the following preparations.

Definition 2. Let $\epsilon > 0$. The ϵ-neighourhood of A denoted by A_ϵ is defined as

$$A_\epsilon = \{y \in E : d(y,A) \leq \epsilon\}$$

where $A \subset E$ and $X,\ d$ have their usual meanings.

Lemma 1. *Let A be bounded. Then A_ϵ is bounded.*

Proof Since A is bounded, there is $c > 0$ such that $\operatorname{diam}(A) \le c$. Let $y \in A_\epsilon$. For any $x, z \in A$,

$$\begin{aligned} d(y,x) &\le d(y,z) + d(z,x) \\ &\le d(y,z) + \operatorname{diam}(A) \\ &\le \inf_{z\in A} d(y,z) + c \\ &\le d(y,A) + c \\ &\le \epsilon + c. \end{aligned}$$

Thus, for any other $p \in A_\epsilon$,

$$\begin{aligned} d(y,p) &\le d(y,x) + d(x,p) \\ &\le 2\epsilon + 2c. \end{aligned}$$

Hence, $\operatorname{diam}(A_\epsilon) \le 2\epsilon + 2c$, for all $p, y \in A_\epsilon$ implies A_ϵ is bounded.

□

Lemma 2. *Let $\epsilon > 0$. Then A_ϵ is a closed set for all $A \subset E$.*

Proof By definition, $A_\epsilon \subset \overline{A_\epsilon}$.
Let $x \in \overline{A_\epsilon}$. Then there is $\{x_n\} \subset A_\epsilon$ such that $d(x_n, x) \to 0$ as $n \to \infty$. By triangle inequality,

$$\begin{aligned} d(x,A) &\le d(x,x_n) + d(x_n,A) \\ &\le d(x,x_n) + \epsilon \\ \Rightarrow d(x,A) &\le \lim_{n\to\infty} d(x,x_n) + \epsilon \end{aligned}$$

So that $x \in A_\epsilon$ implies A_ϵ is closed.

□

Lemma 3. *Let* $\epsilon > 0$. *Then* $H(A,B) \le \epsilon$ *if and only if* $A \subset B_\epsilon$ *and* $B \subset A_\epsilon$.

Proof

$$\begin{aligned} H(A,B) \le \epsilon &\Leftrightarrow \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \le \epsilon \\ &\Leftrightarrow \sup_{a\in A} d(a,B) \le \epsilon \ \text{ and } \ \sup_{b\in B} d(b,A) \le \epsilon \\ &\Leftrightarrow \forall\, a \in A,\ d(a,B) \le \epsilon \ \text{ and } \ \forall\, b \in B,\ d(b,A) \le \epsilon \\ &\Leftrightarrow \forall\, a \in A,\ a \in B_\epsilon \ \text{ and } \ \forall\, b \in B,\ b \in A_\epsilon \\ &\Leftrightarrow A \subset B_\epsilon \ \text{ and } \ B \subset A_\epsilon \end{aligned}$$

The following lemma from elementary analysis will be useful also.

Lemma 4. *Let* $\{x_n\}$ *be a sequence in* E *such that for* $n \ge 1$,

$$d(x_n, x_{n+1}) < \frac{1}{2^n}.$$

Then, $\{x_n\}$ *is a Cauchy sequence in* X.

Proof Follows from triangle inequality and geometric series.

We are now ready to prove

Theorem 3. $(\mathbf{P}_b(E), H)$ *is a complete metric space.*

Proof Let $\{A_n\} \subset \mathbf{P}_b(E)$ be Cauchy. Let

$$A := \bigcap_{j\ge 1} \overline{\bigcup_{n\ge j} A_n}. \tag{3.1}$$

To prove Theorem 3, it suffices to justify

i). $A \ne \emptyset$,

ii). A is closed,

iii). A is bounded,

iv). $H(A_n, A) < \epsilon$ for all $n \geq N$ for some N.

i). Let $\epsilon > 0$. $\{A_n\}$ Cauchy implies there is N such that

$$H(A_n, A_m) < \epsilon \ \ \forall\ m, n \geq N.$$

In particular, for $\epsilon = \frac{1}{2^{k+20}},\ k \geq 0$

$$H(A_{n_k}, A_{n_{k+1}}) < \frac{1}{2^{k+20}}, \quad \forall\ n_k \geq N.$$

Thus, $\forall\ x_{n_k} \in A_{n_k}\ \exists\ x_{n_{k+1}}\ \in A_{n_{k+1}}$ such that

$$d\left(x_{n_k}, x_{n_{k+1}}\right) < \frac{1}{2^k}, \quad \forall\ k \geq 0.$$

By Lemma 4, we conclude x_{n_k} is a Cauchy sequence in X. Since E is complete, there is $x \in E$ such that $x_{n_k} \to x$ as $k \to \infty$. Let $j \geq 1$. Then there is k_0 such that for $n_{k_0} \geq j,\ x_{n_{k_0}} \in A_{n_{k_0}} \subset \bigcup\limits_{n \geq j} A_n$. So that for $k \geq 1$ such that $n_k \geq n_{k_0}$, we have $x_{n_k} \in \bigcup\limits_{n \geq j} A_n$.
This implies $j \geq 1$,

$$\begin{aligned} x = \lim_{n\to\infty} x_{n_k} \in cl\left(\bigcup_{n\geq j} A_n\right) \ &\Rightarrow x \in \overline{\bigcup_{n\geq j} A_n},\ j \geq 1 \\ &\Rightarrow x \in \bigcap_{j\geq 1} \overline{\bigcup_{n\geq j} A_n} \ = \ A. \end{aligned}$$

Hence, $\quad A \neq \emptyset$.

ii). A is closed since arbitrary intersections of closed sets is closed.

iii). Since $\{A_n\}$ is Cauchy, there is N such that for $m, n \geq N$,

$H(A_m, A_n) < \epsilon$. Take $m = N$, then for $n \geq N$,

$$\begin{aligned} H(A_N, A_n) < \epsilon &\Rightarrow A_n \subset (A_N)_\epsilon \\ &\Rightarrow \bigcup_{n\geq N} A_n \subset (A_N)_\epsilon \\ &\Rightarrow \overline{\bigcup_{n\geq N} A_n} \subset (A_N)_\epsilon \\ &\Rightarrow A = \bigcap_{j\geq 1} \overline{\bigcup_{n\geq j} A_n} \subset \overline{\bigcup_{n\geq N} A_n} \subset (A_N)_\epsilon \\ &\Rightarrow A \subset (A_N)_\epsilon . \end{aligned}$$

Since by Lemma 1, $(A_N)_\epsilon$ is bounded, it follows A is bounded. Notice we applied Lemma 2 to conclude $\overline{\bigcup_{n\geq N} A_n} \subset (A_N)_\epsilon$.

iv). To prove $H(A_n, A) < \epsilon$, by Lemma 3 it suffices to prove

(a) $A \subset (A_n)_\epsilon$

(b) $A_n \subset A_\epsilon$

(a). Let $x \in A := \bigcap_{j\geq 1} \overline{\bigcup_{n\geq j} A_n}$. Then for $j \geq 1$,

$$\begin{aligned} x \in \overline{\bigcup_{n\geq j} A_n} &\Rightarrow \forall\ j \geq 1,\ \exists\ x_n \in \bigcup_{n\geq j} A_n :\ x_n \to x \ as\ n \to \infty \\ &\Rightarrow \forall\ j \geq 1,\ \exists\ x_{n_k} \in A_{n_k} :\ x_{n_k} \to x \ as\ k \to \infty \\ &\Rightarrow d(x_{n_k}, x) < \frac{\epsilon}{2}, \quad \forall\ k \geq N_1 \ for\ some\ N_1 \end{aligned}$$

Since $\{A_n\}$ is Cauchy, there is N_2 such that

$$d(x_{n_k}, A_n) \leq H(A_{n_k}, A_n) < \frac{\epsilon}{2}, \quad \forall\ k, n \geq N_2.$$

Take $N_3 = \max\{N_1, N_2\}$, then

$$d(x, A_n) \leq d(x, x_{n_k}) + d(x_{n_k}, A_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Implies for $n \geq N_3$, $x \in (A_n)_\epsilon$ and consequently, $A \subset (A_n)_\epsilon$ for all $n \geq N_3$ which justifies (a).

(a). $\{A_n\}$ Cauchy implies there is N_4 such that for $k, n \geq N_4$,

$$H\left(A_n, A_{n_k}\right) < \frac{\epsilon}{2^{k+100}}.$$

By definition of H, this implies for all $x_n \in A_n$, there is $x_{n_k} \in A_{n_k}$ such that for $n, k \geq N_4$,

$$d(x_n, x_{n_k}) < \frac{\epsilon}{2^{k+1}}$$

Take $k = 1$, then

$$d(x_n, x_{n_k}) < \frac{\epsilon}{2^2} \tag{3.2}$$

Also, for $n = n_{k+k} > n_k > k \geq N_4$,

$$d(x_{k+1}, x_{n_k}) < \frac{\epsilon}{2^{k+1}} \quad \Rightarrow \sum_{n=1}^{\infty} d(x_{k+1}, x_{n_k}) \leq \frac{\epsilon}{2}. \tag{3.3}$$

But $x = \lim_{k\to\infty} x_{n_k}$ for some $x \in A$ as proved in (i) above. Thus, for $x_n \in A_n$, $n \geq N_4$ with 3.2 and 3.3 and triangle inequality, we have

$$\begin{aligned}
d(x_n, A) &\leq d(x_n, x) \\
&\leq d(x_n, x_{n_1}) + d(x_{n_1}, x_{n_2}) + \cdots + d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x) \\
&\leq d(x_n, x_{n_1}) + \sum_{i=1}^{k} d(x_{n_i}, x_{n_{i+1}}) + d(x_{n_{k+1}}, x) \\
&\leq d(x_n, x_{n_1}) + \sum_{i=1}^{\infty} d(x_{n_i}, x_{n_{i+1}}) + \lim_{k\to\infty} d(x_{n_{k+1}}, x) \quad as\ k \to \infty \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Thus, $x_n \in A_\epsilon$, $n \geq N_4$ implies $A_n \subset A_\epsilon$ and this completes the proof of (b).

Let $N = \max(N_3, N_4)$. Then for all $n \geq N$, we have proved that $A \subset (A_n)_\epsilon$ and $A_n \subset A_\epsilon$. $\square$

Theorem 4. $\mathbf{P}_k(E)$ *is a closed subset in* $(\mathbf{P}_b(E), H)$.

Theorem 4 is an application of the following

Lemma 5. *Let* $A \in \mathbf{P}(E)$ *and* $B \in \mathbf{P}_k(E)$. *Let* $\epsilon > 0$. *If* $A \subset B_\epsilon$, *then* $A \in \mathbf{P}_k(E)$.

Proof Since E is complete, it suffices to show A is totally bounded. B compact implies B is totally bounded. Thus, there is $\{x_1, x_2, \ldots, x_n\} \subset B$ such that $B \subset \bigcup_{i=1}^{n} B_\epsilon(x_i)$ implies $A \subset \left(\bigcup_{i=1}^{n} B_\epsilon(x_i)\right)_\epsilon$.
By hypothesis, (*compactness of* B *and* $A \subset B_\epsilon$) for each $a \in A$ there is $b \in B$ such that $d(a, b) \leq \epsilon$.
Hence,

$$d(a, x_i) \leq d(a, b) + d(b, x_i) \leq 2\epsilon. \tag{3.4}$$

Implies $a \in B_{2\epsilon}(x_i)$ and $B_{2\epsilon}(x_i) \cap A \neq \emptyset$ for some $i : 1 \leq i \leq n$.
We reorder the set $\{i\}_{i=1}^{n}$ to get rid of i such that $B_{2\epsilon}(x_i) \cap A = \emptyset$.
Let $p \leq n$ such that $B_{2\epsilon}(x_i) \cap A \neq \emptyset$ for $j : 1 \leq j \leq p$ and $B_{2\epsilon}(x_j) \cap A = \emptyset$ for $p < j \leq n$. Let

$$y_j \in B_{2\epsilon}(x_j) \cap A, \quad 1 \leq j \leq p. \tag{3.5}$$

Then, from 3.4 and 3.5,

$$d(a, y_j) \leq d(a, x_j) + d(x_j, y_j) \leq 4\epsilon$$

Implies $a \in B_{4\epsilon}(y_j)$ and $A \subset \bigcup_{j=1}^{p} B_{4\epsilon}(y_j)$.
Hence, A is totally bounded and since A is a subset of a complete Banach space, E, we conclude that A is compact. $\square$

Proof of Theorem 4

Since every compact set in a metric space is closed and bounded, we have that $\mathbf{P}_k(E) \subset \mathbf{P}_b(E)$. We now prove $\mathbf{P}_k(E)$ is closed in $(\mathbf{P}_b(E), H\ )$

Let $\{A_n\} \subset \mathbf{P}_k(E)$, $\epsilon > 0$ such that there is n_ϵ and for $n \geq n_\epsilon$, $H(A_n, A) < \epsilon$. Take $n = n_\epsilon$. Then $H(A_{n_\epsilon}, A) < \epsilon$ implies $A \subset (A_{n_\epsilon})_\epsilon$. By completeness of $\mathbf{P}_b(E)$, $A \in \mathbf{P}_b(E)$ and this implies A is closed. Hence, by Lemma 5 it follows A is compact so that $A \subset \mathbf{P}_k(E)$. □

In the same spirit, we establish

Theorem 5. $\mathbf{P}_{kc}(E)$ *and* $\mathbf{P}_{bc}(E)$ *are closed subsets in* $\mathbf{P}_b(E)$.

