Characteristic Inequalities in Banach Spaces and Applications

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A THESIS APPROVED

BY

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Dedication

To my beloved parents:
Abdulrashid Muhammad,
Amina Dalhatu, and Salima Yusuf.

To my fiancee:
Fatima Yusuf Sadah

Preface

The contribution of this project falls within the general area of nonlinear functional analysis and applications. We focus on an important topic within this area: *Inequalities in Banach spaces and applications.*

As is well known, among all infinite dimensional Banach spaces, Hilbert spaces generally have simple geometric structures. This makes problems posed in them easier to handle, this is as a result of the existence of inner product, the proximity map, and the two characteristic identities which we state below.

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + 2\langle y, x\rangle + \|y\|^2, && (1)\\ \|\lambda x + (1-\lambda)y\|^2 &= \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, && (2)\end{aligned}$$

for any $x, y \in H$, and $\lambda \in (0,1)$ where H is a real Hilbert space. These are some of the properties which characterize inner product space and also make certain problems posed in Hilbert spaces more manageable to handle than those in the general Banach spaces.

Another important tool which characterizes Hilbert spaces is the fact that the proximity map P_K from a Hilbert space H onto a nonempty, closed, convex subset K of H is nonexpansive, i.e., if $P_K : H \to K$ is defined by $P_K x = z$, where $\|x - z\| = \inf\limits_{u\in K}\|x-u\|$, then

$$\|P_K u - P_K v\| \le \|u - v\| \ \forall u, v \in H.$$

This property of P_K which is central in solving numerous problems in Hilbert spaces, does not hold in all Banach spaces more general than Hilbert spaces.

However, in applications, many problems do not naturally live in Hilbert spaces, therefore to extend some of the Hilbert space techniques to more general Banach spaces, analogue of the inner product, the proximity map, and the two identities (1) and (2) have to be developed.

For this development, the duality mapping

$$J : X \to 2^{X^*}$$

of a Banach space X defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2, \quad \|x\| = \|x^*\|\},$$

which we shall see in the next chapter, has become one of the most important tool in nonlinear functional analysis. It serves as the replacement for inner product in a Banach space more general than Hilbert space. Sunny nonexpansive retractions in Banach spaces, when they exist, generalize the so called proximity map which exists in Hilbert spaces as we shall discuss in chapter three of this work. We study inequalities obtained in various Banach spaces more general than Hilbert spaces as analogues of (1) and (2) and their applications.

Lastly, we constructed explicitly the sunny nonexpansive retraction in certain Banach space. As application of sunny nonexpansive retraction, we approximate a fixed point of nonself nonexpansive map.

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Contents

CHAPTER 1

Preliminaries

1.1 Basic notions of functional analysis

In this chapter, we recall some definitions and results from linear functional analysis.

Proposition 1.1.1 *(The Parallelogram Law) Let X be an inner product space. Then for arbitrary $x, y \in X$,*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Theorem 1.1.1 *(The Riesz Representation Theorem) Let H be a Hilbert space and let f be a bounded linear functional on H. Then there exists a unique vector $y_0 \in H$ such that*

$$f(x) = \langle x, y_0 \rangle \text{ for each } x \in H \text{ and } \|y_0\| = \|f\|.$$

Theorem 1.1.2 *Let X be a reflexive and strictly convex Banach space, K be a nonempty, closed, and convex subset of X. Then for any fixed $x \in X$ there exists a unique $m^* \in K$ such that*

$$\|x - m^*\| = \inf_{k \in K} \|x - k\|.$$

Proof. Let $x \in X$ be fixed, and define $P_x : X \to \mathbb{R} \cup \{\infty\}$ by

$$P_x(k) = \begin{cases} \|x-k\|, & \text{if } k \in K, \\ \infty, & \text{if } k \notin K. \end{cases}$$

Clearly P_x is convex. Indeed, let $\lambda \in (0,1)$ and $k_1, k_2 \in X$. If any of k_1 or k_2 is not in K, then $P_x(\lambda k_1 + (1-\lambda)k_2) \leq \lambda P_x(k_1) + (1-\lambda)P_x(k_2)$ since the right handside is ∞. Now suppose both elements are in K. Then

$$\begin{aligned} P_x(\lambda k_1 + (1-\lambda)k_2) &= \|\lambda(k_1 - x) + (1-\lambda)(k_2 - x\| \\ &\leq \|\lambda(k_1 - x)\| + \|(1-\lambda)(k_2 - x\| \\ &= \lambda P_x(k_1) + (1-\lambda)P_x(k_2). \end{aligned}$$

We next show that P_x is lower semicontinuous. By the continuity of the map

$$k \mapsto \|x - k\|, \quad k \in K,$$

we have P_x is lower semicontinuous on K. We now show P_x is lower semicontinuous on K^c. Let $x_0 \in K^c$ and $\alpha \in \mathbb{R}$ such that $\alpha < P_x(x_0)$. Since K is closed, K^c is an open neighbourhood of x_0 and $\alpha < P_x(y)\ \forall y \in K^c$. Hence P_x is lower semicontinuous on K^c and therefore on the whole X. Obviously P_x is proper. Next, we show that P_x is coercive. Let $y \in X$. Then

$$\begin{aligned} P_x(y) &\geq \|y\| - \|x\| \\ &\geq \|y\| - \frac{\|y\|}{2} = \frac{\|y\|}{2} \quad \text{provided} \quad \|y\| \geq 2\|x\|. \end{aligned}$$

This implies that $P_x(y) \to \infty$ as $\|y\| \to \infty$. Thus, P_x is lower semicontinuous, convex, proper, and coersive. Hence there exists $m^* \in X$ such that

$$P_x(m^*) \leq P_x(m)\ \forall m \in X.$$

Since $P_x(y) = \infty$ for all $y \in K^c$ and $K \neq \emptyset$, we must have $m^* \in K$. Furthermore $\|x - m^*\| = P_x(m^*) \leq P_x(k) = \|x - k\|,\ \forall k \in K$. This completes the proof. We now show that $m^* \in K$ is unique. Indeed, if $x \in K$ then $m^* = x$ and hence it is unique. Suppose $x \in K^c$ and $m \neq n$ such that $\|x - m\| = \|x - n\| \leq \|x - k\|\ \forall k \in K$, then $\frac{1}{\|x - m\|}\|\frac{1}{2}((x - m) + (x - n))\| < 1$. This implies that $\|x - \frac{1}{2}(m + n)\| < \|x - m\|$ and this contradict the fact that m is a minimizing vector in K. Therefore $m^* \in K$ is unique.

Corollary 1.1.1 *Let X be a uniformly convex Banach space and K be any nonempty, closed and convex subset of X. Then for arbitrary $x \in X$ there exists a unique $k^* \in K$ such that*

$$\|x - k^*\| = \inf_{k \in K} \|x - k\|.$$

Remark If H is a real Hilbert space and M is any nonempty, closed, and convex subset of H then in view of the above corollary, then there exists a unique map $P_M : H \to M$ defined by $x \mapsto P_M x$, where $\|x - P_M x\| = \inf_{m \in M} \|x - m\|$. This map is called the projection map. The following properties of projection map P_M of H onto M are well known.

(1) $z = P_M x \Leftrightarrow \langle x - z, m - z\rangle \leq 0\ \forall m \in M$.

(2) $\|P_M x - P_M y\|^2 \leq \langle x - y, P_M x - P_M y\rangle\ \forall x, y \in H$, which implies that $\|P_M x - P_M y\| \leq \|x - y\|\ \forall x, y \in H$, i.e., P_M is nonexpansive.

(3) $P_M(P_M x + t(x - P_M x)) = P_M x\ \forall t \geq 0$.

1.1.1 Differentiability in Banach spaces

Let X and Y be two real normed linear spaces and U be a nonempty open subset of X.

Definition 1.1.1 *(Directional Differentiability) Let $f : U \longrightarrow Y$ be a map. Let $x_0 \in U$ and $v \in X\backslash\{0\}$. We say that f has directional derivative at x_0 in the direction of v if*

$$\lim_{t\to 0}\frac{f(x_0+tv)-f(x_0)}{t},$$

exists in the normed linear space Y. We denote by $f^{'}(x_0; v)$ to be the directional derivative of f at x_0 in the direction of v.

Example Let f be the function defined from $\mathbb{R}^2$ into $\mathbb{R}$ by

$$f(x_1,x_2) \ = \ \begin{cases} \frac{x_1x_2^2}{x_1^2+x_2^2}, & \text{if } (x_1,x_2)\neq(0,0) \\ 0, & \text{otherwise.}\end{cases}$$

Then f has directional derivative at $(0, 0)$ in any direction.
To see this, let $v = (v_1, v_2) \in \mathbb{R}^2\backslash\{0, 0\}$, $t \neq 0$; then

$$\frac{f(0+tv)-f(0)}{t}=\frac{f(tv)}{t}=\frac{v_1v_2^2}{v_1^2+v_2^2}.$$

Thus,

$$\lim_{t\to 0}\frac{f(0+tv)-f(0)}{t}=\frac{v_1v_2^2}{v_1^2+v_2^2}=f^{'}(0;v).$$

So f has directional derivative at $(0, 0)$ in any direction.

Example Let f be the function defined from $\mathbb{R}^2$ into $\mathbb{R}$ by

$$f(x_1,x_2) \ = \ \begin{cases} \frac{x_1x_2}{x_1^2+x_2^2}, & \text{if } (x_1,x_2)\neq(0,0) \\ 0, & \text{otherwise.}\end{cases}$$

Then, this function f has no directional derivative at $(0, 0)$ in any direction.
To see this, let $v = (v_1, v_2) \in \mathbb{R}^2\backslash\{0, 0\}$, and $t \neq 0$; then

$$\frac{f(0+tv)-f(0)}{t}=\frac{f(tv)}{t}=\frac{v_1v_2}{t(v_1^2+v_2^2)}$$

and so the limit does not exists in $\mathbb{R}$. Therefore, the directional derivative of the function f does not exists at $(0, 0)$ in any direction.

Definition 1.1.2 *(Gateaux Differentiability) Let $f : U \longrightarrow Y$ be a map . Let $x_0 \in U$. The function f is said to be Gâteaux Differentiable at x_0 if :*

1. *f has directional derivative at x_0 in every direction $v \in X \setminus \{0\}$ and*

2. *there exists a bounded linear map $A \in \mathcal{B}(X, Y)$ (depending on x_0) such that $f'(x_0; v) = A(v)$ for all v element of $X \setminus \{0\}$.*

In this case the map $f'(x_0, .)$ is called the Gâteaux differential of f at x_0 and is denoted by $D_G f(x_0)$ or $f'_G(x_0)$.

In other words, f is Gâteaux differentiable at x_0 if there exists a bounded linear map $A \in \mathcal{B}(X, Y)$ such that

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = A(v), \qquad \forall v \in X \setminus \{0\}$$

Remark From the above definition, it is obvious that if a function is Gâteaux differentiable at a point, then it has a directional derivative in all directions at that point.

However, the converse is not true in general. We refer to the above example, it is clear that the directional derivative of the function f exists at (0,0) in any direction but $f^{'}(0; \cdot)$ is not linear.

Definition 1.1.3 *A Banach space X is said to have Gateaux differentiable norm if the limit*

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in X$ with $\|x\| = \|y\| = 1$.

Definition 1.1.4 *A Banach space X is said to have uniformly Gateaux differentiable norm if for each $y \in X$ with $\|y\| = 1$, the limit*

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

is attained uniformly in $x \in X$ with $\|x\| = 1$.

