Spectral Theory of Compact Linear Operators and Applications

A Thesis
Submitted to the African University of Science and Technology
Abuja-Nigeria
in partial fulfilment of the requirements for

MASTER DEGREE IN PURE AND APPLIED MATHEMATICS
By

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December, 2011

Epigraph

"Problems can not be solved by the same level of thinking that created them"Albert Einstein

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Preface

This Project primarily falls into the field of Linear Functional Analysis and its Applications to Eigenvalue problems. It concerns the study of Compact Linear Operators (i.e., bounded linear operators which map the closed unit ball onto a relatively compact set) and their spectral analysis applicable to Fourier Analysis and to the solvability of Fredholm Integral Equations, linear elliptic Partial Differential Equations (PDEs) with the Dirichlet boundary condition, Sturm-Liouville problems, and of Optimization problems.

In this work we do :

- consider the complexification of real Banach spaces (allowing immediate extensions of many results on real Banach spaces and their operators to complex Banach spaces),

- present the Riesz-Schauder spectral theory of compact linear operators which shows that a compact linear operator has a countable set of eigenvalues having no limit points except possibly 0, and that each nonzero eigenvalue has finite multiplicity.

- prove the Fredolhm alternative that states that for every compact linear operator T of a Banach space X and every $\lambda \neq 0$, $T - \lambda I$ is either surjective or non injective, (this extends the theory of linear operators of finite dimensional spaces),

- develop the theory of compact linear self-adjoint operators of Hilbert spaces, which provides the spectral decomposition of compact linear self-adjoint operators and shows that the norm of any compact linear self-adjoint operator is the maximum of the absolute values of its eigenvalues,

- give examples of compact linear operators which include the Hilbert-Schmidt operators $T : L^2(]0,1[) \longrightarrow L^2(]0,1[)$ with kernel $k \in L^2(]0,1[\times]0,1[)$

and defined by $f \mapsto Tf$ with

$$(Tf)(x) = \int_0^1 k(x,y)f(y)dy \quad \text{for a. e. } x \in]0,1[\ ,$$

as well as the Green operator $K : L^2(\Omega) \longrightarrow L^2(\Omega)$ where Ω is a non-empty bounded open set of $\mathbb{R}^n$; $n \in \mathbb{N}$, and for each $g \in L^2(\Omega)$, $u = Kg$ is the unique weak solution of the Dirichlet problem

$$\begin{cases} -\Delta u &= g \quad \text{in} \quad \Omega \\ \quad u &= 0 \quad \text{on} \quad \partial\Omega \,, \end{cases}$$

- and use spectral decomposition to solve a Sturm-Liouville problem.

Acknowledgement

My profound gratitude goes to **Almighty God**, for He has done all things well, He has made everything beautiful in his own time, thank you Jesus.

I remain ever thankful to my supervisor **Doctor Guy Degla** for his patience, encouragement both in words and actions. I do appreciate, may God reward you abundantly.
I would like to use this opportunity to say a big word of thanks to my sponsors, **Petroleum Technology Development Fund (PTDF)** for making my dreams a reality.
My sincere gratitude goes to The Acting President of African University of Science and Technology, Abuja **Professor Charles Chidume Ejikeme** for his leadership role and words of wisdom.
My appreciation also goes to my **Professors Djitte Ngalla, Theodore Bulbacao, Leonard Thibault, Wole Soboyejo** who have especially impacted me. You have in no doubt made me better than I was, likewise to **Dr Bashir Ali and Mr Maaruf Minjibir** for their concern and contributions.

I am thankful to my parents **Elder Benjamin Audu Onoja and Deaconess Alice Onoja** and siblings **Sarah, Samson, Stephen, Emmanuel and Deborah** for their understanding, prayers and support. I love you all.

This acknowledgement would be incomplete without mentioning wonderful friends that have contributed immensely to my success in this program. This includes **Salako Rachidi, Adebayo Fashina, Edwin Soh Mukiyawa, Temmitope Adedokun, Patrice, Usman, Cyril, Olaniyi, Maduabuchi, Chinedu, Uche, Edith, Diadie, Bachar**
My Prayer is that God will bless you all in Jesus Name.

Dedication

This Project is dedicated to **Jesus Christ**

and

Elder B.A Onoja and Deaconess Alice Onoja

CHAPTER 1

FOUNDATION

1.1 Basic notions and results from Functional Analysis

The purpose of this section is to refresh our minds on some basic fundamentals required to facilitate a smooth understanding of the study of compact linear operators and its applications.

Definition 1.1.1. *A non-negative function* $||\cdot||$ *on a vector space* X *is called a norm on* X *if and only if*

i) $||x|| \geq 0$ *for every* $x \in X$ *(Positivity).*

ii) $||x|| = 0$ *if and only if* $x = 0$ *(Nondegeneracy).*

iii) $||\lambda(x)|| = |\lambda|||x||$ *for every* $x \in X$ *(Homogeneity).*

iv) $||x + y|| \leq ||x|| + ||y||$ *for every* $x, y \in X$ *(subadditivity).*

A vector space X with a norm $||\cdot||$ is denoted by $(X, ||\cdot||)$ and is called a normed linear space (or just a normed space).

A sequence $(x_n)_{n\in\mathbb{N}}$ of elements in a normed linear space X is called Cauchy if $\forall\epsilon > 0, \exists N \in \mathbb{N}$, such that for $m, n \in \mathbb{N}$, $||x_m - x_n|| \leq \epsilon$, $m, n \geq \mathbb{N}$

A Banach space is a normed linear space $(X, ||\cdot||)$ that is complete in the canonical metric defined by $\rho(x, y) = ||x - y||$ for $x, y \in X$ i.e every Cauchy sequence in X for the metric ρ converges to some point in X.

Remark. Every normed linear space has a completion [3]

Definition 1.1.2. *Let X and Y be $\mathbb{K}$-linear spaces ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). A map $T : X \longrightarrow Y$ is called linear if*

$$T(\lambda f + \mu g) \ = \lambda T(f) \ + \ \mu T(g) \quad for\ all\ f,\, g \in X, \ and\ all\ \lambda,\, \mu \in \mathbb{K}.$$

More generally, we can consider linear maps defined on a sub-space $D(T)$ of X and with values in Y. The Subspace $D(T) \subset X$ is called the domain of T. We denote the image (or Range) of T by $R(T)$ and is defined by

$$R(T) \ := \ \big\{ y \in Y \ : \ y = Tx \text{ for some } x \in D(T) \ \big\}.$$

We define the Kernel (or Null space) of T denoted by $N(T)$ to be the subspace of X defined by

$$N(T) \ := \ \{x \in D(T) : \ Tx = 0\}$$

T is said to be injective (or one to one) if $N(T) = \{0\}$.

T is said to be surjective (or onto) if $R(T) = Y$.

Definition 1.1.3. *A mapping $T : X \longrightarrow Y$ is called a continuous linear operator, if T is linear and is continuous at each point in $a \in X$, that is*

$$\lim_{x \to a} Tx \ = \ Ta \quad for\ all\ a \in X\,.$$

The space of continuous linear operators from a Banach space X into a Banach Space Y is denoted by $\mathcal{B}(X, Y)$.

Proposition 1.1.4. *[2]*
Let X, Y be two normed spaces and B_X be the closed unit ball of X. Let $T : X \longrightarrow Y$ be a linear map. Then the following are equivalent :

i) *T is a bounded linear operator, i.e there exists a constant $M > 0$ such that $||Tx||_Y \leq M||x||_X$ for all $x \in X$.*

ii) *T is continuous.*

iii) *T is continuous at the origin (in the sense that if $\{x_n\}$ is a sequence in X. such that $x_n \to 0$ as $n \to 0$, then $Tx_n \to 0$ in Y as $n \to \infty$).*

iv) *T is Lipschitz.*

v) *$T(B_X)$ is bounded (i.e there exists a constant $C > 0$ such that $||Tx|| \leq C$ for all $x \in B_X$).*

Furthermore, $\mathcal{B}(X,Y)$ becomes naturally endowed with the operator norm,

$$||T|| := \sup_{x\in X, x\neq 0} \frac{||Tx||_Y}{||x||_X} = \sup_{||x||_X \leq 1} ||Tx||_Y \,,$$

and if Y is a Banach space then so is $\mathcal{B}(X,Y)$.

We also recall that every linear operator of a finite dimensional space is bounded.

Proposition 1.1.5. *Every bounded linear map between normed linear space has a unique extension between their completions [4].*

Theorem 1.1.6. *Given a continuous complex function G on $[a,b]\times[a,b]$, let T be defined on $X = C[a,b]$ at each f by*

$$(Tf)(x) = \int_a^b G(x,y)f(y)dy \quad \text{for all} \quad x \in [a,b].$$

T is called an integral operator with the kernel function G. Then $T \in \mathcal{B}(X,Y)$ and

$$||T|| = \max_{a\leq x\leq b} \int_a^b |G(x,y)|dy$$

Proof.
Firstly, we show that T is well defined, i.e., for every $f \in C[a,b]$, $Tf \in C[a,b]$. Let $x_1, x_2 \in [a,b]$, then

$$\begin{aligned} |Tf(x_1) - Tf(x_2)| &\leq \int_a^b |G(x_1,t) - G(x_2,t)||f(t)|dt \\ &\leq \max_{t\in[a,b]} |G(x_1,t) - G(x_2,t)|||f||_\infty (b-a) \end{aligned}$$

Since G is continuous on the compact set $C([a,b]\times[a,b])$ by assumption, it follows that G is uniformly continuous and so we deduce from the above inequality that Tf is uniformly continuous. Thus $Tf \in C[a,b]$.

The linearity of T follows from the linearity of the integral.

Now we investigate the boundedness of T. We have,

$$|(Tf)(x)| \leq \int_a^b |G(x,y)|dy \tag{1.1.1}$$

and so,

$$||Tf||_\infty \;=\; \max_{a\leq x\leq b} |(Tf)(x)| \;\leq\; \max_{a\leq x\leq b} \int_a^b |G(x,y)|dy||f||_\infty$$

showing that T is bounded and

$$\|T\| \;\leq\; \max_{a\leq x\leq b} \int_a^b |G(x,y)|dy\,.$$

Hence

$$\|T\| \;\le\; M, \quad \text{where} \quad M := \int_a^b |G(x,y)|dy.$$

Now we define,

$$S(x) := \int_a^b |G(x,y)|dy.$$

S is continuous on $[a,b]$ and therefore it attains its maximum at some $x_0 \in [a,b]$.

$$\text{Let} \quad g(y) \;:=\; \begin{cases} \frac{|G(x_0,y)|}{G(x_0,y)} & \text{if } \; G(x_0,y) \neq 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

Then the function g is bounded and Lebesgue measurable and so belongs to $L_1([a,b])$. It follows from the fact that $C[a,b]$ is dense in $L_1([a,b])$ that there exists a sequence $(g_n)_n$ of elements of $C[a,b]$ such that $\|g_n\| = \max\limits_{y\in[0,1]} |g_n(y)| \le 1$ and (g_n) converges in $L_1([a,b])$ to g. Hence we have ,

$$\|T\| \;=\; \sup_{||f||\le 1} \|Tf\| \;\ge\; \|Tg_n\| \;=\; \max_{a\le x\le b} |(Tg_n)x| \;\ge\; (Tg_n)(x_0)$$

$$\Rightarrow \|T\| \;\ge\; (Tg_n)(x_0) \;=\; \int_a^b K(x_0,y)g_n(y)dy \longrightarrow \int_a^b K(x_0,y)g(y)dy$$

$$\text{therefore,} \quad \|T\| \;\ge\; \int_a^b K(x_0,y)g(y)dy \;=\; \int_a^b |K(x_0,y)|dy \;=\; M$$

It follows that $||T|| \le M$. Hence,

$$\|T\| \;=\; M$$

□

Corollary 1.1.7. *Given a continuous complex valued function G on $[0,1]\times[0,1]$, let T be defined on $X = C[0,1]$ by*

$$(Tf)(x) = \int_0^1 G(x,y)f(y)dy\,, \quad \forall f \in C[a,b].$$

Then T is a bounded linear map with

$$||T|| = \max_{0\le x\le 1} \int_0^1 |G(x,y)|dy\,.$$

Definition 1.1.8. *A map T defined from a Banach space X into a Banach space Y is called closed if its graph*

$$G(T) \;=\; \{(x,y) :\, x \in D(T)\}$$

is closed in $X \times Y$. In other words, T is closed if whenever $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$, we have $x \in D(T)$ and $Tx = y$

Remark. Every $T \in \mathcal{B}(X, Y)$ is closed

Theorem 1.1.9. ***Closed Graph Theorem***
A closed linear map which maps a Banach space into a Banach space is continuous

Definition 1.1.10. *A map $T \in \mathcal{B}(X, Y)$ is invertible if there is a bounded linear map $T^{-1} \in \mathcal{B}(Y, X)$ such that $T^{-1}T = I_X$ (the identity of X) and $TT^{-1} = I_Y$ (the identity operator of Y)*

Corollary 1.1.11. *Every continuous bijection between Banach spaces has a continuous inverse.*

Proof.
Let $T \in \mathcal{B}(X, Y)$. Then $G(T) = \{(x, Tx) : x \in X\}$ is closed in $X \times Y$,
$G(T^{-1}) = \{(Tx, x) : x \in X\}$ is closed in $Y \times X$
T^{-1} is a closed linear operator mapping Y into X.Therefore T^{-1} is bounded by the closed graph theorem. □

Definition 1.1.12. *Let A be a subset of a Banach space X. A is said to be precompact if any sequence of A has a Cauchy subsequence. A is totally bounded if for every $\epsilon > 0$, there exists a finite cover for A of open balls of same radius ϵ*

Definition 1.1.13. *A is said to be compact if any sequence of A has a subsequence that converges to some point of A.*

Remark. A set is said to be precompact in a Banach X if and only if its closure is compact in X.

Proposition 1.1.14. *Let A be a subset of a Banach space X. A is said to be precompact if and only if it is totally bounded.*

Proof.

Assume that A is a precompact subset of X, we show that A is totally bounded.
Let $\epsilon > 0$ and $x \in \bar{A}$

$$\begin{aligned} x \in \bar{A} &\Rightarrow B(x, \epsilon) \bigcap A \neq \emptyset \\ &\Rightarrow \exists\, a_x \in A : a_x \in B(x, \epsilon) \\ &\Rightarrow \exists\, a_x \in A : d(a_x, x) < \epsilon \\ &\Rightarrow \exists\, a_x \in A : x \in B(a_x, \epsilon) \subset \bigcup_{a \in A} B(a, \epsilon) \end{aligned}$$

since x was arbitrarily chosen, we have that $\bar{A} \subset \bigcup\limits_{a \in A} B(a, \epsilon)$, using the compactness of $\bar{A}, \exists\, a_1, a_2 ... a_n \in A$ such that $\bar{A} \subset \bigcup\limits_{i=1}^{n} B(a_i, \epsilon) \Rightarrow A \subset \bigcup\limits_{i=1}^{n} B(a_i, \epsilon)$

since $A \subset \bar{A}$.Hence A is totally bounded.

Conversely, suppose A is totally bounded, we show that A is precompact (i.e $\bar{A}$ is compact).
Let $\{a_n\}_n \subset \bar{A}$, we show that $\{a_n\}_n$ has a Cauchy subsequence
Case I. $\{a_n, n \geq 1\}$ is finite
Then there exists $n_1, .., n_p$ elements of $\mathbb{N}$ such that

$$\{a_n : n \geq 1\} = \{a_{n_1}, .., a_{n_p}\}.$$

Define for each $i \in \{1, ..p\}$, $E_i := \{n \in \mathbb{N} : a_n = a_{n_i}\}$.
So we have $\mathbb{N} = \bigcup_{i=1}^{p} E_i$.
Since $\mathbb{N}$ is infinite, then one of the sets E_i is infinite. Choose $i_0 \in \{1., , .p\}$ such that E_{i_0} is infinite.
Since $E_{i_0} \subset \mathbb{N}$, then it has a minimum element. Let $m_1 := \min E_{i_0}$
For $k \geq 1$, $m_{k+1} := min\{E_{i_0} \setminus \{m_1, m_2...m_k\}\}$
Clearly, $m_k < m_{k+1}, \forall k \in \mathbb{N}$, also $a_{m_k} = a_{ni_0}$ since $m_k \in E_{i_0}$
$\{a_{m_k}\}_{k\geq 1}$ is a constant subsequence of $\{a_n\}$ which is convergent.

Case II. $\{a_n, n \geq 1\}$ is infinite.
For $\epsilon = \frac{1}{2^2} > 0$, $\exists\, x_1, x_2...x_m$ such that $A \subset \bigcup_{i=1}^{m} B(x_i, \frac{1}{2^2})$ since A is totally bounded.
This implies that $A \subset \bigcup_{i=1}^{m} \bar{B}(x_i, \frac{1}{2^2})$ since $B(x_i, \frac{1}{2^2}) \subset \bar{B}(x_i, \frac{1}{2^2})$.
So $\bar{A} \subset \bigcup_{i=1}^{m} \bar{B}(x_i, \frac{1}{2^2})$ since $\bigcup_{i=1}^{m} \bar{B}(x_i, \frac{1}{2^2})$ is closed.
It follows that $\bar{A} \subset \bigcup_{i=1}^{m} B(x_i, \frac{1}{2})$ which implies that
$\{a_n\}_n \subset \bigcup_{i=1}^{m} B(x_i, \frac{1}{2})$, since $\{a_n\}_n \subset \bar{A}$
Therefore, $\exists\, i_0 \in \{1, 2...m\}$ such that $B(x_{i_o}, \frac{1}{2})$ contains infinitely many terms of the sequence.

$$I_1 := \{n \in \mathbb{N} : a_n \in B(x_{i_o}, \frac{1}{2})\}.$$

I_1 is infinite and for any $\{a_n\}_{n \in I_1}$, $d(a_n, a_m) < 1 \quad \forall m, n \in I_1$.