Proof Let $\{A_n\} \subset \mathbf{P}_{kc}(E)$, $\epsilon > 0$ such that there is n_ϵ and for $n \geq n_\epsilon$, $H(A_n, A) < \epsilon$. By Theorem 4 it is enough to prove A is convex to conclude $A \in \mathbf{P}_{kc}(E)$. Moreover, $\{A_n\} \subset \mathbf{P}_{bc}(E)$ since every compact set is bounded. So, proving A is convex suffices to conclude $A \in \mathbf{P}_{bc}(E)$.

Let $x, y \in A$. Then we can find $\{x_n\}$, $\{y_n\} \subset A_n$, $n \geq 1$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Let $\lambda \in [0,1]$. By convexity of A_n, $n \geq 1$ and properties of limit,

$$\lambda x_n + (1-\lambda) y_n \longrightarrow \lambda x + (1-\lambda) y \in A \quad as \quad n \to \infty.$$

□

We end this chapter with proof of a theorem that characterizes elements of $\mathbf{P}_k(E)$ in terms of sequences.

Theorem 6. *Let* E *be separable . Then* $(\mathbf{P}_k(E), H)$ *is a separable metric space.*

Proof Since E is separable, let D be a countable dense subset of E. Then, $\mathcal{D}$, the family of all finite subsets of D is countable. We aim to show $\mathcal{D}$ is dense in $\mathbf{P}_k(E)$. i.e., $\overline{\mathcal{D}} = \mathbf{P}_k(E)$.
Let $A \in \mathbf{P}_k(E)$. Since compactness implies total boundedness, there is $\{x_1, x_2, \dots, x_n\} \subset A$ such that $A \subset \bigcup_{i=1}^{n} B_{\frac{\epsilon}{2}}(x_i)$. By density of D in X, for any $\epsilon > 0$, $D \cap B_{\frac{\epsilon}{2}}(x_i) \neq \emptyset$. Let $y_i \in D \cap B_{\frac{\epsilon}{2}}(x_i) \neq \emptyset$. Then $d(y_i, x_i) < \frac{\epsilon}{2}$.
Define

$$B \equiv A^\epsilon := \{y_1, y_2, \dots, y_n\}, \quad y_i \in D \cap B_{\frac{\epsilon}{2}}(x_i).$$

It follows $B \in \mathcal{D}$. Let $y_i \in B$. By triangle inequality,

$$d(y_i, A) \leq d(y_i, x_i) + d(x_i, A) < \frac{\epsilon}{2}$$

Implies

$$y_i \in A_{\frac{\epsilon}{2}} \quad \textit{and} \quad B \subset A_{\frac{\epsilon}{2}} \subset A_\epsilon \tag{3.6}$$

Similarly, for $x \in A$, there is $x_i \in A$ such that $d(x, x_i) < \epsilon$ and

$$d(x, B) \leq d(x, y_i) + d(y_i, B) < \epsilon$$

Implies

$$x \in B_\epsilon \quad \textit{and} \quad A \subset B_\epsilon \tag{3.7}$$

From 3.6 and 3.7 it follows that $H(A, B) < \epsilon$ or equivalently, $H(A, A^\epsilon)$. □

CHAPTER 2

Convergences in Hyperspaces

In section 2, chapter 1 we defined Hausdorff distance and gave some illustrative examples. We rerkmark that those examples were carefully selected without computational difficulties just to illustrate how the Hausdorff distance works. Many researchers have noted the computational complexity of the Hausdorff distance **?**.

1 Characterizations of Hausdorff Metric

We now consider equivalent forms or characterizations of the Hausdorff distance that are of great importance analytically and computationally. Moreover, the characterizations will lead us to the different convergences of a sequence of sets.

We begin with the characterization whose parameters depend only on tools we have used so far.

Theorem 7. *Let* $A, B \in \mathbf{P}_b(E)$. *Then*

$$H(A,B) = \sup_{x \in E} |d(x,A) - d(x,B)|$$

To prove this theorem, we need

Lemma 6. *Let* $g : X \to \mathbb{R}$ *be bounded on* $A \subset E$ *where* E *is a metric space. Then*

$$\sup_{a \in A} |g(a)| = \max\left\{\sup_{a \in A} g(a),\ \sup_{a \in A} (-g(a))\right\}.$$

Proof By definition of absolute value function, for $a \in A$,

$$g(a) \leq |g(a)| \quad \Rightarrow \quad \sup_{a\in A} g(a) \leq \sup_{a\in A} |g(a)|$$

and

$$-g(a) \leq |g(a)| \quad \Rightarrow \quad \sup_{a\in A}(-g(a)) \leq \sup_{a\in A} |g(a)|$$

So that

$$\max\left\{\sup_{a\in A} g(a),\ \sup_{a\in A}(-g(a))\right\} \leq \sup_{a\in A} |g(a)| \tag{1.1}$$

On the other hand, for $a \in A$, we have that

$$|g(a)| \leq \max\left\{\sup_{a\in A} g(a),\ \sup_{a\in A}(-g(a))\right\}$$

Implies

$$\sup_{a\in A} |g(a)| \leq \max\left\{\sup_{a\in A} g(a),\ \sup_{a\in A}(-g(a))\right\} \tag{1.2}$$

From 1.1 and 1.2, it follows

$$\sup_{a\in A} |g(a)| = \max\left\{\sup_{a\in A} g(a),\ \sup_{a\in A}(-g(a))\right\}.$$

$\square$

Proof of Theorem 7 Let $a \in A$. Then $d(a, A) = 0$.

Thus, for $a \in A$,

$$\begin{aligned} d(a, B) = d(a, B) - d(a, A) &\leq \sup_{a\in A} [d(a, B) - d(a, A)] \\ &\leq \sup_{x\in E} [d(x, B) - d(x, A)] \end{aligned}$$

Taking supremum over $a \in A$, we have

$$d(A, B) \leq \sup_{x\in E} [d(x, B) - d(x, A)] \tag{1.3}$$

Repeating the same argument with roles of A and B interchanged, we get

$$d(B,A) \ \leq \ \sup_{x\in E}\left[d(x,A) - d(x,B)\right] \tag{1.4}$$

To get the other inequality, let $a \in A$, $b \in B$ and $x \in E$. By triangle inequality,

$$d(x,b) \ \leq \ d(x,a) + d(a,b) \tag{1.5}$$

Taking infimum over $b \in B$,

$$d(x,B) \ \leq \ d(x,a) + d(a,B) \ \leq \ d(x,a) + \sup_{a\in A} d(a,B)$$

Taking infimum over $a \in A$,

$$d(x,B) \ \leq \ d(x,A) + d(A,B)$$

Implies for $x \in E$,

$$d(x,B) - d(x,A) \ \leq \ d(A,B)$$

So that

$$\sup_{x\in E}\left[d(x,B) - d(x,A)\right] \ \leq \ d(A,B) \tag{1.6}$$

Taking infimum in 1.5 over $a \in A$ and repeating same procedures as before, we get

$$\sup_{x\in E}\left[d(x,A) - d(x,B)\right] \ \leq \ d(B,A) \tag{1.7}$$

Hence, from 1.3 and 1.6 we get

$$d(A,B) = \sup_{x\in E}\left[d(x,B) - d(x,A)\right] \tag{1.8}$$

From 1.4 and 1.7 we also get

$$d(B,A) = \sup_{x\in E}\left[d(x,A) - d(x,B)\right] \tag{1.9}$$

Taking maximum of 1.8 and 1.9 and applying Lemma 6, it follows

$$H(A,B) = \max\{d(A,B),\ d(B,A)\} = \sup_{x\in E}[d(x,A) - d(x,B)]$$

□

Let E be a normed linear space. We recall a subset, A is closed convex if and only if it is the intersection of all closed half spaces containing A, generated by supporting hyperplanes of A **?**. Moreover, supporting hyperplanes are generated by bounded linear functionals on E. We, therefore, seek to characterize Hausdorff distance of closed bounded convex subsets in terms of $f \in E^*$. The following lemma gives the distance between any point of E and special subsets of E called *hyperplanes.* Every hyperplane is convex and closed hyperplanes are level sets of $f \in E^*$. For more on hyperplanes, see **?**.

Lemma 7. *Let* $f \in E^*$ *be nonzero and let* $\alpha \in \mathbb{R}$.
If $H_{\alpha,f} \equiv f^{-1}(\alpha) := \{x \in E : \langle f, x\rangle = \alpha\}$ *is the hyperplane generated by* α *and* f, *then for any* $x_0 \in E$,

$$d(x_0, H_{\alpha,f}) = \frac{|\langle f, x_0\rangle - \alpha|}{\|f\|_{E^*}} \tag{1.10}$$

Proof Let $x_0 = 0$. If $\alpha = 0$, then $\langle f, x\rangle = 0$ implies $0 \in H_{\alpha,f}$
So that

$$d(x_0, H_{\alpha,f}) = d(0, H_{\alpha,f}) = 0 = \frac{|\langle f, 0\rangle - 0|}{\|f\|_{E^*}}.$$

If $\alpha \neq 0$, take $y = \frac{\alpha x}{\langle f,x\rangle}$. Then $x = \frac{\langle f,x\rangle y}{\alpha}$
and by linearity of f, $\langle f, y\rangle = f\left(\frac{\alpha x}{\langle f,x\rangle}\right) = \alpha$ implies $y \in H_{\alpha,f}$.

Thus, by definition and for this $y \in H_{\alpha,f}$,

$$\|f\|_{E^*} = \sup_{x\neq 0} \frac{|\langle f, x\rangle|}{\|x\|} = \sup_{\frac{\langle f,x\rangle y}{\alpha}\neq 0} \frac{\left|\langle f, \frac{\langle f, x\rangle y}{\alpha}\rangle\right|}{\left\|\frac{\langle f, x\rangle y}{\alpha}\right\|} = \sup_{y\neq 0} \frac{|\langle f, y\rangle|}{\|y\|}$$

$$= \sup_{y\in H_{\alpha,f}} \frac{|\alpha|}{\|y\|}$$

$$= \frac{|\alpha|}{\inf_{y\in H_{\alpha,f}} \|y\|}$$

$$= \frac{|\alpha|}{d(0, H_{\alpha,f})} \qquad (1.11)$$

Hence for $x_0 = 0$,

$$d(0, H_{\alpha,f}) = \frac{|\langle f, 0\rangle - \alpha|}{\|f\|_{E^*}}.$$

Let $x_0 \neq 0$. To establish 1.10 in this case, we need the following

Fact 1. With same notations as in Lemma 7,

$$x_0 - H_{\alpha,f} = f^{-1}\left(\langle f, x_0\rangle - \alpha\right)$$

Proof of Fact 1 $a \in x_0 - H_{\alpha,f}$ implies there is $x \in H_{\alpha,f}$ such that $a = x_0 - x$.

Applying f to both sides, we have

$$\langle f, a\rangle = \langle f, x_0 - x\rangle = \langle f, x_0\rangle - \alpha$$

So that

$$a \in f^{-1}\left(\langle f, a\rangle\right) = f^{-1}\left(\langle f, x_0\rangle - \alpha\right)$$

Implies

$$x_0 - H_{\alpha,f} \subset f^{-1}\left(\langle f, x_0\rangle - \alpha\right) \qquad (*)$$

Similarly,

$$a \in f^{-1}\left(\langle f, x_0\rangle - \alpha\right) \text{ implies } \langle f, a\rangle = \langle f, x_0\rangle - \alpha$$

Thus,

$$\langle f, x_0 - a\rangle = \alpha \qquad \text{and} \qquad x_0 - a \in f^{-1}\left(\langle f, x_0 - a\rangle\right) = f^{-1}\left(\alpha\right)$$

Implies

$$a \in x_0 - H_{\alpha,f} \qquad \text{and} \qquad f^{-1}\left(\langle f, x_0\rangle - \alpha\right) \subset x_0 - H_{\alpha,f} \qquad (**)$$

Combining $(*)$ and $(**)$, it follows

$$x_0 - H_{\alpha,f} = f^{-1}\left(\langle f, x_0\rangle - \alpha\right)$$

and this ends the proof of Fact 1.

Now, using the fact that $d(x_0, H_{\alpha,f}) = d(0, x_0 - H_{\alpha,f})$, the result of Fact 1 and 1.11, we have

$$d(x_0, H_{\alpha,f}) = d(0, x_0 - H_{\alpha,f}) = \frac{|\langle f, x_0\rangle - \alpha|}{\|f\|_{E^*}}$$

which completes the proof of 1.10 for all values of $x_0 \in E$.

□

In what follows, for any $c \in \mathbb{R},\ c > 0$ let

$$K^* := \{f \in E^* : \|f\|_{E^*} = 1\} \quad \text{and} \quad S^* := \{f \in E^* : \|f\|_{E^*} \le c\}\,.$$

Lemma 8. *Let* $A \in \mathbf{P}_{bc}(E)$ *and* $\beta = d(0, A) > 0$. *Then there exists* $f \in K^*$ *such that*

$$\sup_{x\in \overline{B}_\beta(0)} \langle f, x\rangle \;=\; \beta \;=\; \inf_{x\in A} \langle f, x\rangle$$

Proof We claim $A \cap \mathrm{B}_\lambda(0) = \emptyset$. Else, we can find $x_0 \in A \cap \mathrm{B}_\lambda(0)$ and $d(0, A) \le \|x_0\| < \lambda$. Consequently, $d(0, A) < \lambda$ which contradicts $\lambda = d(0, A)$.

