

**PRICING AND MODELING OF BONDS
AND INTEREST RATE DERIVATIVES**

BY

AKINADE, SAMUEL ENIOLA

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SUPERVISOR:

Professor G.O.S. EKHAGUERE

PhD, DIC (London), FAAS

Department of Mathematics,

University of Ibadan,

Ibadan, Nigeria.

MAY 2013

A THESIS APPROVED

BY

SUPERVISOR _____
Professor G.O.S. Ekhaguere

MEMBER _____
Professor C.E. Chidume

MEMBER _____
Dr. Guy Degla

Epigraph

"If you want to confirm the price of a security, use the price of another security that is as similar to it as possible. All the rest is modeling ."

Emmanuel Derman

Dedication

I dedicate this thesis work to my family members and to my late mum, Mrs Taiwo Akinade. May her gentle soul rest in peace.

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Abstract

Let $P(t, T)$ denote the price of a zero-coupon bond at initial time t with maturity T , given the stochastic interest rate $(r_t)_{t \in \mathbb{R}_+}$ and a Brownian filtration $\{\mathcal{F}_t : t \geq 0\}$. Then,

$$P(t, T) = \mathbb{E}^Q \left[e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right]$$

under some martingale (risk-neutral) measure Q . Assume the underlying interest rate process is solution to the stochastic differential equation (SDE)

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$$

where $(W_t)_{t \in \mathbb{R}}$ is the standard Brownian motion under Q , with $\mu(t, r_t)$ and $\sigma(t, r_t)$ of the form,

$$\begin{cases} \mu(t, r_t) &= a - br \\ \sigma(t, r_t) &= \sqrt{\sigma^2} \end{cases}$$

where $r(0)$, a , b and σ are positive constants.

Then, the bond pricing PDE for $P(t, T) = F(t, r_t)$ written as

$$\mu(t, r_t) \frac{\partial}{\partial x} F(t, r_t) + \frac{\partial}{\partial t} F(t, r_t) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} F(t, r_t) - r(t) F(t, r_t) = 0$$

subject to the terminal condition $F(t, r_t) = 1$ which yield the Riccati equations,

$$\begin{cases} \frac{dA(s)}{ds} &= aB(s) + \frac{\sigma^2}{2} B(s)^2 \\ \frac{dB(s)}{ds} &= -bB(s) - 1 \end{cases}$$

with solution of the PDE in analytical form as the Price for zero-coupon bond is given by,

$$P(t, T) = \exp[A(T-t) + B(T-t)r_t]$$

where,

$$A(T-t) = \left(\frac{\sigma^2 - ab}{b^3} \right) e^{-b(T-t)} + \left(-\frac{\sigma^2}{4b^3} \right) e^{-2b(T-t)} + \left(\frac{\sigma^2 - 2ab}{2b^2} \right) (T-t) + \frac{4ab - 3\sigma^2}{4b^3}$$

$$B(T-t) = \frac{1}{b} (e^{-b(T-t)} - 1)$$

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Chapter 1

Introduction

In Financial Mathematics, one of the most important areas of research where considerable developments and contributions have been recently observed is the pricing of interest rate derivatives and bonds. Interest rate derivatives are financial instruments whose payoff is based on an interest rate. Typical examples are swaps, options and Forward Rate Agreements (FRA's). The uncertainty of future interest rate movements is a serious problem which most investors (commission broker and locals) gives critical consideration to, before making financial decisions. Interest rates are used as tools for investment decisions, measurement of credit risks, valuation and pricing of bonds and interest rate derivatives. As a result of these, the need to profer solution to this problem, using probabilistic and analytical approach to predict future evolution of interest need to be established.

Mathematicians are continually challenged to real world problems, especially in finance. To this end, Mathematicians develop tools to analyze; for example, the changes in interest rates corresponding to different periods of time. The tool designed is a mathematical representation to replicate and solve a real world problem. These models are designed to produce results that are sufficiently close to reality, which are dependent on unstable real life variables. In rare situations, financial models fail as a result of uncertain changes that affect the value of these variables and cause extensive loss to financial institutions and investors, and could potentially affect the economy of a country.

Interest rates depends on several factors such as size of investments, maturity date, credit default risk, economy i.e inflation, government policies, LIBOR (London Inter Bank Offered Rate), and market imperfections. These factors are responsible

for the inconsistency of interest rates, which have been the subject of extensive research and generate lots of chaos in the financial world. To mitigate against this inconsistency, financial analysts develop an instrument to hedge this risk and speculate the future growth or decline of an investment. A financial instrument whose payoff depends on an interest rate of an investment is called interest rate derivative.

Interest rate derivatives are the most common derivatives that have been traded in the financial markets over the years. According to [17] interest rate derivatives can be divided into different classifications; such as interest rate futures and forwards, Forward Rate Agreements(FRA's), caps and floors, interest rate swaps, bond options and swaptions. Generally, investors who trade on derivatives are categorised into three groups namely: hedgers, speculators and arbitrageurs. Hedgers are risk averse traders who uses interest rate derivatives to mitigate future uncertainty and inconsistency of the market, while speculators use them to assume a market position in the future, thereby trading to make gains or huge losses when speculation fails. Arbitrageurs are traders who exploit the imperfections of the markets to take different positions, thereby making risk less profits.

Investors minimize risk of loss by spreading their investment portfolio into different sources whose returns are not correlated. Due to uncertainties in the market, investing in different portfolios of bonds, stocks, real estate and other financial securities reduces risk and provides financial security. Many investors hold bonds in their investment portfolios without knowing what a bond is and how it works. A bond is a form of loan to an entity (i.e financial institution, corporate organization, public authorities or government for a defined period of time where the lender (bond holder) receives interest payments (coupon) annually or semiannually from the (debtor) bond issuer who repay the loaned funds (Principal) at the agreed date of refund (maturity date). Bonds are categorized based on the issuer, considered into four groups: corporate bonds, government bonds (treasury), municipal bonds also called mini bonds and agency bonds.

Bonds are risk-free kind of investment compared to stock, for instance treasury bonds commonly called T-bills are credit default risk free investments, since the bonds are issued by the government, also the mini bond are free of federal or State taxes. Investing in bonds, preserve capital and yield profit with a predictable income stream from such indenture and bond can even be sold before maturity date. Al-

though bonds carry also risk, such as credit default risk, interest rate risk, liquidity risk, exchange rate risk, economic risk and market fluctuations risk. Understanding the characteristics of each kind of bond can be used to control exposure to these forms of risk.

Bonds and shares have a similar property of price fluctuation, for bonds interest rate has an inverse relationship with bond price: when bond price goes up, interest rate go down and when bond price go down, interest rate go up. Investors who trade on bonds frequently ask brokers this question:

What is the total return on a bond and the current market value of the bond ?

For example, an investor who buys a bond from a secondary market at a discount (price below the bond's price) and collects coupons on same bond and at the maturity date, would collect same par value of the bond, but while holding the bond before the maturity date, suppose the interest rate of same bond in the market increases, which result to depreciation of value of the bond below the discount prize that he bought the bond. At this stage, the investor wants to sell his old bond to obtain the bond with higher interest rate and consult his investment broker from whom he bought the bond with same question.

The investment broker analysis to decide expected return and market value of the bond is determined using suitable models for the pricing of bonds and other forms of interest rate derivatives.

1.1 Literature Review

In the past three decades, there have been a phenomenal growth in the trading of interest rate derivatives, leading to a surge in research on derivative pricing theory. Even before the upsurge of active trading of derivatives, considerable research had been devoted to the valuation of interest rate. Several models of the term structure have been proposed in the literatures. Examples are Black's Scholes (1973), Dothan (1978), Brennan and Schwartz (1979), Richard (1979), Langetieg (1980), Courtadon (1982), Cox, Ingersoll, Ross (1985b), Ho and Lee (1986), Longstaff (1989), Longstaff and Schwartz (1992) and Koedijk, Nissen, Schotman, and Wolff (1997).

All these models have the advantage that they can be used to value interest rate derivatives in a consistent way. Most practitioners often use Black's Scholes (1973) model for valuing options on commodity futures where forward bond prices rather than forward interest rates are assumed to be lognormal. Elliot and Baier (1979) in their work, studied six different econometric interest rate models to explain and predict interest rates, tested the accuracy of the models fitted to US monthly data over a sample period of 7 years. The results obtained indicated that four out of the models predict current interest rates movements quite accurately but their ability to forecast future interest rates by applying actual information is seemed to be inaccurate. Further work done by Brennan and Schwartz (1982) focused on modelling and pricing of US government bonds from 1948 to 1979, with the objective of evaluating the ability of the pricing model to detect underpriced or overpriced bonds. In their result obtained over this period, it indicates no relationship between future values of the short term interest rate and the long term interest rate, signifying that the valuation model is consistent for short periods of time.

To improve previous valuation models, Cox, Ingersoll and Ross (1985) developed an intertemporal general equilibrium asset pricing model to study the term structure of interest rates. The model takes into consideration key factors for determining the term structure of interest rates; which include anticipation of future events, risk preferences, investment alternatives and preferences about the timing of consumption. In contribution, these model is able to eliminate negative interest rates in Vasicek (1977) models and able to predict how changes in a diverse range of underlying variables will affect the term structure.

The inconsistency in the volatility parameters of different models is a concern to practitioners on the choice of suitable model for different situations. As a result there have been extension of existing models as new improved models to replace the old ones. In 1990, John Hull and Alan White extended interest rate models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985b) so that they are consistent with both the current term structure of interest rates and either the current volatilities of all spot interest rates or the current volatilities of all forward interest rates. Chan, Karolyi, Longstaff and Sanders (1992) compare eight models of short term interest rate within same framework to determine which model best fits the short term Treasury bill yield data. A comparison of these models indicates that models

which best describe the dynamics of interest rates over time are those that allow the conditional volatility of interest rate on the level of the interest rates. It is found that of Vasicek and Cox-Ingersoll-Ross Square Root models, perform poorly in comparison with Dothan and Cox-Ingersoll-Ross Variable Rate models.

Ho and Lee (1986) pioneered a new approach by showing how an interest-rate model can be designed so that it is consistent with any specified initial term structure. Their work has been extended by a number of researchers, including Black, Derman, and Toy (1990), Dybvig (1988), and Milne and Turnbull (1989). Heath, Jarrow, and Morton (HJM) (1987) present a general multifactor interest rate model consistent with the existing term structure of interest rates and any specified volatility structure. In the extensions, they use forward rate instead of bond prices, incorporate continuous trading and replace the one factor model of Ho-Lee with multiple random factors, broadening insight into the theoretical and practical approach. The HJM model provide practitioners with a general framework within which a no arbitrage model can be developed for the pricing and hedging of interest rate derivatives and bonds. For this reason, it was widely accepted by both the academics and practitioners. Although it has flaws with difference in dimension with short rate models, positive probability of instantaneous forward rate and recovery of caplets, which led to the use of Monte Carlo Simulation Method named after Monte Carlo, which is a time consuming approach used in rare cases when other options fail.

Longstaff and Schwartz (1992) develop a two factor general equilibrium model of the term structure of interest rates. The model is applied to derive closed form expressions for discount bond prices and discount bond option prices. Factors used are the short term interest rate and volatility of short term interest rates. The model is able to determine the value of interest rate contingent claims as well as hedging strategies of interest rate contingent claims. The model demonstrates advantages over two factor models which include endogenous determination of interest rate risk and a simplified version of the term structure of interest rates. Johansson (1994) models a continuous time stochastic process on short term interest rates based on sample results of the average interest rate for overnight loans on the interbank market for the five largest Swedish banks from 1986 to 1991. Results suggest that accuracy on parameters is dependent on sample's time length. Brenner, Harjes and Kroner (1996) also analyze two different interest rate models; LEVELS and GARCH models to develop an alternative class of model which improve on the inadequacy of the two

models. By comparison, LEVELS models put much emphasis on the dependence of volatility on interest rate levels and neglect serial correlation in variances, while GARCH models depend extensively on serial correlation in variances and neglect the relationship between interest rates and volatility.

Furthermore, Koedijk, Nissen, Schotman, and Wolff (1997) compare their model against a single factor model, GARCH model, and to a level GARCH model for one month Treasury bill rates. Quasi-maximum likelihood method was used to estimate these models with results that demonstrate both models determines interest rate volatility whereas GARCH models are non stationary in variance. Also In 1997 Brace, Gatarek and Musiela (BGM) presented a suitable approach that solves the HJM problems. Further research explored this approach to develop new inventive models suitable for pricing interest rate derivatives and models of these forms are called LIBOR market models. In 2001, Linus Kajsajuntti considered pricing of interest rate derivatives with the LIBOR Market Model. Treepongkaruna and Gray (2003) compares various interest rate derivatives by applying closed form solutions, a trinomial tree procedure and a Monte Carlo simulation technique and also provide an accurate description on how to use Monte Carlo simulation to value interest rate derivatives when the short rate follows arbitrary time series process.

Recent years have seen considerable contribution to interest rate derivatives and bonds due to its market demand. In [12], the Fourier transform approach was applied in the pricing of interest rate derivatives based on a technique introduced by Lewis (2001) for equity options. In the books of James and Webber (2000), Hunt and Kennedy (2000), Rebonato (2002), Cairns (2004) and Peter-Kohl Landgraf (2007), extensive work was done on these subject relating different models and suitable techniques to relatively price and hedge interest rate derivatives and bonds, which serve as a guide for further research to solve the problem of pricing and hedging these products.

Chapter 2

Background

2.1 Introduction

Financial mathematics employs different terms, notations, theories and theorems derived from concepts in both mathematics and finance. In this chapter, we introduce terminologies useful in our work, derived from concepts of mathematics and finance giving their mathematical interpretation and notation.

Definition 2.1.1

A bond is a form of loan to an entity (i.e financial institution, corporate organization, public authorities or government) for a defined period of time where the lender (bond holder) receives interest payments (coupon) annually or semiannually from the (debtor) bond issuer who repay the loaned funds (principal) at the agreed date of refund (maturity date)

2.2 Key concepts of bonds

We shall employ the following terms in the description of bonds.

1. Face value or Par

Face value or Par is the amount a bond holder receives at the maturity date of the bond.

2. Coupon

Coupon is the amount the bond holder receives annually or semiannually from the bond issuer as compensation for holding the bond.

3. Coupon rate

Coupon rate is the agreed rate of interest payment on the par value.

4. Maturity date

Maturity date (T) is the date of contract expiration.

5. Time to maturity

Time to maturity ($T - t$) is the amount of time (in years) from the present time t to the maturity time $T > t$.

6. Discount

Discount (D) is the purchase price of a bond in the secondary market, below the face value.

7. Premium

Premium(P) is the purchase price of a bond in the secondary market above the face value.

Definition 2.2.1 Bank Account

Let $B(t)$ denote the value of a bank account at time $t \geq 0$. Then $B(t)$ is assumed to satisfy the initial value ordinary differential equation:

$$\begin{cases} dB(t) = r(t)B(t)dt \\ B(0) = 1. \end{cases} \quad (2.2.1)$$

where $r(t)$ is a positive function of time, called the interest rate at time t .

Then

$$B(t) = \exp\left(\int_0^t r(s)ds\right), \quad (2.2.2)$$

If $r(t)$ is a constant interest rate,

$$B(t) = e^{rt} \quad (2.2.3)$$

The bank account at time t is related to the bank account at a future time T by

$$B(t) = B(T)\exp\left(-\int_t^T r(s)ds\right) \quad (2.2.4)$$

Definition 2.2.2 Stochastic Discount Factor

The stochastic discount factor between two time instants t and T denoted as $D(t, T)$ is the amount at time $t > 0$ that accrues to one unit of currency at time T and is defined by

$$D(t, T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r(s)ds\right) \quad (2.2.5)$$

It follows that the interest rate $r(t)$ influence the value of the investment at each time instant.

Definition 2.2.3 Zero Coupon Bond

Zero coupon bond with maturity time T also called T -bond, is a bond with coupon rate of zero, which guarantees the payment of one unit of currency to the holder at time T .

Let $P(t, T)$ denote the price of default free zero coupon bond price at initial time t with maturity date T . By definition, $P(T, T) = 1$.

Assumptions

1. Existence of arbitrage-free market for T -bond (i.e $P(t, T) \leq 1$).
2. $P(t, T)$ is continuously differentiable on $T \geq t$.

Definition 2.2.4 Continuously compounded spot interest rate

Continuously compounded spot interest rate, or Yield To Maturity (YTM), from initial time t up to maturity date T denoted by $R(t, T)$ is the constant rate at which an investment of $P(t, T)$ units of currency at time t accrues continuously to yield a unit amount of currency at maturity T (i.e $P(T, T) = 1$).