Definition 1.1.5 *Let (M, ρ) be a metric space. A mapping $T : M \to M$ is called a contraction if there exists $k \in [0, 1)$ such that $\rho(Tx, Ty) \leq k\rho(x, y)$ for all $x, y \in M$. If $k = 1$, then T is called non-expansive.*

Theorem 1.1.3 *(Banach Contraction Mapping Principle). Let (M, ρ) be a complete metric space and $T : M \to M$ be a contraction. Then T has a unique fixed point, i.e., there exists a unique $x^* \in M$ such that $Tx^* = x^*$.*

Definition 1.1.6 *A mapping $L : l_\infty \to \mathbb{R}$ is called a Banach limit if*
(a) L is linear and continuous.
(b) $L(x) \geq 0$ if $x \geq 0$, where $x = (x_n)_n$ with $x_n \geq 0$ $\forall n$.
(c) $L(x) = L(\tau x)$; where τ denotes the shift operator defined by

$$\tau(x_1, x_2, x_3, ...) = (x_2, x_3, x_4, ...) \text{ i.e. } \tau(x_n) = (x_{n+1}).$$

(d) $L(1) = 1$

1.1.2 Duality mapping in Banach spaces

In order to find the analogue of the identities (1) and (2) in Banach spaces, we need to have a suitable replacement for the inner product. In this section, we give the definition and properties of duality mapping in an arbitrary normed space.

Definition 1.1.7 *A continuous and strictly increasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$ is called a gauge function.*

Definition 1.1.8 *Given a gauge function ϕ, the mapping $J_\phi : X \longrightarrow 2^{X^*}$ defined by*

$$J_\phi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\|x^*\|; \ \|x^*\| = \phi(\|x\|)\}$$

is called the duality map with gauge function ϕ where X is any normed space. The normalized duality map J is a particular case of J_ϕ where $\phi(t) = t, \ t \in \mathbb{R}^+$.

Lemma 1.1.1 *Let ϕ be a gauge function and*

$$\varphi(t) = \int_0^t \phi(s)ds,$$

then φ is a convex function on $\mathbb{R}^+$.

Proposition 1.1.2 *In a normed linear space X, for every gauge function ϕ, $J_\phi(x)$ is not empty for any $x \in X$.*

Proof: If $x = 0$ then trivially we have $J_\phi(x) \neq \emptyset$ by taking $x^* = 0$.
For $x \neq 0$ in X, Hahn-Banach theorem guarantee the existence of $f \in X^*$ such that $\|f\| = 1$ and $\langle x, f\rangle = \|x\|$. Take $x^* = \phi(\|x\|)f$ then clearly $\|x^*\| = \phi(\|x\|)$ and $\langle x, x^*\rangle = \phi(\|x\|)\langle x, f\rangle = \phi(\|x\|)\|x\|$. Hence the proof.

Proposition 1.1.3 *In a real Hilbert space H, the normalized duality map is the identity map.*

Proof: Since H is a Hilbert space, we identify $H = H^*$. Let $x \in H$, since $\langle x, x\rangle = \|x\|^2$, then $x \in J(x)$. If $y \in J(x)$, then $\langle x, y\rangle = \|x\|\|y\|$ and $\|x\| = \|y\|$.
So that $\|x - y\|^2 = \langle x - y, x - y\rangle = \langle x, x\rangle - \langle y, x\rangle - \langle x, y\rangle + \langle y, y\rangle = 0$. Therefore $y = x$. Hence the proof.

Proposition 1.1.4 *In a Banach space X, let J_ϕ be a duality map of gauge function ϕ. Then for every x in X, $x \neq 0$ and every λ in $\mathbb{R}$, we have*

$$J_\phi(\lambda x) = sign(\lambda)\frac{\phi(|\lambda|\|x\|)}{\phi(\|x\|)}J_\phi(x).$$

Proof: Let $x \in X$ such that $x \neq 0$ and $\lambda \in \mathbb{R}$. We need to show that

$$J_\phi(\lambda x) \subseteq sign(\lambda)\frac{\phi(|\lambda|\|x\|)}{\phi\|x\|}J_\phi(x) \ \textit{and}$$

$$sign(\lambda)\frac{\phi(|\lambda|\|x\|)}{\phi(\|x\|)}J_\phi(x) \subseteq J_\phi(\lambda x).$$

Now, let $u^* \in J_\phi(x)$. Then

$$\begin{aligned}\left\langle \lambda x, sign(\lambda)\frac{\phi(|\lambda|\|x\|)}{\phi(\|x\|)}u^*\right\rangle &= |\lambda|\frac{\phi(|\lambda|\|x\|)}{\phi(\|x\|)}\langle x, u^*\rangle \\ &= |\lambda|\|x\|\phi(|\lambda|\|x\|) \\ &= \|\lambda x\|\phi(\|\lambda x\|) \ \ and\end{aligned}$$

$\|sign(\lambda)\frac{\phi(\|\lambda x\|)}{\phi(\|x\|)}u^*\| = \phi(\|\lambda x\|)$. Thus we have, $sign(\lambda)\frac{\phi(\|\lambda x\|)}{\phi(\|x\|)}J_\phi(x) \subseteq J_\phi(\lambda x)$.
To see the other inclusion, we take $y = \lambda x$, $\mu = \frac{1}{\lambda}$ and use the above result, we get $sign(\mu)\frac{\phi(\|\mu y\|)}{\phi(\|y\|)}J_\phi(y) \subseteq J_\phi(\mu y)$. This implies $sign(\frac{1}{\lambda})\frac{\phi(\|x\|)}{\phi(|\lambda|\|x\|)}J_\phi(\lambda x) \subseteq J_\phi(x)$, ie., $J_\phi(\lambda x) \subseteq sign(\lambda)\frac{\phi|\lambda|\|x\|}{\phi\|x\|}J_\phi(x)$. Hence, we conclude that $J_\phi(\lambda x) = sign(\lambda)\frac{\phi(|\lambda|\|x\|)}{\phi(\|x\|)}J_\phi(x)$.

Corollary 1.1.2 *Let X be a real Banach space and J be the normalized duality map on X. Then $J(\lambda x) = \lambda J(x)$, $\forall \lambda \in \mathbb{R}$, $\forall x \in X$.*

Proof: For the normalized duality map, $\phi(t) = t$, $t \in \mathbb{R}^+$. It follows from the above proposition that
$J(\lambda x) = J_\phi(\lambda x) = sign(\lambda)\frac{|\lambda|\|x\|}{\|x\|}J_\phi(x) = \lambda J_\phi(x) = \lambda J(x)$.

Lemma 1.1.2 *Let X be a normed linear space and J be a normalized duality map on X. Then for any $x \in X$*

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2, \ \ \|x^*\| \le \|x\|\}.$$

Proof: For $x = 0$, the results holds trivially. For $x \neq 0$, it is immediate that $J(x) \subseteq \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2; \ \ \|x^*\| \le \|x\|\}$.
Suppose that $u^* \in X^*$ such that $\langle x, u^*\rangle = \|x\|^2$ *and* $\|u^*\| \le \|x\|$. We need to show $\|x\| \le \|u^*\|$.
$\|u^*\| = \sup\limits_{\|x\|\neq 0}\frac{\langle x,u^*\rangle}{\|x\|}$ which implies that $\|x\| = \frac{\langle x,u^*\rangle}{\|x\|} \le \|u^*\|$. Hence $\|u^*\| = \|x\|$ and so $u^* \in J(x)$.

Proposition 1.1.5 *Let X be a real Banach space and J be the duality map on X, then*
(a) For every $x \in X$, the set $J(x)$ is convex and weak* closed in X^*.
(b) J is monotone in the sense that $\langle x-y, x^*-y^*\rangle \ge 0 \ \ \forall x, y \in X$ and $x^* \in J(x), y^* \in J(y)$.

Proof:(a) Let $x \in X$; $x^*, y^* \in J(x)$ and $\lambda \in (0,1)$. We need to show that $\lambda x^* + (1-\lambda)y^* \in J(x)$.

$$\begin{aligned}\langle x, \lambda x^* + (1-\lambda)y^*\rangle &= \lambda\langle x, x^*\rangle + (1-\lambda)\langle x, y^*\rangle \\ &= \lambda\|x\|^2 + (1-\lambda)\|x\|^2 \\ &= \|x\|^2.\end{aligned}$$

Also,$\|\lambda x^* + (1-\lambda)y^*\| \leq \|\lambda x^*\| + (1-\lambda)\|y^*\| = \|x\|$. Therfore by lemma (1.1.2) $\lambda x^* + (1-\lambda)y^* \in J(x)$. Hence $J(x)$ is convex.
To show $J(x)$ is weak* closed, define for each $x \in X$, a map $\phi_x : X^* \longrightarrow \mathbb{R}$ by $\phi_x(f) = \langle x, f\rangle$. Then $J(x) = \phi_x^{-1}(\|x\|^2) \cap B^*(0, \|x\|)$ and so is $weak^*$ closed in X^* since ϕ_x is continuous if we put the weak* topology on X^*.

(b) Let $x, y \in X$ and $x^* \in J(x), y^* \in J(y)$. Then

$$\begin{aligned}\langle x-y, x^*-y^*\rangle &= \langle x, x^*\rangle - \langle x, y^*\rangle - \langle y, x^*\rangle + \langle y, y^*\rangle \\ &= \|x\|^2 + \|y\|^2 - \langle x, y^*\rangle - \langle y, x^*\rangle \\ &\geq \|x\|^2 - 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \geq 0.\end{aligned}$$

Hence, J is monotone.

1.1.3 The signum function

Definition 1.1.9 *The signum function denoted by sgn, is the function* $sgn : \mathbb{R} \to \mathbb{R}$ *defined as*

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

We now state and prove the following properties:
(a) For all $x \in \mathbb{R}$, $sgn(-x) = -sgn(x)$.
(b) For all $x \in \mathbb{R}$, $|x| = sgn(x)x$.
(c) For all $x \in \mathbb{R}$, $x \neq 0$ $\frac{d}{dx}|x| = sgn(x)$.

Proof.
(a) Let $x \in \mathbb{R}$.
Case 1. $x > 0$. In this case $sgn(x) = 1$, therefore $sgn(-x) = -1 = -sgn(x)$.
Case 2. $x < 0$. In this case $sgn(x) = -1$, therefore $sgn(-x) = 1 = -(-1) = -sgn(x)$.
Case 3. $x = 0$, then $sgn(-x) = 0 = -sgn(x)$.

(b) Let $x \in \mathbb{R}$.
If $x > 0$, then $|x| = x$ and $sgn(x) = 1$ so that $|x| = x = sgn(x)x$.
If $x < 0$, then $|x| = -x$ and $sgn(x) = -1$ so that we obtain $|x| = -x = sgn(x)x$.
If $x = 0$. Then $|x| = 0$ and $sgn(x) = 0$, so we have $|x| = 0 = sgn(x)x$.

(c) Let $x \in \mathbb{R}$ such that $x \neq 0$.
Case 1. $x > 0$. In this case $|x| = x$ and $sgn(x) = 1$. Thus $\frac{d}{dx}|x| = 1 = sgn(x)$.
Case 2. $x < 0$. In this case $|x| = -x$ and $sgn(x) = -1$. So $\frac{d}{dx}|x| = -1 = sgn(x)$.

