PRICING OF COMPOUND OPTIONS

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Dedication

To my parents who are the vanguard of a stable home, hence enable me to reach this academic height. Malam Tijjani Sa'ad Usman (may his soul rest in peace, amin), and Hajiya Hadiza Shehu Danbatta for her honest, and gallant effort towards the family, and her material interest towards my educational career.

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Contents

Chapter 1

INTRODUCTION AND PRELIMINARIES

1.1 Preliminaries

1.1.1 σ-algebra:

Let Ω be a non empty set, and β a non empty collection of subsets of Ω. Then β is called a σ-algebra if the following properties hold:
(i) $\Omega \in \beta$
(ii) If $A \in \beta$, then $A^{'} \in \beta$
(iii) If $\{A_j : j \in J\} \subset \beta$, then

$$\bigcup_{j\in J} A_j \in \beta$$

for any finite or infinite countable subset of $\mathbb{N}$.

1.1.2 Borel σ-algebra:

Let X be a non empty set and τ a topology on X i.e. τ is the collection of subsets of X. Then $\sigma(\tau)$ is called the Borel σ-algebra of the topological space (X, τ)

1.1.3 Probability Space:

Let Ω be a non-empty set and β be a σ-algebra of subsets of Ω. Then the pair (Ω,β) is called a **measurable space**, and a member of β is called a **measurable set**. Let (Ω,β) be a measurable space and μ be a real-valued map on β. Then μ is called a probability measure on (Ω,β) if the following properties hold:
I $\mu(A) \geq 0, \forall A \in \beta$
II $\mu(\Omega)=1$
III For $\{A_n\}_{n\in\mathbb{N}} \subset \beta$ with $A_j \cap A_k = \emptyset$, and $i \neq j$, then

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

i.e. μ is σ-additive (or countably additive).
Now if (Ω,β) is a measurable space and μ is a probability on (Ω,β), then the triple (Ω, β, μ) is called a **probability space**.

1.1.4 Measurable Map:

Let (Ω, β) and (Γ, ζ) be two measurable spaces. Then a map $X : \Omega \longrightarrow \Gamma$ is called measurable if the set $X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\}$ is in β whenever $A \in \zeta$. In particular, we take (Γ, ζ) to be $(\mathbb{R}, \beta(\mathbb{R}))$ or $(\mathbb{R}^n, \beta(\mathbb{R}^n))$ where $n \in \mathbb{N}$ and $\beta(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$.

1.1.5 Random variables/vectors:

Let (Ω, β, μ) be an arbitrary probability space and $(\mathbb{R}^n, \beta(\mathbb{R}^n))$ be the n-dimensional Borel measurable space. Then a measurable map $X : \Omega \longrightarrow \mathbb{R}^n$ is called a **random vector**. If $n = 1$, then X is called a **random variable.** We denote by $L^{\circ}(\Omega, \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued random vectors on Ω, and $L^1(\Omega, \beta, \mu)$ the space of random variables.

1.1.6 Probability Distribution:

Let (Ω, β, μ) be a probability space, $(\mathbb{R}^n, \beta(\mathbb{R}^n))$ be the n-dimensional Borel measurable space, and $X : \Omega \longrightarrow \mathbb{R}^n$ a random vector. Then the map $\mu_X : \beta(\mathbb{R}^n) \longrightarrow [0, 1]$ defined by $\mu_X(A) = \mu(X^{-1}(A)), A \in \beta(\mathbb{R}^n)$ is called the probability distribution of X.

1.1.7 Mathematical Expectation:

Let (Ω,β,μ) be a probability space. If $X \in L^1(\Omega, \beta, \mu)$, then

$$\mathbb{E}(X) = \int_\Omega X(\omega)d\mu(\omega)$$

is called the mathematical expectation or expected value or mean of X.

1.1.8 Variance and Covariance of random variables:

Let (Ω, β, μ) be a probability space and X an $\mathbb{R}$-valued random variable on Ω, such that $X \in L^2(\Omega, \beta, \mu)$. Then X is automatically in $L^1(\Omega, \beta, \mu)$ (because in general if $p \leq q$, then $L^q(\Omega, \beta, \mu) \subset L^p(\Omega, \beta, \mu)$ for all $p \in [1, \infty) \cup \{\infty\}$.) The **variance** of X is defined as

$$Var(X) = \mathbb{E}((X - \mathbb{E}(X))^2).$$

The number $\sigma_X = \sqrt{Var(X)}$ is called the **standard deviation/error.** Now let X, $Y \in L^2(\Omega, \beta, \mu)$. Then the **covariance** of X and Y is given by:

$$Cov(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y)))$$

1.1.9 Stochastic Process:

Let (Ω,β,μ) be a probability space. A stochastic process X indexed by a totally ordered set T (time), is a collection $X = \{X(t) : t \in T\}$, where each $X(t)$ or X_t is a random variable on Ω. We denote $X(t)$ by X_t and write the value of $X(t)$ or X_t at $\omega \in \Omega$ by $X(t, \omega)$ or $X_t(\omega)$. Thus, a stochastic process or random process is a collection of random variables, often used to represent the evolution of some random value, or system overtime.

1.1.10 Brownian Motion:

The Brownian motion refers to the ceaseless, irregular random motion of small particles immersed in a liquid or gas, as observed by R. Brown in 1827. The phenomena can be explained by the perpetual collisions of the particles with the molecules of the surrounding medium. Mathematically, let (Ω, β, μ) be a probability space, and $W = \{W(t) \in \; L^\circ(\Omega, \mathbb{R}^n) : t \in T\}$, where $T \subseteq \mathbb{R}_+ = [0, \infty)$, be an $\mathbb{R}^n$-valued stochastic process on Ω with the

following properties:
(i) $W(0) = 0$, almost surely.
(ii) W has continuous sample paths. i.e. If X is a stochastic process and $\omega \in \Omega$ then the map $t \longmapsto X(t,\omega) \in \mathbb{R}^n$ is called a sample path or trajectory of X. Now if the map is continuous we say X has a continuous sample paths.
(iii) $W(t) - W(s)$ is an $N(0,(t-s)\mathbb{T})$ random vector for all $t > s \geq 0$, where $\mathbb{T}$ is the $n \times n$ identity map.
(iv) W has a stochastically independent increments i.e. For every $0 < t_1 < t_2 < \cdot < t_k$, the random vectors $W(t_1)$, $W(t_2) - W(t_1), \cdot, W(t_k) - W(t_{k-1})$ are stochastically independent.
Then W is called the standard n-dimensional Brownian motion or n-dimensional Wiener process.

For the n-dimensional Brownian motion $W(t) = (W_1(t), \cdot \cdot \cdot , W_n(t))$ we have the following useful properties:
(I) $\mathbb{E}(W_j(t)) = 0, \quad j = 1, 2, 3, \cdot \cdot \cdot, n$
(II) $\mathbb{E}(W_j(t)^2) = t, \quad j = 1, 2, 3, \cdot \cdot \cdot, n$
(III) $\mathbb{E}(W_j(t)W_k(s)) = min(t,s)$ for t, s $\in \mathbb{T}$.

To show the result in III above, we assume $t > s$ (without loss of generality) and consider

$$\mathbb{E}[W_j(t)W_k(s)] = \mathbb{E}[(W_j(t) - W_k(s))W_k(s) + W_k(s)^2]$$

$$= \mathbb{E}[(W_j(t) - W_k(s))W_k(s)] + \mathbb{E}[W_k(s)^2]$$

(because $\mathbb{E}$ is linear). Then, since $W_j(t) - W_k(s)$ and $W_k(s)$ are independent and both $W_j(t) - W_k(s)$ and $W_k(s)$ have zero mean, so

$$\mathbb{E}[W_j(t)W_k(s)] = \mathbb{E}[W_k(s)^2] = s = min(t,s)$$

1.1.11 Filtrations and Filtered Probability space:

Let (Ω, β, μ) be a probability space and consider $\mathbb{F}(\beta) = \{\beta_t : t \in T\}$ a family of σ-algebras of β with the following properties:
(i) For each $t \in T$, β_t contains all the μ-null members of β,
(ii) $\beta_s \subseteq \beta_t$ whenever $t \geq s$, $s, \ t \in T$.
Then $\mathbb{F}(\beta)$ is called a **Filtration** of β and $(\Omega, \beta, \mathbb{F}(\beta), \mu)$ is called a **Filtered Probability Space** or **Stochastic Basis.**

1.1.12 Adaptedness:

A Stochastic process $X = \{X(t) \in L^{\circ}(\Omega, \mathbb{R}^n) : t \in T\}$ is said to be adapted to the filtration $\mathbb{F}(\beta) = \{\beta_t : t \in T\}$ if $X(t)$ is measurable with respect to β_t for each t∈T. It is plain that every stochastic process is adapted to its natural filtration.

1.1.13 Conditional Expectation:

Let (Ω, β, μ) be a probability space, X a real random variable in $L^1(\Omega, \beta, \mu)$ and ξ a σ-subalgebra of β. Then the conditional expectation of X given ξ written $E(X \mid \xi)$ is defined as any random variable Y such that:
(i) Y is measurable with respect to ξ i.e. for any $A \in \beta(\mathbb{R})$, the set $Y^{-1}(A) \in \xi$.
(ii)$\int_B X(\omega) d\mu(\omega) = \int_B Y(\omega) d\mu(\omega)$ for arbitrary $B \in \xi$.
A random variable Y which satisfies (i) and (ii) is called a version of $E(X \mid \xi)$.

