

**EFFECT OF LINEAR DRIFT ON RADIAL
TRANSPORT OF TRACER IN
HOMOGENEOUS POROUS MEDIA**

BY

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**EFFECT OF LINEAR DRIFT ON RADIAL TRANSPORT OF TRACER IN
HOMOGENEOUS POROUS MEDIA**

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ABSTRACT

Analytical models can be valuable tools to investigate solute transport in porous media. The application of analytical solutions is limited by the perception that they are too cumbersome to derive while their implementation rests on assumptions that are too restrictive. This research is aimed at understanding the effect of linear drift on radial transport of tracer in porous media. It provides an analytical Solution that expresses the Concentration Distribution around the source of tracer injection as a function of Location in two dimensional Cartesian Coordinates and Time.

Linear Drift is considered as a scalar velocity field in the positive horizontal axis direction and the concentration distribution is described using a derived Advection Dispersion Equation (ADE). The derived ADE is a time dependent homogeneous PDE that assumes incompressible flow and consists of dispersion and advection components as well as a first order decay term.

The concentration distribution is described for an injection well located at the origin in an infinite plane (one that stretches to infinity along both spatial axis in positive and negative directions) and a positive semi finite time domain.

DEDICATION

This work is dedicated to my Parents, Mr. and Mrs. Benson OFOMANA for their love and support throughout the period of this MSc program and also to the UZIM's that includes my beloved Sister, her Husband and Son and finally to other members of my immediate family as well as Mr. and Mrs. Kriss ODJEMU.

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1.00

INTRODUCTION

For groundwater flow or hydrocarbon recovery processes, tracers are often the only source of internal information about the system (Koplik, Redner and Hinch 1994). Solute transport in consolidated rocks is of central importance in tracer studies, oil recovery, the remediation of contaminated groundwater, and radioactive waste disposal (Branko and Martin, 2006). To simply put it, tracer test analysis is an investigative processes that uses fluids to investigate the properties of fluids and porous rock containing these fluids. This makes it a reliable investigative technique for analyses of underground fluids and manner by which it flows through porous media. Like Reservoir Simulation and Well Testing, Tracer Test is also on reservoir scale (which makes it very useful to reservoir engineers) but unlike reservoir simulation, its results are unique and unlike Well Testing that interprets pressure transient, it is not an interpretation and more so because the tracer passes through the reservoir being investigated just like the reservoir fluids. As a result, tracer test are usually carried out to confirm predictions made by Well Tests and/or Reservoir Simulations.

Tracers are fluids that are easily detectible in extremely low concentration as a result, tracer test usually involves the use of low concentration fluids. Using low concentration fluids, needing very little additional surface equipment and the ability to carry out some tracer test and hydrocarbon production simultaneously, makes tracer tests economically feasible.

Tracer Test Analysis is a very powerful tool for diagnostic, comparative, qualitative or confirmative study; however before it can be used for a quantitative study, equations that accurately describes the systems through which it flows must be developed and solved. To this effect, a considerable amount of experimental data, theoretical modelling and physical insight is available, but largely in situations where the flow is one dimensional on average (Koplik Redner and Hinch 1994). However, in many underground processes the flow is more likely to be radial or multipolar than linear (Koplik Redner and Hinch 1994).

In addition to radial or multipolar flow, Linear Drift is also a phenomenon encountered in underground flow processes. Linear Drift can arise as a result of aquifer movement during production, or due to production in a distant reservoir with which there exist hydraulic connectivity or a combination of these and other factors.

1.10 AIM

- To derive an ADE that models radial flow of tracer in porous media under the influence of linear drift.
- To obtain an analytical solution of the derived ADE in Cartesian Coordinates.

2.00

LITERATURE REVIEW

The advection (or convection) – dispersion (or diffusion) equation (or the general transport equation) occurs in many physical problems such as porous media, engineering, geophysics etc. It can model dispersion of a pollutant in a river estuary, or groundwater transport, atmospheric pollution, concentration of electron inducing an electric current, heat transfer in a heated body (Jacky, Isabelle and Pierre, 2011) or as in this case, the radial transport of tracer in porous media. The general form of the Convection Diffusion Equation (CDE) or ADE can be directly obtained from the continuity equation by applying the appropriate constitutive relationship. ADEs are very powerful tools for modelling transport phenomena because its general form can be easily modified to more accurately describe the physical transport system it models; however as with all Partial Differential Equation (PDE) it is subject to specific boundary conditions, and its solutions are not easily transferable.

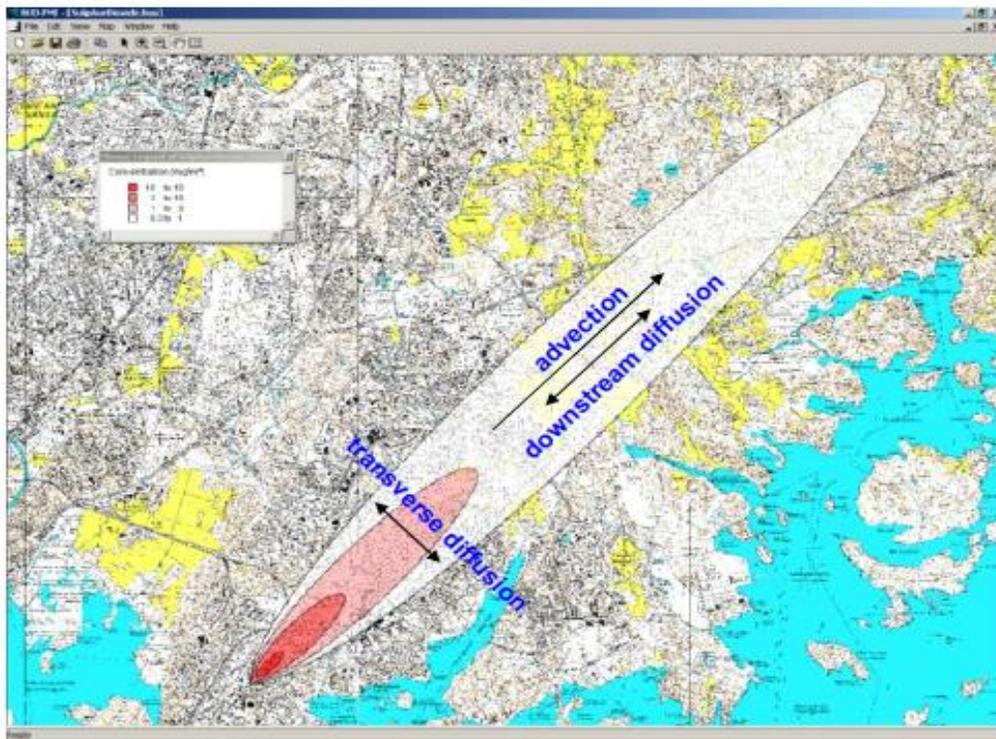


Figure 2.00: Simulation of Advection – Diffusion from a Point Source in the Atmosphere: (Source – Finish Metrological Institute).

As with many other PDEs, the analytical solution of ADEs exists under certain given conditions, but it's often not known explicitly. To obtain a closed form, analytical or exact solution to complex

PDEs like ADEs in porous media, certain assumptions like homogeneity, incompressibility, constant flow field and steady state flow are usually made to simplify the equation, when they are of little practical importance. Numerical methods offer an alternative to analytical methods for solving Complex PDEs (an example of which is reservoir simulation) however, even for linear ADEs numerical schemes are not always well understood. It is still a challenging problem to obtain efficient and robust numerical schemes to solve the ADE, due to the mixing between two different types of behavior, namely the convective and diffusive regimes (Koplik Redner and Hinch 1994).

Analytical solutions have an important role to play because they offer fundamental insight into governing physical processes and provide useful tools for validating numerical approaches. Analytical solutions have other advantages: they are expressed in a closed form, the programs based on this kind of solutions require less processing time, since there is a reduction of the number of operations to be performed, the amount of memory required to execute the routines decreases significantly (Cristiana Jorge and Michelle, 2006). Besides, the source codes based on closed-form solutions are short and easy to debug.

There is a considerable body of work on the analytical solution of the ADE. Milestones in the solution of ADEs include, solution of the diffusion equation by Einstein (1905) and later by Carslaw (1906) who employed Fourier analysis. Matano (1933) solved the same problem using Boltzmann's transformation to transform the PDE to an ODE. Carslaw and Crank by mid Twentieth Century covered almost all knowledge regarding the solution techniques for the diffusion equation. Their works covered cases of sink/sources, moving boundaries, variable coefficients and non-Cartesian coordinate systems (Carslaw and Jaeger, 1946; Crank, 1956). In addition to physicist and mathematicians, geohydrologists (Barry and Sposito, 1989; Parlange, 1998; Ogata and Banks, 1961; Remesikova, 2004), and researchers in the atmospheric sciences (Stockie, 2011; Hundsdorfer and Verwer 2003) have contributed significantly to the analytical and numerical solutions of the general transport equation. Several researchers recast basic solution of diffusion equation by Einstein, from the Lagrangian framework to Eulerian reference to obtain an Advection-Diffusion Equation solution for constant flow field (Bear, 1972). Laplace transform was also used to solve ADEs (Singh et al., 2009). Another approach for solving ADEs is based on Green's function which conveniently handles different initial and boundary conditions as well as multi-dimensional problems. Ellsworth and Butters (1993) employed Green's function for

generalized 3D solution of the transport equation with an arbitrary coordinate system subjected to any given boundary condition. Leij and Van Genuchten (2000) applied the Green's function method to analytically model multi-dimensional transport from persistent solute sources. Their specific analytical solution derived for transport from a rectangular source of persistent contamination used first, second, or third-type boundary conditions. Leij et al. (2000) provided the general frame for the solution of the transport equation in infinite, semi-infinite and finite media. Guerrero et al. (2009) provided analytical solutions for the ADE in finite domain subjected to both transient and steady flow fields. They employed the Generalized Integral Transform Technique (GITT) combined with change of variables.

There are numerous works on the ADE with spatial and temporal varying diffusion coefficients, and velocity. Crank (1956) introduced a transform that could convert an ADE with temporally dependent coefficients to a steady-state coefficient ADE. Subsequently, several time-dependent ADE exact solutions were provided utilizing the Crank's technique and other techniques. Basha and El-Habel (1993) and Kumar et al. (2009, 2010) solved the one-dimensional transport equation with temporally dependent coefficients in an initially solute free domain. They provided exact solutions for both finite and semi-infinite media.

The focus of this work is to derive and solve a two dimensional time dependent advection dispersion equation with spatially varying diffusion coefficient and velocity, which radially describes the tracer concentration distribution around an injection source, under the influence of linear drift.

2.10 GOVERNING EQUATIONS

The PDE governing concentration C under the influence of advection and diffusion is:

$$(\nabla \cdot (D\nabla C)) - \nabla \cdot ((\bar{u})C) \pm l = \frac{\partial C}{\partial t} \quad \dots (2.00) \text{ (Singh, 2013)}$$

Where \bar{u} is the velocity, D the diffusivity and l , the source or sink term.

Furthermore, $(\nabla \cdot (D\nabla C))$ represents the dispersive or diffusive part, $\nabla \cdot ((\bar{u})C)$ represents the advection or convective part and $\frac{\partial C}{\partial t}$ represents accumulation or time dependence of the equation.

Equation 2.00 is also referred to as the general transport equation and assumes the presence of every phenomenal associated with fluid transport. This general form has to be broken, reduced or customized to a specific case to better describe a transport phenomenon. The solution to the customized or reduced version is then determined for specific boundary and/or initial conditions.

Reducing equation 2.00 to a specific case requires basic understanding of the physics describing the flow conditions of the system being modeled. This involves (but not limited to) the nature of: the flow velocity, diffusivity, source and sink term as well as the manner by which they vary across the system from the origin to the boundary. It also involves a basic understanding of flow compressibility effects, nature of the flow field and media through which flow occur.

2.11 Well – Posed Problem Criteria

Since the ADE is a PDE, it requires considerable understanding of mathematics to properly model and solve the equation describing the model. Hadamard (1902) has proposed the following three criteria for a well posed mathematical model equation:

- i. A solution should exist
- ii. The solution should be unique and
- iii. The solution should depend continuously on the given data

2.14 Peclet Number

The solution of ADE also depends upon a non-dimensional number called Peclet Number, Pe .

$$Pe = \left(\frac{uL}{D} \right) \quad \dots (2.01) \text{ (Singh, 2013)}$$

Here L is the length scale. For purely diffusive transport, $Pe = 0$. For values of $Pe < 1$, diffusion dominates and for $Pe > 1$, advection dominates as the transport processes.

In addition to distinguishing between advection and dispersion dominated transport, it can also distinguish all the dispersion - diffusion regimes, including the diffusive regime, the transition regime at low Pe , as well as the power-law dispersion and the mechanical dispersion regime at high Pe . (Branko and Martin, 2006). A solution technique that involves using the Peclet Number, solves the ADE without the diffusive or convective part.

For diffusion, the relative error committed by so doing is expected to be on the order of the Peclet number, and the smaller the Pe , the smaller the error (Benoit Cushman, 2014). The solutions established with diffusion only are thus valid as long as $Pe \ll 1$.

For advection, If $Pe \gg 1$ (in practice, if $Pe > 10$); the advection term is significantly bigger than the diffusion term (Benoit Cushman, 2014). Physically, advection dominates and diffusion is negligible, and spreading is almost inexistent, with the patch of pollutant being simply moved along by the flow. The relative error committed by so doing is expected to be on the order of the inverse of the Peclet number and the larger Pe , the smaller the error (Benoit Cushman, 2014).

Why this can greatly reduce the complexity of the overall solution, it gives no indication of the dynamics between these two transport processes and thus it is most suitable to solving equations that models flow with constant velocity and diffusivity, thus dominated by one transport process.

2.12 River-Like Flow

River-like flow describe flow processes where advection, takes place in one direction, the flow is incompressible and the properties of the system in every direction is somewhat constant or can be adequately represented by an average value. This description can model the spread of pollutants in the atmosphere, a river estuary and some special cases of ground water flow.

The concentration of pollutant for river-like flow in one dimension is governed by an advection diffusion equation given in one horizontal dimension as:

$$D_x \frac{d^2c}{dx^2} - u \frac{dc}{dx} - kC = R \frac{\partial c}{\partial t} \quad \dots (2.02) \text{ (Kubare, Mutsvangwa and Masuku, 2010).}$$

Here D_x , u , k and R are constants. R is called the retardation constant and k the decay constant.

2.13 River-Like Flow in Several Dimensions

For river like flow, the transport problems in two and three dimensions are in many cases a direct extension of the one-dimensional solutions. For a three-dimensional problem in Cartesian coordinates (x,y,z) with uniform flow, the governing ADE is given as:

$$\frac{\partial c}{\partial t} = D_x \frac{\partial^2 c}{\partial x^2} - u \frac{\partial c}{\partial x} + D_y \frac{\partial^2 c}{\partial y^2} + D_z \frac{\partial^2 c}{\partial z^2} \quad \dots (2.03) \text{ (Van Genuchten et al, 2013)}$$

Where D_y and D_z are constant dispersion coefficients in the transverse y and z directions. The boundary conditions may vary depending upon the assumed geometry of the system, but normally include zero-gradient conditions at the transverse boundaries (Van Genuchten et al, 2013).

2.14 Radial Flow

Radial flow describes outward flow from a source or inward flow towards a sink. For radial flow advection is in every direction and for a constant injection rate, the advection velocity at each point in the system is inversely proportional to the distance from the injection point (Falade, Emilio and Brigham, 1987). Thus unlike river-like flow the system will have a velocity gradient. Although every flow geometry can be modelled by any coordinate system, simple radial flow situations are more easily expressed using radial coordinates. When there is no directional variation in properties of a system, a two dimensional Cartesian system can be expressed in one dimension using radial

coordinates. For instance, the two dimensional ADE describing concentration distribution for reactive tracer flow in a porous medium without directional variation of its properties is given as:

$$\left(\frac{1}{r} \frac{\partial}{\partial r} \left(r D_r \frac{\partial C}{\partial r}\right)\right) - \frac{1}{r} \frac{\partial}{\partial r} (r U_r C) - R k C = R \frac{\partial C}{\partial t} \quad \dots (2.04) \text{ (Falade and Brigham, 1989)}$$

2.15 Linear Drift and Radial Flow

Linear Drift is very similar to river-like flow in that the drift velocity is constant and moves in a single direction. Thus the effect of linear drift on radial flow of tracer in porous medium is a combination of river-like flow and radial flow. The choice of Cartesian coordinates for this work was influenced by the river-like nature of linear drift, and radial flow influenced the use of variable coefficients of velocity and diffusivity as well as advection in every direction. The derivation of the ADE that accounts for the effect of linear drift on radial flow of tracer in porous medium is covered in chapter three. It is a variable coefficient linear PDE in three independent variable describing the concentration distribution over time in two dimensional space.

2.20 SOLUTION TECHNIQUES

Over time, various techniques have appeared in literature that solves various versions of the general equation (2.00), by making certain assumptions and imposing boundary and/or initial conditions. The methods highlighted in this sections is not an exhaustive list but considers some popular approach used in obtaining solutions to ADEs. Some of this methods can be used independently but some complex ADEs, will require a combination of more than one methods.

2.21 Separation of Variables

Suppose we seek a solution $U(x, y, z, t)$ to some PDE (expressed in Cartesian coordinates). If it is expressed as a product form given as:

$$U(x, y, z, t) = X(x)Y(y)Z(z)T(t) \quad \dots (2.05) \text{ (Riley, Hobson and Bence, 2006)}$$

A solution that has this form is said to be separable in x, y, z and t, and seeking solutions of this form is called the method of separation of variables.

Separation of variables is not an explicit method of solving differential equations however it is an important step to finding many solutions. It can be used to reduce a separable differential equation whose dependent variable is a function of more than one independent variable, into a series of differential equations whose dependent variable is a function of a single independent variable or lesser number of independent variables. It involves expressing the original dependent

variable as a product of a new set of dependent variables and solving this new set of differential equations using any preferred method. The solution of the original equation is then a product of all the solutions to the new set of equations. Its advantage lies in the fact that it can be used to reduce a complex PDE into a series of simpler PDEs or ODEs whose solutions are already known or can be more easily determined. For a PDE, it is likely that a separable solution is impossible (Riley, Hobson and Bence, 2006), however several important PDEs have separable solutions the most popular of which are Eigen-value problems.

2.22 Conversion of ADE to Diffusion Equation

Conversion of ADEs into an equation that is easily solvable or already has a solution is not a strategy limited to this method, however because diffusion is part of the ADE, its conversion to the diffusion equation means solutions to the diffusion equation can be used to solve the ADE. For instance the ADE in one dimensional Cartesian coordinates with constant diffusivity and velocity, can be reduced to a diffusion equation by substituting:

$$C = We^{\left[\frac{u}{2D}x - \frac{u^2}{4D}t\right]} \dots (2.06) \text{ (Ozisik 1993).}$$

For an isotropic medium and homogeneous medium in three dimensions it's given as:

$$C = We^{\left[\frac{u}{2D}(x+y+z) - \frac{3u^2}{4D}t\right]} \dots (2.07) \text{ (Avhale and Kiwne, 2014)}$$

The general case in Cartesian Coordinates for constant velocity and diffusivity is given as:

$$C = We^{\left[\frac{xu}{2D_x} - \frac{u^2}{4D_x}t + \frac{yv}{2D_y} - \frac{v^2}{4D_y}t + \frac{zw}{2D_z} - \frac{w^2}{4D_z}t\right]} \dots (2.08)$$

After the original equation has been converted and a solution determined, the solution to the original ADE is gotten using equation 2.06, 2.07 or the transformation equation used. This method is tied to finding an appropriate transformation equation and not limited to the case of constant coefficients as will be seen in the next section.

2.23 Variable and Integral Transform Techniques

Variable transform is not by itself a method of solving PDEs. It refers to every method that involves conversion of a PDE into a form that is more easily solvable using change of variable. Section 2.22 is an explicit example of variable transform techniques. However, the technique is not restricted to the diffusion equation alone and can be used to transform a PDE into any form. This involves the use of many kind of transformation equations (including integral and differential equations)

and it transforms the original equation into similar, related or even totally unrelated forms like special equations. Unlike Variable Transforms, Integral Transform Techniques are established methods like Laplace and Fourier Transforms applied to actually solve differential equations. More specifically, the integral transform method is used to convert any given partial differential equation (e.g., Equation (2.00)) to a set of ordinary differential equations, whereupon this set of ordinary differential equations is analytically solved as a relatively straightforward linear system (Chongxuan, William and Hugh Ellis, 1998). For most Integral transform techniques, standard reference tables exist (for many commonly encountered functions) that can be used in transforming these function as well as obtaining inverse transformations. Both variable and integral transform techniques, involves performing an inverse transform as a final step, in order to express the solution in terms of its original independent variable.

Integral transform methods, such as Laplace and Fourier transform methods are frequently used in deriving analytical solutions to ADE's in porous media. However, because of the continuity requirement for both concentration and mass flux, it is difficult (in the problem described by Equation 2.00) to apply integral transforms to space variables. When one applies an integral transform to time variables, the resulting solution, even for the relatively simple cases proves problematic at the inversion step (i.e., transformation back into the time domain) (Chongxuan, William and Hugh Ellis, 1998). One method that avoids this inversion problem is the generalized integral transform method. In essence the generalized integral transform method derives a pair of integral transforms which have the attribute of relatively easy subsequent inverse transformation to the domain of interest (Cotta, 1993; Ozisik, 1993).

2.24 Greens Function Method

Green's function method (GFM) supplies a powerful tool to solve linear partial differential equations. A lot of applications of the GFM can be found in literatures. For example, a problem of solute transport in porous media is solved by using the GFM (Leij et al., 2000). The GFM is applicable only when differential operator satisfies superposition principle. In fact, most operators encountered in engineering can meet the basic requirement, e.g., the diffusion equation, wave equation, Laplace equation, actually all the linear second order partial differential equations and so on (Xu, Travis, and Breitung, 2007). Green's function is the impulse response of an inhomogeneous differential equation defined on a domain, with specified initial conditions or

boundary conditions. Via the superposition principle, the convolution of a Green's function with an arbitrary function on that domain is the solution to the inhomogeneous differential equation (Wikipedia¹). To illustrate how it works, consider a general linear second order, three dimensional ADE for river-like flow with constant velocity and diffusivity:

$$L(C) = C_t + uC_x - D(C_{xx} + C_{yy} + C_{zz}) = \varphi(x, y, z, t) \quad \dots (2.09) \quad (\text{Xu, Travis \& Breitung, 2007})$$

Where: $L(C)$ is a differential operator

Having the boundary general robin boundary conditions as:

$$B(C) = \alpha C(x, y, z, t) + \beta C_n(x, y, z, t) \quad \dots (2.10)$$

Where: n denotes the space – time dimensions.