Definition 1.1.10 *Let $C \subseteq X$ be nonempty subset of a Banach space X and $D \subseteq C$ be non-empty. A retraction Q from C to D is a mapping $Q : C \to D$ such that $Qx = x$ for $x \in D$. Q is nonexpansive if $\|Qx - Qy\| \leq \|x - y\|,\ x, y \in C$.*

Definition 1.1.11 *A retraction Q from C to D is sunny if Q satisfies the property: $Q(Qx + t(x - Qx)) = Qx$ for $x \in C$ and $t > 0$ whenever $Qx + t(x - Qx) \in C$. A retraction Q from C to D is sunny nonexpansive if Q is both sunny and nonexpansive.*

Lemma 1.1.3 [8] *Let C be a nonempty closed convex subset of a smooth Banach space E, $D \subset C$ nonempty, $j : E \to E^*$ the normalized duality mapping of E, and $Q : C \to D$ a retraction. Then the following are equivalent:*

(i) $\langle x - Qx, j(y - Qx)\rangle \leq 0$ *for all* $x \in C$ *and* $y \in D$;

(ii) Q *is both sunny and nonexpansive.*

1.1.4 Convex functions and sub-differentials

In this section, we present the basic notion of convex functions and sub-differential of a convex function.

Definition 1.1.12 *Let D be a non-empty subset of a normed linear space X. The set D is called convex if for each $x, y \in D$ and for any $\lambda \in (0, 1)$ we have $\lambda x + (1 - \lambda)y \in D$.*

Definition 1.1.13 *Let $f : X \longrightarrow \mathbb{R}$ be a map. Then $D(f) = \{x \in X : f(x) < +\infty\}$ is called the effective domain of f. The function f is proper if $D(f) \neq \emptyset$ i.e $\exists x_0 \in X : f(x_0) \in \mathbb{R}$.*

Definition 1.1.14 *Let D be a non-empty convex subset of X. Let $f : D \longrightarrow \mathbb{R} \cup \{+\infty\}$, then f is said to be convex if for any $\lambda \in (0, 1)$ and for all $x, y \in D$ we have*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Definition 1.1.15 *A convex function f on a convex domain $D \subseteq X$ is said to be uniformly convex on D if there exists a function $\mu : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\mu(t) = 0 \Leftrightarrow t = 0$ such that*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \lambda(1 - \lambda)\mu(\|x - y\|), \quad \forall \lambda \in [0, 1].$$

If whenever $\lambda = \frac{1}{2}$ then f is uniformly convex at centre on D.

Definition 1.1.16 *The sub-differential of a convex function f is a map $\partial : X \longrightarrow 2^{X^*}$ defined by*

$$\partial f(x) = \{x^* \in X^* :\ f(y) - f(x) \geq \langle y - x, x^*\rangle, \forall y \in X\}.$$

Definition 1.1.17 *For $p > 1$, let $\phi(t) = t^{p-1}$ be the gauge function. Then we define the generalized duality map $J_p : X \longrightarrow 2^{X^*}$ by*

$$J_p(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\|x^*\|;\ \|x^*\| = \phi(\|x\|) = \|x\|^{p-1}\}.$$

Observe that for $p = 2$, we have $J_p = J_2 = J$ which is the normalized duality map defined in the previous section.

Proposition 1.1.6 *For every $x \neq 0$ in a Banach space X,*

$$\partial\|x\| = \{u^* \in X^* : \langle x, u^* \rangle = \|x\| = \|u^*\|; \|u^*\| = 1\}.$$

Proof. We note that

$$\partial\|x\| = \{u^* \in X^* : \langle y - x, u^* \rangle \leq \|y\| - \|x\| \quad \forall y \in X\}.$$

Now, let $u^* \in X^* : \langle x, u^* \rangle = \|x\| = \|u^*\|, \|u^*\| = 1$. Then, for arbitrary $y \in X$ we have

$$\langle y - x, u^* \rangle = \langle y, u^* \rangle - \langle x, u^* \rangle \leq \|y\| - \|x\|.$$

This implies that $u^* \in \partial\|x\|$.
Conversely, if u^* is in $\partial\|x\|$ then

$$\langle y, u^* \rangle = \langle (y + x) - x, u^* \rangle \leq \|x + y\| - \|x\| \leq \|y\|.$$

From this we get that $\|u^*\| \leq 1$ and with $y = 0$ in the definition of $\partial\|x\|$, we have $\|x\| \leq \langle x, u^* \rangle \leq \|x\|\|u^*\|$. Therefore $\|u^*\| = 1, \|x\| = \langle x, u^* \rangle$ and the result holds.

Lemma 1.1.4 *$J_\phi(x) = \partial\psi(\|x\|)$ for each x in a Banach space X, where $\psi(\|x\|) = \int_0^{\|x\|} \phi(s)ds$.*

Theorem 1.1.4 *For $p \geqslant 1$, J_P is the sub-differential of the functional $\frac{1}{p}\|x\|^p$.*

Proof From the definition of J_p, we note that the gauge function is given by $\phi(t) = t^{p-1}, p \geqslant 1$. By the theorem above, we get

$$J_p = J_{\phi(t)}(x) = \partial \int_0^{\|x\|} \phi(t)dt = \partial \int_0^{\|x\|} t^{p-1}dt = \partial \left(\frac{1}{p}\|x\|^p\right),$$

completing the proof.

CHAPTER 2

Characteristic Inequalities

In this chapter, we present the inequalities obtained as analogues of the identities (1) and (2) in Banach space.

2.1 Uniformly convex spaces

Definition 2.1.1 *A normed linear space X is said to be uniformly convex if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $x, y \in X$ with $\|x\| = 1, \|y\| = 1$ and $\|x - y\| \geq \epsilon$, then $\|\frac{x+y}{2}\| \leqslant 1 - \delta$.*

Lemma 2.1.1 *(Clarkson's Inequality) For every $f, g \in L_p$, we have*

$$\|\frac{1}{2}(f+g)\|_p^p + \|\frac{1}{2}(f-g)\|_p^p \leqslant \frac{1}{2}\left(\|f\|_p^p + \|g\|_p^p\right), \qquad 2 \leqslant p < \infty,$$

$$\|\frac{1}{2}(f+g)\|_p^p + \|\frac{1}{2}(f-g)\|_p^p \leq \left(\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p\right)^{\frac{1}{p-1}}, \qquad 1 < p \leq 2.$$

Theorem 2.1.1 *L_p spaces, $1 < p < \infty$, are uniformly convex.*

Proof Let $\epsilon > 0$, $f, g \in L_p$, $1 < p < \infty$ such that $\|f\|_p = 1, \|g\|_p = 1$ *and* $\|f - g\|_p \geq \epsilon$. Then for $2 \leq p < \infty$, we get from Clarkson's first inequality that

$$\|\frac{1}{2}(f+g)\|_p^p \leq 1 - \|\frac{1}{2}(f-g)\|_p^p \leq 1 - 2^{-p}\epsilon^p.$$

Choose $\delta = 1 - (1 - 2^{-p}\epsilon^p)^{\frac{1}{p}}$ so that $\|\frac{1}{2}(f+g)\|_p \leq 1 - \delta$.
Hence the proof.

Theorem 2.1.2 *Let X be a uniformly convex space. Then, for any $d > 0, \epsilon > 0$ and arbitrary vectors $x, y \in X$ with $\|x\| \leq d, \|y\| \leq d, \|x - y\| \geq \epsilon$, there exists a $\delta > 0$ such that $\|\frac{1}{2}(x + y)\| \leq \left[1 - \delta(\frac{\epsilon}{d})\right] d$.*

Proof For arbitrary $x, y \in X$, let $z_1 = \frac{x}{d}, z_2 = \frac{y}{d}$ and set $\bar{\epsilon} = \frac{\epsilon}{d}$. Clearly $\bar{\epsilon} > 0$. Moreover, $\|z_1\| \leq 1, \|z_2\| \leq 1$ and $\|z_1 - z_2\| = \frac{1}{d}\|x - y\| \geq \frac{\epsilon}{d} = \bar{\epsilon}$. Now, by uniform convexity of X, we have for some $\delta = \delta(\frac{\epsilon}{d}) > 0$,

$$\|\frac{z_1 + z_2}{2}\| \leq 1 - \delta(\frac{\epsilon}{d}).$$

This implies that $\|\frac{x+y}{2}\| \leq \left[1 - \delta(\frac{\epsilon}{d})\right] d$.

Theorem 2.1.3 *Let X be a uniformly convex space and let $\alpha \in (0, 1)$ and $\epsilon > 0$. Then for any $d > 0$, if $x, y \in X$ are such that $\|x\| \leq d$ and $\|y\| \leq d, \|x - y\| \geq \epsilon$, then there exists $\delta = \delta(\frac{\epsilon}{d})$ such that*

$$\|\alpha x + (1 - \alpha)y\| \leq \left[1 - 2\delta(\frac{\epsilon}{d}) min\{\alpha, 1 - \alpha\}\right] d.$$

Proof Without loss of generality we shall assume that $\alpha \in (0, \frac{1}{2}]$, we observe that $\|\alpha x + (1-\alpha)y\| = \|\alpha(x+y) + (1-2\alpha)y\| \leq 2\alpha\|\frac{x+y}{2}\| + (1-2\alpha)\|y\| \leq 2\alpha\|\frac{x+y}{2}\| + (1-2\alpha)\|y\|$. Thus by the uniform convexity of X we have for some $\delta > 0$,

$$\begin{aligned}
\|\alpha x + (1 - \alpha)y\| &= 2\alpha\|\frac{x + y}{2}\| + (1 - 2\alpha)\|y\| \\
&= 2\alpha(1 - \delta(\frac{\epsilon}{d}))d + (1 - 2\alpha)d \\
&= (1 - 2\alpha\delta(\frac{\epsilon}{d}))d \\
&\leq (1 - 2\delta(\frac{\epsilon}{d}) min\{\alpha, 1 - \alpha\})d.
\end{aligned}$$

Example Inner product spaces are uniformly convex. In particular, $\mathbb{R}^n$ with the euclidean norm is uniformly convex.

To see this we shall apply parallelogram law which is valid in any inner product space. That is for all $x, y, \in H$, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Now let $\varepsilon \in (0, 2]$ be given, $x, y \in H$ such that $\|x\| \leq 1, \|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$. Then from the above identity we have

$$\left\|\frac{1}{2}(x + y)\right\|^2 \leq \frac{1}{4}\left[2(2) - \|x - y\|^2\right] = 1 - \left\|\frac{1}{2}(x - y)\right\|^2 \leq 1 - \frac{1}{4}\varepsilon^2.$$

So that

$$\left\|\frac{1}{2}(x + y)\right\| \leq \sqrt{1 - \frac{1}{4}\varepsilon^2}.$$

To complete the proof we choose $\delta = 1 - \sqrt{1 - \frac{1}{4}\varepsilon^2} > 0$.

2.1.1 Strictly convex spaces

Definition 2.1.2 *A normed linear space X is said to be* strictly convex *if for all $x, y \in X$ $x \neq y$, $\|x\| = \|y\| = 1$, we have*

$$\|\alpha x + (1-\alpha)y\| < 1 \text{ for all } \alpha \in (0,1).$$

Theorem 2.1.4 *Every uniformly convex space is strictly convex.*

Proof. Suppose X is uniformly convex, since $x \neq y$, set $\varepsilon = \|x - y\| > 0$ and $d = 1$. Then in view of Theorem (2.5) we see that for each $\alpha \in (0,1)$, $\|\alpha x + (1-\alpha)y\| < 1$, which gives the desired result.