1.1.14 Martingales:

The term martingale has its origin in gambling. It refers to the gambling tactic of doubling the stake when losing in order to recoup oneself. In the studies of stochastic processes, martingales are defined in relation to an adapted stochastic process. Let $X = \{X(t) \in L^1(\Omega, \beta, \mu) : t \in T\}$ be a real-valued stochastic process on a filtered probability space $(\Omega, \beta, \mathbb{F}(\beta), \mu)$. Then X is called a
(i) **Supermartingale** if $E(X(t) \mid \beta_s) \leq X(s)$ almost surely whenever t $\geq$ s.
(ii) **Submartingale** if $E(X(t) \mid \beta_s) \geq X(s)$ almost surely whenever t $\geq$ s.
(iii) **martingale** if X is both a submartingale and a supermartingale i.e. If E(X(t)| β_s) $= X(s)$ almost surely whenever t $\geq$ s.

1.1.15 Ito Calculus:

Let $(\Omega, \beta, \mathbb{F}(\beta), \mu)$ be a filtered probability space and W a Brownian motion relative to this space. We define an integral of the form

$$W(f, t) = \int_0^t f(s) dW(s), t \in \mathbb{R}_+$$

where f belongs to some class of stochastic processes adapted to $(\Omega, \beta, \mathbb{F}(\beta, \mu)$.

1.1.16 Quadratic Variation:

Let X be a stochastic process on a filtered probability space $(\Omega, \beta, \mathbb{F}(\beta), \mu)$. Then the quadratic variation of X on $[0, t]$, t $>$ 0, is the stochastic process $\langle X \rangle$ defined by

$$\langle X \rangle(t) = \lim_{|\mathbb{P}| \to\ 0} \sum_{j=0}^{n-1} |X(t_{j+1}) - X(t_j))|^2$$

where $\mathbb{P} = \{t_\circ, t_1, \cdot,\ t_n\}$ is any partition of $[0, t]$ i.e. $0\ =\ t_1 < t_2 < \cdot < t_n = t$ and $|\mathbb{P}| = max_{0 \leq j \leq n-1}|t_{j+1} - t_j|$
If X is a differentiable stochastic process, then ⟨X⟩=0.

1.1.17 Stochastic Differential Equations:

These are equations of the form

$$dX(t) = g(t, X(t))dt + f(t, X(t))dW(t)$$

with initial condition $X(t_\circ) = x_\circ$

1.1.18 Ito Formula and Lemma:

Let $(\Omega, \beta, \mathbb{F}(\beta), \mu)$ be a filtered probability space, X an adapted stochastic process on $(\Omega, \beta, \mathbb{F}(\beta), \mu)$ whose quadratic variation is ⟨X⟩ and $U \in C^{1,2}([0, 1] \times \mathbb{R})$.
Then,

$$U(t, X(t)) = U(s, X(s)) + \int_s^t \frac{\partial U}{\partial t}(\tau, X(\tau))ds + \int_s^t \frac{\partial U}{\partial x}(\tau, X(\tau))dX(\tau) + \frac{1}{2}\int_s^t \frac{\partial^2 U}{\partial x^2}(\tau, X(\tau))d\langle X \rangle(\tau)$$

which may be written as

$$dU(t,x) = \frac{\partial U}{\partial t}(t,X(t))dt + \frac{\partial U}{\partial x}(t,X(t))dX(t) + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}(t,X(t))d\langle X\rangle(t)$$

The equation above is normally referred to as the **Ito formula.** If X satsifies the stochastic differential equation (SDE)

$$dX(t) = g(t,X(t))dt + f(t,X(t))dW(t)$$
$$X(t_\circ) = x_\circ;$$

then

$$dU(t,X(t)) = g_u(t,X(t))dt + f_u(t,X(t))dW(t)$$
$$U(t_\circ, X(t_\circ)) = U(t_\circ, x_\circ)$$

where

$$g_u(t,x) = \frac{\partial U}{\partial t}(t,x) + g(t,x)\frac{\partial U}{\partial x}(t,x) + \frac{1}{2}(f(t,x))^2\frac{\partial^2 U}{\partial x^2}(t,x);$$
$$f_u(t,x) = f(t,x)\frac{\partial U}{\partial x}(t,x)$$

We obtain a particular case of the Ito formula called the **Ito lemma**, if we take $X = W$, where $g \equiv 0$ and $f \equiv 1$ on $\mathbb{T}\times\mathbb{R}$. Then

$$dU(t,W(t)) = [\frac{\partial U}{\partial t}(t,W(t)) + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}(t,W(t))]dt + \frac{\partial U}{\partial x}(t,W(t))dW(t)$$

The equation above is referred to as the **Ito lemma**.

1.1.19 Risk-neutral Probabilities:

These are probabilities for future outcomes adjusted for risk, which are then used to compute expected asset values. The benefit of this risk-neutral pricing approach is that once the risk-neutral probabilities are calculated, they can be used to price every asset based on its expected payoff. These theoretical risk-neutral probabilities differ from actual real world probabilities; if the latter were used, expected values of each security would need to be adjusted for its individual risk profile. A key assumption in computing risk-neutral probabilities is the absence of arbitrage. The concept of risk-neutral probabilities is widely used in pricing derivatives.

1.1.20 Log-normal Distribution:

A random variable X is said to have a lognormal distribution if its logarithm has a normal distribution. i.e. $ln(X) \sim N(\mu, \sigma)$, meaning logrithim of X is distributed normal with mean μ and variance σ.

1.1.21 Bivariate Normal Density Function:

The bivariate normal density function is given by:

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp[- \frac{x^2 - 2xy\rho + y^2}{2(1-\rho^2)}$$

1.1.22 Cumulative Bivariate Normal Distribution Function:

The standardised cumulative normal distribution function returns the probability that one random variable is less than "a", and that a second random variable is less than "b" when the correlation between the two variables is ρ and is given by:

$$M(a, b; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{a} \int_{-\infty}^{b} exp[- \frac{x^2 - 2xy\rho + y^2}{2(1-\rho^2)}] dxdy$$

1.1.23 Markov Process:

A Markov process is a stochastic process satisfying a certain property, called the Markov property. Let (Ω, β, μ) be a probability space with a filtration $\mathbb{F}(\beta) = \{\beta_t : t \in T\}$ for some totally ordered set T, and let (S, κ) be a measurable space. An s-valued stochastic process $X = \{X_t : t \in T\}$ adapted to the filtration is said to posses the **Markov property** with respect to the filteration $\mathbb{F}(\beta)$ if, for each $A \in \kappa$ and s, $t \in$ T with s $<$ t,

$$P(X_t \in A | \beta_s) = P(X_t \in A | X_s)$$

A **Markov process** is a stochastic process which satisfies the Markov property with respect to its natural filtration.

1.1.24 Backward Kolmogorov Equation:

The Kolmogorov backward equation (diffusion) is a partial differential eqaution (PDE) that arises in the theory of continuous-time Markov processess. Assume that the system state $X(t)$ evolves according to the stochastic differential erquation

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW(t)$$

then the **Kolmogorov backward equation** is as follows

$$-\frac{\partial}{\partial t}p(x,t) = \mu(x,t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2}p(x,t)$$

for $t \leq$ s, subject to the final condition $p(x, s) = u_s(x)$. This can be derived using Ito's lemma on $p(x, t)$ and setting the dt term equal to zero.

1.1.25 Forkker-Planck Equation:

The Fokker-Planck equation describes the time evolution of the of the velocity of a particle, and can be generalised to other observables as well. It is also known as the Kolmogorov forward equation (diffusion). In one spatial dimension X, for an Ito process given by the stochastic differential equation

$$dX_t = \mu(X_t, t)dt + \sqrt{2D(X_t, t)}dW_t$$

with drift $\mu(X_t, t)$ and diffusion coefficient $D(X_t, t)$, the **Fokker-Planck equation** for the probability density $f(x, t)$ of the random variable X_t is

$$\frac{\partial}{\partial t}f(x,t) = -\frac{\partial}{\partial x}[\mu(x,t)f(x,t)] + \frac{\partial^2}{\partial x^2}[D(x,t)f(x,t)]$$

The Fokker-Planck also exist in many dimensions, but we are going to restrict ourselves to one dimension only.

1.1.26 Diffusion Process:

A diffusion process is a solution to a stochastic differential equation. It is a continuous-time Markov process with almost surely continuous sample paths. Mathematically, it is a Markov process with continuous sample paths for which the Kolmogorov forward equation is the Forkker-Planck equation. Brownian motion, reflected Brownian motion and Ornstein-Uhlenbeck processes are examoles of diffusion process.

1.2 Introduction

An option is a financial instrument that specifies a contract between two parties for a future transaction at a reference price. This transaction can be to buy or sell an underlying assets such as stocks, bonds, an interest rate e.t.c. The option holder has the right but not the obligation to carry out the specific transaction (i.e. to buy if it is a "call option" or to sell if it is a "put option") at or by a specified date (reference time).

A European option give the holder the right but not the obligation to buy, (if it is a call) or to sell (if it is a put), an underlying asset on the specified time or maturity date at the specified price. While an American option, give the holder the right but not the obligation to buy, or sell an underlying asset on or prior to the specified time or maturity date at the specified price.

A compound option is an option on an option. Hence, the compound option, or the mother option gives the holder the right but not the obligation to buy, or sell another underlying option, the daughter option; for a certain strike price K_1 at a specified time T_1. The daughter option then gives the holder another right to buy or sell a financial asset for another strike price K_2 at a later point in time T_2. So, a compound option has two strike prices, and two expiration dates. Also, Compound options are very frequently encountered in capital budgeting problems when projects require sequential decisions. For example, when dealing with development projects, the initial development expense allows one later to make a decision to wait or, to engage in further development expenses eventually leading to a final capital investment project. All R&D expenditures involve a sequence of decisions. In the mining and extraction industries, one conducts geological surveys that will lead to the opening of a mine, or to the decision to drill. Then, the owner of the mine, or the drilling platform can any day stop operations, and begin them again later. An investment in the production of a movie, might lead to sequels. The value of a sequel is the value of a compound option.