For $\frac{1}{2^3} > 0, \exists x_1, x_2...x_r :\ A \subset \bigcup_{i=1}^{r} B(x_i, \frac{1}{2^2})$ so, $\exists i_1 \in \{1, 2...r\}$ such that $B(x_{i_1}, \frac{1}{2^2})$ contains infinitely many terms of the subsequence $\{a_n\}_{n \in I_1}$.

$$\text{Now define } I_2 := \left\{ n \in I_1 :\ a_n \in B(x_{i_1}, \frac{1}{2^2}) \right\}.$$

$I_2 \subset I_1$ and for $n, m \in I_2$,

$$d(a_n, a_m) \leq d(a_n, a_{i_1}) + d(a_m, a_{i_1}) \leq \frac{1}{2^2} + \frac{1}{2^2} = \frac{1}{2}.$$

Iteratively, for any I_k infinite we can get I_{k+1} infinite with $I_{k+1} \subset I_k$ and $\forall m, n \in I_{k+1}$

$$d(a_n, a_m) < \frac{1}{2^k}$$

Given n_k choose $n_{k+1} \in I_{k+1}$ such that $n_{k+1} > n_k$ (this is possible because I_{k+1} is infinite). Now for $j > k$ the index n_j belongs toI_k (because$I_1 \supset I_2 \supset I_3 \supset$ ...is a nested sequence of sets)
Now for $k < j$

$$\begin{aligned} d(a_{n_k}, a_{n_j}) &\leq d(a_{n_k}, a_{n_{k+1}}) + \ldots + d(a_{n_{j-1}}, a_{n_j}) \\ &\leq \tfrac{1}{2^k} + \tfrac{1}{2^{k+1}} + \ldots + \tfrac{1}{2^{j-1}} \\ &= \tfrac{1}{2^k}(1 + \tfrac{1}{2} + \ldots + \tfrac{1}{2^{j-k-1}}) \\ &\leq \tfrac{1}{2^{k-1}} \longrightarrow 0, \quad \text{as k} \longrightarrow \infty \end{aligned}$$

$\{a_{n_k}\}_{k\geq 1}$ is a Cauchy subsequence of $\{a_n\}$.Its convergence is guaranteed by the completeness X. Hence A is precompact

□

Theorem 1.1.15. ***(Arzela-Ascoli)****[6]*
A subset $\mathcal{A}$ of the space of continuous functions $C(K)$, where K is a non-empty compact subset of $\mathbb{R}^{\mathbb{N}}$, is relatively compact if and only if the two following conditions are satisfied

i) *$\mathcal{A}$ is uniformly bounded, i.e there exists $M > 0$ such that $\forall x \in X$,*
$|f(x)| \leq M, \forall f \in K$.

ii) *$\mathcal{A}$ is equicontinuous i.e $\forall \epsilon > 0$, $\exists \delta > 0$:*

$$\forall x, y \in X, \|x - y\| \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon, \ \forall f \in K$$

.

Definition 1.1.16. *An inner product (or scalar product) on a vector space X is a scalar valued function, $\langle \ , \ \rangle$ on $X \times X$ such that*

i) *for each $y \in X$ fixed, the functional $x \mapsto \langle x, y\rangle$ is linear..*

ii) *$\overline{\langle x, y\rangle} = \langle y, x\rangle$, where the bar the complex conjugation.*

iii) *$\langle x, x\rangle \geq 0, \quad \forall x \in X$.*

iv) *$\forall x \in X$, $\langle x, x\rangle = 0$ if and only if $x = 0$.*

The pair $(X, \langle \ , \ \rangle)$ is called Pre-hilbertian (or inner product) space.

The function $\|\cdot\|_X = \sqrt{\langle x, x\rangle}$ defines a canonical norm on X.

Definition 1.1.17. *A Pre-hilbertian space $(H, \langle \cdot\cdot \rangle)$ is called a Hilbert space if it is complete when equipped with the corresponding canonical norm.*

Definition 1.1.18. ***Cauchy-Schwarz Inequality and parallelogram law***
Let $\langle x, y\rangle$ be an inner product on a vector space X.
Then

$$|\langle x, y\rangle| \leq \|x\|\|y\| \quad for\ all\ x, y \in X. \tag{1.1.2}$$

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad for \quad all \quad x, y \in X \tag{1.1.3}$$

Proposition 1.1.19. ***The polarization identity.***
Let X be an inner product space. Then for arbitrary $x, y \in X$,

$$\langle x, y\rangle = \frac{1}{4}\{||x+y||^2 - ||x-y||^2 + i||x+iy||^2 - i||x-iy||^2\} \tag{1.1.4}$$

where $i^2 = -1$.

Theorem 1.1.20. ***Jordan-Von Neumann.***
The norm of a normed linear space X is given by an inner product if and only if this norm satisfies the parallelogram law, i.e, if and only if,

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad \forall\, x, y \in X.$$

Definition 1.1.21. *Let H be a Hilbert space and $x, y \in H$. We say that x is orthogonal to y denoted by $x \perp y$ if $\langle x, y\rangle = 0$.*
For $M \subset H$, we say that x is orthogonal to M and write $x \perp M$, if x is orthogonal to every vector y in M.
The subset of vectors of H ortogonal to M is denoted by

$$M^{\perp} \ = \ \{x \in H : \ x \perp M\}$$

and is called the orthogonal complement of M in H.

Proposition 1.1.22. *[1] Let M and N be arbitrary subspaces of a Hilbert space H.Then the following holds*

i) *$M^{\perp}$ is a closed subspace of H*

ii) *$M \subset M^{\perp\perp}$,*

iii) *if $M \subset N$ then $N^{\perp} \subset M^{\perp}$*

iv) *$(M^{\perp})^{\perp} = \overline{M}$.*

Definition 1.1.23. *Given a closed subspace M of H, an operator P defined on H is called the orthogonal projection onto M if*

$$P(m+n) = m, \text{ for all } m \in M \text{ and } n \in M^{\perp}$$

Theorem 1.1.24. ***The projection Theorem***
Let H be a Hilbert space and M be a closed subspace of H. For an arbitrary given vector $x \in H$, there exists a unique vector $m^* \in M$ such that

$$||x - m^*|| \ \leq \ ||x - m|| \quad for\ all\, m \in M.$$

Furthermore, $z \in M$ is the unique vector m^* if and only if

$$(x - z) \perp M.$$

Corollary 1.1.25. ***Direct Sum Decomposition***
Let M be a closed subspace of a Hilbert Space, H. Then $H = M \oplus M^{\perp}$.

Definition 1.1.26. *An orthonormal system is a family $\{\varphi_i\}_{i\in\mathbb{I}}$ of elements of H such that $\langle \varphi_i, \varphi_j \rangle = \delta_{i\,j}$, where, $\delta_{i\,j}$ is the kronecker delta defined by*

$$\delta_{i\,j} \ = \ \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Example 1.1.27.

$$\left\{e^{i2\pi nx} \quad ; n \in \mathcal{Z}\right\} \quad is \quad an \quad othonormal \quad system \quad for \quad L^2([0,1])$$

It is easy to see that

$$\langle e_n, e_m \rangle = \int_{[0,1]} e^{i2\pi nx} e^{-i2\pi mx} dx$$

$$= \int_{[0,1]} e^{i2\pi(n-m)x} dx = \begin{cases} 1 & if \quad n = m \\ 0 & if \quad n \neq m. \end{cases}$$

Definition 1.1.28. *[1] A Hilbert space H is separable if H contains a countable dense subset. Equivalently, a Hilbert space H is said to be separable if there exists a sequence of vectors $v_1, v_2, ..., v_k, ...$ which span a dense subspace of H.*

Theorem 1.1.29.
A Hilbert space admits a countable orthonormal basis if and only if it is separable.

Theorem 1.1.30. ***Riesz representation theorem***
Let f be a continuous linear form on Hilbert space i.e $f \in H^*$. Then there exist a unique $u_f \in H$ such that $\langle f, v \rangle = \langle v, u_f \rangle$ for all $v \in H$.
Furthermore, we have $\|f\|_{H^*} = \|u_f\|_H$

Definition 1.1.31. *Let X and Y be Banach spaces and $T \in \mathcal{B}(X,Y)$, define the dual (also called adjoint) operator as a map $T^* : Y^* \longrightarrow X^*$ defined by*

$$T^* f \;=\; f \circ T$$

that is,

$$(T^* f)(x) \;=\; f(T(x)) \quad for\ all\ x \in X.$$

T^* is called the (topological) dual or adjoint operator of T.

Remark. $T^* \in B(Y^*, X^*)$

Definition 1.1.32. ***Adjoint operators on Hilbert spaces***
Let $T \in \mathcal{B}(H_1, H_2)$, the adjoint of T is the unique map $T^* :\ H_2 \longrightarrow H_1$ such that

$$\langle Tx, y\rangle \;=\; \langle x, T^* y\rangle \qquad for\ all\, x \in H_1\ and\ all\ y \in H_2$$

Example 1.1.33. *Let α be a bounded complex valued Lebesque measurable function on $[a,b]$. Let*

$$T :\ L_2([a,b]) \longrightarrow L_2([a,b])$$

be the bounded linear operator defined by $T(f) \;=\; \alpha \cdot f$, that is,

$$(Tf)(t) \;=\; \alpha(t) \cdot f(t) \quad for\ a.e.\ t \in [a,b].$$

For all $f, g \in L_2([a,b])$ we have

$$\langle Tf, g\rangle \;=\; \int_a^b (Tf)(t)\overline{g(t)}dt \;=\; \int_a^b \alpha(t) f(t)\overline{g(t)}dt \;=\; \langle f, \bar{\alpha} g\rangle.$$

$$Thus \quad T^*(g) \;=\; \overline{\alpha} \cdot g\,.$$

Theorem 1.1.34. *Let $T \,:\, L_2([a,b]) \longrightarrow L_2([a,b])$ be the bounded operator defined by*

$$(Tf)(t) = \int_a^b G(t,s) f(s) ds$$

where G is in $L_2([a,b] \times [a,b])$.
For all $g \in L_2([a,b])$,

$$(T^* g)(t) \;=\; \int_a^b \overline{G(s,t)} g(s) ds\,.$$

Proof.

$$
\begin{aligned}
\langle Tf, g\rangle &= \int_a^b \Big(\int_a^b G(t,s)f(s)ds\Big)\overline{g(t)}dt \\
&= \int_a^b f(s)\overline{\Big(\int_a^b \overline{G(t,s)}g(t)dt\Big)}ds \quad \text{by Fubini's thorem} \\
&= \langle f, g^*\rangle
\end{aligned}
$$

where

$$
g^*(s) = \int_a^b \overline{G(t,s)}g(t)dt
$$

□

1.2 Complexification of real Banach spaces

Many of the classical Banach functions spaces exist in real or complex-valued versions. Examples are the $L_p(\mu)$-spaces and $C(K)$-spaces. Usually one is interested in knowing whether a theory carried for real Banach spaces also holds for complex Banach spaces (or vice-versa). An approach of solution is given by the Complexification theory of real Banach spaces. Complexification preserves norm and allows us to extend all basic notions on any arbitrary real Banach space to Complex Banach space.

Definition 1.2.1. *A complex vector space $E_{\mathbb{C}}$ is a complexification of a real vector space E if the two following conditions holds.*

a) *There is a one-to-one real linear map $j : E \longrightarrow E_{\mathbb{C}}$*

b) *complex-span $(j(E)) = E_{\mathbb{C}}$.*

There are, however various alternative concrete descriptions, some of which include Ordered pair, Tensor and Linear operator descriptions of complexification.

Hence T is well defined ,infact $T \in \mathcal{B}(\ell_2)$

Ordered pair description of a complexification.
If E is a real vector space, we can make $E \times E$ a vector space by defining

$$
\begin{aligned}
(x,y) + (u,v) &:= (x+u, y+v) && \forall x,y,u,v \in E \\
(\alpha + i\beta)(x,\, y) &:= (\alpha x - \beta y,\, \beta x + \alpha y) && \forall x,y \in E,\ \forall \alpha,\beta \in \mathbb{R}.
\end{aligned}
$$

Consider the map

$$\begin{array}{rccc} j: & E & \longrightarrow & E\times E \\ & x & \mapsto & (x,0). \end{array}$$

Clearly,

$$j(\lambda x+y) \;=\; \lambda j(x)+j(y) \quad \text{for any } x,y\in E \;\text{ and } \lambda\in\mathbb{R},$$

$$Ker(j) \;=\; \{x\in E : j(x)=(0,0)\} \;=\; \{0\},$$

and

$$\text{Complex} - \text{span}(j(E)) \;=\; E\times E.$$

The map j *satisfies the conditions a)* $\subset \overline{F(\overline{T(B)})} \Rightarrow \overline{F(T(B))} andb)above, andsothiscomplexvec$ $E\oplus iE.andalsosuppressrefrencetojbywrittingz = x+iy$ for the element $z = (x,y) = j(x)+ij(y).Itisnaturaltowritex{=}\Re\mathrm{e}\, z$ *and* $y = \Im\mathrm{m}\, z$.
For other two descriptions, we refer to[4].

Definition 1.2.2. *Let* E *be a real Banach space and* $E_{\mathbb{C}} := E\oplus iE$. $\big(E_{\mathbb{C}},\, ||\cdot||\big)$ *is called a complexification of* E *if* $\big(E_{\mathbb{C}},\, ||\cdot||\big)$ *is a complex Banach space,* $||\cdot|||_E$ *is the original norm of* E *(i.e* $||x+i0|| = ||x||, \forall x\in E$*) and*

$$||x+iy|| \;=\; ||x-iy|| \quad \forall x,y\in E.$$

Now we might ask the **Question***: Is there a norm on* $E_{\mathbb{C}}$ *which makes* $E_{\mathbb{C}}$ *a complex Banach space and induces the original norm on*E *?*
The **answer is affirmative** *and there are infinitely many ways to do so.[4].*

Proposition 1.2.3. *Let* $E_{\mathbb{C}}$ *be a complexification of the real space* E *endowed with a norm* $||\cdot||$ *such that* $(E, ||\cdot||)$ *is Banach. Then* $||\cdot||_T$ *as defined below defines a norm on* $E_{\mathbb{C}}$.

$$||x+iy||_T \;:=\; \sup_{0\le t\le 2\pi} ||(\cos t)x \;-\; (\sin t)y||$$

All other complexification norms $||\cdot||$ *on* $E_{\mathbb{C}}$ *are equivalent to* $||\cdot||_T$*. Indeed*

$$||x+iy||_T \;\le\; ||x+iy|| \;\le\; 2||x+iy||_T \quad \forall\, x,y\in E. \tag{1.2.1}$$

Definition 1.2.4. *Let* E *be a real Banach space. We say that a norm on the complexification* $E_{\mathbb{C}}$ *is reasonable if*

c) $||j(x)|| = ||x|| \quad \forall x\in E$

d) $||x+iy|| = ||x-iy|| \quad x,y\in E$

When $E_{\mathbb{C}}$ *is equipped with such a norm, we call it a reasonable complexification of* E

Proposition 1.2.5. *Let $E_{\mathbb{C}}$ be a reasonable complexification of the real Banach space E. for any $x, y \in E$ we have $||x||_E \leq ||x+iy||_{E_{\mathbb{C}}}$ and $||y||_E \leq ||x+iy||_{E_{\mathbb{C}}}$*

Proof. By property (c),

$$2||x||_E = ||(x+iy)+(x-iy)||_{E_{\mathbb{C}}} \leq ||x+iy||_{E_{\mathbb{C}}} + ||x-iy||_{E_{\mathbb{C}}}$$

An application of property d) gives $||x||_E \leq ||x+iy||_{E_{\mathbb{C}}}$
Similarly we have the other inequality.

□

Proposition 1.2.6. *Let $E_{\mathbb{C}}$ be a complexification of the real Banach space E For any $x, y \in E$ we have,*

$$\sup_{0\leq t\leq 2\pi} ||(\cos t)x - (\sin t)y||_E \ \leq \ ||x+iy||_{E_{\mathbb{C}}}$$

and

$$||x+iy||_{E_{\mathbb{C}}} \ \leq \ \inf_{0\leq t\leq 2\pi}(||(\cos t)x - (\sin t)y||_E + ||(\sin t)x + (\cos t)y||_E).$$

Proof. For each $0 \leq t \leq 2\pi$

$$||x+iy||_{E_{\mathbb{C}}} = ||e^{it}(x+iy)||_{E_{\mathbb{C}}} = ||((\cos t)x-(\sin t)y)+i((\sin t)x+(\cos t)y)||_{E_{\mathbb{C}}}.$$

Using proposition (1.2.5) on the left and the triangle inequality on the right, we have
$||(\cos t)x - (\sin t)y||_E \leq ||x+iy||_{E_{\mathbb{C}}}$ and $||x+iy||_{E_{\mathbb{C}}} \leq ||(\cos t)x - (\sin t)y||_E + ||(\sin t)x + (\cos t)y||_E$.
Hence, the result follows immediately. □

Let us check verify that $||\cdot||_T$ is a reasonable complexification norm.
i)For any $x \in E$, $||x||_T = ||x+i0||_T = \sup_{0\leq t\leq 2\pi} ||xcost - 0sint|| = ||x||$
ii) $||xcost - ysint|| = ||xcos(-t) + ysin(-t)||$ *and the function*
$t \mapsto xcost - ysint$ is periodic with period of 2π for all $x, y \in E$.Therefore,

$$\begin{aligned} ||x+iy||_T = & \sup_{0\leq t\leq 2\pi} ||xcost - ysint|| \\ = & \sup_{t\in R} ||xcos(-t) + ysin(-t)|| \\ = & \sup_{t\in R} ||xcos(t) + ysin(t)|| \\ = & \sup_{0\leq t\leq 2\pi} ||xcost + ysint|| \\ ||x+iy||_T = & ||x-iy||_T \end{aligned}$$

We have shown that property c) and d) are satisfied. Hence $||\cdot||_T$ is a reasonable complexification norm.
Let us also verify the inequality in (1.2.1)
From proposition (1.2.6)

$$\begin{aligned}
||x+iy||_T \leq ||x+iy|| \leq & \inf_{0\leq t\leq 2\pi}(||xcost - ysint||_E + ||xsint + ycost||_E) \\
\leq & \sup_{0\leq t\leq 2\pi}||xcost - ysint||_E + \sup_{0\leq t\leq 2\pi}||xsint + ycost||_E \\
= & 2||x+iy||_T \\
||x+iy||_T \leq ||x+iy|| \leq & 2||x+iy||_T
\end{aligned}$$

The norm $||\cdot||_T$ was first considered by A.Y Taylor [ad]. $(E_{\mathbb{C}}, ||\cdot||_T)$ is known as Taylor complexification of E.
There is a useful alternative description of $||x+iy||_T$:

$$\begin{aligned}
||x+iy||_T = & \sup_{0\leq t\leq 2\pi}||xcost - ysint|| \\
= & \sup_{0\leq t\leq 2\pi}\sup_{||f||_{E^*}\leq 1}|f(x)cost - f(y)sint| \\
= & \sup_{||f||_{E^*}\leq 1}\sqrt{f(x)^2 + f(y)^2} \quad , \qquad \forall x, y \in E
\end{aligned}$$

Another feature of the Taylor complexification, is that it is a general complexification whose definition is not tied to any specific characteristic of the real Banach space E which is being complexified. Moreover, this procedure allows us to extend continuous linear maps between real Banach space to complex linear maps between their complexifications without increasing the norm. If $L : E \longrightarrow F$ is a linear map between real vector spaces E and F, there is a unique complex-linear extension $\tilde{L} : E_{\mathbb{C}} \longrightarrow F_{\mathbb{C}}$ given by

$$\tilde{L}(x+iy) = L(x) + iL(y)$$

Proposition 1.2.7. *Let E and F be real Banach spaces. If $L \in \mathcal{L}(E, F)$, then $\tilde{L} \in \mathcal{L}\Big((E_{\mathbb{C}}, ||\cdot||_T); (F_{\mathcal{C}}, ||\cdot||_T)\Big)$ and $||\tilde{L}|| = ||L||$*

Proof. Since $\tilde{L}$ extends L, we have $||\tilde{L}|| \geq ||L||$.
On the other hand, if $x, y \in E$ then

$$\begin{aligned}
||\tilde{L}(x+iy)||_T = & ||L(x) + iL(y)||_T = \sup_{0\leq t\leq 2\pi}||L(x)cost - L(y)sint||_F \\
= & \sup_{0\leq t\leq 2\pi}||L(xcost - ysint)||_F \\
\leq & ||L|| \sup_{0\leq t\leq 2\pi}||xcost - ysint||_E \\
||\tilde{L}(x+iy)||_T \leq & ||\tilde{L}||||x+iy||_T \Longrightarrow ||\tilde{L}|| \leq ||L||
\end{aligned}$$

Hence,

$$||\tilde{L}|| = ||L||$$

□

Taylor's procedure is just one of infinitely many procedures with similar properties.