Since A is closed and $\mathrm{B}_\lambda(0)$ is nonempty and open, by Hahn Banach theorem Chidume (2014), there is $f \in K^*$ and $\lambda \in \mathbb{R}$ such that

$$\sup_{x\in \overline{\mathrm{B}}_\beta(0)} \langle f, x\rangle \;\le\; \lambda \;\le\; \inf_{x\in A} \langle f, x\rangle. \tag{1.12}$$

Let $x \in \overline{\mathrm{B}}_\beta(0)$. Then $y = \frac{x}{\lambda} \in \overline{\mathrm{B}}_1(0)$.

So that

$$\begin{aligned}1 = \|f\|_{E^*} &= \sup_{\|y\|\leq 1} |\langle f, y\rangle| \\ &= \sup_{\|\frac{x}{\lambda}\|\leq 1} \left|\langle f, \frac{x}{\lambda}\rangle\right| \\ &= \frac{1}{\lambda} \sup_{\|x\|\leq 1} |\langle f, x\rangle|\end{aligned}$$

Hence,

$$\sup_{x\in\overline{\mathrm{B}}_\beta(0)} |\langle f, x\rangle| = \lambda$$

and symmetry of balls centered at the origin, we have

$$\lambda = \sup_{x\in\overline{\mathrm{B}}_\beta(0)} |\langle f, x\rangle| = \sup_{x\in\overline{\mathrm{B}}_\beta(0)} \langle f, x\rangle \tag{1.13}$$

By Lemma 7,

$$d(0, f^{-1}(\lambda)) = \frac{|\langle f, 0\rangle - \lambda|}{\|f\|_{E^*}} = \lambda$$

and

$$\beta = d(0, A) \geq d(0, f^{-1}(\lambda)) = \lambda \tag{1.14}$$

By boundedness of f, for any $x \in A$, we have that

$$\langle f, x\rangle \leq \|f\|_{E^*}\|x\| = \|x\|$$

Implies

$$\inf_{x\in A}\langle f, x\rangle \leq \inf_{x\in A} \|x\| = d(0, A) = \beta \tag{1.15}$$

Combining 1.12, 1.13, 1.14, and 1.15, it follows

$$\sup_{x\in\overline{\mathrm{B}}_\beta(0)} \langle f, x\rangle = \beta = \inf_{x\in A}\langle f, x\rangle. \tag{†}$$

□

Remarks 2.

(i). By properties of infimum and supremum, if we multiply equation † above by negative, we obtain

$$\sup_{x\in A}\langle g, x\rangle = \beta = \inf_{x\in \overline{\mathrm{B}}_\beta(0)} \langle g, x\rangle$$

where $g =$ -f and $\lambda =$ -β

(ii). If $\beta = d(0, A) = 0$, (i.e., $0 \in A$) then equation † clearly holds for any $f \in E^*$.

These developments lead us to

Definition 3.

1). Let $A \subset E$. The convex hull of A, denoted by coA is the set of all convex combinations of A. Mathematically,

$$coA = \left\{ x \in E : x = \sum_{i=1}^{n} \alpha_i x_i,\ x_i \in A,\ \sum_{i=1}^{n} \alpha_i = 1,\ \alpha_i \in [0,1],\ n \in \mathbb{N} \right\}.$$

$\overline{co}A$ will denote the closure of coA, popularly known as closed convex hull of A. and is defined as

$$\overline{co}A = \{x \in E : \exists\ \{x_n\},\ \{y_n\} \subset A,\ \alpha_n \in [0,1],$$
$$\lim_{n\to\infty} (\alpha_n x_n + (1-\alpha_n)y_n) = 0\}.$$

2). Let $A \in \mathbf{P}_{bc}(E)$. The *support function* of A, denoted by $h_f(A)$ is defined as

$$h_f(A) \ = \ \sup_{x\in A}\langle f, x\rangle, \quad f \in E^*.$$

We remark that for $A \in \mathbf{P}(E)$, $h_f(A)$ may exists and may be achieved in, on or outside the subset A so that the support function can be defined on any $A \in \mathbf{P}(E)$ whenever it makes sense.

Also, using properties of supremum, it is easy to derive

$$h_f(A \oplus B) = \sup_{x \in A} \langle f, x \rangle + \sup_{x \in B} \langle f, x \rangle = h_f(A + B)$$

$$h_f(\alpha A) = \alpha \sup_{x \in A} \langle f, x \rangle$$

for any $A, B \in \mathbf{P}_{bc}(E)$ and for all $\alpha \geq 0$.

The following is an important characterization of $\overline{co}A$ that will be useful in the sequel.

Theorem 8. *Let $A \in \mathbf{P}(E)$. An element $x \in E$ belongs to $\overline{co}A$ if and only if*

$$\langle f, x \rangle \leq h_f(A) \quad \textit{forall} \ \ f \in E^*.$$

Proof

$\Rightarrow$). Let $x \in \overline{co}A$. Then there are $\{x_n\}$, $\{y_n\} \subset A$, $\alpha_n \in [0,1]$ such that $coA \ni (\alpha_n x_n + (1-\alpha_n) y_n) \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned}
\langle f, \alpha_n x_n + (1-\alpha_n) y_n \rangle &= \alpha_n \langle f, x_n \rangle + (1-\alpha_n) \langle f, y_n \rangle \\
&\leq \alpha_n \sup_{x \in A} \langle f, x \rangle + (1-\alpha_n) \sup_{x \in A} \langle f, x \rangle \\
&\leq [\alpha_n + (1-\alpha_n)] \sup_{x \in A} \langle f, x \rangle \\
&\leq \sup_{x \in A} \langle f, x \rangle
\end{aligned}$$

As $n \to \infty$, it follows

$$\langle f, x \rangle = \lim_{n \to \infty} \langle f, \alpha_n x_n + (1-\alpha_n) y_n \rangle \leq \sup_{x \in A} \langle f, x \rangle = h_f(A).$$

$\Leftarrow$). We proceed by contra-position.

$x \in (\overline{co}A)^c$ implies there is $r > 0$ such that $\overline{\mathrm{B}}_r(x) \subset (\overline{co}A)^c$ so that $\overline{\mathrm{B}}_r(x) \cap \overline{co}A = \emptyset$. By geometric form of Hahn Banach theorem, Chidume (2014) there is $f \in E^*$, $\beta \in \mathbb{R}$ such that

$$\sup_{x \in \overline{co}A} \langle f, x \rangle \ \leq \ \beta \ \leq \ \inf_{y \in \overline{\mathrm{B}}_r(x)} \langle f, y \rangle$$

But $A \subset \overline{co}A$ implies $\sup_{x \in A} \langle f, x \rangle \leq \sup_{x \in \overline{co}} \langle f, x \rangle$ and $\mathrm{B}_r(x) \subset \overline{\mathrm{B}}_r(x)$ Implies

$$\inf_{y \in \overline{\mathrm{B}}_r(x)} \langle f, y \rangle \leq \inf_{y \in \mathrm{B}_r(x)} \langle f, y \rangle < \langle f, y \rangle - \epsilon, \quad y \in \mathrm{B}_r(x)$$

for any sufficiently small $\epsilon > 0$.

Hence,

$$h_f(A) = \sup_{x \in A} \langle f, x \rangle < \langle f, x \rangle.$$

□

We are now ready to state and prove another equivalent form of the Hausdorff distance.

Theorem 9. *Let* $A, B \in \mathbf{P}_{bc}(E)$. *Then*

$$H(A, B) = \sup_{f \in S^*} |h_f(A) - h_f(B)| \, .$$

Proof Let $a \in A$. By Lemma 8,

$$\begin{aligned} d(a, B) = d(0, a - B) = \inf_{b \in B} \langle f, a - b \rangle &= \inf_{b \in B} (\langle f, a \rangle - \langle f, b \rangle) \\ &\leq \langle f, a \rangle - \sup_{b \in B} \langle f, b \rangle \\ &=\leq \sup_{a \in A} \langle f, a \rangle - \sup_{b \in B} \langle f, b \rangle \\ &= \sup_{f \in S^*} (h_f(A) - h_f(B)) \end{aligned}$$

Implies

$$d(A, B) = \sup_{a \in A} d(a, B) \leq \sup_{f \in S^*} (h_f(A) - h_f(B)) \tag{1.16}$$

Let $\alpha = h_f(A) - \beta = \sup_{a \in A}\langle f, a\rangle - \beta$ where $\beta = h_f(B)$.

By definition of supremum, for each $\epsilon > 0$, there is $a \in A$ such that

$$\alpha - \epsilon < \langle f, a\rangle - \beta \leq \alpha$$

In particular, for $\epsilon : 0 < \epsilon < \alpha$, we have

$$0 < \alpha - \epsilon < \langle f, a\rangle - \beta \quad \text{and} \quad \beta < \langle f, a\rangle \tag{1.17}$$

Thus, the hyperplane $f^{-1}(\beta)$ separates B and a

Using Lemma 7, we have

$$d(A, B) \geq d(a, B) \geq d(a, f^{-1}(\beta)) = \frac{|\langle f, a\rangle - \beta|}{\|f\|_{E^*}} > \alpha - \epsilon \quad \text{from } 1.17$$

Implies $\alpha - \epsilon < d(A, B)$ and $\alpha \leq d(A, B)$ since $0 < \epsilon < \alpha$ is arbitrary. Hence,

$$\sup_{f \in S^*}(h_f(A) - h_f(B)) \leq d(A, B) \tag{1.18}$$

From 1.17 and 1.18, we get

$$d(A, B) = \sup_{f \in S^*}(h_f(A) - h_f(B)).$$

Repeating the same arguments, but now with $b \in B$, we also obtain

$$d(B, A) = \sup_{f \in S^*}(h_f(B) - h_f(A))$$

Combining these with the result of Lemma 6, it follows

$$\begin{aligned} H(A, B) &= \max\{d(A, B), d(B, A)\} \\ &= \max\left\{\sup_{f \in S^*}(h_f(A) - h_f(B)), \sup_{f \in S^*}(\text{-}\,[(h_f(A) - h_f(B)])\right\} \\ &= \sup_{f \in S^*}|(h_f(A) - h_f(B))|. \end{aligned}$$

□

Corollary 1. *Let* $A \in \mathbf{P}_{bc}(E)$. *Then*

$$\|A\|_{\mathbf{P}} = \sup_{f \in S^*} |h_f(A)|$$

Proof Take $B = \{0\}$ in Theorem 9. Then

$$\|A\|_{\mathbf{P}} = H(A, \{0\}) = \sup_{f \in S^*} |h_f(A) - h_f(\{0\})|$$
$$= \sup_{f \in S^*} |h_f(A)|$$

□

2 Hausdorff and Related Convergences

We now consider what it means for a sequence, say, $\{A_n\} \subset \mathbf{P}_b(E)$, $n \geq 1$ to go to $A \in \mathbf{P}_b(E)$ as $n \to \infty$. Equivalent forms of the Hausdorff distance as treated in the previous section suggest different convergences of this sequence. We will, therefore, take our reference point to be convergence in H.

Definition 4. Let $\{A_n\} \subset \mathbf{P}(E)$, $n \geq 1$ and $A \in \mathbf{P}(E)$.

1). (*H-Convergence*) We say that $\{A_n\}$ converges to A in *Hausdorff*, denoted by $H : A_n \to A$ as $n \to \infty$ or $\lim\limits_{H:n\to\infty} A_n = A$ if

$$H(A_n, A) \to 0 \text{ as } n \to \infty.$$

2). (*We-Convergence*) The sequence, $\{A_n\}$ converges *weakly* to A, denoted by We:$A_n \to A$ as $n \to \infty$ or $\lim\limits_{We:n\to\infty} A_n = A$ if for any $f \in E^*$,

$$h_f(A_n) \to h_f(A) \text{ as } n \to \infty.$$

Furthermore, we say $\{A_n\}$ converges to A *uniformly on a bounded subset of* E^* if

$$\sup_{f\in S^*} |h_f(A_n) - h_f(A)| \to 0 \text{ as } n \to \infty$$

3). (Wj-*Convergence*) $\{A_n\}$ converges to A in *Wijsman*, denoted by Wj : $A_n \to A$ as $n \to \infty$ or $\lim\limits_{\mathrm{Wj}:n\to\infty} A_n = A$ if for any $x \in E$,

$$d(x, A_n) \to d(x, A) \text{ as } n \to \infty.$$

Furthermore, we say $\{A_n\}$ converges to A *uniformly on* E if

$$\sup_{x\in E} |d(x, A_n) - d(x, A)| \to 0 \text{ as } n \to \infty$$

We will now explore sequences that converge according to these definitions and make some comparisons between the convergences.