This project is divided into five chapters; chapter one is the preliminaries and introduction, chapter two is the Literature Review. Chapter three will consist of financial derivatives and compound options, where we'll give a full explanation of what compound option is all about. As in the case of pricing and valuation of other financial instruments (bonds or stocks) or derivatives (futures or swaps), options too can be priced to avoid underestimates or overestimates of the prices. As such, option pricing theory is one of the cornerstones, and most successful theory in finance and economics as described by Ross. Therefore, chapter four will deal with pricing of compound options, where we are going to give some methods that are used in pricing compound options, which is the main work of the project. Black-Scholes formula for pricing compound options, forward valuation of compound options will also be discussed, where we use the Forkker-Planck equation and backward Kolmogorov equation to obtain the formula for pricing compound options. We will also discuss the binomial lattice model or binomial tree model for pricing sequential compound options. Finally, chapter five will deal with applications.

Chapter 2

LITERATURE REVIEW

Black and Scholes (1973) mentioned in their seminal paper that, most corporate liabilities may be seen as options. As well as their famous pricing formulas for vanilla European call and put options. They also discussed how to evaluate the equity of a company that has coupon bonds outstanding. They suggested that the equity can be viewed as a "Compound option", because the equity "is an option on an option on . . . an option on the firm. Geske (1977) derived formulas for valuing coupon bonds and subordinated debt as compound options; while Ross (1977) has used this technique to value American options on stocks paying constant dividend. Myer in the same year, also suggested that corporate investment opportunities may be represented as options. In that settling, common stock is again a compound option. Insurance policies with sequential premiums offer another application policies with compound options.

As we knew earlier, compound options are options with other options as underlying assets. The fold number of a compound option counts the number of option layers attached to directly onto underlying options. Geske (1979) developed the original closed-form formula of 2-fold compound option (or vanilla European call on a European call), and shows that the standard Black and Scholes framework is a special case of such a formula. Rubbistein (1991) generalises this result to all four possible combinations: Call on a call, call on a put, put on a call, and put on a put, and includes techniques for American options.

Specific multi-fold compound option pricing formula were proposed by Geske and Johnson (1984) and Carr (1988), while the pricing formula for sequential compound call is proved by Thomassen and Van Wouwe (2001).

Gukhal (2003) derives analytical valuation formulas for compound options when the underlying asset follows a Jump-diffusion process, applying these results to value extendible options, American call options on stock that pay discrete dividends, and American options on assets that pay continuous proportional dividends.

Chen (2002) and Lajeri-Chaherli (2002) simultaneously derive the price formula for 2-fold compound options through risk neutral method. Agliardi and Agliardi (2003) generalise the result to 2-fold compound options calls with time dependent parameters, while Agliardi and Agliardi (2005) extend the multi-fold compound calls to parameters varying with time. Roll (1977), Whalley (1981) and Selby and Hodges (1987) also study compound options.

It turns out that a wide variety of important problems are closely related to the valuation of compound options. Some include pricing American puts & hedging volatility risk by trading options on straddles in Brenner & Zhang (2006). Han (2003) in his thesis and Fuoque & Han (2005) introduce a fast, and robust approximation to compute the prices of compound options, such as call-on-call options, within the context of stochastic volatility models. However, they only consider the case of a European option on a European option.

Furthermore, their method relies on certain expansions so its range of validity is not entirely clear. The main difficulties in using the Black-Scholes differential equation when dealing with compound options is that it assumes that the variance of the return/volatility on the stock is constant. However, with Compound options, this variance is not constant, but depends on the level of the stock price, or more fundamentally, on the value of the firm.

Several papers focused on issues related to Compound options, particularly on their valuation. One reason for such an interest is that Compound options form an important foundation for the pricing of many of options. First and foremost, in a straightforward manner, for options whose payoff is the function of a European vanilla option at a future point in time, such as chooser, forward start and cliquet options.

In addition, compound options can also be used as building blocks to options with a more exotic payoff, so that their valuation is instrumental for the valuation of the exotic option. Moreover compound options can be employed to approximate options involving a sequence of exercise decisions, like Bermudan options, instalment options and American options.

Most importantly, compound options can provide a useful instrument to traders for hedging volatility risk in practice. Additionally, compound options widely consist in staged investments, such as investment in new technologies, pharmaceutical drug development program, and investments into technology platforms.

It is difficult to find a literature studying the Compound option pricing problem under both stochastic volatility and stochastic interest rates. Some authors though, discuss the American option pricing problem under these dynamics. Boyarchenko and Levendorski (2007), formulate the option pricing problem by a Partial Differential Equation (PDE) approach, and they calculate the option prices with the help of an iteration method based on Wiener-Hopf factorization. Medvedev and Scallet (2010), introduce a new analytical approach. After using an explicit and intuitive proxy for the exercise rule, the derive tractable pricing formulate using short-maturity asymptotic expansion. Depending on model parameters, this method can accurately price options with time-to-maturity up to several years.

Considering European options on European options under Geometric Brownian Motion (GBM) dynamics, there exists almost explicit integral-form solution. Nevertheless, in situation involving more general dynamics like stochastic volatility, either explicit solutions do not exist, or the integrals become difficult to evaluate. In contrast, it turns out that the PDE approach provides a very accurate, efficient and flexible way to compute prices of compound options. One interesting thing is that, the use of this approach is not restricted to European type options, and can also include American type, Asian type or other exotic options.

Chapter 3

FINANCIAL DERIVATIVES AND COMPOUND OPTIONS

3.1 FINANCIAL DERIVATIVES

A financial derivative is a contract between individuals, or institution whose value at the maturity date (or expiry date), is uniquely determine by the value of the underlying assets at time T, or until time T. In practice, it is a financial contract between two parties that specifies (conditions especially the dates, resulting values, and definitions of the underlying variables, the parties contractual obligations, and the national amount) under which payments are to be made between the parties. Common examples of underlying assets are stocks, bonds, commodities, and currencies.

A derivative can also be regarded as a kind of asset, the ownership of which entitles the holder to receive from the seller a cash payment, or possibly a series of each payments at some point in time in the future; depending in some pre-specified way on the behaviour of the underlying assets over the relevant time interval. In some instances, instead of a cash payment another asset might be delivered. For example, a basic stock option allows the holder to purchase shares at some point in the future for a specified price. In derivatives transactions, one party's loss is always another party's gain. As such, the main aim of derivatives is to transfer risk from one person or firm to another, that is, to provide insurance. If a farmer before planting can guarantee a certain price he will receive, he is more likely to plant. Derivatives improve overall performance of the economy.

3.2 CATEGORIES OF DERIVATIVES

3.2.1 Forwards

A forward, or forward contract, is an agreement between a buyer, and a seller to exchange a commodity, or a financial asset, for a pre-specified price called the delivery price, on a prearranged future date called the delivery time (T). One of the parties to a forward contract assumes a long position, and agree to buy the underlying asset on a certain specified future date for a certain specified price. The other party assumes a short position, and agrees to sell the asset on the same date for the same price. At the time the contract is entered into, the delivery price is chosen so that the value of the forward contract to both parties is zero. This means that it costs nothing to take either a long or a short position.

The payoff from a long position in a forward contract on one unit of an asset is $S(T) - K$. This is because the holder of the contract is oblige to buy an asset worth $S(T)$ for K. Similarly, the payoff from a short position in a forward contact on one unit of an asset is $K - S(T)$. These payoffs can be positive or negative. Since it costs nothing to enter into a forward contract, the payoff from the contract is also the investor's total gain or loss from the contract.

3.2.2 Futures

A futures contract like a forward contract , is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price. Unlike forward contracts, futures contracts are normally traded on exchange. This kind of contract is marked to market daily, i.e. it is revalued daily to reflect the current values of relevant market variables. A futures contract generally has commodity as its underlying asset, consequently if specialization is often more detailed.

To make trading possible, the exchange specifies certain standardized features of the contract. As the two parties to the contract do not necessarily know each other, the exchange also provides a mechanism which gives the two parties a guarantee that the contract will be honoured.

3.2.3 Swaps

A swap is a contract in which the two investors undertake to exchange at known dates in the future, various financial assets. An example is a currency swap; commodity swaps and interest rate swaps.

3.2.4 Options

An option is a contract that gives the holder the right but not the obligation to exercise a certain transaction on the maturity date T, or until the maturity date at a fixed price K, the so-called exercise price, (or strike price). Options are agreements between two parties. The seller of the option is called the option writer, while the buyer of the option is called an option holder. The buyer of the option gains the right but not the obligation, to engage in that transaction, while the seller incurs the corresponding obligation to fulfil the transaction. There are two types of option namely: Call option and Put option.

CALL OPTION

A call option gives a holder the right to buy an asset at a predetermined price (strike price), on or before a specific date. The holder is said to hold a long position on the option. If the asset price at maturity date T is higher than the strike price i.e. $S(T) > K$, then the option is in-the-money. If the asset price at maturity date T is exactly the strike price i.e. $S(T) = K$, then the option is at-the-money, and If the asset price at maturity date T is lower than the strike price i.e. $S(T) < K$, then the option is out-of-the-money. Obviously would not exercise an option that is out of the money.

PUT OPTION

A put option gives a holder the right to sell an asset at a predetermined price (strike price), on or before a specific date. The financial agent is said to hold a short position on the option. If the asset price at maturity date T is lower than the strike price i.e. $S(T) < K$, then the option is in-the-money. If the asset price at maturity date T is exactly the strike price i.e. $S(T) = K$, then the option is in-the-money, and If the asset price at maturity date T is higher than the strike price i.e. $S(T) > K$, then the option is out- of-the-money. Which is the opposite of a call option.