Let the kernel $G(x^*, y^*, z^*, t^*, x, y, z, t)$ be called the Green's function which is, in fact, a characteristic of the operator $L(C)$ for any finite number of independent variables.

Upon multiplication of $L(C)$ by G and integration by parts gives:

$$\int \iiint GL(C)dx^*dy^*dz^*dt^* = \text{Boundary_Terms} + \int \iiint CL^*(G)dx^*dy^*dz^*dt^* \quad \dots (2.11)$$

Where: $L^*()$ is an adjoint operator given as:

$$L^*() = -\frac{\partial}{\partial t} - u\frac{\partial}{\partial x} - D\nabla^2() \quad \dots (2.12)$$

G is required to satisfy:

$$L^*(G) = \delta(x^* - x, y^* - y, z^* - z, t^* - t) \quad \dots (2.13)$$

Where: $\delta(x^* - x, y^* - y, z^* - z, t^* - t)$ is the Dirac Delta Function.

Substituting equation 2.13 into 2.11 makes the last term equivalent to $C(x, y, z, t)$ i.e.:

$$\int \iiint C(x^*, y^*, z^*, t^*)\delta(x^* - x, y^* - y, z^* - z, t^* - t)dx^*dy^*dz^*dt^* = C(x, y, z, t) \quad \dots (2.14)$$

$C(x, y, z, t)$ is made subject of formula in equation 2.11, the greens function G is evaluated using equation 2.13 and it is substituted into 2.11 to obtain a solution.

The essential point of the Greens Function Method is that the boundary value problem governing G is in general somewhat simpler than the original one governing C . It is worth mentioning that the running variables in Green's functions are x^*, t^* , etc. (the dummy variables) instead of x, t , etc. (the real variables). However the latter are active in the function $C(x, y, z, t)$ (Xu, Travis, and Breitung, 2007). Using Green's functions offers several advantages over Eigen function and series expansion. First, its integral representation provides a direct way of describing the general analytical structure of a solution that may be obscured by an infinite series representation. Second, from a practical point of view, the evaluation of a solution from an integral representation may

prove simpler than finding the sum of an infinite series, particularly near rapidly-varying features of a function, where the convergence of an Eigen function expansion is expected to be slow. Third, in view of the Gibbs phenomenon, the integral representation seems to be less stringent requirements on the functions that describe the initial conditions or the values of a solution are required to assume on a given boundary than expansions based on Eigen functions (Tyn Myint-U and Lokenath Debnath, 2007). Integral transform methods are found to be very useful for finding solutions. The application of GFM generally requires understanding of certain functions like the Delta – Dirac Function, Heaviside Function, Fourier Transforms, Laplace Transforms, Error Function and Complementary Error Function among others.

2.25 Series Method, (Frobenius and Power Series Method)

The power series of a function around a given point a is given as:

$$f(x) = \sum_{m=0}^{\infty} A_m (x - a)^m \quad \dots (2.15) \text{ (Riley, Hobson and Bence, 2006)}$$

Where: A_m are coefficients to be determined.

This sort of approximation is often used to simplify equations into manageable forms. It may seem imprecise at first but is perfectly acceptable so long as it matches the experimental accuracy that can be achieved. (Riley, Hobson and Bence, 2006)

The idea of a power series can be extended to more than one variable given by the equation.

$$f(x, y) = \sum_{m,n=0}^{\infty} A_{mn} (x - a)^m (y - b)^n \quad \dots (2.16) \text{ (James Nearing, 2003)}$$

Where: A_{mn} are coefficients to be determined.

This method of solving differential equations involves looking for a solution in form of a power series. To do this; the dependent variable is expanded as a power series of the independent variable(s) around an ordinary point, regular or irregular singular point; the series expansion is differentiated as required by the equation; like terms in terms of the powers of the independent variable are collected; and finally the coefficients are determined by equating coefficients of similar powers and using the given initial and/or boundary conditions. For a solution centered at the origin the ordinary or singular point can be taken as zero.

For certain differential equations, the power series method will fail to produce a solution. Under such circumstances, a more generalized form of the power series method called Frobenius method might still be able to provide a solution. The Frobenius series is given as:

$$f(x) = \sum_{m=0}^{\infty} A_m x^{m+r} \quad \dots (2.17) \text{ (Richard, 2008)}$$

By substituting $f(x)$ into say a second order ordinary differential equation, a quadratic equation in terms of m is obtained. This quadratic equation is referred to as the indicial equation and the two values of m as the roots of the indicial equation. The series solution of such a differential equation is then dependent on the value of the indicial roots.

Richard H. Rand (2008) also provided a classification scheme that tells which equations can be solved by these two methods, which out of the two methods should be used, and what type of solution to expect. He grouped them into the following rules.

Consider a general homogenous second order ordinary differential equation

$$A(x)Y'' + B(x)Y' + E(x)Y = 0 \quad \dots (2.18)$$

Where $A(x)$, $B(x)$, and $C(x)$ have a power series expansion about $x=0$ given as:

$$\left. \begin{aligned} A(x) &= A_0 + A_1x + A_2x^2 + A_3x^3 + \dots && \dots a \\ B(x) &= B_0 + B_1x + B_2x^2 + B_3x^3 + \dots && \dots b \\ E(x) &= E_0 + E_1x + E_2x^2 + E_3x^3 + \dots && \dots c \end{aligned} \right\} \quad \dots (2.19)$$

Definition: If $A_0 \neq 0$ then $x = 0$ is called an ordinary point. If $A_0 = 0$, $B_0 \neq 0$ and $C_0 \neq 0$ then $x = 0$ is called a singular point.

Let: $P(x) = x \frac{B(x)}{A(x)}$ and $Q(x) = x^2 \frac{C(x)}{A(x)}$

Definition: Let $x=0$ be a singular point. If $P(x)$ and $Q(x)$ don't blow up as x approaches zero, then $x=0$ is called a regular singular point. A singular point which is not regular is called an irregular singular point.

Rule 1: If $x=0$ is an ordinary point, then two linearly independent solutions can be obtained by the method of power series expansions about $x=0$.

Rule 2: If $x=0$ is a regular singular point, then at least one solution can be obtained by the method of Frobenius expanded about $x=0$.

Rule 3: If $x=0$ is a regular singular point, and if the two indicial roots r_1 and r_2 are not identical and do not differ by an integer, then the method of Frobenius will yield two linearly independent solutions.

Rule 4: If $x=0$ is a regular singular point, and if the two indicial roots r_1 and r_2 are either identical or differ by an integer, then one solution can be obtained by the method of Frobenius.

The series method is a midpoint between numerical and analytical methods and can be a time saving approach to both methods in importunate situations. It can become an analytical method if the derived series expression is that of some recognized function, a combination of functions or

can be converted into integral representations. If a long enough series is used, its accuracy is determined by the processing power of the computer, just like exact solutions; however like numerical solutions, it will require considerably more time and computer resources. The power series method will give solutions only to initial value problems (opposed to boundary value problems), however this might not be an issue when dealing with linear differential equations since the solution may turn up multiple linearly independent solutions which may be combined (by superposition) to solve boundary value problems as well (Wikipedia²). This also means its application to non – linear partial differential equations is somewhat limited.

2.26 Special Equations and Functions

Special equations refers to some set of equations that occur with such frequency in physical sciences and engineering that solutions to them, which obey particular commonly occurring boundary conditions, have been extensively studied and given special names referred to as special functions (Riley, Hobson and Bence, 2006). In other words, for every special equation there is a standard method of deriving a solution and/or a standard solution exist which are referred to as special functions. If a variable coefficient ODE or PDE can be transformed, re-written or exist in form of a special equation, the special function to that special equation can serve as the solution to the ODE or PDE in most cases. Some special equations and functions that have appeared in literature involving the derivation of analytical solutions to ADE are briefly treated below.

Gamma Function

The gamma function is an integral equation defined as

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \text{ for } x > 0 \quad \dots (2.20) \text{ (Tyn Myint-U and Lokenath Debnath, 2007)}$$

It has the recurrence relations:

$$\Gamma(x) = (x - 1)\Gamma(x - 1) \quad \dots (2.21)$$

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \quad \dots (2.22)$$

It also have the following properties among many others.

$$\Gamma(1) = \int_0^{\infty} e^{-t} dt = 1 \quad \dots (2.23)$$

Thus for any integer n:

$$\Gamma(n) = (n - 1)! \quad \dots (2.24)$$

$$\Gamma(1/2) = \sqrt{\pi} \quad \dots (2.25)$$

Beta Function

The beta function is an integral equation defined as:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \quad \dots (2.26) \text{ (Tyn Myint-U and Lokenath Debnath, 2007)}$$

Let:

$$t = u / (1 + u) \quad \dots (2.27)$$

Substituting 2.27 in 2.28 gives:

$$B(x, y) = \int_0^1 u^{y-1}(1+u)^{x+y} du \quad \dots (2.28)$$

It also follows that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \dots (2.29)$$

Error Function and Complementary Error Function

The error function is an integral equation defined as:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \text{ for } -\infty < x < \infty \quad \dots (2.30) \text{ (Tyn Myint-U \& Lokenath Debnath, 2007)}$$

It has the following properties:

$$\text{erf}(-x) = -\text{erf}(x) \quad \dots (2.31)$$

$$\frac{d}{dx}(\text{erf}(x)) = \frac{2}{\sqrt{\pi}} e^{-x^2} \quad \dots (2.32)$$

$$\text{erf}(0) = 0 \quad \dots (2.33)$$

$$\text{erf}(\infty) = 1 \quad \dots (2.34)$$

The complementary error function is also an integral equation defined as:

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \quad \dots (2.35)$$

It has the following properties:

$$\text{erfc}(x) = 1 - \text{erf}(x) \quad \dots (2.36)$$

$$\text{erfc}(-x) = 2 - \text{erfc}(x) \quad \dots (2.37)$$

$$\text{erfc}(0) = 1 \quad \dots (2.38)$$

$$\text{erfc}(\infty) = 0 \quad \dots (2.39)$$

Cauchy Euler Equation

The Cauchy Euler equation of order n for Y(x) is given as:

$$a_n x^n Y^{(n)} + a_{n-1} x^{n-1} Y^{(n-1)} + \dots + a_0 Y = 0 \quad \dots (2.40) \text{ (Wikipedia}^3\text{)}$$

Where the notation $^{(n)}$ in the equation 2.40 represents the order of differentiation:

This can be written in two dimensions as:

$$x^2 Y'' + axY' + bY = 0 \quad \dots (2.41)$$

Substituting:

$$Y = x^m \quad \dots (2.42) \text{ or}$$

$$x = e^m \Leftrightarrow \ln x = m \quad \dots (2.43)$$

Reduces equation 2.40 to a linear ODE with constant coefficients, which can be solved using standard methods for solving ODEs.

Bessel Function

Bessel's functions are solutions of the Bessel's differential equation. For $Y(x)$ it's given as:

$$x^2 Y'' + xY' + (x^2 - n^2)Y = 0 \quad \dots (2.44) \text{ (Dass, 2000)}$$

Where: n is a real constant.

It's solved using the Frobenius Method which assumes a solution in the form:

$$Y(x) = \sum_{r=0}^{\infty} A_r x^{m+r} \quad \dots (2.45)$$

The particular solution is called the Bessel function of the first kind of order n and it's given as:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!} \left(\frac{x}{2}\right)^{n+2r} \quad \dots (2.46)$$

Replacing n with $-n$ in equation 2.46 gives:

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r} \quad \dots (2.47)$$

For integer values n $J_n(x)$ also has the following integral representation:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(nt - x \sin t)} dt \quad \dots (2.48a) \text{ (Wikipedia⁴)}$$

Equation 2.48a can be extended to none integer values by one of Schlafli's integrals ($\text{Re}(x) > 0$).

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(nt - x \sin t) dt - \frac{\sin(n\pi)}{\pi} \int_0^{\infty} e^{-x \sinh(t) - nt} dt \quad \dots (2.48b) \text{ (Wikipedia⁴)}$$

When n is an integer, the solution of the Bessel equation is given as:

$$Y(x) = AJ_n(x) + BJ_{-n}(x) \quad \dots (2.49) \text{ (for integral values of } n \text{ only)}$$

Let the Bessel function of the second kind of order n , $Y_n(x)$ be defined as:

$$Y_n(x) = \frac{J_n(x) \cos(n\pi) - J_{-n}(x)}{\sin(n\pi)} \quad \dots (2.50a)$$

Then the general solution of the Bessel Equation is given as:

$$Y(x) = AJ_n(x) + BY_n(x) \quad \dots (2.50b) \text{ (for all } n \text{)}$$

The Bessel Function has some of the following recurrence relations:

$$\left. \begin{aligned} xJ'_n &= nJ_n - xJ_{n+1} && \dots a \\ xJ'_n &= -nJ_n + xJ_{n-1} && \dots b \\ 2J'_n &= J_{n-1} - J_{n+1} && \dots c \\ 2nJ'_n &= x(J_{n-1} + J_{n+1}) && \dots d \\ \frac{d}{dx}(x^{-n}J_n) &= -x^{-n}J_{n+1} && \dots e \\ \frac{d}{dx}(x^nJ_n) &= -x^nJ_{n-1} && \dots f \end{aligned} \right\} \dots (2.51)$$

Modified Bessel Function

The modified Bessel Functions are solutions to the equation:

$$x^2Y'' + xY' - (x^2 + n^2)Y = 0 \quad \dots (2.52) \text{ (Wikipedia}^4\text{)}$$

It also has two linearly independent solutions, called the Modified Bessel Function of the first kind and second kind and given by equations 2.53 and 2.54 respectively:

$$I_n(x) = i^{-n}J_n(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n} \quad \dots (2.53) \text{ (First kind)}$$

$$K_n(x) = \frac{\pi I_{-n}(x) - I_n(x)}{2 \sin(n\pi)} \quad \dots (2.54) \text{ (Second kind)}$$

These two linearly independent solutions (2.53 and 2.54) also have integral forms.

Airy Equation

The General form of Airy's equation is given as:

$$Y''(x) \pm \eta^2 xY(x) = 0 \quad \dots (2.55a) \text{ (Wolfram Mathworld}^1\text{)}$$

The equation can be changed from its positive to negative version (signified by the \pm sign) by simply changing the dependent variable from x to $-x$ and vice versa. When $\eta = 1$ the general form reduces to the conventional Airy Equation, whose negative version is given as:

$$Y''(x) - xY(x) = 0 \quad \dots (2.55b) \text{ (Wikipedia}^5\text{)}$$

The Airy equation has two linearly independent solutions given as:

$$\left. \begin{aligned} Ai(x) &= \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{y^3}{3} - xy\right) dy \equiv \frac{1}{\pi} \lim_{b \rightarrow \infty} \int_0^b \cos\left(\frac{y^3}{3} - xy\right) dy && \dots a \\ Bi(x) &= \frac{1}{\pi} \int_0^{\infty} e^{\left(-\frac{y^3}{3} + xy\right)} + \sin\left(\frac{y^3}{3} + xy\right) dy && \dots b \end{aligned} \right\} \dots (2.56)$$

They have some of the following properties.

$$\left. \begin{aligned} Ai(0) &= \frac{1}{3^{\frac{2}{3}}\Gamma(2/3)} & \dots a \\ Ai'(0) &= -\frac{1}{3^{\frac{1}{3}}\Gamma(1/3)} & \dots b \\ Bi(0) &= \frac{1}{3^{\frac{1}{6}}\Gamma(2/3)} & \dots c \\ Bi'(0) &= \frac{1}{3^{\frac{1}{6}}\Gamma(2/3)} & \dots d \end{aligned} \right\} \dots (2.57)$$

$$\begin{vmatrix} Ai(x) & Bi(x) \\ Ai'(x) & Bi'(x) \end{vmatrix} = \frac{1}{\pi} \dots (2.58)$$

$$\int_{-\infty}^{\infty} Ai(t+x)Ai(t+y) dt = \delta(x-y) \dots (2.59)$$

It also have the following relationship to the Bessel and Modified Bessel Function

$$\left. \begin{aligned} Ai(x) &= \frac{1}{\pi} \sqrt{\frac{x}{3}} K_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) & \dots a \\ Bi(x) &= \sqrt{\frac{x}{3}} \left(I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) \right) & \dots b \\ Ai(-x) &= \sqrt{\frac{x}{9}} \left(J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) \right) & \dots c \\ Bi(-x) &= \sqrt{\frac{x}{3}} \left(J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) - J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) \right) & \dots d \end{aligned} \right\} \dots (2.60)$$

The solution to the general form of the Airy's Equation is given as:

$$\left. \begin{aligned} \sqrt{\frac{x}{9}} \left(AJ_{\frac{1}{3}}\left(\frac{2}{3}\eta x^{\frac{3}{2}}\right) + BJ_{-\frac{1}{3}}\left(\frac{2}{3}\eta x^{\frac{3}{2}}\right) \right) & \text{ for the + ve version } \dots a \\ \sqrt{\frac{x}{9}} \left(AI_{-\frac{1}{3}}\left(\frac{2}{3}\eta x^{\frac{3}{2}}\right) - BI_{\frac{1}{3}}\left(\frac{2}{3}\eta x^{\frac{3}{2}}\right) \right) & \text{ for the - ve version } \dots b \end{aligned} \right\} \dots (2.61) \text{ (Wolfram Mathworld}^1) \text{)}$$

Where: I and J are the Bessel and Modified Bessel function respectively, both of the first kind respectively, while A and B are constants of integration to be determined.

Lengendre Equation

The Lengendre Equation for $Y(x)$ it's given as:

$$\begin{aligned} (1-x^2)Y'' - 2xY' + n(n+1)Y &= 0 \dots a \\ \frac{d}{dx} \left\{ (1-x^2) \frac{dY}{dx} \right\} + n(n+1)Y &= 0 \dots b \end{aligned} \dots (2.62) \text{ (Dass, 2000)}$$

It's solved using the Frobenius Method and assumes a solution in ascending or descending powers of x . The assumed solution in descending powers of x is given as:

$$Y(x) = \sum_{r=0}^{\infty} A_r x^{m-r} \dots (2.63)$$

The solution of the first kind in descending powers of x $P(x)$ is given as:

$$P(x) = A_0 \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \times 4(2n-1)(2n-3)} x^{n-4} \dots \right] \quad \dots (2.64)$$

If n is a positive integer and $A_0 = \frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{n!}$, equation 2.51 becomes:

$$P_n(x) = \frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \times 4(2n-1)(2n-3)} x^{n-4} \dots \right] \quad \dots (2.65)$$

The solution of the second kind Q(x) is given as:

$$Q(x) = A_0 \left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \dots \right] \quad \dots (2.66)$$

If $A_0 = \frac{n!}{1 \times 3 \times 5 \times \dots \times (2n+1)}$, equation 2.53 becomes:

$$Q_n(x) = \frac{n!}{1 \times 3 \times 5 \times \dots \times (2n+1)} \left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \dots \right] \quad \dots (2.67)$$

Since $P_n(x)$ and $Q_n(x)$ are independent solutions, the general solution is given as:

$$Y(x) = aP_n(x) + bQ_n(x) \quad \dots (2.68)$$

Where the arbitrary constants b and a can be determined using the appropriate boundary condition.

$P_n(x)$ has the following recurrence relations.

$$\left. \begin{aligned} nP_n &= (2n-1)P_{n-1} - (n-1)P_{n-2} & \dots a \\ xP'_n - P'_{n-1} &= nP_n & \dots b \\ P'_n - xP'_{n-1} &= nP_{n-1} & \dots c \\ P'_{n+1} - P'_{n-1} &= (2n+1)P_n & \dots d \\ (x^2-1)P'_n &= n[xP'_n - P'_{n-1}] & \dots e \\ (x^2-1)P'_n &= (n+1)(P_{n+1} - xP_n) & \dots f \end{aligned} \right\} \quad \dots (2.69)$$

Laguerres Differential Equation

The Laguerres differential equation for a function Y(x) is given as:

$$xY'' + (1-x)Y' + nY = 0 \quad \dots (2.70) \text{ (Dass, 2000)}$$

Where: n is an integer.

Its solution is called the Laguerres polynomial is given as:

$$L_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x}) \quad \dots (2.71)$$

For n = 0

$$L_0(x) = e^x (x^0 e^{-x}) = 1$$

For n = 3

$$L_3(x) = e^x \frac{d^3}{dx^3} (x^3 e^{-x}) = [-x^3 + 9x^2 - 18x + 6]e^{-x} e^x = -x^3 + 9x^2 - 18x + 6$$

Confluent Hypergeometric Equation

The General Confluent Hypergeometric Equation of degree N is given as:

$$xY'' + (\gamma - x)Y' - (\sum_{n=0}^N \alpha_n x^n)Y(x) = 0 \quad \dots (2.72)$$

Where: n and N are integers.

For n = 0 equation 2.72 reduces to the regular Confluent Hypergeometric Equation.

The confluent hypergeometric equation has two standard forms. One is the Kummer type and the other is the Whittaker type. The Kummer type is given as:

$$xY'' + (\gamma - x)Y' - \alpha Y(x) = 0 \quad (2.73) \text{ (Encyclopedia of Mathematics)}$$

The Whittaker type is given as:

$$Z'' + Z' + \left(\frac{\gamma}{x} + \frac{\frac{1}{4} - \alpha^2}{x^2}\right)Z(x) = 0 \quad \dots (2.74) \text{ (Wolfram Mathworld}^2)$$

With the substitution that:

$$Z = w_{\gamma,\alpha} e^{-x/2} \quad \dots (2.75)$$

Equation 2.74 can be written as:

$$w'' + \left(-\frac{1}{4} + \frac{\gamma}{x} + \frac{\frac{1}{4} - \alpha^2}{x^2}\right)w = 0 \quad \dots (2.76) \text{ (Wolfram Mathworld}^2)$$

Equation 2.76 is also a popular form of the Whittaker Equation.