1.3 Some function spaces (L^p, Sobolev spaces)

Definition 1.3.1. *Let $1 \le p < \infty, \Omega$ be an open bounded subset of $\mathbb{R}^n$. We define*
$L^p(\Omega)$ as the set of measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that

$$\int_\Omega |f(x)|^p dx < +\infty$$

$L^\infty(\Omega)$ as the set of measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that $esup|f| < \infty$ where,

$$esup|f| = inf\{k > 0, |f(x)| \le k \qquad a.e \qquad x \in \Omega\}$$

For $f \in L^p(\Omega)$, we define,

$$||f||_p = \Big(\int_\Omega |f(x)|^p\Big)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

$$||f||_\infty = esup|f|, if \quad p = \infty.$$

Theorem 1.3.2. *The following properties holds for L^p space*

i) *L^p-space is Banach for $1 \le p \le \infty$*

ii) *L^p-space is Reflexive for $1 < p < \infty$*

iii) *L^p-space is Separable for $1 \le p < \infty$*

$\subset \overline{F(\overline{T(B)})} \Rightarrow \overline{F(T(B))} Itisalsoexpedienttorecallsomenotationsandbasicresultsfromdistribu$

Definition 1.3.3. *The Space $L^1_0(\Omega)$ is the space of all Lebesgue measurable functions in Ω having absolute value integrable on each compact subset of Ω*
A multi-index α is a vector $(\alpha_1, \alpha_2...\alpha_n) \in \mathbb{N}^n$.
The length of α is given by $|\alpha| = \alpha_1 + ... + \alpha_n$
We also define the generalized derivative

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \quad . \quad . \quad .\partial^{\alpha_n} x_n}$$

Definition 1.3.4.
A locally integrable function v i.e element of $L^1_0(\Omega)$ is called the $\alpha - th$ weak derivative of $u \in L^1_0(\Omega)$, if it satisfies

$$\int_\Omega u(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x)\phi(x) dx, \qquad \forall \phi \in \mathcal{D}(\Omega)$$

Where $D(\Omega)$ denotes the set of C^∞-functions on Ω with compact support in Ω

Let $x \in R^n$, we write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$, $x' = (x_1, x_2, ...x_{n-1})$. We consider the following notations[5].

$$\begin{aligned}
\mathbb{R}^n_+ =& \quad \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\} \\
\mathcal{B} =& \quad \{x = (x', x_n) \in \mathbb{R}^n : ||x'|| < 1, |x_n| < 1\} \\
\mathcal{B}_+ =& \quad \{x = (x', x_n) \in \mathcal{B} : x_n > 0\} \\
\mathcal{B}_0 =& \quad \{x = (x', x_n) \in \mathcal{B} : x_n = 0\}
\end{aligned}$$

Definition 1.3.5. *We say that an open subset $\Omega \subset \mathbb{R}^n$ is of class $C^m(\Omega)(m,\ integer)$ if for every $x \in \partial\Omega$, there exist an open neighbourhood U of x in $\mathbb{R}^n$ and a map $\Phi : \mathcal{B} \longrightarrow U$ such that,*

i) Φ *is a bijection*

ii) $\Phi \in C^m(\bar{\mathcal{B}}, U), \Phi^{-1} \in C^m(\bar{U}, \mathcal{B})$

iii) $\Phi(\mathcal{B}_+) = \Omega \cap U, \Phi(\mathcal{B}_0) = \partial\Omega \cap U$

Definition 1.3.6.
Let $1 \ \leq \ p \leq +\infty$, $m \in \mathbb{N}$,. The Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \quad | \quad D^\alpha u \in L^P(\Omega) \ for\ all\ |\alpha| \leq \ m\}$$

We shall be working with the case $p = 2$ .The Sobolev space $W^{m,p}(\Omega)$ are denoted by $H^m(\Omega)$. $H^1_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$
. Finally, we shall consider two important results which are very instrumental to the application of spectral theorem of compact self adjoint operators to elliptic partial differential equations.

Proposition 1.3.7. *Poincare Inequality*
Let $1 \leq p < \infty$ and Ω a bounded open subset of $\mathbb{R}^N$. Then there exist a constant $C(\Omega, p)$ such that

$$||u||_{L^p(\Omega)} \leq C||\nabla u||_{L^p(\Omega)} \qquad \forall u \in W^{1,p}_0(\Omega).$$

If Ω is connected and satisfies a C^1 boundary condition, then there exists a constant $C(\Omega, p)$ such that

$$||u - \bar{u}||_{L^{p(\Omega)}} \leq C||\nabla u||_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega)$$

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)dx$$

, is the mean value of u on Ω

Definition 1.3.8.
Let E and F be two normed vector spaces such that $E \subset F$.We say $Ec \hookrightarrow F$ is compact embeddings if any bounded subset of E is precompact in F ,or equivalently any bounded sequence of E has a subsequence that converges in F.

CHAPTER 2

Linear Compact Operators on Banach Spaces

Definition 2.0.9.
Let X and Y be arbitrary Banach Spaces. A linear operator, $T : X \longrightarrow Y$ is called compact if the image of the closed unit ball $B_X(0,1) = \{x \in X : \|x\|_X \leq 1\}$ by T is a relatively compact subset of Y. In other words, T is compact if $\overline{T(B_X)}$ is compact.
This definition is equivalent to each of the following properties.

i) *For each bounded $B \subset X$, the image $T(B)$ is relatively compact in Y.*

ii) *For every bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, $\{Tx_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in Y.*

We introduce the following notations

$$\mathcal{K}(X,Y) := \{T : X \longrightarrow Y \;\; | \; T \textit{ is linear and compact}\}$$

$$\mathcal{K}(X) = \mathcal{K}(X,X)$$

.

lemma 2.0.10.

$$\mathcal{K}(X,Y) \subset \mathcal{B}(X,Y).$$

Proof. Suppose $T \in \mathcal{K}(X,Y)$, we show that $T \in \mathcal{B}(X,Y)$ i.e for all $x \in X$ there exist $M > 0$ such that $||Tx|| \leq M||x||$
Let $x \in X$. if $x = 0$, it holds trivially.
Assume $x \neq 0$, $\frac{x}{||x||} \in B_X$
$\overline{T(B_X)}$ is compact since $T \in \mathcal{K}(X,Y)$,so $T(B_X)$ is bounded.
Therefore there exist $M > 0$ such that $T(\frac{x}{||x||}) \leq M$, which implies that

$$||Tx|| \leq M||x||$$

□

Remark. Recall that Riesz theorem characterizes the compactness of the closed unit ball of a Banach space X by the finiteness of the dimension of X.
Thus for an infinite dimensional Banach space X, we have $I \in \mathcal{B}(X) \setminus \mathcal{K}(X)$ and so the inclusion $\mathcal{K}(X) \subset \mathcal{B}(X)$ is strict, that is,

$$\mathcal{K}(X) \subsetneq \mathcal{B}(X) \quad \text{whenever } \dim X = +\infty.$$

2.1 Properties of compact linear maps

Theorem 2.1.1.
Let X, Y and Z be Banach spaces, and let $T : X \longrightarrow Y$, $S : Y \longrightarrow Z$ and $F : Z \longrightarrow X$ be bounded linear operators.

i) *If the $R(T)$ is finite dimensional (i.e $\dim R(T) < +\infty$), then T is compact.*

ii) *If T is compact, then $T \circ S$ and $F \circ T$ are compact.*

iii) *For every T_1, $T_2 \in \mathcal{K}(X,Y)$ and every scalar α and β, we have $\alpha T_1 + \beta T_2 \in \mathcal{K}(X,Y)$. That is $\mathcal{K}(X,Y)$ is a linear subspace of $\mathcal{B}(X,Y)$.*

Proof.
i) Suppose that $R(T)$ is finite dimensional and let B be any bounded subset of X. We show that $\overline{T(B)}$ is compact in Y.
$T(B)$ is a bounded subset of Y, since T is bounded. So $\overline{T(B)}$ is compact as a closed and bounded subset of a finite dimensional space. Hence T is compact.

ii) Let B be any bounded set in X. We need to show that $\overline{T(S(B))}$ and $\overline{F(T(B))}$ are compact.
$S(B)$ is bounded since S is bounded. Therefore $\overline{T(S(B))}$ is compact since T is compact. This shows that $T \circ S$ is compact.
Now $(F \circ T)(B) = F(T(B)) \subset F(\overline{T(B)})$, since $T(B) \subset \overline{T(B)}$.

$$(F \circ T)(B) \subset F(\overline{T(B)}).$$

And since T is compact, $\overline{T(B)}$ is compact, and so $F(\overline{T(B)})$ is compact as a continuous image of compact set. Therefore $\overline{F(T(B))} \subset F(\overline{T(B)})$ is compact as a closed subset of a compact set. Thus $F \circ T$ is compact.

iii) Clearly $\alpha T_1 + \beta T_2 \in \mathcal{B}(X,Y)$. Let B_X be the closed unit ball of X, we show that $\overline{(\alpha T_1 + \beta T_2)(B_X)}$ is compact in Y.

$$(\alpha T_1 + \beta T_2)(B_X) \subseteq \alpha T_1(B_X) + \beta T_2(B_X) \quad \text{by linearity of } T_1 \text{ and } T_2.$$

Thus $(\alpha T_1 + \beta T_2)(B_X) \subset \overline{\alpha T_1(B_X)} + \overline{\beta T_2(B_X)}$ since $\alpha T_1(B_X)$ and $\beta T_2(B_X)$ are subsets of $\overline{\alpha T_1(B_X)}$ and $\overline{\beta T_2(B_X)}$ respectively. $\overline{\alpha T_1(B_X)} + \overline{\beta T_2(B_X)}$ is

compact as the sum of compact sets since αT_1, $\beta T_2 \in \mathcal{K}(X,Y)$
Now

$$\overline{(\alpha T_1 + \beta T_2)(B_X)} \;\subset\; \overline{\overline{\alpha T_1(B_X)} + \overline{\beta T_2(B_X)}},$$

that is,

$$\overline{(\alpha T_1 + \beta T_2)(B_X)} \;\subset\; \overline{\alpha T_1(B_X)} + \overline{\beta T_2(B_X)},$$

and therefore $\overline{(\alpha T_1 + \beta T_2)(B_X)}$ is compact as a closed subset of a compact set. Hence

$$\alpha T_1 + \beta T_2 \in \mathcal{K}(X,Y).$$

□

Remark. The linear combination of compact operators is compact, according to iii).

Proposition 2.1.2. *Let $T_n : X \longrightarrow Y$ be a compact linear operator for each $n \geq 1$. Assume that $(T_n)_n$ converges to some T in $\mathcal{K}(X,Y)$. Then T is compact.*

Proof. Let (B_X) denote the Unit ball in X. To show that T is compact, it suffices to show that $T(B_X)$ is totally bounded from (1.1.14). Since

$$\lim_{n\to\infty} \|T_n - T\|_{\mathcal{B}(X,Y)} \;=\; 0\,,$$

$$\forall \epsilon > 0 \quad \exists\, N \in \mathbb{N}, \text{ such that } \quad \|T_N - T\|_{\mathcal{B}(X,Y)} < \epsilon/2 \tag{2.1.1}$$

. T_N being compact, $T_N(B_X)$ is totally bounded and so it can be covered by a finite number of balls of radius $\epsilon/2$. Thus for some y_1, ..., $y_m \in Y$, we have

$$T_N(B_X) \;\subset\; \bigcup_{i=1}^{m} B\left(y_i, \frac{\epsilon}{2}\right) \tag{2.1.2}$$

Let $x \in B_X$ then

$$\|T_N(x) - T(x)\| \;<\; \epsilon/2 \text{ from equation (2.1.1).}$$

$$\text{Also, } T_N(x) \;\in\; T_N(B_X) \;\subset\; \bigcup_{i=1}^{m} B\left(y_i, \frac{\epsilon}{2}\right)\;.$$

$$\text{Hence } \exists i_0 \in \{1, 2, ..., m\} \quad \text{such} \quad T_N(x) \in B\left(y_{i_0}, \frac{\epsilon}{2}\right) \tag{2.1.3}$$

and so $\|T_N(x) - y_{i_0}\| \leq \frac{\epsilon}{2}$ which implies that

$$\|T(x) - y_{i_0}\| \leqslant \|T(x) - T_N(X)\| + \|T_N(x) - y_{i_0}\| \leqslant \epsilon \tag{2.1.4}$$

$$T(x) \in B(y_{i_0}, \epsilon) \subset \bigcup_{i=1}^{m} B(y_i, \epsilon)$$

$$\Rightarrow \quad \forall\, x \in B_X, \quad T(B_X) \;\subset\; \bigcup_{i=1}^{m} B(y_i, \epsilon).$$

Hence T is compact. □

Theorem 2.1.3. *Let T be a linear and continuous operator from a reflexive Banach space X into a Banach space Y. Then T is compact if and only if it maps weakly convergent sequence in X to a strongly convergent sequence in Y, i.e,*

$$\big(x_n \rightharpoonup x \quad in \quad X\big) \quad \Longrightarrow \quad \big(Tx_n \to Tx \quad in \quad Y\big).$$

Proof.
Assume $T \in \mathcal{K}(X,Y)$ and let $(x_n) \subset X$ such that $x_n \rightharpoonup x$. We show that $Tx_n \to Tx$.
Suppose by the way of contradiction that $(Tx_n)_n$ does not converge strongly to Tx. Then $\exists\, \epsilon > 0$ and $\exists \{Tx_{n_k}\}$ subsequence of $\{Tx_n\}$ such that

$$\|Tx_{n_k} - Tx\| \geq \epsilon\,. \tag{2.1.5}$$

$\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, so $x_{n_k} \rightharpoonup x$. Thus $\{x_{n_k}\}$ is bounded. Since T is compact, there exists $\{x_{n_{k_m}}\}$, a subsequence of $\{x_{n_k}\}$, such that $Tx_{n_{k_m}} \to y$ in Y and so $Tx_{n_{k_m}} \rightharpoonup y$ (since strong convergence implies weak convergence). Moreover the weak convergence $x_{n_{k_m}} \rightharpoonup x$ implying that $Tx_{n_{k_m}} \rightharpoonup T(x)$, we conclude that $T(x) = y$. So $\{Tx_{n_{k_m}}\}$ converges to Tx and is a subsequence of $\{Tx_{n_k}\}$ which satisfies equation (2.1.5). Therefore

$$\epsilon \;\leq\; \|Tx_{n_{k_m}} - Tx\| \;\to\; 0, \quad \text{as } m \to \infty \tag{2.1.6}$$

which is a contradiction. Hence $Tx_n \to Tx$.