The process of activating an option and thereby trading the underlying asset at the agree-upon price is referred to as exercising it. Most options has an expiration date. So, if the option is not exercised by expiration date, it becomes void and worthless.

EUROPEAN AND AMERICAN OPTIONS

EUROPEAN OPTIONS

A European option gives the holder the right, but no obligation to buy (if it is a call option), or to sell (if it is a put option) only on the expiry (or maturity) date at the specified price.

AMERICAN OPTIONS

An American option gives holder the right, but no obligation to buy (if it is a call option), or to sell (if it is a put option), on or prior to the expiry (maturity) date. i.e. any time before the maturity date, at the specified price.

Some race track terms have slipped into the options vocabulary: An option finishes in-the-money; if it has a positive value at expiration. It finishes out-of-the-money if its exercise value is negative at expiration. Before expiration, options would be in-the-money , at-the-money , or out-of-the-money, if they, when exercised immediately, resulted in a positive, zero, or negative value, respectively.

A hedge portfolio is a riskless portfolio with respect to changes in the price of its components. Holding a long position of a security is a strategy that involves owning the security itself and leads to profits when prices increase. Holding a short position of a security is equivalent to selling a security that is not owned. It involves borrowing the security and is profitable when prices fall.

PAYOFF OF A EUROPEAN OPTION

Since an option gives the holder a right, it has a value which is called option price.

Call Option:

We denote by $C(t)$ the value of a call option at time t, and by $S(t)$ the value of the financial asset at time t. We distinguish the following two cases:

1. At the maturity date T, the value $S(t)$ of the asset is higher than the strike price K. The call option is then exercised; i.e. the holder buys

the asset for market price $S(T)$ and sells it to the writer at price K. The holder realizes the profit $V(S(T), T) = C(T) = S(T) - K$.

2. At the maturity date T, the value $S(T)$ of the asset is less than or equal to the strike price. In this case, the holder does not exercise the call option. i.e. the option expires worthless with $V(S(T), T) = C(T) = 0$.

   In summary, at the maturity date T, the value of the call is given by the payoff function

$$V(S(T), T) = (S(T) - K)^+ = max(S(T) - K, 0)$$

Put Option: We denote by $P(t)$ the value of a put option at time t, and by $S(t)$ the value of the financial asset at time t. We distinguish the following cases as well:

1. At the maturity date T, the value $S(t)$ of the asset is less than the strike price K. The put option is then exercise. i.e. the holder buys the asset for market price $S(T)$ and sells it to the writer at price K. The holder realizes the profit $V(S(T), T) = P(T) = K - S(T)$.

2. At the maturity date T, the value $S(T)$ of the asset is greater than or equal to the strike price. In this case, the holder does not exercise the put option. i.e. the option expires worthless with $V(S(T), T) = P(T) = 0$.

In summary, at the maturity date T, the value of the put is given by the payoff function

$$V(S(T), T) = (K - S(T))^+ = max(K - S(T), 0)$$

PUT-CALL PARITY

Put-call parity gives a relationship between the price of a European call option, and a European put option, both with identical strike price, and expiry; namely that a portfolio of a long a call option and short a put option is equivalent to (and hence has the same value as) a single forward contract at this strike price, and expiry.
Theorem: (Put-call parity)
A European call $C(S(T), T)$ and a European put $P(S(T), T)$ with the same

strike price K and maturity T on an underlying asset S paying no dividend are related as follows:

$$S(t) + P(S(T), T) - C(S(T), T) = Ke^{(-r(T-t))}$$

where r is the riskless interest rate and market price is arbitrage-free.

3.2.5 FINANCIAL MARKETS

A market is an actual, or nominal place where forces of demand, and supply operates, and where buyers, and sellers interact (directly or through intermediaries) to trade goods, services, contracts or instruments, for money or barter. Markets include mechanisms, or means for determining price of the traded item, communicating the price information, facilitating deals, and transactions, and effecting distribution. The market for a particular item is made up of existing, and potential customers who need it, and have the ability, and willingness to pay for it.

Derivatives market is the financial market for derivatives, financial instruments like futures contracts, or options which are derived from other assets. Financial derivatives are traded in two kinds of markets namely: Derivative exchange and Over-the-counter (OTC) markets.

Derivative exchange:

These are regulated markets where investors trade standardised contracts.

Over-the-counter (OTC) markets:

This kind of market is done by telephone, or computer, and is often between two financial institutions.

3.2.6 TYPES OF TRADERS

Hedgers

Hedgers use futures, forwards, options, and swap to reduce the risk that they face from potential future movements in market variables. They are interested in reducing risk that they already face. Hedgers prefer to forgo the chance to make exceptional profits, even if future uncertainty appears to work to their advantage; by protecting themselves against exceptional loss.

Speculators

Whereas hedgers want to eliminate an exposure to movements in the price of an asset, speculators wish to take a position in the market. They use derivative securities to bet on the future direction of a market. They take the opposite position to hedgers, in that they are always out to make opportunistically high profits. Either they are betting that a price will go up, or they are betting that it will go down.

Speculators are needed in financial markets to make hedging possible. Since a hedger wishing to lay off risk cannot do so unless someone is willing to take it on. Forward contracts can be used for speculation. An investor who thinks that sterling pounds will increase in value relative to the US dollar, can speculate by taking a long position in a forward contract on sterling pounds.

There is an important difference between speculating using forward markets, and speculating by buying the underlying asset (in this case, a currency) in the spot market. Buying a certain amount of the asset in the spot market requires no initial cash payment equal to the total value of what is bought. Entering into a forward contract on the same amount of the assets, require no initial cash payment. Speculating using forward markets therefore provides an investor with a much higher level of leverage than speculating using spot markets. Option too gives extra leverage when used for speculation. Sellers of futures bet on price decreases, while buyers of futures bet on price increases.

Arbitrageurs

Practice of simultaneously buying, and selling financial instruments to benefit from temporary price difference is called arbitrage. They try to lock in risk less profit by simultaneously engaging themselves into transactions in two or more markets. In other words, Arbitrageurs borrows funds to buy an instrument(s), and sells the futures contract, get the difference in prices as instant profits. For example, an instrument can be bought in Kano at one price, and sold at a slightly higher price in Abuja. Market eliminates such opportunities. i.e. there are no arbitrage opportunities. This eliminates the presence of Arbitrageurs.

Financial derivatives are used for a number of purposes including: Risk management, trading efficiency, speculation. e.t.c.

3.2.7 EXOTIC OPTIONS

Derivatives securities with more complicated payoffs than the standard European or American calls, and puts are sometimes referred to as Exotic options. Some of the Exotic options include:

Asian Options:

An Asian option is a special type of option contract. It is like the European option, but with the difference that the final underlying asset price is taken as the average of the underlying asset price over the period of the option. The payoff P of an Asian option is given by:

$$P = max\{\frac{1}{T}\int_0^T S(t)dt - K, 0\}$$

Barrier Option

A Barrier option differs from a vanilla option in that part of the option contract is triggered if the asset price hits some barrier, $S = X$, say, at any time prior to expiry. Being either calls or puts, barrier options are categorized as follows:

Up and In

The option expires worthless unless the barrier $S = X$ is reached from below before expiry.

Down and In

The option expires worthless unless the barrier $S = X$ is reached from above before expiry.

Up and out

The option expires worthless if the barrier $S = X$ is reached from below before expiry.

Down and out

The option expires worthless if the barrier $S = X$ is reached from above before expiry.
Some barrier options specify a rebate, usually a fixed amount paid to the holder if the barrier is reached in the case of outbarriers or not reached in the case of in-barriers.

Lookback option

This is an option whose payoffs depend on the maximum, or minimum price reached during the life of the option. If S_1 is the minimum price reached, S_2 is the maximum price reached, and $S(T)$ is the final price reached, the payoff from a lookback call is $max(0, S(T) - S_1)$, and the payoff from a lookback put is $max(0, S_2 - S(T))$.

Basket option

A type of financial derivative where the underlying asset is a group of commodities, securities, or currencies. Like other options, a basket option gives the holder the right, but not the obligation, to buy, or sell an underlying asset at a specific price, on or before a certain date (the holder has the option to buy or sell, or to let the option expire worthless). With a basket option, however, the holder has the right, but not the obligation, to buy, or sell a group of underlying assets.

A currency basket option provides a more cost effective method for multinational corporations to manage multi-currency exposures on a consolidated basis. For example, a global corporation such as McDonald's might buy a basket option involving Indian rupees, and British pounds, in exchange for U.S. dollars. The currency basket option has all the characteristics of a standard option, but the strike price is based on the weighted value of the component currencies (calculated in the holder's base currency). A basket option often costs less than multiple single options.

Real options

Real options can include opportunities to expand, and cease projects if certain conditions arise, amongst other options. They are referred to as "real" because they usually pertain to tangible assets such as capital equipment,

rather than financial instruments. Taking into account real options can greatly affect the valuation of potential investments. Often times, however, valuation methods, such as NPV, do not include the benefits that real options provide. Note that this kind of option is not a derivative instrument, but an actual option (in the sense of "choice") that a business may gain by undertaking certain endeavours. For example, by investing in a particular project, a company may have the real option of expanding, downsizing, or abandoning other projects in the future. Other examples of real options may be opportunities for R&D, M&A, and licensing.
Some types of real options are:
I **Option to expand:** Here the project is built with capacity in excess of the expected level of output so that it can produce at higher rate if needed. Management then has the option (but not the obligation) to expand i.e. exercise the option should conditions turn out to be favourable. A project with the option to expand will cost more to establish, the excess being the option premium, but is worth more than the same without the possibility of expansion. This is equivalent to a call option.
II **Option to abandon:** When market conditions, or operation performance became worse, and cash flows are far below expectations, it is useful to have the option to bail out, and recover the value of the project's plant, equipment, or other assets (Brealey et al, 2006). This option is called option to abandon.