The solution to equation 2.73, called the Kummer Function of the first kind; it's given as:

$$Y_{\alpha,\gamma}(x) = {}_1F_1(\gamma; \alpha; x) = \sum_{n=0}^{\infty} \frac{\alpha^{(n)} x^n}{\gamma^{(n)} n!} \quad \dots (2.77) \text{ (Wikipedia}^6)$$

Where the notation $a^{(n)}$ in the equation 2.76 represents the rising factorial defined as:

$$\left. \begin{aligned} a^{(0)} &= 1 && \dots a \\ a^{(1)} &= a && \dots b \\ a^{(n)} &= a(a+1)(a+2)\dots(a+n-1) && \dots c \end{aligned} \right\} \quad \dots (2.78)$$

Equation 2.65 can be expressed in integral form as:

$$Y_{\alpha,\gamma}(x) = {}_1F_1(\alpha; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 e^{xu} u^{a-1} (1-u)^{\alpha-\gamma-1} du \quad \dots (2.79) \text{ (Wikipedia}^6)$$

It can also be expressed as the Barnes integrals given as:

$$Y_{\alpha,\gamma}(x) = \frac{1}{2\pi i} \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \int_{-i\infty}^{i\infty} \frac{\Gamma(-s)\Gamma(\alpha+s)}{\Gamma(\gamma+s)} (-x)^s ds \quad \dots (2.80) \text{ (Wikipedia}^6)$$

The solution to equation 2.64 is given as:

$$w = C_1 W_{\alpha,\gamma}(x) + C_2 W_{\alpha,-\gamma}(-x) \quad \dots (2.81)$$

Where the notation $W_{\alpha,\gamma}(x)$ represents the Whittaker function defined as:

$$W_{\alpha,\gamma}(x) = x^{\alpha+1/2} e^{-x/2} \left[1 + \frac{\frac{1}{2}+\alpha-\gamma}{(2\alpha+1)} x + \frac{(\frac{1}{2}+\alpha-\gamma)(\frac{3}{2}+\alpha-\gamma)}{2!(2\alpha+1)(2\alpha+2)} x^2 + \dots \right] \dots (2.82)$$

$$W_{\alpha,\gamma}(x) = x^{\alpha+1/2} e^{-x/2} \sum_{n=0}^{\infty} \frac{(\alpha-\gamma+\frac{1}{2})^{(n)}}{n!(2\alpha+1)^{(n)}} x^n \dots (2.83) \text{ (Mathsworld)}$$

Where the notation $^{(n)}$ in the equation 2.83 represents the rising factorial.

Equation 2.83 can also be expressed in integral form as:

$$W_{\gamma,\alpha}(x) = \frac{x^\gamma e^{-x/2}}{\Gamma(\frac{1}{2}-\alpha+\gamma)} \int_0^\infty t^{\alpha-\gamma-1/2} \left(1 + \frac{t}{x}\right)^{\alpha-\gamma-1/2} e^{-t} dt \dots (2.84) \text{ (Mathsworld)}$$

For $N \geq 1$ equation 2.72 becomes:

$$xY'' + (\gamma - x)Y' - (\alpha + \sum_{n=1}^N \alpha_n x^n)Y(x) = 0 \dots (2.85) \text{ (Campos, 2000)}$$

Equation 2.85 is referred to as the Extended Confluent Hypergeometric Equation.

Confluent Hypergeometric Functions are very important because they can be used to solve many second order differential equation including special equations (Wikipedia⁶). This is because many second order differential equations can be easily converted into the confluent hypergeometric equation in its regular or extended form. A special case is the Extended Confluent Hypergeometric Equation of degree 1, which can be used to solve linear second order differential equations whose variable coefficients are linear functions of the independent variable.

The Extended Hypergeometric Equation of degree 1 is given as:

$$xY'' + (\gamma - x)Y' - (\alpha + \alpha_1 x)Y(x) = 0 \dots (2.86)$$

Consider the linear second order differential equation:

$$(A + Bx)Y'' + (C + Dx)Y' + (E + Fx)Y(x) = 0 \dots (2.87)$$

With the substitution that:

$x_1 = A + Bx$, and using chain rule to obtain the derivatives, equation 2.87 can be written as:

$$x_1 Y'' + (C_1 + D_1 x_1) Y' + (E_1 + F_1 x_1) Y(x_1) = 0 \dots (2.88)$$

With the substitution that:

$x_2 = D_1 x_1$, and using the chain rule to obtain the derivatives, equation 2.88 can be written as:

$$x_2 Y'' + (C_1 + x_2) Y' + (E_2 + F_2 x_2) Y(x_2) = 0$$

Which can be written as:

$$x_2 Y'' + [C_1 - (-x_2)] Y' - [(-E_2) + (-F_2) x_2] Y(x_2) = 0 \dots (2.89)$$

Thus equation 2.87 has been converted into equation 2.89 which is in the form of equation 2.86, the Extended Confluent Hypergeometric Equation of Degree 1.

Two integral solutions of equation 2.86 exist when certain conditions are met, referred to as solutions of the first and second kind respectively. Consider Equation 2.86

Let:

$$\sigma = \sqrt{1 + 4\alpha_1} \dots (2.90)$$

If $\text{Real}(\gamma) > \text{Real}(\alpha) > 0 \dots (2.91)$ (Campos, 2000).

Then the function of the solution of the first kind is given as:

$$Y(x) \equiv {}_1F_1^1(\alpha; \gamma; \alpha_1; x) = \left[\frac{e^{\frac{x}{2}(1-\sigma)}}{B\left(\frac{\gamma}{2}, \frac{\gamma}{2\sigma} + \frac{\alpha}{\sigma}, \frac{\gamma}{2} + \frac{\gamma}{2\sigma} \frac{\alpha}{\sigma}\right)} \right] \int_0^1 \left[(z)^{\frac{\gamma}{2}\left(1-\frac{1}{\sigma}\right) + \frac{\alpha}{2} - 1} (1-z)^{\frac{\gamma}{2}\left(1+\frac{1}{\sigma}\right) - \frac{\alpha}{2} - 1} e^{\sigma x z} \right] dz \dots (2.92)$$

(Campos, 2000).

If $\text{Real}(x) > 0; \text{Real}(\alpha) > 0 \dots (2.93)$ (Campos, 2000).

Then the solution of the second kind is given as:

$$Y(x) \equiv {}_1G_1^1(\alpha; \gamma; \alpha_1; x) = \left[\frac{e^{\frac{x}{2}\left(1-\frac{1}{\sigma}\right)}}{\Gamma(\alpha)} \right] \int_0^\infty \left[(z)^{\frac{\gamma}{2}\left(1-\frac{1}{\sigma}\right) + \frac{\alpha}{\sigma} - 1} (1+z)^{\frac{\gamma}{2}\left(1+\frac{1}{\sigma}\right) - \frac{\alpha}{\sigma} - 1} e^{-\sigma x z} \right] dz \dots (2.93)$$

(Campos, 2000).

2.27 Method of Characteristics

The Method of Characteristics can be used to reduce certain types of linear / semi-linear PDEs to ODEs. This makes use of the general philosophy that ODEs are easier to solve than PDEs (Evy Kersale', 2004). It is effective against many variable coefficients linear / semi-linear First Order and Second Order PDEs in two independent variables. For PDEs in more than two independent variables, it is not usually possible to reduce the equation to a simple canonical form; however, such a reduction is possible if the equation has constant coefficients (Evy Kersale', 2004).

In essence the method of characteristics is just a change of variable technique that is done in a specific way. Consider the general form of a second order PDE in two variables:

$$a(x, t)U_{xx} + 2b(x, t)U_{xt} + c(x, y)U_{tt} + d(x, t)U_x + e(x, t)U_t + f(x, t)U = d(x, y) \dots (2.94)$$

Three possibilities exist.

Table 2.00: Three Types of Second Order PDE in Two Variables.

The Hyperbolic Case	The Parabolic Case	The Elliptic Case
$(b^2 - 4ac) > 0$	$(b^2 - 4ac) = 0$	$(b^2 - 4ac) < 0$
$\frac{\partial t}{\partial x} = -\frac{b}{2a} \pm \left(\frac{\sqrt{b^2 - 4ac}}{2a} \right)$	$\frac{\partial t}{\partial x} = \frac{b}{a}$	$\frac{\partial t}{\partial x} = -\frac{b}{2a} \pm \left(\frac{\sqrt{b^2 - 4ac}}{2a} \right) i$
$\eta = t$ or $\eta = x$	$\eta = t$ or $\eta = x$	$\eta = t$ or $\eta = x$

With one of the new variable already defined as either x or t (as in table 2.00), all that is left is to find a second linearly independent new variable ξ . The second linearly independent solution involves finding a family of curves with constant characteristics. This is straight forward in the parabolic case where only one characteristic curve exist. The integration with respect to the independent variables of the equation in the second column (under the parabolic case columns) will give an expression that can be equated to a constant.

For instance, let:

$$\left. \begin{aligned} a &= t^2 & \dots a) \\ b &= -xt & \dots b) \\ c &= x^2 & \dots c) \end{aligned} \right\} \dots (2.95)$$

Then: $b^2 - ac = 0$

Thus: $\frac{\partial t}{\partial x} = \frac{b}{a} = -\frac{x}{t}$

$$t dt = x dx \Rightarrow x^2 + t^2 = C \dots (2.95d)$$

Where: C is the constant of integration.

Thus the second linearly independent variable is given as:

$$\xi = x^2 + t^2 \dots (2.95e)$$

The procedure is similar in the hyperbolic case, just that two family of characteristic curves exist which can be gotten upon subsequent integration to produce two different expression equal to a constants. This is just like in the parabolic case only that the work is two times more.

In the hyperbolic case, there also exist two family of curves, however the discriminant is a complex number. Thus after finding ξ and choosing η a second transformation is carried out to keep the transformation real. This transformation is given as:

$$\left. \begin{aligned} \alpha &= \xi + \eta & \dots a) \\ \beta &= \xi - \eta & \dots b) \end{aligned} \right\} \dots (2.96)$$

After suitable new independent variables have been found, a variable transformation process is applied to the original (parent equation) using chain rule and upon subsequent substitution, the equation can be reduced to just one variable ODE which should be simpler to solve.

Although this method is very effective, it breaks down when:

$$b^2 = ac = 0 \dots (2.97)$$

As a result it cannot be used to solve many important PDEs like the heat equation or some one dimensional ADEs. Many more instances might exist where it is ineffective.

2.28 When One Solution is Known

When one solution is known, a second linearly independent solution (and thus the general solution) might be obtained via the reduction of order method (using the Wronskian and Abel's theorem) for homogeneous higher order ODEs or via variation of parameter method or Greens Function, for non-homogeneous higher order ODEs (David, 2009).

2.29 Numerical Methods

Numerical methods are now finding increasing applications for realistic media properties and geometries. Even the computation of results from analytical solutions require numerical techniques of function calculations and series summations. In numerical methods, the differential equation is reduced to a system of algebraic equations. For solutions to ADEs, two main classes of numerical methods have been proposed: Finite Difference and Finite Element method (Singh, 2013). In the finite difference method, the difference forms of the derivatives are substituted in the equations. These difference forms are obtained by using the functional values at the grids spanning the media using Taylor's series expansion. The linear ordinary differential equation is reduced to a system of linear algebraic equations. The solution is obtained by solving this system of equations. Several efficient algorithms are available to obtain direct or iterative solution of these equations. The linear partial differential equations are also reduced to solving a set of linear equations. In the finite element method.

Cellular Automata

Recently cellular automata methods are also used to simulate transport phenomena. Here, in addition to discretization of space and time variables, the dependent variables such as chemical concentration / temperature are also discretized in finite number of levels. Earlier the dependent variables needed infinite number of points, but now these require only finite number of values. Recently such methods have been used to describe a vast range of physical, chemical and biological phenomena (Wolfram, 2002).

3.00 DERIVATION AND PROBLEM DEFINITION

Consider an arbitrary control volume ΔV as shown in Figure 1 below

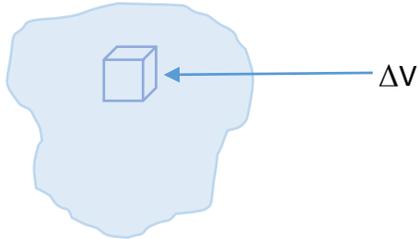


Figure 3.00: Tracer Concentration in a Closed System

Let N be amount of tracer molecules in the system

C be the concentration per unit volume of tracer molecules within the system and

V be the volume of the system

Then the number of tracer molecules within the control volume ΔN is given as

$$\Delta N = C \Delta V \quad \dots (3.00) \text{ (Omololu 2014)}$$

This implies

$$N = \sum_i C \Delta V_i \quad \dots (3.01)$$

For a continuous system

$$N = \iiint_v C dV \quad \dots (3.02)$$

$$\frac{\partial N}{\partial t} = \iiint_v \frac{\partial C}{\partial t} dV \quad \dots (3.03)$$

Let l be the number of tracer molecules lost within the control volume

Then the amount of tracer molecules lost within the entire system L is given as

$$L = \iiint_v l dV \quad \dots (3.04)$$

Let \bar{P} be the net flux of the system

$$\bar{P} = P_i + P_j + P_k \quad \dots (3.05)$$

Where i, j and k denotes components in the three principal space coordinates.

If the flow of tracer molecules out of the system is greater than the inflow then

$$\frac{\partial N}{\partial t} = - \iiint_v l dV - \iint_s \bar{P} ds \text{ applying the divergence theorem,}$$

$$\frac{\partial N}{\partial t} = - \iiint_V l dV - \iiint_V (\nabla \cdot \bar{P}) dV \quad \dots (3.06)$$

Equating 3.03 and 3.06 gives:

$$\iiint_V \left((\nabla \cdot \bar{P}) + l + \frac{\partial C}{\partial t} \right) dV = 0 \quad \dots (3.07)$$

Since the volume is arbitrary, it can be assumed that

$$(\nabla \cdot \bar{P}) + l + \frac{\partial C}{\partial t} = 0 \quad \text{or}$$

$$-(\nabla \cdot \bar{P}) - l = \frac{\partial C}{\partial t} \quad \dots (3.08) \text{ (Wikipedia}^7\text{) (The Continuity Equation)}$$

Molecules can be transported within porous media via two methods.

1. By Bulk Transfer: this involves bulk transfer processes such as convection, drift, etc.

$$\bar{P}_b = (\bar{u})C \quad \dots (3.09a) \text{ (Omololu 2014) (Advection or Convection)}$$

Where \bar{u} is advection velocity defined as $\bar{u} = u_i + v_j + w_k$

2. By Molecular Diffusion or Dispersion (usually driven by concentration gradients) which can be folded into the hydrodynamic dispersion term and expressed as a Fickian process, given as:

$$P_m = -D\nabla C \quad \dots (3.09b) \text{ (Omololu 2014) (According to Fick's law)}$$

Where D is the hydrodynamic Dispersion.

When both processes takes place simultaneously.

$$\bar{P} = P_m + \bar{P}_b \quad \dots (3.10)$$

Substituting for \bar{P} using equation 3.10 gives:

$$(\nabla \cdot (D\nabla C)) - \nabla \cdot ((\bar{u})C) - l = \frac{\partial C}{\partial t} \quad \dots (3.11) \text{ (The Advection Diffusion Equation)}$$

Equation 3.11a is the Advection (or Convection) – Diffusion (or Dispersion) Equation.

For reactive tracer flow, the source and sink term is given as:

$$l = \frac{Q(C, C_a)}{S_m} \quad \dots (3.12) \text{ (Falade 2014)}$$

Where: C_a is the absorbed chemical tracer concentration (that is lost within the system)

C is the unabsorbed tracer concentration (present in the mobile fluid phase)

S_m is the tracer saturation of the mobile fluid phase

And the accumulation or time dependence is modified as:

$$\frac{\partial}{\partial t} \left(\frac{S_m C + (1-S_m)C_a}{S_m} \right) \quad \dots (3.13) \text{ (Falade 2014)}$$

If it's assumed that the reactive tracer undergoes an irreversible first order reaction in the mobile fluid phase only, the flow incompressible (i.e. $\nabla \cdot \bar{u} = 0$); and the medium is not heterogeneous.

Equation 3.11 becomes:

$$(\nabla \cdot (D \cdot \nabla C)) - \bar{u} \cdot \nabla C - \frac{Q(C, C_a)}{S_m} = \frac{\partial}{\partial t} \left(\frac{S_m C + (1 - S_m) C_a}{S_m} \right) \dots (3.14)$$

Chemical tracer operations usually involve tracer solutions of very low concentration such that the tracer reaction equation can be approximated by a first order reaction equation given as:

$$Q(C, C_a) = k(S_m C + (1 - S_m) C_a) \dots (3.15a) \text{ (Falade and Brigham 1989)}$$

Where: k is a first order chemical reaction constant.

The equilibrium relation between the very low concentration adsorbed tracer phase, C_a , and the mobile tracer phase concentrations, C can be represented by a Langmuir isotherm expressed as:

$$C_a = \frac{k_1 C}{1 + k_2 C} \dots (3.15b) \text{ (Falade and Brigham 1989)}$$

Where k_1 and k_2 are constants. For systems of dilute concentrations usually encountered in tracer experiments, the constant k_2 is approximately zero simplifying equation 15b to

$$C_a \cong k_1 C \dots (3.15c) \text{ (Falade and Brigham 1989)}$$

Substituting equations 3.15a and 3.15c into 3.14 gives:

$$(\nabla \cdot (D \cdot \nabla C)) - \bar{u} \cdot \nabla C - \frac{k(S_m C + k_1(1 - S_m)C)}{S_m} = \frac{\partial}{\partial t} \left(\frac{S_m C + k_1(1 - S_m)C}{S_m} \right)$$

$$(\nabla \cdot (D \cdot \nabla C)) - \bar{u} \cdot \nabla C - kC \left(\frac{S_m + k_1(1 - S_m)}{S_m} \right) = \left(\frac{S_m + k_1(1 - S_m)}{S_m} \right) \frac{\partial C}{\partial t}$$

$$\text{Let } R = \left(\frac{S_m + k_1(1 - S_m)}{S_m} \right) \text{ (Falade and Brigham 1989)}$$

This simplifies the above equation to

$$(\nabla \cdot (D \cdot \nabla C)) - \bar{u} \cdot \nabla C - RkC = R \frac{\partial C}{\partial t} \dots (3.16)$$

Thus to take into account the effect of linear drift on the radial flow of tracers in porous medium all that needs to be done is define linear drift and take into account the modifications it makes to components of equation 3.16 (i.e. D and \bar{u}) that is affected by velocity.

$$(\nabla \cdot (D_{\#} \cdot \nabla C)) - (\bar{u} + \text{linear drift}) \cdot \nabla C - l_0 - RkC = R \frac{\partial C}{\partial t} \dots (3.17)$$

A cursory look at equation 3.17 reveals that it is dimensionally stable;

$\nabla \cdot (D_{\#} \cdot \nabla C)$ Represents the diffusion part of the equation.

$(\bar{u} + \text{linear drift}) \cdot \nabla C$ Represents the advection part of the equation.

$\frac{\partial C}{\partial t}$ Represents the accumulation term or time dependence of the equation.

k is a first order decay constant (whose unit is the inverse of time) and represents the amount of tracer lost within the system that may be as a result of adsorption or chemical reaction.

R Represents the systems retardation constant and it's dimensionless.

3.10 FLOW GEOMETRY

Since the porous medium is not heterogeneous and the fluid containing the tracer is simultaneously injected across the entire thickness of the medium, the fluid flows radially outward. For a constant injection rate, the propagation velocity in a particular direction should decrease with increasing distance from the line source. Because geologically porous media are usually highly stratified and flow is often restricted to roughly planer permeable layers (Koplik, Redner and Hinch, 1994) (i.e. having large lateral extent as compared to their thickness), a two dimensional system can serve as a plausible approximation to a three dimensional underground flow geometry so long as the porous medium is not multilayered. Thus the medium can be represented by a two dimensional model where it's assumed that whatever goes on in one plane is replicated across the entire thickness of the medium.

For radial flow in a two dimensions, \bar{u} becomes the radial convective velocity such that:

$$\left. \begin{aligned} \bar{u}^2 &= u^2 + v^2 & \dots a) \\ v &= \bar{u} \sin \theta & \dots b) \\ u &= \bar{u} \cos \theta & \dots c) \end{aligned} \right\} \dots 3.18$$

For a stationary velocity field, \bar{u} is not constant but decreases at a constant rate with increasing distance from the well, for a constant injection rate (q). From pure material balance consideration:

$$\bar{u} = \frac{pq}{2\pi r h \phi S_m} \dots (3.19) \text{ (Falade, Emilio and Brigham, 1987)}$$

Where: q is the injection rate

h is height of porous medium

r radial distance from the well

p is a unit conversion factor (that is unity in SI units)

ϕ is the systems porosity

When the injection rate is constant, equation 3.19 implies:

$$\bar{u}r = \varepsilon = \text{Constant} \dots (3.20a) \text{ (Falade, Emilio and Brigham, 1987)}$$

$$\bar{u} = \frac{\varepsilon}{r} \dots (3.20b)$$

Equation 3.20b makes it possible to express the advection velocity in degenerate form. This can be done in two ways both of which are products of the injection velocity at the well, with either a function of time or distance from the well serving as the other part of the product.

From equation 3.20b, the injection velocity at the well is given as:

$$u_0 = \frac{\varepsilon}{r_w} \dots (3.20c)$$

Where: r_w is radius of the well.

Multiplying equation 3.20b by $\frac{r_w}{r}$, the following conclusion can be reached.

$$\left. \begin{array}{l} \bar{u} = \frac{\varepsilon r_w}{r r_w} = u_0 \frac{r_w}{r} \\ \text{for } r > r_w \end{array} \right\} \dots d$$

$$\left. \begin{array}{l} \bar{u} = u_0 \\ \text{for } r \leq r_w \end{array} \right\} \dots e$$

... (3.20)

$\bar{u} = \frac{u}{\cos\theta}$ and $\frac{1}{r} = \frac{\cos\theta}{x}$ similarly $\bar{u} = \frac{v}{\sin\theta}$ and $\frac{1}{r} = \frac{\sin\theta}{y}$ individually substituting these expressions for the x and y cases into Equation 3.20d gives:

$$\left. \begin{array}{l} u = \frac{1}{x} u_0 r_w \cos^2\theta \quad \dots a \\ v = \frac{1}{y} v u_0 r_w \sin^2\theta \quad \dots b \end{array} \right\} \dots (3.21)$$

Where: v is the isotropy factor and it is strictly applicable to homogeneous systems. It is the ratio of permeability in the other principal axes of direction without linear drift to the permeability in the principal axis of direction along which linear drift is acting. It is always a positive number, whose value is unity only when a porous medium is isotropic and homogeneous.

$$\left. \begin{array}{l} \beta = r_w \cos^2\theta \quad \dots a \\ \lambda = v r_w \sin^2\theta \quad \dots b \end{array} \right\} \dots (3.22)$$

This implies:

$$\left. \begin{array}{l} u = u_0 \beta \frac{1}{x} \quad \dots a \\ v = u_0 \lambda \frac{1}{y} \quad \dots b \end{array} \right\} \dots (3.23)$$

3.20 LINEAR DRIFT

Working in Cartesian Coordinates equation 3.16 becomes

$$\nabla \cdot \left((D_x + D_y) \cdot \left(\frac{\partial C}{\partial x} + \frac{\partial C}{\partial y} \right) \right) - (u + v) \cdot \nabla C - RkC = R \frac{\partial C}{\partial t}$$

$$\nabla \cdot \left(D_x \frac{\partial C}{\partial x} + D_y \frac{\partial C}{\partial y} \right) - \left(u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} \right) - RkC = R \frac{\partial C}{\partial t}$$

$$\left(\frac{\partial}{\partial x} \left(D_x \frac{\partial C}{\partial x} \right) + \frac{\partial}{\partial y} \left(D_y \frac{\partial C}{\partial y} \right) \right) - \left(u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} \right) - RkC = R \frac{\partial C}{\partial t} \dots (3.24)$$

Linear drift can be considered as a coordinate-independent scalar field (or zero order tensor field) that associates a constant value of linear velocity to every point within the porous medium. Coordinate-independence, implies that any two observers using the same units will agree on the value of the scalar field at the same point in space (or space-time) irrespective of their coordinate system (Wikipedia⁸). Thus linear drift can be considered as inflation solely in the positive x

direction that modifies the x components, $u(x, t)$ and $D_x(x, t)$ in equation 3.24. Where u_d denotes linear drift, this modification is given by equation 3.25 and 3.27 below.

$$u_* = u + u_d \quad \dots (3.25)$$

The diffusivity coefficient is given as:

$$\left. \begin{aligned} D_x &= D_m + D_{\#}|u| & \dots a \\ D_y &= D_m + D_{\#}|v| & \dots b \end{aligned} \right\} \quad \dots (3.26) \text{ (Falade, Emilio and Brigham, 1987)}$$

Where: D_m is the molecular diffusion coefficient.