By definition,

$$P(t, T) = e^{(t-T)R(t,T)} \quad (2.2.6)$$

Take natural log and divide by $T - t$ to obtain,

$$R(t, T) = \left(\frac{1}{t-T}\right)\ln P(t, T) \quad (2.2.7)$$

Definition 2.2.5 Simple compounded spot interest rate

A simple compounded spot interest rate from time t up to the maturity T , denoted by $L(t, T)$, is the constant rate at which an investment of $P(t, T)$ units of currency at time t has to be made to produce an amount of one unit of currency at maturity T defined by

$$P(t, T) + P(t, T)L(t, T)(T - t) = 1$$

Thus,

$$L(t, T) = \frac{1 - P(t, T)}{P(t, T)(T - t)} \quad (2.2.8)$$

The bond price $P(t, T)$ is expressed in terms of $L(t, T)$ as

$$P(t, T) = \frac{1}{1 + L(t, T)(T - t)}. \quad (2.2.9)$$

2.2.1 Forward Rates

A zero coupon bond price $P(t, T)$ at initial time t with maturity T is related to interest rates of two future dates S and T where $S < T$. The interest rate at the different time instants $t < S < T$ is called the forward rate .

Definition 2.2.6 Continuously compounded forward rate

A continuously compounded forward rate at initial time t for the period between the exercise date S and the maturity date T denoted as $R_t(S, T)$ satisfies:

$$P(t, S) = P(t, T)e^{R_t(S, T)(T - S)} \quad \forall t < S < T$$

By solving for $R_t(S, T)$ we obtain

$$R_t(S, T) = \frac{1}{T - S}(\ln P(t, S) - \ln P(t, T))$$

Setting $S = t$, we obtain the continuous compounded spot interest rate

$$R(t, T) = \frac{1}{t - T} \ln P(t, T) \quad (2.2.10)$$

Definition 2.2.7 Simple compounded forward rate

The simple compounded forward rate at initial time t for the period between the exercise date S and the maturity date T is denoted by $L_t(S, T)$ and satisfies:

$$P(t, T) + P(t, T)L_t(S, T)(T - S) = P(t, S), \quad \forall t < S < T$$

By solving for $L_t(S, T)$ we obtain,

$$L_t(S, T) = \frac{1}{T - S} \left(\frac{P(t, S)}{P(t, T)} - 1 \right), \quad \forall t < S < T$$

Setting $S = t$, we obtain the simple compounded spot interest rate

$$L(t, T) = \frac{1 - P(t, T)}{P(t, T)(T - t)}$$

Definition 2.2.8 Instantaneous forward rate

The instantaneous forward rate at time t with maturity T is denoted by $f(t, T)$. This is a forward interest whose maturity is very close to its expiry S and is defined by

$$\begin{aligned} f(t, T) &= \lim_{S \rightarrow T^-} L_t(S, T) \text{ (by assumption of differentiability of bond price)} \\ &= - \lim_{S \rightarrow T^-} \frac{1}{P(t, T)} \frac{P(t, T) - P(t, S)}{T - S} \\ &= - \frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T} \\ &= - \frac{\partial \ln P(t, T)}{\partial T} \end{aligned}$$

Definition 2.2.9 Instantaneous spot rate

The instantaneous spot rate at time t is the rate attainable on the spot and is defined by

$$\begin{aligned} r(t) &= \lim_{T \rightarrow t^+} L(t, T) \\ &= - \lim_{T \rightarrow t^+} \frac{1}{P(t, T)} \frac{P(t, t) - P(t, T)}{t - T} \\ &= - \frac{\partial \ln P(t, t)}{\partial t} = \lim_{T \rightarrow t^+} R(t, T) \\ &= f(t, t) \end{aligned}$$

2.2.2 Fixed and floating coupon bonds

For coupon bonds, there are two types of interest payments, fixed and floating interest payments.

Fixed coupon bonds

A coupon bond with fixed interest rate as coupon payments C_1, C_2, \dots, C_n based on nominal value N over a consecutive period of future dates T_1, T_2, \dots, T_n is called a fixed coupon bond.

The price $P(t)$ of a fixed coupon bond at time $t \leq T$ is the sum of the discounted cash flows given by

$$P(t) = \sum_{i=1}^n P(t, T_i)C_i + P(t, T_n)N$$

where $C_i = K\delta_i N$ for some interest rate K and $\delta_i = T_{i+1} - T_i$

Substituting for C_i , we obtain

$$P(t) = (K \sum_{i=1}^n \delta_i P(t, T_i) + P(t, T_n))N \quad (2.2.11)$$

Floating coupon bonds

Coupon bonds with no fixed interest rate payments at different future date T_1, T_2, \dots, T_n , where coupon payment is determined by the LIBOR, reset at the previous instants T_0, T_1, \dots, T_{n-1} between every coupon period are called floating coupon bonds.

The coupon payment at time T_i , determined by the LIBOR at time T_{i-1} for the floating coupon bond is

$$C_i = NL(T_{i-1}, T_i)(T_i - T_{i-1})$$

Recall that $L(T_{i-1}, T_i)$ is the simple compounded spot interest rate for the interval (T_{i-1}, T_i) determined at T_{i-1} , with payoff at time T_i given as

$$L(T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\frac{1}{P(T_{i-1}, T_i)} - 1 \right)$$

Thus,

$$C_i = N \left(\frac{1}{P(T_{i-1}, T_i)} - 1 \right) \quad (2.2.12)$$

We then calculate the time t value of C_i , by relating it to the price of an asset at time t , with same value at C_i .

Consider the net cash flow of these asset (zero coupon bond) at time T_i .

1. At time t : buy N T_{i-1} -bonds
2. At time T_{i-1} : receives N unit of currency (\mathbb{N}) and buy $\frac{N}{P(T_{i-1}, T_i)}$ T_i bonds (Zero profit)
3. At time T_i : receives $\mathbb{N} \frac{N}{P(T_{i-1}, T_i)}$.

Hence, the cash flow of these asset at T_i is equal to:

$$N \left(\frac{1}{P(T_{i-1}, T_i)} - 1 \right) \quad (2.2.13)$$

From (1): the time t value of N T_{i-1} bond that pays \mathbb{N} at time T_{i-1} is $NP(t, T_{i-1})$.

From (2-3): the time t value of $\frac{N}{P(T_{i-1}, T_i)}$ T_i -bonds, that pays $\mathbb{N} \left(\frac{N}{P(T_{i-1}, T_i)} \right)$ is $NP(t, T_i)$.

Hence, the time t value of C_i , is obtained as:

$$P(t, T_i) C_i = NP(t, T_{i-1}) - NP(t, T_i) \quad (2.2.14)$$

The price $P(t)$ of a Floating coupon bond at time $t \leq T_0$, is also the sum of the discounted cash flows given by

$$\begin{aligned} P(t) &= \sum_{i=1}^n P(t, T_i) C_i + P(t, T_n) N \\ &= \sum_{i=1}^n NP(t, T_{i-1}) - NP(t, T_i) + P(t, T_n) N \\ &= NP(t, T_0) \end{aligned}$$

$$i.e \quad P(t) = NP(t, T_0) \quad (2.2.15)$$

At time $t = T_0$, the price of a floating Coupon bond $P(t) = N$

Definition 2.2.10 Swap

A swap is an agreement between two parties to exchange fixed rate of interest known as Swap rate(R) for a future floating rate known as LIBOR based on a nominal amount (N) or equivalently Swaps a floating rate for a fixed rate. Swap constitute of fixed leg and floating leg. Fixed leg is the set of fixed interest payments and floating leg is the set of floating interest payment based on LIBOR rate. Interest rate swap based on the fixed leg or floating leg is classified as Payer Interest rate Swap (Payer IRS) and Receiver interest rate Swap(Receiver IRS) respectively. A Payer IRS holder pays fixed rate to receive floating rate while a Receiver IRS holder pays floating rate to receive fixed rate.

To illustrate, consider a set of payments dates T_i for $i = 1, 2, \dots, n$ with fixed LIBOR rates on a coupon bond that swaps a fixed interest rate for a floating rate, to obtain a swap value for the Payer IRS.

Solution

At time T_i the Payer IRS holder

Pays: $N(T_i - T_{i-1})R$

Receive: $N(T_i - T_{i-1})L(T_i, T_{i-1})$

Net cash-flow at time T_i is

$$C_i = N(T_i - T_{i-1})(L(T_i, T_{i-1}) - R) \quad (2.2.16)$$

By definition

$$L(T_i, T_{i-1}) = \frac{1}{T_i - T_{i-1}} \left(\frac{1}{P(T_i, T_{i-1})} - 1 \right)$$

Thus,

$$C_i = N \left(\frac{1}{P(T_i, T_{i-1})} - (1 + R(T_i - T_{i-1})) \right) \quad (2.2.17)$$

Applying similar approach used in Floating rate bond, we obtain the time t value of C_i as:

$$N (P(t, T_{i-1}) - P(t, T_i(1 + R(T_i - T_{i-1}))) \quad (2.2.18)$$

Thus, the total Swap value for the Payer IRS with maturity (T_n) given as:

$$\begin{aligned}
\Pi_P(t, T_0, T_n, R) &= N \left(\sum_{k=1}^n P(t, T_{i-1}) - P(t, T_i)(1 + R(T_i - T_{i-1})) \right) \\
&= N \left(\sum_{k=1}^n P(t, T_{i-1}) - P(t, T_i) - \sum_{k=1}^n P(t, T_i)(T_i - T_{i-1})R \right) \\
&= N \left(P(t, T_0) - P(t, T_n) - \sum_{k=1}^n P(t, T_i)(T_i - T_{i-1})R \right)
\end{aligned}$$

Using same approach, the swap value for the Receiver IRS with maturity (T_n) is obtained as:

$$\begin{aligned}
\Pi_R(t, T_0, T_n, R) &= -N \left(P(t, T_0) - P(t, T_n) - \sum_{k=1}^n P(t, T_i)(T_i - T_{i-1})R \right) \\
&= -\Pi_P(t, T_0, T_n, R)
\end{aligned}$$

To obtain an arbitrage-free value for interest rate swap we set $\Pi_p(t, T_0, T_n, R) = 0$, the fixed rate R is called the *Par Swap Rate* defined by:

$$R = \frac{P(t, T_0) - P(t, T_n)}{\sum_{k=1}^n P(t, T_i)(T_i - T_{i-1})} \quad (2.2.19)$$

Interest rate swap are used to exploit the future interest rate fluctuation, that takes place in Financial market to get a positive payoff in an investment. In other instance, this interest rate fluctuation might be unfavorable, resulting in huge loss.

2.2.3 Interest Rate Derivatives [20]

Interest rate derivatives are financial instruments whose payoff on the underlying security traded in the derivative market is based on interest rate. Interest rate derivatives traded in the derivative market are classified into 3 main products namely:

1. Bond Options
2. Caps and Floors
3. Swaptions

1. Bond Options

An option is a financial agreement acquired at a cost between two parties, with no obligation on the contract. The price in the contract is called the Strike or Exercise price (K). Options are generalized into two types: Call and Put options. In terms of call option, these option provides its holder with the right to buy an underlying asset for a fixed price at an agreed date called the maturity or exercise date, while put options provides its holder with the right to sell an underlying asset for a fixed price at the maturity date. The two common forms of options traded in the exchange market, is the American and European option. American option is preferred to the European option, as an option which gives its holder the provision to exercise his right before or on the expiry date, compared to European option which can be only be exercised on the expiry date.

Bond option as the name implies, is an option with coupon bond as its underlying asset. Similarly, an option on a coupon bond can either be a call or put option in the form of either American or European.

Consider a coupon bond of maturity date T as the underlying asset of an European call option (ECO) of strike price K and maturity S where $S < T$ has a payoff of

$$\begin{aligned}\Phi_{ECO}(P(T_1, T_2)) &= \text{Max}(P(S, T) - K, 0) \\ &= (P(S, T) - K, 0)^+\end{aligned}$$

Similarly, the payoff for an European put option is defined as

$$\begin{aligned}\Phi_{ECP}(P(T_1, T_2)) &= \text{Min}(K - P(S, T), 0) \\ &= -(P(S, T) - K, 0)^+ \\ &= -\text{Max}(P(S, T) - K, 0) \\ &= -\Phi_{ECO}(P(T_1, T_2))\end{aligned}$$

Example 2.2.1 *An investor who buys a coupon bond call (CBC) option of strike price K with option maturity T_1 and bond maturity T_2 , where $T_1 < T_2$ gets a payoff of*

$$\begin{aligned}\Phi_{CBCP}(t, T_1, T_2) &= \text{Max}(P(t, T_2) - K) \\ &= \text{Max}\left(\sum_{k=1}^n P(t, T_k)(T_{k+1} - T_k)R - K, 0\right)\end{aligned}$$

2. Caps and Floors

Cap

A cap is a contract similar to payer IRS with positive payoff. Cap is an interest rate option used to protect against rate of interest on the floating rate note rising above a predetermined set rate called the cap rate \mathbb{k} . The time interval between the period when the rate of interest on the floating rate note is readjusted equal to LIBOR is called *Tenor*.

Consider a cap with future dates $T_0 < T_1 \dots < T_n$ with T_n as the maturity of the cap, a notional N , and a cap rate of \mathbb{k} . At time $T_i (i = 1, 2, \dots, n)$ the cap holder receives a payoff of

$$\Phi_{Cap}(t, T_i, T_{i+1}, N) = N\delta_i(L(T_i, T_{i+1}) - \mathbb{k})^+$$

where

$$\delta_i = T_{i+1} - T_i$$

$L(T_i, T_{i+1})$ is the interest rate observed at T_i for the tenor between T_i and T_{i+1}

Setting $n = 1$, the cap for the interval (T_1, T_2) , preferably say (S, T) with notional (N) and cap rate (\mathbb{k}) is called caplet. Similarly, the payoff for the caplet holder is

$$\Phi_{Cpl}(t, S, T, N) = N(T - S)(L(S, T) - \mathbb{k})^+$$

By definition

$$L(S, T) = \frac{1}{T - S} \left(\frac{1}{P(S, T)} - 1 \right)$$

Thus,

$$\begin{aligned}
\Phi_{Cpl}(t, S, T, N) &= N(T - S) \left[\frac{1}{T - S} \left(\frac{1}{P(S, T)} - 1 \right) - \mathbb{k} \right]^+ \\
&= N \left[\frac{1}{P(S, T)} - (\mathbb{k}(T - S) + 1) \right]^+ \\
&= \frac{N(\mathbb{k}(T - S) + 1)}{P(S, T)} \left[\frac{1}{\mathbb{k}(T - S) + 1} - P(S, T) \right]^+
\end{aligned}$$

The expression above implies $N(\mathbb{k}(T - S) + 1)$ as units of European put option with exercise price $K = \frac{1}{\mathbb{k}(T - S) + 1}$ at time S , characterize interest rate cap as a portfolio of European put option on zero coupon bonds with maturity T .

Interest rate cap is defined in terms of caplet for the future dates $T_0 < T_1 \dots < T_n$ as the sum of caplets over the tenor $(T_i, T_{i+1})_{0 \leq i \leq n}$ with the same notional N , and cap rate \mathbb{k} .

The cap price at time $t \leq T_0$ within the tenor (T_i, T_{i+1}) is defined as

$$Cap(t) = \sum_{k=1}^n Cpl(i, t)$$

where

$Cpl(i, t)$ denote the i th caplet within the Tenor (T_i, T_{i+1}) , with (T_i) as the reset date and T_{i+1} as the payment date.

Floors

A floor is the converse of cap, typically it is similar to Receiver IRS with positive payoff. Floor is also an interest rate option, used rather to protect against interest rates falling below a predetermined rate, also called the cap rate \mathbb{k} .

Using similar notion from cap, a floor with future dates $T_0 < T_1 \dots < T_n$ with T_n as the maturity of the cap, a notional N , and a cap rate of \mathbb{k} gives the floor holder at time $T_i (i = 1, 2, \dots, n)$, receives a payoff of

$$\Phi_{Fl}(t, T_i, T_{i+1}, N) = N\delta_i(\mathbb{k} - L(T_i, T_{i+1}))^+$$

Setting $i = 1$, the floor for the interval (T_1, T_2) , preferably say (S, T) with notional (N) and cap rate (\mathbb{k}) is called floorlet. Similarly, the payoff for the floorlet holder is

$$\Phi_{Fl}(t, S, T, N) = N(T - S)(\mathbb{k} - (L(S, T)))^+$$

Thus,

$$\begin{aligned} \Phi_{Fl}(t, S, T, N) &= N(T - S)\left[\mathbb{k} - \frac{1}{T - S}\left(\frac{1}{P(S, T)} - 1\right)\right]^+ \\ &= N\left[(\mathbb{k}(T - S) + 1) - \frac{1}{P(S, T)}\right]^+ \\ &= \frac{N(\mathbb{k}(T - S) + 1)}{P(S, T)}\left[P(S, T) - \frac{1}{\mathbb{k}(T - S) + 1}\right]^+ \end{aligned}$$

The expression above implies $N(\mathbb{k}(T - S) + 1)$ as units of European call option with exercise price $K = \frac{1}{\mathbb{k}(T - S) + 1}$ at time S which characterize interest rate floor as a portfolio of European call option on zero coupon bonds with maturity T .

Interest rate floor is also defined in terms of floorlet for the future dates $T_0 < T_1 \dots < T_n$ as the sum of caplets over the tenor $(T_i, T_{i+1})_{0 \leq i \leq n}$ with the same notional N , and cap rate \mathbb{k} .

The floor price at time $t \leq T_0$ within the tenor (T_i, T_{i+1}) is defined as

$$Fl(t) = \sum_{k=1}^n Fl(i, t)$$

where

$Fl(i, t)$ denote the i th floorlet within the Tenor (T_i, T_{i+1}) , with (T_i) as the reset date and T_{i+1} as the payment date.