Conversely, let $\{x_n\}$ be a bounded sequence in X. Then there exists $\{x_{n_k}\}$ subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$ (since X is reflexive), this implies that $Tx_{n_k} \to Tx$ in Y (from hypothesis). Hence T is compact. □

Proposition 2.1.4. *Let $T \in \mathcal{K}(X,Y)$ be an injective compact linear operator. Then T^{-1} is not continuous unless X is finite dimensional.*

Proof.
First, we recall that for Banach spaces X,Y and Z, if $S \in \mathcal{K}(Y,Z)$, and $T \in \mathcal{B}(X,Y)$ then $ST \in \mathcal{K}(X,Z)$. Assume that $T^{-1} : R(T) \longrightarrow X$ is bounded. It follows immediately that $T^{-1}T = I : X \longrightarrow X$ is compact, which is possible only if X is finite dimensional (by Riesz theorem). □

2.2 Some examples of compact linear operators

Example 2.2.1. *Let $\{\lambda_n\}$ be a sequence of positive real number decreasing to zero. Define*

$$T : \ell_2(\mathbb{R}) \longrightarrow \ell_2(\mathbb{R})$$

by

$$T(x) \;=\; (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3 \ldots)\,.$$

Then T is a compact linear operator.
Indeed it is clear that T is well-defined and linear. Moreover we have

$$\|T(x)\|_{\ell_2} = \|(\lambda_1 x_1,\, \lambda_2 x_2,\, \lambda_3 x_3,\, ...)\|_{\ell_2}$$

which implies that

$$\|T(x)\|_{\ell_2}^2 \;=\; \sum_{i=1}^{+\infty} |\lambda_i|^2 |x_i|^2 \;\leq\; (\max_{i\in\mathbb{N}} |\lambda_i|)^2 ||x||^2 \qquad \forall\, x \in \ell_2$$

and so T is bounded with

$$\|T\| \;\leq\; |\lambda_1|\,.$$

To conclude that T is compact, it suffices to show that T is the limit, with respect to the operator norm, of a sequence of compact linear operators. For this end, define $T_n : \ell_2(\mathbb{R}) \longrightarrow \ell_2(\mathbb{R})$ by

$$T_n(x) \;=\; (\lambda_1 x_1, \lambda_2 x_2,\, \lambda_3 x_3,\, ...,\, \lambda_n x_n,\, 0,\, 0...)\ .$$

T_n is bounded and $\dim(T_n(\ell_2(\mathbb{R})) < \infty$*, so T_n is compact by theorem* (2.1.1).

claim. : $\|T_n - T\| \;\longrightarrow 0, \quad as \quad n \longrightarrow \infty$

Proof.

$$\|(T - T_n)(x)\| \;=\; \|(\lambda_{n+1} x_{n+1}, \lambda_{n+2} x_{n+2}, \lambda_{n+3} x_{n+3} ...)\|$$

$$\begin{aligned} \|(T - T_n)(x)\|^2 = \sum_{j=n+1}^{+\infty} |\lambda_j x_j|^2 = &\quad \sum_{j=n+1}^{+\infty} |\lambda_j|^2 |x_j|^2 \\ \leq &\quad \lambda_{n+1}^2 \sum_{j=n+1}^{+\infty} |x_j|^2 \quad \text{and so} \\ \leq &\quad \lambda_{n+1}^2 \sum_{j=1}^{+\infty} |x_j|^2 \\ = &\quad \lambda_{n+1}^2 \|x\|^2 \end{aligned}$$

$$\|T - T_n\| \;\leq\; \lambda_{n+1} \;\;\rightarrow\;\; 0, \quad as \quad n \rightarrow +\infty.$$

We conclude from theorem (2.1.1) that T is compact. □

Example 2.2.2. *Let $X = L_2[0,1]$ and $G \in L_2([0,1] \times [0,1])$. Define the operator*

$$T : L_2[0,1] \longrightarrow L_2[0,1] \quad by$$

$$(Tf)(x) = \int_0^1 G(x,t) f(t) dt \quad is \quad Compact.$$

Proof.
Firstly, we show that the map T is well defined i.e for any $f \in L_2[0,1]$, $Tf \in L_2[0,1]$. Let $f \in L_2[0,1]$. Clearly Tf is measurable and

$$\begin{aligned}
\left(\int_0^1 |(Tf)(x)|^2 dx\right)^{\frac{1}{2}} &= \left(\int_0^1 |\int_0^1 G(x,t)f(t)dt|^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^1 \left(\int_0^1 |G(x,t)f(t)dt|\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^1 \left(\int_0^1 |G(x,t)|^2 dt \int_0^1 |f(t)|^2 dt\right) dx\right)^{\frac{1}{2}} \\
&= \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^1 \int_0^1 |G(x,t)|^2 dt dx\right)^{\frac{1}{2}} \\
&= \|f\| \left(\int_0^1 \int_0^1 |G(x,t)|^2 dt dx\right)^{\frac{1}{2}} < \infty \quad \text{since,} \quad G \in L_2([0,1]\times[0,1]).
\end{aligned}$$

Thus

$$Tf \in L_2[0,1], \quad T \in \mathcal{B}(L_2[0,1]) \quad \textit{and} \quad \|T\| \leq \left(\int_0^1 \int_0^1 |G(x,t)|^2 dt dx\right)^{\frac{1}{2}}.$$

We show that T is compact.
First case: G is continuous, i.e., $G \in C([0,1]\times[0,1])$.
Let $f \in B_{L_2}$

$$\begin{aligned}
|(Tf)(x)| &\leq \int_0^1 |G(x,t)f(t)| dt \\
&\leq \left(\int_0^1 |G(x,t)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}} \\
&= \left(\int_0^1 |G(x,t)|^2 dt\right)^{\frac{1}{2}} \|f\| \\
&\leq \left(\int_0^1 |G(x,t)|^2 dt\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^1 M^2 dt\right)^{\frac{1}{2}}, \quad \textit{where} \quad M := \textit{Sup}\{|G(x,t)| : (x,t) \in [0,1]\times[0,1]\} < \infty
\end{aligned}$$

which implies that $|(Tf)(x)| \leq M$. Therefore $T(B_{L_2})$ is bounded (uniformly) in $L_2[0,1]$.
G is uniformly continuous on $[0,1]\times[0,1]$ by continuity of G on the compact set $[0,1]\times[0,1]$. Therefore, given any $\epsilon > 0$ $\exists\, \delta > 0$ such that for $x_1, x_2 \in [0,1]$ with $|x_1 - x_2| < \delta$, we have

$$\text{for every} \quad t \in [0,1], \quad |G(x_1,t) - G(x_2,t)| < \epsilon. \qquad (2.2.1)$$

$$\begin{aligned}
|(Tf)(x_1)-(Tf)(x_2)| \leq & \int_0^1 |G(x_1,t)-G(x_2,t)||f(t)|dt \\
\leq & \left(\int_0^1 |G(x_1,t)-G(x_2,t)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^1 |f(t)|^2 dt\right)^{\frac{1}{2}} \\
\leq & \left(\int_0^1 \epsilon^2 dt\right)^{\frac{1}{2}} \|f\| \\
\leq & \ \epsilon
\end{aligned}$$

$$\Rightarrow |(Tf)(x_1)-(Tf)(x_2)| \leq \epsilon, \quad \forall x_1, x_2 \in [0,1], f \in B_{L_2} \tag{2.2.2}$$

Therefore,$T(B_{L_2})$ is uniformly bounded by M and equicontinuous subset of $C[0,1]$, hence by Arzela-Ascoli (1.1.15), $T(B_{L_2})$ is relatively compact in $C[0,1]$. However the norm topology of $L_2([0,1])$ is weaker than the topology of $C[0,1]$ $\big($because, $\|f\|_{L_2[0,1]} \leq \|f\|_\infty\big)$, so $\overline{T(B_{L_2})}$is compact in $L_2[0,1]$,Hence T is compact.

Second case (general). G is square-integrable but not continuous. Then there exists a sequence (G_n) of continuous functions on $[0,1]\times[0,1]$ that converges to G in $L_2([0,1]\times[0,1])$, since $C([0,1]\times[0,1])$ is dense in $L_2([0,1]\times[0,1])$. Therefore,

$$\Big(\int_0^1 \int_0^1 (G(x,t)-G_n(x,t))^2 dt dx\Big)^{\frac{1}{2}} \longrightarrow 0, \quad n \longrightarrow \infty$$

For each $n \in \mathbb{N}$, define

$$T_n : L_2[0,1] \longrightarrow L_2[0,1]$$

$$(T_n f)(x) = \int_0^1 G_n(x,t) f(t) dt$$

T_n is compact for each n

$$\begin{aligned}
||T-T_n||_{L_2} = & \Big(\int_0^1 |(Tf-T_nf)(t)|^2 dx\Big)^{\frac{1}{2}} \\
\leq & \Big(\int_0^1 \int_0^1 |G(x,t)-G_n(x,t)|^2 dt dx\Big)^{\frac{1}{2}} \longrightarrow 0, \quad n \longrightarrow \infty
\end{aligned}$$

Hence T is compact from proposition (2.1.2) □

Example 2.2.3. *Now let $X = C[a,b]$ and $G \in C([a,b]\times[a,b])$ the operator*

$$T : C[a,b] \longrightarrow C[a,b] \quad by$$

$$(Tf)(x) = \int_a^b G(x,t) f(t) dt \quad is \quad also \quad Compact.$$

Proof. Clearly T is well-defined and linear. Moreover for any $f \in B_{C[a,b]}$,

$$\begin{aligned}
|(Tf)(x)| = & |\int_a^b G(x,t)f(t)dt| \\
\leq & \int_a^b |G(x,t)||f(t)|dt \\
\leq & \sup_{t\in[0,1]} |f(t)| \int_a^b |G(x,t)|dt \\
= & \|f\|_\infty \int_a^b |G(x,t)|dt \\
\leq & \max_{(x,t)\in[a,b]\times[a,b]} |G(x,t)| \int_a^b dt \\
\leq & \max_{(x,t)\in[a,b]\times[a,b]} |G(x,t)|(b-a)
\end{aligned}$$

$\Rightarrow \|Tf\|_\infty \leq \max_{(x,t)\in[a,b]\times[a,b]} |G(x,t)|(b-a) < \infty$

Therefore, $T(B_{C[a,b]})$ is uniformly bounded. By same approach as in example (2.2.2), $T(B_{C[a,b]})$ is an equicontinuous subset of $C[a,b]$, thus $\overline{T(B_{C[a,b]})}$ is compact in $C[a,b]$. We conclude by (1.1.15) that T is compact. □

Example 2.2.4. ***Compact Embedding theorems***
suppose that Ω *is* C^1*- bounded open subset of* $\mathbb{R}^n(\Omega)$. *Let* $k \geq 1$ *and* $1 \leq p \leq \infty$, *the following embeddings are compact.*

i) *if* $kp < n$, *then* $H^{k,p} \hookrightarrow\hookrightarrow L^q(\Omega)$, *for all* $1 \leq q < np/(n-kp)$,

ii) *if* $kp = n$, *then* $H^{k,p} \hookrightarrow\hookrightarrow L^q(\Omega)$, *for all* $q \in [1,\infty)$,

iii) *if* $kp > n$, *then* $H^{k,p} \hookrightarrow\hookrightarrow C(\bar{\Omega})$.

Theorem 2.2.5. *For* $T \in \mathcal{B}(H_1,H_2)$, *the following dual properties holds[1]*

i) $KerT = R(T^*)^\perp$

ii) $KerT^* = R(T)^\perp$

iii) $\overline{R(T)} = kerT^{*\perp}$

iv) $\overline{R(T^*)} = kerT^\perp$

Theorem 2.2.6. *Let* H_1 *and* H_2 *be Hilbert spaces.*
An operator $T \in \mathcal{B}(H_1,H_2)$ *is compact if and only if its adjoint* T^* *is compact.*

Proof.
Suppose $T \in \mathcal{K}(H_1,H_2)$
observe that, $T^* \in \mathcal{B}(H_2,H_1)$ so $TT^* \in \mathcal{K}(H_2)$ by (2.1.1).
Let $\{x_n\} \subset H_2$ such that $\|x_n\| = 1$. We have that $\{T^*x_n\}$ is bounded in H_1. Therefore there exists $\{T^*x_{n_k}\}$ subsequence of $\{T^*x_n\}$ such that $\{TT^*x_{n_k}\}$ converges in H_2.since $T \in \mathcal{K}(H_1,H_2)$

$$\begin{aligned}\|T^*x_{n_k} - T^*x_{n_l}\|^2 &= \langle T^*x_{n_k} - T^*x_{n_l}, T^*x_{n_k} - T^*x_{n_l}\rangle \\ &= \langle T^*x_{n_k} - T^*x_{n_l}, T^*(x_{n_k} - x_{n_l})\rangle \\ &= \langle TT^*(x_{n_k} - x_{n_l}) - x_{n_k} - x_{n_l}\rangle \\ &\leq \|TT^*(x_{n_k} - x_{n_l})\|\|x_{n_k} - x_{n_l}\| \\ &\leq 2\|TT^*(x_{n_k} - x_{n_l})\| \longrightarrow 0, k, l \longrightarrow \infty \quad \text{since} \quad \{TT^*x_{n_k}\} \quad converges.\end{aligned}$$

Consequently $\{T^*x_{n_k}\}$ is Cauchy in H_1 ,its convergence is ensured by the completeness of H_1. Hence, T is compact.
Conversely, Suppose T^* is compact. Then by the forward direction of this theorem, $T^{**} = T$ is compact. □

lemma 2.2.7. ***(Riesz)***
Let X be a Banach space and M be a closed proper subspace of X. Then $\forall \epsilon > 0$ there exist $x \in X$ with $\|x\| = 1$ such that $d(x, M) \geq 1 - \epsilon$.

Proof.
Let $\epsilon \in (0, 1)$.
choose $x \in X \setminus M$ then d:=$dist(x, M) > 0$,
since$M$is closed.
Now set $\epsilon^* = \frac{\epsilon d}{1-\epsilon} > 0$ and d=$\inf\limits_{y \in M} \|x - y\|$.

$$\text{So there exists } y \in M \text{ such that} \quad d \leq \|x - y\| < d + \epsilon^* = \frac{d}{1 - \epsilon} \tag{2.2.3}$$

Set $v = \frac{x-y}{\|x-y\|}$,thus $v \in X$, $\|v\| = 1$ and
Given $m \in M$,we have,

$$\begin{aligned}||v - m|| &= \left|\left|\frac{x-y}{||x-y||} - m\right|\right| \\ &= \left|\left|\frac{x-(y+||x-y||)m}{||x-y||}\right|\right| \\ ||v - m|| &= \frac{1}{||x-y||}\left|\left|x - \underbrace{(y + \|x - y\|m)}\right|\right| \\ dist(v, M) &= \inf_{m \in M} ||v - m|| \\ &= \frac{1}{||x-y||}\inf_{m \in M}\left|\left|x - \underbrace{(y + \|x - y\|m)}\right|\right| \\ dist(v, M) &= \frac{1}{||x-y||}dist(x, M) \geq 1 - \epsilon\end{aligned}$$

□

Theorem 2.2.8. ***Fredholm Alternatives***
Let X be a Banach space and $T \in \mathcal{K}(X)$. Then for any $\lambda \neq 0$,the following holds,

i) *$Ker(\lambda I - T)$ is finite dimensional*

ii) *$R(\lambda\ I - T)$ is closed*

iii) $Ker(\lambda I - T) = 0 \Longleftrightarrow R(\lambda I - T) = X$

Proof.
i) For each λ, define $N_\lambda = Ker(\lambda I - T)$.
For all $x \in N_\lambda$, $Tx = \lambda x$.
Let B_λ be the unit ball in N_λ
$\forall x \in B_\lambda$, $Tx = \lambda x \Rightarrow T(B_\lambda) = \lambda B_\lambda \Rightarrow \overline{T(B_\lambda)} = \bar{\lambda}\overline{B_N}$
Since T is compact, it follows that $\overline{B_N}$ is compact.
Hence $\dim(N_\lambda) < \infty$
ii) Let $y_n \in R(\lambda I - T)$, such that $y_n \to y$, we show that $y \in R(\lambda I - T)$.
$y_n \in R(\lambda I - T) \Rightarrow \exists x_n \in X : (\lambda I - T)x_n = y_n \Rightarrow$

$$\lambda x_n - Tx_n = y_n \tag{2.2.4}$$

$Ker(\lambda I - T)$ is finite dimensional, so there exists $v_n \in Ker(\lambda I - T)$ such that

$$dist(v_n, Ker(\lambda I - T)) = \|x_n - v_n\| \tag{2.2.5}$$

$v_n \in Ker(\lambda I - T) \Rightarrow \lambda v_n = Tv_n$ so (2.2.4) becomes

$$\lambda x_n - \lambda v_n + Tv_n - Tx_n = Tv_n = y_n$$

$$\Rightarrow \lambda(x_n - v_n) - T(x_n - v_n) = y_n. \tag{2.2.6}$$

Assume $\{\|x_n - v_n\|\}_n$ is bounded. Since T is compact, $T(x_{n_k} - v_{n_k}) \to l \in X$.
From (2.2.6), we have that $\lambda(x_{n_k} - v_{n_k}) \to y + l \Rightarrow x_{n_k} - v_{n_k} \to \frac{1}{\lambda}(y + l)$
By continuity of T, from (2.2.6) we have, $(I - \frac{1}{\lambda}T)(y + l) = y$ and so $y \in R(\lambda I - T)$.
Now, suppose that $\{\|x_n - v_n\|\}_n$ is not bounded. From (2.2.6), we have

$$\lambda\frac{(x_n - v_n)}{\|x_n - v_n\|} - T\frac{(x_n - v_n)}{\|x_n - v_n\|} = \frac{y_n}{\|x_n - y_n\|} \tag{2.2.7}$$

$$u_n = \frac{x_n - v_n}{\|x_n - v_n\|}$$

There exists a subsequence of $\{\|x_{n_k} - v_{n_k}\|\}_k$ of $\{\|x_n - v_n\|\}_n$ such that

$$\|x_{n_k} - v_{n_k}\| \to \infty, .$$

but equation (2.2.7) gives rise to

$$\lambda u_{n_k} - Tu_{n_k} = \frac{y_{n_k}}{\|x_{n_k} - v_{n_k}\|} \tag{2.2.8}$$

which implies that $\lambda u_{n_k} - Tu_{n_k} \longrightarrow 0 \quad \text{since } y_{n_k}$ converges.
Since (u_n) is bounded and T is compact, $Tu_{n_k} \to z \in X$, so $\lambda u_{n_k} \to z$ and then $u_{n_k} \to \frac{1}{\lambda}z$. Thus by continuity of T, $T(u_{n_k}) \to \frac{1}{\lambda}T(z), \quad k \to \infty$.
Since the strong topology is Haursdoff by uniqueness of limit, we have that, $\frac{1}{\lambda}T(z) = z \Rightarrow T(z) = \lambda z$
Now,

$$\begin{aligned} dist(u_n, ker(\lambda I - T)) &= \inf_{w \in ker(\lambda I - T)} \|u_n - w\| \\ &= \inf_{w \in ker(\lambda I - T)} \Big\| \frac{x_n}{\|x_n - v_n\|} - (\frac{v_n}{\|x_n - v_n\|} + w) \Big\| \\ &= \tfrac{1}{\|x_n - v_n\|} \inf_{t \in ker(\lambda I - T)} \|x_n - t\|, \quad t = v_n + \|x_n - v_n\| w \\ &= \tfrac{1}{\|x_n - v_n\|} dist(x_n, ker(\lambda I - T)) \\ &= \tfrac{1}{\|x_n - v_n\|} \|x_n - v_n\| = 1 \end{aligned}$$

Therefore,

$$1 = dist(u_n, ker(\lambda I - T)) = \inf_{w \in ker(\lambda I - T)} \|u_n - w\| \leq \|u_n - \frac{1}{\lambda} z\| \to 0 \quad (2.2.9)$$

This is a contradiction.
The proof is complete.
$iii)$ suppose $ker(\lambda I - T) = 0$. We show that $R(\lambda I - T) = X$
By contradiction,we suppose that $R(\lambda I - T) \neq X$

$$X_1 := (\lambda I - T)X \quad is \quad closed \quad proper \quad subspace \quad of \quad X$$
$$X_2 := (\lambda I - T)X_1 = (\lambda I - T)^2 X$$
$$\vdots$$
$$X_n := (\lambda I - T)^n X$$

X_{n+1} is a closed proper subspace of X_n $(n = 0, 1, 2...)$ since $(\lambda I - T)$ is one-to-one.
Therefore by Reiz lemma (2.2.7) $\exists\ x_n \in X_n, \|x\| = 1$, $dist(x_n, X_{n+1}) \geq \frac{1}{2}$.
For $n > m$, $X_n \subsetneq X_m$

$$\begin{aligned} \|Tx_n - Tx_m\| &= \|\lambda x_m - \lambda x_m + \lambda x_n - \lambda x_n + Tx_n - Tx_m\| \\ &= \|(\lambda I - T)x_m - (\lambda I - T)x_n + \lambda x_n - \lambda x_m\| \end{aligned}$$

Observe that $(\lambda I - T)x_m - (\lambda I - T)x_n + \lambda x_n \in X_{m+1}$

$$so \quad \|Tx_n - Tx_m\| \geq |\lambda| dist(x_m, X_{m+1}) \geq \frac{|\lambda|}{2} \quad (2.2.10)$$

claim. $\{Tx_n\}$ has no convergent subsequence.