$D_{\#}$ is the dispersion coefficient.

To take into account the effect of linear drift, u_* is substituted for u in equation 3.23.

$$\left. \begin{aligned} D_* &= D_m + D_{\#}|u + u_d| & \dots a \\ D_y &= D_m + D_{\#}|v| & \dots b \end{aligned} \right\} \quad \dots (3.27)$$

For all practical purpose the molecular diffusion coefficient is small as compared to the dispersion (or shear mixing coefficient). Thus equation 3.27 reduces to:

$$\left. \begin{aligned} D_* &\cong D_{\#}|u + u_d| & \dots a \\ D_y &\cong D_{\#}|v| & \dots b \end{aligned} \right\} \quad \dots (3.28)$$

Since u_d is always positive, If:

$$\left. \begin{aligned} -D_{\#} &\text{ is used for negative x when } u > -u_d & \dots a \\ -D_{\#} &\text{ is used for negative y} & \dots b \\ D_{\#} &\text{ is always used for the positive x and y axis} & \dots c \end{aligned} \right\} \quad \dots 3.29$$

Then equation 3.28 can be written as:

$$\left. \begin{aligned} D_* &\cong D_{\#}(u + u_d) & \dots a \\ D_y &\cong D_{\#}v & \dots b \end{aligned} \right\} \quad \dots (3.30)$$

Substituting equation 3.30 into 3.24 gives:

$$D_{\#} \left(\frac{\partial}{\partial x} \left((u + u_d) \frac{\partial C}{\partial x} \right) + \frac{\partial}{\partial y} \left(v \frac{\partial C}{\partial y} \right) \right) - \left((u + u_d) \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} \right) - RkC = R \frac{\partial C}{\partial t} \quad \dots (3.31)$$

Expanding equation 30 gives:

$$D_{\#}(u + u_d) \frac{\partial^2 C}{\partial x^2} + D_{\#}v \frac{\partial^2 C}{\partial y^2} + D_{\#} \left(\frac{\partial u}{\partial x} \frac{\partial C}{\partial x} \right) + D_{\#} \left(\frac{\partial v}{\partial y} \frac{\partial C}{\partial y} \right) - (u + u_d) \frac{\partial C}{\partial x} - v \frac{\partial C}{\partial y} - RkC = R \frac{\partial C}{\partial t}$$

$$D_{\#}(u + u_d) \frac{\partial^2 C}{\partial x^2} + D_{\#}v \frac{\partial^2 C}{\partial y^2} + D_{\#}((\nabla C) \cdot (\nabla \cdot \bar{u})) - (u + u_d) \frac{\partial C}{\partial x} - v \frac{\partial C}{\partial y} - RkC = R \frac{\partial C}{\partial t}$$

For incompressible flow $\nabla \cdot \bar{u} = 0$ Thus:

$$D_{\#}(u + u_d)C_{xx} - (u + u_d)C_x + D_{\#}vC_{yy} - vC_y - RkC = RC_t \quad \dots (3.32a)$$

$$D_0 \left(\frac{\beta}{x} + d \right) \frac{\partial^2 C}{\partial x^2} - u_0 \left(\frac{\beta}{x} + d \right) \frac{\partial C}{\partial x} + D_0 \frac{\lambda}{y} \frac{\partial^2 C}{\partial y^2} - u_0 \frac{\lambda}{y} \frac{\partial C}{\partial y} - RkC = R \frac{\partial C}{\partial t} \quad \dots (3.32b)$$

$$D_0 \left(\frac{\beta}{x} + d \right) C_{xx} - u_0 \left(\frac{\beta}{x} + d \right) C_x + D_0 \frac{\lambda}{y} C_{yy} - u_0 \frac{\lambda}{y} C_y - RkC = RC_t \quad \dots (3.32b)$$

Where the conditions for the use of $D_{\#}$ (equation 3.29a) applies to D_0 and d and D_0 are given as:

$$\left. \begin{aligned} D_0 &= D_{\#}u_0 & \dots a \\ d &= \frac{u_d}{u_0} & \dots b \end{aligned} \right\} \dots (3.33) \text{ } d \text{ is the linear drift ratio.}$$

3.30 BOUNDARY CONDITIONS

Equation 3.32 has the following boundary conditions.

$$\left. \begin{aligned} C(x, y, t = 0) &= C_i(x, y) & (-\infty \leq x \leq \infty, -\infty \leq y \leq \infty,) & \dots a \\ C(x = \pm\infty, y, t) &= 0 & (-\infty \leq y \leq \infty, t > 0) & \dots b \\ C(x, y = \pm\infty, t) &= 0 & (-\infty \leq x \leq \infty, t > 0) & \dots c \\ C(x = x_w, y = y_w, t) &= C_0 & (t > 0) & \dots d \end{aligned} \right\} \dots (3.34)$$

Where:

$$\left. \begin{aligned} x_w &= r_w \cos \theta & \dots a \\ y_w &= r_w \sin \theta & \dots b \end{aligned} \right\} \dots (3.35)$$

The concentration within the wellbore doesn't change throughout the period of the tracer test, therefore the solution $C(x, y, t)$ within the wellbore is already known. All that remains is to solve equation 3.31 (for $t > 0$) in the region outside the wellbore.

4.00

SOLUTION

This section is devoted to finding an analytical solution to equation 3.32 subject to the boundary conditions in equation 3.33, for $t > 0$.

Using Separation of Variables,

$$C(x, y, s) = X(x)Y(y, t) \quad \dots (4.00)$$

Substituting $X(x)Y(y, t)$ into equation 3.31b and dividing through by $X(x)Y(y, t)$ gives:

$$\frac{D_0\left(\frac{\beta}{x}+d\right)X''-u_0\left(\frac{\beta}{x}+d\right)X'}{X(x)} + \frac{D_0\frac{\lambda}{y}Y_{yy}-u_0\frac{\lambda}{y}Y_y-RY_t}{Y(y,t)} - Rk = 0$$

Rearranging gives:

$$\frac{D_0\left(\frac{\beta}{x}+d\right)X''-u_0\left(\frac{\beta}{x}+d\right)X'}{X(x)} = -\frac{D_0\frac{\lambda}{y}Y_{yy}-u_0\frac{\lambda}{y}Y_y-RY_t}{Y(y,t)} + Rk \quad \dots (4.01)$$

Since the right hand side and left hand side depends on different independent variables, it must imply that they are both equal to a constant, thus equation 4.03 can be separated as:

$$\left. \begin{array}{l} \frac{D_0\left(\frac{\beta}{x}+d\right)X''-u_0\left(\frac{\beta}{x}+d\right)X'}{X(x)} = \mu \quad \dots a \\ \frac{D_0\frac{\lambda}{y}Y_{yy}-u_0\frac{\lambda}{y}Y_y-RY_t}{Y(y,t)} - Rk = -\mu \quad \dots b \end{array} \right\} \quad \dots (4.02)$$

Where μ is the constant of separation; rearranging gives:

$$\left. \begin{array}{l} D_0\left(\frac{\beta}{x}+d\right)X''-u_0\left(\frac{\beta}{x}+d\right)X'-\mu X(x) = 0 \quad \dots a \\ D_0\frac{\lambda}{y}Y_{yy}-u_0\frac{\lambda}{y}Y_y-RY_t-(Rk-\mu)Y(y,t) = 0 \quad \dots b \end{array} \right\} \dots (4.03)$$

Equation 4.03a is steady state in time meaning its time independent and constant with respect to time, while equation 4.03b is time dependent. Thus writing equation 4.03 in Laplace Space, gives:

$$\left. \begin{array}{l} \frac{1}{s}\left[D_0\left(\frac{\beta}{x}+d\right)X''-u_0\left(\frac{\beta}{x}+d\right)X'-\mu X(x)\right] = 0 \quad \dots a \\ D_0\frac{\lambda}{y}Y''-u_0\frac{\lambda}{y}Y'-(Rs+Rk-\mu)Y(y,s) = 0 \quad \dots b \end{array} \right\} \dots (4.04)$$

$$\left. \begin{array}{l} D_0\left(\frac{\beta}{x}+d\right)X''-u_0\left(\frac{\beta}{x}+d\right)X'-\mu X(x) = 0 \quad \dots a \\ D_0\frac{\lambda}{y}Y''-u_0\frac{\lambda}{y}Y'-(Rs+Rk-\mu)Y(y,s) = 0 \quad \dots b \end{array} \right\} \dots (4.05)$$

Where “s” is the Laplace parameter.

Taking Equation 4.05a:

$$D_0\left(\frac{\beta}{x}+d\right)X''-u_0\left(\frac{\beta}{x}+d\right)X'-\mu X = 0 \quad \dots (4.04a) \text{ (recalled)}$$

Multiplying through by the independent variable x and dividing by D_0 gives:

$$(\beta + xd)X'' - \frac{u_0}{D_0}(\beta + xd)X' - \frac{\mu}{D_0}xX = 0 \quad \dots (4.06)$$

Let:

$$\left. \begin{aligned} x &= \frac{x_1 - \beta}{d} && \dots a \\ x_1 &= \beta + xd && \dots b \\ \frac{d}{dx} &= \frac{dx_1}{dx} \frac{d}{dx_1} = d \frac{d}{dx_1} && \dots c \\ \frac{d^2}{dx^2} &= \frac{d}{dx} \left(\frac{d}{dx} \right) = d \frac{d}{dx} \frac{d}{dx_1} = d^2 \frac{d^2}{dx_1^2} && \dots d \end{aligned} \right\} \quad \dots (4.07)$$

The independent variable is changed from x to x_1 by substituting equation 4.07 into 4.06.

$$d^2 x_1 X'' - \frac{u_0 d}{D_0} x_1 X' - \frac{\mu}{D_0} \left(\frac{x_1 - \beta}{d} \right) X = 0$$

Dividing through by d^2 gives:

$$x_1 X'' - \frac{u_0}{D_0 d} x_1 X' - \left(\frac{\mu}{d^3 D_0} x_1 - \frac{\mu \beta}{d^3 D_0} \right) X = 0 \quad \dots 4.08$$

Let:

$$\left. \begin{aligned} x_1 &= \frac{D_0 d}{u_0} x_2 && \dots a \\ x_2 &= \frac{u_0}{D_0 d} x_1 && \dots b \\ \frac{d}{dx_1} &= \frac{dx_2}{dx_1} \frac{d}{dx_2} = \frac{u_0}{D_0 d} \frac{d}{dx_2} && \dots c \\ \frac{d^2}{dx_1^2} &= \frac{d}{dx_1} \left(\frac{d}{dx_1} \right) = \left(\frac{u_0}{D_0 d} \right)^2 \frac{d^2}{dx_2^2} && \dots d \end{aligned} \right\} \quad \dots (4.09)$$

The independent variable is changed from x_1 to x_2 by substituting equation 4.09 into 4.08.

$$\frac{u_0}{D_0 d} x_2 X'' - \frac{u_0}{D_0 d} x_2 X' - \left(\frac{\mu}{u_0 d^2} x_2 - \frac{\mu \beta}{D_0 d^3} \right) X = 0$$

Multiplying through by $\frac{D_0 d}{u_0}$ gives:

$$x_2 X'' - x_2 X' - \left(\frac{\mu D_0}{u_0^2 d} x_2 - \frac{\mu \beta}{u_0 d^2} \right) X = 0$$

Which can be written as:

$$x_2 X'' + (0 - x_2) X' - \left(-\frac{\mu \beta}{u_0 d^2} + \frac{\mu D_0}{u_0^2 d} x_2 \right) X = 0 \quad \dots (4.10)$$

Equation 4.10 is in the form of equation 2.86 (the extended confluent hypergeometric equation of degree 1). Comparing both equations shows that:

$$\left. \begin{aligned} \gamma &= 0 && \dots a \\ \alpha &= -\frac{\mu \beta}{u_0 d^2} && \dots b \\ \alpha_1 &= \frac{\mu D_0}{u_0^2 d} && \dots c \end{aligned} \right\} \quad \dots (4.11)$$

The conditions in equation 2.91 and 2.93 implies that solutions exist for positive values of α . Since u_0 , β and d^2 are always positive numbers, it implies that μ must be a negative number. Which can be expressed as:

$$\mu = -\omega^2 \quad \dots (4.12a)$$

$$\therefore \omega^2 = |\mu| \quad \dots (4.12b)$$

So that:

$$\left. \begin{array}{l} \gamma = 0 \quad \dots a \\ \alpha = \frac{\omega^2 \beta}{u_0 d^2} \quad \dots b \\ \alpha_1 = -\frac{\omega^2 D_0}{u_0^2 d} \quad \dots c \end{array} \right\} \dots (4.13)$$

The conditions in equation 2.91 is not satisfied because $\gamma = 0$ but that of equation 2.93 is satisfied meaning the solution to equation 4.10 is in the form of the extended confluent hypergeometric function of the second kind (${}_1G_1^1$).

Thus:

$$\text{For } \text{Re}(x_2) > 0, \text{Re}(\alpha) > 0 \quad \dots (4.14)$$

$$X(x_2) = {}_1G_1^1(\alpha; \gamma; \alpha_1; x_2) \quad \dots (4.15a)$$

$$X(x_2) = \left[\frac{e^{-\frac{1}{2}(\frac{1}{\sigma}-1)x_2}}{\Gamma(\alpha)} \right] \int_0^\infty \left[(z)^{\left(\frac{\alpha}{\sigma}-1\right)} (1+z)^{-\left(\frac{\alpha}{\sigma}+1\right)} e^{-\sigma x_2 z} \right] dz \dots (4.15b)$$

$$\sigma \equiv \sqrt{1 + 4\alpha_1} = \sqrt{1 - \frac{4\omega^2 D_0}{u_0^2 d}} \quad \dots (4.16)$$

$$x_1 = \beta + xd \quad \dots (4.06b) \text{ (recalled)}$$

$$x_2 = \frac{u_0}{D_0 d} x_1 \quad \dots (4.08b) \text{ (recalled)}$$

Thus:

$$x_2 = \frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x \quad \dots (4.17)$$

Using equation 4.17, 4.14 and 4.15 can now be written in terms of x :

$$\text{Re} \left(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x \right) > 0 \Rightarrow \frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x > 0$$

Thus:

$$\text{For } \frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x > 0 \quad \dots (4.18a)$$

$$X(x) = \left[\frac{e^{-\frac{1}{2}(-1+\frac{1}{\sigma})\left(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x\right)}}{\Gamma(\alpha)} \right] \int_0^\infty \left[(z)^{\left(\frac{\alpha}{\sigma}-1\right)} (1+z)^{-\left(\frac{\alpha}{\sigma}+1\right)} e^{-\sigma \left(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x\right) z} \right] dz \quad \dots (4.18b)$$

The Tricomi Kummer $U(a,b,x)$ function is defined as:

$$U(a, b, x) = \frac{1}{\Gamma(a)} \int_0^\infty z^{a-1} (1+z)^{-(a+1-b)} e^{-zx} dz \quad \dots (4.19a) \text{ (Mathworks)}$$

Rearranging equation 4.19a gives:

$$\int_0^\infty z^{a-1} (1+z)^{-(a+1-b)} e^{-zx} dz = \Gamma(a) U(a, b, x) \quad \dots (4.19b)$$

Comparing the integral part of equation 4.18b with 4.19b gives:

$$X(x) = \frac{\Gamma(\frac{\alpha}{\sigma})}{\Gamma(\alpha)} U\left(\frac{\alpha}{\sigma}, 0, \sigma \left[\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x \right]\right) e^{-\frac{1}{2}(-1+\frac{1}{\sigma})\left(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x\right)} \quad \dots (4.18c)$$

Taking equation 4.05b,

$$D_0 \frac{\lambda}{y} Y'' - u_0 \frac{\lambda}{y} Y' - (Rk + Rs - \mu)Y = 0 \quad \dots (4.05b) \text{ (recalled)}$$

Substituting the value of μ from equation 4.12 gives:

$$D_0 \frac{\lambda}{y} Y'' - u_0 \frac{\lambda}{y} Y' - (Rs + Rk + \omega^2)Y = 0$$

Multiplying through by $\frac{y}{\lambda D_0}$ gives:

$$Y'' - \frac{u_0}{D_0} Y' - \frac{Rs+Rk+\omega^2}{D_0 \lambda} y Y = 0 \quad \dots (4.20)$$

Let:

$$\left. \begin{aligned} Y &= \psi e^{\frac{u_0 y}{2D_0}} && \dots a \\ Y' &= \psi' e^{\frac{u_0 y}{2D_0}} + \frac{u_0}{2D_0} \psi e^{\frac{u_0 y}{2D_0}} && \dots b \\ Y'' &= \psi'' e^{\frac{u_0 y}{2D_0}} + \frac{u_0}{D_0} \psi' e^{\frac{u_0 y}{2D_0}} + \frac{1}{4} \left(\frac{u_0}{D_0}\right)^2 \psi e^{\frac{u_0 y}{2D_0}} && \dots c \end{aligned} \right\} \dots (4.21)$$

$$\psi'' e^{\frac{u_0 y}{2D_0}} + \frac{u_0}{D_0} \psi' e^{\frac{u_0 y}{2D_0}} + \frac{1}{4} \left(\frac{u_0}{D_0}\right)^2 \psi e^{\frac{u_0 y}{2D_0}} - \frac{u_0}{D_0} \psi' e^{\frac{u_0 y}{2D_0}} - \frac{u_0}{D_0} \frac{u_0}{2D_0} \psi e^{\frac{u_0 y}{2D_0}} - \frac{Rs+Rk+\omega^2}{D_0 \lambda} y \psi e^{\frac{u_0 y}{2D_0}} = 0$$

Rearranging gives:

$$\psi'' - \left[\frac{1}{4} \left(\frac{u_0}{D_0}\right)^2 + \frac{Rs+Rk+\omega^2}{D_0 \lambda} y \right] \psi = 0 \quad \dots (4.22)$$

Let:

$$\left. \begin{aligned} y &= \left(y_1 - \frac{1}{4} \left(\frac{u_0}{D_0}\right)^2 \right) \frac{D_0 \lambda}{Rs+Rk+\omega^2} && \dots a \\ y_1 &= \left[\frac{1}{4} \left(\frac{u_0}{D_0}\right)^2 + \frac{Rs+Rk+\omega^2}{D_0 \lambda} y \right] && \dots b \\ \frac{d}{dy} &= \frac{dy_1}{dy} \frac{d}{dy_1} = \frac{Rs+Rk+\omega^2}{D_0 \lambda} \frac{d}{dy_1} && \dots c \\ \frac{d^2}{dy^2} &= \frac{d}{dy} \left(\frac{d}{dy} \right) = \left[\frac{Rs+Rk+\omega^2}{D_0 \lambda} \right]^2 \frac{d^2}{dy_1^2} && \dots d \end{aligned} \right\} \dots (4.23)$$

The independent variable is changed from y to y_1 by substituting equation 4.23 in 4.22.

$$\left[\frac{Rs+Rk+\omega^2}{D_0\lambda}\right]^2 \psi'' - y_1 \psi = 0$$

Multiplying through by $\left[\frac{D_0\lambda}{Rs+Rk+\omega^2}\right]^2$ gives:

$$\psi'' - \left[\frac{D_0\lambda}{Rs+Rk+\omega^2}\right]^2 y_1 \psi = 0 \quad \dots (4.24)$$

Equation 4.24 is the general form of the Airy Equation 2.55a, whose solution is given by equation 2.61b. Comparing equation 4.24 with 2.55a yields the general solution to equation 4.24 as:

$$\psi(y_1, s) = \frac{1}{3} y_1^{\frac{1}{2}} \left[I_{-1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] y_1^{3/2} \right) - I_{1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] y_1^{3/2} \right) \right] \quad \dots (4.25)$$

Where: I is the Modified Bessel function of the first kind.

Substituting the expression of y_1 from equation 4.23b gives:

$$\psi(y, s) = \frac{1}{3} \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y \right]^{\frac{1}{2}} \left[I_{-1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y \right]^{3/2} \right) - I_{1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y \right]^{3/2} \right) \right] \quad \dots (4.26)$$

Substituting equation 4.26 into 4.21a gives the expression for $Y(y, s)$ as:

$$Y(y, s) = \frac{1}{3} \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y \right]^{\frac{1}{2}} e^{\frac{u_0 y}{2D_0}} \left[I_{-1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y \right]^{3/2} \right) - I_{1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y \right]^{3/2} \right) \right] \quad \dots (4.27)$$

4.10 SOLUTION SUMMARY

To proceed to the Results and Discussion section where the solution is to be applied, it's necessary to summarize the derived solution and highlight some of its behaviors, in order to properly understand its physical nature.