Theorem 10. *Let E be a metric space. Suppose $\{A_n\} \subset \mathbf{P}_k(E)$, $n \geq 1$ is a non-increasing sequence and $A = \bigcap\limits_{n\geq 1} A_n$. Then*
1). *$A_n \to A$ as $n \to \infty$ in Hausdorff;*
2). *$A_n \to A$ as $n \to \infty$ in Wijsman;*
3). *If A_n is convex for all $n \geq 1$, then $A_n \to A$ as $n \to \infty$ weakly.*

Proof A is compact if A_n's are compact since arbitrary intersections of compact sets is compact. Also, by non-decreasingness of $\{A_n\}$, A is convex if A_n's are convex. Hence, $A \in \mathbf{P}_k(E)$ and $A \in \mathbf{P}_{kc}(E)$ respectively.
1). Let $\epsilon > 0$. By definition, $A = \bigcap\limits_{n\geq 1} A_n \subset A_n$ for all $n \geq 1$ implies $A \subset A_n \subset (A_n)_{\frac{\epsilon}{2}}$
Similarly, $A = \bigcap\limits_{n\geq 1} A_n$ implies $A^c = \bigcup\limits_{n\geq 1} A_n^c$; so that E can be covered as follows

$$X = A_{\frac{\epsilon}{2}} \cup \left(\bigcup_{n\geq 1} A_n^c \right).$$

In particular,

$$A_1 \subset A_{\frac{\epsilon}{2}} \cup \left(\bigcup_{n \geq 1} A_n^c \right)$$

By compactness of A_1, we can find $\{n_1, n_2, \ldots, n_k\} \subset \mathbb{N}$ such that

$$A_1 \subset A_{\frac{\epsilon}{2}} \cup \left(\bigcup_{i=1}^{k} A_{n_i}^c \right)$$

Since A_n's are non-increasing, there is $i_0 : 1 \leq i_0 \leq k$ for which $A_{n_i}^c \subset A_{n_{i_0}}^c$ for all $i \in \{1, 2, \ldots, k\}$. Thus, $A_1 \subset A_{\frac{\epsilon}{2}} \cup A_{n_{i_0}}^c$ implies $A_1 \cap A_{\frac{\epsilon}{2}} \cap A_{n_{i_0}}^c = \emptyset$ and $A_n \cap A_{\frac{\epsilon}{2}}^c = \emptyset$ for all $n \geq n_{i_0}$ from which $A_n \subset A_{\frac{\epsilon}{2}}$, $n \geq n_{i_0}$ Taking $n_\epsilon = \max\{1, n_{i_0}\}$, it follows from Lemma 3 that $H(A_n, A) < \epsilon$ for all $n \geq n_\epsilon$.

2). Using Theorem 7, we have

$$\sup_{x \in E} |d(x, A_n) - d(x, A)| = H(A_n, A)$$

Since $A_n \to A$ as $n \to \infty$ in Hausdorff as proved in Theorem 10 (1), it follows

$$\sup_{x \in E} |d(x, A_n) - d(x, A)| \to 0 \quad \text{as} \quad n \to \infty$$

Hence, $d(x, A_n) \to d(x, A)$ as $n \to \infty$ for all $x \in E$ as required.

3). Same lines of arguments as in Theorem 10 (2) but now, using the result of Theorem 9.

□

Theorem 11. *Let E be a metric space and $\{A_n\} \subset \mathbf{P}_b(E)$. If*
1). $A_n \to A$ as $n \to \infty$ in Hausdorff then $A_n \to A$ as $n \to \infty$ in Wijsman.

2). *$A_n \to A$ as $n \to \infty$ uniformly on E then $A_n \to A$ as $n \to \infty$ in Hausdorff.*

Proof

1). Since $H(A_n, A) \to 0$ as $n \to \infty$, from Theorem 7, we have

$$\sup_{x \in E} |d(x, A_n) - d(x, A)| = H(A_n, A) \to 0 \quad \text{as} \quad n \to \infty$$

Implies $d(x, A_n) \to d(x, A)$ as $n \to \infty$ for all $x \in E$ as required.

2). This follows also from Theorem 7. i.e.,

$$H(A_n, A) = \sup_{x \in E} |d(x, A_n) - d(x, A)| \to 0 \text{ as } n \to \infty$$

since $A_n \to A$ as $n \to \infty$ uniformly on E.

□

Theorem 12. *Let E be a metric space and $\{A_n\} \subset \mathbf{P}_{bc}(E)$. If*
1). $A_n \to A$ as $n \to \infty$ in Hausdorff then $A_n \to A$ as $n \to \infty$ Weakly.
2). $A_n \to A$ as $n \to \infty$ uniformly on any bounded subset of E^* then $A_n \to A$ as $n \to \infty$ in Hausdorff.

Proof

1). Since $H(A_n, A) \to 0$ as $n \to \infty$, from Theorem 9, we have

$$\sup_{f \in S^*} |h_f(A_n) - h_f(A)| = H(A_n, A) \to 0 \quad \text{as} \quad n \to \infty$$

Implies $h_f(A_n) \to h_f(A)$ as $n \to \infty$ for all $f \in E^*$ as required.

2). This follows also from Theorem 9. i.e.,

$$H(A_n, A) = \sup_{f \in S^*} |h_f(A_n) - h_f(A)| \to 0 \text{ as } n \to \infty$$

since $A_n \to A$ as $n \to \infty$ uniformly on any bounded subset of E^*.

□

2.1 Kuratowski - Mosco Convergence

The Hausdorff definition of a metric for hyperspaces has been generalized by many scholars. While some introduced metrics, some proposed pseudometrics and others simply use convergences that are comparable to the Hausdorff metric, see Michael (1951), Kuratowski (1966). For convex closed subsets of a normed linear space, E (with dual E^*), Umberto Mosco(1971) extended the work of Kuratowski to include the weak topology on E, leading to what is known today as Kuratowski - Mosco convergence. It is evident convexity on hyperspaces is required so that limits of weakly convergent sequences will be closed.
We now define the convergence and make general comparisons with other convergences we have established.

Let E be a Banach space and E^* its dual. By $x = \lim_{n\to\infty} x_n$ we mean $\|x_n - x\| \to 0$ as $n \to \infty$ where $\|\cdot\|$ is the norm on E. On the other hand, w-$\lim_{n\to\infty} x_n = x$ or w:$x_n \to x$ as $n \to \infty$ will mean x is the weak limit of $\{x_n\} \subset E$. We recall $x =$ w-$\lim_{n\to\infty} x_n$ if and only if $f(x_n) \to f(x)$ as $n \to \infty$ for any $f \in E^*$.

Definition 5. Let $\{A_n\} \subset \mathbf{P}_c(E)$ $n \geq 1$.
1). The *Kuratowski limit inferior* of $\{A_n\}$, denoted by $\underline{\mathrm{Lim}} A_n$ is defined as

$$\underline{\mathrm{Lim}} A_n = \left\{ x \in E : \exists\, n_0 \in \mathbb{N},\ \forall\, n \geq n_0,\ x_n \in A_n,\ \ x = \lim_{n\to\infty} x_n \right\}.$$

2). The *Kuratowski limit superior* of $\{A_n\}$, denoted by $\overline{\text{Lim}}A_n$ is defined as

$$\overline{\text{Lim}}A_n = \left\{x \in E : \ \forall\ k \geq 1,\ \exists\ x_{n_k} \in A_{n_k}, \quad x = \lim_{k\to\infty} x_{n_k}\right\}.$$

3). The *Mosco limit superior* of $\{A_n\}$, denoted by w-$\overline{\text{Lim}}A_n$ is defined as

$$\text{w-}\overline{\text{Lim}}A_n = \left\{x \in E : \ \forall\ k \geq 1,\ \exists\ x_{n_k} \in A_{n_k}, \quad x = \text{w-}\lim_{k\to\infty} x_{n_k}\right\}.$$

4). The sequence $\{A_n\}$ is said to converge to A in *Kuratowski - Mosco*, denoted by $\text{km} : A_n \to A$ or $\lim\limits_{\text{km}:n\to\infty} A_n = A$ if

$$\underline{\text{Lim}}A_n = A = \text{w-}\overline{\text{Lim}}A_n$$

Remarks 3. 1). It is evident from definition that $\underline{\text{Lim}}A_n \subset \overline{\text{Lim}}A_n \subset \text{w-}\overline{\text{Lim}}A_n$. Hence, to prove $A_n \to A$ as $n \to \infty$ in Kuratowski - Mosco, it suffices to show

$$\text{w-}\overline{\text{Lim}}A_n \subset A \subset \underline{\text{Lim}}A_n.$$

2). The Kuratowski limit inferior, $\underline{\text{Lim}}A_n$, Kuratowski limit superior, $\overline{\text{Lim}}A_n$ and Mosco limit superior, w-$\overline{\text{Lim}}A_n$ are quite different from the usual definitions of limit inferior and superior denoted by $\underline{\lim}A_n$ and $\overline{\lim}A_n$ respectively.

But $x \in \underline{\lim}A_n$ if and only if $x \in A_n^c$ finitely often (f.o.) and $x \in \overline{\lim}A_n$ if and only if $y \in A_n$ infinitely often (i.o.) Lo (2017a). So taking $x_n = x$ and $y_{n_k} = y,\ k \geq 1$ we see that $\{x_n\}$ converges strongly to x and $\{y_n\}$ converges strongly to y. This implies $x \in \underline{\text{Lim}}A_n$ and $y \in \overline{\text{Lim}}A_n$.

Hence, $\underline{\lim}A_n \subset \underline{\text{Lim}}A_n$ and $\overline{\lim}A_n \subset \overline{\text{Lim}}A_n \subset \text{w-}\overline{\text{Lim}}A_n$.

Theorem 13. *Let E be a Banach space.*

1). *If* $\{A_n, A\} \subset \mathbf{P}_c(E)$ *and* $H(A_n, A) \to 0$ *as* $n \to \infty$, *then* $\mathrm{km} : A_n \to A$;

2). *If* $\dim E < \infty$, $\{A_n, A\} \subset \mathbf{P}_k(E)$ *and* $\mathrm{km} : A_n \to A$, *as* $n \to \infty$ *then* $H(A_n, A) \to 0$.

Proof Let $x \in A$. Then $d(x, A) = 0$. By definition of $H(A, A_n)$, there is $x_n \in A_n$, $n \geq 1$ such that

$$\|x - x_n\| \leq d(x, A_n) + \frac{1}{n}.$$

1). Since $H(A_n, A) \to 0$ as $n \to \infty$ and Hausdorff convergence implies Wijsman convergence Theorem 10 (2), we have that $d(x, A_n) \to d(x, A)$ implies $x_n \to x$ as $n \to \infty$. Consequently, $x \in \underline{\mathrm{Lim}} A_n$ and $A \subset \underline{\mathrm{Lim}} A_n$.

Let $x \in$ w-$\overline{\mathrm{Lim}} A_n$. Then for $k \geq 1$, there exists $x_{n_k} \in A_{n_k}$ such that w:$x_{n_k} \to x$ as $k \to \infty$ and $\langle f, x_{n_k} \rangle \leq h_f(A)_{n_k}$ for any $f \in E^*$ from Theorem 8. But from Theorem 10 (3), $H(A_n, A) \to 0$ implies $h_f(A_n) \to h_f(A)$ as $n \to \infty$. Hence, $h_f(A_{n_k}) \to h_f(A)$ implies

$$\langle f, x \rangle = \lim_{k \to \infty} \langle f, x_{n_k} \rangle \leq \lim_{k \to \infty} h_f(A)_{n_k} = h_f(A).$$

It follows from Theorem 8 again that $x \in A$ so that w-$\overline{\mathrm{Lim}} A_n \subset A$ which completes the proof of the first part.

2). $\dim E < \infty$ implies strong and weak topologies are equivalent, so that $\overline{\mathrm{Lim}} A_n =$ w-$\overline{\mathrm{Lim}} A_n$. This gives us km:$A_n \to A$ if and only if $\underline{\mathrm{Lim}} A_n = A = \overline{\mathrm{Lim}} A_n$.

Let $\epsilon > 0$. Since A is compact and, therefore, totally bounded, we can find $\{x_1, x_2, \ldots, x_k\} \subset A$ such that $A \subset \cup_{i=1}^{k} \mathrm{B}_{\frac{\epsilon}{4}}(x_i)$. Now, $x_i \in A = \underline{\mathrm{Lim}} A_n$ implies there is $x_i^n \in \underline{\mathrm{Lim}} A_n$, $n_i \in \mathbb{N}$ such that for each $n \geq n_i$,

$$d(x_i, A_n) \leq \|x_i^n - x_i\| \leq \frac{\epsilon}{4} \text{ for all } i \in \{1, 2, \ldots, k\}.$$

Let $n_0 = \max\{n_i : i \in \{1, 2, \ldots, k\}\}$. Then $\{x_1, x_2, \ldots, x_k\} \subset (A_n)_{\frac{\epsilon}{4}}$ for all $n \geq n_0$. Thus, $A \subset \cup_{i=1}^{k} \mathrm{B}_{\frac{\epsilon}{4}}(x_i) \subset (A_n)_{\frac{\epsilon}{2}}$.