Example The space l_1 is not strictly convex. To see this, take $\epsilon = 1$ and choose $x = (1, 0, 0, 0, ...)$ and $y = (0, -1, 0, 0, ...)$. Clearly $x, y \in l_1$ and $\|x\|_{l_1} = 1 = \|y\|_{l_1}, \|x - y\|_{l_1} = 2 > \epsilon$. However, $\|\frac{1}{2}(x + y)\| = 1$, showing that l_1 is not strictly convex.

Example $\mathbb{R}^n$ *with* $\|\cdot\|_1$ *is not strictly convex.* To see this we take any two vectors in the canonical basis e_1, e_2 in $\mathbb{R}^n$ and take $\lambda = \frac{1}{2}$. Clearly $\|e_1\| = \|e_2\| = 1$, $e_1 \neq e_2$ and

$$\left\|\frac{1}{2}e_1 + \frac{1}{2}e_2\right\| = \frac{1}{2}\|e_1 + e_2\| = 1.$$

Thus we have $\mathbb{R}^n$ with $\|\cdot\|_1$ is not strictly convex.

Example *The space $C[a, b]$ of all real valued continuous functions on the compact interval $[a, b]$ endowed with the "sup norm" is not strictly convex.* To see this we choose two functions such that

$$f(t) := 1 \text{ for all } t \in C[a, b], \quad g(t) := \frac{b - t}{b - a} \text{ for all } t \in C[a, b].$$

Take $\varepsilon = \frac{1}{2}$. Clearly, $f, g \in C[a, b]$, $\|f\| = \|g\| = 1$ and $\|f - g\| = 1 > \varepsilon$. But $\|\frac{1}{2}(x + y)\| = 1$. Thus, $C[a, b]$ is not strictly convex.

Theorem 2.1.5 *Let X be a real norm space and J be the duality mapping on X. If X^* is strictly convex then J is single valued.*

Proof. Assume X^* is strictly convex, and suppose that there exit $x^*, y^* \in Jx$ such that $x^* \neq y^*$. Since Jx is convex then $\lambda x^* + (1 - \lambda)y^* \in Jx$ for any $\lambda \in (0, 1)$, this implies that $\|x\| = \|\lambda x^* + (1 - \lambda)y^*\|$. So by strict convexity of X^* we get $\|\lambda x^* + (1 - \lambda)y^*\| < \|x\|$ which is a contradiction. Therefore $x^* = y^*$ and so J is single valued.

We now give the definition of a function called the modulus of convexity of a normed linear space X.

Definition 2.1.3 *Let X be a normed space with $dim X \geq 2$. The modulus of convexity of X is a function $\delta_X : (0,2] \to [0,1]$ defined by*

$$\delta_X(\epsilon) := \inf\left\{1 - \left\|\frac{(x+y)}{2}\right\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\right\}.$$

In the particular case of an inner product space H, we have

$$\delta_H(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}.$$

Theorem 2.1.6 *For every normed space X, the function $\frac{\delta_X(\epsilon)}{\epsilon}$ is non-decreasing on (0,2].*

Theorem 2.1.7 *The modulus of convexity of a normed space X, δ_X, is a convex and continuous function.*

Theorem 2.1.8 *A normed space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0,2]$.*

Proof. If X is uniformly convex, given $\epsilon > 0$ there exists $\delta > 0$ such that $\delta \leq 1 - \|\frac{x+y}{2}\|$ for every x and y such that $\|x\| = \|y\| = 1$ and $\epsilon \leq \|x-y\|$. Therefore $\delta_X(\epsilon) > 0$. For the converse, assume $0 < \delta_X(\epsilon)$ for every $\epsilon \in (0,2]$. Fix $\epsilon \in (0,2]$ and x, y *with* $\|x\| = \|y\| = 1$ and $\epsilon \leq \|x-y\|$, then

$$0 < \delta_X(\epsilon) \leq 1 - \|\frac{x+y}{2}\|$$

and therefore $\|\frac{x+y}{2}\| \leq 1 - \delta$ with $\delta = \delta_X(\epsilon)$ which does not defend on x or y.

Corollary 2.1.1 *In a uniformly convex space X, the modulus of convexity is a strictly increasing function.*

2.1.2 Inequalities in uniformly convex spaces

In this section, we give the analogue of the identities (1) and (2) in a uniformly convex Banach space. We begin with the following definition.

Definition 2.1.4 *Let $p > 1$ be a real number. Then a normed space X is said to be p-uniformly convex if there is a constant $c > 0$ such that*

$$\delta_X(\epsilon) \geq c\epsilon^p.$$

Example Let $X = L_p$ (orl_p), $1 < p < \infty$, then

$$\delta_X(\epsilon) \geq \frac{1}{2^{p+1}}\epsilon^2 \qquad 1 < p < 2,$$

$$\delta_X(\epsilon) \geq \epsilon^p \qquad 2 \leq p < \infty.$$

Theorem 2.1.9 *Let X be a real normed space, and $J_p : X \to 2^{X^*}$, $1 < p < \infty$ be the generalized duality map. Then, for any $x, y \in X$, the following inequality holds*

$$\|x+y\|^p \leq \|x\|^p + p\langle y, j_p(x+y)\rangle$$

for all $j_p(x+y) \in J_p(x+y)$. In particular, if $p = 2$, then

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$.

Proof. Since J_p is the sub-differential of the function $\frac{1}{p}\|\cdot\|^p$, if $p > 1$. Hence by the sub-differential inequality, for all $x, y \in X$ and $j_p(x+y) \in J_p(x+y)$, we obtain that

$$\frac{1}{p}\|x\|^p - \frac{1}{p}\|x+y\|^p \geq \langle x - (x+y), j_p(x+y)\rangle,$$

so that

$$\|x+y\|^p \leq \|x\|^p + p\langle y, j_p(x+y)\rangle,$$

as required. In particular, if $p = 2$, we have from the definition of normalise duality map that

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, j(x+y)\rangle \\
&\leq \|x\|\|x+y\| + \langle y, j(x+y)\rangle \\
&\leq \frac{1}{2}\left(\|x\|^2 + \|x+y\|^2\right) + \langle y, j(x+y)\rangle,
\end{aligned}$$

so that $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$, as required.

Lemma 2.1.2 *Let X be a real Banach space. Then, $\delta_X(\epsilon) \geq c.\epsilon^p$ if and only if there exists a constant $c > 0$ such that*

$$\frac{1}{2}\left(\|x+y\|^p + \|x-y\|^p\right) \geq \|x\|^p + c\|y\|^p \quad \forall x, y \in X.$$

Proposition 2.1.1 *Let X be a Banach space. Then for some constant $c > 0$,*

$$\frac{1}{2}\left(\|x+y\|^p + \|x-y\|^p\right) \geq \|x\|^p + c\|y\|^p \quad \forall x, y \in X$$

if and only if $\|\cdot\|^p$ is uniformly convex at center on X.

The theorem below gives us analogue of the identities (1) and (2) in p-uniformly convex spaces.

Theorem 2.1.10 *Let $p > 1$ be a fixed real number. Then the functional $\|\cdot\|^p$ is uniformly convex on the Banach space X if and only if X is p-uniformly convex.*

Corollary 2.1.2 *Let $p > 1$ be a given real number. Then the following are equivalent in a Banach space X.*

(i) X is p-uniformly convex.
(ii) There is a constant $c > 0$ such that for every $x, y \in X$, $j_p(x) \in J_p(x)$, the following inequality holds:

$$\|x + y\|^p \geq \|x\|^p + p\langle y, j_p(x)\rangle + c\|y\|^p.$$

(iii) There is a constant $d > 0$ such that for every $x, y \in X$, $j_p(x) \in J_p(x), j_p(y) \in J_p(y)$, the following inequality holds:

$$\langle x - y, j_p(x) - j_p(y)\rangle \geq d\|x - y\|^p.$$

Theorem 2.1.11 *Let $p > 1$ and $r > 0$ be two fixed real number. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing and convex function*

$$g : \mathbb{R}^+ \to \mathbb{R}^+, g(0) = 0$$

such that for all $x, y \in B_r$ and $0 \leq \lambda \leq 1$,

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|),$$

where $W_p(\lambda) := \lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p$ and $B_r := \{x \in X : \|x\| \leq r\}$.

Corollary 2.1.3 *Let $p > 1$ and $r > 0$ be two fixed real numbers and X be a Banach space. Then the following are equivalent.*

(i) X is uniformly convex.
(ii) There is a continuous, strictly increasing and convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+, g(0) = 0$$

such that

$$\|x + y\|^p \geq \|x\|^p + p\langle y, j_p(x)\rangle + g(\|y\|)$$

for every $x, y \in B_r$ and $j_p(x) \in J_p(x)$.
(iii) There is a continuous, strictly increasing and convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+, g(0) = 0$$

such that

$$\langle x - y, j_p(x) - j_p(y)\rangle \geq g(\|x - y\|)$$

for every $x, y \in B_r$ and $j_p(x) \in J_p(x), j_p(y) \in J_p(y)$.

Remark Theorem (2.1.11) and corollary (2.1.3) give us the analogue of the identities (1) and (2) in a uniformly convex spaces.

2.2 Uniformly smooth spaces

Definition 2.2.1 *A normed space X is called smooth if for every $x \in X, \|x\| = 1$, there exists a unique x^* in X^* such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.*

Theorem 2.2.1 *If X^* is strictly convex then X is smooth, and if X^* is smooth then X is strictly convex.*

Proof. If X is not smooth then there exists an $x_0 \in S_X$ and two functional $x^* \neq y^* \in S_{X^*}$ with $\langle x_0, x^* \rangle = \langle x_0, y^* \rangle = 1$ but this means that

$$\|x^* + y^*\| \geq \langle x_0, x^* + y^* \rangle = 2,$$

which implies that X^* is not strictly convex.
If X is not strictly convex then there exists $x \neq y \in S_X$ such that $\|\lambda x + (1 - \lambda)y\| = 1$, for all $\lambda \in [0, 1]$. So let $x^* \in S_{X^*}$ such that

$$x^* \left(\frac{x + y}{2} \right) = 1.$$

But this implies that

$$1 = x^* \left(\frac{x + y}{2} \right) = \frac{1}{2} x^*(x) + \frac{1}{2} x^*(y) \leq \frac{1}{2} + \frac{1}{2} = 1,$$

which implies that $x^*(x) = x^*(y) = 1$, which by viewing x and y to be elements in X^{**}, implies that X^* is not smooth.

Definition 2.2.2 *Let X be a normed space with $dimX \geq 2$. The modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by*

$$\rho_X(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau \right\}.$$

Observe that $\rho_X(0) = 0$.