4.11 The X(x) Component

The X(x) component is given by the equation 4.18b and 4.18c and subject to the conditions in 4.18a as well as the conditions for the use of positive or negative D_0 especially 3.29a these conditions and equations are:

$-D_{\#}$ is used for negative x when $u > -u_d$ (3.29a) (recalled)

$$\frac{\beta u_0}{dD_0} + \frac{u_0}{D_0} x > 0 \quad \dots (4.18a) \text{ (recalled)}$$

$$X(x) = \left[\frac{e^{-\frac{1}{2}(-1+\frac{1}{\sigma})(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x)}}{\Gamma(\alpha)} \right] \int_0^\infty \left[(z)^{\frac{\alpha}{\sigma}-1} (1+z)^{-\frac{\alpha}{\sigma}+1} e^{-\sigma(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x)z} \right] dz \quad \dots (4.18b) \text{ (recalled)}$$

$$X(x) = \frac{\Gamma(\frac{\alpha}{\sigma})}{\Gamma(\alpha)} U\left(\frac{\alpha}{\sigma}, 0, \sigma \left[\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x \right]\right) e^{-\frac{1}{2}(-1+\frac{1}{\sigma})(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x)} \dots (4.18c) \text{ (recalled)}$$

The last two equations suggests that the solution is might not be global and terminates somewhere along negative x axis. Rearranging equation 4.18 gives:

$$x > -\frac{\beta}{d} \Rightarrow x > -\frac{u_0 \beta}{u_d} \quad \dots (4.28)$$

Equation 3.29a is concerned about the point along the negative x axis where the resultant direction of convection changes, which in turn determines the use of positive or negative D_0 . Dividing both sides by u_d gives:

$$\frac{u}{u_d} < -1$$

Substituting the value of u gives:

$$\frac{u_0 \beta}{x u_d} < -1 \Rightarrow \frac{u_0 \beta}{u_d} < -x \text{ or } x > -\frac{u_0 \beta}{u_d} \quad \dots (4.28) \text{ (recalled)}$$

Thus the region where equation 4.19 correctly predicts the $X(x)$ component of the concentration distribution is the same as the region of interest where the resultant direction of the x component of convection has not changed and so only positive $D_\#$ (and by implication D_0) would be used.

In addition to the above, substituting $x = \infty$ in equation 4.18b) gives:

$$X(\infty) = \left[\frac{e^{-(-1+\frac{1}{\sigma})\infty}}{\Gamma(\alpha)} \right] \int_0^\infty \left[(z)^{\frac{\alpha}{\sigma}-1} (1+z)^{-\frac{\alpha}{\sigma}+1} e^{-\infty} \right] dz = 0 \quad \dots (4.29)$$

$$\text{For } 0 < \sigma < 1 \quad \dots (4.30)$$

The range in 4.30 are the only real values of σ obtainable from equation 4.16.

Thus the original boundary condition for x reduces from $-\infty < x \leq \infty$ to:

$$-\frac{u_0 \beta}{u_d} < x \leq \infty \quad \dots (4.31)$$

4.12 **The Y(v,s) Component**

The $Y(y,s)$ component is given by equation 4.27 and its only subject to the condition for using positive or negative $D_\#$ especially 3.29b. These equation and condition are:

$-D_\#$ is used for negative y ... 3.29b (recalled)

$$Y(y, s) = \frac{1}{3} \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs + Rk + \omega^2}{D_0 \lambda} y \right]^{\frac{1}{2}} e^{\frac{u_0 y}{2D_0}} \left[I_{-1/3} \left(\frac{2}{3} \left[\frac{D_0 \lambda}{Rs + Rk + \omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs + Rk + \omega^2}{D_0 \lambda} y \right]^{3/2} \right) - I_{1/3} \left(\frac{2}{3} \left[\frac{D_0 \lambda}{Rs + Rk + \omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs + Rk + \omega^2}{D_0 \lambda} y \right]^{3/2} \right) \right] \quad \dots (4.27) \text{ (recalled)}$$

Because the Modified Bessel functions of the first kind as used in the solution rapidly decays to zero and it's in fact zero at infinity so, the positive half from zero to infinity is properly described.

The negative half from zero to negative infinity is explained thus:

Equation 4.27 is the general solution to equation 4.05b which can be written in full as:

$$D_{\#} u_0 \frac{\lambda}{y} Y'' - u_0 \frac{\lambda}{y} Y' - (Rs + Rk + \omega^2) Y = 0 \quad \dots (4.05b) \text{ (recalled)}$$

Let:

$$\left. \begin{aligned} y &= -y && \dots a \\ \frac{d}{dy} &= -\frac{d}{d(-y)} && \dots b \\ \frac{d^2}{dy^2} &= -\frac{d}{d(-y)} \left(-\frac{d}{d(-y)} \right) = \frac{d^2}{d(-y)^2} && \dots c \end{aligned} \right\} \quad \dots (4.32)$$

Changing the dependent variable of 4.05b from $+y$ to $-y$ by substituting 4.32 into 4.05b gives:

$$-D_{\#} u_0 \frac{\lambda}{y} Y'' - u_0 \frac{\lambda}{y} Y' - (Rs + Rk + \omega^2) Y(-y, s) = 0 \quad \dots (4.33)$$

However the condition in 3.29b says $-D_{\#}$ must be used with negative y ; substituting $-D_{\#}$ for $D_{\#}$ in equation 4.33 gives:

$$D_{\#} u_0 \frac{\lambda}{y} Y'' - u_0 \frac{\lambda}{y} Y' - (Rs + Rk + \omega^2) Y(-y, s) = 0 \quad \dots (4.34a)$$

$$D_0 \frac{\lambda}{y} Y'' - u_0 \frac{\lambda}{y} Y' - (Rs + Rk + \omega^2) Y(-y, s) = 0 \quad \dots (4.34b) \text{ (in terms of } D_0)$$

Upon Comparing equation 4.34b with 4.05b, it can be concluded that:

$$Y(-y, s) = Y(y, s) \quad \dots (4.35)$$

Equation 4.35 implies that a global solution exist for the $Y(y,s)$ component because a solution for the positive half $Y(+y,s)$ exist. Furthermore the concentration distribution of the $Y(y,s)$ component in the negative half, is a mirror image of its positive half.

4.13 **The Concentration Distribution $C(x, y, t)$**

The concentration distribution in Laplace Space $C(x,y,s)$ is given as:

$$C(x, y, s) = X(x)Y(y, s) \quad \dots (4.36)$$

$$X(x) = \frac{\Gamma(\frac{\alpha}{\sigma})}{\Gamma(\alpha)} U \left(\frac{\alpha}{\sigma}, 0, \sigma \left[\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x \right] \right) e^{-\frac{1}{2} \left(-1 + \frac{1}{\sigma} \right) \left(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x \right)} \quad \dots (4.18c) \text{ (recalled)}$$

$$Y(y, s) = \frac{1}{3} \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y \right]^{\frac{1}{2}} e^{\frac{u_0 y}{2D_0}} \left[I_{-1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y \right]^{3/2} \right) - I_{1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y \right]^{3/2} \right) \right] \quad \dots (4.27) \text{ (recalled)}$$

Thus:

$$C(x, y, s) = AX(x)Y(y, s) \quad \dots (4.37)$$

Where A is a constant.

Since both components, $X(x)$ and $Y(y,s)$ already satisfy the boundary conditions at their respective terminal points, only one constant will be evaluated at the wellbore; the point connecting these two components. The maximum concentration of the system C_0 is encountered at the wellbore, it is constant and independent of time. Thus at the wellbore;

$$\mathcal{L}\{C_0\} = AX(x_w)Y(y_w, s)$$

$$\frac{C_0}{s} = AX(x_w)Y(y_w, s)$$

$$A = \frac{C_0}{s X(x_w)Y(y_w, s)} \quad \dots (4.38)$$

Substituting equation 4.37 into 4.36 gives:

$$C(x, y, s) = \frac{X(x)}{X(x_w)} \frac{Y(y,s)}{Y(y_w,s)} \frac{C_0}{s} \quad \dots (4.39)$$

Thus:

$$C(x, y, t) = \mathcal{L}^{-1} \left\{ \frac{X(x)Y(y,s)}{X(x_w)Y(y_w,s)} \frac{C_0}{s} \right\}$$

$$C(x, y, t) = \frac{X(x)}{X(x_w)} C_0 \mathcal{L}^{-1} \left\{ \frac{Y(y,s)}{Y(y_w,s)} \frac{1}{s} \right\} \quad \dots (4.40)$$

Equation 4.40 is always C_0 at the wellbore.

$C(x, y, t)$ is defined within the boundaries:

$$\left. \begin{array}{l} -\frac{u_0\beta}{u_d} < x \leq \infty \quad \dots a \\ -\infty \leq y \leq \infty \quad \dots b \\ t > 0 \quad \dots c \end{array} \right\} \quad \dots (4.41)$$

4.14 The Drift Ratio d

The drift ratio is defined as:

$$d = \frac{u_d}{u_0} \quad \dots (3.33b) \text{ (recalled)}$$

Equation 3.33b implies that the value of the drift ratio can be adjusted by changing the injection velocity. Equation 4.28 defines the region where the X(x) solution is valid as:

$$x > -\frac{\beta}{d} \quad \dots (4.28) \text{ (recalled)}$$

Where x and β are defined as:

$$x = r \cos(\theta)$$

$$\beta = r_w \cos^2(\theta) \quad \dots (3.22a)$$

Substituting this into equation 4.28 gives:

$$r \cos(\theta) > -\frac{r_w \cos^2(\theta)}{d} \quad \dots (4.42a)$$

The minimum value of x is gotten when $\cos(\theta) = -1$ substituting this value into 4.41a gives:

$$-r > -\frac{r_w}{d} \quad \text{Which implies that}$$

$$r < \frac{r_w}{d} \quad \dots (4.42b)$$

Equation 4.42b gives the maximum radial distance along the negative x axis where the X(x) solution is valid in terms of two known parameters namely the drift ratio and the well radius.

Rearranging equation 4.42b gives:

$$d < \frac{r_w}{r} \quad \dots (4.42c)$$

$$\text{But } r > r_w \quad \dots (3.20d) \text{ (recalled)}$$

Substituting equation 3.20d into 4.42c gives the range of d as:

$$0 < d < 1 \quad \dots (4.42d)$$

4.15 **The Separation Constant ω^2**

For the right constant of separation equation 4.03 must be satisfied i.e.

$$\left. \begin{aligned} D_0 \left(\frac{\beta}{x} + d \right) X'' - u_0 \left(\frac{\beta}{x} + d \right) X' - \mu X(x) &= 0 \quad \dots a \\ D_0 \frac{\lambda}{y} Y_{yy} - u_0 \frac{\lambda}{y} Y_y - RY_t - (Rk - \mu) Y(y, t) &= 0 \quad \dots b \end{aligned} \right\} \quad \dots (4.03) \text{ (recalled)}$$

Since both equations are equal to the same value, they are equal. Therefore:

$$D_0 \left(\frac{\beta}{x} + d \right) X'' - u_0 \left(\frac{\beta}{x} + d \right) X' - \mu X(x) = D_0 \frac{\lambda}{y} Y_{yy} - u_0 \frac{\lambda}{y} Y_y - RY_t - (Rk - \mu) Y(y, t) \text{ or}$$

$$D_0 \left(\frac{\beta}{x} + d \right) X'' - u_0 \left(\frac{\beta}{x} + d \right) X' - \mu X(x) - \left[D_0 \frac{\lambda}{y} Y_{yy} - u_0 \frac{\lambda}{y} Y_y - RY_t - (Rk - \mu) Y(y, t) \right] = 0$$

Which can then be written as:

$$D_0 \left(\frac{\beta}{x} + d \right) X'' - u_0 \left(\frac{\beta}{x} + d \right) X' - \mu X(x) - \mathcal{L}^{-1} \left[D_0 \frac{\lambda}{y} Y'' - u_0 \frac{\lambda}{y} Y' - (Rs + Rk - \mu) Y(y, s) \right] = 0 \quad \dots (4.43)$$

Where: X(x) is given by equation 4.19a and 4.19b and Y(y,s) by equation 4.27.

$$\mu = -\omega^2 \quad \dots (4.12a) \text{ (recalled) } (\mu \text{ is a negative number})$$

$$\omega^2 = |\mu| \quad \dots (4.12b) \text{ (recalled) } (\omega^2 \text{ is a positive number})$$

$$\sigma \equiv \sqrt{1 + 4\alpha_1} = \sqrt{1 - \frac{4\omega^2 D_0}{u_0^2 d}} \quad \dots (4.16) \text{ (recalled)}$$

$$0 < \sigma < 1 \quad \dots (4.30) \text{ (recalled)}$$

Rearranging equation 4.16 gives:

$$\omega^2 = \frac{u_0^2(1-\sigma^2)d}{4D_0} \quad \dots (4.44a)$$

Substituting the value of σ from equation 4.30 into 4.44a gives:

$$0 < \omega^2 < \frac{u_0^2 d}{4D_0} \quad \dots (4.44b)$$

The range of values for ω^2 already satisfy equation 4.16, 4.12a and 4.12b and in reality any value of ω^2 that satisfy equation 4.44b will result in a solution. Thus the nature of ω^2 , can then be investigated using equation 4.43 and conditions at the wellbore.

At the wellbore, Equation 4.43a can be rewritten as:

$$D_0 \left(\frac{r_w(\cos\theta)^2}{r_w \cos\theta} + d \right) X'' - u_0 \left(\frac{r_w(\cos\theta)^2}{r_w \cos\theta} + d \right) X' + \omega^2 X(x_w) - \mathcal{L}^{-1} \left[D_0 \frac{vr_w(\sin\theta)^2}{r_w \sin\theta} Y'' - u_0 \frac{vr_w(\sin\theta)^2}{r_w \sin\theta} Y' - (Rs + Rk + \omega^2) Y(y_w, s) \right] = 0, \text{ which reduces to}$$

$$D_0(\cos\theta + d)X'' - u_0(\cos\theta + d)X' + \omega^2 X(x_w) - \mathcal{L}^{-1}[D_0(v\sin\theta)Y'' - u_0(v\sin\theta)Y' - (Rs + Rk + \omega^2)Y(y_w, s)] = 0 \quad \dots (4.45)$$

Equation 4.45 implies that ω^2 might be directionally dependent because of the presence of θ .

Thus after every constant or parameter needed to obtain the solution $X(x)$ and $Y(y,t)$ solution have been determined, a positive number guess that falls within the range of equation 4.42 is chosen for ω^2 . This guess can then be refined through successive iteration until equation 4.45 is approximately true. During each round of iteration, the angle θ must be kept constant; so that for every angle θ a corresponding ω^2 will be determined. After which an empirical relationship between θ and ω^2 can then be established using regression analysis.

5.00

RESULTS AND DISCUSSIONS

Using practical considerations, certain values were adopted for the parameters needed to implement a model of the solution in equation 4.40. Equation 4.40 is an expression for the tracer concentration in the mobile fluid phase and for the purpose of this thesis, the mobile fluid phase is assumed to be Water with a saturation of about 80% while oil is the immobile fluid phase. In practice values of these parameters are usually obtained from formation evaluation, core analysis (including special core analysis), logs, well testing, etc. The system is assumed to be isotropic (i.e. having the same properties in the x and y principal coordinate axes) with $\nu = 1$, and homogeneous.

5.01 Given Parameters

Table 5.00: Summary of Parameters.

Description	SYM	Range			Model		Source
		Field Units	MAX	MIN	Field Units	SI Units	
Dispersion	D_0	m ² /day	2299	0.0114	120	1.389×10^{-3} m ² /s	Kubare et al (2010)
Well Radius	r_w	inches	10	2	$\cong 5.000$	0.127 m	Bourgoyne et al (1984)
Injection Rate	q	bbl./day	> 10,000	250	750.0	1.380×10^{-3} m ³ /s	Kubare et al (2010)
Medium Height	h	ft			30.00	9.114 m	Assumed
Porosity	ϕ		0.467	0.020	0.250	0.250	Taigbenu and Rusinga (2005)
Water Saturation	S_m		1.00	0.100	0.800	0.800	Kubare et al (2010)
Decay Constant	k	year ⁻¹	9.85×10^{-10}	4.95×10^{-11}	3×10^{-10}	9.506×10^{-18} s ⁻¹	Faure (1986)
Retardation	R		10,000	1	1.200	1.200	Kubare et al (2010)
Drift Ratio	d		1	> 0	0.250	0.250	Introduced in this work
Isotropy factor	ν			> 0	1.000	1.000	Introduced in this work

5.02 Calculated and other Required Parameters

The Injection Velocity:

$$\bar{u} = \frac{pq}{2\pi r h \phi S_m} \dots (3.19) \text{ (recalled)}$$

$$u_0 = \frac{pq}{2\pi r_w h \phi S_m} = \frac{1.380 \times 10^{-3}}{2 \times 3.142 \times 0.127 \times 9.114 \times 0.25 \times 0.80} = 9.4876 \times 10^{-4} \text{ m s}^{-1} \dots (5.00)$$

The next set of required parameters given below, are dependent on θ whose range is given as:

$$0 \leq \theta \leq 360 \dots (5.01)$$

$$\left. \begin{aligned} \beta &= r_w \cos^2 \theta \dots a) \\ \lambda &= \nu r_w \sin^2 \theta \dots b) \end{aligned} \right\} \dots (3.22) \text{ (recalled)}$$

$$\left. \begin{aligned} x_w &= r_w \cos \theta & \dots a \\ y_w &= r_w \sin \theta & \dots b \end{aligned} \right\} \dots (3.35) \text{ (recalled)}$$

Finally the last set of required parameters, are dependent on the constant of separation, they are:

$$\left. \begin{aligned} \gamma &= 0 & \dots a \\ \alpha &= \frac{\omega^2 \beta}{u_0 d^2} & \dots b \\ \alpha_1 &= -\frac{\omega^2 D_0}{u_0^2 d} & \dots c \end{aligned} \right\} \dots (4.13) \text{ (recalled)}$$

$$\sigma = \sqrt{1 - \frac{4\omega^2 D_0}{u_0^2 d}} \quad \dots (4.16) \text{ (recalled)}$$

5.03 Calculating The Constant of Separation

The directionally dependent constant of separation is itself dependent on the angle θ . Its range is given as:

$$0 < \omega^2(\theta) < \frac{u_0^2 d}{4D_0} \quad \dots (4.42) \text{ (recalled)}$$

For the right separation constant equation 4.45 is satisfied i.e.

$$D_0(\cos\theta + d)X'' - u_0(\cos\theta + d)X' + \omega^2 X(x_w) - \mathcal{L}^{-1}[D_0(v\sin\theta)Y'' - u_0(v\sin\theta)Y' - (Rs + Rk + \omega^2)Y(y_w, s)] = 0 \quad \dots (4.45) \text{ (recalled)}$$

$X(x_w)$, $X'(x_w)$ and $X''(x_w)$

The $X(x_w)$ component of the concentration distribution is given as:

$$X(x_w) = \frac{1}{\Gamma(\alpha)} \left[e^{-\frac{1}{2}(-1+\frac{1}{\sigma})(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w)} \int_0^\infty \left[z^{\frac{\alpha}{\sigma}-1} (1+z)^{-\frac{\alpha}{\sigma}+1} e^{-\sigma(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w)z} \right] dz \right]$$

$$X(x_w) = \frac{\Gamma(\frac{\alpha}{\sigma})}{\Gamma(\alpha)} U\left(\frac{\alpha}{\sigma}, 0, \sigma \left[\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w \right]\right) e^{-\frac{1}{2}(-1+\frac{1}{\sigma})(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w)} \quad \dots (5.02a)$$

Differentiating $X(x_w)$ with respect to x_w gives:

$$X'(x_w) = -\frac{u_0}{D_0 \Gamma(\alpha)} e^{-\frac{1}{2}(-1+\frac{1}{\sigma})(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w)} \left\{ \left(-\frac{1}{2} + \frac{1}{2\sigma} \right) \int_0^\infty \left[z^{\frac{\alpha}{\sigma}-1} (1+z)^{-\frac{\alpha}{\sigma}+1} e^{-\sigma(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w)z} \right] dz + \right.$$

$$\left. \sigma \int_0^\infty \left[z^{\frac{\alpha}{\sigma}} (1+z)^{-\frac{\alpha}{\sigma}+1} e^{-\sigma(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w)z} \right] dz \right\}$$

Comparing the integral part with 4.19b gives:

$$X'(x_w) = -\frac{u_0}{D_0 \Gamma(\alpha)} e^{-\frac{1}{2}(-1+\frac{1}{\sigma})(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w)} \left\{ \left(-\frac{1}{2} + \frac{1}{2\sigma} \right) \Gamma\left(\frac{\alpha}{\sigma}\right) U\left(\frac{\alpha}{\sigma}, 0, \sigma \left[\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w \right]\right) + \right.$$

$$\left. \sigma \Gamma\left(\frac{\alpha}{\sigma} + 1\right) U\left(\frac{\alpha}{\sigma} + 1, 1, \sigma \left[\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w \right]\right) \right\} \quad \dots (5.02b)$$

Differentiating $X'(x_w)$ with respect to x_w gives:

$$X''(x_w) = \frac{u_0^2}{D_0^2 \Gamma(\alpha)} e^{-\frac{1}{2}(-1+\frac{1}{\sigma})(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w)} \left\{ \left[\left(-\frac{1}{2} + \frac{1}{2\sigma} \right)^2 \int_0^\infty \left[z^{\left(\frac{\alpha}{\sigma} - 1 \right)} (1+z)^{-\left(\frac{\alpha}{\sigma} + 1 \right)} \times \right. \right. \right. \\ \left. \left. e^{-\sigma \left(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w \right) z} \right] dz + (-\sigma + 1) \int_0^\infty \left[z^{\frac{\alpha}{\sigma}} (1+z)^{-\left(\frac{\alpha}{\sigma} + 1 \right)} e^{-\sigma \left(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w \right) z} \right] dz + \right. \\ \left. \left. \sigma^2 \int_0^\infty \left[z^{\left(\frac{\alpha}{\sigma} + 1 \right)} (1+z)^{-\left(\frac{\alpha}{\sigma} + 1 \right)} e^{-\sigma \left(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w \right) z} \right] dz \right\}$$

Comparing the integral part with 4.19b gives:

$$X''(x_w) = \frac{u_0^2}{D_0^2 \Gamma(\alpha)} e^{-\frac{1}{2}(-1+\frac{1}{\sigma})(\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w)} \left\{ \left(-\frac{1}{2} + \frac{1}{2\sigma} \right)^2 \Gamma\left(\frac{\alpha}{\sigma}\right) U\left(\frac{\alpha}{\sigma}, 0, \sigma \left[\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w \right]\right) + (-\sigma + 1) \times \right. \\ \left. \Gamma\left(\frac{\alpha}{\sigma} + 1\right) U\left(\frac{\alpha}{\sigma} + 1, 1, \sigma \left[\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w \right]\right) + \sigma^2 \Gamma\left(\frac{\alpha}{\sigma} + 2\right) U\left(\frac{\alpha}{\sigma} + 2, 2, \sigma \left[\frac{\beta u_0}{D_0 d} + \frac{u_0}{D_0} x_w \right]\right) \right\} \dots (5.02c)$$

The main definitions used to obtain values for Tricomi Kummer $U(a,b,z)$ are given below:

$$\left. \begin{aligned} U(a,0,z) &= \frac{1}{\Gamma(a+1)} \left\{ 1 + az \ln(z) {}_1F_1(a+1;2;z) - \sum_{c=1}^{\infty} \left[\frac{(a)_c (2\psi(c) - \psi(a+c) + \frac{1}{c}) z^c}{c!(c-1)!} \right] \right\} & \dots a \\ U(a,n,z) &= \frac{(-1)^n}{\Gamma(a-n+1)} \left\{ \frac{\ln(z)}{(n-1)!} {}_1F_1(a;n;z) + \sum_{c=0}^{\infty} \left[\frac{(a)_c (\psi(a+c) - \psi(c+1) - \psi(c+n)) z^c}{c!(c+n-1)!} \right] - \sum_{c=1}^{n-1} \left[\frac{(c-1)! z^{-c}}{(1-a)_c (n-c-1)!} \right] \right\} & \dots b \\ U(a,n,z) &= \lim_{f \rightarrow a} U(f,n,z) \text{ for } a \in \mathbb{Z} \text{ and } f \notin \mathbb{Z} & \dots c \\ U(a,b+1,z) &= \left(\frac{a}{z} + 1\right) U(a,b,z) + \frac{a}{z} (b-a-1) U(a+1,b,z) & \dots d \end{aligned} \right\} \dots (5.03)$$

(Wolfram), (Abramowitz and Stegun, 1970).