4. Swaption

Swaption or swap option are options on interest rate swap that gives the holder the

right to enter into either Payer IRS or Receiver IRS with no obligation to exercise this right, at a given future date called swaption maturity. Swaption that allows holder to enter into payer IRS (with no obligation) is called payer or call swaption, while swaption that allows holder to enter into receiver IRS (with no obligation) is called receiver or put swaption. Similarly, options on cap and floor are called caption and floortion respectively.

In a Swaption, the fixed rate swapped against the floating rate is called strike rate \mathbf{R}_s . Swaption for future dates $T_0 < T_1 \dots < T_n$ has a length of time $T_n - T_0$, called the **tenor** of the swaption.

Exemplary, consider a payer swaption with strike rate \mathbf{R}_s on a swap with reset dates $T_0 < T_1 \dots < T_{n-1}$, a nominal N and a swap rate \mathbf{R} .

Recall, at time T_0 the value of the payer swap is:

$$\Pi_P(T_0, T_n, R) = N \left(\sum_{i=1}^n P(t, T_{i-1}) - P(t, T_i)(1 + R_s(T_i - T_{i-1})) \right)$$

Therefore, the payoff of the swaption at time T_0 with fixed strike rate (R_s) is obtained as:

$$\begin{aligned} \Phi_*(T_0, T_n, R) &= N \left(\sum_{i=1}^n P(t, T_{i-1}) - P(t, T_i)(1 + R_s(T_i - T_{i-1})) \right)^+ \\ &= N \left(\sum_{i=1}^n P(t, T_{i-1}) - P(t, T_i)(1 + R_s(T_i - T_{i-1})) \right) \\ &\quad - N \left(\sum_{i=1}^n (P(t, T_{i-1}) - P(t, T_i)(1 + R(T_i - T_{i-1}))) \right)^+ \\ &= N \left((R - R_s) \sum_{i=1}^n P(t, T_i)(T_i - T_{i-1}) \right)^+ \end{aligned}$$

Chapter 3

Stochastic Processes [4]

3.1 Introduction

In this chapter, we introduce our readers to concepts of stochastic process that are useful in our work, we assume that notions of probability theory and stochastic processes [4] are already known to the reader.

3.2 Stochastic Processes

Let (Ω, \mathcal{A}, P) be a probability space.

A stochastic process is a collection of parametrized random variables $\{X_t\}_{t \in I}$ indexed by a time interval $I = [0, \infty)$ defined on the probability space and assuming values in \mathbb{R}^d , i.e. $X_t: \Omega \rightarrow \mathbb{R}^d$.

The value of the stochastic process X_t at $\omega \in \Omega$ is denoted by $X_t(\omega)$ or $X(t, \omega)$.

Remark 3.2.1

1. $X_t(\omega)$ represents the result at time t of the possible outcome ω in the sample space (Ω) .
2. For fixed $\omega \in \Omega$, the map

$$t \rightarrow X(t, \omega) \in \mathbb{R}^d; \quad t \in I$$

is called a sample path or trajectory of X_t .

Example 3.2.1

1. For $n = 1, 2, 3, \dots$. Let X_n denote the number of trading events that occur in an exchange on day n
2. Let X_t for $t \geq 0$, represent the number of oscillations in a centrifuge after time t .

3.2.1 Classes of Stochastic Processes

Stochastic processes are separated into two classes; the Markov process and Martingale process. The stochastic process of great interest in this work which belongs to both the class of Markov process and the class of martingale processes is the Brownian motion. The Brownian motion was discovered in 1828 by scottish botanist Robert Brown.

3.2.2 Brownian Motion

Let (Ω, \mathcal{A}, P) be a probability space and $W = \{W(t) \in L^0(\Omega, \mathbb{R}^d) : t \in I\}$ be an \mathbb{R}^d valued stochastic process on Ω , with the following properties

- i. $W(0) = 0$ almost surely,
- ii. W has continuous sample paths
- iii. For any times $0 \leq s < t$, $W_t - W_s$ is normally distributed with mean zero and variance $t - s$ i.e $W_t - W_s \sim N(0, t - s)$
- iv. W has stochastically independent increments i.e for any finite sequence of times $0 < t_1 < t_2 < \dots < t_n$, the random vectors $W(t_1), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$ are stochastically independent.

Then W is called the **standard Brownian motion or Wiener process**.

3.2.3 Filtration and Adapted Process

Let (Ω, \mathcal{A}, P) be a probability space and $\mathbb{F}(\mathcal{A}) = \{\mathcal{A}_t : t \in I\}$ be a σ -subalgebra of \mathcal{A} satisfying the following properties

1. $\mathcal{A}_s \subseteq \mathcal{A}_t$ for any $0 \leq s \leq t$ where $s, t \in I$ i.e \mathcal{A}_t is increasing $\forall t \in I$,

2. for each $t \in I$, \mathcal{A}_t contains all the \mathbb{P} – null members of \mathcal{A} .

Then the family $\mathbb{F}(\mathcal{A})$ is called the **filtration** of \mathcal{A} and $(\Omega, \mathcal{A}, \mathbb{F}(\mathcal{A}), P)$ is called a **filtered probability space or stochastic basis**.

Definition 3.2.1 Let $(\Omega, \mathcal{A}, \mathbb{F}(\mathcal{A}), P)$ be a filtered probability space. Then $\mathbb{F}(\mathcal{A})$ is called **right-continuous** if

$$\mathcal{A}_t = \mathcal{A}_{t+} := \bigcap_{\epsilon > 0} \mathcal{A}_{t+\epsilon}$$

$\mathbb{F}(\mathcal{A})$ is said to be **left-continuous** if

$$\mathcal{A}_t = \mathcal{A}_{t-} := \bigcup_{\epsilon > 0} \mathcal{A}_{t-\epsilon}$$

Remark 3.2.2

1. \mathcal{A}_t is the information available up to the known time t
2. Associated with a stochastic process $X = \{X(t) \in L^0(\Omega, \mathbb{R}^d) : t \in I\}$ is its **natural filtration** $\mathbb{F}^X(\mathcal{A}) = \{\mathcal{A}_t^X : t \in I\}$, where $\mathcal{A}_t^X = \sigma\{X(s) : s \leq t\}$.

Definition 3.2.2

A stochastic process $X = \{X(t) \in L^0(\Omega, \mathbb{R}^d) : t \in I\}$ is said to be adapted to a filtration $\mathbb{F}(\mathcal{A}) = \{\mathcal{A}_t : t \in I\}$, if $X(t)$ is \mathcal{A}_t measurable for each $t \in I$. Then X is called an **adapted process**.

Remark 3.2.3

1. Every stochastic process is adapted to its natural filtration.
2. A stochastic process $X = \{X(t) \in L^0(\Omega, \mathbb{R}^d) : t \in I\}$ is adapted to a filtration $\mathbb{F}(\mathcal{A}) = \{\mathcal{A}_t : t \in I\}$ iff $\mathcal{A}_t^X \subset \mathcal{A}_t$, $t \in I$.

Proposition 3.2.1 For the d -dimensional brownian motion, we have the following

1. $\mathbb{E}(W_j(t)) = 0$ where $j = 1, 2, \dots, d$

$$2. \mathbb{E} \left((W_j(t) - W_j(s))^2 \right) = |t - s|.$$

$$3. \mathbb{E} (W_j(t) W_k(s)) = \delta_{jk} \min \{t, s\}.$$

Proof.

Fix $j, k = 1, 2, \dots, d$

Proof of (1) and (2) follows directly from definition of a Brownian motion

To prove 3., let $0 \leq s < t$. Then

$$\begin{aligned} \mathbb{E}(W_j(t) W_k(s)) &= \mathbb{E}((W_j(t) - W_j(s) + W_j(s)) W_k(s)) \\ &= \mathbb{E}(W_j(t) - W_j(s) W_k(s)) + \mathbb{E}(W_j(s) W_k(s)) \\ &= 0 + \mathbb{E}(W_j(s) W_k(s)) \\ &= \begin{cases} 0 & \text{if } j \neq k \\ s = \min \{t, s\} & \text{if } j = k \end{cases} \\ &= \delta_{jk} \min \{t, s\} \\ &= \delta_{jk} t \wedge s. \end{aligned}$$

3.3 Martingale

Definition 3.3.1 Let $X = \{X(t) \in L^1(\Omega, \mathcal{A}, P) : t \in I\}$ be a real valued stochastic process on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}(\mathcal{A}), P)$. Then X is called a

(i) submartingale, if $\mathbb{E}(X_t | \mathcal{A}_s) \geq X_s$ for $t \geq s$.

(ii) supermartingale if $\mathbb{E}(X_t | \mathcal{A}_s) \leq X_s$ for $t \geq s$.

(iii) martingale if X is both a submartingale and a supermartingale i.e if $\mathbb{E}(X_t | \mathcal{A}_s) = X_s$ for $t \geq s$.

Example 3.3.1

Consider successive tosses of a fair coin, and let $\mathcal{E}_n = 1$ if the n th toss is heads and $\mathcal{E}_n = -1$ if the n th toss is tails. Let $S_n = \mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_n$ and $\mathcal{A}_n = \sigma(\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n)$. Then S_n is a martingale with respect to \mathcal{A}_n .

Proof.

By hypothesis, we have $S_n = \mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_n$, $\mathbb{E}(S_n) < \infty$

Thus we have,

$$\begin{aligned}\mathbb{E}(S_{n+1} | \mathcal{A}_n) &= \mathbb{E}(S_n + \mathcal{E}_{n+1} | \mathcal{A}_n) \\ &= \mathbb{E}(S_n | \mathcal{A}_n) + \mathbb{E}(\mathcal{E}_{n+1} | \mathcal{A}_n)\end{aligned}$$

Since S_n is measurable with respect to \mathcal{A}_n and \mathcal{E}_{n+1} is independent of \mathcal{A}_n , then

$$\begin{aligned}\mathbb{E}(S_{n+1} | \mathcal{A}_n) &= S_n + \mathbb{E}(\mathcal{E}_{n+1}) \\ &= S_n\end{aligned}$$

Thus $\{S_n : n \in \mathbb{N}\}$ is a martingale with respect to \mathcal{A}_n .

Example 3.3.2

Let $(\Omega, \mathcal{A}, \mathbb{F}(\mathcal{A}), P)$ be a filtered probability space with $\mathbb{F}(\mathcal{A}) = \{\mathcal{A}_n : n \in \mathbb{N}\}$. Let $X \in L^1(\Omega, \mathcal{A}, P)$ and define X_n by $X_n = \mathbb{E}(X | \mathcal{A}_n)$. Then $\{X_n : n \in \mathbb{N}\}$ is a martingale.

Proof. Let $n \geq m$, then

$$\begin{aligned}\mathbb{E}(X_n | \mathcal{A}_m) &= \mathbb{E}(\mathbb{E}(X | \mathcal{A}_n) | \mathcal{A}_m) \\ &= \mathbb{E}(X | \mathcal{A}_m) \quad \text{using tower property} \\ &= X_m\end{aligned}$$

This shows that the sequence $\{X_n : n \in \mathbb{N}\}$ is a martingale.

Example 3.3.3

Let $n \in \mathbb{N}$, \mathcal{E}_n a positive variable on $(\Omega, \mathcal{A}, \mathbb{F}(\mathcal{A}), P)$ with $\|\mathcal{E}_n\| = 0$ and $X_n = \mathcal{E}_1 + \mathcal{E}_2 \dots + \mathcal{E}_n$ (pointwise). Suppose that $\{\mathcal{E}_j : j \in \mathbb{N}\}$ is a set of stochastically independent random variables, then $\{X_n : n \in \mathbb{N}\}$ is a martingale.

Proof. Let $n \geq m$, then

$$\begin{aligned}\mathbb{E}(X_n | \mathcal{A}_m) &= \mathbb{E}(\mathcal{E}_1 + \mathcal{E}_2 + \dots + \mathcal{E}_{m-1} + \mathcal{E}_m + \mathcal{E}_{m+1} + \dots + \mathcal{E}_n | \mathcal{A}_m) \\ &= \mathcal{E}_1 + \mathcal{E}_2 + \dots + \mathcal{E}_m + \mathbb{E}(\mathcal{E}_{m+1} + \dots + \mathcal{E}_n | \mathcal{A}_m) \\ &= X_m + \|\mathcal{E}_{m+1}\| + \dots + \|\mathcal{E}_n\| \\ &= X_m\end{aligned}$$

This shows that $\{X_n : n \in \mathbb{N}\}$ is a martingale.

Proposition 3.3.1 For any martingale (\mathcal{E}_n) , $\mathbb{E}(\mathcal{E}_n) = \mathbb{E}(\mathcal{E}_0)$

Proof. Let \mathcal{G} be the trivial σ algebra i.e $\mathcal{G} = \{\Omega, \emptyset\}$
Since $\mathcal{G} \subset \mathcal{A}_n$ for all $n \in \mathbb{N}$, we apply the tower property

$$\begin{aligned}\mathbb{E}(\mathcal{E}_n) &= \mathbb{E}(\mathcal{E}_n | \mathcal{G}) \\ &= \mathbb{E}(\mathbb{E}(\mathcal{E}_n | \mathcal{A}_0) | \mathcal{G}) \\ &= \mathbb{E}(\mathcal{E}_0 | \mathcal{G}) \\ &= \mathbb{E}(\mathcal{E}_0)\end{aligned}$$

Proposition 3.3.2 Any Brownian motion W_t is a martingale with respect to $\mathcal{A}_t = \sigma\{W_s : s \leq t\}$.

Proof.

By definition of Brownian motion, W_t has continuous sample path and $W_t \sim \mathbb{N}(0, t)$, hence $\mathbb{E} |W_t| < \infty$ for all t .

Since $\mathcal{A}_t = \sigma \{W_s : s \leq t\}$, then W_t is measurable with respect to \mathcal{A}_t .

Let $0 \leq s < t$. Then

$$\begin{aligned}\mathbb{E}(W_t | \mathcal{A}_s) &= \mathbb{E}(W_t - W_s + W_s | \mathcal{A}_s) \\ &= \mathbb{E}(W_t - W_s | \mathcal{A}_s) + \mathbb{E}(W_s | \mathcal{A}_s)\end{aligned}$$

Since $W_t - W_s$ is independent of \mathcal{A}_s and W_s is measurable with respect to \mathcal{A}_s , we have

$$\begin{aligned}\mathbb{E}(W_t | \mathcal{A}_s) &= \mathbb{E}(W_t - W_s) + W_s \\ &= W_s\end{aligned}$$

This shows that $\{W_t : t \geq 0\}$ is a martingale.

3.4 Brownian Stochastic Integral

Let $(\Omega, \mathcal{A}, \mathbb{F}(\mathcal{A}), P)$ be a filtered probability space and W be a Brownian motion relative to this space. In this section, we construct a stochastic integral with respect to the Brownian motion.

Consider the function $f : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$. If f is continuous and deterministic, the integral $\int f(s)dW(s)$ cannot be defined as $\int f(s)\frac{dW(s)}{ds}ds$, because $W(s)$ is not differentiable on any interval \mathbb{R}_+ .

Remark 3.4.1

A stochastic integral with respect to the Brownian motion is called an Itô Integral.

Definition 3.4.1

A stochastic process f on $(\Omega, \mathcal{A}, \mathbb{F}(\mathcal{A}), P)$ is called simple, if f is of the form

$$f(t, \omega) = f_0(\omega)\mathcal{X}_{\{0\}} + \sum_{j=0}^{\infty} f_j(\omega)\mathcal{X}_{[t_j, t_{j+1}]}, \quad t \in \mathbb{R}_+, \omega \in \Omega \quad (3.4.1)$$

for some strictly increasing sequence of numbers $\{t_j\}_{j=0}^\infty$ with $t_0 = 0$, $\lim_{j \rightarrow \infty} t_j = \infty$ and a sequence $\{f_j\}_{j=0}^\infty$ of random variables satisfying $\sup_{j \geq 0} |f_j(w)| < C$, for a positive constant C , such that f_j is $\mathcal{A}_j(t)$ measurable for each $j \geq 0$.

Notation:

- (1) Let \mathcal{S} denote the set of all simple stochastic processes.
- (2) In the above \mathcal{X}_A denotes the characteristic (indicator) function of the set A .

Definition 3.4.2 For $f \in \mathcal{S}$ of the form of (3.4.1) and $t \in \mathbb{R}_+$. The stochastic integral with respect to the Brownian motion (Ito Integral) of the simple stochastic process f is defined by

$$W(f, t) = \int_0^t f(s) dW(s) = \sum_{j=0}^{\infty} f_j (W(t \wedge t_{j+1}) - W(t \wedge t_j)) \quad (3.4.2)$$

Properties of Ito Integral of Simple Stochastic process

The Ito integral of a simple stochastic process has some properties for the map $(f, t) \mapsto W(f, t)$. These properties are also shared by the Ito Integrals of general processes.