Proposition 2.2.1 *For every normed linear space X, the modulus of smoothness, ρ_X, is a convex and continuous function.*

Proposition 2.2.2 *Let X be a Banach space. For every $\tau > 0$, x in X, $\|x\| = 1$ and x^* in X^* with $\|x^*\| = 1$ we have*

$$\rho_{X^*}(\tau) = \sup \left\{ \frac{\tau \epsilon}{2} - \delta_X(\epsilon) : 0 < \epsilon \leq 2 \right\},$$

$$\rho_X(\tau) = \sup \left\{ \frac{\tau \epsilon}{2} - \delta_{X^*}(\epsilon) : 0 < \epsilon \leq 2 \right\}.$$

Corollary 2.2.1 *For every Banach space X, the function $\frac{\rho_X(t)}{t}$ is non-decreasing and $\rho_X(t) \leq t$.*

Definition 2.2.3 *A normed spaces X is said to be uniformly smooth whenever given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = 1$ and $\|y\| \leq \delta$, then*

$$\|x + y\| + \|x - y\| < 2 + \epsilon\|y\|.$$

Theorem 2.2.2 *A normed space X is uniformly smooth if and only if*

$$\lim_{t \to 0^+} \frac{\rho_X(t)}{t} = 0.$$

Proof. If X is uniformly smooth and $\epsilon > 0$ then, there exists $\delta > 0$ such that

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 < \frac{\epsilon}{2}\|y\|,$$

for every $x, y \in X$ such that $\|x\| = 1, \|y\| = \delta$. This implies that $\rho_X(t) < \frac{\epsilon}{2}t$ for every $t < \delta$.

Conversely, let $\epsilon > 0$, and suppose there exists $\delta > 0$ such that $\rho_X(t) < \frac{1}{2}\epsilon t$, for every $t < \delta$. Let $\|x\| = 1, \|y\| = \delta$. Then with $t = \|y\|$ we have $\|x + y\| + \|x - y\| < 2 + \epsilon\|y\|$ and the space is uniformly smooth.

Theorem 2.2.3 *Every uniformly smooth normed space X is smooth.*

Proof. Suppose that X is not smooth, then there exists an x_0 in X with $\|x_0\| = 1$ and x^*, y^* in X^* such that $x^* \neq y^*$, $\|x^*\| = \|y^*\| = 1$ and $\langle x_0, x^* \rangle = \|x_0\| = \langle x_0, y^* \rangle$. Let $y_0 \in X$ be such that $\|y_0\| = 1$ and $\langle y_0, x^* - y^* \rangle > 0$. For every $t > 0$ we have

$$\begin{aligned} 0 &< t\langle y_0, x^* - y^* \rangle \\ &= t\left(\langle y_0, x^* \rangle - \langle y_0, y^* \rangle\right) \\ &= \frac{\langle x_0 + ty_0, x^* \rangle + \langle x_0 - ty_0, y^* \rangle}{2} - 1 \\ &\leq \frac{\|x_0 + ty_0\| + \|x_0 - ty_0\|}{2} - 1, \end{aligned}$$

therefore, $0 < \langle y_0, x^* - y^* \rangle \leq \frac{\rho_X(t)}{t}$ for any $t > 0$ and then X is not uniformly smooth.

The theorem below gives us a duality between uniformly convex and uniformly smooth spaces.

Theorem 2.2.4 *Let X be a Banach space.*
(a) X is uniformly convex if and only if X^* is uniformly smooth.
(b) X is uniformly smooth if and only if X^* is uniformly convex.

Proof. (a)($\Rightarrow$) If X^* is not uniformly smooth, then

$$\lim_{t\to 0^+} \frac{\rho_{X^*}(\tau)}{\tau} \neq 0.$$

This means there exists $\epsilon_0 \in (0, 2]$ such that for every $\delta > 0$ we can find τ_δ with $0 < \tau_\delta < \delta$ and $\frac{\rho_{X^*}(\tau_\delta)}{\tau_\delta} \geq \epsilon_0$. In particular $\delta = \frac{1}{n}$ there exists $(\tau_n)_{n\geq 1}$ such that $0 < \tau_n < \frac{1}{n}, \tau_n \to 0$ and $\rho_{X^*}(\tau_n) > \frac{\tau_n \epsilon_0}{2}$. By the first formula of proposition (2.2.2) we have for every $n \geq 1$ there exists $\epsilon_n \in (0, 2]$ such that

$$\frac{\tau_n \epsilon_n}{2} - \delta_X(\epsilon_n) \geq \frac{\tau_n \epsilon_0}{2}$$

which implies

$$0 < \delta_X(\epsilon_n) \leq \frac{\tau_n}{2}(\epsilon_n - \epsilon_0),$$

thus, we get $\epsilon_0 < \epsilon_n$ and $\delta_X(\epsilon_n) \to 0$. Given the fact that δ_X is a non-decreasing function we have $\delta_X(\epsilon_0) \leq \delta_X(\epsilon_n) \to 0$. Therefore X is not uniformly convex which is a contradiction.

($\Leftarrow$) Suppose X is not uniformly convex, then there exist $\epsilon_0 \in (0, 2]$ with $\delta_X(\epsilon_0) = 0$, and by the first formula of proposition (2.2.2) we obtain for every $\tau > 0$,

$$0 < \frac{\epsilon_0}{2} \leq \frac{\rho_{X^*}(\tau)}{\tau}$$

which means that X^* is not uniformly smooth.

Observe that, interchanging the roles of X and X^* in this proof, and using the second formula of proposition (2.2.2) we get the proof of part (b).

Corollary 2.2.2 *Every uniformly smooth space is reflexive.*

Proof. Suppose X is uniformly smooth space. Then by theorem 2.27, X^* is uniformly convex and therefore it's reflexive by Milman-Petti's Theorem. Thus X is reflexive.

2.2.1 Inequalities in uniformly smooth spaces

In this section, we give the analogue of the identities (1) and (2) obtained in a uniformly smooth Banach space.

Definition 2.2.4 *Let $f : X \to \overline{\mathbb{R}}$ be a convex function. Then, the conjugate of f, $f^* : X^* \to \mathbb{R}$, defined by*

$$f^*(x^*) = \sup\{\langle x, x^*\rangle - f(x) : x \in X\}.$$

Definition 2.2.5 *For $q > 1$, a Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that*

$$\rho_X(t) \leq c\epsilon^q, \quad t > 0.$$

Theorem 2.2.5 *Let $q > 1$ be a fixed real number and X be a smooth Banach space. Then X is q-uniformly smooth if and if only there exists a constant $c_q > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^q \geq \lambda\|x\|^q + (1-\lambda)\|y\|^q - W_p(\lambda)c_q\|x-y\|^q$$

for all $x, y \in X; 0 \leq \lambda \leq 1$.

Corollary 2.2.3 *Let $q > 1$ be a fixed real number and X be a smooth Banach space. Then the following statements are equivalent:*

(i) X is q-uniformly convex.
(ii) There is a constant $c > 0$ such that for every $x, y \in X$, the following inequality holds:

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c\|y\|^q.$$

(iii) There is a constant $d > 0$ such that for every $x, y \in X$, the following inequality holds:

$$\langle x-y, J_q(x) - J_q(y)\rangle \leq d\|x-y\|^q.$$

Theorem 2.2.6 *Let $q > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space X is uniformly smooth if and only if there exists a continuous, strictly increasing and convex function*

$$g^* : \mathbb{R}^+ \to \mathbb{R}^+, g^*(0) = 0$$

such that for all $x, y \in B_r$ and $0 \leq \lambda \leq 1$,

$$\|\lambda x + (1-\lambda)y\|^q \geq \lambda\|x\|^q + (1-\lambda)\|y\|^q - W_q(\lambda)g^*(\|x-y\|),$$

where $W_q(\lambda) := \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$ and $B_r := \{x \in X : \|x\| \leq r\}$.

Corollary 2.2.4 *Let $q > 1$ and $r > 0$ be two fixed real numbers and X be a smooth Banach space. Then the following are equivalent.*

(i) X is uniformly smooth.
(ii) There is a continuous, strictly increasing and convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+, g(0) = 0$$

such that

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + g(\|y\|)$$

for every $x, y \in B_r$.
(iii) There is a continuous, strictly increasing and convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+, g(0) = 0$$

such that

$$\langle x-y, J_q(x) - J_q(y)\rangle \leq g(\|x-y\|)$$

for every $x, y \in B_r$.

Remark Theorem 3.15 gives us the analogues of the identities (1) and (2) in uniformly smooth spaces.

2.2.2 Characterization of uniformly smooth spaces by the duality maps

In this subsection we give a characteristics of uniformly smooth Banach spaces in terms of the normalized duality map.

Theorem 2.2.7 *If X^* is a strictly convex normed linear space, then the normalized duality map is single valued.*

Proof. *Assume X^* is strictly convex, and suppose that there exit $x^*, y^* \in Jx$ such that $x^* \neq y^*$, then $\|x^*\| = \|y^*\| = \|x\|$ and by the strict convexity of X we have that for any $\lambda \in (0,1)$, $\|\lambda x^* + (1-\lambda)y^*\| < \|x\|$. In particular taking $\lambda = \frac{1}{2}$, we have $\|\frac{1}{2}(x^* + y^*)\| < \|x\|$, which contradicts the fact that $\|\frac{1}{2}(x^* + y^*)\| = \|x\|$ (since Jx is convex).*

Theorem 2.2.8 *Let X be a real uniformly smooth Banach space. Then*
(i)$J : X \to X^*$ is single-valued.
(ii)j is norm-to-norm uniformly continuous on bounded subsets of X.

Corollary 2.2.5 *Let $E = L_p$ $(1 < p < \infty)$, $p \neq 2$. Then the normalize duality map $j : E \to E^*$ is single valued and norm-to-norm uniformly continuous on bounded subsets of E.*

Problem. Let $A : L_p(\Omega) \to L_p^*(\Omega) = L_q(\Omega)$, $1 < p < \infty$, $\mu(\Omega) < \infty$. Suppose $\langle Ax - Ay, x - y\rangle \geq 0$. It is known that if $q < p$, then $L_p(\Omega) \subset L_q(\Omega)$. Consider the restriction $A' : L_p(\Omega) \to L_p(\Omega)$ of A (assuming it is well defined). The question of interest is whether we can relate $\langle A'x - A'y, j(x-y)\rangle$ $\forall x, y \in L_p(\Omega)$ to $\langle A'x - A'y, x - y\rangle$ $\forall x, y \in L_p(\Omega)$?

CHAPTER 3

Sunny Nonexpansive Retraction

3.1 Construction of sunny nonexpansive retraction in Banach spaces

It is known that in a Hilbert space H, the proximity map P_K from H onto a nonempty, closed, and convex subset K of H is nonexpansive. This property of P_K which is central in the methods of solving numerous problems in Hilbert spaces, does not hold in all Banach spaces more general than Hilbert space. In this section we constructed the sunny nonexpansive retraction in a Banach space more general than Hilbert.

Lemma 3.1.1 [7] *Let α be a real number and let $a = (a_1, a_2, ...) \in l_\infty$ be such that*

$$LIM(a) \leq \alpha,$$

for all Banach limits LIM, and

$$\limsup_{n\to\infty}(a_{n+1} - a_n) \leq \alpha.$$

Then

$$\limsup_{n\to\infty} a_n \leq \alpha.$$

Before we state our main theorem, let us make some assumptions on the space, mappings, and parameters which we are going to consider in the theorem.

Let X be a Banach space, C be a non-empty, closed and convex subset of X, G an unbounded subset of $\mathbb{R}_+$ such that

$$t + h \in G \quad \forall t, h \in G,$$

$$t - h \in G \quad \forall t, h \in G \quad \text{with} \quad t \geq h, \tag{3.1}$$

and $\Gamma = \{T_t : \; t \in G\}$ a family of non-expansive self-mappings of C such that the set F of the common fixed points of Γ is non-empty.