Where: n is an integer, f is not an integer, while a , b and z can be either integers or non-integers.

$$(a)_c \text{ is the rising factorial which can also be expressed as } (a)_k = \frac{\Gamma(a+c)}{\Gamma(a)} \dots (5.04)$$

ψ is the digamma or psi function.

${}_1F_1$ is the Generalized Hypergeometric Function.

$\Gamma(c+1) = c! \dots (5.05)$, was used to express factorials involving none integer values.

Using the recurrence relation in 5.03d was more accurate than 5.03b or 5.03c in estimating $U(a,1,z)$, however it was less accurate than the mentioned equations in when estimating $U(a,2,z)$ from $U(a,1,z)$.

For $b = 0$ and using equation 5.03d,

$$U(a,1,z) = \left(\frac{a}{z} + 1\right) U(a,0,z) - \frac{a}{z} (a+1) U(a+1,0,z) \dots (5.06)$$

These formulas (equation 5.04 to 5.06) were used to create the $U(a,b,z)$ function in MATLAB, for “ a ” & “ z ” greater than zero and integer values of “ b ” from zero to two (0, 1, 2). The MATLAB code can be seen in the Appendix A.

$Y(y_w, s), Y'(y_w, s)$ and $Y''(y_w, s)$

The $Y(y_w, s)$ component of the concentration distribution is given as:

$$Y(y_w, s) = \frac{1}{3} \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{\frac{1}{2}} e^{\frac{u_0 y_w}{2D_0}} \left[I_{-1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{3/2} \right) - I_{1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{3/2} \right) \right] \quad \dots (5.07a)$$

Using MATLAB Symbolic Differentiation,

$$Y'(y_w, s) = \frac{1}{3} e^{\frac{u_0 y_w}{2D_0}} \left\{ \left[I_{-1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{3/2} \right) - I_{1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{3/2} \right) \right] \left[\frac{1}{2} \left(\frac{u_0}{D_0} \right) \left(\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right)^{\frac{1}{2}} \right] - \left[I_{-2/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{3/2} \right) - I_{2/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{3/2} \right) \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{1/2} \right\} \quad \dots (5.07b)$$

$$Y''(y_w, s) = \frac{1}{3} e^{\frac{u_0 y_w}{2D_0}} \left\{ \left[I_{-1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{3/2} \right) - I_{1/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{3/2} \right) \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{3/2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{\frac{3}{2}} + \frac{3}{4} \frac{(Rs+Rk+\omega^2)^2}{\lambda^2 \left(\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right)^{3/2}} \left(\frac{1}{D_0} - \frac{1}{D_0^2} \right) + \frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 \left(\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right)^{\frac{1}{2}} \right] - \left[I_{-2/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{3/2} \right) - I_{2/3} \left(\frac{2}{3} \left[\frac{D_0\lambda}{Rs+Rk+\omega^2} \right] \left[\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right]^{3/2} \right) \right] \left[\frac{u_0}{D_0} \left(\frac{1}{4} \left(\frac{u_0}{D_0} \right)^2 + \frac{Rs+Rk+\omega^2}{D_0\lambda} y_w \right) \right] \right\} \quad \dots (5.07c)$$

Thus by substituting equation 5.02 and 5.07 into 4.45, values of ω^2 that best satisfies equation 4.45 for every angle θ can be identified.

Taking a hundred different values for each angle, and using Gaver-Stehfest Algorithm for Laplace Inversion, ω^2 was obtained at two different times (for t=30 days and t=50 days), to check if it is time dependent. Plots of ω^2 versus θ , as well as the corresponding error associated with each value of ω^2 are given below, while the MATLAB Code can be seen in Appendix A.

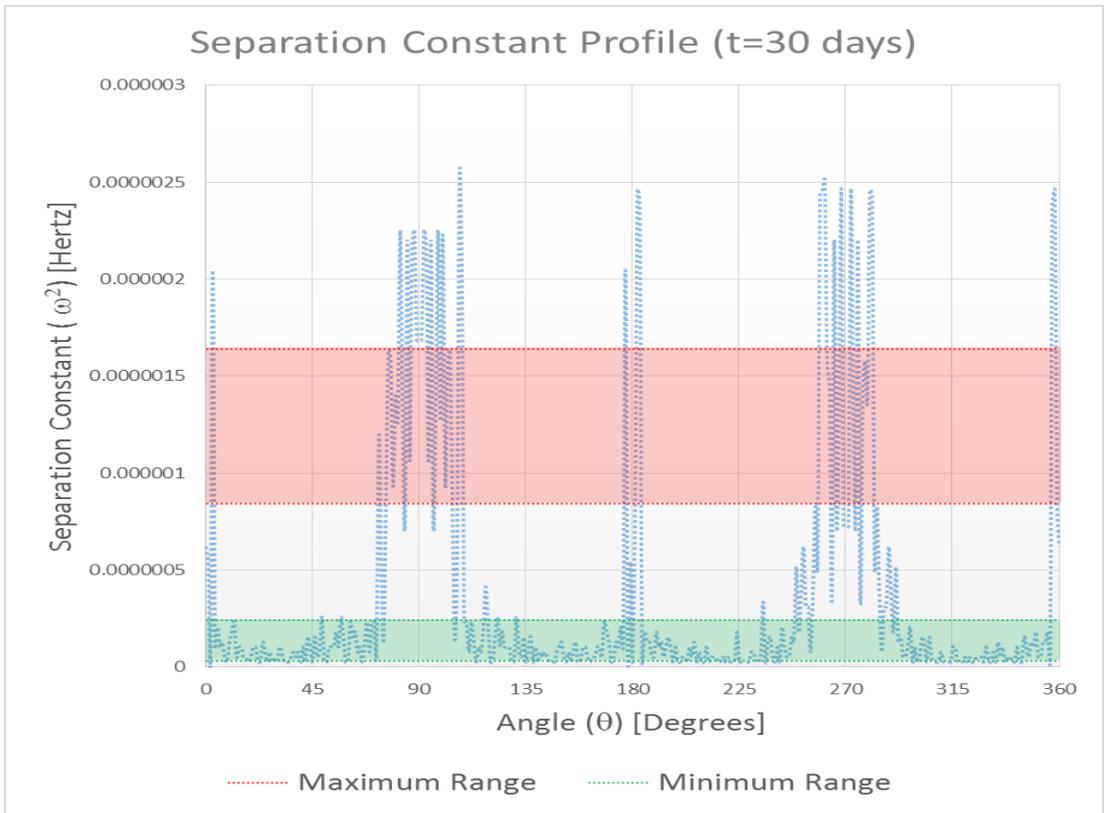


Figure 5.00 Simulated Profile of Separation Constant (t=30 days)

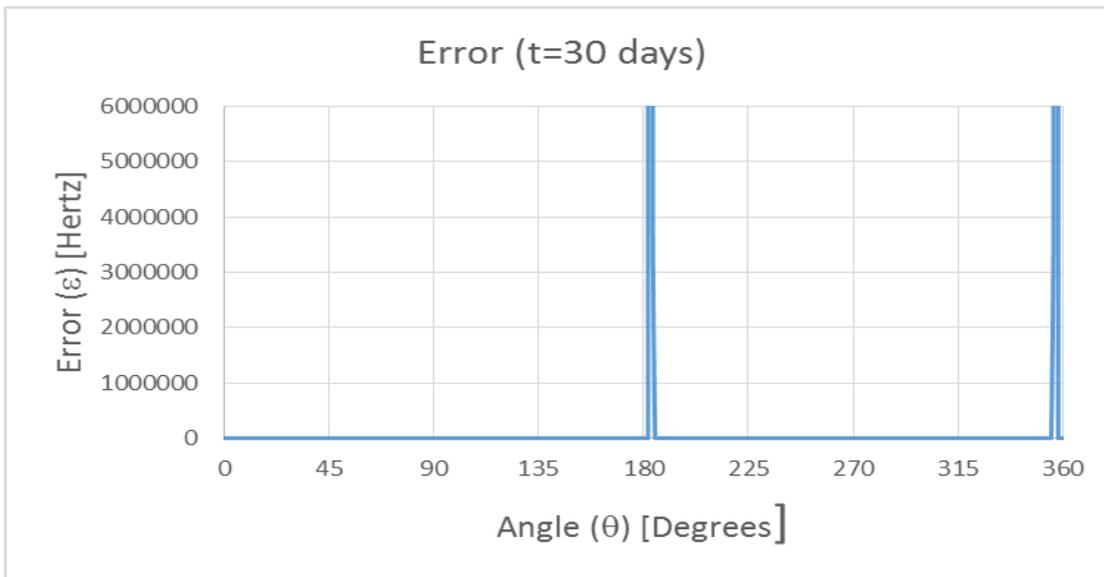


Figure 5.01 Error Profile (t=30 days)

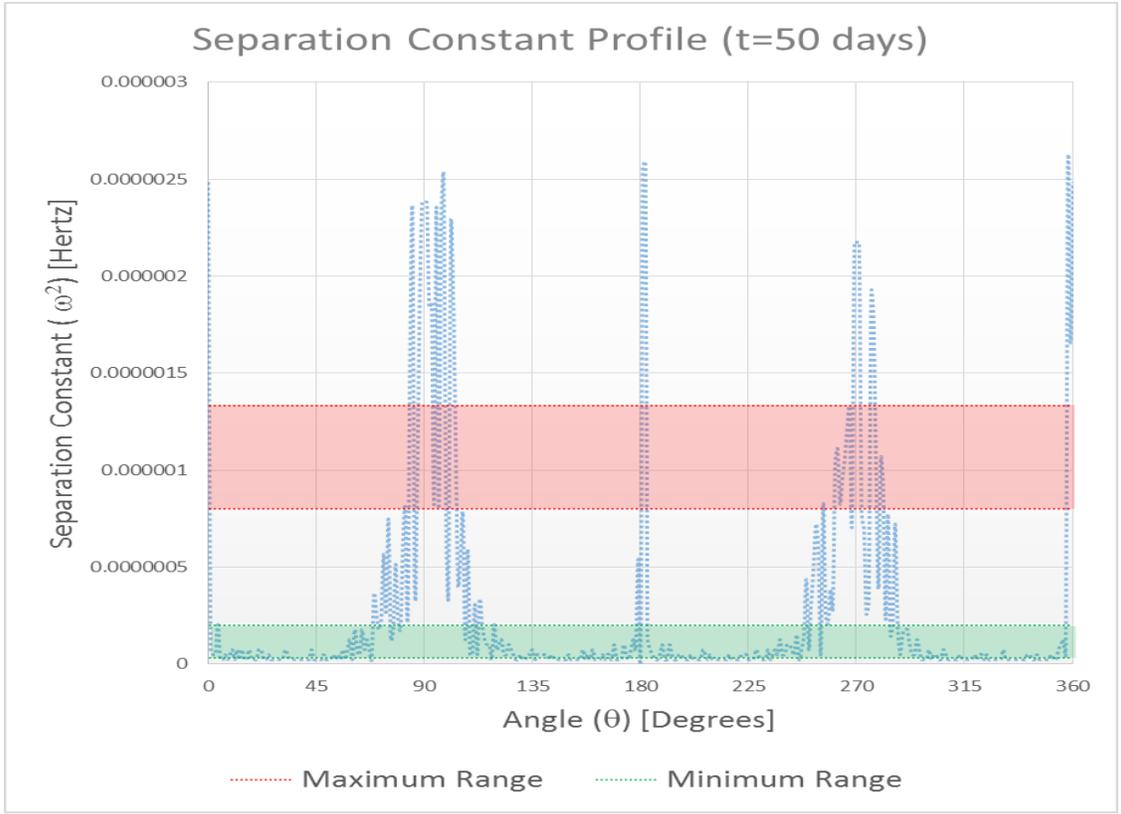


Figure 5.02 Simulated Profile of Separation Constant (t=50 days)

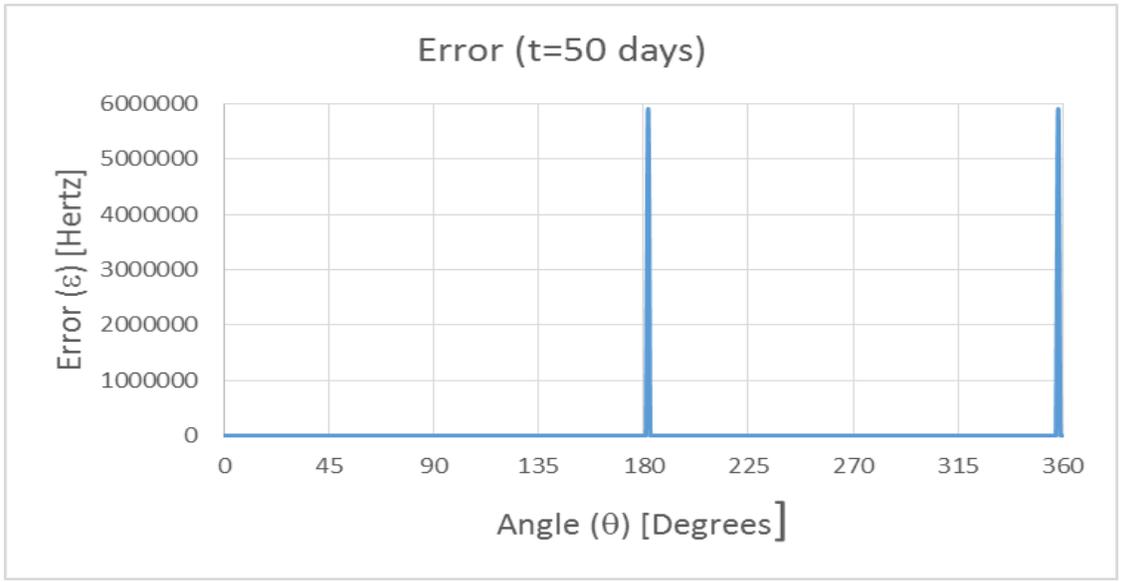


Figure 5.03 Error Profile (t=50 days)

One important conclusion that can be reached from figure 5.00 and 5.02 is the fact that ω^2 is time dependent and perhaps less obvious is the fact that it is inversely proportional to time (as its unit suggest). Thus ω^2 is a three dimensional variable that depends on angle (which is a two dimensional parameter) and time.

As with many numerical simulation, the result showed some instability which is connected to the Laplace Inversion Algorithm, however it provided an idea of what ω^2 should be. Furthermore, based on the error profiles (figure 5.01 and 5.03), the spikes at 180° , 360° (and 0° by extension) in figure 5.00 and 5.02 are as a result of singularity encountered when taking the inverse of the coordinate point y . y is 0 when θ is either 0, 180 or 360. Thus those points in figure 5.00 and 5.02 can be ignored. ω^2 was modelled using two values taken from the charts. For this work;

$$\omega^2 = (\omega_{max}^2) - (\omega_{max}^2 - \omega_{min}^2)|\cos(\theta)| \quad \dots (5.08)$$

Where: ω_{max}^2 is a numeric value gotten from the maximum range in figure 5.00 and 5.02.

ω_{min}^2 is a numeric value gotten from the minimum range in figure 5.00 and 5.02.

Plots of ω^2 versus θ used in this work for both time periods is can be seen in figure 5.04 and 5.05.

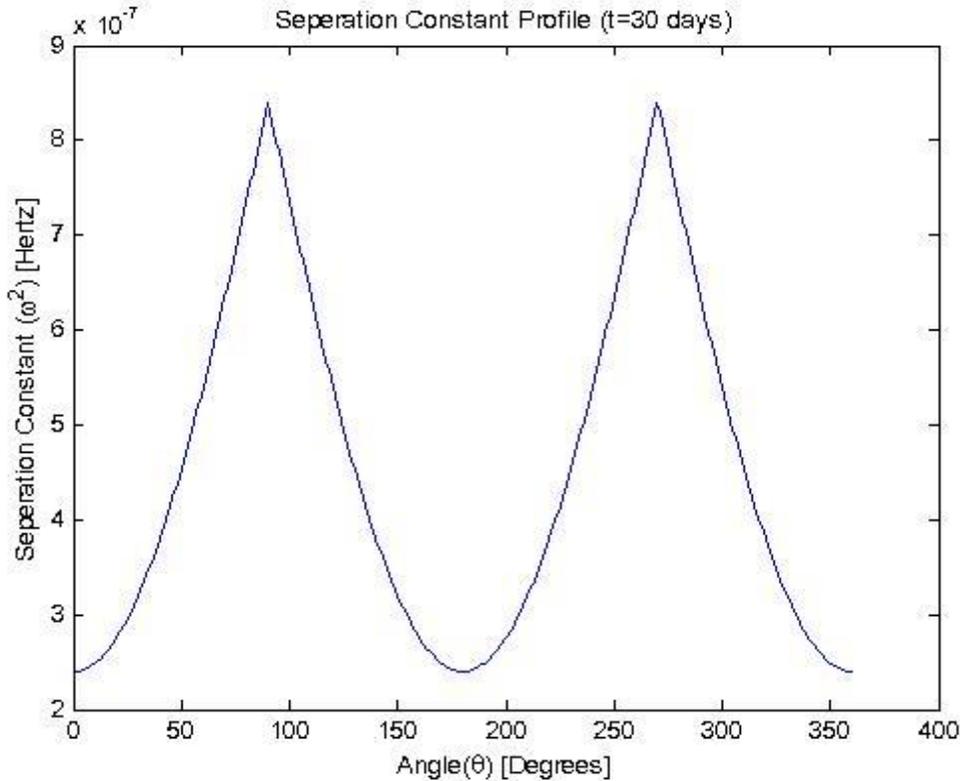


Figure 5.04 Separation Constant Profile (t=30 days)

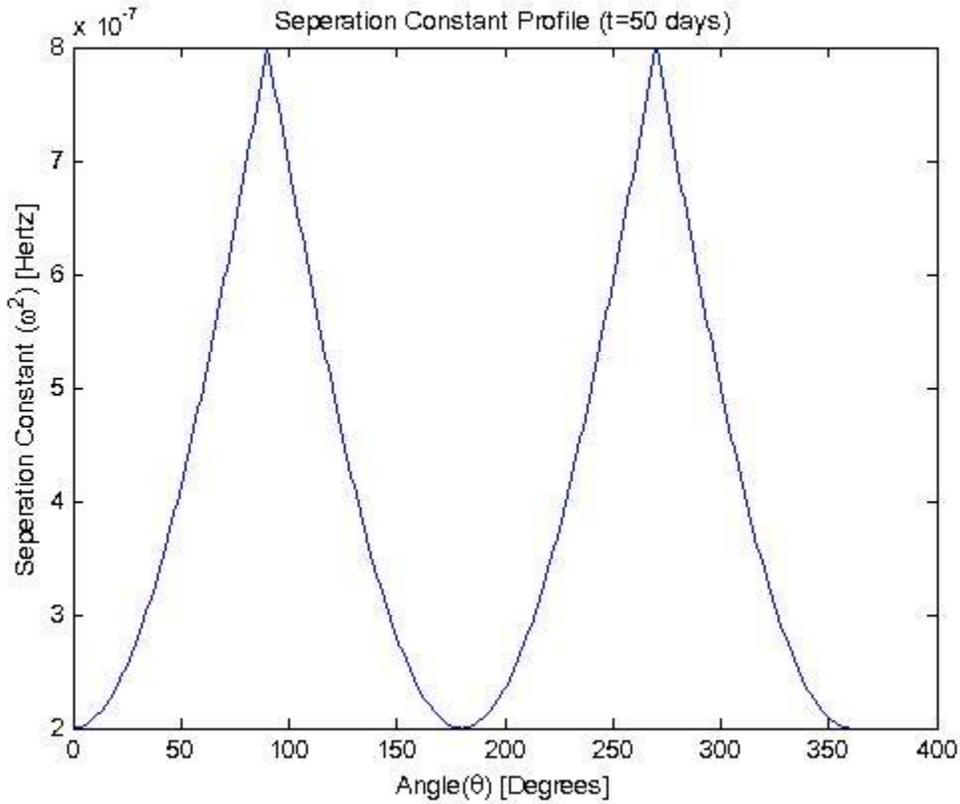


Figure 5.05 Separation Constant Profile (t=50 days)

Table 5.01: Values of ω_{\max}^2 and ω_{\min}^2 used in the Figures Above

	ω_{\max}^2	ω_{\min}^2
t = 30 days	8.4	2.4
t = 50 days	8.0	2.0

As table suggests, the minimum value from the maximum value range was taken for ω_{\max}^2 and the maximum value from the minimum value range was taken for ω_{\min}^2 .

5.03 Result: C_r

The Concentration Distribution is given as:

$$C(x, y, t) = \frac{X(x)}{X(x_w)} C_0 \mathcal{L}^{-1} \left\{ \frac{Y(y, s)}{Y(y_w, s)} \frac{1}{s} \right\} \dots (4.40) \text{ (recalled)}$$

The maximum value of $C(x, y, t)$ is C_0 which is obtainable at the wellbore. Dividing through by C_0 gives a concentration ratio function, which is a value always between one and zero. i.e.

$$\frac{C(x, y, t)}{C_0} = \frac{X(x)}{X(x_w)} \mathcal{L}^{-1} \left\{ \frac{Y(y, s)}{Y(y_w, s)} \frac{1}{s} \right\}$$

Which can then be written as:

$$C_r = X_1 Y_r \quad \dots (5.13a)$$

Where: C_r is the concentration ratio distribution of the medium at a particular time t_r and

$$0 \leq C_r \leq 1 \quad \dots (5.13b)$$

$$X_1 = \frac{X(x)}{X(x_w)} \quad \dots (5.13c) \text{ (time independent)}$$

$$Y_i = \mathcal{L}^{-1} \left\{ \frac{Y(y,s)}{Y(y_w,s)} \frac{1}{s} \right\} \quad (5.13d) \text{ (time dependent)}$$

Equation 5.13c and 5.13d are both numeric values between zero and one; 5.13c is a steady state equation with respect to time why 5.13d is time dependent and thus would require Laplace Inversion. The Gaver-Stehfest Laplace Inversion Algorithm was used throughout this work, it was downloaded from mathworks; the link can be seen in the reference section.