Proof. suppose $\exists\{x_{n_k}\}$ subsequence of $\{x_n\}$ such that Tx_{n_k} converges.Then from equation (1.3.16) we have

$$\|Tx_{n_k} - Tx_{n_l}\| \geq \frac{|\lambda|}{2} \tag{2.2.11}$$

This is a contradiction, since $\{Tx_{n_k}\}$ converges □

Conclusively, $\{x_n\}$ is bounded but there exist no subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ converges.This contradicts the fact that T is compact.
Hence $R(\lambda I - T) = X$ Conversely, Suppose $R(\lambda I - T) = X$ we show that $Ker(\lambda I - T) = 0$

claim. : $Ker(\lambda I - T^*) = \{0\}$

Proof. Suppose that $Ker(\lambda I - T^*) \neq 0$, then there exists $f \in X^* \setminus \{0\}$ such that $(\lambda I - T^*)f = 0$.

$$\begin{aligned}
(\lambda I - T^*)f = 0 \Rightarrow &\quad \lambda f(x) - (f \circ T)(x) = 0, \quad \forall x \in X \\
\Rightarrow &\quad f(\lambda x - T(x)) = 0, \quad \forall x \in X \\
\Rightarrow &\quad f((\lambda I - T)(x)) = 0, \quad \forall x \in X \\
\Rightarrow &\quad f((\lambda I - T)(X)) = \{0\} \\
\Rightarrow &\quad f(X) = \{0\} \quad (\text{ Since } R(\lambda I - T) = X) \\
\Rightarrow &\quad f = 0 \quad \text{contrdict the fact that } f \in X^* \setminus \{0\}.
\end{aligned}$$

Therefore $Ker(\lambda I - T^*) = \{0\}$. □

Applying the forward direction of this theorem to T^* which is compact, gives that $R(\lambda I - T^*) = X^*$,Hence $Ker(\lambda I - T) = \{0\}$ □

2.3 Spectrum of Linear compact operators

Definition 2.3.1. ***Spectrum of linear operators***
Let X be a Banach space over a scalar field $\mathbb{K}$ for $T \in \mathcal{B}(X, Y)$, the spectrum $\sigma(T)$ of T is defined by

$$\sigma(T) = \{\lambda \in \mathbb{K} : \lambda I - T \quad \textit{is not invertible} \quad \textit{in } \mathcal{B}(X)\}$$

The resolvent set $\rho(T)$ is defined by

$$\rho(T) \ = \ \mathbb{K} \setminus \sigma(T).$$

The points of $\rho(T)$ are called the regular values of T.
If $\lambda \in \rho(T)$, then

$$R_\lambda(T) = (\lambda I - T)^{-1} \quad \text{is called the resolvent of} \quad T \quad \text{at} \quad \lambda$$

The spectrum is decomposed into the disjoint union of the following three sets.

a) The point spectrum of T :

$$\sigma_p(T) = \{\lambda \in \mathbb{K} \quad | \quad Ker(\lambda I - T) \neq \{0\}\}.$$

b) The continuous spectrum of T :

$$\sigma_c(T) = \{\lambda \in \mathbb{K} : Ker(\lambda I - T) = \{0\}, \overline{R(\lambda I - T)} = X \text{ but } R(\lambda I - T) \neq X\}.$$

c) The residual spectrum of T :

$$\sigma_r(T) = \{\lambda \in \mathbb{K} \quad | \quad Ker(\lambda I - T) = \{0\}, \quad \overline{R(\lambda I - T)} \neq X\}.$$

Remark: An element of $\sigma_p(T)$ is called an eigenvalue of T and a non-zero vector f such that $Tf = \lambda f$ is called an eigenvector of T associated to the eigenvalue λ.

Theorem 2.3.2. *Let X be a Banach space over $\mathbb{C}$ and $T \in \mathcal{B}(X)$. Then the following holds.*

i) *The spectrum, $\sigma(T)$ is a closed subset of $\mathbb{C}$*

ii) *The spectrum, $\sigma(T) \subset B(0, ||T||_{\mathcal{B}(X)})$*

iii) *The spectrum, $\sigma(T)$ is a compact subset of $\mathbb{C}$*

Proof.
i) It suffices to show that $\rho(T)$ is open. Let $\lambda_0 \in \rho(T)$.

$$\begin{aligned} R_{\lambda_0}(T) = & \quad (\lambda_0 I - T)^{-1} \\ R^{-1}_{\lambda_0}(T) = & \quad \lambda_0 I - T \Rightarrow T = \lambda_0 I - R^{-1}_{\lambda_0}(T) \\ \Rightarrow I\lambda - T = & \quad R^{-1}_{\lambda_0}(T) - (\lambda_0 - \lambda)I \end{aligned}$$

$$\Rightarrow \lambda I - T = R^{-1}_{\lambda_0}(T)[I - (\lambda_0 - \lambda)R_{\lambda_0}] \tag{2.3.1}$$

If $|\lambda_0 - \lambda| < \frac{1}{||R_{\lambda_0}||}$, then $I - (\lambda_0 - \lambda)R_{\lambda_0}(T)$is invertible by lemma (2.3.3).
Therefore $(I\lambda - T)^{-1} = [I - (\lambda_0 - \lambda)R_{\lambda_0}]^{-1}R_{\lambda_0}(T)$

$$R_\lambda(T) = (I\lambda - T)^{-1} = R_{\lambda_0}(T)\sum_{k=0}^{+\infty}(\lambda - \lambda_0)^k R^k_{\lambda_0}(T) = \quad \sum_{k=0}^{+\infty}(\lambda - \lambda_0)^k R^{k+1}_{\lambda_0}(T)$$

$$\Rightarrow R_\lambda(T) = \sum_{k=0}^{+\infty}(\lambda - \lambda_0)^k R^{k+1}_{\lambda_0}(T) \tag{2.3.2}$$

This implies that R_λ is invertible in the neighbourhood of $\lambda = \lambda_0$ such that $\{|\lambda - \lambda_0| < \frac{1}{||R_{\lambda_0}||}\} \in \rho(T)$. Thus $B(\lambda_0, \frac{1}{||R_{\lambda_0}||}) \subset \rho(T)$

Hence $\rho(T)$ is open,it follows immediately that $\sigma(T)$ is closed.

$ii)$Let $\lambda \in \mathbb{C} : |\lambda| > ||T||$
$\frac{||T||}{\lambda} < 1$, therefore $(I - \frac{T}{\lambda})^{-1}$ exist and $(I - \frac{T}{\lambda})^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^k}$

$$(I\lambda - T)^{-1} = \frac{1}{\lambda}(I - \frac{T}{\lambda})^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}$$

This implies that $(I\lambda - T)$ is invertible, thus $\lambda \in \rho(T)$.
Observe that we have shown that $B(0, ||T||)^c \subset \sigma(T)^c \Rightarrow \sigma(T) \subset B(0, ||T||)$.
$iii)$ It follows immediately from $i)$ and $ii)$ that the spectrum, $\sigma(T)$ is a closed and bounded subset of $\mathbb{C}$. Therefore it is compact.

Remark: For real Banach space X with $A \in \mathcal{B}(X)$, the spectrum $\sigma(T)$ may be empty. For example, with $X = \mathbb{R}^2$ and $A \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we have $\lambda I - A$ is invertible for all $\lambda \in \mathbb{R}$.
However the spectrum of the complexification of T is nonempty. In fact $\sigma(T_{\mathbb{C}}) = \{\lambda \in \mathbb{C} \quad | \quad \lambda I - T \quad \text{is not invertible}\} = \{-i, i\}$.

lemma 2.3.3. *Let X be a Banach space and $T \in \mathcal{B}(X, Y)$. If $\|T\| < 1$, then $(I_X - T)$ is invertible and*

$$(I_X - T)^{-1} = \sum_{k=0}^{+\infty} T^k$$

where the series converge absolutely.

Proof. $S_n := \sum_{k=0}^{+\infty} T^k$
for $n, m \in \mathbb{N}$, such that $n > m$, we have that,

$$\begin{aligned} ||S_n - S_m|| = ||\sum_{k=0}^{n} T^k - \sum_{k=0}^{m} T^k|| = & \quad ||\sum_{k=m+1}^{n} T^k|| \\ \leq & \quad \sum_{k=m+1}^{n} ||T^k|| \\ \leq & \quad \sum_{k=m+1}^{n} ||T||^k \end{aligned}$$

(S_n) is Cauchy, the completeness of X guarantees the convergence of $\sum_{k=0}^{+\infty} T^k$.

$$(I - T)\sum_{k=0}^{n} T^k = \sum_{k=0}^{n} (T^k - T^{k+1}) = \sum_{k=0}^{n} T^k(I - T) = I - T^{k+1} \qquad (2.3.3)$$

passing to the limit as $n \to \infty$ and taking into account that

$$||T^{k+1}|| \leq ||T||^k \longrightarrow 0, \quad \text{since} \quad ||T|| < 1,$$

we obtain

$$(I-T)\sum_{k=0}^{\infty} T^k = \sum_{k=0}^{\infty} T^k (I-T) = I.$$

Hence, the operator $(I-T)^{-1}$ does exist and $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$ $\square$

Theorem 2.3.4. ***Riesz-Schauder***
Let X be an infinite dimensional Banach space, $T \in \mathcal{K}(X)$. Then The following holds.

i) $0 \in \sigma(T)$

ii) $\sigma(T) \setminus \{0\}$ *consist of eigenvalues of finite multiplicity i.e the dimension of the λ -eigenspace($Ker(\lambda I - T)$) has finite dimension $\forall \lambda \in \sigma \setminus \{0\}$*

iii) $\sigma(T) \setminus \{0\}$ *is either empty, finite or a sequence converging to* 0 *(i.e., it is a discrete set with no limit point other than* 0*).*

Proof.
i) If $0 \notin \sigma(T)$,then $0 \in \rho(T) \Rightarrow T$ is invertible. Since X is Banach, $T^{-1} \in \mathcal{B}(Y, X)$. Therefore, $TT^{-1} = I$ is compact. This is a contradiction, because X is infinite dimensional. Thus $0 \in \sigma(T)$

Remark: In an infinite dimensional space, a compact operator is never invertible.
ii) Let $\lambda \in \sigma(T) \setminus \{0\}$, then $\lambda \neq 0$, we show that $\lambda \in \sigma_p(T)$.
Assume that $\lambda \notin \sigma_p(T) \Rightarrow Ker(\lambda I - T) = \{0\}$. It follows immediately from (2.2.8) that $R(\lambda I - T) = X$. Thus $(\lambda I - T)$ is invertible. This contradicts our choice of λ.
Hence $\lambda \in \sigma_p(T)$, i.e $Ker(\lambda I - T) \neq \{0\}$. So λ is an eigenvalue of T. Also From (2.2.8) $\dim(Ker(\lambda I - T)) < \infty$.
iii) It suffices to show that for each $k \in \mathbb{N}$,

$$S_k = \{\lambda \in \sigma(T) : |\lambda| \geq \tfrac{1}{k}\} \quad \text{is finite}$$

since,

$$\sigma(T) \setminus \{0\} = \bigcup_{k \in \mathbb{N}} S_k.$$

Assume $\exists k \in \mathbb{N}$ such that S_k contains infinitely many eigenvalues $\lambda_1, \lambda_2, \lambda_3...$
Let v_j be the eigenvectors corresponding to each λ_j.

$$X_n := span\{v_1, v_2, v_3...v_n\}$$

claim. $v_i, 1 \leq i \leq n$ are linearly independent

Proof: By Induction, assume that $\{v_1, v_2, v_3, ..., v_n\}$ are not linearly independent. Then $\exists\, v_k, k \in \{1, 2, ..., n\}$ such that $v_1, ...v_{k-1}$ are linearly independent.

$$v_k = \sum_{j=1}^{k-1} \alpha_j v_j$$

$$\begin{aligned} Tv_k = \lambda_k v_k \Longrightarrow 0 = & \quad \lambda_k v_k - Tv_k \\ \Longrightarrow 0 = & \quad \sum_{j=1}^{k-1} \lambda_k \alpha_j v_j - T(\sum_{j=1}^{k-1} \alpha_j v_j) \\ \Longrightarrow 0 = & \quad \sum_{j=1}^{k-1} \lambda_k \alpha_j v_j - \sum_{j=1}^{k-1} \lambda_j \alpha_j v_j \\ \Longrightarrow 0 = & \quad \sum_{j=1}^{k-1} \alpha_j (\lambda_k - \lambda_j) v_j \end{aligned}$$

Since $\lambda_k \neq \lambda_j$ and v_j for $i \leq j \leq k-1$ are linearly independent we have that $\alpha_j = 0$, $1 \leq j \leq k-1$. Therefore $v_k = 0$, which is impossible. Hence $\{v_1, v_2 ... v_n\}$ are linearly independent.

X_{n-1} is a closed proper subspace of X_n, therefore by lemma (2.2.7) $\exists x_n \in X_n \quad : ||x|| = 1$ and $dist(x_n, X_{n-1}) \geq \frac{1}{2}$

For $n, m \in \mathbb{N}$ with $n > m$,

$$\begin{aligned} ||Tx_n - Tx_m|| = & \quad ||\lambda_n x_n - (\lambda_n x_n - Tx_n + Tx_m)|| \\ = & \quad ||\lambda_n (x_n - (x_n - \tfrac{1}{\lambda_n} Tx_n + \tfrac{1}{\lambda_n} T_m)|| \\ = & \quad |\lambda_n| ||(x_n - (x_n - \tfrac{1}{\lambda_n} Tx_n + \tfrac{1}{\lambda_n} T_m)|| \end{aligned}$$

$$\begin{aligned} \text{observe that} \quad x_n - \tfrac{1}{\lambda_n} Tx_n = & \quad \sum_{k=1}^{n} \alpha_k v_k - \frac{1}{\lambda_n} T(\sum_{k=1}^{n} \alpha_k v_k) \\ = & \quad \sum_{k=1}^{n} \alpha_k v_k - \frac{1}{\lambda_n} \sum_{k=1}^{n} \lambda_k \alpha_k v_k \\ = & \quad \sum_{k=1}^{n} (1 - \frac{\lambda_k}{\lambda_n}) \alpha_k v_k \\ = & \quad \sum_{k=1}^{n-1} (1 - \frac{\lambda_k}{\lambda_n}) \alpha_k v_k \\ \Rightarrow x_n - \tfrac{1}{\lambda_n} Tx_n = & \quad \sum_{k=1}^{n-1} (1 - \frac{\lambda_k}{\lambda_n}) \alpha_k v_k \in X_{n-1} \\ \Rightarrow ||Tx_n - Tx_m|| \geq & \quad |\lambda_n| dist(x_n, X_{n-1}) \\ \Rightarrow ||Tx_n - Tx_m|| \geq & \quad \tfrac{1}{2k} \qquad \star \end{aligned}$$

claim. $\{Tx_n\}$ has no convergent subsequence

proof: Suppose that $\{Tx_n\}$ is a convergent subsequence of $\{Tx_{n_j}\}$, then by equation $(\star)$ we have that $||Tx_{n_j} - Tx_{n_l}|| \geq \frac{1}{2k}$.
This is a contradiction, since every convergent sequence is Cauchy. □

In Conclusion,$\{x_n\}$ is bounded , and there is no subsequence $\{Tx_{n_j}\}$ that converges , this contradicts the fact that T is compact. Hence S_k is finite for each k. This implies that $\sigma(T) \setminus \{0\}$ is a countable union of finite sets. Thus, it is countable.

□

2.4 Spectral theory of compact linear self-adjoint operators

One of the main motivations of considering self-adjoint compact linear operators, is the guaranteed existence of the eigensystem (the eigenvalues and eigenvectors)and the simple form of the spectral decomposition. This result can be viewed as a generalization of the finite dimensional case. In the finite dimensional context, every linear operator may be identified with a representation matrix for suitable bases of X and Y. We recall an elementary result concerning diagonalization of symmetric matrices

Example 2.4.1. *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Then the following fact is well known: If T is a real symmetric matrix, then it has real eigenvalues and there exists an orthonormal basis $\{e_1, e_2 ... e_n\}$ of $\mathbb{R}^n$ such that e_i is an eigenvector of T. In other words, T is diagonalisable over an orthonormal basis.*

In this chapter, we generalized this result for a compact and self-adjoint operator in a infinite dimensional Hilbert space H.

Definition 2.4.2. *Let H be a Hilbert space.An operator $T \in \mathcal{B}(H)$ is called self-adjoint or hermitian if $T^* = T$ that is,*

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in H$$

Remark. TT^* and T^*T are self-adjoint for any $T \in \mathcal{B}(H)$

Proposition 2.4.3. *Let $T \in \mathcal{B}(H)$ be a self-adjoint operator. Then the following holds.*

i) *Any eigenvalue of T is real.*

ii) *The eigenvectors corresponding to different eigenvalues are orthogonal.*

iii) *If λ is an eigenvalue of $T \in \mathcal{B}(H)$, then $|\lambda| \leq ||T||$.*

iv) *$\langle Tx, x\rangle$ is real for all $x \in H$*

Proof.
Let λ be an eigenvalue of T. Then exists nonzero $x \in H$ such that $Tx = \lambda x$

i) $\lambda||x||^2 = \lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \quad \langle x, Tx\rangle = \langle x, \lambda x\rangle = \bar{\lambda}||x||^2$

$$\Rightarrow \lambda = \quad \bar{\lambda}$$

ii) Let λ, μ be distinct eigenvalues of T with the corresponding eigenvectors x_1, x_2. $\Rightarrow Tx_1 = \lambda x_1$ and $Tx_2 = \mu x_2$ with $x_1, x_2 \neq 0$

$$\langle Tx_1, x_2\rangle = \langle \lambda x_1, x_2\rangle = \lambda\langle x_1, x_2\rangle$$

$$\langle Tx_1, x_2\rangle = \langle x_1, Tx_2\rangle = \mu\langle x_1, x_2\rangle$$

$$\Rightarrow \lambda\langle x_1, x_2\rangle = \mu\langle x_1, x_2\rangle$$

$$\Rightarrow (\lambda - \mu)\langle x_1, x_2\rangle = 0$$

$$\Rightarrow \langle x_1, x_2\rangle = 0 \quad \text{since} \quad \lambda \neq \mu$$

$$\Rightarrow x_1 \perp x_2$$

iii) $||T||||x|| \geq ||Tx|| = \underline{|\lambda|||x||} \Rightarrow |\lambda| \leq ||T||$
iv) $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$

□

2.5 Problems of existence of eigenvalues

Every linear operator on a finite dimensional Hilbert space has an eigenvalue. The situation is not the same for infinite dimensional space.Lets us check out some of such instances .