Similarly, Since compactness implies boundedness and A and A_n are compact, we can find $p > 0$ $N \in \mathbb{N}$ such that $A \subset \mathrm{B}_p(0)$ and $A_n \subset \mathrm{B}_p(0)$ for each $n \geq N$. We claim $A_n \cap \left(A_{\frac{\epsilon}{2}}\right)^c = \emptyset$ for all $\epsilon > 0$, $n \geq N$. Suppose this is not true. Then, there exists a subsequence, $\{A_{n_j}\} \subset \{A_n\}$ such that $A_{n_j} \subset A_n \cap \left(A_{\frac{\epsilon}{2}}\right)^c \neq \emptyset$. Take $x_{n_j} \in A_{n_j}$ and let $j \geq N$. Then $d(x_{n_j}, A) > \frac{\epsilon}{2}$ and

$$x_{n_j} \in A_{n_j} \subset A_n \cap \left(A_{\frac{\epsilon}{2}}\right)^c \subset \mathrm{B}_p(0).$$

This implies $\{x_{n_j}\}_{j=1}^{\infty}$ is a bounded sequence. By Bolzano-Weierstrass theorem, there exists $\{x_{n_{j_i}}\}_{i=1}^{\infty} \subset \{x_{n_j}\}$ such that $x_{n_{j_i}} \to x$ as $i \to \infty$. Hence, $0 = d(x, A) = \lim_{i\to\infty}(x_{n_{j_i}}, A) > \frac{\epsilon}{2}$ which is a contradiction. We conclude $A_n \cap \left(A_{\frac{\epsilon}{2}}\right)^c = \emptyset$ from which we get $A_n \subset A_{\frac{\epsilon}{2}}$ for all $n \geq N$. By Lemma 3, it follows that $H(A_n, A) < \epsilon$ for all $n \geq \max n_0, N$.

□

To round up our comparison between these convergences, we need the following lemmas

Lemma 9. *Let $\{A_n, A\} \subset \mathbf{P}_{kc}(E)$. If $A_n \to A$ as $n \to \infty$ weakly. Then $\{A_n\}$ is uniformly bounded.*

Proof Let $\epsilon > 0$. By hypothesis, we can find $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$, $|h_f(A_n) - h_f(A) < \epsilon$. By triangle inequality,

$$|h_f(A_n)| - |h_f(A)| \leq |h_f(A_n) - h_f(A)| < \epsilon$$

So that $|h_f(A_n)| < \epsilon + |h_f(A)|$ for all $n \geq n_\epsilon$.

Let $M = \max\{|h_f(A_1), \cdots, |h_f(A_{n_\epsilon - 1})\}$. Using Corollary 1 we have for each $n \in \mathbb{N}$,

$$\|A_n\|_{\mathbf{P}} = \sup_{f \in S^*} |h_f(A_n)| < \epsilon + \sup_{f \in S^*} |h_f(A)| + M \leq C \text{ for some } C \in \mathbb{R}.$$

Hence, $\sup_{n \in \mathbb{N}} \|A_n\|_{\mathbf{P}} \leq C$.

□

Lemma 10. *Let* $\{f_n, f\} \subset E^*$ *and* $A \subset E$ *be bounded. If* $\|f_n - f\|_{E^*} \to 0$ *as* $n \to \infty$, *then* $h_{f_n}(A) \to h_f(A)$.

Proof Let $A \subset E$ be as stated. Then $\|A\|_{\mathbf{P}} < \infty$. Hence, for $f_n, f \in E^*$,

$$\begin{aligned} |h_{f_n}(A) - h_f(A)| &= |\sup_{x \in A} \langle f_n, x\rangle - \sup_{x \in A} \langle f, x\rangle| \\ &\leq |\sup_{x \in A} \langle f_n, x\rangle - \langle f, x\rangle| \quad \text{for all} \quad x \in A \\ &\leq \sup_{x \in A} |\langle f_n - f, x\rangle| \\ &\leq \|f_n - f\|_{E^*} \|A\|_{\mathbf{P}} \longrightarrow 0 \text{ as } n \to \infty \end{aligned}$$

So that $|h_{f_n}(A) - h_f(A)| \to 0$ as $n \to \infty$.

□

Theorem 14. *Let* $\{A_n, A\} \subset \mathbf{P}_{kc}(E)$ *and* $\dim E < \infty$. *Then the following assertions are equivalent:*

(1). H : $A_n \to A$;

(2). Wj : $A_n \to A$;

(3). km : $A_n \to A$;

(4). We : $A_n \to A$.

Proof From Theorem 11, (1) $\Rightarrow$ (2). To see (2) $\Rightarrow$ (3), suppose that Wj:$A_n \to A$. Then for each $x \in A$, we have $\lim_{n\to\infty} d(x, A_n) = d(x, A) = 0$. But by definition of infimum, for all $n \in \mathbb{N}$ we can find $x_n \in A_n$ such that $\|x - x_n\| < d(x, A_n) + \frac{1}{n}$. Thus, $x_n \to x$ as $n \to \infty$ so that $x \in \underline{\text{Lim}} A_n$.

On the other hand, let $x \in$ w-$\overline{\text{Lim}} A_n$. Then for each $k \geq 1$, there is $x_{n_k} \in A_{n_k}$ such that w:$x_{n_k} \to x$. Equivalently, $x_{n_k} \to x$ since $\dim E < \infty$. Hence,

$$d(x, A_{n_k}) \leq \|x - x_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty$$

Implies

$$d(x, A) = \lim_{n\to\infty} d(x, A_n) = \lim_{n\to\infty} d(x, A_{n_k}) = 0$$

Thus, we get $x \in A$ and $\underline{\text{Lim}} A_n =$ w-$\overline{\text{Lim}} A_n$ as required.

From Theorem 13(2), (3) $\Rightarrow$ (1). So we have shown the equivalence of (1), (2) and (3).

(1) $\Rightarrow$ (4) from Theorem 12 (1). To complete the chain of equivalence, it suffices to prove (4) implies (1). To see this, we proceed by contradiction.

Suppose $h_f(A_n) \to h_f(A)$ but $\{A_n\}$ does not converge in Hausdorff. Then from Theorem 9, we can find $\epsilon_0 > 0$ such that for all $N \in \mathbb{N}$, there exists $n \geq N$ for which

$$\sup_{f\in S^*} |h_f(A_n) - h_f(A)| = H(A_n, A) \geq \epsilon_0.$$

By definition of supremum, for each $k \geq 1$ there exists $f_k \in S^*$ such that

$$\epsilon_0 - \frac{1}{k} \leq \sup_{f\in S^*} |h_f(A_n) - h_f(A)| - \frac{1}{k} < |h_{f_k}(A_{n_k}) - h_{f_k}(A)|$$

Thus,

$$\liminf_{i\to\infty} |h_{f_k}(A_{n_k}) - h_{f_k}(A)| \geq \epsilon_0 > 0. \tag{2.1}$$

Since $\dim E < \infty$ implies unit closed balls, $S^* \in E^*$ are compact, for each $i \geq 1$ we can find $\{f_{k_i}\} \subset \{f_k\}$ and $f \in S^*$ such that $\|f_{k_i} - f\|_{E^*} \to 0$ as $k \to \infty$

Since weak convergence of $\{A_n\}$ implies $\{A_{n_{k_i}}\}$ converges weakly to the same limit, we conclude by Lemma 9 that $\{A_{n_{k_i}}\}$ is uniformly bounded over $i \in \mathbb{N}$. So by Lemma 10, $h_{f_{k_i}}(A_{n_{k_i}}) \to h_f(A_{n_{k_i}})$ and $h_{f_{k_i}}(A) \to h_f(A)$ as $i \to \infty$.

Hence, by 2.1 and triangle inequality, we have

$$\begin{aligned}
0 < \liminf_{k\to\infty} |h_{f_{k_i}}(A_{n_{k_i}}) - h_{f_{k_i}}(A)| &= \liminf_{k\to\infty} |h_{f_{k_i}}(A_{n_{k_i}}) - h_f(A_{n_{k_i}}) \\
&\qquad + h_f(A_{n_{k_i}}) - h_f(A) \\
&\qquad\qquad + h_f(A) - h_{f_{n_{k_i}}}(A)| \\
&\leq \liminf_{k\to\infty} |h_{f_{k_i}}(A_{n_{k_i}}) - h_f(A_{n_{k_i}})| \\
&\qquad + \liminf_{k\to\infty} |h_f(A_{n_{k_i}}) - h_f(A)| \\
&\qquad\qquad + \liminf_{k\to\infty} |h_f(A) - h_{f_{n_{k_i}}}(A)| \\
&= \liminf_{i\to\infty} |h_f(A_{n_{k_i}}) - h_f(A)|
\end{aligned}$$

So that

$$\liminf_{i\to\infty} |h_f(A_{n_{k_i}}) - h_f(A)| > 0$$

contradicts the hypothesis that $\{A_n\}$ converges weakly to A.

□

CHAPTER 3

Measurability of Set-Valued Maps and Classes of Integrable Maps

1 Set-Valued Maps

Recent developments in set analysis have given hope to some mathematical, physics, engineering and economics models whose solutions are not generally unique. Moreover, advances in scientific computing paved way for deeper study of scientific experiments whose outcomes are generally set-valued, for instance in image recovery, signal processing, artificial intelligence and control theory.
The study of *set-valued maps* Aubin and Frankowska (1990), *multi-valued maps* Robinson (1976), *multifunctions* **?**, *correspondences* Aliprantis and Border (2007), as different authors call it, has attracted the attention of many scholars and has been developed extensively, with applications to mathematical economics and optimal control problems Hiai and Umegaki (1977).
We will look at the peripheral of the general set-valued maps and then focus more on how it is used in probability theory.

Definition 6. Let X and E be metric spaces. If for each $x \in X$ we can find a nonempty set $F(x) \subset E$, then F is called a set-valued map from X to $\mathcal{P}(E)$, and we write $F : X \longrightarrow \mathcal{P}(E)$. It is evident if we consider $f : X \longrightarrow E$ and let $F(x) := \{f(x)\}$ then F is a set-valued map.

2 Sources of Set-valued Maps

Many practical set-valued maps originate from our day to day mathematics. Examples include:

(1). an attempt to find an inverse image for a non-injective function

Consider $f : \mathbb{R} \longrightarrow [0, \infty)$ given by $f(x) = x^2$. Then $f^{-1}(y) = \{-\sqrt{y}, \sqrt{y}\}$.

(2). solution sets of metric projections

Let $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The map $P_A : x \longmapsto P_A(x)$ defined as

$P_A(x) = \{y \in A : d(x, y) = d(x, A)\}$

called the metric projection of $\mathbb{R}^n$ onto A is a set-valued map.

(3). the subdifferential map of a convex function

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex function. Then for each $x \in \mathbb{R}^n$, the map

$$\partial f(x) := \{z \in \mathbb{R}^n : \langle z, y - x\rangle + f(x) \leq f(y) \ \forall \ y \in \mathbb{R}^n\},$$

called the subdifferential map of f, is a set-valued map.

(4). the solution set of an optimization problem.

(5). assignment of sets to outcomes of random experiments.

Complete exposure of set-valued maps on topological spaces can be found in Aubin and Frankowska (1990), **?**

We now confine the scope of our set-valued maps to those defined on measurable spaces.

3 Basic Definitions

Let $(\Omega, \mathcal{A}, \mu)$ be an abstract measure space where $\mathcal{A}$ is a σ-algebra and μ, a measure, both on the set Ω. We recall $\mathcal{A}$ is a σ-algebra on the set Ω if

(i). $\emptyset,\ \Omega \in \mathcal{A}$;

(ii). $A \in \mathcal{A} \ \Rightarrow\ A^c = \Omega \setminus A \in \mathcal{A}$;

(iii). $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A} \ \Rightarrow\ \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Also, μ is a measure on Ω if $\mu : \mathcal{A} \to \mathbb{R}$ is non-negative, proper and σ-additive. If $\mu(A) = 0$, then $A \subset \Omega$ is called a null set. If $\mu(\Omega) < \infty$, then μ is said to be a finite measure on Ω. On the other hand, if we can find $\{A_n\} \subset \mathcal{A}$ such that for each $n \in \mathbb{N},\ \mu(A_n) < \infty$ and $\Omega = \bigcup_{n=1}^{\infty} A_n$ then we say that μ is σ-finite (see Lo (2017a)).
For each $A \subset \Omega$, the outer measure μ^* on Ω is defined as

$$\mu^*(A) = \inf\{\mu(D) : \quad A \subset D, \quad D \in \mathcal{A}\}.$$

The measure, μ is said to be complete if $\mu = \mu^*$. We remark that from this definition, if a measure is complete, then every subset of a null set is measurable and has measure zero.

Let (E, d) be a metric space and $F : \Omega \to \mathcal{P}(E)$, a set-valued map. The domain of F, denoted by $\operatorname{dom}(F)$ is given as

$$\operatorname{dom}(F) := \{\omega \in \Omega : F(\omega) \neq \emptyset\}$$

The range of F, denoted by $\operatorname{rang}(F)$ is given as

$$\operatorname{rang}(F) := \bigcup_{\omega \in \operatorname{dom}(F)} F(\omega)$$

The graph of F, denoted by $\mathcal{G}_F$ is defined as

$$\mathcal{G}_F := \{(\omega, x) \in \Omega \times E : x \in F(\omega),\ \omega \in \mathrm{dom}(F)\}$$

Consequently, since $x \in F(\omega)$ implies $d(x, F(\omega)) = 0$,

$$\mathcal{G}_F = \{(\omega, x) \in \Omega \times E : d(x, F(\omega)) = 0\}.$$

The preimage of $A \in \mathcal{P}(E)$, denoted by $F^{-1}(A)$ is defined as

$$F^{-1}(A) := \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\} = \bigcup_{y \in A} F^{-1}(\{y\}).$$

It is easy to see that the preimage preserves union.