Using MATLAB, a code was developed that implemented equation 5.13 subject to every condition guiding the $X(x)$, $Y(y,s)$ and $C(x,y,t)$. Only the solution along the positive half of the x axis was implemented as the $X(x)$ solution derived in this work does not cover the entire negative half of the x axis. A “PCOLOR”, “SURFACE” and “CONTOURF” plot for were obtained. The MATLAB code can be seen in Appendix A.

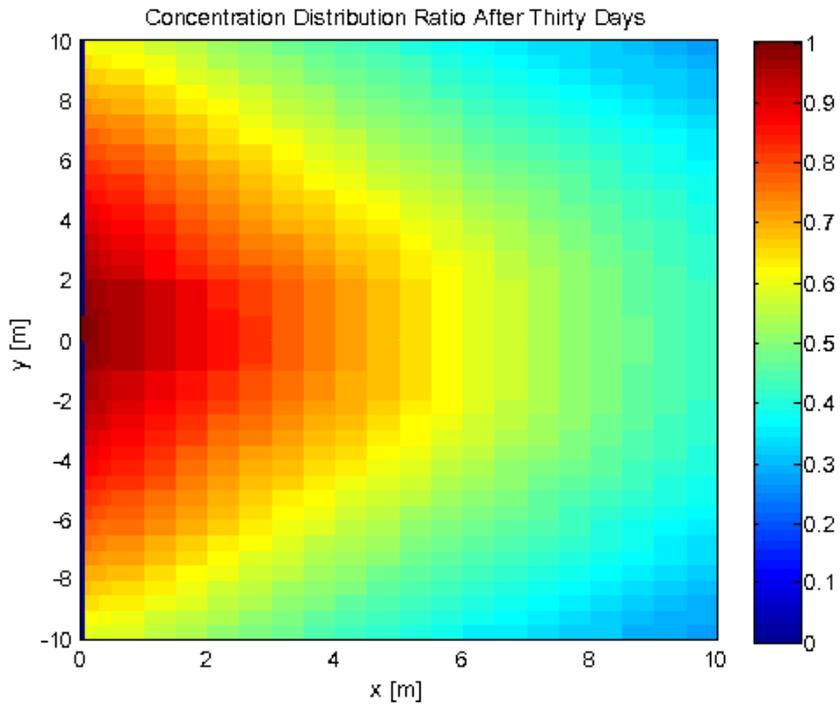


Figure 5.06 PCOLOR Plot of C_r After Thirty Days

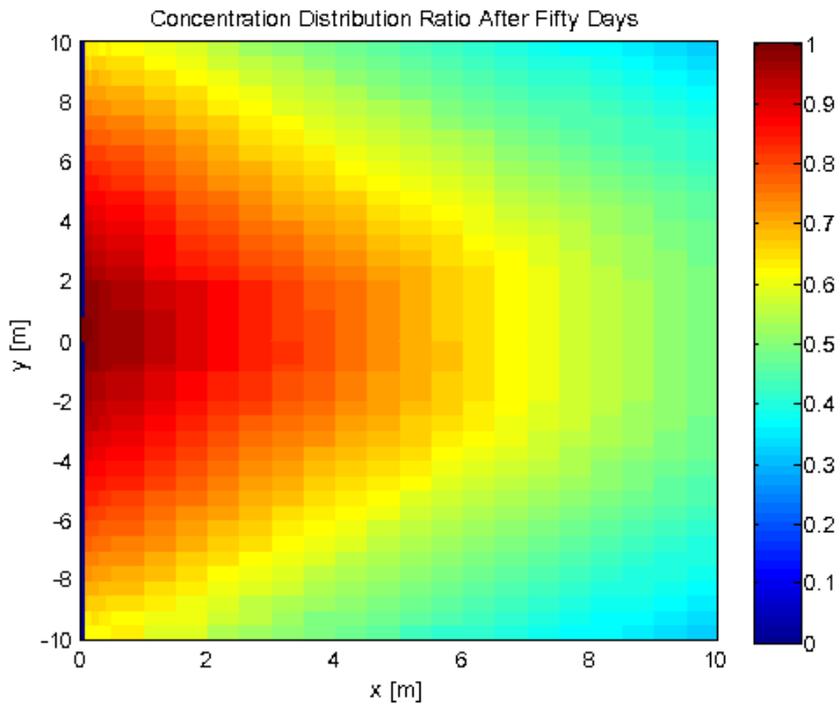


Figure 5.07 PCOLOR Plot of C_r After Fifty Days

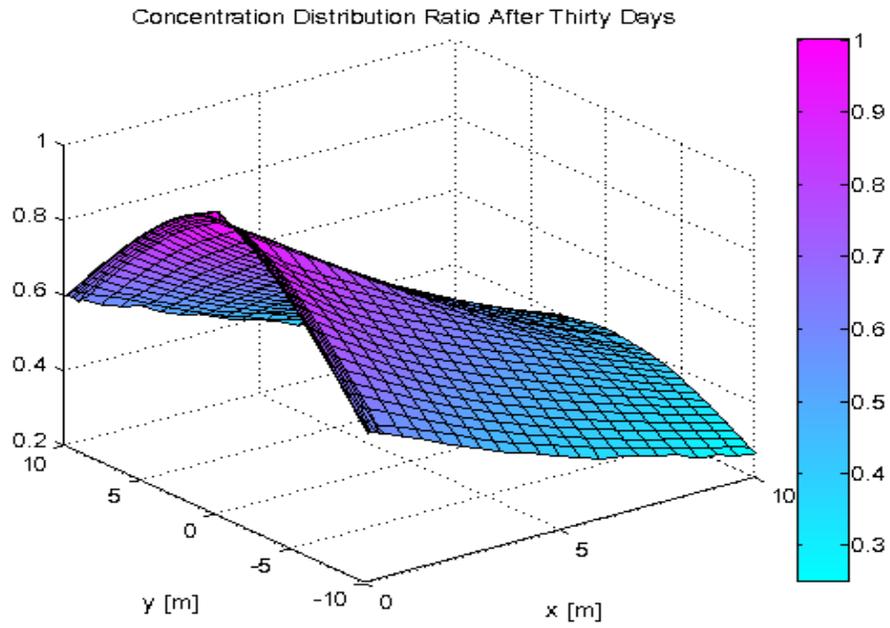


Figure 5.08 Surface Plot of C_r After Thirty Days

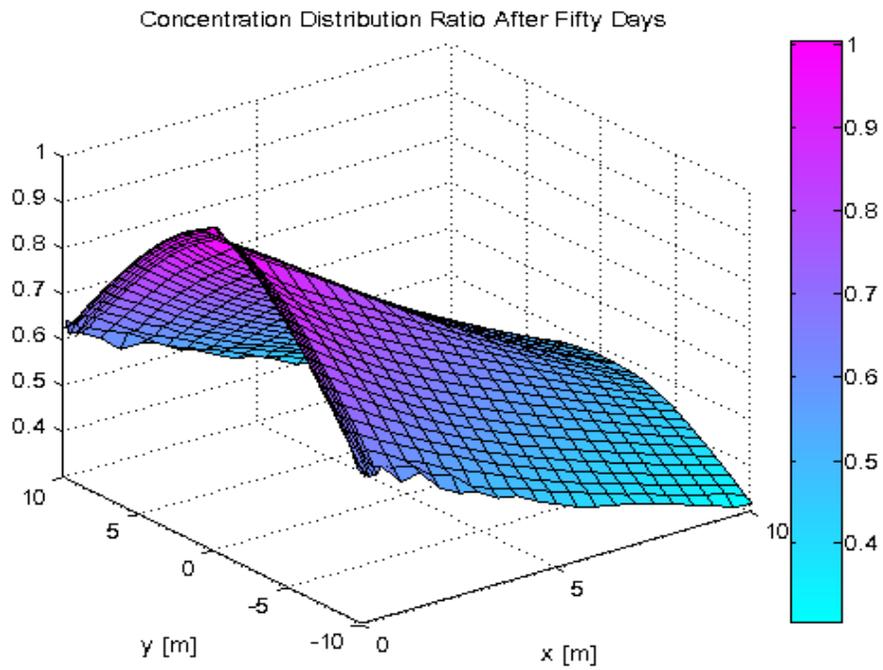


Figure 5.09 Surface Plot of C_r After Fifty Days

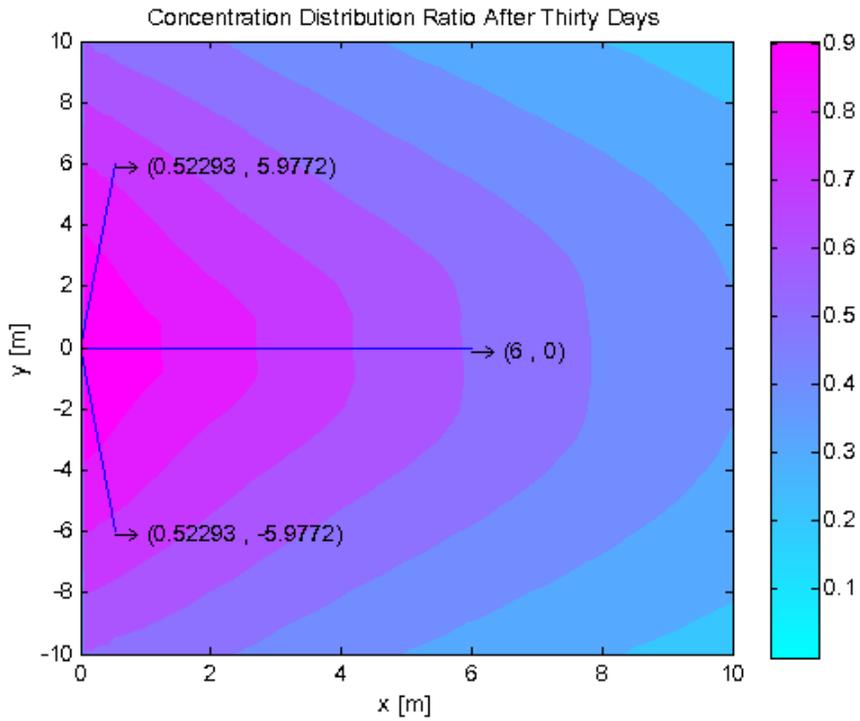


Figure 5.10 Contour Plot of C_r After Thirty Days

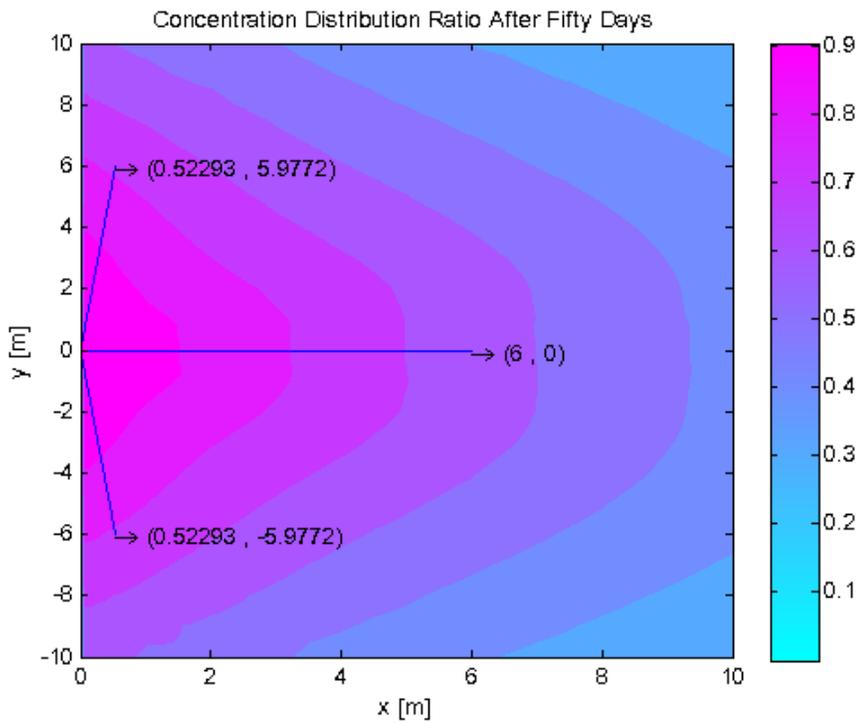


Figure 5.11 Contour Plot of C_r After Fifty Days.

Figure 5.06 to 5.11 shows the Concentration Distribution Ratio C_i After Thirty and Fifty Days of continuous tracer injection. All three type of plots shows an increase in C_i from when the thirty day plot its corresponding fifty day plot.

In a system without linear drift or any sort of natural convection, the propagation profile is expected to be cyclic. The concentration distribution will be equal at equal radial distance from the injection well, provided the medium is isotropic and homogeneous.

Linear Drift along the x-axis causes an unequal distribution in such a manner that the concentration gradient is at its minimum along the x axis. Since Convection and hydrodynamic dispersion is maximum along the positive x-axis the concentration of any point along the positive x-axis will be lower when compared to other points of equal radial distance. As the angle increases in the positive and negative y directions, so does the convection and hydrodynamic dispersion and thus the higher concentration. Consequently the concentration distribution gradient is highest along the y-axis in all the plots of C_i .

Three points, with equal radial distance of six meters were indicated to show this behavior.

- The point along the x-axis has the lowest concentration of the three points indicated.
- The other two points are both having equal concentration because they are at equal angles away from the x axis in the positive and negative y-directions. This shows that the angular effect of the concentration distribution under the effect of linear drift, is similar to the cyclic case for situations, just that for cyclic concentration distribution, it is also equal for every other angle.

5.03 Discussion

The velocity of the system is of paramount importance as it is through the velocity that properties of the system can be introduced. Anisotropy for instance was modelled by the inclusion of an isotropy factor in the definition of the systems velocity. Thus for the tracer concentration distribution profile of a system can only be understood if the systems velocity is properly modelled because both the resultant convection and hydrodynamic dispersion depends on velocity.

In this work, it was assumed that the velocity varies as the inverse of radial distance, which is only possible in homogeneous systems that could be either isotropic or anisotropic. Extensions to isotropic or anisotropic systems that are non-homogeneous will require another definition of velocity. Furthermore, the solution is only applicable to cases where radial convection disrupts the

natural linear convection (or linear drift) of the system and not cases like spread of pollutant in a river or diffusion of contaminants into underground water where there exist radial diffusion with no radial convection, because in such a case the drift ratio would be indeterminate thereby invalidating the $X(x)$ solution.

Since the drift is along the x axis alone, extension to three dimension is possible by assuming a solution in the z direction similar to that in the y direction and a similar interaction between the x and z dimensions to that between the x and y direction, provided the medium is not multilayered.

For continuous injection and/or production periods, this Solution can be used to simulate Concentration Distribution in Single and Multiple Wells, i.e. combining injection and production periods for a Single Well, or injection and production wells in the case of Multiple Wells. For a single point, this can be achieved by the addition or subtraction two or more solutions at that point, which can be extended to two or three dimensions using addition or subtraction of similar matrices that have the same size and reference the same coordinates; but contains independent solutions, of the injection period and the production period for a Single Well or the individual wells in the case of Multiple Wells.

In conjunction with a second solution that estimates the concentration dispersion of tracers (or pollutants) in homogeneous underground porous medium, under natural linear convection alone; they can be used to simulate Slug Tracer Test, which involves break periods (i.e. periods of no-injection of tracer or any other fluid) between periods of continuous tracer injection. The concentration distribution for the period of continuous injection will be handled by this solution, while the second solution will handle the break periods during which the system returns to its natural flow pattern.

Unlike Reservoir Simulation, it can instantly give results for individual points at various time locations and like Reservoir Simulation it can be used to obtain an estimate for the time of injection. True to its nature as an analytic solution, the response of the solution to a change in any term can be properly understood.

6.00

SUMMARY AND CONCLUSION

Although this is an analytic solution and possesses many of its properties, the inclusion of numerical Laplace Inversion weakens the idea of an analytic solution. Even though many efficient Laplace Inversion Algorithms exist, their misuse remains a possibility, they require additional time and computer resources, and there are always issues with instability. Furthermore the use of Laplace Inversion Algorithms in solutions to ADEs in general, makes it difficult to; understand the role played by the time variable, and in this work it also made it difficult to obtain an analytic expression for the separation constant, to which the accuracy of this solution is tied. Many possibilities exist for functions with two peaks at 90 and 270 degrees and some of them can give closely related solutions that are qualitatively correct. Thus it might require Lab experiments to verify which model is best applicable to various cases. Lastly, because the solution is presented in terms of certain special functions, its accuracy is also tied to the accuracy with which a computer calculates those special functions. This also implies significant computer resources and high processor requirements are necessities. In today's world where superfast computers are commonplace this might not be a challenge.

To understand the effect of linear drift on radial transport of tracer in porous medium, the problem was described as an Advection – Dispersion Equation in Cartesian Coordinates where linear drift and radial convection was properly defined and their interactions properly expressed. Subsequently a solution to the ADE was derived and implemented to show the effect linear drift. The derived solution covered the entire positive and negative y-axis and the positive x-axis but only the immediate regions of negative x-axis where the resultant direction of convection has not changed.

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ABBREVIATIONS

ADE	– Advection Dispersion (Diffusion) Equation.
CDE	– Convection Dispersion (Diffusion) Equation.
PDE	– Partial Differential Equation.
GFM	– Green Function Method.
GITT	– Generalized Integral Transform Technique.
MAX	– Maximum.
MIN	– Minimum.
ODE	– Ordinary Differential Equation.
SI	– Standard International.
SYM	– Symbol.

NOMENCLATURE

This section applies to this work therefore it's only applicable to Chapter Three, Four, and Five.

A – Constant of Integration.

a – As defined in text or used as part of numbering.

b – As defined in text or used as part of numbering.

C – Unabsorbed Tracer Concentration (per unit volume) in the Mobile Fluid Phase.

C_0 – Concentration at the Wellbore.

C_a – Chemical Tracer Concentration Lost (per unit volume) within the system.

C_i – Concentration Ratio with respect to Concentration at the Wellbore.

c – As defined in text or used as part of numbering.

D – Hydrodynamic Diffusivity or Dispersion.

$D_{\#}$ – Shear Mixing Coefficient or Hydrodynamic Dispersion Coefficient.

D_* – Dispersion along the x axis with Linear Drift Included.

D_0 – Hydrodynamic Dispersion in the Wellbore.

D_m – Molecular Diffusion Coefficient.

D_x – Dispersion along the x axis without Linear Drift.

D_y – Dispersion along the y axis.

d – Drift Ratio or used as part of equation numbering.

e - used as part of equation numbering.

f – As defined in text.

${}_1F_1$ – Generalized Hypergeometric Function.

${}_1G_1^1$ – Solution to The extended Confluent Hypergeometric Equation of The Second Kind.

h – Height of Porous Medium.

I – Modified Bessel Function of the First Kind.

i – Unit vector along the x axis.

j – Unit vector along the y axis.

k – Unit vector along the z axis.

k – Decay Constant.

k_1 – First Langmuir Isotherm Constant.

k_2 – Second Langmuir Isotherm Constant.
 L – Amount of tracer molecules lost in the entire system.
 l – Number of tracer molecules lost within the control volume.
 \ln – Natural Logarithm.
 m – Meter (unit of distance).
 m – Integer or as used in text.
 N – Number of Molecules.
 n – Integer or as used in text.
 p – Unit Conversion Factor.
 \bar{P} – The net Flux of the system.
 \bar{P}_b – Bulk Transfer Flux.
 P_m – Molecular Diffusion or Dispersion Flux.
 $Q(C, C_d)$ – Tracer Reaction Term.
 q – Injection Rate.
 R – Retardation Constant.
 $Re()$ – Real Part of the parameter contained within the bracket.
 r – Radial distance from the well.
 r_w – Well Radius.
 S_m – Saturation of the Mobile Fluid Phase.
 s – Seconds (unit of time).
 s – Laplace Parameter.
 t – Time Variable.
 U – Tricomi Kummer U Function.
 u – Velocity along the x axis without Linear Drift.
 u_* – Velocity along the x axis with Linear Drift Included.
 u_0 – Velocity at the wellbore.
 u_d – Linear Drift Velocity.
 \bar{u} – Advection Velocity of the System.
 V – Volume.
 v – Velocity along the y axis.
 w – Velocity along the z axis.

X – Time independent x component of Concentration C .
 x – x coordinate variable.
 x_w – x Component of Wellbore Radius.
 Y – Time dependent y component of Concentration C .
 y – y coordinate variable.
 y_w – y Component of Wellbore Radius.
 z – As defined in text.
 α – Constant related to the $X(x)$ solution.
 α_1 – Constant related to the $X(x)$ solution.
 β – Parameter related to the velocity along the x direction.
 Γ – Gamma Function.
 γ – Constant related to the $X(x)$ solution.
 ϵ – A constant that is the product of Velocity and radial distance from the well.
 θ – Angle.
 λ - Parameter related to the velocity along the y direction.
 μ – Constant of Separation.
 ν – Isotropy Factor.
 σ – Constant related to the $X(x)$ solution.
 φ – Porosity.
 ω^2 – Absolute Value of the Separation Constant.
 \in – Set Notation- A member of or belongs to:
 \notin – Set Notation- Not a member of or does not belong to:
 \mathbb{Z} – Set Notation- Set of Integers.
 \mathcal{L} - Laplace Operator.
 ∞ – Infinity.