Theorem 3.4.1

Let $(f, t) \in \mathcal{S} \times \mathbb{R}_+$, with the Ito Integral $\int_0^t f(s) dW(s)$ of a simple stochastic process f satisfies the following properties,

- (1) Linearity: For fixed $t \in \mathbb{R}_+$, the map $(f, t) \mapsto W(f, t)$ is linear i.e for any $f, g \in \mathcal{S}$ and some α and β constants, then

$$W(\alpha f + \beta g, t) = \alpha W(f, t) + \beta W(g, t),$$

which is also written as

$$\int_0^t (\alpha f(s) + \beta g(s)) dW(s) = \alpha \int_0^t f(s) dW(s) + \beta \int_0^t g(s) dW(s)$$

.

(2) Adaptedness of the Ito integral: For fixed $f \in \mathcal{S}$, the map $(f, t) \mapsto W(f, t)$ is an adapted stochastic process on $(\Omega, \mathbb{A}, \mathbb{F}(\mathcal{A}), P)$ such that $W(f, 0) = 0$.

(3) Zero mean property: $\mathbb{E}(W(f, t)) = 0$.

(4) The Ito isometry:

$$\mathbb{E}(W(f, t)^2) = \int_0^t \mathbb{E}(|f(s)|^2) ds$$

(5) Martingale property: For fixed $f \in \mathcal{S}$, the map $(f, t) \mapsto W(f, t)$ is a martingale on $(\Omega, \mathcal{A}, \mathbb{F}(\mathcal{A}), P)$ i.e. for $s \leq t$, $\mathbb{E}(W(f, t) | \mathcal{A}_s) = W(f, s)$ almost surely.

(6) Doob's inequality for the Ito integral:

$$\mathbb{E}(\sup_{0 \leq s \leq t} |W(f, s)|^2) \leq 4(|W(f, t)|^2), \quad t \in \mathbb{R}_+.$$

Proof.

Proof of property (1) and (2) follows directly from the definition (3.4.2)

To prove property (3), since $f'_i s$ are square Integrable, using Cauchy-Schwartz inequality we have,

$$\mathbb{E}|f_j W(t_{j+1}) - W(t_j)| \leq (\mathbb{E}(f_j^2))^{1/2} (\mathbb{E}(W(t_{j+1}) - W(t_j))^2)^{1/2} < \infty$$

which implies that

$$\mathbb{E}|\sum_{j=0}^{n-1} f_j (W(t_{j+1}) - W(t_j))| \leq \sum_{j=0}^{n-1} \mathbb{E}|f_j W(t_{j+1}) - W(t_j)| < \infty$$

and the stochastic integral has expectation. By the martingale property of Brownian motion, using that $f'_j s$ are \mathcal{A}_{j_i} -measurable

$$\mathbb{E}(f_j (W(t_{j+1}) - W(t_j)) | \mathcal{A}_{j_i}) = f_j \mathbb{E}(W(t_{j+1}) - W(t_j) | \mathcal{A}_{j_i}) = 0$$

Thus, $\mathbb{E}(f_j (W(t_{j+1}) - W(t_j))) = 0$ and the result follows.

To prove property (4), write the square as the double sum

$$\begin{aligned}
\mathbb{E} \left[\left(\sum_{j=0}^{n-1} f_j (W_{t_{j+1}} - W_{t_j}) \right)^2 \right] &= \mathbb{E} \left[\sum_{j,k=0}^{n-1} f_j f_k (W_{t_{j+1}} - W_{t_j})(W_{t_{k+1}} - W_{t_k}) \right] \\
&= \mathbb{E} \left[\sum_{j=0}^{n-1} |f_j|^2 (W_{t_{j+1}} - W_{t_j})^2 \right] \\
&\quad + 2\mathbb{E} \left[\sum_{j < k} (f_j f_k (W_{t_{j+1}} - W_{t_j})(W_{t_{k+1}} - W_{t_k})) \right].
\end{aligned}$$

using the martingale property of Brownian motion

$$\begin{aligned}
\mathbb{E} \left[\left(\sum_{j=0}^{n-1} f_j (W_{t_{j+1}} - W_{t_j}) \right)^2 \right] &= \sum_{j=0}^{n-1} \mathbb{E}[\mathbb{E}[|f_j|^2 (W_{t_{j+1}} - W_{t_j})^2 | \mathcal{A}_{t_j}]] \\
&\quad + 2 \sum_{j < k} \mathbb{E}[\mathbb{E}[f_j f_k (W_{t_{j+1}} - W_{t_j})(W_{t_{k+1}} - W_{t_k}) | \mathcal{A}_{t_j}]]. \\
&= \sum_{j=0}^{n-1} \mathbb{E}[|f_j|^2 \mathbb{E}[(W_{t_{j+1}} - W_{t_j})^2 | \mathcal{A}_{t_j}]] \\
&\quad + 2 \sum_{j < k} \mathbb{E}[f_j f_k (W_{t_{k+1}} - W_{t_k}) \mathbb{E}[(W_{t_{j+1}} - W_{t_j}) | \mathcal{A}_{t_j}]]. \\
&= \sum_{j=0}^{n-1} \mathbb{E}[|f_j|^2 \mathbb{E}[(W_{t_{j+1}} - W_{t_j})^2]] + 0 \\
&= \sum_{j=0}^{n-1} \mathbb{E}[|f_j|^2 (t_{j+1} - t_j)] \\
&= \int_0^t \mathbb{E}[|f(s)|^2] ds
\end{aligned}$$

Thus, $\mathbb{E}(W(f, t)^2) = \int_0^t \mathbb{E}(|f(s)|^2) ds$.

To prove property (5), let $s \leq t$

$$\begin{aligned}\mathbb{E}(W(f_t)|\mathcal{A}_s) &= \mathbb{E}\left(\int_0^t f_u dW(u)|\mathcal{A}_s\right) \\ &= \mathbb{E}\left(\int_0^s f_u dW(u) + \int_s^t f_u dW(u)|\mathcal{A}_s\right) \\ &= \mathbb{E}\left(\int_0^s f_u dW(u)|\mathcal{A}_s\right) + \mathbb{E}\left(\int_s^t f_u dW(u)|\mathcal{A}_s\right) \\ &= \mathbb{E}\left(\int_0^s f_u dW(u)|\mathcal{A}_s\right) \\ &= \int_0^s f_u dW(u) \\ &= W(f, s)\end{aligned}$$

This result shows that the Ito integral is a martingale.

To prove property (6), using Doob's inequality substitute $\alpha = 2$, the result follows

3.5 Stochastic Differential Equation (SDE)

Definition 3.5.1

[5] An equation of the form

$$dX(t) = \mu(X(t), t)dt + \alpha(X(t), t)dW(t) \quad (3.5.1)$$

where $\mu(X(t), t)$ and $\alpha(X(t), t)$ are given and $X(t)$ is the unknown process, is called a stochastic differential equation (SDE) driven by a Brownian motion W .

3.5.1 Itô formula

The Itô formula is an important tool for solving stochastic differential equations.

Theorem 3.5.1 Let $(\Omega, \mathcal{A}, \mathbb{F}(\mathcal{A}), \mu)$ be a filtered probability space, X be an adapted stochastic process on $(\Omega, \mathcal{A}, \mathbb{F}(\mathcal{A}), \mu)$ and $\langle X \rangle$ its quadratic variation. Let $U \in C^{1,2}[0, T] \times \mathbb{R}$, then the following holds :

$$(a) \quad U(t, X_t) = U(s, X_s) + \int_s^t \frac{\partial u}{\partial t}(\tau, X_\tau) ds + \int_s^t \frac{\partial u}{\partial x}(\tau, X_\tau) dX_\tau + \frac{1}{2} \int_s^t \frac{\partial^2 u}{\partial x^2}(\tau, X_\tau) d\langle X_\tau \rangle \quad (3.5.2)$$

which may be written as

$$dU(t, x) = \frac{\partial u}{\partial t}(t, X_t) dt + \frac{\partial u}{\partial x}(t, X_t) dX_t + \frac{1}{2} \int_s^t \frac{\partial^2 u}{\partial x^2}(t, X_t) d\langle X_t \rangle \quad (3.5.3)$$

(b) If X satisfies the stochastic differential equation (S.D.E)

$$dX(t) = g(t, X_t) dt + f(t, X_t) dW(t); X(t_0) = x_0 \quad (3.5.4)$$

Then

$$dU(t, X(t)) = g_u(t, X_t) dt + f_u(t, X_t) dW_t; U(t_0, X_{t_0}) = U(t_0, x_0) \quad (3.5.5)$$

Where

$$g_u(t, X_t) = \frac{\partial u}{\partial t}(t, x) + g(t, x) \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} f(t, x)^2 \frac{\partial^2 u}{\partial x^2}(t, x) \quad (3.5.6)$$

$$f_u(t, x) = f(t, x) \frac{\partial u}{\partial x}(t, x) \quad (3.5.7)$$

Proof

See proof in [6]

Ito Multiplication Table

- 1) $dt \times dt = 0$
- 2) $dt \times dW = 0$
- 3) $dW \times dW = dt$

Lemma 3.5.1

Ito Lemma

Let $X = W$, whence $g \equiv 0$ and $f \equiv 1$ then,

$$du(t, W_t) = \left[\frac{\partial u}{\partial t}(t, W_t) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \right] dt + \frac{\partial u}{\partial x}(t, W_t) dW(t) \quad (3.5.8)$$

The Ito lemma is used to solve S.D.E and stochastic integration, which shows that the Ito integral is different from the Riemann integral.

Example 3.5.1

(1) Evaluate $\int_0^t W(s)dW(s)$

Proof

Let $U(t, x) = x^2$, using the Ito Formula

$$W_t^2 = W_0^2 + \int_0^t 2W_s dW(s) + \frac{1}{2} \int_0^t 2d\langle W \rangle(s)$$

Hence, we obtain the result

$$\int_0^t W(s)dW(s) = \frac{1}{2}W_t^2 - \frac{t}{2} \quad (3.5.9)$$

Thus, the Ito integral is different from the Riemann integral, as it includes the summand $-\frac{t}{2}$ called the Ito correction.

(2) Evaluate $\int_0^t s dW(s)$

Let $U(t, x) = tx$, so we find that $\partial_t U = x$, $\partial_x U = t$, and $\partial_x^2 U = 0$ then using Ito formula

$$U(t, W_t) = U(0, W_0) + \int_0^t W(s)ds + \int_0^t s dW(s)$$

Then we have

$$\int_0^t s dW(s) = tW(t) - \int_0^t W(s)ds \quad (3.5.10)$$

Thus, $\int_0^t W(s)ds$ is the Ito correction

(3) Evaluate $d(e^{tW(t)})$, where W is a Brownian motion

Let $U(t, x) = e^{tx}$, hence $\partial_t U = xe^{tx}$, $\partial_x U = te^{tx}$, and $\partial_x^2 U = t^2 e^{tx}$

Using the Ito Lemma, then we have

$$d(e^{tW(t)}) = \left(W_t e^{tW_t} + \frac{t^2}{2} e^{tW_t} \right) dt + t e^{tW_t} dW(t) \quad (3.5.11)$$

$$= e^{tW_t} \left[\left(W(t) + \frac{t^2}{2} \right) dt + t dW(t) \right] \quad (3.5.12)$$

3.5.2 Existence and uniqueness of solution

In solving SDE, various conditions exist to guarantee existence and uniqueness of solution.

Definition 3.5.2 $X(t)$ is called a strong solution of the SDE with initial value $X(0)$ if for all $t > 0$, $X(t)$ is a function $F(t, X(0), (W(s), s \leq t))$ of the given Brownian motion $W(t)$ and $X(0)$, $\int_0^t \mu(X(s), s)ds$ and $\int_0^t \sigma(X(s), s)dW(s)$ exist, and the integral equation is satisfied by

$$X(t) = X(0) + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s). \quad (3.5.13)$$

Example 3.5.2

Ornstein-Uhlenbeck Process

Consider the SDE

$$dX(t) = -\alpha X(t)dt + \sigma dW(t) \quad (3.5.14)$$

where α , σ are constants

Solution Case 1

If $\alpha = 0$ the solution of (3.5.14) is

$$X(t) = X(0) + \int_0^t \sigma dW(s) \quad (3.5.15)$$

$$= X(0) + \sigma W(t). \quad (3.5.16)$$

Case (2)

If $\sigma = 0$, then the solution is

$$X(t) = X(0)e^{-\alpha t}.$$

In the general case of arbitrary α and σ , let $U(t) = e^{\alpha t}X(t)$ since

$$\begin{aligned} d(e^{\alpha t}X(t)) &= e^{\alpha t}(\alpha X(t)dt + dX(t)) \\ &= e^{\alpha t}\sigma dW(t) \end{aligned}$$

Integrating both sides gives

$$U(t) = U(0) + \int_0^t e^{\alpha s}\sigma dW(s)$$

Substituting $U(t) = e^{\alpha t} X(t)$, we obtain

$$X(t) = e^{-\alpha t} \left(X(0) + \int_0^t e^{\alpha s} \sigma dW(s) \right)$$

Using integration by parts, we evaluate $\int_0^t e^{\alpha s} \sigma dW(s)$,

$$\int_0^t e^{\alpha s} \sigma dW(s) = \sigma e^{\alpha t} W(t) + \int_0^t B(s) d(e^{\alpha t})$$

Thus

$$X(t) = e^{-\alpha t} \left(X(0) + \sigma e^{\alpha t} W(t) + \int_0^t W(s) d(e^{\alpha t}) \right) \quad (3.5.17)$$

Therefore as $X(t)$ is a function of $W(s)$ for $s \leq t$, it follows that $X(t)$ is a strong solution of the SDE.

Theorem 3.5.2 Existence and Uniqueness

Let $(\Omega, \mathcal{A}, \mathbb{F}(\mathcal{A}), P)$ be a filtered probability space. Consider the SDE

$$dX(t) = \mu(t, X(t))dt + \alpha(t, X(t))dB(t), \quad X_{t_0} = x_0 \quad (3.5.18)$$

if the following conditions are satisfied

1. Coefficients are locally Lipschitz in x uniformly in t , that is for every T and N , there is a positive constant C depending only on T and N such that for all $|x|, |y| \leq N$ and all $0 \leq t \leq T$,

$$|\alpha(t, x) - \alpha(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq C|x - y| \quad (3.5.19)$$

2. Coefficients satisfy the linear growth condition

$$|\alpha(t, x)| \leq C(1 + |x|) \quad (3.5.20)$$

$$|\sigma(t, x)| \leq C(1 + |x|) \quad (3.5.21)$$

3. $X(0)$ is independent of $(W(t), 0 \leq t \leq T)$, and $\exp(X^2)(0) < \infty$.

Then (3.5.18) possess a unique strong solution with almost surely continuous sample paths.

Example 3.5.3

Solve the SDE

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t); X(t_0) = x_0 \quad (3.5.22)$$

assuming α and σ are constants

Solution

Divide through the SDE by $X(t)$, to obtain that

$$\frac{dX(t)}{X(t)} = \alpha dt + \sigma dW(t) \quad (3.5.23)$$

We apply Ito formula to $U(t, x) = \ln x$.

Then

$$d \ln X(t) = \frac{dX(t)}{X(t)} - \frac{1}{2} \frac{d\langle X \rangle(t)}{X^2(t)} = \frac{dX(t)}{X(t)} - \frac{1}{2} \sigma^2 dt$$

Integrating both sides

$$\ln X(t) - \ln X(0) = \left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)$$

Thus, the solution of the SDE is

$$X(t) = X(0) \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right) \quad (3.5.24)$$

Remark 3.5.1 *The process X is called the exponential Brownian motion*

Theorem 3.5.3 Feynman-Kac Theorem [2]

Suppose that the function $P(t, x)$ satisfies the following partial differential differential equation

$$\frac{\partial P}{\partial t} + f(t, x) \frac{\partial P}{\partial x} + \frac{1}{2} \rho^2(t, x) \frac{\partial^2 P}{\partial X^2} - R(x)P + h(t, x) = 0 \quad (3.5.25)$$

subject to the boundary condition $P(T, X) = \psi(X)$. Then there exist a process $W(t)$ and a measure Q under which $W(t)$ is a standard Brownian and where $P(T, x)$ has the solution

$$P(t, x) = \mathbb{E}^Q \left[\int_t^T V(t, u) h(u, X(u)) du + V(t, T) \psi(X, (T) | \mathcal{F}_t \right] \text{ for } t < T \quad (3.5.26)$$

where $dX = f(t, X(t)dt + \rho(t, X(t))dW(t)$ and $V(t, x) = \exp\left(-\int_t^u R(X(s)) ds\right)$ provided

$$\int_t^T \mathbb{E} \left[(\rho(S, X(s)) \frac{\partial P}{\partial x}(s, X(s))^2 \mid \mathcal{F}_t) \right] ds < \infty$$

Proof

See proof in [\[14\]](#)

Chapter 4

Pricing of bonds and interest rate derivatives

4.1 Introduction

In this chapter, we highlight two main approaches, the martingale approach and the approach based on the use of partial differential equations, for deriving the formula for pricing bonds and interest rate derivatives. The martingale method is based on the risk valuation principle, which uses the theory of martingales to establish the price of a derivative security.

4.1.1 Basic Setup

Let $(\Omega, \mathcal{A}, \mathcal{F}, P)$ be a filtered probability space with a finite number of stochastic process $S_0, S_1 \dots S_N$ under the assumption that the stochastic process are semi-martingales. Using this setup, we consider $N + 1$ traded assets where $S_i(t)$ is the price of one unit of asset i at time t with S_0 as the numeraire asset (asset that define the units in which security prices are measured).