4 Random Sets

Definition 7. Let (E, d) be a metric space and $F : \Omega \to \mathbf{P}(E)$, a set-valued map.

(i). F is said to be *strongly measurable* if for every $A \subset E$ closed, we have $F^{-1}(A) = \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\} \in \mathcal{A}$.

(ii). F is said to be *weakly measurable* or simply *measurable* if for every $A \subset E$ open, we have $F^{-1}(A) = \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\} \in \mathcal{A}$.

Weakly measurable set-valued maps are called *set-valued random variables or random sets*. We remark that by definition of measurability of F,

$$\mathrm{dom}(F) := \{\omega \in \Omega : F(\omega) \neq \emptyset\} \in \mathcal{A}.$$

To see the relationship that exists between strong and weak measurabilities of set-valued maps, we need

Lemma 11. *Let (E,d) be a metrizable space, assuming d is the metric that induces a topology on E. If $A \subset E$ is open, then for each $n \in \mathbb{N}$, there exists $B_n \subset E$ closed such that $A = \bigcup_{n=1}^{\infty} B_n$ (i.e., A is a countable union of closed subsets).*

Proof If $A = \emptyset$, take $B_n = \emptyset$ and we are done.
If $A \neq \emptyset$, consider $B_n := \{x \in E : d(x, A^c) \geq \frac{1}{n}\}, n \in \mathbb{N}$. Since A^c is closed and $d(\cdot, A^c)$ is continuous, it follows that B_n is closed for each $n \in \mathbb{N}$. Since A^c is closed,

$$\begin{aligned} x \in A &\iff x \notin A^c \\ &\iff d(x, A^c) \geq \frac{1}{n} \qquad \text{for some} \quad n \in \mathbb{N} \\ &\iff x \in B_n \qquad \text{for some} \quad n \in \mathbb{N} \end{aligned}$$

Implies $A = \bigcup_{n=1}^{\infty} B_n$. □

Theorem 15. *Let E be a metric space and $F : \Omega \to \mathbf{P}(E)$, a strongly measurable set-valued map. Then F is a set-valued random variable.*

Proof Let A be open in E. If $A = E$, then $F^{-1}(A) = \Omega \in \mathcal{A}$. It follows trivially if $A = \emptyset$.
Suppose $\emptyset \neq A \neq E$. By Lemma 11, for each $n \in \mathbb{N}$, we can find $B_n \subset E$ closed such that $A = \bigcup_{n=1}^{\infty} B_n$.
Hence,

$$\begin{aligned} F^{-1}(A) &= F^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) \\ &= \bigcup_{n=1}^{\infty} F^{-1}(B_n) \in \mathcal{A}. \end{aligned}$$

□

Let $\mathcal{B}_E$ be the Borel σ-algebra generated by open subsets of E. Since every closed set belongs to $\mathcal{B}_E$, it is evident if $A \in \mathcal{B}_E$ implies $F^{-1}(A) \in \mathcal{A}$ then F is strongly measurable. Also, F is said to be *graph measurable* if and only if

$$\mathcal{G}_F := \{(\omega, x) \in \Omega \times E : x \in F(\omega), \omega \in \mathrm{dom}(F)\} \in \mathcal{A} \times \mathcal{B}_E.$$

We summarize the measurability conditions of F in

Theorem 16. *Let $(\Omega, \mathcal{A})$ be a measurable space, (E, d) a separable metric space and $F : \Omega \to \mathbf{P}(E)$, a set-valued map. Suppose*
(a) *for each $B \in \mathcal{B}_E$, $F^{-1}(B) \in \mathcal{A}$;*
(b) *for each closed set $C \subset E$, $F^{-1}(C) \in \mathcal{A}$;*
(c) *for each open set $O \subset E$, $F^{-1}(O) \in \mathcal{A}$;*
(d) *$\omega \longmapsto d(F(\omega), x)$ is a measurable function for each $x \in E$;*
(e) *$\mathcal{G}_F$ is $\mathcal{A} \times \mathcal{B}_E$-measurable.*

Then
(1) *(a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e);*
(2) *If E is complete and $\mathcal{A}$ is complete with respect to some σ-finite measure, then all the conditions (a)-(e) are equivalent.*

Proof: (1). (a) $\Rightarrow$ (b) follows from definition of $\mathcal{B}_E$ and (b) $\Rightarrow$ (c) is proved in Theorem 15. To see (c) $\Rightarrow$ (d), assume for any open set $O \subset E$, $F^{-1}(O) \in \mathcal{A}$. Then for $\beta > 0$,

$$\begin{aligned}\{\omega \in \Omega : d(x, F(\omega)) < \beta\} &= \{\omega \in \Omega : F(\omega) \cap \mathrm{B}_\beta(x) \neq \emptyset\} \\ &= F^{-1}\left(\mathrm{B}_\beta(x)\right) \in \mathcal{A}\end{aligned}$$

Implies $\omega \longmapsto d(x, F(\omega))$ is measurable.

For (d) $\Rightarrow$ (c), suppose $\{\omega \in \Omega : d(x, F(\omega)) < \beta\} \in \mathcal{A}$ and let O be open in E. Since E is separable and consequently, second countable, for each $n \in \mathbb{N}$, we can find $\alpha_n > 0,\ x_n \in E$ such that $O = \bigcup\limits_{n=1}^{\infty} \mathrm{B}_{\alpha_n}(x_n)$. Thus,

$$\begin{aligned} F^{-1}(O) &= F^{-1}\left(\bigcup_{n=1}^{\infty} \mathrm{B}_{\alpha_n}(x_n)\right) \\ &= \bigcup_{n=1}^{\infty} F^{-1}\left(\mathrm{B}_{\alpha_n}(x_n)\right)) \\ &= \bigcup_{n=1}^{\infty} \{\omega \in \Omega : F(\omega) \cap \mathrm{B}_{\beta}(x) \neq \emptyset\} \\ &= \bigcup_{n=1}^{\infty} \{\omega \in \Omega : d(x_n, F(\omega)) < \alpha_n\} \in \mathcal{A}. \end{aligned}$$

Hence, (c) $\Leftrightarrow$ (d).

(d) $\Rightarrow$ (e). Since E is separable, let $\{x_n\}$ be dense in X. For $\beta \geq 1$, consider $d_\beta : \Omega \times E \to \mathbb{R}$ defined by $(w, x) \mapsto d_\beta(\omega, x) = d(x_n, F(\omega))$ whenever

$$x \in \left(\mathrm{B}_{\frac{1}{\beta}}(x_n) \setminus \left[\bigcup_{k=1}^{\infty} \mathrm{B}_{\frac{1}{\beta}}(x_k)\right]^c\right) \equiv C_\beta(x_n) \text{ for } n \geq 2$$

or $x \in \mathrm{B}_{\frac{1}{\beta}}(x_1) \equiv C_\beta(x_1)$ for $n = 1$.

For $\lambda > 0$,

$$\begin{aligned} &\{(\omega, x) \in \Omega \times E : d_\beta(\omega, x) < \lambda\} \\ &\qquad = \{(\omega, x) \in \Omega \times E : d(x_n, F(\omega)) < \lambda\} \cap \left(\Omega \times \bigcup_{n=1}^{\infty} C_\beta(x_n)\right) \\ &\qquad = \bigcup_{n=1}^{\infty} \left(\{\omega \in \Omega : d(x_n, F(\omega)) < \lambda\} \times C_\beta(x_n)\right) \in \mathcal{A} \times \mathcal{B}_E \end{aligned}$$

Implies d_β is $\mathcal{A} \times \mathcal{B}_E$ - measurable; consequently,

$$d(x, F(\omega)) = \lim_{\beta \to \infty} d_\beta(\omega, x)$$

is $\mathcal{A} \times \mathcal{B}_E$ - measurable. Also, by closedness of $F(\omega)$,

$$d(x, F(\omega)) = 0 \Longleftrightarrow x \in F(\omega)$$

Thus,

$$\begin{aligned}\mathcal{G}_F &= \{(\omega, x) \in \Omega \times E : x \in F(\omega)\} \\ &= \{(\omega, x) \in \Omega \times E : d(x, F(\omega)) = 0\} \in \mathcal{A} \times \mathcal{B}_E\end{aligned}$$

and this completes the proof of (1).

(2). Assuming E is complete and we can define a complete σ-finite measure on $\mathcal{A}$, to prove the equivalence of (a) - (e), following the lines of proof above it suffices to prove (e) $\Rightarrow$ (a). To this end, let $G \in \mathcal{A} \times \mathcal{B}_E$, then the projection $P_\Omega(G)$ of G on Ω belongs to $\mathcal{A}$.
Thus, for $A \in \mathcal{A} \times \mathcal{B}_E$,

$$\begin{aligned}F^{-1} &= \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\} \\ &= \{\omega \in \Omega : (\omega, x) \in \mathcal{G}_F \cap (\Omega \times A)\} \\ &= P_\Omega(\mathcal{G}_F \cap (\Omega \times A)) \in \mathcal{A}.\end{aligned}$$

□

5 Selection of a Random Set

Properties of set-valued maps derive from properties of single-valued maps; interestingly, these properties agree when the set-valued map takes on singleton values. As such, we desire to characterize a set-valued map in terms of single-valued ones. This leads us to the concept of *selection*.

Definition 8. Let $(\Omega, \mathcal{A})$ be a measurable space, E a metric space and $F : \Omega \to \mathbf{P}(E)$ a set-valued map.

(1). The E-valued map $f : \Omega \to E$ is said to be a *selection* for the map, F if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$;

(2). If f is measurable, then f is called a *measurable selection* for the set-valued map F;

(3). Suppose, in addition, $(\Omega, \mathcal{A}, \mu)$ is a measure space, then f is said to be *almost everywhere (a.e.) selection* of F if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$, μ-a.e.

It is interesting to know the link between measurability of the set-valued map, F and that of its selection, f. Fortunately, we not only have the existence of measurable selection for the random set, F under suitable conditions; we can completely characterize F in terms of its measurable selections.

Theorem 17. *Let $(\Omega, \mathcal{A})$ be a measurable space and (X, d), a complete separable metric space. Suppose $F : \Omega \longrightarrow \mathbf{P}(E)$ is a set-valued random variable. Then, F has a measurable selection.*

Proof: Let $\{x_n\}_{n\geq 1}$ be dense in X. For $\omega \in \Omega$, let $n \geq 1$ be the smallest integer such that for each $m \in \mathbb{N}$ fixed,

$$C_{m,n} := \left\{ \omega \in \Omega : F(\omega) \cap \mathrm{B}_{\frac{1}{2^m}}(x_n) \neq \emptyset \right\}.$$

Define $f_m : \Omega \to E$ by $f_m(\omega) = x_n$ for $\omega \in C_{m,n}$. Then, by measurability of F, we have

$$\begin{aligned} f_m^{-1}(x_n) &= \{\omega \in \Omega : f_m(\omega) = x_n\} \\ &= \{\omega \in \Omega : \omega \in C_n\} \\ &= \{\omega \in \Omega : F(\omega) \cap \mathrm{B}_{\frac{1}{2^m}}(x_n) \neq \emptyset\} \\ &= F^{-1}\left(B_{\frac{1}{2^m}}(x_n)\right) \in \mathcal{A} \end{aligned} \tag{5.1}$$

But for any $A \subset E$ open, A contains countably many or no points of $\{x_n\}$. So,

$$f_m^{-1}(A) = \begin{cases} \emptyset\ , & \text{if } A \text{ contains no points of } \{x_n\} \\ \\ \bigcup\limits_{k=1}^{\infty} f_m^{-1}(x_{n_k}), & \text{if } A \text{ contains } \{x_{n_k}\} \subset \{x_n\} \end{cases}$$

Implies f_m is measurable for each $m \in \mathbb{N}$, by equation 5.1 above. Moreover,

$$\begin{aligned} \omega \in f_m^{-1}(x_n) &\Longleftrightarrow d(x_n, F(\omega)) < \frac{1}{2^m} \\ &\Longleftrightarrow d(f_m(\omega), F(\omega)) < \frac{1}{2^m} \end{aligned} \tag{5.2}$$

Thus,

$$\begin{aligned} d(f_m(\omega), f_{m+1}(\omega)) &\leq d(f_m(\omega), F(\omega)) + d(F(\omega), f_{m+1}(\omega)) \\ &< \frac{1}{2^m} + \frac{1}{2^{m+1}} \\ &\leq \frac{1}{2^m} + \frac{1}{2^m} \\ &= \frac{1}{2^{m-1}} \end{aligned}$$

Implies for each $\omega \in \Omega$, $\{f_m(\omega)\}$ is a Cauchy sequence in E. By completeness of E and the closedness of $F(\omega)$, we can find $f : \Omega \to E$

such that

$$d(f(\omega), F(\omega)) = \lim_{m\to\infty} d(f_m(\omega), F(\omega)) = 0 \quad \text{from equation (5.2)}$$

Implies $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$ and measurability of f follows from that of $f_m,\ m \in \mathbb{N}$. $\square$

Theorem 18. *Suppose* $(\Omega, \mathcal{A})$ *is a measurable space,* E *is a complete separable metric space and* $F : \Omega \to \mathbf{P}(E)$ *is a set-valued map. Then the following statements are equivalent:*