Example 2.5.1. *Consider* $T : L_2([0,1], \mathbb{R}) \longrightarrow L_2([0,1], \mathbb{R})$ $f \mapsto Tf$ *defined by*

$$(Tf)(t) = tf(t), \qquad for\ a.e.\ t \in [0,1]\,,$$

has no eigenvalue.

Claim 1: $T \in \mathcal{B}(L_2[0,1])$

Proof.
It is easy to see that T is well-defined and is linear. Moreover for every $f \in L_2[0,1]$, we have

$$||Tf||^2_{L_2} = \int_0^1 |(Tf)(t)|^2 dt = \int_0^1 |t|^2|f(t)|^2 dt$$
$$\leq \int_0^1 |f(t)|^2 dt$$
$$\text{that is} \quad ||Tf||_{L_2[0,1]} \leq \quad ||f||_{L_2[0,1]}$$

which implies that T is bounded and $||T|| \leq 1$.
Hence $T \in \mathcal{B}(L_2[0,1])$

$\square$

Now let us find the adjoint operator T^* of T.
For all $f, g \in L_2[0,1]$, we have

$$\langle Tf, g\rangle = \int_0^1 (Tf)(t)\overline{g(t)}dt = \int_0^1 tf(t)\overline{g(t)}dt$$

$$\Rightarrow \langle Tf, g\rangle = \int_0^1 f(t)t\overline{g(t)}dt = \langle f, Tg\rangle$$

Thus T is self-adjoint on $L_2[0,1]$
Claim 2: $\sigma(T) = [0,1]$.

Proof. Suppose $\lambda \in \rho(T)$. Then $(\lambda I - T)$ is invertible in $\mathbf{B}(L_2[0,1])$.
Consider $f_0 : t \mapsto 1$, $t \in [0,1]$. Therefore there exists $g \in L_2[0,1]$ such that:

$$\begin{array}{rrl} & (\lambda I - T)^{-1} f_0 & = g \\ \Rightarrow & f_0 & = (\lambda I - T)(g) \\ & f_0 & = \lambda g - T(g) \\ & 1 & = g(t) - (Tg)(t) \quad \text{for a.e.} \quad t \in [0,1] \\ & 1 & = \lambda g(t) - tg(t) \quad \text{for a.e.} \quad t \in [0,1] \\ & g(t) & = \frac{1}{\lambda - t} \quad \text{a.e.} \quad t \in [0,1] \end{array}$$

$g \in L_2[0,1]$ implies that

$$\int_0^1 \frac{1}{|\lambda - t|^2}\, dt < \infty.$$

Making the change of variable $y := \lambda - t$, we have that

$$\int_{\lambda-1}^{\lambda} \frac{1}{y^2} dy < \infty\,.$$

It follows that $0 \notin [\lambda - 1, \ \lambda]$, which implies that $\lambda \in \mathbb{R} \setminus [0,1]$.
Thus, $[0,1] \subset \sigma(T)$.
Now we show that $\sigma(T) \subset [0,1]$.
Let $\lambda \in \mathbb{R} \setminus [0,1]$. We show that $\lambda I - T$ has a bounded inverse.
For every $f \in L_2[0,1]$,

$$(\lambda I - T)g = f \iff g(t) = \frac{f(t)}{\lambda - t} \quad \text{a.e } t \in [0,1]\,.$$

And since

$$\frac{1}{|\lambda - t|} \leq \gamma = \frac{1}{\min\{|\lambda|, \, |\lambda - 1|\}} < +\infty \quad \forall\, t \in [0,1],$$

it follows that $|g| \leq \gamma |f|$.
Therefore $\lambda I - T$ is invertible and

$$\|(\lambda I - T)^{-1}\| \leq \gamma .$$

Thus $\lambda \in \rho(T)$ and so $\sigma(T) \subset [0,1]$. □

claim. T has no eigenvalue

Proof. Assume that T has eigenvalue λ.
Let φ be the eigenvector corresponding to λ, then

$$\begin{aligned} T\varphi &= \lambda\varphi \\ (T\varphi)(t) &= (\lambda\varphi)(t) \qquad for\ a.e \quad t \in [0,1] \\ t\varphi(t) &= \lambda\varphi(t) \\ (t-\lambda)\varphi(t) &= 0 \qquad \text{a.e} \\ \varphi &= 0 \qquad \text{a.e} \end{aligned}$$

This means that $\varphi = 0 \quad a.e$, when considered as a vector in $L_2([0,1])$, which contradicts the fact that φ is an eigenvector.
Remark: More generally, for $T : L_2([a,b]) \longrightarrow L_2([a,b])$ defined by

$$(Tf)(t) = tf(t), \quad t \in [a,b]$$

T has no eigenvalue and $\sigma(T) = [a,b]$.

□

Example 2.5.2. *We also consider* $T : L_2([0,1]) \longrightarrow L_2([0,1])$ *defined by*

$$(Tf)(t) = \int_0^t f(s)ds$$

T can be re defined as $(Tf)(t) = \int_0^1 G(t,s)f(s)ds$, *where* $G(t,s) := \begin{cases} 1 & 0 \leq s \leq t \\ 0 & otherwise \end{cases}$
Now from example (2.2.2), *we deduce that T is compact.*
Now let find the adjoint of T. let $f, g \in L_2[0,1]$

$$\begin{aligned} \langle Tf, g\rangle &= \int_0^1 Tf(t)\overline{g(t)}dt \\ &= \int_0^1 \Big[\int_0^t f(s)ds\Big]\overline{g(t)}dt \\ &= \int_0^1 \Big[\int_0^t f(s)\overline{g(t)}ds\Big]dt \\ &= \int_0^1 \Big(\int_s^1 f(s)\overline{g(t)}dt\Big)ds \quad \textit{by Fubini's theorem} \\ &= \int_0^1 f(s)\Big(\int_s^1 \overline{g(t)}dt\Big)ds \\ &= \langle f, T^*g\rangle \end{aligned}$$

Therefore the adjoint operator is given by

$$T^*g(s) = \int_s^1 g(t)dt$$

claim. T has no eigenvalue.

Proof. Assume T has an eigenvalue λ. Then there exists a nonzero $\varphi \in L_2([0,1])$ such that

$$T\varphi = \lambda\varphi$$

. So,

$$\int_0^t \varphi(s)ds = \lambda\varphi(t) \qquad a.e \tag{2.5.1}$$

Since $L_2([0,1])$ consist of equivalence classes of square integrable functions which are equal almost every where. (2.5.1) gives rise to the differential equation

$$\varphi(t) = \lambda\varphi'(t)$$
$$\varphi(0) = 0$$

observe that λ can not be zero or else φ would be zero when considered as vector in $L_2([0,1])$.
For $0 \neq \lambda$, the solution to the above differential equation is given by

$$\varphi(t) \;=\; Ce^{\frac{1t}{\lambda}}.$$

Using the initial condition, we have that $\varphi = 0$,this contradicts our choice of φ. Hence T has no eigenvalue. □

From examples (2.5) and (2.5.2) , it is evident that compactness alone or self-adjoint alone does not quarantee the existence of basic system.

2.6 Spectral property of the Norm of A linear compact and self adjoint Operator

Here, it is shown that that every compact, self adjoint operator on a Hilbert space has a basic system.

Theorem 2.6.1. *If $T \in \mathcal{B}(H)$ is self adjoint, then*

$$||T|| = \sup_{||x||=1} |\langle Tx, x\rangle|$$

Proof. Let $M := \sup_{||x||=1} |\langle Tx, x\rangle|$
Then for $x \in H$ with $||x|| = 1$, we have,

$$|\langle Tx, x\rangle| \leq ||T||||x||^2 = ||T||$$

$$\Rightarrow \sup_{||x||=1} |\langle Tx, x\rangle| \leq ||T||$$

$$\Rightarrow M \leq ||T||$$

Now let x and y be arbitrary vectors in H, then

$$\begin{aligned} \langle T(x+y), x+y\rangle = & \langle Tx + Ty, x+y\rangle \\ = & \langle Tx + Ty, x\rangle + \langle Tx + Ty, y\rangle \\ = & \langle Tx, x\rangle + 2Re\langle Tx, y\rangle + \langle Ty, y\rangle \end{aligned}$$

$$\langle T(x+y), x+y\rangle = \langle Tx, x\rangle + 2Re\langle Tx, y\rangle + \langle Ty, y\rangle \tag{2.6.1}$$

Similarly,

$$\langle T(x-y), x-y\rangle = \langle Tx, x\rangle - 2Re\langle Tx, y\rangle + \langle Ty, y\rangle \tag{2.6.2}$$

From (2.6.1) and (2.6.2) we deduce that ,

$$4Re\langle Tx, y\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle \tag{2.6.3}$$

Suppose $x + y \neq 0$

$\langle T(\frac{x+y}{||x+y||}), \frac{(x+y)}{||x+y||}\rangle \leq M$

$$\Rightarrow \langle T(x+y), (x+y)\rangle \leq M||x+y||^2$$

Similarly, $\langle T(x-y), (x-y)\rangle \leq M||x-y||^2$
Therefore, (2.6.3) becomes

$$\begin{aligned} 4Re\langle Tx, y\rangle \leq & M||x+y||^2 + M||x-y||^2 \\ = & M(||x+y||^2 + ||x-y||^2) \\ \leq & 2M(||x||^2 + ||y||^2) \qquad \text{from} \qquad (1.1.3) \end{aligned}$$

$$\Rightarrow 4Re\langle Tx, y\rangle \leq 2M(||x||^2 + ||y||^2) \tag{2.6.4}$$

$$\text{But,} \quad \langle Tx, y\rangle = |\langle Tx, y\rangle|e^{i\theta} \quad \text{therefore,}$$

$$4Re|\langle Tx, y\rangle|e^{i\theta} \leq 2M(||x||^2 + ||y||^2), \quad \text{for each} \quad x \in H$$

In particular, for $e^{-i\theta}x$ we have that

$$4Re|\langle Te^{-i\theta}x, y\rangle|e^{i\theta} \leq 2M(||x||^2 + ||y||^2)$$

$$\Rightarrow Re|\langle Tx, y\rangle| \leq \frac{M}{2}(||x||^2 + ||y||^2)$$

$$\Rightarrow |\langle Tx, y\rangle| \leq \frac{M}{2}(||x||^2 + ||y||^2) \quad (2.6.5)$$

If $T = 0$, (2.6.5) holds trivially.
Suppose $Tx \neq 0$, $y := \frac{||x||}{||Tx||}Tx$.Substituting y in (2.6.5) gives ,

$$\begin{aligned} \langle Tx, \tfrac{||x||}{||Tx|}|Tx\rangle| \leq & \ \tfrac{M}{2}(||x||^2 + ||x||^2) \\ \tfrac{||x||}{||Tx|}\langle Tx, Tx\rangle \leq & \ \tfrac{M}{2}(||x||^2 + ||x||^2) \\ ||x||||Tx|| \leq & \ M||x||^2 \\ ||Tx|| \leq & \ M||x|| \\ ||T|| \leq & \ M \end{aligned}$$

We can therefore conclude that $||T|| = \sup_{||x||=1} |\langle Tx, x\rangle| = M$. □

Corollary 2.6.2. *If $T \in \mathcal{B}(H)$ and $\langle Tx, x\rangle = 0$ for all $x \in H$, then $T = 0$ and symmetric*

Theorem 2.6.3. *If $T \in \mathcal{B}(H)$ is compact and self-adjoint, then at least one of the numbers $||T||$ or $-||T||$ is an eigenvalue of T.*

Remark. This theorem shows the existence of a non-zero eigenvalue for any non-zero linear compact and self adjoint operator. It also implies that the norm of any semi-definite positive self-adjoint compact linear operator is its largest eigenvalue

Proof. If $T = 0$ then $KerT = H$
Recall that $\sigma_p(T)$=$\{\lambda \in \mathbb{C} : Ker(\lambda I - T) \neq \{0\}\}$.
Clearly,$||T|| = 0 \in \sigma_p(T)$.

Now ,assume $T \neq 0$.This implies that$||T|| \neq 0$

$$M := ||T|| := \sup_{||x||=1} |\langle Tx, x\rangle| := |\lambda|$$

$\Rightarrow \exists\{x_n\} \subset H$ with $||x_n|| = 1$ such that $\langle Tx_n, x_n\rangle \longrightarrow \lambda$,as $n \longrightarrow \infty$ since $\langle Tx_n, x_n\rangle$ is real for each n
To prove that λ is an eigenvalue of T, we first note that ,

$$\begin{aligned} 0 \leq ||Tx_n - \lambda x_n||^2 = & \ \langle Tx_n - \lambda x_n, Tx_n - \lambda x_n\rangle \\ = & \ \langle Tx_n, Tx_n\rangle - 2\lambda\langle Tx_n, x_n\rangle + \langle \lambda x_n, \lambda x_n\rangle \\ = & \ ||Tx_n||^2 - 2\lambda\langle Tx_n, x_n\rangle + \lambda||x_n||^2 \\ \Rightarrow 0 \leq ||Tx_n - \lambda x_n||^2 \leq & \ ||T||^2||x_n||^2 - 2\lambda\langle Tx_n, x_n\rangle + \lambda^2||x_n||^2 \\ \Rightarrow 0 \leq ||Tx_n - \lambda x_n||^2 \leq & \ \lambda^2 - 2\lambda\langle Tx_n, x_n\rangle + \lambda^2 \\ \Rightarrow 0 \leq ||Tx_n - \lambda x_n||^2 \leq & \ 2\lambda^2 - 2\lambda\langle Tx_n, x_n\rangle \longrightarrow 0. \quad \text{since} \quad \langle Tx_n, x_n\rangle \longrightarrow \lambda \end{aligned}$$

$$\Rightarrow ||Tx_n - \lambda x_n||^2 \longrightarrow 0$$

$$\Rightarrow Tx_n - \lambda x_n \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty \quad (2.6.6)$$

Since T is compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \to y \in H$.
From equation (2.6.6), we have that $y - \lambda x_{n_k} \to 0, k \to 0 \Rightarrow x_{n_k} \to \frac{1}{\lambda}y$

$$y = \lim_{k\to\infty} Tx_{n_k}$$

By continuity of T, $y = T \lim_{k\to\infty} x_{n_k} \Rightarrow y = T\frac{1}{\lambda}y \Rightarrow Ty = \lambda y$

and $y \neq 0$, since $||y|| = \lim_{k\to\infty} ||\lambda x_{n_k}|| = |\lambda| = ||T|| \neq 0$.
Hence λ is an eigenvalue of T.

□

Corollary 2.6.4. *For every compact self-adjoint operator T on a Hilbert space , we have that $\sigma_p(T) \neq \emptyset$*

2.7 Spectral theorem of Linear compact and self-adjoint operators

In this section, we shall prove the main theorems of this work.

lemma 2.7.1. *Let Y be a closed subspace of a normed space X.If $x \notin Y$ then there exist $f \in S_{X^*}$ such that $f(y) = 0$ for all $y \in Y$ and $f(x) = dist(x, Y)$*

Proof.
Let $x \in X \setminus Y$.

$$d := dist(x, Y) > 0, \quad \text{since} \quad Y \quad \text{is closed}$$

$$Z := span\{Y, x\}$$

$$\text{Define} \quad f : Z \longrightarrow \mathbb{K} \quad \text{by}$$

$$f(y + tx) = td, y \in Y \quad \text{and} \quad k \in \mathbb{K}$$

For any $u := y + tx$, where $y \in Y$ and t is a scalar in $\mathbb{K}$
$f(u) = td \in \mathbb{K}$, since $d \in \mathbb{R}_+$.
Hence f is well defined
claim. $f \in \mathcal{B}(Z)$

Proof.
Obviously, f is linear.
$u := y + tx$, where $y \in Y$and t is a scalar such that $u \neq 0$. Then,

$$|f(u)| = |f(y+tx)| = |t|d = \frac{|t|||u||d}{||u||} = \frac{|t|||u||d}{||y+tx||}$$

$$= \frac{|t|||u||d}{|t|||\frac{y}{t}+x||} = \frac{||u||d}{||x-(\frac{-y}{t})||} \leq \frac{||u||d}{dist(x,Y)} = \frac{||u||d}{d} = ||u||$$

$$|f(u)| = ||u|| \Rightarrow ||f|| \leq 1$$

Hence, $f \in \mathcal{B}(Z)$

□

Now ,$d = \inf_{y \in Y} ||x - y|| \Rightarrow \exists y_n \in Y : ||y_n - x|| \longrightarrow d, n \longrightarrow \infty$

$$d = |f(y_n) - f(x)| \leq ||f||||y_n - x||$$

Passing to the limit as $n \longrightarrow \infty$, we have that $d \leq ||f||d \Rightarrow 1 \leq ||f||$

Therefore, $||f|| = 1 \Rightarrow f \in S_{Z^*}$
Clearly, $f|_Y = 0$ and $f(x) = d = dist(x, Y)$
In conclusion, Z is a subspace of X and f is a bounded linear functional defined on Z. Therefore by Hahn Banach theorem there exists a bounded linear functional F defined on X , which extends f with $||F|| = ||f|| = 1$. This proves the required result. □

Definition 2.7.2. *Let M be a closed subspace of a Hilbert space H.M is said to be T-invariant If and only if $TM \subset M$, i.e. $\forall x \in M, Tx \in M$*

lemma 2.7.3. *Let T be a self-adjoint operator on a Hilbert space H and let λ be a scalar. Then $\lambda \in \sigma(T)$ if and only if $\inf_{||x||=1} ||(\lambda I - T)x|| = 0$*

Proof.
Suppose $\lambda \in \sigma(T)$, we show that $\inf_{||x||=1} ||(\lambda I - T)x|| = 0$
By contrapositive, assume that $\inf_{||x||=1} ||(\lambda I - T)x|| > C$, for some $C > 0$.
Therefore,$||(\lambda I - T)x|| > c||x||, \qquad \forall x \in H$

$R(\lambda I - T)$ is closed in H by theorem (2.2.8), we will also show that $R(\lambda I - T)$ is dense in H. By contradiction assume that$R(\lambda I - T)$ is not dense in H. Picking $y_0 \in \overline{R(\lambda I - T)}^{\perp}$ then $\langle (\lambda I - T)(x), y_0 \rangle = 0, \forall x \in H$

$\langle (\lambda I - T)(x), y_0 \rangle = \langle x, (\bar{\lambda} I - T)(y_0) \rangle = 0, \forall x \in H,$ since T is self -adjoint.
$\langle x, (\bar{\lambda} I - T)(y_0) \rangle = 0, \forall x \in H$
$\Longrightarrow (\bar{\lambda} I - T)(y_0) = 0$
$\Longrightarrow \bar{\lambda} y_0 = T y_0$.