APPENDIX A

MATLAB Code for The Tricomi Kummer U function Used in This Thesis

```
function [u] = kummerU(a,b,z)

% This function computes the Kummer Tricomi function U(a,b,z)
%When a>0, z>0, b=0,1,2.
% the formulas are given at
  "http://functions.wolfram.com/HypergeometricFunctions/HypergeometricU/06/01/03/
  02/0001/"
% however the summation cannot run to infinity
% Since the maximum number my computer can process is about 10^308,
%it implies that the maximum factorial that could be processed by
%my computer is 170!. Thus the summation in the equation only ran
%till the 100th term.
%the result was compared with the output at
%"http://functions.wolfram.com/webMathematica/FunctionEvaluation.jsp?name=Hyperge
  ometricU"
%and found to be completely identical.

%The purpose of this code was for my Msc Thesis, part of which requires
%the evaluation of U(a,0,z), U(a,1,z) and U(a,2,z) for positive values of %"a"
  and "z". The code was stable and adequate for every condition %encountered
  around in my Thesis. However its stability at "a>2" and "z>22" % was not
  considered.

s=10^-10; %Adjusting this value might change Result accuracy when "b=2".

S=100; %Adjusting this value changes the number of iteration, and might
  %have to be reduced for large values of "a" and "z" before any
  %result can be gotten at all, especially when the answer is "NaN".

%Some Special Cases First
if a==0
  u=1;
elseif and(a~=0, and(mod(a,1)~=0, and(b==0, z==0)))
  u=1/gamma(a+1);
elseif and(a~=0, and(mod(a,1)==0, and(a>0, and(b==0, z==0))))
  u=1/gamma(a+1);
elseif and(a~=0, and(mod(a,1)==0, and(a<0, and(b==0, z==0))))
  u=(1/gamma(a+1+s))+(1/gamma(a+1-s))/2;
elseif and(and(a>0, real(b)<1), and(z==0, and(a~=0, b~=0)))
  u=(gamma(1-b))/(gamma(a-b+1));
elseif and(real(b)>=1, and(z==0, and(a~=0, b~=0)))
  u='ComplexInfinity';
elseif and(a>0, and(z>0, or(b==0, or(b==1, b==2))))
  if or(b==0, b==1)
    A=0;
  for k=1:S
    B=((gamma(a+k)/gamma(a))/(prod(1:k)*prod(1:k-1)))*(2*(psi(k))- ...
    (psi(a+k)+(1/k))*z^k);
    A=B+A;
  end
  c=(1/gamma(a+1))*((1+(a*z)*(log(z))*hypergeom(a+1,2,z))-A);
```

```

if b==0, u=c;
elseif b==1,
C=0; a=a+1;
for k=1:S
D=(((gamma(a+k))/gamma(a))/((prod(1:k))*prod(1:k-1)))*(2*(psi(k))- ...
(psi(a+k)+(1/k))*z^k);
C=C+D;
end
d=(1/gamma(a+1))*((1+(a*z)*(log(z))*hypergeom(a+1,2,z))-C);
a=a-1;
u=((a/z)+1)*c-((a/z)*(a+1))*d;
end
elseif b==2
J=0;
for k=0:S
K=(((gamma(a+k))/gamma(a))*((psi(a+k))-psi(k+1))-psi(k+b))*z^k ...
/((prod(1:k))*gamma(k+b));
J=K+J;
end
E=0;
if mod(a,1)==0, e=(a-s);
for k=1:b-1
F=(((prod(1:k-1))*z^-k)/((gamma(-e+1+k))/gamma(-e+1))*gamma(b-k));
E=E+F;
end
u=((( (-1)^b)/gamma(e-b+1))*(((log(z))/gamma(b))*hypergeom(a,b,z))+J-E);
else
for k=1:b-1
L=(((prod(1:k-1))*z^-k)/((gamma(-a+1+k))/gamma(-a+1))*gamma(b-k));
E=E+L;
end
u=((( (-1)^b)/gamma(a-b+1))*(((log(z))/gamma(b))*hypergeom(a,b,z))+J-E);
end
end
else
u='NotConsidered';
end
end
% Designed by OFOMANA Emmanuel
% For my MSc thesis at African University of Science and Technology, Abuja.

```

MATLAB Code for Calculating the Constant of Separation ω^2

```

%Initialization
d=0.016; % The Drift Ratio
u=9.4876*10^(-4); % The Injection Velocity
D=1.389 *10^(-3); % The Diffusivity
rw=0.127; % The Wellbore Radius
nu=1; %Isotropy Factor
R=1.2; %Systems Retardation
k=9.506e-18; %Decay Constant
w_max=(d*u^2)/(4*D); % Maximum value of the Separation Constant (omega)
w=0.009*w_max:0.01*w_max:0.999*w_max; % Creates a single row matrix of 100
predicted omegas

```

```

O=0:1:360; % Creates a single row matrix of 361 angles (theta) from 0 to 360
al=zeros(361,100); a=al;
si=zeros(1,100);
be=zeros(1,361); xw=be; zw=be;
la=be; yw=be;
t=50*86400;
X=a; X1=a; X11=a; err=a;
syms s; Y=a; Y1=a; Y11=a;

%Calculation of subsidiary parameters
for j=1:100
    si(j)=sqrt(1-((4*((w(j))^2)*D)/(d*u^2))); %Calculates Sigma
    for i=1:361
        if j==1,
            be(i)=(rw*cosd(O(i)))^2); % Calculates Beta
            la(i)=(nu*rw*(sind(O(i)))^2); %Calculates Lamda
            xw(i)=rw*cosd(O(i)); % Calculates The x component of Radius
            yw(i)=rw*sind(O(i)); % Calculates The y component of Radius
            zw(i)=((u*be(i))/(D*d))+(u/D)*xw(i); %calculates "z"
        end
        if i~=1
            if i~=91
                if i~=271
                    if i~=361 % excludes four Angles
                        al(i,j)=(be(i)*(w(j)))/(u*d^2);
                        a(i,j)=al(i,j)/si(j); % Calculates "a" as a 361 rows by 100 columns
                    end % matrix using every combination of theta and
                end
            end
        end
    end
end

%omega
end
end
end

for i=1:361
    if and(i~=1, and(i~=91, and(i~=271, i~=361)))
        for j=1:100
            X(i,j)=(((gamma(a(i,j)))/gamma(al(i,j)))*(kummerU(a(i,j),0,(si(j)*zw(i)))))...
            *exp(-0.5*((1/si(j))-1)*zw(i)));

            X1(i,j)=(((u/D)/gamma(al(i,j)))*(exp(-0.5*((1/si(j))-1)*(zw(i)))))...
            *(((0.5*((1/si(j))-1))*(gamma(a(i,j)))*(kummerU(a(i,j),0,(si(j)*zw(i)))))...
            *(si(j)*gamma(1+a(i,j))*(kummerU(a(i,j),1,(si(j)*zw(i)))))));

            X11(i,j)=((((u/D)^2)/gamma(al(i,j)))*(exp(-0.5...
            *((1/si(j))-1)*(zw(i)))))...
            *(((0.5*((1/si(j))-1))^2)*(gamma(a(i,j)))...
            *(kummerU(a(i,j),0,(si(j)*zw(i)))))...
            +((1-si(j))*kummerU(a(i,j),1,(si(j)*zw(i))))...
            +((si(j)^2)*(gamma(2+a(i,j)))*(kummerU(a(i,j),2,(si(j)*zw(i))))));

            Y(i,j)=ilt(36,@(s) (w(j) + R*k + R*s)*...
            ((exp((u*yw(i))/(2*D))*(besseli(-1/3, (2*D*la(i))*(u^2/(4*D^2)) ...
            + (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))^(3/2))...
            /(3*(w(j) + R*k + R*s))) ...
            - besseli(1/3, 2*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i))*yw(i))...

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*(w(j) + R*k + R*s))^(3/2)...
/(3*w(j) + 3*R*k + 3*R*s))* (u^2/(4*D^2) + (yw(i)*(w(j) ...
+ R*k + R*s))/(D*la(i)))^(1/2))/3),t);

Y1(i,j)=ilt(32,@(s) ((u*exp((u*yw(i))/(2*D))...
*(besseli(-1/3, (2*D*la(i)*(u^2/(4*D^2) ...
+ (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))^(3/2))...
/(3*(w(j) + R*k + R*s))) ...
- besseli(1/3, 2*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2))...
/(3*w(j) + 3*R*k + 3*R*s))* (u^2/(4*D^2) + (yw(i)*(w(j) + R*k + R*s))...
/(D*la(i)))^(1/2))/(6*D) - (exp((u*yw(i))/(2*D))* (u^2/(4*D^2) ...
+ (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))^(1/2))* (u^2/(4*D^2) ...
+ (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))^(1/2))...
*besseli(-2/3, (2*D*la(i)*(u^2/(4*D^2) ...
+ (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))^(3/2))/...
(3*(w(j) + R*k + R*s))) - (u^2/(4*D^2) ...
+ (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))^(1/2))...
*besseli(2/3, (2*D*la(i)*(u^2/(4*D^2) ...
+ (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))^(3/2))/(3*(w(j) + R*k + R*s))) ...
+ (besseli(-1/3, (2*D*la(i)*(u^2/(4*D^2) + (yw(i)*(w(j) + R*k + R*s))...
/(D*la(i)))^(3/2))...
/(3*(w(j) + R*k + R*s)))*(w(j) + R*k + R*s))/(2*D*la(i)*(u^2/(4*D^2) ...
+ (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))) ...
- (besseli(1/3, (2*D*la(i)*(u^2/(4*D^2) ...
+ (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))^(3/2))/(3*(w(j) + R*k + R*s)))...
*(w(j) + R*k + R*s))...
/(2*D*la(i)*(u^2/(4*D^2) + (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))))/3 ...
+ (exp((u*yw(i))/(2*D))* (besseli(-1/3, (2*D*la(i)*(u^2/(4*D^2) ...
+ (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))^(3/2))/(3*(w(j) + R*k + R*s))) ...
- besseli(1/3, 2*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2))...
/(3*w(j) + 3*R*k + 3*R*s))* (w(j) + R*k + R*s))...
/(6*D*la(i)*(u^2/(4*D^2) ...
+ (yw(i)*(w(j) + R*k + R*s))/(D*la(i)))^(1/2))),t);

Y11(i,j)=ilt(28,@(s) (1/3*exp(1/2/D*u*yw(i))...
*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(1/2)...
*((1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(1/2)...
*besseli(-1/3, 2/3*D*la(i)*(1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s)) ...
- 1/D/la(i)/(1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))...
*besseli(2/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s))* (w(j) + R*k + R*s))...
*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(1/2) ...
- ((1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(1/2)...
*besseli(1/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2))...
/(w(j) + R*k + R*s)) - 1/D/la(i)/(1/4/D^2*u^2 + 1/D/la(i)...
*yw(i)*(w(j) + R*k + R*s))...
*besseli(-2/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2))...
/(w(j) + R*k + R*s))* (w(j) + R*k + R*s))*(1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(1/2) ...
+ 1/2/D^2/la(i)^2/(1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^2 ...

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*besseli(-1/3, 2/3*D*la(i)*(1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s))...
*(w(j) + R*k + R*s)^2 ...
- 1/2/D^2/la(i)^2/(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^2 ...
*besseli(1/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i) ...
*(w(j) + R*k + R*s))^(3/2)...
/(w(j) + R*k + R*s))*(w(j) + R*k + R*s)^2 + 1/2/D/la(i)*((1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(1/2) ...
*besseli(-2/3, 2/3*D*la(i)*(1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s)) ...
- 1/2/D/la(i)/(1/4/D^2*u^2 + 1/D/la(i)*yw(i))*yw(i)*(w(j) + R*k + R*s))...
*besseli(1/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) ...
+ R*k + R*s))^(3/2)...
/(w(j) + R*k + R*s))*(w(j) + R*k + R*s)/(1/4/D^2*u^2 + 1/D/la(i)...
*yw(i)*(w(j) + R*k + R*s))...
*(w(j) + R*k + R*s) - 1/2/D/la(i)*((1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(1/2)...
*besseli(2/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)...
*yw(i)*(w(j) + R*k + R*s))^(3/2)...
/(w(j) + R*k + R*s)) - 1/2/D/la(i)/(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))...
*besseli(-1/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)...
/(w(j) + R*k + R*s))*(w(j) + R*k + R*s)/(1/4/D^2*u^2 + 1/D/la(i)...
*yw(i)*(w(j) + R*k + R*s))...
*(w(j) + R*k + R*s) - 1/2/D/la(i)/(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(1/2)...
*besseli(-2/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)...
/(w(j) + R*k + R*s))*(w(j) + R*k + R*s) + 1/2/D/la(i)/(1/4/D^2 ...
*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(1/2)*besseli(2/3, 2/3*D*la(i)*(1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s))...
*(w(j) + R*k + R*s)) ...
- 1/3/D*u*exp(1/2/D*u*yw(i))*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(1/2)...
*((1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(1/2)...
*besseli(-2/3, 2/3*D*la(i)...
*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(3/2)/...
(w(j) + R*k + R*s)) ...
- (1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(1/2)...
*besseli(2/3, 2/3*D*la(i)...
*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(3/2)...
/(w(j) + R*k + R*s)) ...
+ 1/2/D/la(i)/(1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))...
*besseli(-1/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)...
/(w(j) + R*k + R*s))*(w(j) + R*k + R*s) - 1/2/D/la(i)/(1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))*besseli(1/3, 2/3*D*la(i)...
*(1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s))...
*(w(j) + R*k + R*s)) ...
+ 1/12/D^2*u^2*exp(1/2/D*u*yw(i))*(besseli(-1/3, 2/3*D*la(i)*(1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s)) ...
- besseli(1/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)...

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/(w(j) + R*k + R*s))* (1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(1/2) ...
- 1/12/D^2/la(i)^2*exp(1/2/D*u*yw(i))*(besseli(-1/3, 2/3*D*la(i)...
*(1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s)) ...
- besseli(1/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)...
/(w(j) + R*k + R*s))/ (1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)...
*(w(j) + R*k + R*s)^2 - 1/3/D/la(i)*exp(1/2/D*u*yw(i))...
/(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(1/2)*(w(j) + R*k + R*s)*((1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(1/2)...
*besseli(-2/3, 2/3*D*la(i)*(1/4/D^2*u^2 ...
+ 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s)) ...
- (1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))^(1/2)...
*besseli(2/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s)) ...
+ 1/2/D/la(i)/(1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))...
*besseli(-1/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s))*(w(j) + R*k + R*s) ...
- 1/2/D/la(i)/(1/4/D^2*u^2 + 1/D/la(i)*yw(i)*(w(j) + R*k + R*s))...
*besseli(1/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)...
/(w(j) + R*k + R*s))*(w(j) + R*k + R*s) + 1/6/D^2/la(i)...
*u*exp(1/2/D*u*yw(i))...
*(besseli(-1/3, 2/3*D*la(i)*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)...
/(w(j) + R*k + R*s)) - besseli(1/3, 2/3*D*la(i)...
*(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(3/2)/(w(j) + R*k + R*s))...
/(1/4/D^2*u^2 + 1/D/la(i)*yw(i)...
*(w(j) + R*k + R*s))^(1/2)*(w(j) + R*k + R*s)),t);

err(i,j) =abs((D*(d+cosd(O(i)))*X11(i,j))-(u*(d+cosd(O(i)))...
*X11(i,j))+(w(j)*X(i,j)))...
-((D*nu*sind(O(i))*Y11(i,j))-(u*nu*sind(O(i))*Y11(i,j))...
-(Y(i,j)));

end
end
for i=1
for j=1:100
err(1,j)=((err(360,j)+err(2,j))/2);
err(361,j)=err(1,j);
end
end
for i=91
for j=1:100
err(i,j)=((err(i-1,j)+err(i+1,j))/2);
end
end
for i=271
for j=1:100
err(i,j)=((err(i-1,j)+err(i+1,j))/2);
end
end
end

```

```

% Sorts the Calculated omega According to the least Error for Each Angle
ER=zeros(1,361); ERR=zeros(1,361); W=zeros(1,361);
for i=1:361
    ER=err(i,:);
    A=sortrows([ER(:),w(:)]);
    ERR(i)=A(1,1); %Error
    W(i)=A(1,2); %Associated Omega
end

```

MATLAB Code for the Concentration Distribution Ratio C_r

```

% INITIALIZATION
%Constants
D=1.389 *10^(-3); %Diffusivity
d=0.016; %Drift Ratio
k=9.506e-18; %Decay Constant
nu=1; %Isotropic and Homogeneous System.
rw=0.127; %Wellbore Radius
R=1.2; %Retardation Constant
u=9.4876*10^(-4); %Injection Velocity
w_max=8.4e-07; %Maximum value of Separation Constant
w_min=2.4e-07;
%Creating the Separation Constant Profile
f=0:1:360;
m=(w_max)-(w_max-w_min)*abs(cosd(f));

%Main Variables Preconditioning
n=-20:0.5:20;
x=[0 0.08 0.1 0.127 0.2 0.3 0.4 n(42:61)]; %x coordinates
y=[n(21:37) -1 -0.8 0 0.8 1 n(45:61)]; %y coordinates
y1=abs(y);
O=zeros(39,27); %Angles
t=30*86400; %Thirty Days.

%Model Variables Pre-conditioning
a=zeros(39,27); %a in U(a,0,z)
al=a; %alpha
be=a; %beta
si=a; %sigma
xw=a; %x component of the Wellbore Radius
yw=a; %y component of the Wellbore Radius
z=a; %z in U(a,0,z)
zw=a;
la=a; %lamda
syms s; %Laplace Parameter
syms ka; %Applies to the Y(y,s) Solution
syms p; %Applies to the Y(y,s) Solution
syms pw;
q=a; %Applies to the Y(y,s) Solution
qw=a;

% Output Preconditioning

```

```

C=zeros(39,27); %C(x,y,t) initially
Ci=C;      %The Concentration Ci at time t=10 days
Cii=C;

X=C;      %X(x) Component
Xw=C;     %X(x) Component at the Wellbore
Xr=C;

Yi=C;     %Y(x,y,t) at First time step
syms Y0;
syms Y;   %Y(y,s) Component
syms Yw;  %Y(y,s) Component at the Wellbore

%Main Calculations
for j=1:39
    for i=1:27
        %Calculating Angles (O(j,i)) from x and y coordinates
        if and(x(i)==0,y(j)==0), O(j,i)=0;
            elseif and(x(i)==0,y(j)>0), O(j,i)=90;
            elseif and(x(i)==0,y(j)<0), O(j,i)=270;
            elseif and(x(i)>0,y(j)==0), O(j,i)=0;
            elseif and(x(i)<0,y(j)==0), O(j,i)=180;
            elseif and(x(i)>0,y(j)>0), O(j,i)=atand(y(j)/x(i));
            elseif and(x(i)>0,y(j)<0), O(j,i)=360 + atand(y(j)/x(i));
            elseif and(x(i)<0,y(j)>0), O(j,i)=180 + atand(y(j)/x(i));
            elseif and(x(i)<0,y(j)<0), O(j,i)=180 + atand(y(j)/x(i));
        end
        if sqrt(((x(i))^2)+((y(j))^2))<=rw % Applying Wellbore Boundary
            Condition.
                X(j,i)=1; Xw(j,i)=1; C(j,i)=1;
                Xr(j,i)=1;
                Ci(j,i)=1; Yi(j,i)=1;
            end
            be(j,i)=rw*(cosd(O(j,i)))^2;
        end
    end
end
w=interp1(f,m,(1.+O)); %The Separation For Every Point
%Calculating Xr - Which shows the Effect of Linear Drift
for i=1:27
    for j=1:39
        if and(C(j,i)==0,x(i)>-be(j,i)/d)
            xw(j,i)=(rw*cosd(O(j,i)));
            si(j,i)=sqrt(1-(4*w(j,i)*D/(d*u^2)));
            z(j,i)=(be(j,i)*u/(d*D)+(u*(x(i))/D));
            zw(j,i)=(be(j,i)*u/(d*D)+(u*xw(j,i)/D));
            al(j,i)=(w(j,i)*be(j,i)/(u*d^2));
            a(j,i)=al(j,i)/si(j,i);
            X(j,i)=(gamma(a(j,i))/gamma(al(j,i)))...
                *(kummerU(a(j,i),0,(si(j,i)...
                    *z(j,i))))*exp(-0.5*(-1+(1/si(j,i)))*z(j,i)));
            Xw(j,i)=(gamma(a(j,i))/gamma(al(j,i)))...
                *(kummerU(a(j,i),0,(si(j,i)...
                    *zw(j,i))))*exp(-0.5*(-1+(1/si(j,i)))*zw(j,i)));
            Xr(j,i)=X(j,i)/Xw(j,i);
        end
    end
end

```

```

end
%Calculating Yi and Ci
for i=1:27
    for j=1:39
        if and(C(j,i)==0,and(x(i)>-be(j,i)/d,y(j)~=0))
            yw(j,i)=abs(rw*sind(O(j,i)));
            la(j,i)=nu*rw*(sind(O(j,i)))^2;
            ka(j,i)=(D*la(j,i))/((R*s)+(R*k)+w(j,i));
            p(j,i)=((0.25*(u/D)^2)+(y1(j)*(1/ka(j,i))));
            pw(j,i)=((0.25*(u/D)^2)+(yw(j,i)*(1/ka(j,i))));
            q(j,i)=(0.5*u*y1(j)/D);
            qw(j,i)=(0.5*u*yw(j,i)*(1/D));
            Y(j,i)=((1/3)*((p(j,i))^0.5)*(exp(q(j,i)))*(besseli(1/3,((2/3)*...
            *ka(j,i)*(p(j,i))^1.5)))-(besseli(1/3,((2/3)*ka(j,i)...
            *(p(j,i))^1.5))));
            Yw(j,i)=((1/3)*((pw(j,i))^0.5)*(exp(qw(j,i)))*(besseli(1/3,((2/3)*...
            *ka(j,i)*(pw(j,i))^1.5)))-besseli(1/3,((2/3)*ka(j,i)...
            *(pw(j,i))^1.5))));
            Y0(j,i)=(eval(Y(j,i))/eval(Yw(j,i)));
            Yi(j,i)=ilt(16,@(s) (1/s)*eval(Y0(j,i)),t);
            Ci(j,i)=Xr(j,i)*Yi(j,i);
        end
    end
end
%Getting an Estimate for the Singularity along the x axis for the Y Ratio
for i=1:27
    for j=1:39
        if and(C(j,i)==0,y(j)==0)
            Ci(j,i)=((Ci(j-1,i))+Ci(j+1,i))/2;
        end
        Cii(j,i)=Ci(j,i);
        pcolor(x,y,Cii), shading flat
        colorbar('eastoutside')
        colormap('cool')
        xlabel('x [m]', ylabel('y [m]')
        title('Concentration Distribution Ratio After Thirty Days');
        pause(0.01)
    end
end
ff=input('Press Enter To Continue','s');

close
surf(x(2:27),y,Ci(:,2:27))
colorbar('eastoutside')
colormap('cool')
xlabel('x [m]', ylabel('y [m]')
title('Concentration Distribution Ratio After Thirty Days');
fg=input('Press Enter To Continue','s');

close
contourf(x,y,Ci), shading flat
hold on
x2 = 6;
y2 = 0;
plot([0,x2],[0,y2]);
strx2=num2str(x2);

```

```
stry2=num2str(y2);
str2 = ['\rightarrow (' strx2 ' , ' stry2 ')'];
text(x2,y2,str2)
x3 = 6*cosd(85);
y3 = 6*sind(85);
plot([0,x3],[0,y3]);
strx3=num2str(x3);
stry3=num2str(y3);
str3 = ['\rightarrow (' strx3 ' , ' stry3 ')'];
text(x3,y3,str3)
x4 = 6*cosd(85);
y4 = -6*sind(85);
plot([0,x4],[0,y4]);
strx4=num2str(x4);
stry4=num2str(y4);
str4 = ['\rightarrow (' strx4 ' , ' stry4 ')'];
text(x4,y4,str4)
colorbar('eastoutside')
colormap('cool')
xlabel('x [m]'), ylabel('y [m]')
title('Concentration Distribution Ratio After Thirty Days')
```