Definition 4.1.1

(1) A portfolio (trading strategy) is a $(N + 1)$ component, locally bounded and predictable vector process of the form:

$$h(t) = [h_0(t), h_1(t) \dots h_N(t)] \tag{4.1.1}$$

(2) The value process corresponding to a portfolio h is defined by

$$V(t; h) = \sum_{i=0}^N h_i(t) S_i(t) \quad (4.1.2)$$

(3) A portfolio is self financing if the following holds

$$dV(t; h) = \sum_{i=0}^N h_i(t) dS_i(t) \quad (4.1.3)$$

where

$h_i(t)$ is the number of investment type in the portfolio at time t and

$V(t; h)$ is the market value of the portfolio h at time t

(4) The discounted price process vector $D(t) = [D_0(t), D_1(t) \dots D_N(t)]$ is defined by

$$D(t) = \frac{S(t)}{S_0(t)} \quad (4.1.4)$$

(5) The discounted value process corresponding to a portfolio h is defined as

$$V^D(t, h) = \frac{V(t; h)}{S_0(t)} \quad (4.1.5)$$

4.1.2 Financial Market

Definition 4.1.2

1. A financial market is a place where investors sell or buy **financial derivatives**.
2. A financial derivative is a financial contract whose value at an expiration date written into the contract is determined by the prices of the underlying financial assets.

Example 4.1.1

A bond option is a financial derivative, whose value is dependent on the price of a bond.

Financial derivative are generally traded in two types of markets.

1. Derivative Exchange

This is a market where investors trade on standardized contracts guided by the exchange regulations.

2. Over-The-Counter (OTC) Markets

An OTC market is a market where trade is done at high volume via telephone and computer (email). The OTC market is exposed to credit default risk which is negligible in a derivative exchange due to the existence of regulations to eliminate credit risk.

4.1.3 Contingent claim, arbitrage and martingale measure

Definition 4.1.3

(1) Contingent T -claim is any random variable $X \in L_+^0(\mathcal{F}_T, P)$ where $L_+^0(\mathcal{F}_T, P)$ denotes the set of all non negative \mathcal{F}_T measurable random variable and $L_{++}^0(\mathcal{F}_T, P)$ denote set of element X of $L_+^0(\mathcal{F}_T)$ with $P(X > 0) > 0$.

(2) A fixed T -claim X is said to be attainable if there exists a self financing portfolio h , such that the corresponding value process has the property that $V(T; h) = X$.

(3) An arbitrage portfolio h is called an admissible self financing portfolio if the corresponding value process satisfy the following properties

(a) $V_h(0) = 0$

(b) $V_h(T) \in L_{++}^0(\mathcal{F}_T, P)$

(4) The market is said to be complete if every claim is attainable.

Example 4.1.2 [1]

Assume we invest in N assets in which asset i cost $P_i(t)$ at time t with no coupon payments. The value process corresponding to our portfolio of x_i units of i assets is given as

$$V(t) = \sum_{i=1}^N x_i P_i(t) \tag{4.1.6}$$

In a market with arbitrage opportunities	In an arbitrage-free market
$V(0) = \sum_{i=1}^N x_i P_i(0) = 0$	$V(0) = \sum_{i=1}^N x_i P_i(0) \neq 0$
$P(V(T) \geq 0) = 1$	$P(V(T) < 0) < 1$
$P(V(T) > 0) < 1$	$P(V(T) > 0) = 1$

Example 4.1.3

Consider an investor who enters into a forward contract with delivery price $\mathbb{N}F$ at time T . At time t (i.e today's date) the price of the asset $\mathbb{N}S$ is called the spot price. At expiry time T the investor makes a profit or a loss of $S(T) - F$. To eliminate the uncertainty of return in the future, the investor enters into a portfolio with zero net position to sell the asset, so as to receive $S(t)$ to invest in a bank which accrues fixed interest, in which at expiry he receives the value $S(t)e^{r(T-t)}$. Therefore the net position of the investment at maturity is $S(t)e^{r(T-t)} - F$. Since the net position is zero, then $F = S(t)e^{r(T-t)}$.

In mathematical terms, the relationship between the forward price and spot price is expressed as $F \propto S(t)$, which means that the forward price is directly proportional to the spot price.

Claim:

Any financial market in which the relation $F = S(t)e^{r(T-t)}$ is violated has arbitrage opportunities.

Solution

W.L.O.G assume $F < S(t)e^{r(T-t)}$

An investor going short in a long forward contract receives $S(t)$ to invest in bank. At maturity T , the investor receives $S(t)e^{r(T-t)}$ and pay F to cancel out the position in the forward contract, to exploit the market and receive a riskless profit of

$$S(t)e^{r(T-t)} - F.$$

If trading continues in this way, investors would exploit the opportunity. The market will adjust and the arbitrage opportunity will be eliminated.

Remark 4.1.1

In this thesis, the financial market is considered to be arbitrage-free market .

The Black Scholes equation [19]

In an arbitrary financial market, there is a positive correlation between the value of call and the value of its underlying asset, while the correlation between the value of a put option and the value of its underlying asset is negative. Given that there exist opposite correlations between the call option and put option with their underlying assets, we construct a portfolio to eliminate arbitrage opportunity.

Let Π denote the value of a long option position and a short option position in some quantity Δ , of the underlying:

$$\Pi = V(S, t) - \Delta S \quad (4.1.7)$$

$V(S, t)$ be the option value and ΔS be the short asset position, where Δ is a constant. Assume the underlying is a log-normal random walk:

$$dS = \mu S dt + \sigma S dX.$$

As a result of changes in the value of the underlying and the option price from time t to $t + dt$, the change in value of the portfolio is obtained as:

$$d\Pi = dV + \Delta dS$$

$$\Leftrightarrow dV = d\Pi - \Delta dS$$

Using the Ito formula

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \sigma^2 \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dt - \Delta dS \quad (4.1.8)$$

To eliminate randomness, we set the random terms to zero to obtain

$$\frac{\partial V}{\partial S} dS - \Delta dS = 0 \quad (4.1.9)$$

Hence, $\Delta = \frac{\partial V}{\partial S}$.

Thus with $\Delta = \frac{\partial V}{\partial S}$, randomness is reduced to zero, where our portfolio is hedged against market risk. So the value of portfolio is:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \delta^2 \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \right) dt \quad (4.1.10)$$

Hence the change $d\Pi$ is completely risk-less. But a portfolio value Π with a completely risk-less change $d\Pi$ has a rate of return as that of a risk free bank account, i.e

$$d\Pi = r\Pi dt.$$

Using (4.1.7), (4.1.8) and (4.1.10) we obtain:

$$\begin{aligned} \left(\frac{\partial V}{\partial t} + \delta^2 \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \right) dt &= r(V - \Delta S) dt \\ \left(\frac{\partial V}{\partial t} + \delta^2 \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \right) dt &= r(V - \frac{\partial V}{\partial S} S) dt \\ \frac{\partial V}{\partial t} + \delta^2 \frac{1}{2} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV &= 0 \end{aligned} \tag{4.1.11}$$

This equation is called the **Black-Scholes equation**.

Theorem 4.1.1

The Black Scholes model is arbitrage free and complete.

Proof

We apply the following Meta-theorem[23]. Let K denote the number of underlying assets in the model, excluding the numeraire asset, and R the number of random sources. Then the following relations hold.

1. The model is arbitrage free if and only if $K \leq R$.
2. The model is complete if and only if $K \geq R$.
3. The model is complete and arbitrage free if and only if $K = R$.

It suffices to show that $K = R$

In the Black-Scholes model we have one underlying asset S and one driving Brownian process. Hence $K = 1 = R$, whence $K = R$

Therefore the Black-Scholes model is complete and arbitrage free.

Definition 4.1.4

A probability measure Q is called a martingale measure if it satisfies the following:

1. $Q \sim P$
2. The discounted price process D is a Q -martingale.

The set of martingale measures is denoted by \mathcal{P} . If D is a Q -local martingale, then Q is called a strong martingale measure.

Proposition 4.1.1

Let Q be a martingale measure. A self financing portfolio h is Q -admissible if $V^D(t; h)$ is a Q -martingale.

Proof.

Assume that there exists a martingale measure \mathbb{Q} . Then the model is free of arbitrage in the sense that there exists no \mathbb{Q} -admissible arbitrage portfolio.

Suppose there exist a \mathbb{Q} -admissible arbitrage portfolio say h .

Then,

$$V_h(T) \in L_{++}(\mathcal{F}_T, \mathbb{Q})$$

Hence

$$V_h^D(T) \in L_{++}(\mathcal{F}_T, \mathbb{Q})$$

Thus,

$$V(0) = V^D(0) = \mathbb{E}^{\mathbb{Q}}[V^D(T)] > 0 \quad (\text{contradiction})$$

Theorem 4.1.2 *If the martingale measure \mathbb{Q} is unique, then the market is complete.*

Proof. Define a \mathbb{Q} martingale M by

$$M(t) = \mathbb{E}^{\mathbb{Q}} \left[\frac{X}{S_0(T)} \mid \mathcal{F}_t \right] \quad (4.1.12)$$

with integral representation of the form

$$M(t) = M(0) + \sum_{i=1}^N \int_0^t h_i(s) dD_i(s), \quad (4.1.13)$$

where h_1, h_2, \dots, h_N are locally bounded and predictable.

Claim: $V^D = M$.

Define (h_1, h_2, \dots, h_N) by (4.1.1) and let h_0 be

$$h_0(t) = M(t) - \sum_{i=1}^N h_i(t) D_i(t), \quad (4.1.14)$$

From (4.1.5) we obtain that $V^D = M$

Also from (4.1.3)

$$dV^D = dM = \sum_{i=1}^N h_i(t) dD_i(t), \quad (4.1.15)$$

Hence the portfolio h is self-financing, clearly we have the result

$$V(T; h) = M(T) = \mathbb{E}^{\mathbb{Q}} \left[\frac{X}{S_0(T)} \mid \mathcal{F}_T \right] = \frac{X}{S_0(T)}, \quad (4.1.16)$$

which shows that the claim X is attainable, i.e. X can be replicated (hedged) by the portfolio h .

Remark 4.1.2

From the previous results, we show that a model is free of arbitrage if there exists a martingale measure, and that the model is complete if the martingale measure is unique.

Proposition 4.1.2 [23]

Let $\Pi(t, X)$ denote the price process for a contingent T-claim X .

Assume there exists a strong martingale measure for the market $[H(X), S_0, S_1 \dots S_N]$.

Then X must be priced according to the formula

$$\Pi(t, X) = S_0(t)\mathbb{E}^Q \left[\frac{X}{S_0(T)} \mid \mathcal{F}_t \right] \quad (4.1.17)$$

where Q is a strong martingale measure for $[\Pi(X), S_0, S_1 \dots S_N]$.

Proof. Let Q be the strong martingale measure for the extended market $[\Pi(X), S_0, S_1 \dots S_N]$

By definition of strong martingale measure

$$\frac{\Pi(t, X)}{S_0(t)} = \mathbb{E}^Q \left[\frac{\Pi(T, X)}{S_0(T)} \mid \mathcal{F}_t \right] = \mathbb{E}^Q \left[\frac{X}{S_0(T)} \mid \mathcal{F}_t \right] \quad (4.1.18)$$

Therefore, we obtain the pricing formula

$$\Pi(t, X) = S_0(t)\mathbb{E}^Q \left[\frac{X}{S_0(T)} \mid \mathcal{F}_t \right] \quad (4.1.19)$$

This formula is called **Fundamental asset pricing formula** which can be used to determine the value of any derivative security $X(T)$

Remark 4.1.3

From the formula, it follows that choices of Q give rise to different price processes.

Proposition 4.1.3 *For an attainable claim, the price process will also be given by the formula*

$$\Pi(t, X) = V(t; h) \quad (4.1.20)$$

Proof.

Assume the claim X is replicated by a portfolio h .

Then the derivative contract and the replicating portfolio are equivalent.

Thus

$$\Pi(t, X) = V(t; h) \quad (4.1.21)$$

Consider the case when X is replicated by a portfolio h , from (4.1.5)

$$\frac{\Pi(t, X)}{S_0(t)} = V^D(t) \quad (4.1.22)$$

By assumption of integrability, $\frac{\Pi(t, X)}{S_0(t)}$ is a Q -martingale
 Thus, for any attainable claim X , we obtain the formula

$$V(t; h) = S_0(t) \mathbb{E}^Q \left[\frac{X}{S_0(T)} \mid \mathcal{F}_t \right] \quad (4.1.23)$$

Remark 4.1.4

The value $V(t; h)$ of the derivative security is independent of the choice of numeraire.

Theorem 4.1.3

Change of Numeraire theorem

Let Q^N be the equivalent martingale measure with respect to the numeraire $N(t)$.
 Let Q^M be the equivalent martingale measure with respect to the numeraire $M(t)$.
 The Radon-Nikodym derivative that change the equivalent measure Q^M into Q^N is given by

$$\frac{dQ^N}{dQ^M} = \frac{N(T)/N(t)}{M(T)/M(t)}$$

Proof. Let N and M be two arbitrary numeraires, then define the martingale measures Q^N and Q^M . Using (4.1.7), the value of any given derivative security $X(T)$ with respect to each martingale measure is

$$V(t; h) = N(t) \mathbb{E}^N \left[\frac{X(T)}{N(T)} \mid \mathcal{F}_t \right] = M(t) \mathbb{E}^M \left[\frac{X(T)}{M(T)} \mid \mathcal{F}_t \right] \quad (4.1.24)$$

Define $G(T) = \frac{X(T)}{N(T)}$ and substitute into (4.1.24) to obtain

$$\mathbb{E}^N [G(T) \mid \mathcal{F}_t] = \mathbb{E}^M \left[G(T) \frac{N(T)/N(t)}{M(T)/M(t)} \mid \mathcal{F}_t \right] \quad (4.1.25)$$

Thus from (4.1.25), the expectation of G with respect to the measure Q^N , is equal to the expectation of G times the random variable $\frac{N(T)/N(t)}{M(T)/M(t)}$ with respect to the measure Q^M . The random variable $\frac{N(T)/N(t)}{M(T)/M(t)}$ is the Radon-Nikodym Derivative

denoted as $\frac{dQ^N}{dQ^M}$ that changes the equivalent measure Q^M into Q^N .

Thus

$$\frac{dQ^N}{dQ^M} = \frac{N(T)/N(t)}{M(T)/M(t)} \quad (4.1.26)$$

Theorem 4.1.4

Girsanov Theorem [2]

For any stochastic process $\kappa(t)$ such that

$$\int \kappa(s)^2 < \infty \quad (4.1.27)$$

with probability one, consider the Radon - Nikodyn derivative $\frac{dQ^*}{dQ} = P(t)$ given by

$$P(t) = \exp \left\{ \int_0^t \kappa(s) dW(s) - \frac{1}{2} \int_0^t \kappa(s)^2 ds \right\}$$

Then,

$$W^*(t) = W(t) - \int_0^t \kappa(s) dW(s) - \frac{1}{2} \int_0^t \kappa(s)^2 ds$$

is a Brownian motion

Similarly

$$dW(t) = dW^*(t) + \kappa(t) dt$$

Example 4.1.4 [2]

Consider the Black Scholes option pricing model, with two asset: B which is a riskless bank account and a stock S.

The prices of the assets are derived from the stochastic differential equation

$$\begin{cases} dB(t) = rB(t)dt & ; & B(0) = 1 \\ dS(t) = \mu S(t)dt + \sigma S(t)dW \end{cases} \quad (4.1.28)$$

where r, μ, σ are the constant interest rate, drift and volatility respectively, under the assumption that S is a geometric Brownian motion.

The relative price $D(t) = \frac{S(t)}{B(t)}$

By Ito's product formula

$$dD(t) = \left(\frac{1}{B(t)}\right)dS(t) + S(t)d\left(\frac{1}{B(t)}\right) + dS(t)d\left(\frac{1}{B(t)}\right). \quad (4.1.29)$$

Since $B(t)$ is differentiable, by quotient rule of differentiation,

$$d\left(\frac{1}{B(t)}\right) = -\frac{1}{B(t)^2}dB(t). \quad (4.1.30)$$

Substituting (4.1.28) and (4.1.30) into (4.1.29),

$$dD(t) = D(t)(\mu - r)dt + \sigma D(t)dW(t) \quad (4.1.31)$$

Hence, observe that since

$$\int_t^T d \log D(t) = \frac{D(T)}{D(t)} \quad (4.1.32)$$

We apply Girsanov's theorem to identify the equivalent martingale measure. To apply Girsanov's theorem we need to identify the process $\kappa(t)$

We want to find the Radon-Nikodym derivative that changes the measure Q^S into Q^B

$$\int_t^T d \log D(\tau) = \frac{D(T)}{D(t)} = \exp \left\{ \int_t^T -\frac{1}{2} [\sigma^2 - 2(\mu - r)] ds + \sigma dW \right\} \quad (4.1.33)$$

To obtain the new measure Q^B , we take the process $\kappa(t) = \frac{\mu - r}{\sigma}$

From Girsanov's theorem, it follows that

$$D(t) = (\mu - r)D(t)dt + \sigma(t)D(t)(dW^B - \frac{\mu - r}{\sigma}dt) = \sigma D(t)dW^B(t) \quad (4.1.34)$$

which is a martingale. The measure Q^B for $\sigma \neq 0$ is the only (unique) measure which makes the relative price a martingale. Hence the Black Scholes model is arbitrage free and complete due to the existence of unique martingale measure.