for $y \neq 0$, we have that $\bar{\lambda}$ is an eigenvalue for T
Since all the eigenvalues of T are real, we have that $(\lambda I - T)y_0 = 0$ and $y_0 \neq 0$ contradicting the fact that $||(\lambda I - T)y_0|| \geq C||y_0||$.
Thus, $R(\lambda I - T)$ is dense in H. Since$R(\lambda I - T)$ is both closed and dense in H, we conclude that $(\lambda I - T)$ is invertible $\Longrightarrow \lambda \in \rho(T)$,which contradicts our hypothesis.
Hence $\inf\limits_{||x||=1} ||(\lambda I - T)x|| = 0$
Conversely, Let λ be a scalar such that $\inf\limits_{||x||=1} ||(\lambda I - T)x|| = 0$, we show that $\lambda \in \sigma(T)$.
Suppose $\lambda \notin \sigma(T)$.
$\Rightarrow \lambda \in \rho(T)$.Then $(\lambda I - T)^{-1}$ exist. Moreover, $(\lambda I - T)^{-1} \in \mathcal{B}(H)$, since H is Banach.
For $x \in H : ||x|| = 1$, we have that

$$1 = ||x|| = ||(\lambda I - T)^{-1}(\lambda I - T)(x)|| \leq ||(\lambda I - T)^{-1}||||(\lambda I - T)(x)||$$

$$\Rightarrow 1 \leq ||(\lambda I - T)^{-1}||||(\lambda I - T)(x)||$$
$$\Rightarrow ||(\lambda I - T)(x)|| \geq ||(\lambda I - T)^{-1}||^{-1}$$
$$\Rightarrow \inf_{||x||=1} ||(\lambda I - T)x|| \geq ||(\lambda I - T)^{-1}||^{-1}$$

This contradicts our hypothesis . Hence $\lambda \in \sigma(T)$

□

lemma 2.7.4. *Let T be a self -adjoint operator on a Hilbert space H. Let M be a closed subapace of H that is T-invariant. Then $N = M^{\perp}$ is invariant under T .Denote $T_1 = T|_M$ and $T_2 = T|_N$.Then T_1, T_2 are self-adjoint operators on M and N respectively and $\sigma(T) = \sigma(T_1) \bigcup \sigma(T_2)$.*

Proof. Firstly, we show that N is invariant under T.
$\forall x \in M, Tx \in M$ since M is T- invariant
Let $y \in N = M^{\perp}$,

$$\langle Tx, y \rangle = 0 \quad \text{since} \quad Tx \in M$$
$$\Rightarrow 0 = \langle Tx, y \rangle = \langle x, Ty \rangle$$
$$\Rightarrow \langle x, Ty \rangle = 0, \forall x \in M$$
$$\Rightarrow Ty \in M^{\perp} = N$$

For an arbitrary $y \in N$, we have shown that $Ty \in N$, therefore, N is T-invariant.

Since both M and N are T-invariant,T_1 and T_2 are self-adjoint on the corresponding subspaces.
It is left to show that $\sigma(T) = \sigma(T_1) \bigcup \sigma(T_2)$.

Let $\lambda \in \sigma(T_1)$ by lemma (2.7.3) we have that $\inf\limits_{x\in M, ||x||=1} ||(\lambda I - T_1)x|| = 0$
Therefore, $\exists x_n \in M$ with $||x|| = 1$ such that $||\lambda x_n - T_1(x_n)|| \longrightarrow 0, n \longrightarrow \infty, \Rightarrow ||\lambda x_n - T(x_n)|| \longrightarrow 0, n \longrightarrow \infty$.
Thus by lemma (2.7.3) $\lambda \in \sigma(T)$
For arbitrary $\lambda \in \sigma(T_1)$, we have shown that $\lambda \in \sigma(T)$, this implies that $\sigma(T_1) \subset \sigma(T)$. similarly, $\sigma(T_2) \subset \sigma(T)$. Therefore we conclude that

$$\sigma(T_1) \bigcup \sigma(T_2) \subset \sigma(T)$$

It is left to show that $\sigma(T) \subset \sigma(T_1) \bigcup \sigma(T_2)$
Assume that $\lambda \notin \sigma(T_1) \bigcup \sigma(T_2)$.Then there exist $C > 0$ such that for every $x \in M$ and for every $y \in N$, we have that $||\lambda x - T_1 x|| \geq C||x||$ and $||\lambda y - T_2 y|| \geq C||y||$.

$z := x + y, z \in H, \lambda x - Tx \in M$ and $\lambda y - Ty \in N$

$$\begin{aligned}
||\lambda z - Tz||^2 = & \ ||\lambda x - Tx + \lambda y - Ty||^2 \\
= & \ \langle \lambda x - Tx + \lambda y - Ty, \lambda x - Tx + \lambda y - Ty \rangle \\
= & \ \langle \lambda x - Tx, \lambda x - Tx \rangle + \langle \lambda y - Ty, \lambda y - Ty \rangle \text{ since } (N = M^{\perp}) \\
||\lambda z - Tz||^2 = & \ ||\lambda x - Tx||^2 + ||\lambda y - Ty||^2 \\
\geq & \ C^2(||x||^2 + ||y||^2) \\
\Rightarrow ||\lambda z - Tz||^2 \geq & \ C^2||z||^2 \\
\Rightarrow \inf_{||z||=1} ||(\lambda I - T_1)z|| \geq & \ C \\
\Rightarrow \lambda \notin & \ \sigma(T)
\end{aligned}$$

$\Rightarrow \sigma(T) \subset \sigma(T_1) \bigcup \sigma(T_2)$.
Hence $\sigma(T) = \sigma(T_1) \bigcup \sigma(T_2)$ □

Theorem 2.7.5. ***Spectral theorem of compact self-adjoint operator***
Let $T \neq 0$ be a compact self-adjoint operator on an infinite dimensional Hilbert spaceH.Then $\sigma(T) = \{0\} \bigcup \{\lambda_i\}$, where λ_i are distinct real non-zero eigenvalues of T.The set $\{\lambda_i\}$ contains $||T||$ and is either finite or a countable sequence convergent to zero. Moreover , the space H has orthonormal basis formed by eigenvectors corresponding to eigenvalues of T.

Proof.

$$\sigma_p(T) := \{\lambda_i\}$$

Step I. we show that $\sigma(T) = \{0\} \bigcup \sigma_p(T)$
claim. $\sigma(T) \subset \{0\} \bigcup \sigma_p(T)$

Proof. Suppose λ is non-zero and not an eigenvalue of T .Then $Ker(\lambda I - T) = \{0\}$. Therefore from theorem (2.2.8) $R(\lambda I - T) = H$. This implies that $\lambda I - T$ is invertible,so $\lambda \notin \sigma(T)$.
For arbitrary $\lambda \notin \{0\} \bigcup \sigma_p(T)$, we have shown that $\lambda \notin \sigma(T)$.Hence $\sigma(T) \subset \{0\} \bigcup \sigma_p(T)$ □

claim. $\{0\} \bigcup \sigma_p(T) \subset \sigma(T)$

Proof. Since$T$ is compact, we have that $0 \in \sigma(T)$ by theorem (2.3.4) and by definition ,$\sigma_p(T) \subset \sigma(T)$.Thus $\{0\} \bigcup \sigma_p(T) \subset \sigma(T)$ □

We can therefore affirm that $\sigma(T) = \{0\} \bigcup \sigma_p(T)$

Step II. We justify that $||T|| \in \sigma_p(T)$

This is evident from theorem (2.6.3) , since T is compact and self-adjoint.

Step III. We argue that $\sigma_p(T) = \sigma(T) \setminus \{0\}$ is at most countable.

This also follows from Reisz-schauder theorem (2.3.4)

Step IV.Finally we show that the eigenvectors corresponding to the eigenvalues of T form orthonormal basis for H
For an eigenvalue λ, $N_\lambda := Ker(\lambda I - T)$
We can form an orthonormal basis B_λ of each N_λ, since N_λ is finite dimensional for each λ.

$$B := \bigcup_\lambda B_\lambda$$

B is an orthonormal set in H, since the eigenvectors are orthogonal . It is left to show that B is orthonormal basis for H.clearly $\overline{span(B)}$ contains all the eigenvectors of H. It is enough to show that $\overline{span(B)} = H$
By contradiction,assume that $\overline{span(B)} \neq H$
Consider $G = \overline{span(B)}^\perp$. Since all the eigenvalues are T-invariant, it follows that $\overline{span(B)}$ is T-invariant and hence by lemma (2.3.4) G is also T-invariant .Moreover,

$$\sigma(T) = \sigma(T|_{\overline{span(B)}}) + \sigma(T|_G)$$

However, $T|_G$ has an eigenvalue (because it is compact and self adjoint), hence it has a non-zero eigenvector v. v must be an eigenvector of H.
Thus $v \in G \bigcap \overline{span(B)}$.

$v \in G \bigcap \overline{span(B)} \Rightarrow v \in \overline{span(B)}^\perp$ and $v \in \overline{span(B)}$
$\Rightarrow \langle x, v\rangle = 0 \quad \forall x \in \overline{span(B)}$ and $v \in \overline{span(B)}^\perp$
In particular, $\langle v, v\rangle = 0 \Rightarrow ||v||^2 = 0 \Rightarrow v = 0$.
This contradicts the fact that v is non-zero. Therefore $\overline{span(B)} = H$
Hence B is an orthonormal basis of H. □

Theorem 2.7.6. ***Decomposition of compact, self- adjoint operators*** *Suppose T is a compact self adjoint operator on H , there exist an orthonormal system $\varphi_1, \varphi_2, \varphi_3, ...$ of eigenvectors of T and corresponding eigenvalues*

$\lambda_1, \lambda_2, \lambda_3...$ such that for all $x \in H$,

$$Tx = \sum_k \lambda_k \langle x, \varphi_k \rangle \varphi_k$$

If $\{\lambda_k\}$ is an infinite sequence ,then it converges to zero.

We shall proof this theorem by successive applications of theorem (2.6.3) and lemma (2.3.4)

Proof. Let $H_1 = H$ and $T_1 = T$
By theorem (2.6.3), there exist an eigenvalue λ_1 of T_1 and a corresponding eigenvector φ_1 such that $||\varphi_1|| = 1$ and $|\lambda_1| = ||T_1||$

$$H_2 := \{\varphi_1\}^\perp$$

H_2 is a closed subspace of H_1 and $TH_2 \subset H_2$ (i.e H_2 is T-invariant).
Now let T_2 be the restriction of T to H_2. Then T_2 is compact self adjoint operator in $\mathcal{B}(H_2)$
If $T_2 \neq 0$,then there exists an eigenvalue λ_2 of T_2 and corresponding eigenvector φ_2 such that $||\varphi_2|| = 1$ and $|\lambda_2| = ||T_2|| \leq ||T_1|| = |\lambda_1|$
$\{\varphi_1, \varphi_2\}$ is orthonormal.

$$H_3 = \{\varphi_1, \varphi_2\}^\perp$$

H_3 is a closed subspace of H and $TH_3 \subset H_3$
Letting T_3 be the restriction of T to H_3, we have that T_3 is a compact self-adjoint operator in $\mathcal{B}(H_3)$. Continuing in this manner, the process stops when $T_n = 0$ or else we get a sequence $\{\lambda_n\}$ of eigenvalues of T and corresponding orthonormal set $\{\varphi_1, \varphi_2, \varphi_3...\}$ of eigenvectors such that

$$|\lambda_{n+1}| = ||T_{n+1}|| \leq ||T_n|| = |\lambda_n| \qquad n = 1, 2, 3... \tag{2.7.1}$$

claim. If $\{\lambda_n\}$ is an infinite sequence, then $\lambda_n \longrightarrow 0, n \longrightarrow \infty$

Proof. Suppose by contradiction , there exist $\epsilon > 0$ such that $|\lambda_n| \geq \epsilon$ for all $n \in \mathbb{N}$
Hence for $n \neq m$, we have that,

$$||T\varphi_n - T\varphi_m||^2 = ||\lambda\varphi_n - \lambda\varphi_m||^2 = \lambda_n^2 + \lambda_m^2 > \epsilon \tag{2.7.2}$$

But this is impossible, since $\{T\varphi_n\}$ has a convergent subsequence due to the compactness of T. We therefore conclude that $\lambda_n \longrightarrow 0, n \longrightarrow \infty$ □

Now, we prove the representation of T as asserted in the theorem.
Case I. $T_n = 0$ for some n

$$x_n := x - \sum_{k=1}^{n} < x, \varphi_k > \varphi_k$$

It is evident that x_n is orthogonal to φ_i for $1 \leq i \leq n$
Therefore, $x_n \in H_n$

$$0 = T_n x_n = Tx - T(\sum_{k=1}^{n} \langle x, \varphi_k \rangle \varphi_k)$$

$$\Rightarrow Tx = \sum_{k=1}^{n} \lambda_k \langle x, \varphi_k \rangle \varphi_k$$

Case II. $T_n \neq 0$ for all $n \in \mathbb{N}$

$$\begin{aligned} ||Tx - \sum_{k=1}^{n} \lambda_k \langle x, \varphi_k \rangle \varphi_k|| &= ||T_n x_n|| \\ &\leq ||T_n|| ||x_n|| \\ &= |\lambda_n| ||x_n|| \\ &\leq |\lambda_n| ||x|| \longrightarrow 0 \\ \Rightarrow ||Tx - \sum_{k=1}^{n} \lambda_k \langle x, \varphi_k \rangle \varphi_k|| &\longrightarrow 0, n \longrightarrow \infty \end{aligned}$$

Hence, $Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, \varphi_k \rangle \varphi_k$

□

CHAPTER 3

Application to Linear Elliptic Boundary value Problems

3.1 Notations and definitions

All funtions and Vector fields used are of class atleast C^2

Definition 3.1.1. *We define the gradient of the scalar function $f \in \mathbb{R}^n$ as the vector field of the partial derivatives of f denoted by ∇f i.e*

$$\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} ... \frac{\partial f}{\partial x_n})$$

Definition 3.1.2. *The divergence of the vector field $F = (f_1, f_2, ..., f_n)$ in the coordinates $(x_1, x_2, ..., x_n)$, is given by*

$$divF = \nabla \cdot F = \frac{\partial f}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + ... + \frac{\partial f_n}{\partial x_n}.$$

Definition 3.1.3. *Let $\phi \in C^k(\Omega), k \geq 2$, where Ω is open in $\mathbb{R}^n$. We define the Laplacian operator of ϕ by*

$$\triangle \phi = div(\nabla \phi)$$

Proposition 3.1.4. *By taking $\phi, \psi \in C^k(\Omega), k \geq 2$, we get,*

i) $\triangle(\phi + \psi) = \triangle\phi + \triangle\psi$

ii) $div(\phi \nabla \psi) = \phi(\triangle\psi) + \langle \nabla\phi, \nabla\psi \rangle$

iii) $\triangle(\phi\psi) = \psi(\triangle\phi) + 2\langle \nabla\phi, \nabla\phi \rangle + \phi(\triangle\psi)$

Proof.

$$
\begin{aligned}
i)\quad \triangle(\phi+\psi) =&\quad div(\nabla(\phi+\psi)) \\
=&\quad div(\tfrac{\partial(\phi+\psi)}{\partial x_1},...,\tfrac{\partial(\phi+\psi)}{\partial x_n}) \\
=&\quad div\Big((\tfrac{\partial\phi}{\partial x_1}+\tfrac{\partial\psi}{\partial x_1}),...,(\tfrac{\partial\phi}{\partial x_n}+\tfrac{\partial\psi}{\partial x_n})\Big) \\
=&\quad \tfrac{\partial}{\partial x_1}(\tfrac{\partial\phi}{\partial x_1}+\tfrac{\partial\psi}{\partial x_1})+\tfrac{\partial}{\partial x_2}(\tfrac{\partial\phi}{\partial x_2}+\tfrac{\partial\psi}{\partial x_2})+...+\tfrac{\partial}{\partial x_n}(\tfrac{\partial\phi}{\partial x_n}+\tfrac{\partial\psi}{\partial x_n}) \\
=&\quad \tfrac{\partial^2\phi}{\partial x_1^2}+\tfrac{\partial^2\psi}{\partial x_2^2}+...+\tfrac{\partial^2\phi}{\partial x_n^2}+\tfrac{\partial^2\psi}{\partial x_n^2} \\
=&\quad \tfrac{\partial^2\phi}{\partial x_1^2}+\tfrac{\partial^2\phi}{\partial x_2^2}+...+\tfrac{\partial^2\phi}{\partial x_n^2}+\tfrac{\partial^2\psi}{\partial x_1^2}+\tfrac{\partial^2\psi}{\partial x_2^2}+...+\tfrac{\partial^2\phi}{\partial x_n^2} \\
\Rightarrow \triangle(\phi+\psi) =&\quad div(\nabla\phi)+div(\nabla\psi) \\
\Rightarrow \triangle(\phi+\psi) =&\quad \triangle\phi+\triangle\psi
\end{aligned}
$$

$$
\begin{aligned}
ii)\quad div(\phi(\nabla\psi)) =&\quad div\Big(\phi(\tfrac{\partial\psi}{\partial x_1},\tfrac{\partial\psi}{\partial x_2},...\tfrac{\partial\psi}{\partial x_n})\Big) \\
=&\quad \tfrac{\partial}{\partial x_1}(\phi\tfrac{\partial\psi}{\partial x_1})+\tfrac{\partial}{\partial x_2}(\phi\tfrac{\partial\psi}{\partial x_2})+...\tfrac{\partial}{\partial x_n}(\phi\tfrac{\partial\psi}{\partial x_n}) \\
=&\quad \phi\Big(\tfrac{\partial^2\psi}{\partial x_1^2}+\tfrac{\partial^2\psi}{\partial x_2^2}+...+\tfrac{\partial^2\psi}{\partial x_n^2}\Big)+\Big(\tfrac{\partial\phi}{\partial x_1},...\tfrac{\partial\phi}{\partial x_1}\Big)\Big(\tfrac{\partial\psi}{\partial x_1},...\tfrac{\partial\psi}{\partial x_1}\Big)
\end{aligned}
$$

$$\Rightarrow div(\phi(\nabla\psi)) = \phi\triangle\psi+\langle\nabla\phi,\nabla\psi\rangle \tag{3.1.1}$$

iii) $\triangle(\phi\psi) = div(\nabla(\phi\psi))$, proceeding as in above we have that

$$\triangle(\phi\psi) = \phi(\triangle\psi)+\psi(\triangle\phi)+2\langle\nabla\phi,\nabla\psi\rangle$$

□

Divervence Theorem
Let F be a continuously differentiable and compactly supported Vector F on Ω. Then

$$\int_\Omega (divF)dx = \int_{\partial\Omega}\langle V,F\rangle dA$$

Where V is an an outward normal vector and dA is a unit surface on the boundary of Ω.