Assumptions on the space. X is a Reflexive Banach space with a uniformly Gateaux differentiable norm such that each non-empty, bounded, closed, and convex subset K of X has the common fixed point property for nonexpansive mappings; that is, any family of commuting non-expansive self-mappings of K has a common fixed point.

Assumption on the mappings. Γ is a uniformly asymptotically regular semi-group on bounded subsets of C, that is

$$T_{s+t}x = T_s T_t x \tag{3.2}$$

for all $t, s \in G$, $x \in C$, and for all bounded subsets K of C there holds

$$\lim_{r \to \infty} \sup_K \|T_s T_r x - T_r x\| = 0, \tag{3.3}$$

uniformly for all $s \in G$.

Assumption on the parameters. $(\lambda_n)_n \subseteq [0, 1)$ is a sequence of numbers with the following properties:

$$\lambda_n \to 0, \tag{3.4}$$

$$\prod_{n=0}^{\infty}(1 - \lambda_n) = 0; \quad \textit{equivalently}, \sum_{n=0}^{\infty} \lambda_n = \infty, \tag{3.5}$$

$$\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty. \tag{3.6}$$

Given points $f \in F$, $u, x_0 \in C$, the set $D = \{x \in C : \; \|x - f\| \leq \max(\|x_0 - f\|, \|u - f\|)\}$ is bounded. From the fact that G is unbounded and using (3.3), there exists a sequence $(r_n)_n \subseteq G$ such that

$$r_0 < r_1 < r_2 < ... < r_n < ..., \; \lim_{n \to \infty} r_n = \infty, \tag{3.7}$$

$$\sup_D \|T_s T_{r_n} x - T_{r_n}\| < \frac{1}{2^n}, \; \forall s \in G.$$

This implies that

$$\sum_{n=0}^{\infty} \sup_D \|T_s T_{r_n} x - T_{r_n}\| < \infty, \tag{3.8}$$

uniformly for all $s \in G$. We now define the sequence $(x_n)_n$ by

$$x_{n+1} = \lambda_n u + (1 - \lambda_n) T_{r_n} x_n, \tag{3.9}$$

where $n \geq 0$.

We now state our main theorem of this section which gives us the sunny non-expansive retraction mapping in Banach space.

Theorem 3.1.1 *If the above assumptions on the space, mappings, and parameters hold, Then the sequence generated by (3.9) converges in norm to Qu, where Q is the unique sunny non-expansive retraction from C into F.*

Proof. We first prove the result for the special case $x_0 = u$ and then extend it to the general case. We divide the proof into a sequence of separate claims.

Claim 1. For all $n \geq 0$ and every $f \in F$,

$$\|x_n - f\| \leq \|u - f\|. \tag{3.10}$$

We proceed by induction on n. Fix $f \in F$. Clearly for $n = 0$ the result holds since $x_0 = u$. If $\|x_n - f\| \leq \|u - f\|$, then

$$\begin{aligned} \|x_{n+1} - f\| &= \|\lambda_n u + (1-\lambda_n)T_{r_n}x_n - \lambda_n f - (1-\lambda_n)f\| \\ &= \|\lambda_n(u - f) + (1-\lambda_))(T_{r_n}x_n - f)\| \\ &\leq \lambda_n\|u - f\| + (1-\lambda_n)\|x_n - f\| \\ &\leq \|u - f\|. \end{aligned}$$

Hence $(x_n)_{n\geq 0}$ is bounded.

Claim 2. The following strong convergence holds:

$$x_{n+1} - T_{r_n}x_n \to 0. \tag{3.11}$$

This is true since $(x_n)_n$ is bounded from (3.10). Also $\{T_{r_n}x_n\}$ is bounded. Indeed for $f \in F$ we have,

$$\begin{aligned} \|T_{r_n}x_n - f\| &= \|T_{r_n}x_n - T_{r_n}f\| \\ &\leq \|x_n - f\|, \end{aligned}$$

which implies that $\{T_{r_n}x_n\}$ is bounded since (x_n) is.
Thus,

$$\begin{aligned} \|x_{n+1} - T_{r_n}x_n\| &= \|\lambda_n u + (1-\lambda_n)T_{r_n}x_n - T_{r_n}x_n\| \\ &= \|\lambda_n(u - T_{r_n}x_n)\| \\ &= \lambda_n\|u - T_{r_n}x_n\| \to 0 \ as\ n \to \infty. \end{aligned}$$

Claim 3. The differences of the consecutive iterates strongly converges to zero,

$$x_{n+1} - x_n \to 0. \tag{3.12}$$

Indeed, it follows from (3.10) that $x_n \in D$ for all $n \geq 0$. By the boundedness of (x_n) and $\{T_{r_n}x_n\}$ there exists some constant $L \geq 0$ such that $\|x_{n+1} - x_n\| \leq L$ and $\|u - T_{r_n}x_n\| \leq L$

for all $n \geq 0$. Therefore, for any $n \geq 1$ we get

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(\lambda_n - \lambda_{n-1})u + (T_{r_n}x_n - T_{r_{n-1}}x_{n-1}) - \lambda_n T_{r_n}x_n + \lambda_{n-1}T_{r_{n-1}}x_{n-1}\| \\
&= \|(\lambda_n - \lambda_{n-1})u + (T_{r_n}x_n - T_{r_{n-1}}x_{n-1})(1-\lambda_n) - \lambda_n T_{r_{n-1}}x_{n-1} \\
&\quad +\lambda_{n-1}T_{r_{n-1}}x_{n-1}\| \\
&= \|(\lambda_n - \lambda_{n-1})(u - T_{r_{n-1}}x_{n-1}) + (1-\lambda_n)(T_{r_n}x_n - T_{r_{n-1}}x_{n-1})\| \\
&= \|(\lambda_n - \lambda_{n-1})(u - T_{r_{n-1}}x_{n-1}) + (1-\lambda_n)(T_{r_n}x_n - T_{r_n}x_{n-1} + T_{r_n}x_{n-1} \\
&\quad -T_{r_{n-1}}x_{n-1})\| \\
&\leq \|(\lambda_n - \lambda_{n-1})(u - T_{r_{n-1}}x_{n-1})\| + \|(1-\lambda_n)(T_{r_n}x_n - T_{r_n}x_{n-1})\| \\
&\quad +\|(1-\lambda_n)(T_{r_n}x_{n-1} - T_{r_{n-1}}x_{n-1})\| \\
&\leq |\lambda_n - \lambda_{n-1}|\|u - T_{r_{n-1}}x_{n-1}\| + (1-\lambda_n)\|x_n - x_{n-1}\| + \|T_{r_n}x_{n-1}T_{r_{n-1}}x_{n-1}\| \\
&\leq |\lambda_n - \lambda_{n-1}|L + (1-\lambda_n)\|x_n - x_{n-1}\| + \|T_{r_n}x_{n-1}T_{r_{n-1}}x_{n-1}\|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq |\lambda_n - \lambda_{n-1}|L + (1-\lambda_n)\|x_n - x_{n-1}\| \\
&+ \|T_{r_n - r_{n-1}}T_{r_{n-1}}x_{n-1} - T_{r_{n-1}}\|
\end{aligned}$$

for all $n \geq 1$. Hence, inductively,

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq L\sum_{k=m}^{n}|\lambda_k - \lambda_{k-1}| + \|x_m - x_{m-1}\|\prod_{k=m}^{n}(1-\lambda_k) \\
&\quad +\sum_{k=m}^{n}\sup_D\|T_{r_k - r_{k-1}}T_{r_{k-1}}x_{k-1} - T_{r_{k-1}}x_{k-1}\|,
\end{aligned}$$

for all $n \geq m \geq 1$. Letting n tends to ∞ we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\|x_{n+1} - x_n\| &\leq L\sum_{k=m}^{\infty}|\lambda_k - \lambda_{k-1}| + L\prod_{k=m}^{\infty}(1-\lambda_k) \\
&\quad +\sum_{k=m}^{\infty}\sup_D\|T_{r_k - r_{k-1}}T_{r_{k-1}}x - T_{r_{k-1}}x\|
\end{aligned}$$

by (3.5). On the other hand, condition and (3.8) implies that

$$\lim_{m\to\infty}\sum_{k=m}^{\infty}|\lambda_k - \lambda_{k-1}| = 0,$$

$$\lim_{m\to\infty}\sum_{k=m}^{\infty}\sup_D\|T_{r_k - r_{k-1}}T_{r_{k-1}}x - T_{r_{k-1}}x\| = 0.$$

Altogether, by letting m tends to ∞, we have

$$\lim_{m\to\infty}\limsup_{n\to\infty}\|x_{n+1} - x_n\| \leq 0,$$

which implies that $\limsup_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and so $x_{n+1}-x_n \to \infty$ as $n\to\infty$.

Claim 4. For each fixed $s \in G$,

$$T_s x_n - x_n \to 0. \tag{3.13}$$

Indeed, let $s \in G$. Then

$$\begin{aligned} \|T_s x_n - x_n\| &= \|T_s x_n - T_s T_{r_n} x_n + T_s T_{r_n} x_n - T_{r_n} x_n + T_{r_n} x_n - x_n\| \\ &\leq \|T_s x_n - T_s T_{r_n} x_n\| + \|T_s T_{r_n} x_n - T_{r_n} x_n\| + \|T_{r_n} - x_n\| \\ &\leq 2(\|x_n - x_{n+1}\| + \|x_{n+1} - T_{r_n} x_n\|) + \sup_D \|T_s T_{r_n} x - T_{r_n} x\|. \end{aligned}$$

It follows from (3.3), (3.11) and (3.12) that $T_s x_n - x_n \to 0$, as asserted.

Let LIM be a Banach limit and let $\{\alpha_s\}_{s\in G}$ be a net in the interval $(0,1)$ such that $\lim_{s\to\infty}\alpha_s = 0$. By Banach's fixed point theorem, for each $s \in G$, there exists a unique point $z_s \in C$ satisfying the equation $z_s = \alpha_s u + (1-\alpha_s)T_s z_s$.

Claim 5.

$$z_s \to Qu, \tag{3.14}$$

where $Q : C \to F$ is the unique sunny non-expansive retraction from C onto F. Indeed, let $\{s_n\}$ be a subsequence of G such that $\lim_{n\to\infty} s_n = \infty$. We can easily check that $\{z_s\}_s \in G$ is bounded and $\|z_s - T_s z_s\| \to 0$. To see this,

$$\begin{aligned} \|z_s - T_s z_s\| &= \|\alpha_s(u - T_s z_s)\| \\ &= \alpha_s \|u - T_s z_s\| \to 0 \; as \; s \to \infty. \end{aligned}$$

Therefore, we can define a functional g on C by

$$g(x) = LIM\left(\{\|z_{s_n} - x\|^2\}\right). \tag{3.15}$$

We can easily check that g is convex and continuous. Indeed, for $u, v \in C$ and $\lambda \in (0,1)$ we have

$$\begin{aligned} g(\lambda u + (1-\lambda)v) &= LIM\left(\{\|z_n - \lambda u - (1-\lambda)v\|^2\}\right) \\ &\leq LIM\left(\{\lambda\|z_n - u\|^2 + (1-\lambda)\|z_n - v\|^2\}\right) \\ &= \lambda LIM\left(\{\|z_n - u\|^2\}\right) + (1-\lambda)LIM\left(\{\|z_n - v\|^2\}\right) \\ &= \lambda g(u) + (1-\lambda)g(v), \end{aligned}$$

showing the result for convexity. To show continuity of g, we can write g as a composition of two functions f and h, that is, $g = f \circ h$ where $h : C \to l_\infty$ and $f : l_\infty \to \mathbb{R}$ defined, respectively, by $h(x) = \{\|z_n - x\|^2\}$ and $f(x) = LIM(x)$. It's clear that f is continuous. We now show h is also continuous. Indeed, for any $x \in C$ and given $\varepsilon > 0$ we have

$$\begin{aligned} \|h(x) - h(x_0)\|_\infty &= \|\left(\|z_n - x\|^2 - \|z_n - x_0\|^2\right)_n\|_\infty \\ &= \sup_n\left[(\|z_n - x\| - \|z_n - x_0\|)(\|z_n - x\| + \|z_n - x_0\|)\right] \\ &\leq \sup_n\left(\|x - x_0\|(M + \|x\|)\right) \quad \text{for some constant M depending on } x_0. \end{aligned}$$

If $\|x - x_0\| < 1$, then $\|h(x) - h(x_0)\| \leq N\|x - x_0\|$ where $N = M + 1 + \|x_0\| > 0$. Choose $\delta = \frac{\varepsilon}{N+\varepsilon}$, so that $\|h(x) - h(x_0)\| < \varepsilon$, therefore h is continuous. Hence g is a composition of two continuous function which is also continuous.