Chooser option

A Chooser option, or an as-you-like-it option, gives its owner the right to purchase, for an amount/price K_1 at time T_1, either a call or a put with exercise price K_2 at time T_2 i.e. call on a call or put.

Compound Options

A compound option is simply an option on an option. It is an option whose underlying is another option. Hence, compound option (the mother option) gives the holder the right, but not the obligation to buy (long), or sell (short) the underlying option (the daughter option). The option holder has the right at time T_1 to pay price K_1 (compound strike) to buy, or sell the daughter option. The daughter option then gives the holder another right to buy, or sell a financial asset for another price K_2 at time T_2. At first exercise date T_1, you must decide whether it is worth exercising the first option (daughter), depending on the strike price K_1 (compound strike) and the current asset price S. If so, you get a further option with strike price K_2 and maturity T_2.

The purchaser is entitled to purchase a call option (put option) with a fixed strike price, and fixed maturity at an agreed price, at a pre-determined future date. In this case, the price at which the agreed option may be purchased at maturity is the strike price (compound strike). At maturity, the current market price is compared with the agreed price. If the market price is lower than the compound strike, the client allows the option to expire; if the compound strike is lower than the market price, the client achieves cost-effective hedging. The higher the price the client is prepared to pay immediately, the more attractive the compound strike. However, the total price is always more expensive than that of a normal option. In the best-case scenario (not exercised), the hedge works out less expensive because of the relatively small advance price. For this reason, this option is particularly suitable for a company that cannot gauge whether, or not a pending transaction (bid deadline) might subsequently entail a degree of currency risk.

Compound options are common in many multiphase projects, such as product, and drug development, where initiation of one phase of the project depends on the successful completion of the preceding phase. For example, launching a product that involves a new technology requires successful testing of the technology; drug approval is dependent on successful phase two trials, which can be conducted only after the end of each phase one tests. With compound options, at the end of each phase, one has the option to continue to the next phase, abandon the project, or defer it to a later time. Each phase becomes an option that is contingent upon the exercise of earlier options. For phased projects, two, or more phases may occur at the same time (parallel options), or in sequence (staged or sequential options).

A compound option provides their owner with the right to buy, or sell another option; and these options create positions with greater leverage than do traditional options. There are four basic types of compound options:

Call on a Call:

A type of compound option in which the holder has the right to exercise a call on the underlying asset, which is an option. A holder who owns a call on a call option has until the expiration date to exercise the compound option.

If exercised, the holder will receive the underlying call option, which will have a set expiration date and a new exercise price. If the underlying option is exercised, the holder receives the underlying assets. The value of the call on a call option (with the underlying good being a stock) increases as the stock's price increases. The holder will exercise the call on a call option if, at the expiration date, the price of the underlying call option is worth more than the exercise price of the option.

In summary, a call on a call option gives the holder the right at time T_1, to buy an underlying call option at price K_1, and then the underlying call option again provides the holder with another right at time T_2 to buy the underlying asset at price K_2.

Call on a Put:

This is a call option on an underlying put option. If the option holder exercises the call option, he or she receives a put option, which is an option that gives the holder the right but not the obligation to sell a specific asset at a set price within a defined time period. The value of a call on a put changes in inverse proportion to the stock price, i.e. it decreases as the stock price increases, and increases as the stock price decreases. Also known as a split-fee option. A call on a put will have therefore two strike prices and two expiration dates, one for the call option and the other for the underlying put option.

As well, there are two option price involved; the initial price is paid upfront for the call option; the additional price is only paid if the call option is exercised, and the option holder receives the put option. The price in this case would generally be higher than if the option owner had only purchased the underlying put option to begin with.

For example, consider a U.S. company that is bidding on a contract for a European project; if the company's bid is successful, it would receive say 10 million Euro upon project completion in one year's time. The company is concerned about the exchange risk posed to it by the weaker Euro if it wins the project. Buying a put option on 10 million Euro expiring in one year would involve significant expense for a risk that is as yet uncertain (since the company is not sure that it would be awarded the bid). Therefore, one hedging strategy the company could use would be to buy, for example, a two-month call on a one-year put on the Euro (contract amount of 10 million Euro). The price in this case would be significantly lower than it would be if it had instead purchased the one-year put option on the 10 million Euro outright. On the two-month expiry date of the call option, the company has two alternatives to consider. If it has won the project contract, or is in a winning position, and still desires to hedge its currency risk, it can exercise the call option, and obtain the put option on 10 million Euro. Note that the put option will now have ten months (i.e. $12 - 2$ months) left to expiry. On the other hand, if the company does not win the contract, or no longer wishes to hedge currency risk, it can let the call option expire unexercised, and walk away.

Put on a Put:

This is a put option on another underlying put option. The buyer of a put on a put has the right but not the obligation to sell the underlying put option - also known as the vanilla option - on the expiration date. This type of option is used when leverage is desired, and the trader is moderately bullish on the underlying asset.

The value of a put on a put changes in direct proportion to the price of the underlying asset, i.e. it increases as the asset price increases, and decreases as the asset price decreases. A put on a put has two strike prices and two expiration dates, one for the initial compound put option, and the other for the underlying vanilla put option. Since one of the variables that determines the cost of an option is the price of the underlying asset, the cost of a put on a put option will generally be lower than the cost of a put on the corresponding asset. It can therefore provide a great deal of leverage to the options trader.

So, a put on a put option, gives the holder the right to sell a put option, and later the put option gives the holder another right to sell the underlying asset.

Put on a Call:

This is a "put" option on an underlying "call" option. The buyer of a put on a call has the right but not the obligation to sell the underlying call option on the expiration date. This type of option is used when leverage is desired, and the trader is bearish on the underlying asset. The value of a put on a call changes in inverse proportion to the price of the underlying asset, i.e. it decreases as the asset price increases, and increases as the asset price decreases. A put on a call has two strike prices, and two expiration dates, one for the initial put option, and the other for the underlying call option. The cost of a put on a call option will generally be much lower than the cost of a put on the corresponding asset. It can therefore provide a great deal of leverage to the options trader.

3.2.8 Simultaneous and Sequential Compound Options

Compound options are options whose value is contingent on the value of other options. There are simultaneous compound options and sequential compound options. In simultaneous compound options, the life of the first option is longer (or equal to) the life of the second option. During the life of the second option, both options are alive simultaneously. Example, Geske's compound option is a simultaneous compound option because the options are alive at the same time, the equity, and the call option on equity are alive simultaneously.

In sequential compound options, the second option is created only when the first option is exercised. Hence, sequential compound options occur when an earlier option must be exercised to keep later options open. In a sense, the first option chronologically is the right to buy the second option. Sequential Compound Options are path-dependent options, where one phase depends on the success of another.

Chapter 4

PRICING COMPOUND OPTIONS

In this chapter, we explain some methods that are used in pricing compound options. But before that, we outline the factors affecting the price of an option.

4.1 FACTORS AFFECTING OPTION PRICES

4.1.1 Exercise Price of the Option

As a key characteristic, the strike price will impact the value of option. Such as, the value of the call option declines as the strike price increases. However, in the case of puts, the value will increase as the strike price increases.

4.1.2 Current value of the Underlying Asset

Options are contracts that derive value from an underlying asset. Moreover, the underlying asset price is positively related to the value of a call option and negatively related to the put option. Since the call option provides the right to purchase the underlying asset at a fixed price, the higher the underlying asset price, the more valuable the call is. On the other hand, the put option will become less valuable as the value of the underlying asset increases.

4.1.3 Time to Expiration on the Option

As the time to expiration increases, both call and put options become more valuable. As a reason, the longer time to expiration provides more time for the value of the underlying asset to move, increasing the value of both types of options (Damodaran,1996).

4.1.4 Variance in Value of underlying asset

Being different from other securities, the higher the variance in the value of the underlying asset will lead greater value of the option. According to the characteristic, the buyers of options will never lose more than the price they pay for them, and potentially earn significant returns form large price movement.

4.1.5 Risk free Interest Rate

Since the buyer of an option pays the price of the option up front, there is an opportunity cost involved. This opportunity cost will depend on the level of interest rates, and the time to expiration on the option. Additionally, risk free interest rate also enters into the valuation of options when the present value of the exercise price is calculated, since the exercise price does not have to be paid (received) until expiration on calls (puts). Therefore, an increase in the interest rate will increase the value of calls and reduce the value of puts (Damodaran, 1996).