(a) *F is a set-valued random variable;*

(b) *There exists a countable family* $\{f_n : n \in \mathbb{N}\}$ *of measurable selections of F such that for all* $\omega \in \Omega$,

$$F(\omega) = \overline{\{f_n(\omega) : n \in \mathbb{N}\}} \tag{5.3}$$

Proof: (a) $\Rightarrow$ (b). Suppose F is a set- valued random variable and let $\{x_n\}$ be dense in E. For each $m \in \mathbb{N}$ fixed, let $n \geq 1$ be the smallest integer such that

$$C_{m,n} := \{\omega \in \Omega : F(\omega) \cap \overline{\mathrm{B}}_{\frac{1}{2^m}}(x_n) \neq \emptyset\}$$

Define a family of closed set-valued maps as follows:

$$F_{m,n}(\omega) = \begin{cases} F(\omega) \cap \overline{\mathrm{B}}_{\frac{1}{2^m}}(x_n), & \omega \in C_{m,n} \\ \\ F(\omega), & \text{otherwise} \end{cases}$$

Then $F_{m,n}(\omega) \subset F(\omega)$ for all $\omega \in \Omega$ and for any $A \subset E$ closed,

$$\begin{aligned} F_{m,n}^{-1}(A) &= \{\omega \in \Omega : F_{m,n}(\omega) \cap A \neq \emptyset\} \\ &= \left\{\omega \in \Omega : F_{m,n}(\omega) \cap \left(\overline{\mathrm{B}}_{\frac{1}{2^m}}(x_n) \cap A\right) \neq \emptyset\right\} \\ &\qquad \cup \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\} \in \mathcal{A} \end{aligned}$$

Implies $F_{m,n}$ is a set-valued random variable. By Theorem 17, we can find $f_{m,n} : \Omega \to E$ measurable such that for all $\omega \in \Omega$, we have $f_{m,n}(\omega) \in F_{m,n}(\omega) \subset F(\omega)$. Thus by letting $m \to \infty$,

$$\overline{\{f_n(\omega) : n \in \mathbb{N}\}} \subset F(\omega) \tag{5.4}$$

On the other hand, by separability of E,

$$\begin{aligned} x \in F(\omega) &\iff x \in \overline{\mathrm{B}}_{\frac{1}{2^m}}(x_n) \text{ for each } m \in \mathbb{N} \text{ fixed and } n \in \mathbb{N} \\ &\iff d(x, x_m) \leq \frac{1}{2^m} \\ &\iff \omega \in C_{m,n}. \end{aligned}$$

Thus,

$$d(x_n, f_{m,n}(\omega)) \leq d(x_n, x) + d(x, F(\omega)) + d(F(\omega), f_{m,n}(\omega)) \leq \frac{1}{2^m}$$

From which

$$\begin{aligned} d(x, f_{m,n}(\omega)) &\leq d(x, x_n) + d(x_n, f_{m,n}(\omega)) \\ &\leq \frac{1}{2^m} + \frac{1}{2^m} = \frac{1}{2^{m-1}} \end{aligned}$$

Implies $f_{m,n}(\omega) \to x$ as $m \to \infty$.

Hence,

$$F(\omega) \subset \overline{\{f_n(\omega) : n \in \mathbb{N}\}} \tag{5.5}$$

Combining 5.4 and 5.5, we obtain 5.3 as required.

(b) $\Rightarrow$ (a). Assume there exists a countable family $\{f_n : n \in \mathbb{N}\}$ of measurable selections of F such that

$$F(\omega) = \{f_n(\omega) : n \in \mathbb{N}\} \quad \text{for all } \omega \in \Omega.$$

Since for each $n \in \mathbb{N}$ and $x \in E$, $\omega \to d(x, f_n(\omega)$ is measurable and

$$d(x, F(\omega)) = \inf_{n \in \mathbb{N}} d(x, f_n(\omega)),$$

it follows from Theorem 16, F is a set-valued random variable.

□

With Theorem 18, we have extended the measurability of set-valued random variables, now summarized in

Theorem 19. *Let $(\Omega, \mathcal{A})$ be a measurable space, (E, d) a separable metric space and $F : \Omega \to \mathbf{P}(E)$, a set-valued map. Suppose*

(a) *for each $B \in \mathcal{B}_E$, $F^{-1}(B) \in \mathcal{A}$;*

(b) *for each closed set $C \subset E$, $F^{-1}(C) \in \mathcal{A}$;*

(c) *for each open set $O \subset E$, $F^{-1}(O) \in \mathcal{A}$;*

(d) *There exists a countable family $\{f_n : n \in \mathbb{N}\}$ of measurable selections of F such that for all $\omega \in \Omega$, $F(\omega) = \overline{\{f_n(\omega) : n \in \mathbb{N}\}}$;*

(e) *$\omega \longmapsto d(F(\omega), x)$ is a measurable function for each $x \in E$;*

(f) *$\mathcal{G}_F$ is $\mathcal{A} \times \mathcal{B}_E$-measurable.*

Then

(1) *(a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (e) $\Rightarrow$ (f);*

(2) *If E is complete and $\mathcal{A}$ is complete with respect to some σ-finite measure, then all the conditions (a)-(f) are equivalent.*

6 Classes of Integrable Maps

Having seen conditions for which the set-valued map $F : \Omega \to \mathbf{P}(E)$ is measurable, of interest to us now are those F's that are *integrable*, in the sense we will soon define.

In what follows, let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space; $(E, \|\cdot\|)$ be a real separable Banach space; and $\mathcal{M}(\Omega, \mathbf{P}(E))$ be the collection of all set-valued random variables $F : \Omega \to \mathbf{P}(E)$. We recall that for $1 \leq p < \infty$,

$$L^p(\Omega, E) := \left\{ f : \Omega \to E \ \textit{measurable} \ \mid \int_\Omega \|f(\omega)\|^p d\mu < \infty \right\}$$

$$L^\infty(\Omega, E) := \{ f : \Omega \to E \ \textit{measurable} \ \mid \ \|f(\omega)\| < \infty \ \ \mu\text{-a.e.} \}$$

are the Banach spaces of equivalent classes of measurable functions $f : \Omega \to E$ such that

$$\|f\|_p = \left\{ \int_\Omega \|f\|^p d\mu \right\}^{\frac{1}{p}}, \quad 1 \leq p < \infty$$
$$\|f\|_\infty = \text{ess sup}_{\omega \in \Omega} \|f(\omega)\|$$

are norms on $L^p(\Omega, E)$ and $L^\infty(\Omega, E)$ respectively.
For a set-valued random variable $F : \Omega \to \mathbf{P}(E)$, the set

$$\mathrm{I}_F^p := \{ f \in L^p(\Omega, E) : f(\omega) \in F(\omega), \quad \mu\text{-a.e.} \}$$

is called the set of p-th integrable selections of F. In particular, for $p = 1$, we have

Definition 9.

(1). A set-valued random variable $F : \Omega \to \mathbf{P}(E)$ is said to be *integrable* if I_F^1 is not empty.

(2). F is called *integrably bounded* if there is $g \in L^1(\Omega, \mathbb{R})$ such that $\|x\| \leq g(\omega)$ for any $x \in E$ and $\omega \in \Omega$ with $x \in F(\omega)$.

By this definition and Remark 1 (2), it follows F is integrably bounded if and only if there is $g \in L^1(\Omega, \mathbb{R})$ such that

$$\|F(\omega)\|_{\mathbf{P}} = \sup_{x \in F(\omega)} \|x\| \leq g(\omega) \quad \mu\text{-a.e. on } \Omega.$$

From Definition 9(1), to show a set-valued random variable is integrable, it suffices to show I_F^1 is non-empty. Consequently, the following theorem characterizes the class of integrable set-valued random variables.

Theorem 20. *Let $F \in \mathcal{M}(\Omega, \mathbf{P}(E))$. I_F^1 is non-empty if and only if the function $\omega \to d(0, F(\omega))$ belongs to $L^1(\Omega, [0, \infty))$.*

Proof:

($\Rightarrow$). Assuming $\mathrm{I}_F^1 \neq \emptyset$. Then for $f \in \mathrm{I}_F^1$,

$$d(0, F(\omega)) \leq d(0, f(\omega)) = \|f(\omega)\|$$

Implies $d(0, \cdot) \in L^1(\Omega, [0, \infty))$

($\Leftarrow$). From Theorem 18, we can find a countable family $\{f_n : n \in \mathbb{N}\}$ of measurable selections of F such that

$$F(\omega) = \overline{\{f_n(\omega) : n \in \mathbb{N}\}} \quad \text{for all } \omega \in \Omega.$$

Thus,

$$d(0, F(\omega)) = \inf_{n \in \mathbb{N}} \|f_n(\omega)\|$$

Consider $g \in L^1(\Omega, \mathbb{R})$ strictly positive and let

$$A_1 = \{\omega \in \Omega : \|f_1(\omega)\| < d(0, F(\Omega)) + g(\omega)\}$$

$$A_n = \{\omega \in \Omega : \|f_n(\omega)\| < d(0, F(\Omega)) + g(\omega)\} \setminus \bigcup_{i=1}^{n-1} A_i$$

Then $\{A_n : n \in \mathbb{N}\}$ is a countable measurable partition of Ω. Let 1_{A_n} be the indicator function of A_n then,

$$f(\omega) = \sum_{n=1}^{\infty} 1_{A_n} f_n(\omega)$$

Implies $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$. Moreover,

$$\begin{aligned}
\int_{\Omega} \|f(\omega)\| d\mu &\leq \int_{\Omega} \sum_{n=1}^{\infty} 1_{A_n} \|f_n(\omega)\| d\mu \\
&= \sum_{n=1}^{\infty} \int_{A_n} \|f_n(\omega)\| d\mu \\
&\leq \sum_{n=1}^{\infty} \left(\int_{A_n} d(0, F(\omega)) d\mu + \int_{A_n} g(\omega) d\mu \right) \\
&= \int_{\Omega} d(0, F(\omega)) d\mu + \int_{\Omega} g(\omega) d\mu < \infty
\end{aligned}$$

Hence, $f \in \mathrm{I}_F^1$ so that $\mathrm{I}_F^1 \neq \emptyset$.

□

Corollary 2. I_F^1 *is non-empty if* F *is integrably bounded.*

Proof: By Theorem 20, it suffices to prove $d(0, \cdot) \in L^1(\Omega, [0, \infty))$. Since F is integrably bounded,

$$d(0, F(\omega)) = \inf_{x \in F(\omega)} \|x\| \leq \sup_{x \in F(\omega)} \|x\| = \|F(\omega)\|_{\mathbf{P}}$$

Implies $d(0, \cdot) \in L^1(\Omega, [0, \infty))$.

Following Corollary 2, we state without proof a theorem that characterizes the class of integrably bounded set-valued random variables.

Theorem 21. *Let $F \in \mathcal{M}(\Omega, \mathbf{P}(E))$. Then F is integrably bounded if and only if I_F^1 is non-empty and bounded in $L^1(\Omega, E)$.*

7 Characterization of Elements of I_F^1

In this section, we seek to give a representation of a measurable selection of a set-valued random variable given some selections.

Definition 10. Let W be a collection of $f : \Omega \to E$ measurable. Then W is said to be *decomposable* if for each $f_1, f_2 \in W$ and $A \in \mathcal{A}$, we have that $1_A f_1 + 1_{A^c} f_2 \in W$ where $A^c = \Omega \setminus A$ and 1_A is the indicator function of A.

From Definition 10, it is evident if W is decomposable, then $\sum_{i=1}^n 1_{A_i} f_i \in W$ for each finite partition $\{A_1, \cdots, A_n\}$ and $\{f_1, \cdots, f_n\} \subset W$.

Theorem 22. *Let $F \in \mathcal{M}(\Omega, \mathbf{P}(E))$. Then I_F^1 is decomposable.*

Proof: Let $f_1, f_2 \in \mathrm{I}_F^1$. Then for any $A \in \mathcal{A}$, we have that $1_A f_1 + 1_{A^c} f_2$ is measurable. Moreover, $f_1, f_2 \in L^1(\Omega, E)$ implies $1_A f_1 + 1_{A^c} f_2 \in L^1(\Omega, E)$. Hence, $1_A f_1 + 1_{A^c} f_2 \in \mathrm{I}_F^1$ and it follows I_F^1 is decomposable.

□

We remark that I_F^1 being decomposable guarantees that with two or finitely many selections, we can always get a new selection by decomposing the given ones.

Example 1. Let $E = \mathbb{R}$ and $F(\omega) = \{6, 10\}$ for each $\omega \in \Omega$. Then $f_1, f_2 : \Omega \to \mathbb{R}$ given by $f_(\omega) = 6$ and $f_2(\omega) = 10$ are selections of F. It is evident $g_A = 1_A f_1 + 1_{A^c} f_2$ is also a selection of F for any $A \in \mathcal{A}$. In fact, for $A = \Omega$ we have that $g_A = f_1$ and for $A = \emptyset$, $g_A = f_2$.

CHAPTER 4

The Aumann Integral

In this chapter, we keep the notations and assumptions of the previous section; $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space, $(E, \|\cdot\|)$ is a real separable Banach space and $\mathcal{M}(\Omega, \mathbf{P}(E))$, a collection of all set-valued random variables $F : \Omega \to \mathbf{P}(E)$.