With respect to the equivalent martingale measure Q^B , the price of S is described by the stochastic differential equation given by

$$dS(t) = \mu S(t)dt + \sigma(t)(dW^B - \frac{\mu - r}{\sigma}dt) \quad (4.1.35)$$

To obtain an expression to derive the price of stock S , we equate the drift μ of the process by the interest rate r

$$dS(t) = rS(t)dt + \sigma(t)(dW^B), \quad (4.1.36)$$

The solution for the expression above is derived as

$$S(t) = e^{(r - \frac{1}{2}\mu^2)t + \sigma(t)W^B(t)}, \quad (4.1.37)$$

where $W^B(t)$ is the Brownian motion at time t under the equivalent martingale measure and has a normal distribution of mean 0 and variance t .

4.2 Martingale Pricing Approach

The martingale approach for pricing a contingent claim is based on the risk valuation principle which uses the theory of martingales to derive the pricing formula for a derivative security, by introducing a risk free asset as a numeraire.

Definition 4.2.1

Let $(\Omega, \mathcal{A}, P, \mathbb{F})$ be a filtered probability space. A martingale measure Q is called a risk neutral martingale measure if it uses the risk free asset as numeraire, i.e. if for every fixed T , the process

$$D(t, T) = \frac{P(t, T)}{B(t)}, \quad 0 \leq t \leq T,$$

is a Q martingale.

Proposition 4.2.1

Let X be a fixed T - claim . Assume that Q is a risk neutral martingale measure. Then the price process is

$$\Pi(t, x) = \mathbb{E}^Q \left[X \cdot e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right] \quad (4.2.1)$$

In particular the price process for a T -bond is given by

$$P(t, T) = \mathbb{E}^Q \left[e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right] \quad (4.2.2)$$

Proof. Using the fundamental asset pricing formula in (4.1.19) with risk free asset B as numeraire,

$$\Pi(X) = \mathbb{E}^Q \left[X \cdot \frac{B(t)}{B(T)} \mid \mathcal{F}_t \right] = \mathbb{E}^Q \left[X \cdot e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right]. \quad (4.2.3)$$

Thus, the price process for a T - bond (i.e $X = 1$) is

$$P(t, T) = \mathbb{E}^Q \left[e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right]. \quad (4.2.4)$$

Example 4.2.1

Consider a European call option with maturity T and strike K written on a unit-principal zero coupon bond with maturity $S > T$ has a pricing formula defined as

$$P_{CE}(t, T, S, K) = \mathbb{E}^Q \left[e^{-\int_t^T r(s) ds} (P(T, S) - K)^+ \mid \mathcal{F}_t \right] \quad (4.2.5)$$

Apply the change of numeraire theorem, with Q as the equivalent martingale measure with respect to the numeraire $B(t)$ and Q^T the equivalent martingale measure with respect to the numeraire $P(t, T)$.

The Radon-Nikodym derivative that changes the equivalent measure Q into Q^T is defined by

$$\frac{dQ}{dQ^T} = \frac{P(T, T)B(0)}{P(0, T)B(T)} = \frac{e^{-\int_0^T r(s)ds}}{P(0, T)} = \frac{D(0, T)}{P(0, T)}$$

we obtain

$$P_{C_E}(t, T, S, K) = P(t, T)\mathbb{E}^T \left[e^{-\int_t^T r(s)ds} (P(T, S) - K)^+ \mid \mathcal{F}_t \right]. \quad (4.2.6)$$

4.2.1 Valuation of Interest rate Derivatives

Swaps

A swap is one of the most used interest rate derivatives to hedge an investment against future uncertainty of change by exchanging (floating) interest rate for a fixed interest rate or vice versa.

Consider a swap contract at fixed time t , with a prespecified fixed interest rate R and notional amount N over the sequential periods $[T_{i-1}, T_i]$, $i = 1, 2, \dots, n$ with payments at time T_i , $i = 1, 2, \dots, n$.

The net cash flow at time T_i is defined by

$$C_i = N(T_i - T_{i-1})(L(T_{i-1}, T_i) - R)$$

The bond price over the period $[T_{i-1}, T_i]$ with the floating rate $L(T_{i-1}, T_i)$ is defined by

$$L(T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\frac{1}{P(T_{i-1}, T_i)} - 1 \right)$$

Using the above definition, W.L.O.G take the nominal amount $N = 1$. Then the price of a swap contract at time t is given by

$$\begin{aligned} \Pi(t) &= \sum_{i=1}^n \mathbb{E}^Q \left[e^{-\int_t^{T_i} r(s)ds} \delta (L(T_{i-1}, T_i) - R) \mid \mathcal{F}_t \right] \\ &= \sum_{i=1}^n \mathbb{E}^Q \left[e^{-\int_t^{T_i} r(s)ds} \left(\frac{1}{P(T_{i-1}, T_i)} - (1 + \delta R) \right) \mid \mathcal{F}_t \right] \\ &= \sum_{i=1}^n \mathbb{E}^Q \left[e^{-\int_t^{T_{i-1}} r(s)ds} e^{-\int_{T_{i-1}}^{T_i} r(s)ds} \left(\frac{1}{P(T_{i-1}, T_i)} - (1 + \delta R) \right) \mid \mathcal{F}_t \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n \mathbb{E}^Q \left[e^{-\int_t^{T_{i-1}} r(s) ds} \mathbb{E}^Q \left[e^{-\int_{T_{i-1}}^{T_i} r(s) ds} \mid \mathcal{F}_{T_{i-1}} \right] \left(\frac{1}{P(T_{i-1}, T_i)} - (1 + \delta R) \mid \mathcal{F}_t \right) \right] \\
&= \sum_{i=1}^n \mathbb{E}^Q \left[e^{-\int_t^{T_{i-1}} r(s) ds} P(T_{i-1}, T_i) \left(\frac{1}{P(T_{i-1}, T_i)} - (1 + \delta R) \mid \mathcal{F}_t \right) \right] \\
&= \sum_{i=1}^n [P(t, T_{i-1}) - (1 + \delta R)P(t, T_i)]
\end{aligned}$$

Thus the price of the swap at initial time t is

$$\Pi(t) = P(t, T_0) - P(t, T_n) - \sum_{i=1}^n \delta R P(t, T_i) \quad (4.2.7)$$

Using the expression above, we obtain the swap rate as

$$R_s = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^n P(t, T_i)} \quad (4.2.8)$$

Remark 4.2.1

The swap price depends on the price of bond at time t , which implies that swap contract can be replicated by a portfolio of bonds.

Caps and Floors

Let $\mathcal{D} = \{d_1, d_2 \dots d_n\}$ denote the set of cap/floor payment dates and let the set of the corresponding time differences in years between d_i and the settlement date t be denoted by $\mathcal{T} = T_0, T_1, \dots T_n$, where T_0 is the first reset date, with N as the cap/floor nominal value.

Thus, the arbitrage free price of the i th caplet is given by

$$\begin{aligned}
Cpl(t, t_i, t_{i-1}, N, K) &= \mathbb{E}^Q \left[e^{-\int_t^{t_i} r(s) ds} N \delta_i (L(T_{i-1}, T_i) - K)^+ \mid \mathcal{F}_t \right] \\
&= N \mathbb{E}^Q \left[e^{-\int_t^{T_{i-1}} r(s) ds} P(T_{i-1}, T_i) \delta_i (L(T_{i-1}, T_i) - K)^+ \mid \mathcal{F}_t \right] \\
&= N \mathbb{E}^Q \left[e^{-\int_t^{T_{i-1}} r(s) ds} P(T_{i-1}, T_i) \left(\frac{1}{P(T_{i-1}, T_i)} - (1 + \delta_i K)^+ \mid \mathcal{F}_t \right) \right]
\end{aligned}$$

$$= N\mathbb{E}^Q \left[e^{-\int_t^{T_{i-1}} r(s)ds} (1 - P(T_{i-1}, T_i)(1 + \delta_i K))^+ \mid \mathcal{F}_t \right]$$

Similarly

$$Flr(t, t_i, t_{i-1}, N, K) = N\mathbb{E}^Q \left[e^{-\int_t^{T_{i-1}} r(s)ds} ((1 + \delta_i K)P(T_{i-1}, T_i) - 1)^+ \mid \mathcal{F}_t \right]$$

Market price of risk [8]

The valuation of the factors that determine the state of the market over the time interval $[0, T]$ which determines the expected return on the asset is called the market price of risk.

Let $S(t)$ be the price of a financial asset derived from the stochastic differential equation (SDE)

$$dS(t) = \mu S(t)dt + \delta S(t)dWt \quad (4.2.9)$$

where $dW(t)$ is a wiener process,

μ is the drift(expected growth), generally dependent on t and $S(t)$,

δ is the volatility, generally dependent on t and $S(t)$.

Assume that f_1 and f_2 are the prices of derivatives on S which satisfy (4.2.9):

$$df_1 = \mu_1 f_1 dt + \delta_1 f_1 dW \quad (4.2.10)$$

$$df_2 = \mu_2 f_2 dt + \delta_2 f_2 dW \quad (4.2.11)$$

Let h be a portfolio consisting of $\delta_2 f_2$ of f_1 and $-\delta_1 f_1$ of f_2 (i.e h is instantaneously a riskless portfolio)

Thus,

$$V(t; h) = \delta_2 f_2 f_1 - \delta_1 f_1 f_2 \quad (4.2.12)$$

$$dV(t; h) = \delta_2 f_2 df_1 - \delta_1 f_1 df_2 \quad (4.2.13)$$

substituting equation(4.2.10) and (4.2.11) into (4.2.13) we obtain

$$dV(t; h) = (\mu_1 \delta_2 - \mu_2 \delta_1) f_1 f_2 dt \quad (4.2.14)$$

Since dW is eliminated, h is an instantaneous risk free portfolio

Therefore,

$$dV(t; h) = rV dt \quad (4.2.15)$$

Substituting from equation (4.2.13) and (4.2.12) into equation(4.2.15) we get

$$\mu_1 \delta_2 - \mu_2 \delta_1 = r(\delta_2 - \delta_1)$$

$$\frac{\mu_1 - r}{\delta_1} = \frac{\mu_2 - r}{\delta_2} = \lambda$$

Hence, the market price of risk λ on S with derivative whose price f satisfies(4.2.9) can be written as

$$\lambda = \frac{\mu - r}{\delta} \quad (4.2.16)$$

Remark 4.2.2

The equation(4.2.16) holds for an investment asset not a consumable asset
Using the above result, in a risk-free market, the SDE satisfied by price of the derivative f is:

$$df = rfdt + \delta fdW \quad (4.2.17)$$

In an arbitrary market with the price of market risk λ , and the assumption on the price of the derivative, we obtain

$$df = (\mu - \lambda\delta)fdt + \delta fdW \quad (4.2.18)$$

4.3 PDE Pricing Approach

In this section, we apply the PDE approach used in the work of Vasicek(1977), to present a general PDE approach for pricing. The PDE approach is more applicable in practise than the martingale approach, since it is a useful tool to develop numerical and analytical method.

This approach is based on the following assumptions

1. $r(t)$ is Markov.
2. $P(t, T)$ is dependent on interest rate movement $r(s)$ for $t \leq s \leq T$.
3. The market is efficient with no transaction costs.

Remark 4.3.1

Since $r(t)$ is solution to the SDE,

$$dr(t) = a(t)dt + b(t)dW \quad (4.3.1)$$

the process for r is markov.

From assumption (1) and (2)

$a(t) = a(t, r(t))$, $b(t) = b(t, r(t))$ and $P(t, T) = P(t, T, r(t))$

Consider the unknown as $P(t, T)$, using Ito formula

$$\begin{aligned} dP(t, T) &= \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial r} dr + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} d\langle r, r \rangle \\ &= \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial r} (adt + bdW) + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} b^2 dt \\ &= \left[\frac{\partial P}{\partial t} dt + a \frac{\partial P}{\partial r} + \frac{1}{2} b^2 \frac{\partial^2 P}{\partial r^2} \right] dt + b \frac{\partial P}{\partial r} dW \end{aligned}$$

This is of the form of bond price dynamics:

$$dP = P(t, T)[m(t, T, r)dt + S(t, T, r)dW] \quad (4.3.2)$$

where

$$m(t, T, r) = \frac{1}{P} \left[\frac{\partial P}{\partial t} dt + a \frac{\partial P}{\partial r} + \frac{1}{2} b^2 \frac{\partial^2 P}{\partial r^2} \right] \quad (4.3.3)$$

$$S(t, T, r) = \frac{b}{P} \frac{\partial P}{\partial r} \quad (4.3.4)$$

Thus, from (4.2.16) the market price of risk $\gamma(t, r(t))$ is

$$\gamma(t, r(t)) = \frac{m(t, T, r) - r(t)}{S(t, T, r)}$$

which can be written as

$$m(t, T, r) = r(t) + \gamma(t, r(t))S(t, T, r) \quad (4.3.5)$$

substituting (4.3.3) and (4.3.4) into (4.3.5), we obtain that

$$\frac{\partial P}{\partial t} + (a - b\gamma) \frac{\partial P}{\partial r} + \frac{1}{2} b^2 \frac{\partial^2 P}{\partial r^2} - rP = 0 \quad (4.3.6)$$

This equation is the term structure equation for the arbitrage free bond price.

Equation (4.3.6) is similar to the following PDE which leads to the Feynman-Kac formula:

$$\frac{\partial P}{\partial t} + f(t, r) \frac{\partial P}{\partial r} + \frac{1}{2} \rho^2(t, r) \frac{\partial^2 P}{\partial r^2} - R(r)P + h(t, r) = 0$$

where $f(t, r) = a(t, r) - b(t, r)\gamma(t, r)$, $\rho(t, r) = b(t, r)$, and $h(t, r) = 0$, where the boundary conditions for the PDE are $P(T, T, r) = \psi(r) = 1 \quad \forall T, r$.

Using Feynman-Kac formula, there exist a suitable probability space (Ω, \mathcal{F}, Q) with filtration $\{\mathcal{F}_t : 0 \leq t < \infty\}$ such that

$$P(t, T, r(t)) = \mathbb{E}^Q \left[\exp \left(- \int_t^T \tilde{r}(s) ds \right) \mid \mathcal{F}_t \right] \quad (4.3.7)$$

The process $\tilde{r}(s)(t \leq s \leq T)$ is a Markov diffusion process with $\tilde{r}(t) = r(t)$ under the measure Q , and \tilde{r} satisfies the SDE

$$d\tilde{r}(u) = f(u, \tilde{r}(u))du + \rho(u, \tilde{r}(u))d\tilde{W}(u), \quad (4.3.8)$$

where $\tilde{W}(u)$ is a standard Brownian motion under Q .

The short rate dynamic is $dr(t) = a(t)dt + b(t)dW(t)$, where $W(t)$ is a Brownian motion under P . To obtain $P(t, T)$ with respect to the measure P , it suffices to show that P and Q are equivalent.

Assume $\gamma(s, r(s))$, satisfies the Novikov condition

$$\mathbb{E}^Q \left[\exp \left(\frac{1}{2} \int_0^T \gamma(s, r(s))^2 ds \right) \right] < \infty$$

Define $\tilde{W}(t) = W(t) + \int_0^t \gamma(s, r(s))ds$

By Girsanov theorem, there exists an equivalent measure Q for which $\tilde{W}(t)$, for $(0 \leq t \leq T)$ is a Brownian motion and with Radon-Nikodym derivative

$$\frac{dQ}{dP} = \exp \left[- \int_0^T \gamma(t, r(t))dW(t) - \frac{1}{2} \int_0^T \gamma(t, r(t))^2 dW(t) \right]$$

Since

$$\begin{aligned} dP(t, T) &= P(t, T)(m(t, T, r(t))dt + S(t, T, r(t))dW) \\ &= P(t, T)(m(t, T, r(t))dt + S(t, T, r(t))d\tilde{W} - \gamma(t, r(t))dt) \\ &= P(t, T)[(m(t, T, r(t)) - \gamma(t, r(t))S(t, T, r(t))dt + S(t, T, r(t))d\tilde{W}] \\ &= P(t, T)(r(t)dt + S(t, T, r(t))dW(t)). \end{aligned}$$

This shows that Q is the risk neutral equivalent measure, since under Q the expected return on any bond is the risk free rate.

In general, interest rate derivative contract can be valued using the Feynman-Kac formula.