4.2 BLACK-SCHOLES-MERTON MODEL

4.2.1 Black-Scholes Option Pricing

The Black-Scholes option pricing model is based on a normal distribution of underlying asset return, which is the same thing as saying that the underlying asset prices themselves are log-normally distributed. A log-normal distribution has a longer right tail compared with a normal, or bell-shaped distribution. The log-normal distribution allows for a stock price distribution of between zero and maturity (i.e. no negative prices), and has an upward bias (representing the fact that a stock price can only drop 100% but can rise by more than 100%) See [3]. European options can be best estimated using

the Black-Scholes-Merton model. Now consider the Black-Scholes-Merton formula;
Let S be the price of the stock
$V(S, t)$ be the price of a derivative as a function of time and stock price
$C(S, t)$ and $P(S, t)$ be the price of European call option and European put option respectively
K be the strike price of the option
r be the annualised risk-free interest rate, continuously compounded
μ be the drift rate of S, annualised
σ be the volatility of the stock's return, this is the square root of the quadratic variation of the stock's log price process
t be the time in years; we generally use: now$= 0$, expiry$= T$.
Finally we will use $N(x)$ which denotes the standard normal cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\omega^2}{2}} d\omega$$

$N^{'}(x)$ which denotes the standard normal probability density function:

$$N^{'}(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

The Black-Scholes model of the market for a particular stock makes the following explicit assumptions:

1. There is no arbitrage opportunity (i.e. there is no way to make risk less profit).

2. It is possible to borrow and lend cash at a known constant risk-free interest rate.

3. It is possible to buy and sell any amount, even fractional, of stock (this includes short selling).

4. The volatility and interest rate are constant throughout the life of the option.

5. The above transactions do not incur any fees or costs (i.e. frictionless market).

6. The underlying security does not pay a dividend.

7. There are no costs associated with buying and selling the stock.

8. There is no risk of default.

Several of these assumptions of the original model have been removed in subsequent extensions of the model. Modern versions account for changing interest rates (Merton, 1976), transaction costs, and taxes (Ingersoll, 1976), and dividend payout. The use of more sophisticated tools, such as stochastic calculus, has proved to be an even better foundation for the theory, and practice of option pricing, and hedging. These tools can be used to derive the famous Black-Scholes (and Merton), or BSM option-pricing model, and formulae. The Black-Scholes equation is a partial differential equation which describes the price of the option over time:
The Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

The price of the underlying asset S (typically a stock) follows a geometric Brownian motion. That is,

$$\frac{dS}{S} = \mu dt + \sigma dW \tag{4.1}$$

where W is a Brownian motion. Note that W, and consequently its infinitesimal increment dW, represents the only source of uncertainty in the price history of the stock. Intuitively, $W(t)$ is a process that "wiggles up and down" in such a random that its expected change over any time interval is 0. (In addition, its variance over time T is equal to T; a good discrete analogue for W is a simple random walk. Thus, equation (4.1) above states that the infinitesimal rate of return on the stock has an expected value of μdt and a variance of $\sigma^2 dt$. The value of a call option for a non-dividend paying underlying stock in terms of the Black-Scholes parameters is:

$$C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)},$$

$$d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

We obtain the value of a European call option by solving the Black-Scholes PDE:

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0$$

SKETCH OF THE PROOF

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0 \tag{4.2}$$

where $C = C(t, S)$ is the European call option. The boundary conditions are:

$$C(t,0)=0$$
$$C(T,S)=max(S-K,0)$$
$$C(t,S)\to S \quad if \quad S\to\infty$$

First the parabolic PDE of the European call option is transformed into the heat equation form. Then, the heat equation is solved using Fourier transform, and hence making some transformations and changes of variables leads to the famous formula of the European call option derived by Black and Scholes. The sketch of the proof is as follows:
We change the original variables to get a simple PDE. Since our goal is to obtain the heat equation which we know its solution; we try to obtain a new PDE in terms of $V(\tau, x)$:

1. $$S=Ke^x,\frac{dS}{dx}=Ke^x\Rightarrow\frac{dx}{dS}=\frac{1}{K}e^{-x} \tag{4.3}$$

2. $$t=T-\frac{\tau}{\frac{1}{2}\sigma^2}\Rightarrow\tau=\frac{1}{2}\sigma^2(T-\tau),\quad \frac{d\tau}{dt}=-\frac{1}{2}\sigma^2 \tag{4.4}$$

3. $$C(t,S)=KV(\tau,x) \tag{4.5}$$

We find the derivatives $\frac{\partial C}{\partial t}$, $\frac{\partial C}{\partial S}$, $\frac{\partial^2 C}{\partial S^2}$; so that we substitute in equation 4.2. Simplifying and collecting like terms we have:

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}+(r-\frac{1}{2}\sigma^2)\frac{\partial V}{\partial x}-\frac{1}{2}\sigma^2\frac{\partial V}{\partial \tau}-rV=0 \tag{4.6}$$

Let $n = \frac{r}{\frac{1}{2}\sigma^2} \Rightarrow r = \frac{1}{2}n\sigma^2$, substituting in 4.6 we have:

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} + (n-1)\frac{\partial V}{\partial x} - nV \tag{4.7}$$

Now we need to change the boundary conditions. According to our transformation, the terminal condition becomes an initial one, so we have:
$t = T \Rightarrow \tau = 0$. Using the substitution above in 4.5, we have an initial condition for the problem in terms of V(.), which is:

$$V(0,x) = max(e^x - 1, 0) \tag{4.8}$$

We are going to make another transformation in the PDE. We have a PDE in $V(\tau,x)$ and we will transform it into a new PDE in $U(\tau,x)$ where both these functions are related as follows:

$$V(\tau,x) = e^{\alpha x + \beta\tau}U(\tau,x) \tag{4.9}$$

where α and β will be defined later. Considering the initial value for the new function U(.):

$$V(0,x) = e^{\alpha x}U(0,x) \Rightarrow U(0,x) = e^{-\alpha x}V(0,x) \tag{4.10}$$

Again we find the derivatives $\frac{\partial V}{\partial \tau}$, $\frac{\partial^2 V}{\partial x^2}$, and $\frac{\partial V}{\partial x}$, so as we substitute in equation 4.7. Using these derivatives into the equation we have the PDE with the function U(.):

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2} + (2\alpha + n - 1)\frac{\partial U}{\partial x} + [(\alpha + n)(\alpha - 1) - \beta]U \tag{4.11}$$

$$(\alpha + n)(\alpha - 1) - \beta = 0 \Rightarrow \beta = \alpha^2 + \alpha(n-1) - n \tag{4.12}$$

$$2\alpha + n - 1 = 0 \Rightarrow \alpha = -\frac{1}{2}(n-1) \tag{4.13}$$

Substituting 4.13 into 4.12, we have:

$$\beta = -\frac{1}{4}(n+1)^2 \tag{4.14}$$

Hence we obtain the value of α and β above. Using the values of α and β obtained, we then substitute in equation 4.9 to get:

$$V(\tau,x) = e^{(-\frac{1}{2}(n-1) - \frac{1}{2}(n+1)^2\tau)}U(\tau,x) \tag{4.15}$$

The final PDE in U(.) is:

$$\frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial x^2}, \quad -\infty < x < \infty \quad and \quad \tau > 0 \tag{4.16}$$

Hence we obtain the heat equation in equation 4.16. Next we find the boundary conditions for the equation. Substituting 4.8 in 4.10 we have:

$U(0,x) = e^{-\alpha x} max(e^x - 1, 0) \Rightarrow U(0,x) = max(e^{(1-\alpha)x} - e^{-\alpha x}, 0)$
Substituting 4.13 we get:

$$U(0,x) = max(e^{\frac{1}{2}(n+1)x} - e^{\frac{1}{2}(n-1)x}, 0)$$

Rewriting the problem we have:

$$\begin{cases} \frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial x^2}, \quad -\infty < x < \infty, \quad \tau > 0 \\ U(0,x) = \phi(x) = max[e^{\frac{1}{2}(n+1)x} - e^{\frac{1}{2}(n-1)x}, 0] \end{cases} \tag{4.17}$$

The problem in 4.17 is the classical heat equation. Its solution is well known from Physics, and is given by:

$$U(\tau,x) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} \phi(y) e^{-\frac{(x-y)^2}{4\tau}} dy \tag{4.18}$$

The reader interested in the solution of the heat equation can consult Churchill (1963) or See[27]. Taking the solution of heat equation we are almost done. We will require a simple algebra to get the closed form solution of Black and Scholes PDE. All we need is to solve the integral above and find the function U(.). After this we come back to function V(.), and finally get C(.). Let's examine the function $\phi(y)$.

$\phi(y) = max[e^{\frac{1}{2}(n+1)y} - e^{\frac{1}{2}(n-1)y}, 0]$

$\phi(y) = e^{\frac{1}{2}(n+1)y} - e^{\frac{1}{2}(n-1)y} \quad if \quad e^{\frac{1}{2}(n+1)y} - e^{\frac{1}{2}(n-1)y} \geq 0$

$\Leftrightarrow \quad \frac{1}{2}(n+1)y \geq \frac{1}{2}(n-1)y \quad \Leftrightarrow \quad n+1 \geq n-1.$

The last inequality is true whatever n, since we have y > 0 . So taking the integral for positive values of y we can write:

$$U(\tau,x) = \frac{1}{\sqrt{4\pi\tau}} \int_{0}^{\infty} \phi(y) e^{-\frac{(x-y)^2}{4\tau}} dy$$

To solve this integral we change variables. So we consider:
$x^{'} = \frac{y-x}{\sqrt{2\tau}} \quad \Rightarrow$ dy=$\sqrt{2\tau}dx^{'}$
Substituting in the integral, and applying the definition of ϕ function, we have:

$$U(\tau, x^{'}) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \phi(\sqrt{2\tau}x^{'} + x) e^{-\frac{1}{2}x^{'2}} dx^{'}$$

$\Rightarrow$
$U(\tau, x^{'}) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} [e^{\frac{1}{2}(n+1)(\sqrt{2\tau}x^{'}+x)} - e^{-\frac{1}{2}(n-1)(\sqrt{2\tau}x^{'}+x)}] e^{-\frac{x^{'2}}{2}} dx^{'}$

Let's break this integral into I_1 and I_2: i.e. $U(\tau, x^{'}) = I_1 - I_2$, where:

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(n+1)x + \frac{1}{2}(n+1)\sqrt{2\tau}x^{'} - \frac{x^{'2}}{2}} dx^{'}$$

and

$$I_2 = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(n-1)x + \frac{1}{2}(n-1)\sqrt{2\tau}x^{'} - \frac{x^{'2}}{2}} dx^{'}$$

Next we compute I_1 and I_2 as follows:

$$I_1 = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}(n+1)x} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(-(n+1)\sqrt{2\tau}x^{'} + x^{'2})} dx^{'}$$

Completing the square in the exponential of the integral we have:

$$I_1 = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}(n+1)x} e^{\frac{1}{4}(n+1)^2\tau} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(x^{'} - \frac{(n+1)\sqrt{2\tau}}{2})^2} dx^{'}$$

We then again change the variable of integration as follows:
Let $\rho = x^{'} - \frac{n+1}{2} \Rightarrow d\rho = dx^{'}$ and for the lower limit of integration;

If $x^{'} = -\frac{x}{\sqrt{2\tau}} \Rightarrow \rho = -\frac{x}{\sqrt{2\tau}} - \frac{(n+1)\sqrt{2\tau}}{2}$
Finally we have:

$$I_1 = e^{\frac{1}{2}(n+1)x} e^{\frac{1}{4}(n+1)^2\tau} [\frac{1}{\sqrt{2\pi}} \int_{-d_1}^{\infty} e^{-\frac{1}{2}\rho^2} d\rho]$$

where $-d_1 = -\frac{x}{\sqrt{2\tau}} - \frac{(n+1)\sqrt{2\tau}}{2}$.
The term in the bracket is the area under the standard normal distribution between $-d_1$ and ∞. And by the symmetry of normal distribution, this area

is the same as the area between $-\infty$ and d_1 which we call $N(d_1)$. N(.) is the cumulative normal distribution function. Hence, we have;

$$I_1 = e^{\frac{1}{2}(n+1)x}e^{\frac{1}{4}(n+1)^2\tau}N(d_1) \tag{4.19}$$

where $d_1 = \frac{x}{\sqrt{2\tau}} + \frac{(n+1)\sqrt{2\tau}}{2}$.
We do similar algebra for I_2, and hence we get:

$$I_2 = e^{\frac{1}{2}(n-1)x}e^{\frac{1}{4}(n-1)^2\tau}N(d_2) \tag{4.20}$$

where $d_2 = \frac{x}{\sqrt{2\tau}} + \frac{(n-1)\sqrt{2\tau}}{2}$.
The function $U(\tau, x)$ is given by:

$$U(\tau, x) = I_1 - I_2 = e^{\frac{1}{2}(n+1)x}e^{\frac{1}{4}(n+1)^2\tau}N(d_1) - e^{\frac{1}{2}(n-1)x}e^{\frac{1}{4}(n-1)^2\tau}N(d_2) \tag{4.21}$$

Substituting 4.21 in 4.15 we have:

$$V(\tau, x) = e^x N(d_1) - e^{-\tau n}N(d_2) \tag{4.22}$$

Also, from 4.3, $x = ln(\frac{S}{K})$. Substituting 4.5, x, and the value of τ in 4.4 we get:

$$C(t, S) = K[e^{ln(\frac{S}{K})}N(d_1) - e^{-r(T-t)}N(d_2)]$$

Therefore we finally have the popular Black and Scholes solution of the PDE described by equation 4.2 as follows:

$$C(t, S) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{4.23}$$

where

$$d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

after the above substitution.
The price of a corresponding put option based on put-call parity is:

$$P(S, t) = Ke^{-r(T-t)} - S + C(S, t) = N(-d_2)Ke^{-r(T-t)} - N(-d_1)S$$

Therefore, the price of put option is;

$$P(S,t) = N(-d_2)Ke^{-r(T-t)} - N(-d_1)S,$$

where;
N($\cdot$) is the cumulative distribution function of the standard normal distribution,
$T - t$ is the time-to-maturity,
S is the spot price of the underlying asset
K is the strike price
r is the risk-free interest rate (annual rate, expressed in terms of continuous compounding)
σ is the volatility of returns of the underlying asset.
If we observe, we will see that the Black-Scholes model has five main inputs, which are as follows:

1. Stock price: The market price of the underlying asset on the valuation date is stock price. This can be a difficult input to estimate for options on illiquid assets; however under normal circumstances the closing market price can usually be used.

2. Strike price: This is the price level at which the option holder has the right to buy, or sell the underlying asset. It is the most straightforward input as it will always be given in the option contract.

3. Time to Maturity: The time (in years) until the option expires, and the holder is no longer entitled to exercise the option.

4. Risk free Interest Rate: The risk free interest rate for the period until the option expires. The risk free rate should typically be a zero coupon government bond yield.

5. Volatility: Volatility is the standard deviation of the continuously compounded return on the stock. It is probably the most important single input to any option pricing model. There are several methods for estimating volatility (See [25]). Historic volatility entails using historic price data for share price movements. A useful rule of thumb is to collect data from as far back as the options term (e.g an option with a 5 year life would require an input of historic volatility calculated from the last 5 years of historic data). Historic volatility is often considered

as flawed as it assumes the past will reflect the future thus several forward-looking measures of volatility can be more powerful, and accurate.

Example:
Consider for example a European call option over a certain stock has a term of six months. The current price of the stock is \$10·00, and the strike of the option is \$11·00. The risk-free interest rate is 3·92% per annum. The volatility of the stock is 20% per annum. What is the value of the option using Black-Scholes formula?
Solution:
Thus, $S = 10 \cdot 00$, $r = 0 \cdot 0392$, $\sigma = 0 \cdot 20$, and $K = 11 \cdot 00$. Now we find $C =?$, which is the value of the European call.
First we find the value of d_1 and d_2. Manually, substituting the values into the equation for d_1 we have:

$$d_1 = \frac{ln(\frac{10}{11}) + (0 \cdot 0392 + \frac{0 \cdot 20^2}{2})0 \cdot 5}{0 \cdot 2\sqrt{0 \cdot 5}} = -0 \cdot 464641$$

$$d_2 = d_1 - \sigma\sqrt{T - t} = -0 \cdot 464641 - 0 \cdot 2\sqrt{0 \cdot 5} = -0 \cdot 606062$$

The calculation of $N(d_1)$ and $N(d_2)$ can be done easily using standard normal tables.
Hence $N(d_1) = 0 \cdot 391024$ and $N(d_2) = 0 \cdot 272237$
Now, we then value the call option:

$$\begin{aligned} C = N(d_1)S - N(d_2)Ke^{-r(T-t)} &= (0 \cdot 391024 \times 10) - (0 \cdot 272237 \\ &\times 11e^{-0 \cdot 0392 \times 0 \cdot 2}) \\ &= 0 \cdot 2744 \end{aligned}$$

Therefore, the value of the European call option is $0 \cdot 2744$. The value of the

European put option (other facts constant) would be:

$$P = -N(-d_1)S + N(-d_2)Ke^{-r(T-t)}$$

where $N(-d_1) = 0 \cdot 678906$ and $N(-d_2) = 0 \cdot 727763$
Then,

$$P = (-0 \cdot 678906 \times 10) + (0 \cdot 727763 \times 11e^{-0 \cdot 0392 \times 0.5}) = 1 \cdot 0610$$

Hence the value of the European put option is $1 \cdot 0610$.
An alternative way to find the Black-Scholes option value is to solve the Black-Scholes partial differential equation (PDE) numerically using several different methods (See[1]).

4.2.2 The Generalised Black-Scholes-Merton option pricing formula

The Black-Scholes-Merton model can be "generalized" by incorporating cost-of-carry rate b. This model can be used to price European options on stocks, stocks paying a continuous dividend yield, options on futures, and currency options: The generalised BSM formula for European call option, and European put option is given by:

$$C_{BSM}(S,t) = Se^{(b-r)(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$P_{BSM}(S,t) = Ke^{-r(T-t)}N(-d_2) - Se^{(b-r)(T-t)}N(-d_1)$$

where

$$d_1 = \frac{ln(\frac{S}{K}) + (b + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

and b is the cost-of-carry rate; which is generally referred to as the risk-free interest rate that could be earned by investing currency in a theoretically safe investment *minus* any future cash-flows that are expected from holding an equivalent instrument with the same risk (generally expressed in percentage). This is the generalised Black-Scholes-Merton PDE, and this PDE is derived using the Ito's lemma (See [1]):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + bS\frac{\partial V}{\partial S} - rV = 0$$

We got the generalised Black-Scholes formula for pricing options, by solving the Black-Scholes-Merton PDE with appropriate boundary conditions in terms of asset price S. An alternative is to solve the PDE numerically (See[1]). The method is slower but more flexible. With the aid of the generalised Black-Scholes-Merton formula, we'll be able to find the formulas for pricing all the four types of compound options.

4.2.3 Compound Options

Call on Call:

If we recall, in the previous Chapter, we explained all the four types of compound options. A call on a call option gives the owner the right to buy an underlying option, which is a call option at a price K_1, at time T_1. Then the call option gives the owner another right to buy the underlying asset at price K_2, at a later point in time T_2. Under the BSM model, the payoff of the call on call is given by:
Payoff=max$(C_{BSM}(S, K_1, T_2) - K_2; 0)$, where K_1 is the strike price of the underlying option, K_2 is the strike price of the option on option, and $C_{BSM}(S, K_1, T_2)$ is the generalised BSM call option formula with strike K_1 and time to maturity T_2. The value of the call on call is given by:

$$C_{call} = Se^{(b-r)T_2}M(z_1, y_1; \rho) - K_1e^{-rT_2}M(z_2, y_2; \rho) - K_2e^{-rT_1}N(y_2)$$

where

$$y_1 = \frac{ln(\frac{S}{I}) + (b + \frac{\sigma^2}{2})T_1}{\sigma\sqrt{T_1}}$$

$$y_2 = y_1 - \sigma\sqrt{T_1}$$

$$z_1 = \frac{ln(\frac{S}{K_1}) + (b + \frac{\sigma^2}{2})T_2}{\sigma\sqrt{T_2}}$$

$$z_2 = z_1 - \sigma\sqrt{T_2}$$

and
$\rho = \sqrt{\frac{T_1}{T_2}}$, where T_2 is the time to maturity on the underlying option, and T_1 is the time to maturity of the option on the option, and $M(a, b; \rho)$ is the cumulative bivariate normal distribution function.