Here, we seek to define and consider some properties of a type of integral called the Aumann integral. As a quick reminder, we recall

1 Bochner Integral

A measurable function $g : \Omega \to E$ is *Bochner integrable* if we can find a sequence of simple measurable functions $g_n : \Omega \to E$ such that

$$\lim_{n\to\infty} \int_\Omega \|g(\omega) - g_n(\omega)\| d\mu = 0$$

For this g, the Bochner integral, denoted by $\int_\Omega g(\omega)d\mu$ is given as

$$\int_\Omega g(\omega)d\mu = \lim_{n\to\infty} \int_\Omega g_n(\omega)d\mu$$

A detailed construction of the Bochner integral and some of its properties can be found in Mikusinski (1978).

2 Definition of the Aumann Integral

Let $F \in \mathcal{M}(\Omega, \mathbf{P}(E))$ and $\mathrm{I}^1_F \neq \emptyset$. Then

$$\int_\Omega F(\omega)d\mu := \left\{ \int_\Omega f(\omega)d\mu : f \in S^1_F \right\} \tag{2.1}$$

is called the Aumann integral of the set-valued random variable $F : \Omega \to \mathbf{P}(E)$ where $\int_\Omega f(\omega)d\mu$ is the Bochner integral of $f \in L^1(\Omega, E)$.

Following the lines of argument of Theorem 20, it is evident 2.1 is well defined. We now give some examples of the Aumann Integral of set-valued random variables.

Example 2. Consider the measure space $(\Omega, \mathcal{A}, \mu)$ where $\Omega = [0,1]$, $\mathcal{A}$ is the Borel σ-algebra on $[0,1]$ and μ is the Lebesgue measure on $[0,1]$; let $F : [0,1] \to \mathbf{P}(\mathbb{R})$ be a set-valued random variable given by $F(\omega) = \{5, 8\}$ for each $\omega \in \Omega$. $f_1, f_2[0,1] \to \mathbb{R}$ given by $f_1(\omega) = 5$ and $f_2(\omega) = 8$ for each $\omega \in \Omega$ are two selections of F. Thus, by characterization of elements of I^1_F,

$$\mathrm{I}^1_F = \{f = 1_A f_1 + (1 - 1_A) f_2 : A \in \mathcal{A}\}$$

and

$$\begin{aligned}\int_{[0,1]} f(\omega)d\mu &= \int_0^1 1_A(\omega) f_1(\omega)d\mu + \int_0^1 (1 - 1_A(\omega)) f_2(\omega)d\mu \\ &= 5\mu(A) + 8 - 8\mu(A) \\ &= 8 - 3\mu(A)\end{aligned}$$

Since Lebesgue measure is atomless and A is any measurable subset of $[0,1]$, it follows that the Aumann integral of F, $\int_{[0,1]} F(w)d\mu = [8,5]$.

We remind that a measure, μ is said to be atomless if for every measurable subset A of Ω with positive measure, we can find a measurable B properly contained in A with a positive measure.

Example 3. Let $\Omega = [5, 89]$; $\mathcal{A}$ be the Borel σ-algebra on $[5, 89]$ and μ be the Lebesgue measure on $[5, 89]$. Consider $F : \Omega \to \mathbf{P}(\mathbb{R}^2)$ defined by $F(\omega) = \{(1,2),(3,4)\}$. $f_1, f_2 : [5,89] \to \mathbb{R}^2$ given by $f_(\omega) = (1,2)$ and $f_2(3,4)$ for each $\omega \in [5,89]$ are selections of F. Thus,

$$\mathrm{I}_F^1 = \{f = 1_A f_1 + (1 - 1_A) f_2 : A \in \mathcal{A}\}$$

and

$$\begin{aligned}
\int_{[5,89]} f(\omega) d\mu &= \int_5^{89} 1_A(\omega) f_1(\omega) d\mu + \int_5^{89} (1 - 1_A(\omega)) f_2(\omega) d\mu \\
&= \mu(A)(1,2) + \int_5^{89} (1 - 1_A(\omega)) f_2(\omega) d\mu \\
&= \mu(A)(1,2) + 84(3,4) - \mu(A)(3,4) \\
&= (252 - 2\mu, 336 - 2\mu(A))
\end{aligned}$$

Since $\max\{\mu(A) : A \subset [5,89],\ A \in \mathcal{A}\} = 84$; $\min\{\mu(A) : A \subset [5,89],\ A \in \mathcal{A}\} = 0$ and Lebesgue measure is atomless, it follows

$$\int_{[5,89]} F(\omega) d\mu = \{(x,y) : 84 \leq x \leq 168,\ 252 \leq y \leq 336\}$$

Example 4. Let $\Omega_1 = [-7,8)$ and $\Omega = \Omega_1 \cup \{8\}$. Assume $\mu_{|\mathcal{B}(\Omega_1)}$ is the Lebesgue measure (where $\mathcal{B}(\Omega_1)$ is the Borel σ-algebra on Ω_1) and $\{8\}$ is an atom with $\mu(\{8\}) = 9$. We define $F \in \mathcal{M}(\Omega, \mathbf{P}(E))$ by $F(\omega) = \{3,5\}$ for each $\omega \in \Omega$. Suppose $f_1, f_2 : \Omega_1 \to \mathbb{R}$ are given by $f_1(\omega) = 3$ and $f_2(\omega) = 5$ for each $\omega \in \Omega_1$. Then, for each $A \in \mathcal{B}(\Omega_1)$,

$$g_1 = 1_A f_1 + 1_{A^c} f_2 + 1_{\{8\}} f_1 \qquad \text{and} \qquad g_2 = 1_A f_1 + 1_{A^c} f_2 + 1_{\{8\}} f_2$$

are the possible measurable selections of F.

Thus,

$$\int_\Omega g_1(\omega)d\mu = \int_\Omega 1_A f_1(\omega)d\mu + \int_\Omega 1_{A^c} f_2(\omega)d\mu + \int_\Omega 1_{\{8\}}(\omega)f_1(\omega)d\mu$$
$$= 3\mu(A) + 5\mu(\Omega) - 5\mu(A) + 3\mu(\{8\})$$
$$= 10 - 2\mu(A)$$

and

$$\int_\Omega g_2(\omega)d\mu = \int_\Omega 1_A f_1(\omega)d\mu + \int_\Omega 1_{A^c} f_2(\omega)d\mu + \int_\Omega 1_{\{8\}}(\omega)f_2(\omega)d\mu$$
$$= 3\mu(A) + 5\mu(\Omega) - 5\mu(A) + 5\mu(\{8\})$$
$$= 120 - 2\mu(A)$$

Since Lebesgue measure is atomless, $\mu|_{\mathcal{B}(\Omega_1)}$ is a Lebesgue measure and $A \in \mathcal{B}(\Omega_1)$ is arbitrary, it follows $\int_\Omega F(\omega)d\mu = [-20, 10] \cup [90, 120]$

3 Properties of Aumann Integral

From Examples 2 and 3 it is evident the Aumann integral can be convex-valued even when $F \in \mathcal{M}(\Omega, \mathbf{P}(E))$ is not. On the other hand, Example 4 shows without the condition of the measure being atomless, the Aumann integral is not convex-valued. More generally, we have

Theorem 23. *If $(\Omega, \mathcal{A}, \mu)$ has no atom and $F \in \mathcal{M}(\Omega, \mathbf{P}(E))$ with $\mathbf{I}^1_F \neq \emptyset$, then $\operatorname{cl}\int_\Omega F(\omega)d\mu$ is convex.*

Proof: It is enough to show for any $f_1, f_2 \in \mathrm{I}^1_F$, ϵ, α, $\beta > 0$, with $\alpha + \beta = 1$, we can find $f \in \mathrm{I}^1_F$ such that

$$\left\| \alpha \int_\Omega f_1(\omega)d\mu + \beta \int_\Omega f_2(\omega)d\mu - \int_\Omega f(\omega)d\mu \right\| < \epsilon$$

Define an $E \times E$-valued measure m by

$$m(A) = \left(\int_A f_1(\omega)d\mu, \int_A f_2(\omega)d\mu \right) \qquad \text{for each } A \in \mathcal{A}.$$

Since μ has no atom, $\text{cl}\{m(A) : A \in \mathcal{A}\}$ is convex in $E \times E$. Furthermore, $m(\emptyset) = (0,0)$ and $m(\Omega) = (\int_\Omega f_1(\omega)d\mu, \int_\Omega f_2(\omega)d\mu)$ implies we can find $A \in \mathcal{A}$ such that

$$\left\| \alpha \int_\Omega f_i(\omega)d\mu - \int_A f_i(\omega)d\mu \right\| < \frac{\epsilon}{2}, \qquad i = 1, 2.$$

Taking $f = 1_A f_1 + 1_{A^c} f_2$ and $\gamma = 1 - \beta$, we have

$$\begin{aligned}
&\left\| \alpha \int_\Omega f_1(\omega)d\mu + \beta \int_\Omega f_2(\omega)d\mu - \int_\Omega f(\omega)d\mu \right\| \\
&\quad = \left\| \alpha \int_\Omega f_1(\omega)d\mu + \beta \int_\Omega f_2(\omega)d\mu - \left(\int_\Omega (1_A f_1(\omega) + 1_{A^c} f_2(\omega))d\mu \right) \right\| \\
&\quad \leq \left\| \alpha \int_\Omega f_1(\omega)d\mu - \int_A f_1(\omega)d\mu \right\| + \left\| \gamma \int_\Omega f_2(\omega)d\mu - \int_A f_2(\omega)d\mu \right\| \\
&\quad < \frac{\epsilon}{2} + \frac{\epsilon}{2}
\end{aligned}$$

Hence, $f \in \mathrm{I}_F^1$.

□

To prove the Aumann integral is closed is not a piece of cake, neither is it trivial to produce a counterexample to show the Aumann integral is not closed in general. However, if the Aumann integral is closed, it will do us a lot of good since, ab initio, we have established a metric on the set of closed (and bounded) subsets of a metric space, E; convergence with respect to this metric; and related convergences (such as Kuratowski, Mosco, Wijsman, Kuratowski-Mosco). For a counterexample, see Li *et al.* (2002). We now make preparations to determine when it suffices for the Aumann integral to be closed.

In what follows, E remains a separable Banach space, E^* its dual and E^{**} its bidual. In Chidume (2014) it is proved that the map (*the canonical embedding*) $J : E \to E^{**}$ defined by $J(x) = \phi_x$ (where

$\phi_x : E^* \to \mathbb{R}$ is given by $\phi_x(f) = \langle f, x \rangle$ for each $f \in E^*$) is an *isometric isomorphism* onto $J(E)$.

Definition 11. The space $(E, \| \cdot \|)$ is said to be *reflexive* if the canonical embedding J is onto (i.e., $J(E) = E^{**}$).

Let μ, ν be measures on $(\Omega, \mathcal{A})$. We remind that ν is *absolutely continuous* with respect to μ if $\nu(A) = 0$ whenever $\mu(A) = 0$, $A \in \mathcal{A}$. Also, in plain terms, a function is said to be of *bounded variation* if the supremum of the variations over all possible partitions is finite.

Definition 12. A Banach space E is said to have the *Radon-Nikodym property (RNP)* with respect to a finite measure space $(\Omega, \mathcal{A})$ if for each μ-absolutely continuous measure $\nu : \mathcal{A} \to E$ of bounded variation, we can find an integrable function $f : \Omega \to E$ such that

$$\nu(A) = \int_A f(\omega) d\mu \qquad \text{for each } A \in \mathcal{A}.$$

It is known that every separable dual space of a separable Banach space and every reflexive space have the RNP Li *et al.* (2002).

With these developments, we now state without proofs two sufficient conditions under which the Aumann integral is closed, as adapted from Li *et al.* (2002).

Theorem 24.
(1). *If E is a reflexive Banach space and $F \in \mathcal{M}^1(\Omega, \mathbf{P}_c(E))$, then the Aumann integral*

$$\int_\Omega F(\omega) d\mu = \left\{ \int_\Omega f(\omega) d\mu : f \in \mathrm{I}_F^1 \right\}$$

is closed in E.

(1). *If E has the RNP and $F \in \mathcal{M}^1(\Omega, \mathbf{P}_{kc}(E))$, then the Aumann integral*

$$\int_\Omega F(\omega)d\mu = \left\{ \int_\Omega f(\omega)d\mu : f \in \mathrm{I}_F^1 \right\}$$

is closed in E.

CHAPTER 5

Conclusion and Future Work

1 Conclusion

We obtained the Aumann integral of a set-valued random variable, which is useful in so many ways. Of interest to us is its application in proving laws of large numbers (LLNs) for set-valued random variables. If the measure, μ defined on the σ-algebra, $\mathcal{A}$ is a probability measure, the Aumann integral of the set-valued random variable F is exactly the expectation of the set-valued random variable F. From LLNs, as the number of samples drawn increases infinitely, the sample mean approaches (in Hausdorff, Kuratowski, Mosco, Wijsman, Kuratowski-Mosco sense) the population mean which is the expectation of the population random variable.

2 Future Work

We hope, in the future, to apply Aumann integral in establishing different laws of large numbers where applicable. Laws of large numbers are in turn, applied in *image processing, artificial intelligence, control theory and mathematical economics.*

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