If we define $\widetilde{g} : X \to \mathbb{R} \cup \{\infty\}$ by

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{if } x \in C \\ +\infty & \text{if } x \in C^c \end{cases}$$

then by similar arguments to those given in the proof of Theorem (1.1.2), we have that the set K define as

$$K := \{x \in C : \ g(x) \leq g(y) \ \forall y \in C\}$$

is not empty.
We also have for all $x \in C$, for each $r \in G$,

$$\begin{aligned} g(T_r x) &= LIM\left(\{\|z_{s_n} - T_r x\|^2\}\right) = LIM\left(\{\|T_r T_{s_n} z_{s_n} - T_r x\|^2\}\right) \\ &\leq LIM\left(\{\|T_{s_n} z_{s_n} - x\|^2\}\right) \\ &= LIM\left(\{\|z_{s_n} - x\|^2\}\right), \end{aligned}$$

by (3.3). In other words, $g(T_r x) \leq g(x)$ for each $r \in G$ and $x \in C$. It follows that K is non-empty, closed, bounded, and convex subset of C with the property that K is invariant under each T_r, i.e., $T_r(K) \subset K$, $r \in G$. Hence K contains a common fixed point of Γ. Let $q \in K \cap F$ be such a common fixed point. Since q is a minimizer of g over C, it follows that for each $x \in C$,

$$\begin{aligned} 0 &\leq \lim_{\lambda \to 0^+} \frac{1}{\lambda}(g(q + \lambda(x - q)) - g(q)) \\ &= LIM\left(\{\lim_{\lambda \to 0^+} \frac{1}{\lambda}(\|(z_{s_n} - q) + \lambda(q - x)\|^2 - \|z_{s_n} - q\|^2)\}\right) \\ &= 2LIM\left(\{\lim_{\lambda \to 0^+} \frac{1}{2\lambda}(\|(z_{s_n} - q) + \lambda(q - x)\|^2 - \|z_{s_n} - q\|^2)\}\right) \\ &= 2LIM\left(\langle q - x, J(z_{s_n} - x)\rangle\right). \end{aligned}$$

Thus

$$LIM(\{\langle x - q, J(z_{s_n} - q)\rangle\}) \leq 0 \tag{3.16}$$

for all $x \in C$. On the other hand, for any $f \in F$,

$$z_{s_n} - f = (1 - \alpha_{s_n})(T_{s_n} z_{s_n} - f) + \alpha_{s_n}(u - f).$$

It follows that

$$\begin{aligned} \|z_{s_n} - f\|^2 &= (1 - \alpha_{s_n})\langle T_{s_n} z_{s_n} - f, J(z_{s_n} - f)\rangle + \alpha_{s_n}\langle u - f, J(z_{s_n} - f)\rangle \\ &\leq (1 - \alpha_{s_n})\|T z_{s_n} - f\|\|J(z_{s_n} - f)\| + \alpha_{s_n}\langle u - f, J(z_{s_n} - f)\rangle \\ &\leq (1 - \alpha_{s_n})\|z_{s_n} - f\|^2 + \alpha_{s_n}\langle u - f, J(z_{s_n} - f)\rangle. \end{aligned}$$

Hence

$$\|z_{s_n} - f\|^2 \leq \langle u - f, J(z_{s_n} - f)\rangle. \tag{3.17}$$

Combining (3.16) and (3.17) we get

$$LIM(\{\|z_{s_n} - q)\|^2\}) \leq 0.$$

Hence there is a subsequence $\{z_{r_j}\}$ of $\{z_{s_n}\}$ such that $\lim\limits_{j\to\infty} \|z_{r_j} - q\| = 0$. Assume that there is another subsequence $\{z_{p_k}\}$ of $\{z_{s_n}\}$ such that $\lim\limits_{k\to\infty} \|z_{p_k} - q_0\| = 0$, where $q_0 \in K \cap F$. Then (3.17) implies that

$$\|z_{r_j} - q_0\|^2 \leq \langle u - q_0, J(z_{r_j} - q_0)\rangle.$$

We can easily check that $\langle u - q_0, J(z_{r_j} - q_0)\rangle \to \langle u - q_0, J(q - q_0)\rangle$. Indeed, since $z_{r_j} - q_0 \longrightarrow q - q_0$ and the normalized duality map J is norm to weak* uniformly continuous on bounded subsets of X, then it follows that $\{J(z_{r_j} - q_0)\}$ converges to $J(q - q_0)$ with respect to weak* topology which implies $\langle u - q_0, J(z_{r_j} - q_0)\rangle \longrightarrow \langle u - q_0, J(q - q_0)\rangle$.
Hence

$$\|q - q_0\|^2 \leq \langle u - q_0, J(q - q_0)\rangle. \tag{3.18}$$

Similarly we have

$$\|q_0 - q\|^2 \leq \langle u - q, J(q_0 - q)\rangle. \tag{3.19}$$

Adding up (3.18) and (3.19) we obtain that $q = q_0$. Therefore $\{z_s\}$ converges in norm to a point in F. Now define a map $Q : C \to F$ by $Qu = \lim\limits_{s\to\infty} z_s$. Then Q is a retraction from C onto F. Indeed, for $v \in F$ and for each $s \in G$ we have a unique point $z_s \in C$ satisfying the equation $z_s = \alpha_s v + (1 - \alpha_s)T_s z_s$, but v satisfies $\alpha_s v + (1 - \alpha_s)T_s v = v$, this implies that $z_s = v$ for each $s \in G$ since z_s is unique.
Moreover, by (3.17) we get for all $f \in F$,

$$\|Qu - f\|^2 \leq \langle u - f, J(Qu - f)\rangle.$$

This implies that

$$\langle u - Qu, J(f - Qu)\rangle \leq 0$$

for all $f \in F$. Therefore by lemma (1.1.3) Q is the unique sunny non-expansive retraction from C onto F.

Claim 6.

$$\limsup_{n\to\infty} \langle u - Qu, J(x_n - Qu)\rangle \leq 0. \tag{3.20}$$

Indeed, since T_s is non-expansive, (3.13) implies that

$$\begin{aligned}
LIM(\{\|x_n - T_s z_s\|^2\}) &= LIM(\{\|x_n - T_s x_n + T_s x_n - T_s z_s\|^2\}) \\
&\leq LIM(\{(\|x_n - T_s x_n\| + \|T_s x_n - T_s z_s\|)^2\}) \\
&= LIM(\{\|T_s x_n - T_s z_s\|^2\}) \\
&\leq LIM(\{\|x_n - z_s\|^2\}).
\end{aligned}$$

Since $(1-\alpha_s)(x_n - T_s z_s) = (x_n - z_s) - \alpha_s(x_n - u)$, we have

$$\begin{aligned}(1-\alpha_s)^2\|x_n - T_s z_s\|^2 &\geq \|x_n - z_s\|^2 - 2\alpha_s\langle x_n - u, J(x_n - z_s)\rangle \\ &= \|x_n - z_s\|^2 - 2\alpha_s\langle x_n - z_s, J(x_n - z_s)\rangle - 2\alpha_s\langle z_s - u, J(x_n - z_s)\rangle \\ &= (1-2\alpha_s)\|x_n - z_s\|^2 + 2\alpha_s\langle u - z_s, J(x_n - z_s)\rangle.\end{aligned}$$

Therefore

$$(1-\alpha_s)^2 LIM(\{\|x_n - T_s z_s\|^2\}) \geq (1-2\alpha_s)LIM(\{\|x_n - z_s\|^2\}) + 2\alpha_s LIM(\{\langle u - z_s, J(x_n - z_s)\rangle\})$$

for each $n \geq 0$. These inequalities yield

$$\frac{\alpha_s}{2} LIM(\{\|x_n - z_s\|^2\}) \geq LIM(\{\langle u - z_s, J(x_n - z_s)\rangle\}).$$

Since

$$\begin{aligned}\langle u - z_s, J(x_n - z_s)\rangle \; - \; &\langle u - Qu, J(x_n - Qu)\rangle = \langle u - z_s - (u - Qu), J(x_n - z_s)\rangle \\ &+ \langle u - Qu, J(x_n - z_s) - J(x_n - Qu)\rangle,\end{aligned}$$

we obtain by letting s tends to ∞ that

$$\lim_{s\to\infty}\langle u - z_s, J(x_n - z_s)\rangle = \langle u - Qu, J(x_n - Qu)\rangle$$

because X has a uniformly Gateaux differentiable norm and $z_s \to Qu$. It follows that

$$LIM(\{\langle u - Qu, J(x_n - Qu)\rangle\}) \leq 0.$$

On the other hand, we have

$$\lim_{n\to\infty}|\langle u - Qu, J(x_{n+1} - Qu)\rangle - \langle u - Qu, J(x_n - Qu)\rangle| = 0$$

by (3.12). Hence we obtain by Lemma (3.1.1)

$$\limsup_{n\to\infty}\langle u - Qu, J(x_n - Qu)\rangle \leq 0,$$

as claimed. Now we can conclude the proof for the special case.

Claim 7.

$$x_n \to Qu. \tag{3.21}$$

Indeed, since

$$(x_{n+1} - Qu) = \lambda_n(u - Qu) + (1-\lambda_n)(T_{r_n}x_n - Qu),$$

we have

$$\begin{aligned}\|x_{n+1} - Qu\|^2 &\leq (1-\lambda_n)^2\|T_{r_n}x_n - Qu\|^2 + 2\lambda_n\langle u - Qu, J(x_{n+1} - Qu)\rangle \\ &\leq (1-\lambda_n)\|x_n - Qu\|^2 + 2(1-(1-\lambda_n))\langle u - Qu, J(x_{n+1} - Qu)\rangle\end{aligned}$$

for each $n \geq 0$. Let $\varepsilon > 0$ be given. By (3.20) there exists $m \geq 0$ such that

$$\langle u - Qu, J(x_n - Qu)\rangle \leq \frac{\varepsilon}{2}$$

for all $n \geq m$. Therefore

$$\|x_{n+m} - Qu\|^2 \leq (\prod_{k=m}^{n+m-1}(1-\lambda_k))\|x_m - Qu\|^2 + (1 - \prod_{k=m}^{n+m-1}(1-\lambda_k))\varepsilon$$

for all $n \geq 1$. Hence by (3.5) we get $\limsup_{n\to\infty}\|x_n - Qu\|^2 = \limsup_{n\to\infty}\|x_{n+m} - Qu\|^2 \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that (x_n) converges to Qu.