Let $V(t)$ be the price of a derivative with expected payoff to the holder of $\psi(r(T))$ at time T that satisfies the PDE

$$\frac{\partial V}{\partial t} + f(t, r)\frac{\partial V}{\partial r} + \frac{1}{2}\rho^2(t, r)\frac{\partial^2 V}{\partial r^2} - R(r)V = 0$$

subject to the boundary condition $V(T) = \psi(r)$, where $f(t, r) = a(t, r) - \gamma(t, r)b(t, r)$ and $R(r) = r$

Applying the Feynman-Kac formula, we obtain the price of the derivative

$$V(t) = \mathbb{E}^Q \left[\exp \left(- \int_t^T \tilde{r}(s) ds \right) \psi(\tilde{r}(T) | \mathcal{F}_t) \right] \quad (4.3.9)$$

where $\tilde{r}(s)$ is as in (4.3.8)

In particular, the price of a zero-coupon bond at initial time t and maturity T , is given by

$$P(t, T) = \mathbb{E}^Q \left[\exp \left(- \int_t^T \tilde{r}(s) ds \right) | \mathcal{F}_t \right]$$

,

4.3.1 Bond Pricing using PDE

Let the risk-neutral process of the short term interest rate be the solution to the SDE

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t) \quad (4.3.10)$$

where $(W_t)_{t \in \mathbb{R}}$ is a standard Brownian motion under Q .

Thus $r(t)$ has Markov property, since all solution of (4.3.10) is a Markov process by Emery's existence theorem [21].

Proposition 4.3.1 [13]

The bond pricing PDE for $P(t, T) = F(t, r(t))$ is

$$xF(t, x) = \mu(t, x) \frac{\partial}{\partial x} F(t, x) + \frac{1}{2} \sigma^2 \frac{\partial^2 F(t, x)}{\partial x^2} + \frac{\partial F(t, x)}{\partial t} \quad (4.3.11)$$

with the terminal condition $F(T, x) = 1$, $x \in \mathbb{R}$, and $P(T, T) = 1$.

Proof

Apply the Ito product law $d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$ to

$$\begin{aligned} d \left(e^{-\int_0^t r(s) ds} P(t, T) \right) &= e^{-\int_0^t r(s) ds} dP(t, T) - r(t) e^{-\int_0^t r(s) ds} P(t, T) dt \\ &= e^{-\int_0^t r(s) ds} dF(t, r_t) - r(t) e^{-\int_0^t r(s) ds} F(t, r_t) dt \\ &= e^{-\int_0^t r(s) ds} \left(\frac{\partial F(t, r_t)}{\partial x} dr_t + \frac{\partial F(t, r_t)}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F(t, r_t)}{\partial x^2} d\langle r_t \rangle \right) \\ &\quad - r(t) e^{-\int_0^t r(s) ds} F(t, r_t) dt \\ &= e^{-\int_0^t r(s) ds} \left(\frac{\partial F(t, r_t)}{\partial x} \mu(t, r_t) + \frac{\partial F(t, r_t)}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 F(t, r_t)}{\partial x^2} - r(t) F(t, r_t) \right) dt \\ &\quad + e^{-\int_0^t r(s) ds} \sigma(t, r_t) \frac{\partial F(t, r_t)}{\partial x} dW(t) \end{aligned}$$

Claim:

$e^{-\int_0^t r(s)ds} P(t, T)$ is a martingale

Let $0 < u < t$, it suffices to show that

$$\mathbb{E}^Q \left[e^{-\int_0^t r(s)ds} P(t, T) \mid \mathcal{F}_u \right] = e^{-\int_0^u r(s)ds} P(u, T) \quad (4.3.12)$$

Now,

$$\begin{aligned} \mathbb{E}^Q \left[e^{-\int_0^t r(s)ds} P(t, T) \mid \mathcal{F}_u \right] &= \mathbb{E}^Q \left[\mathbb{E}^Q \left[e^{-\int_0^T r(s)ds} \mid \mathcal{F}_t \right] \mid \mathcal{F}_u \right] \\ &= \mathbb{E}^Q \left[e^{-\int_0^T r(s)ds} \mid \mathcal{F}_u \right] \\ &= \mathbb{E}^Q \left[e^{-\int_0^u r(s)ds} e^{-\int_u^T r(s)ds} \mid \mathcal{F}_u \right] \\ &= e^{-\int_0^u r(s)ds} \mathbb{E}^Q \left[e^{-\int_u^T r(s)ds} \mid \mathcal{F}_u \right] \\ &= e^{-\int_0^u r(s)ds} P(u, T). \end{aligned}$$

Since $e^{-\int_0^t r(s)ds} P(t, T)$ is a martingale, this implies that the coefficient of dt (drift) is equal to zero.

Hence,

$$\frac{\partial F(t, r_t)}{\partial x} \mu(t, r_t) + \frac{\partial F(t, r_t)}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 F(t, r_t)}{\partial x^2} - r(t) F(t, r_t) = 0, \quad (4.3.13)$$

The result follows immediately.

Also, from the result obtained in (4.3.13), it implies that

$$d \left(e^{-\int_0^t r(s)ds} P(t, T) \right) = e^{-\int_0^t r(s)ds} \sigma(t, r_t) \frac{\partial F(t, r_t)}{\partial x} dW(t) \quad (4.3.14)$$

Since,

$$d \left(e^{-\int_0^t r(s)ds} P(t, T) \right) = e^{-\int_0^t r(s)ds} dP(t, T) - r(t) e^{-\int_0^t r(s)ds} P(t, T) dt \quad (4.3.15)$$

combining (4.3.14) and (4.3.15), we obtain that

$$\frac{dP(t, T)}{P(t, T)} = r(t) dt + \frac{\partial \log F(t, r_t)}{\partial x} dW(t) \quad (4.3.16)$$

Remark 4.3.2

In the next chapter, we obtain the solution of the PDE (4.3.11), using both probabilistic and analytical approaches.

Chapter 5

Modelling of Interest Rate Derivatives and Bonds

5.1 Introduction

In the previous chapters, we discussed some approaches to the pricing of interest rate derivatives and bonds. This chapter presents a model for the pricing of interest rate derivatives and bonds.

There are several models, for the pricing and hedging of interest rate derivatives and bonds. Some of the models are widely used in practice. In this work, we would give an analysis of a model for the valuation of bonds and interest rate derivatives.

5.2 Short Rate Model

The short rate model is the oldest existing interest rate model, it assumes that the 1) short rate r at time t is the instantaneous spot rate $r(t)$, under a risk neutral measure Q and satisfies the SDE

$$dr(t) = \alpha(t, r_t)dt + \sigma(t, r_t)dW(t) \quad (5.2.1)$$

2) drift α and the volatility σ satisfy the usual Lipschitz and boundedness conditions for the existence and uniqueness of a strong solution of the SDE.

By assumption (1) and the existence of a risk-neutral measure Q , the arbitrage-free price at time t of a contingent claim with payoff $X(T)$ at time T is given by

$$\Pi(t, X) = S_0(t)\mathbb{E}^Q \left[\frac{X(T)}{S_0(T)} | \mathcal{F}_t \right] \quad (5.2.2)$$

thus, the price at time t of a zero coupon bond maturing at time T is given as

$$P(t, T) = \mathbb{E}^Q \left[\exp \left(- \int_t^T r(s) ds \right) | \mathcal{F}_t \right]; \text{ since } P(T) = 1 \quad (5.2.3)$$

Claim:

Instantaneous spot rate at time t is equal to the forward rate (i.e $r(t) = f(t, t)$)

Proof.

The forward rate at time t with maturity T is given by

$$f(t, T) = - \frac{\partial}{\partial T} \ln P(t, T)$$

To solve for $P(t, T)$, we integrate both sides from t to T to obtain,

$$P(t, T) = \exp \left(- \int_t^T f(t, s) ds \right). \quad (5.2.4)$$

Differentiating equation (5.2.3)

$$\frac{\partial}{\partial T} P(t, T) = \mathbb{E}^Q \left[- r(t) \exp \left(- \int_t^T r(s) ds \right) | \mathcal{F}_t \right]. \quad (5.2.5)$$

Substituting $T = t$ in equation (5.2.5), we obtain that

$$\mathbb{E}^Q [- r(t) | \mathcal{F}_t] = - r(t).$$

Also differentiate (5.2.4) to obtain

$$\frac{\partial}{\partial T} P(t, T) = - f(t, T) \exp \left(- \int_t^T f(t, s) ds \right),$$

similarly set $T = t$, then

$$r(t) = f(t, t)$$

Relationship between bonds, short rates and forward rates [9]

We analyse the theoretical relationship between bonds, short rates and forward rates under the assumption that they satisfy the following dynamics:

Short rate dynamics

$$dr(t) = a(t)dt + b(t)dW(t) \quad (5.2.6)$$

Bond price dynamics

$$dp(t, T) = p_t(T)m(t, T)dt + p_t(T)n(t, T)dW(t) \quad (5.2.7)$$

Forward rate dynamics

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW(t) \quad (5.2.8)$$

Proposition 5.2.1

If $P(t, T)$ satisfies (5.2.7), then the forward rate dynamics is

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW(t)$$

where α , and σ are given by

$$\begin{cases} \alpha(t, T) &= n(t, T)\frac{\partial n(t, T)}{\partial T} - \frac{\partial m(t, T)}{\partial T} \\ \sigma(t, T) &= -\frac{\partial n(t, T)}{\partial T} \end{cases}$$

Proof

Apply the Ito formula to the process $\log P(t, T)$

$$d \log P(t, T) = m(t, T)dt + n(t, T)dW(t) + \left(-\frac{1}{2}n^2(t, T)\right)dt$$

Write in integral form and differentiate with respect to T

$$\log P(t, T) = \log P(0, T) + \int_t^T m(s, T)ds + \int_t^T n(s, T)dW(s) + \int_t^T \left(-\frac{1}{2}n^2(s, T)\right)ds$$

$$\frac{\partial}{\partial T} \log P(t, T) = \frac{\partial}{\partial T} \int_t^T m(s, T)ds + \frac{\partial}{\partial T} \int_t^T n(s, T)dW(s) + \frac{\partial}{\partial T} \int_t^T \left(-\frac{1}{2}n^2(s, T)\right)ds$$

Hence by definition of $f(t, T)$,

$$f(t, T) = -\frac{\partial}{\partial T} \int_t^T m(s, T)ds - \frac{\partial}{\partial T} \int_t^T n(s, T)dW(s) + \frac{\partial}{\partial T} \int_t^T \frac{1}{2}n^2(s, T)ds$$

Then,

$$\begin{aligned} df(t, T) &= -\frac{\partial}{\partial T} m(t, T)dt - \frac{\partial}{\partial T} n(t, T)dW(t) + n(t, T)\frac{\partial}{\partial T} n(t, T)dt \\ &= \left(n(t, T)\frac{\partial}{\partial T} n(t, T) - \frac{\partial}{\partial T} m(t, T) \right) dt - \frac{\partial}{\partial T} n(t, T)dW(t) \end{aligned}$$

Compare with (5.2.7), the result follows.

Proposition 5.2.2

If $f(t, T)$ satisfies (5.2.8), then $r(t)$ satisfies (5.2.6) with

$$\begin{aligned} a(t) &= \frac{\partial f(t, T)}{\partial T} \Big|_{T=t} + \alpha(t) \\ b(t) &= \sigma(t) \end{aligned}$$

Proof

Integrate the forward rate dynamics for any $t \leq T$

$$f_t(T) = f(0, T) + \int_0^t [\alpha(s, T)ds + \sigma(s, T)dW(s)] \quad (5.2.9)$$

$$\partial f(t, T) = \partial f(0, T) + \int_0^t [\partial \alpha(s, T)ds + \partial \sigma(s, T)dW(s)] \quad (5.2.10)$$

$$\begin{cases} \alpha(s, T) = \alpha(s) + \int_s^T \partial \alpha(s, u)du \\ \sigma(s, T) = \sigma(s) + \int_s^T \partial \sigma(s, u)du \\ f(0, T) = r(0) + \int_0^T \partial f(0, u)du \end{cases} \quad (5.2.11)$$

substitute (5.2.11) into (5.2.9) with $T=t$

$$r(t) = r(0) + \int_0^t [\alpha(s)ds + \sigma(s)dW(s)] + \int_0^t \left(\partial f_0(u) + \int_0^u [\partial \alpha(s, u)ds + \partial \sigma(s, u)dW(s)] \right) du \quad (5.2.12)$$

Using (5.2.10)

$$r(t) = r(0) + \int_0^t [\partial f(s)ds + \alpha(s)ds + \sigma(s)dW(s)] \quad (5.2.13)$$

To obtain the required result, we differentiate (5.2.13)

$$dr(t) = \left(\frac{\partial f(t, T)}{\partial T} \Big|_{T=t} + \alpha(t) \right) dt + \sigma(t)dW(t) \quad (5.2.14)$$

Compare the result obtained with (5.2.6), thus we have

$$\begin{cases} a(t) = \frac{\partial f(t, T)}{\partial T} \Big|_{T=t} + \alpha(t) \\ b(t) = \sigma(t) \end{cases} \quad (5.2.15)$$

Proposition 5.2.3

If $f(t, T)$ satisfies (5.2.8), then $P(t, T)$ satisfies

$$dp(t, T) = p(t, T) \left\{ r(t) + A(t, T) + \frac{1}{2} \left| \int_t^T \sigma(s, t) ds \right|^2 \right\} dt + p(t, T) S(t, T) dW(t)$$

$$\begin{cases} A(t, T) &= - \int_t^T \alpha(s, t) ds \\ S(t, T) &= - \int_t^T \sigma(s, t) ds. \end{cases} \quad (5.2.16)$$

Proof

Using the definition of forward rate

$$P(t, T) = \exp(Y(t, T)) \quad (5.2.17)$$

where Y is given by

$$Y(t, T) = \left(- \int_t^T f(t, s) ds \right) \quad (5.2.18)$$

write (5.2.8) in integrated form,

$$f(t, s) = f(0, s) + \int_0^t \alpha(u, s) du + \int_0^t \sigma(u, s) dW(u). \quad (5.2.19)$$

Inserting this expression into (5.2.18), splitting integrals and changing the order of integration gives us

$$\begin{aligned} Y(t, T) &= - \int_t^T f(0, s) ds - \int_0^t \int_t^T \alpha(u, s) ds du - \int_0^t \int_t^T \sigma(u, s) ds dW(u) \\ &= - \int_0^T f(0, s) ds - \int_0^t \int_u^T \alpha(u, s) ds du - \int_0^t \int_u^T \sigma(u, s) ds dW(u) + \\ &\quad \int_0^t f(0, s) ds + \int_0^t \int_u^t \alpha(u, s) ds du + \int_0^t \int_u^t \sigma(u, s) ds dW(u) \\ &= Y(0, T) - \int_0^t \int_u^T \alpha(u, s) ds du - \int_0^t \int_u^T \sigma(u, s) ds dW(u) + \\ &\quad \int_0^t f(0, s) ds + \int_0^t \int_0^s \alpha(u, s) ds du + \int_0^t \int_0^s \sigma(u, s) ds dW(u). \end{aligned}$$

Integrating the forward rate dynamics over the interval $[0, s]$, we obtain

$$Y(t, T) = Y(0, T) - \int_0^t \int_u^T \alpha(u, s) ds du - \int_0^t \int_u^T \sigma(u, s) ds dW(u) + \int_0^t r(s) ds \quad (5.2.20)$$

differentiate (5.2.20) with respect to t ,

$$dY(t, T) = \{r(t) + A(t, T)\} dt + S(t, T)dW(t) \quad (5.2.21)$$

Apply Ito formula to the process $p(t, T) = \exp\{Y(t, T)\}$

$$dp(t, T) = \exp(Y(t, T)) dY_t(T) + \frac{1}{2} \exp(Y(t, T)) (dY_t(T))^2 \quad (5.2.22)$$

Substituting (5.2.21) into (5.2.22), we obtain

$$dp(t, T) = P(t, T) \left\{ r(t) + A(t, T) + \frac{1}{2} \left| \int_t^T \sigma(s, t) ds \right|^2 \right\} dt + P(t, T) S(t, T) dW(t). \quad (5.2.23)$$

Types of short rate models

Short rate models are generally categorised into two types namely; equilibrium models and no-arbitrage models.

Equilibrium model

The equilibrium model specifies that the short rate is a stochastic process, upon which the prices of bonds, options and interest rate derivatives depend (i.e initial term structure of interest rate is an output).

One Factor Equilibrium Model

A one-factor equilibrium model comprises two or more state variables with a single source of uncertainty for the interest rate r .

There are different forms of equilibrium models obtained under short rate Q dynamics.

1. Vasicek Model-

$$dr(t) = (a - br(t))dt + \sigma dW(t) \quad (5.2.24)$$

2. Cox, Ingersoll Model-

$$dr(t) = (a - br(t))dt + \sigma \sqrt{r(t)} dW(t) \quad (5.2.25)$$

3. Randleman and Bartter model-

$$dr(t) = \alpha(t)r(t)dt + \sigma r(t)dW(t) \quad (5.2.26)$$

No-arbitrage Model

A no-arbitrage model is designed to be consistent with the initial term structure of interest rate (i.e initial term structure of interest rate is input). The short rate no-arbitrage model is a function of time that is consistent with the initial term structure of interest rate. An equilibrium model can also be converted into a no arbitrage model. Different forms of no-arbitrage models are obtained under certain assumptions for the short rate dynamics of each model, with the coefficient of dt dependent on time.