Green's Formula
Applying the divergence theorem and definition of divergence ,such that atleast $\phi, \psi \in C^2(\Omega)$. We get

$$\int_\Omega div(\phi(\nabla\psi))dx = \int_\Omega (\phi(\nabla\psi) + \langle\nabla\psi, \nabla\phi\rangle)dx = \int_{\partial\Omega} \langle V, \phi(\nabla\psi)\rangle dA$$

This gives rise to the **first Green's formula**

$$\int_\Omega (\phi(\triangle\psi) + \langle\nabla\psi, \nabla\phi\rangle)dx = \int_{\partial\Omega} \phi\langle V, \nabla\psi\rangle dA \qquad (3.1.2)$$

In particular, if $\phi = 0$ on the boundary. The formula becomes

$$\int_\Omega (\phi(\nabla\psi) + \langle\nabla\psi, \nabla\phi\rangle)dx = 0$$

The **second Green's formula** is given as

$$\int_\Omega \Big(\psi(\triangle\phi) - \phi(\triangle\psi)\Big)dx = \int_{\partial\Omega} \Big(\psi\langle V, \nabla\phi\rangle - \phi\langle V, \nabla\psi\rangle\Big)dA$$

3.2 Elliptic Boundary Value Problem

Let Ω be a bounded domain (i.e open connected) in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let L denote the differential operator

$$L = \sum_{1\leq i,j\leq n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)$$

where $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$
L is uniformily elliptic if there exist $\alpha > 0$ such that

$$\sum_{1\leq i,j\leq n} a_{ij}(x)\xi_i\xi_j \;\geq\; \alpha||\xi||^2, \quad \forall\xi \in \mathbb{R}^n\,.$$

Consider the Dirichlet Boundary Value Problem

$$\begin{aligned} -Lu &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \qquad (3.2.1)$$

Where f is a given function on Ω.
Let $f \in L^2(\Omega)$
A weak solution of (3.2.1) is a $u \in H_0^1(\Omega)$ such that

$$\sum_{1\leq i,j\leq n} \int_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} = \int_\Omega fv \quad \text{for all} \quad v \in H_0^1$$

In other words, a weak solution of (3.2.1)is a function $u \in H_0^1(\Omega)$ such that

$$B_0[u,v] = \langle f, v\rangle_{L^2}, \qquad \forall v \in H_0^1(\Omega) \tag{3.2.2}$$

Where $B_0 : H_0^1 \times H_0^1 \longrightarrow \mathbb{R}$ is a continuous bilinear form.

$$B_0[u,v] = \sum_{1\le i,j\le n} \int_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx$$

Theorem 3.2.1. ***Solution of the elliptic Boundary Value Problem***
Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let the operator L be uniformly elliptic, with the coefficients $a_{ij} \in L^\infty(\Omega)$. Then for every $f \in L^2(\Omega)$, the boundary Value problem (3.2.1) has a unique weak solution $u \in H_0^1(\Omega)$. The corresponding solution operator denoted as $L^{-1} : f \mapsto u$ is a continuous operator from $L^2(\Omega)$ into $H_0^1(\Omega) \cap H^2(\Omega)$ and compact from $L^2(\Omega)$ into $L^2(\Omega)$

Proof.
The existence and uniqueness of weak solution to the elliptic boundary value problem (3.2.1) will be achieved by using Lax-Milgram theorem.
1) The continuity of B_0

$$|B_0[u,v]| \le \sum_{1\le i,j\le n} \int_\Omega |a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}|dx \le \sum_{1\le i,j\le n} ||a_{ij}||_{L^\infty}\left\|\frac{\partial u}{\partial x_i}\right\|_2\left\|\frac{\partial v}{\partial x_j}\right\|_2 \le C||u||_{H^1}||v||_{H^1}$$

2) we claim that B_0 is coercive, i.e. there exists $\beta > 0$ such that

$$B_0[u,v] \ge \beta||u||_{H^1} \quad \text{for all} \quad u \in H_0^1(\Omega) \tag{3.2.3}$$

Indeed, since Ω is bounded, Poincaré inequality yields the existence of a constant K such that

$$||u||_{L^2}^2 \le K\int_\Omega |\nabla u|^2 dx \qquad \forall u \in H_0^1 \tag{3.2.4}$$

On the other hand, the uniform ellipticity condition implies that

$$B_0[u,u] = \int_\Omega \sum_{1\le i,j\le n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx \ge \int_\Omega \alpha\sum_{i=1}^n(\frac{\partial u}{\partial x_i})^2 = \alpha\int_\Omega |\nabla u|^2 dx \tag{3.2.5}$$

With the two inequalities above, we have,

$$\begin{aligned} ||u||_{H^1(\Omega)}^2 =& \ ||u||_{L^2\Omega)}^2 + ||\nabla u||_{L^2(\Omega)}^2 \\ \le& \ K||\nabla u||_{L^2(\Omega)}^2 + ||\nabla u||_{L^2(\Omega)}^2 \\ ||u||_{H^1(\Omega)}^2 \le& \ (K+1)||\nabla u||_{L^2(\Omega)}^2 \\ ||u||_{H^1(\Omega)}^2 \le& \ \tfrac{K+1}{\alpha}B_0[u,u] \\ \Longrightarrow B_0[u,u] \ge& \ \tfrac{\alpha}{K+1}||u||_{H^1(\Omega)}^2 \end{aligned}$$

This proves (3.15) by taking $\beta = \frac{\alpha}{K+1}$

By Lax-Milgram for every $f \in H_0^1(\Omega)$ there exists a unique element $u \in H_0^1(\Omega)$ such that

$$B_0[u, v] = \langle f, v\rangle_{H^1} \quad \forall v \in H_0^1(\Omega)$$

Furthermore, we show that the map $\Lambda : f \mapsto u$ is continuous.

$$\begin{aligned} B_0[u,u] = & \quad \langle f, u\rangle, \quad u \in H_0^1(\Omega) \\ \beta||u||_{H^1}^2 \leq B_0[u,u] = & \quad \langle f, u\rangle_{H^1}, \quad u \in H_0^1(\Omega) \\ \beta||u||_{H^1}^2 \leq & \quad ||f||_{L^2}||u||_{H^1} \\ ||u||_{H^1} \leq & \quad \beta^{-1}||f||_{L^2} \end{aligned}$$

Therefore

$$||L^{-1}(f)||_{H^1} \leq C||f||_{L^2}, \quad C = \beta^{-1} \tag{3.2.6}$$

Hence map $\Lambda : f \mapsto u$ is continuous.
Now, we prove that the solution operator $L^{-1} : f \mapsto u$ is compact From $L^2(\Omega)$ into $L^2(\Omega)$
Let S be a bounded subset of $L^2(\Omega)$, we show that $L^{-1}(S)$ is precompact in $L^2(\Omega)$.
From (3.2.6), we have that $L^{-1}(S)$ is bounded in $H_0^1(\Omega)$. By Rellich-Kondrachov, the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ is compact. Hence $L^{-1}(S)$ is precompact in $L^2(\Omega)$ □

We saw earlier that the existence of basic systems is guaranteed only for Compact and self-adjoint operator. However, the Laplace operator

$$\triangle : \; H_0^1(\Omega) \cap H^2(\Omega) \;\longrightarrow\; L^2(\Omega)$$

is not compact (since it is unbounded) but yet it has a basic system due to the properties of its inverse. This can be achieved by taking advantage of spectral theorem of compact self-adjoint operators and theorem(3.2.1)

lemma 3.2.2. *The Laplace operator is* $\triangle : H_0^1(\Omega) \cap H^2(\Omega) \longrightarrow L^2(\Omega)$ *is self-adjoint.*

Proof. Let $\phi \in H_0^1(\Omega)$,$\psi \in H^2(\Omega)$

$$\langle \phi, \triangle\psi\rangle = \int_\Omega \phi\triangle\psi dx = \int_{\partial\Omega} \phi(\nabla\psi)V dA - \int_\Omega \nabla\phi dx\nabla\psi \quad \text{from} \quad (3.1.2)$$

$\phi = 0$ on the boundary, therefore

$$\langle \phi, \triangle\psi\rangle = -\int_\Omega \nabla\phi \cdot \nabla\psi$$

Similarly,

$$\langle \psi, \triangle\phi\rangle = -\int_\Omega \nabla\phi \cdot \nabla\psi$$

We deduce that $\langle \triangle\phi, \psi\rangle = \langle \phi, \triangle\psi\rangle$.
Hence the Laplace operator defined above is self-adjoint. $\square$

Now from (3.2.1) taking $Ł = \triangle, u = \phi$ we have

$$\begin{aligned} -\triangle\phi &= f \quad \text{in} \quad \Omega \quad . \\ \phi &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{3.2.7}$$

Multiplying the first equation in (3.2.1) by $\psi \in H_0^1$ and integrating both sides over Ω, we have,

$$\int_\Omega -\psi(\triangle\psi)dx = \int_\Omega \psi f dx, \qquad \phi \in H^2$$

By Green's formula we have,

$$\int_\Omega (\nabla\phi\dot{\nabla}\psi)dx = \int_\Omega f\psi dx \qquad \phi, \psi \in H_0^1(\Omega)$$

Here, we shall explore the Lax-Milgram theorem.
H_0^1 is closed subspace of a Hilbert space H^1, so it is Hilbert. Define

$$\begin{aligned} B : H_0^1 \times H_0^1 &\longrightarrow R \\ [\phi, \psi] &\mapsto B[\phi, \psi] = \textstyle\int_\Omega (\nabla\phi.\nabla\psi)dx \end{aligned}$$

B is linear, which follows from the linearity of integral and divergence.

claim. B is bounded

Proof.

$$\begin{aligned} |B[\phi, \psi]| \leq \int_\Omega |\langle\nabla\phi, \nabla\psi\rangle| dx \leq & \int_\Omega ||\nabla\phi|| ||\nabla\psi|| dx \\ \leq & \left(\textstyle\int_\Omega ||\nabla\phi(x)||^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega ||\nabla\psi(x)||^2 dx\right)^{\frac{1}{2}} \\ = & ||\nabla\phi||_{L^2} ||\nabla\psi||_{L^2} \\ \leq & ||\phi||_{H^1} ||\psi||_{H^1} \\ \Longrightarrow |B[\phi, \psi]| \leq & ||\phi||_{H^1} ||\psi||_{H^1} \end{aligned}$$

$\square$

claim. B is H_0^1-elliptic i.e there exists $\beta > 0$ such that

$$B[\phi, \phi] \geq \beta ||\phi||_{H_0^1(\Omega)}, \quad \forall \phi \in H_0^1(\Omega)$$

Proof.

$$B[\phi,\phi]=\int_\Omega \nabla\phi.\nabla\phi=\langle\nabla\phi,\nabla\phi\rangle=||\nabla\phi||^2_{L^2}$$

From Poincare inequality, there exist a constant κ such that

$$\begin{aligned} ||\phi||_{L^2(\Omega)} \leq &\ \kappa||\nabla\phi||_{L^2}, \qquad \phi\in H^1_0\\ \Rightarrow ||\phi||^2_{L^2(\Omega)} \leq &\ \kappa^2||\nabla\phi||^2_{L^2} \qquad \phi\in H^1_0,\\ \Rightarrow ||\phi||^2_{H^1} \leq &\ (\kappa^2+1)||\nabla\phi||^2_{L^2}\\ \Rightarrow ||\phi||^2_{H^1} \leq &\ (\kappa^2+1)B[\phi,\phi]\\ \Rightarrow B[\phi,\phi] \geq &\ \tfrac{1}{\kappa^2+1}||\phi||^2_{H^1} \end{aligned}$$

Hence setting $\beta=\frac{1}{\kappa^2+1}$ proves our claim. □

Now, for each $f\in L^2(\Omega)$, we define

$$\begin{aligned} L_f : H^1_0(\Omega) &\longrightarrow \mathbb{R} \quad \text{by}\\ \psi &\mapsto L_f(\psi)=\int_\Omega f\psi dx \end{aligned}$$

i) For each f, L_f is linear

$$\begin{aligned} ii)\quad |L_f(\psi)| \leq \int_\Omega |f\psi|dx \leq &\ \Big(\int_\Omega |f(x)|^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |\psi(x)|^2dx\Big)^{\frac{1}{2}}\\ = &\ ||f||_{L^2(\Omega)}||\psi||_{L^2(\Omega)}\\ |L_f(\psi)| \leq &\ ||f||_{H^1}||\psi||_{H^1} \end{aligned}$$

It follows that for each $f\in L^2(\Omega)$,L_f is bounded.
Hence by Lax-Milgram theorem, for each $f\in L^2(\Omega)$, problem(3.2.7) is assured of a unique solution say ϕ_f. We define the solution operator

$$D(\triangle) := \{u\in L_2(\Omega) : \triangle u\in L_2(\Omega)\}$$

$$\begin{aligned} \triangle^{-1} : L^2(\Omega) &\longrightarrow D(\triangle) \quad \text{defined by}\\ f &\mapsto \triangle^{-1}(f)=\phi_f, \quad \text{where} \quad \phi_f \quad \text{solves} \quad (3.2.7) \end{aligned} \tag{3.2.8}$$

claim. *i)*$\triangle^{-1}$ is positive.

Proof. Let $f\in L^2(\Omega)$ and $\triangle^{-1}f=\phi_f$ such that ϕ_f is a solution of (3.2.7)
$\langle\triangle^{-1}f,f\rangle=-\langle\phi_f,\triangle\phi_f\rangle=\langle\nabla\phi_f,\nabla\phi_f\rangle\geq 0$ □

claim. *ii)*$\triangle^{-1}$ is self-adjoint.

Proof. Let $f,g\in L^2(\Omega)$ and $\triangle^{-1}f=\phi_f$, $\triangle^{-1}g=\psi_g$

$$\begin{aligned} \langle\triangle^{-1}f,g\rangle = &\ \langle\phi_f,-\triangle\psi_g\rangle\\ = &\ \langle\nabla\phi_f,\nabla\psi_g\rangle\\ = &\ -\langle\triangle\phi_f,\psi_g\rangle\\ \langle\triangle^{-1}f,g\rangle = &\ \langle f,\triangle^{-1}g\rangle \end{aligned}$$

□

claim. *iii)*$\triangle^{-1}\in\mathcal{B}(L^2(\Omega),H^1_0)$

Proof.
Obviously, $\triangle^{-1}$ is linear.

$$B[\phi_f, \phi_f] = \int_\Omega f\phi_f dx, \qquad \phi_f \in H_0^1(\Omega)$$

Using the H^1-ellipticity of B, we have that

$$\beta||\phi_f||_{H^1}^2 \leq B[\phi_f, \phi_f] = \int_\Omega f\phi_f dx \qquad \phi_f \in H_0^1(\Omega)$$

$$\begin{aligned} \Rightarrow \beta||\phi_f||_{H^1}^2 \leq & \quad |f||_{L^2(\Omega)}||\phi_f||_{H^1} \\ \Rightarrow ||\phi_f||_{H^1} \leq & \quad \tfrac{1}{\beta}||f||_{L(\Omega)} \\ \Rightarrow ||\triangle^{-1}||_{H^1} \leq & \quad \gamma, \quad \text{with} \quad \gamma = \tfrac{1}{\beta} > 0 \end{aligned}$$

Hence $\triangle^{-1} \in \mathcal{B}(L^2(\Omega), H_0^1)$ □

The compactness of $\triangle^{-1}$ follows from theorem (3.2.8).
As a consequence of the spectral theorem, there exists a Hilbert basis $\{\phi_n\}$ of $L^2(\Omega)$, and a positive sequence $\{\nu_n\}$ converging to zero such that for every n,

$$\triangle^{-1}\phi_n = \nu_n\phi_n \tag{3.2.9}$$

Observe that $\nu_n \neq 0$ for all n, otherwise there exist $n_0 \in \mathbb{N}$ such that ϕ_{n_0} is zero, which is a contradiction. It follows from (3.2.7) and (3.2.8) that there exists an orthonormal basis $\{\phi_n\}$ for $H_0^1(\Omega)$ such that

$$-\triangle\phi_n = \lambda_n\phi_n \quad \text{with} \quad \lambda_n = \frac{1}{\nu_n} \tag{3.2.10}$$

Remark. The eigenvalues of $-\triangle$ are positive and form a sequence converging to $+\infty$

Example 3.2.3. *Let* $\Omega =]0, \pi[\subset \mathbb{R}$,

$$\text{Dom}(L) = \left\{u \in H_0^1(]0, \pi[) : \; u'' \in L^2(]0, \pi[)\right\}$$

and $Lu = -u_{xx}$.
Given $f \in L^2(]0, \pi[)$, *consider the elliptic boundary value problem*

$$\begin{cases} -u_{xx} &= f \quad \text{in }]0, \pi[\\ u(0) &= 0 \\ u(1) &= 0. \end{cases}$$

First step, we compute the eigenfunctions of L. By solving the boundary value problem.

$$-u_{xx} = \mu u, \qquad u(0) = u(\pi) = 0$$

The eigenvalues and the normalized eigenfunctions are given by

$$\mu_k = k^2 \quad \text{and} \quad \phi_k(x) \ = \ \sqrt{\frac{2}{\pi}}\,\sin(kx), \quad k \in \mathbb{N}\setminus\{0\}\,.$$

Of course, the inverse operator Ł$^{-1}$ has the same eigenfunctions $\phi_k(x)$, with eigenvalues $\lambda_k = \frac{1}{k^2}$. By spectral theorem

$$\begin{aligned} u(x) = L^{-1}f = & \sum_{k=1}^{\infty} \lambda_k \langle f, \phi_k\rangle_{L^2}\phi_k \\ = & \sum_{k=}^{\infty} \frac{1}{k^2}\Big(\int_0^{\pi} f(y)\sqrt{\frac{2}{\pi}}\sin(ky)\,dy\Big)\sqrt{\frac{2}{\pi}}\sin(kx) \\ = & \sum_{k=1}^{\infty} \frac{2}{\pi k^2}\Big(\int_0^{\pi} f(y)\sin(ky)\,dy\Big)\sin(kx). \end{aligned}$$

In conclussion we have presented linear compact operators on arbitrary Banach spaces, investigated the spectral properties of compact operators, shown that neither compactness nor self-adjointness guarantees the existence of basic system for a bounded linear operator on an infinite dimensional Hilbert space. We have also shown that the spectral decomposition theorem could be extended to justify the existence of basic system for unbounded differential operators, specially those which are self-adjoint and have compact inverses.

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