Finally, we extend the proof to the general case. Let (x_n) be the sequence generated by (3.9) with an initial point x_0 (possibly different from u) and let (y_n) be another sequence generated by (3.9) with an initial point $y_0 = u$. On the one hand, by the special case,

$$y_n \to Qu.$$

On the other hand, it follows by induction that

$$\|x_n - y_n\| \leq \|x_0 - y_0\| \prod_{k=0}^{n-1}(1-\lambda_k)$$

for all $n \geq 1$. Thus, we get $x_n - y_n \to 0$ and, altogether, we obtain

$$\|x_n - Qu\| \leq \|x_n - y_n\| + \|y_n - Qu\| \longrightarrow 0.$$

Hence

$$x_n \to Qu.$$

CHAPTER 4

An Application of Sunny Nonexpansive Retraction

As application of sunny nonexpansive retraction, we approximate a fixed point of nonself nonexpansive map. We shall need the following definition.

Definition 4.0.1 *Let K be a nonempty subset of a Banach space E. For $x \in K$, the inward set of x, $I_K x$, is defined by $I_K x = \{x + \alpha(u - x) : \ u \in K, \alpha \geq 1\}$. A mapping $T : K \to E$ is called weakly inward if $Tx \in cl[I_K(x)]$ $\forall x \in K$, where $cl[I_K(x)]$ denotes the closure of the inward set.*

Example Every self-map is trivially weakly inward.

The following result is needed in the main theorem of this section.

Lemma 4.0.2 *Let K be a nonempty, closed and convex subset of a real Banach space E possessing a uniformly Gateaux differentiable norm. Let $T : K \to E$ be weakly inward, then $F(T) = F(QT)$, where Q is a sunny nonexpansive retraction of E onto K.*

Proof. We first show that $F(QT) \subseteq F(T)$.
Let $x \in F(QT)$. Since T is weakly inward then $Tx \in cl(I_K x)$. This implies that $\exists(\lambda_n)_n \subset [1, \infty)$, $(u_n)_n \subset K$ with $Tx = \lim_{n\to\infty}(x + \lambda_n(u_n - x))$. Let $y_n = x + \lambda_n(u_n - x)$, $n \geq 1$. Since Q is sunny nonexpansive retraction then $\langle Tx - QTx, j(y - QTx)\rangle \leq 0$ $\forall y \in K$. In particular $\lambda_n\langle Tx - x, j(u_n - QTx)\rangle \leq 0$ $\forall n \geq 1$. This implies that $\langle Tx - x, j(y_n - x)\rangle \leq 0$ $\forall n \geq 1$, and since j is $\| \cdot \|$-w* continuous then we have that $\|Tx - x\|^2 = \langle Tx - x, j(Tx - x)\rangle \leq 0$. So $x = Tx$. It remains to show that $F(T) \subseteq F(QT)$. Let $x \in F(T)$. This implies that $Tx = x$ and since Q is a retraction, we have that $Q(Tx) = Tx = x$. Hence $x \in F(QT)$.

Lemma 4.0.3 *Let (a_n) be a sequence of nonnegative real numbers satisfying the following relation*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n\alpha_n + \gamma_n, \ n \geq 0,$$

where $\alpha_n \to 0,\ 0 < \alpha_n < 1,\ \sum_{n=1}^{\infty} \alpha_n = \infty$*. Suppose* $\limsup \sigma_n \le 0$*;* $\sum_{n=1}^{\infty} \gamma_n < \infty$*; then* $a_n \to 0$ *as* $n \to \infty$*.*

We now state and prove the following convergence theorem.

Theorem 4.0.2 *Let K be a nonempty closed and convex subset of a real Banach space E which has a uniformly Gateaux differentiable norm, and $T : K \to E$ be a nonexpansive mapping satisfying weakly inward condition with $F(T) \neq \emptyset$. Assume K is sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Assume that $\{z_t\}$ converges strongly to a fixed point z of QT as $t \to 0$, where $0 < t < 1$, z_t is the unique element of K which satisfies $z_t = tx + (1-t)QTz_t$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ which satisfies the following conditions: (i) $\lim\limits_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and either (iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, or (iii)* $\lim\limits_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$.*
For fixed $u, x_0 \in K$, let the sequence (x_n) be defined iteratively by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)QTx_n,\ n \ge 0. \tag{4.1}$$

Then, (x_n) converges to a fixed point of T.

Proof. Let $x^* \in F(T)$. It follows by induction that $\|x_n - x^*\| \le \max\{\|x_0 - x^*\|, \|u - x^*\|\}$, for all $n \ge 0$. Indeed, for $n = 0$ we have $\|x_0 - x^*\| \le \max\{\|x_0 - x^*\|, \|u - x^*\|\}$, which is true. Assume the result holds for $n = k$, i.e. $\|x_k - x^*\| \le \max\{\|x_0 - x^*\|, \|u - x^*\|\}$. We now show it is true for $n = k+1$. We compute as follows:

$$\begin{aligned}
\|x_{k+1} - x^*\| &\le \|\alpha_k(u - x^*) + \|(1-\alpha_k)(QTx_k - QTx^*)\| \\
&\le \alpha_k\|u - x^*\| + (1-\alpha_k)\|x_k - x^*\| \\
&\le \max\{\|x_0 - x^*\|, \|u - x^*\|\}.
\end{aligned}$$

Hence, by induction the result is true for any $n \ge 0$. This implies that (x_n) and (QTx_n) are bounded. Also, we have from (4.1) that

$$\|x_{n+1} - QTx_n\| = \alpha_n\|u - QTx_n\| \to 0 \tag{4.2}$$

since $\{\|u - QTx_n\|\}$ is bounded and $\alpha_n \to 0$. Furthermore, for some constant $M > 0$,

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(\alpha_n - \alpha_{n-1})(u - QTx_{n-1}) + (1-\alpha_n)(QTx_n - QTx_{n-1})\| \\
&\le M|\alpha_n - \alpha_{n-1}| + (1-\alpha_n)\|x_n - x_{n-1}\|.
\end{aligned}$$

We now consider two cases.

Case 1. Condition (iii)* is satisfied. Then, $\|x_{n+1} - x_n\| \le (1-\alpha_n)\|x_n - x_{n-1}\| + \sigma_n$, where $\sigma_n = \alpha_n\beta_n$ and $\beta_n = \frac{M|\alpha_n - \alpha_{n-1}|}{\alpha_n}$ so that $\sigma_n = o(\alpha_n)$.

Case 2. Condition (iii) is satisfied. Then $\|x_{n+1} - x_n\| \le (1-\alpha_n)\|x_n - x_{n-1}\| + \sigma_n$, where $\sigma_n = M|\alpha_n - \alpha_{n-1}|$ so that $\sum_{n=1}^{\infty} \sigma_n < \infty$.

In either case, we have by lemma (4.0.3) that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Combining this with (4.2), we obtain that

$$\|x_n - QTx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - QTx_n\|$$

which implies that $\|x_n - QTx_n\| \to 0$ as $n \to \infty$. For each integer $n \geq 0$, let $t_n \in (0,1)$ be such that $t_n \to 0$ and $\frac{\|x_n - QTx_n\|}{t_n} \to 0$. (Such t_n exist. Indeed, there is $N \geq 1$ such that $\|QTx_n - x_n\| \in (0,1)$, $\forall n \geq N$ since $\|QTx_n - x_n\| \to 0$. So take $t_n = \sqrt{\|QTx_n - x_n\|}$ for $n \geq N$). Let $z_{t_n} \in K$ be the unique fixed point of the contraction mapping T_{t_n} given by $T_{t_n}x = t_n u + (1-t_n)QT_{t_n}x$ $x \in K$. Then, $z_{t_n} - x_n = t_n(u - x_n) + (1-t_n)(QTz_{t_n} - x_n)$. Moreover, we have

$$\begin{aligned}
\|z_{t_n} - x_n\|^2 &\leq (1-t_n)^2\|QTz_{t_n} - x_n\|^2 + 2t_n\langle u - x_n, j(z_{t_n} - x_n)\rangle \\
&\leq (1-t_n)^2(\|QTz_{t_n} - QTx_n\| + \|QTx_n - x_n\|)^2 + 2t_n(\|z_{t_n} - x_n\|^2 \\
&\quad + \langle u - z_{t_n}, j(z_{t_n} - x_n)\rangle) \\
&\leq (1+t_n^2)\|z_{t_n} - x_n\|^2 + \|QTx_n - x_n\|(2\|z_{t_n} - x_n\| + \|QTx_n - x_n\|) \\
&\quad + 2t_n\langle u - z_{t_n}, j(z_{t_n} - x_n)\rangle
\end{aligned}$$

and hence,

$$\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \leq \frac{t_n}{2}\|z_{t_n} - x_n\|^2 + \frac{\|QTx_n - x_n\|}{t_n}(2\|z_{t_n} - x_n\| + \|QTx_n - x_n\|).$$

Since $(x_n), (z_{t_n})$, and (Tx_n) are bounded and $\frac{\|x_n - QTx_n\|}{t_n} \to 0$ as $n \to \infty$, it follows from the last inequality that

$$\limsup_{n\to\infty}\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \leq 0. \tag{4.3}$$

Moreover, we have that

$$\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle = \langle u - z, j(x_n - z)\rangle + \langle u - z, j(x_n - z_{t_n}) - j(x_n - z)\rangle + \langle z - z_{t_n}, j(x_n - z_{t_n})\rangle \tag{4.4}$$

But by hypothesis, $z_{t_n} \to z \in F(QT)$ as $n \to \infty$ and by lemma (4.0.2) we have that $QTz = z = Tz$. Thus $|\langle z - z_{t_n}, j(x_n - z_{t_n})\rangle| \leq \|z - z_{t_n}\|\|x_n - z_{t_n}\|$. This implies that $\langle z - z_{t_n}, j(x_n - z_{t_n})\rangle \to 0$ as $n \to \infty$ (since (x_n) is bounded). Also, $\langle u - z, j(x_n - z_{t_n}) - j(x_n - z)\rangle \to 0$ as $n \to \infty$ (since j is $\|\cdot\|$-w* uniformly continuous on bounded subsets of E). Therefore, we obtain from (4.3) and (4.4) that

$$\limsup_{n\to\infty}\langle u - z, j(x_n - z)\rangle \leq 0.$$

Now from (4.1) we get $x_{n+1} - z = \alpha_n(u - z) + (1-\alpha_n)(QTx_n - z)$. It follows that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)^2\|QTx_n - z\|^2 + 2\alpha_n\langle u - z, j(x_{n+1} - z)\rangle \\
&\leq (1-\alpha_n)\|x_n - z\|^2 + \sigma_n,
\end{aligned}$$

where $\sigma_n = 2\alpha_n\langle u - z, j(x_{n+1} - z)\rangle$ so that $\sigma_n \leq 2\alpha_n\|u - z\|\|x_{n+1} - z\|^2$ which implies that $\limsup_{n\to\infty}\sigma_n \leq 0$ (since (x_n) is a bounded sequence). Thus by lemma (4.0.3) we have $\|x_n - z\| \to 0$ which implies $x_n \to z \in F(T)$.
Hence the proof.

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