1. Ho-Lee Model

$$dr(t) = \theta(t)dt + \sigma dW(t) \quad (5.2.27)$$

2. Hull and White

$$dr(t) = (\theta(t) - \alpha(t)r(t))dt + \sigma dW(t) \quad (5.2.28)$$

Remark 5.2.1

In our work, we shall study the Vasicek model as an example of a one factor short-rate model.

5.3 Vasicek Model [25]

The Vasicek(1977) model was the first one factor model proposed in the literature, under the assumption that the risk neutral process for the interest rate $r(t)$ is the Orstein Uhlenbeck process which satisfies:

$$dr(t) = (a - br(t))dt + \sigma dW(t) \quad (5.3.1)$$

where $r(0)$, a , b and σ are positive constants

Similarly the risk-neutral process for the the interest rate $r(t)$ can be written as

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t) \quad (5.3.2)$$

where $r(0)$, k , θ are positive constants.

To solve the equation,

Let $X_t = e^{bt}r_t$, we use Ito lemma for $t \geq 0$

$$\frac{\partial X}{\partial r} = e^{bt}, \quad \frac{\partial X}{\partial t} = e^{bt}br(t) \quad \text{and} \quad \frac{\partial^2 X}{\partial r^2} = 0$$

Thus,

$$X_t = X(0) + \int_0^t \frac{\partial X}{\partial r} dr + \int_0^t \frac{\partial X}{\partial t} dt + \frac{1}{2} \int_0^t \frac{\partial^2 X}{\partial r^2} d\langle r, r \rangle_t$$

Then,

$$\begin{aligned} X_t &= X(0) + \int_0^t e^{bs} dr + \int_0^t e^{bs} br_s ds \\ &= X_0 + \int_0^t (a - br(s))e^{bs} ds + \int_0^t e^{bs} br_s ds + \int_0^t e^{bs} \sigma dW \end{aligned}$$

since $dr(t) = (a - br(t))dt + \sigma dW(t)$. It follows that

$$r_t = r_0 e^{-bt} + \int_0^t a e^{b(s-t)} ds + \int_0^t e^{b(s-t)} \sigma dW$$

Integrating the first integral

$$r_t = r_0 e^{-bt} + \frac{a}{b} [1 - e^{-bt}] + \int_0^t e^{b(s-t)} \sigma dW \quad (5.3.3)$$

since the expectation of an Ito integral is zero, hence

$$\mathbb{E}[r_t] = r_0 e^{-bt} + \frac{a}{b} [1 - e^{-bt}] \quad (5.3.4)$$

Thus

$$\lim_{t \rightarrow \infty} \mathbb{E}[r_t] = \frac{a}{b}$$

Therefore, the Vasicek model has a mean reverting property since,

1. If $r(t) = \frac{a}{b}$, the drift term is zero
2. If $r(t) > \frac{a}{b}$, the drift term is negative, with movement back toward $\frac{a}{b}$
3. If $r(t) < \frac{a}{b}$, the drift term is positive, with movement back toward $\frac{a}{b}$.

Combining (5.3.3) and (5.3.4), we obtain that

$$\int_0^t e^{b(s-t)} \sigma dW = r_t - \mathbb{E}[r_t] \quad (5.3.5)$$

since $Var[r_t] = \mathbb{E}[r_t - \mathbb{E}[r_t]]^2$, thus

$$Var[r_t] = \mathbb{E} \left[\int_0^t e^{b(s-t)} \sigma dW \right]^2$$

Applying Ito's isometry

$$\begin{aligned}
\text{Var}[r_t] &= \sigma^2 e^{(-2bt)} \int_0^t e^{2b(s)} ds \\
&= \frac{-\sigma^2}{2b} e^{(-2bt)} \left[e^{2b(t)} - 1 \right] \\
&= \frac{\sigma^2}{2b} \left[1 - e^{(-2bt)} \right]
\end{aligned}$$

Thus,

$$\lim_{t \rightarrow \infty} \text{Var}[r_t] = \frac{\sigma^2}{2b}$$

Remark 5.3.1

The Vasicek model is a Gaussian mean reverting model with mean of $\frac{a}{b}$ as $t \rightarrow \infty$ and variance of $\frac{\sigma^2}{2b}$ as $t \rightarrow \infty$.

Probabilistic solution of PDE [13]

We can solve the PDE in (4.3.13), using the probabilistic bond pricing formula,

$$P(t, T) = \mathbb{E}^Q \left[e^{\int_t^T r(s) ds} \mid \mathcal{F}_t \right]$$

under the assumption that the short rate $(r_t)_{t \in \mathbb{R}^+}$ is given by

$$r_t = g(t) + \int_0^t h(t, s) dW(s) \tag{5.3.6}$$

In the Vasicek model, the interest rate $(r_t)_{t \in \mathbb{R}^+}$ satisfies (5.3.3) where

$$\begin{cases} g(t) &= r_0 e^{-bt} + \frac{a}{b} [1 - e^{-bt}] \\ h(t, s) &= \sigma e^{b(s-t)} \end{cases} \tag{5.3.7}$$

Using the properties of conditional expectation,

Then,

$$\begin{aligned}
P(t, T) &= \mathbb{E}^Q \left[e^{-\int_t^T r_s} | \mathcal{F}_t \right] \\
&= \mathbb{E}^Q \left[e^{-\int_t^T (g(s)ds + \int_0^s h(s, u)dB_u)ds} | \mathcal{F}_t \right] \\
&= e^{-\int_t^T g(s)ds} \mathbb{E}^Q \left[e^{-\int_t^T \int_0^s h(s, u)dB_u ds} | \mathcal{F}_t \right] \\
&= e^{-\int_t^T g(s)ds} \mathbb{E}^Q \left[e^{-\int_0^T \int_{u \vee t}^T h(s, u)ds dB_u} | \mathcal{F}_t \right] \\
&= e^{-\int_t^T g(s)ds} e^{-\int_0^t \int_{u \vee t}^T h(s, u)ds dB_u} \mathbb{E}^Q \left[e^{-\int_t^T \int_{u \vee t}^T h(s, u)ds dB_u} | \mathcal{F}_t \right] \\
&= e^{-\int_t^T g(s)ds} e^{-\int_0^t \int_t^T h(s, u)ds dB_u} \mathbb{E}^Q \left[e^{-\int_t^T \int_u^T h(s, u)ds dB_u} | \mathcal{F}_t \right] \\
&= e^{-\int_t^T g(s)ds} e^{-\int_0^t \int_t^T h(s, u)ds dB_u} \mathbb{E}^Q \left[e^{-\int_t^T \int_u^T h(s, u)ds dB_u} \right] \\
&= e^{-\int_t^T g(s)ds} e^{-\int_0^t \int_t^T h(s, u)ds dB_u} e^{\frac{1}{2} \int_t^T \left(\int_u^T h(u, s)ds \right)^2 du}
\end{aligned}$$

Substituting (5.3.7) into the expression above

$$\begin{aligned}
P(t, T) &= \mathbb{E}^Q \left[e^{-\int_t^T r_s} | \mathcal{F}_t \right] \\
&= e^{-\int_t^T (r_0 e^{bs} + \frac{a}{b}(1 - e^{-bs}))ds} e^{-\sigma \int_0^t \int_t^T e^{-b(s-u)} ds dB_u} e^{\frac{\sigma^2}{2} \int_t^T \left(\int_u^T e^{-b(s-u)} ds \right)^2 du} \\
&= e^{-\int_t^T (r_0 e^{bs} + \frac{a}{b}(1 - e^{-bs}))ds} e^{-\frac{\sigma}{b}(1 - e^{-b(T-t)}) \int_0^t e^{-b(t-u)} dB_u} e^{\frac{\sigma^2}{2} \int_t^T e^{2bu} \left(\frac{e^{-bu} - e^{-bT}}{b} \right)^2 du} \\
&= e^{-\frac{r_t}{b}(1 - e^{-b(T-t)}) + \frac{1}{b}(1 - e^{-b(T-t)}) (r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}))} \\
&\times e^{-\int_t^T (r_0 e^{-bs} + \frac{a}{b}(1 - e^{-bs}))ds + \frac{\sigma^2}{2} \int_t^T e^{2bu} \left(\frac{e^{-bu} - e^{-bT}}{b} \right)^2 du} \\
&= e^{B(T-t)r_t + A(T-t)}
\end{aligned}$$

where

$$B(T-t) = -\frac{1}{b} \left(1 - e^{-b(T-t)} \right)$$

and

$$\begin{aligned}
A(T-t) &= \frac{1}{b} \left(1 - e^{-b(T-t)}\right) \left(r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt})\right) - \int_t^T \left(r_0 e^{-bs} + \frac{a}{b}(1 - e^{-bs})\right) ds \\
&+ \frac{\sigma^2}{2} \int_t^T e^{2bu} \left(\frac{e^{-bs} - e^{-bT}}{b}\right)^2 du \\
&= \frac{1}{b} \left(1 - e^{-b(T-t)}\right) \left(r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt})\right) - \frac{r_0}{b} \left(e^{-bt} - e^{-bT}\right) - \frac{a}{b}(T-t) \\
&+ \frac{a}{b^2} \left(e^{-bt} - e^{-bT}\right) + \frac{\sigma^2}{2b^2} \int_t^T \left(1 + e^{-2b(T-t)} - 2e^{-b(T-u)}\right) du \\
&= \frac{a}{b^2} \left(1 - e^{-b(T-t)}\right) \left(1 - e^{-bt}\right) - \frac{a}{b}(T-t) + \frac{a}{b^2} \left(e^{-bt} - e^{-bT}\right) \\
&+ \frac{\sigma^2}{2b^2}(T-t) + \frac{\sigma^2}{2b^2} e^{-2bT} \int_t^T e^{2bu} du - \frac{\sigma^2}{b^2} e^{-bT} \int_t^T e^{bu} du \\
&= \frac{a}{b^2} \left(1 - e^{-2b(T-t)}\right) + \frac{\sigma^2 - 2ab}{2b^2}(T-t) + \frac{\sigma^2}{4b^3} \left(1 - e^{-2b(T-t)}\right) - \frac{\sigma^2}{b^3} \left(1 - e^{-b(T-t)}\right) \\
&= \frac{4ab - 3\sigma^2}{4b^3} + \frac{\sigma^2 - 2ab}{2b^2}(T-t) + \frac{\sigma^2 - ab}{b^3} e^{-(T-t)} - \frac{\sigma^2}{4b^3} e^{-2b(T-t)}.
\end{aligned}$$

Using Vasicek model, the bond pricing formula is obtained as

$$P(t, T) = \exp(A(T-t) + B(T-t)r_t) \quad (5.3.8)$$

where

$$\begin{cases} A(T-t) &= \frac{4ab-3\sigma^2}{4b^3} + \frac{\sigma^2-2ab}{2b^2}(T-t) + \frac{\sigma^2-ab}{b^3}e^{-(T-t)} - \frac{\sigma^2}{4b^3}e^{-2b(T-t)} \\ B(T-t) &= \frac{1}{b}(e^{-b(T-t)} - 1) \end{cases} \quad (5.3.9)$$

Definition 5.3.1

Affine Model [9]

A one-factor short rate model will be called an affine model, if the price of a zero coupon bond at initial time t with maturity T can be written as

$$P(t, T) = \exp[A(t, T) + B(t, T)r(t)] \quad (5.3.10)$$

for some deterministic functions A and B .

Proposition 5.3.1**Existence of Affine Model**

Assume that the Q dynamics for the short rate $r(t)$ is given by

$$dr(t) = \mu(t, r_t)dt + \sigma(t, r_t)dW_t \quad (5.3.11)$$

with μ and σ of the form

$$\mu(t, r_t) = \alpha(t)r + \beta(t) \quad (5.3.12)$$

$$\sigma(t, r(t)) = \sqrt{\theta(t)r + \lambda(t)} \quad (5.3.13)$$

for some deterministic functions $\alpha, \beta, \theta, \lambda$. Then the model is affine and the functions A and B satisfy the Riccati equations

$$\begin{cases} \frac{dB}{dt} = -\alpha(t)B - \frac{1}{2}\theta(t)B^2 + 1 \\ \frac{dA}{dt} = -\beta(t)B - \frac{1}{2}\lambda(t)B^2 \end{cases} \quad (5.3.14)$$

for $0 \leq t < T$, with boundary conditions $A(T, T) = B(T, T) = 0$.

Proof

Assume that the short rate model is affine with the Q dynamics for the short rate $r(t)$ given by (5.3.11)

Then,

$$P(t, T) = \exp[A(t, T) + B(t, T)r_t] \quad (5.3.15)$$

for deterministic functions A and B .

Then,

$$\frac{\partial}{\partial t}P(t, T) = P(t, T) \left(\frac{dA}{dt} + r \frac{dB}{dt} \right), \quad \frac{\partial}{\partial r}P(t, T) = P(t, T)B \quad \text{and} \quad \frac{\partial^2}{\partial r^2}P(t, T) = P(t, T)B^2$$

substitute the expression above into (4.3.13) with $F(t, r_t) = P(t, T)$, $a(t, r_t) = \mu(t, r_t)$ and $\sigma(t, r_t) = b(t, r_t)$ to obtain the result below,

$$P(t, T) \left(\left[\frac{dA}{dt} + r \frac{dB}{dt} \right] + (\alpha(t)r + \beta(t))B + \frac{1}{2}(\theta(t)r + \lambda(t))B^2 - r \right) = 0 \quad (5.3.16)$$

Since $P(t, T) \neq 0$, equate coefficients of each term with the R.H.S to obtain,

$$\frac{dA}{dt} = -\beta(t)B - \frac{1}{2}\lambda(t)B^2 \quad (5.3.17)$$

$$\frac{dB}{dt} = -\alpha(t)B - \frac{1}{2}\theta(t)B^2 + 1 \quad (5.3.18)$$

5.3.1 Pricing of zero-coupon bonds

In the Vasicek model, the price of a zero-coupon bond with initial time t and maturity T is given by

$$P(t, T) = \exp(A(T - t) + B(T - t)r) \quad (5.3.19)$$

where A and B are deterministic functions, with the boundary conditions $A(0) = B(0) = 0$.

Thus, with the bond price in the form of (5.3.19), this implies that the Vasicek model is affine, since $\mu(t, r) = a - br$ and $\sigma(t, r) = \sqrt{\sigma^2}$, which ensures the existence of the affine model.

To obtain the Riccati equations of (5.3.15), we substitute (5.3.19) into (4.3.13)

$$\frac{dA(s)}{ds} = aB(s) + \frac{\sigma^2}{2}B(s)^2 \quad (5.3.20)$$

$$\frac{dB(s)}{ds} = -bB(s) - 1 \quad (5.3.21)$$

We obtain the solution of the linear equation (5.3.17), using an integrating factor.

Thus,

$$\frac{d}{ds}(B(s)e^{bs}) = -e^{bs} \quad (5.3.22)$$

whence

$$B(s) = Ke^{-bs} - \frac{1}{b} \quad (5.3.23)$$

Substituting $B(0) = 0$ into (5.3.23), we get $K = \frac{1}{b}$

Then,

$$B(s) = \frac{1}{b}(e^{-bs} - 1) \quad (5.3.24)$$

To solve the Riccati equation

$$\frac{dA(s)}{ds} = aB(s) + \frac{\sigma^2}{2}B^2(s) \quad (5.3.25)$$

substitute $B(s) = \frac{1}{b}(e^{-bs} - 1)$ into (5.3.25), to obtain

$$dA = \frac{a}{b}(e^{-bs} - 1) + \frac{\sigma^2}{2bs}(e^{-2bs} - 2e^{-bs} + 1) ds$$

Then,

$$A(s) = -\frac{a}{b^2}e^{-bs} - \frac{as}{b} - \frac{\sigma^2}{4b^3}e^{-2bs} + \frac{\sigma^2 e^{-bs}}{b^3} + \frac{\sigma^2 s}{2b^2} + K \quad (5.3.26)$$

Substitute the boundary condition $A(0) = 0$, to obtain

$$K = \frac{4ab - 3\sigma^2}{4b^3}.$$

Then

$$A(s) = \left(\frac{\sigma^2 - ab}{b^3}\right) e^{-bs} + \left(-\frac{\sigma^2}{4b^3}\right) e^{-2bs} + \left(\frac{\sigma^2 - 2ab}{2b^2}\right) s + \frac{4ab - 3\sigma^2}{4b^3} \quad (5.3.27)$$

In particular,

$$A(T-t) = \left(\frac{\sigma^2 - ab}{b^3}\right) e^{-b(T-t)} + \left(-\frac{\sigma^2}{4b^3}\right) e^{-2b(T-t)} + \left(\frac{\sigma^2 - 2ab}{2b^2}\right) (T-t) + \frac{4ab - 3\sigma^2}{4b^3} \quad (5.3.28)$$

and also from (5.3.24)

$$B(T-t) = \frac{1}{b}(e^{-b(T-t)} - 1) \quad (5.3.29)$$

Remark 5.3.2

We have used an analytical approach for pricing zero-coupon bond by solving the PDE in (4.3.13). This is suitable for use in numerically simulations.

5.4 Conclusion

Pricing and modeling of interest rate derivatives and bonds have proven to be a complex area where extensive research needs to be done, to develop consistent techniques (models) using theoretical and numerical tools.

In this work, we have described approaches for the valuation of interest rate derivatives and bonds.

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