Put on call:

Under the BSM model, the payoff of the put on call is given by:
Payoff=max$(K_2 - C_{BSM}(S, K_1, T_2); 0)$. Then the value of the put on call is:

$$\begin{aligned} P_{call} &= K_1 e^{-rT_2} M(z_2, -y_2; -\rho) - S e^{(b-r)T_2} M(z_1, -y_1; -\rho) \\ &\quad + K_2 e^{-rT_1} N(-y_2) \end{aligned}$$

where the value of I called the critical value in the formula for valuing both call on call option, and put on call option is found by solving the equation

$$C_{BSM}(I, K_1, T_2 - T_1) = K_2$$

Call on Put:

Under the BSM model, the payoff of the call on put is given by:
$Payoff = max(P_{BSM}(S, K_1, T_2); 0)$. Now the value of the call on put is given by:

$$\begin{aligned} C_{put} &= K_1 e^{-rT_2} M(-z_2, -y_2; \rho) - S e^{(b-r)T_2} M(-z_1, -y_1; \rho) \\ &\quad - K_2 e^{-rT_1} N(-y_2) \end{aligned}$$

Put on put:

Under the BSM model, the payoff of the put on call is given by:
Payoff=max$(K_2 - P_{BSM}(S, K_1, T_2); 0))$. Then the value of the put on put is given by:

$$\begin{aligned} P_{put} &= S e^{(b-r)T_2} M(-z_1, y_1; -\rho) - K_1 e^{-rT_2} M(-z_2, -y_2; \rho) \\ &\quad + K_2 e^{-rT_1} N(y_2), \end{aligned}$$

where the critical value I in the formula for valuing both call on put option, and put on put option is found by solving the equation

$$P_{BSM}(I, K_1, T_2 - T_1) = K_2.$$

In summary, we show how to price European call option, and European put option using the Black-Scholes-Merton model. Whereby, certain assumptions has to be met for Black-Scholes option pricing model to be applicable, one of which is that, the volatility σ has to be constant throughout the life of the option, so is the risk-free interest rate r. Also, we saw that the Black-Scholes-Merton model has five inputs: the underlying price, the strike price (exercise price), the risk-free interest rate, the time to expiration, and the volatility. Having laid that foundation, we also give the generalised Black-Scholes-Merton formula by incorporating the cost-of-carry rate "b"; which is generally referred to as the risk-free interest rate that could be earned by investing currency in a theoretically safe investment *minus* any future cash-flows that are expected from holding an equivalent instrument with the same risk (generally expressed in percentage). When we obtained the generalised Black-Scholes-Merton model, we used it to give the formula for pricing all the four types of compound options.

4.2.4 Put-Call Parity Compound Options

Schilling (2001) gives the Put-Call parity between options on options.
Theorem 1.1: The European call on call C_{call}, and European put on call P_{call}, with the same strike prices K_1 and K_2, and maturities T_1 and T_2 on an underlying asset are related as follows:

$$\begin{aligned} C_{call}(S, K_1, K_2, T_1, T_2, r, b, \sigma) + K_2 e^{-rT_1} \\ = P_{call}(S, K_1, K_2, T_1, T_2, r, b, \sigma) \\ + C_{BSM}(S, K_1, T_2, r, b, \sigma) \end{aligned}$$

where r, σ, b are the interest rate, volatility, and cost-of-carry rate respectively. And $C_{BSM}(S, K_1, T_2, r, b, \sigma)$ is the generalised BSM formula for a European call option.

In other words, the theorem is saying; a **call on a call** plus the discounted strike price of the compound option, is equal in value to a **put on a call** plus a standard call with strike K_1, and time to maturity T_2. Hence, if we know the value of a call on a call, we can use the put-call parity to obtain the value of a put on a call, and vice versa.

Similarly, we have a relationship between a call on a put, and a put on a put given below:
Theorem 1.2: The European put on put P_{put}, and European call on put C_{put}, with the same strike prices K_1 and K_2, and maturities T_1 and T_2 on an underlying asset are related as follows:

$$C_{put}(S, K_1, K_2, T_1, T_2, r, b, \sigma) + K_2 e^{-rT_1}$$
$$= P_{put}(S, K_1, K_2, T_1, T_2, r, b, \sigma)$$
$$+ P_{BSM}(S, K_1, T_2, r, b, \sigma)$$

where r, σ, b are the interest rate, volatility, and cost-of-carry rate respectively. And $P_{BSM}(S, K_1, T_2, r, b, \sigma)$ is the generalised BSM formula for a European put option.

In other words, the theorem is also saying; a **call on a put** plus the discounted strike price of the compound option, is equal in value to a **put on a put** plus a standard put with strike K_1, and time to maturity T_2. Hence, if we know the value of a call on a put, we can use the put-call parity to obtain the value of a put on a put, and vice versa.

4.3 BINOMIAL LATTICE MODEL

4.3.1 Compound option model in a two period Binomial tree

The binomial pricing model popularly known as binomial tree, observes the evolution of the options price in discrete time. This technique, introduced by Cox. et al in 1979 is applied using binomial lattice for a number of time steps between the valuation, and expiration dates, where each node represents a possible value at a given time. The accuracy of the result increases with an increase in the number of time-steps.

In a compound option analysis, the value of the option depends on the value of another option. For example, exercise of the first option give the holder the right to acquire the second option, and the second option gives the holder the right to buy or sell the underlying asset. Thus, the value of the first option is dependent on the second option. The typical compound model based on binomial lattice approach has three valuation steps: first, value underlying asset (underlying lattice); second, value second option on the underlying asset (equity lattice); finally, value the first option (valuation lattice).

Consider an underlying asset with value (asset price) S, a second call with strike price K_2, and a first call with strike K_1. The value of the underlying asset after one period say year one will either be uS ("up", S multiply by the variable u), or dS ("down", S multiply by variable d) with probability P and $1 - P$, respectively. We also assume $u > 1 > d$, so the value uS, and dS represent the up-movement, and down-movement of the of the asset's price, respectively. At year two (period two), uS can take two more values: uuS ("up up", uS multiply by variable u), and udS ("up down", uS multiply by variable d). The same applies dS: values udS ("up down", dS multiply by variable u), and ddS ("down down", dS multiply by variable d). The whole scenario is transferred into a binomial tree below:

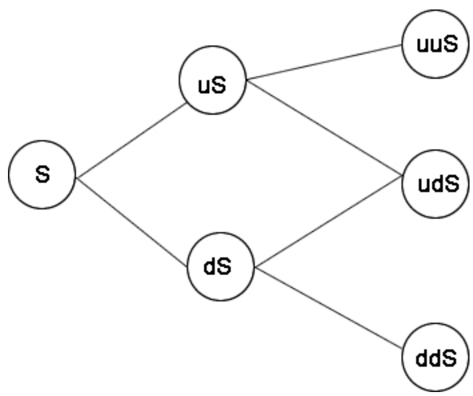


Fig: Binomial lattice of the underlying asset based on the up and down factors.

The probability (risk-neutral probability) P, the parameters u, and d are given by the following:

$$u = e^{\sigma\sqrt{\frac{T}{n}}}$$

$$d = 1/u$$

$$P = \frac{e^{r_f T} - d}{u - d}$$

where r_f is the risk-free interest rate, σ is the volatility, n is the number of binomial lattice steps or periods, and T is the option maturity. Some text usually write dt, or Δt instead of $\frac{T}{n}$ meaning the same thing. The risk-free interest rate r_f, and the volatility σ are assumed to be constant throughout the life of the option. Now, we consider year 1 to be the maturity of the first option, and year 2 to be the maturity of the second option.

At year two, maturity of the second option, the call can take three values:

$$C^{''}uu = max[uuS - K_2, 0]$$

$$C^{''}ud = max[udS - K_2, 0]$$

$$C^{''}dd = max[ddS - K_2, 0]$$

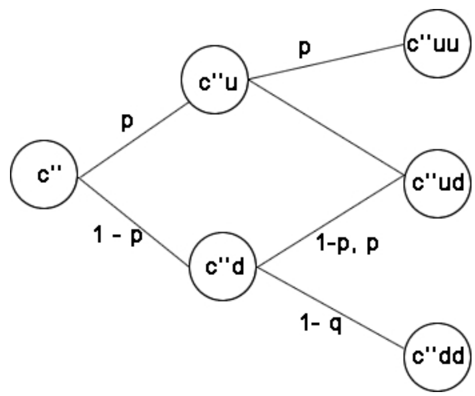


Fig: Binomial lattice for the second call option.

At year one, price of the second call $C^{''}u$ is a discounted value of $C^{''}uu$, and $C^{''}ud$ weighted by the probability P, and $(1-P)$. Also, $C^{''}d$ is a discounted value of $C^{''}ud$, and $C^{''}dd$ weighted by the probability P, and $(1-P)$. i.e.

$$C^{''}u = \frac{PC^{''}uu + (1-P)C^{''}ud}{1+r_f}$$

and

$$C^{''}d = \frac{PC^{''}ud + (1-P)C^{''}dd}{1+r_f}$$

The price of the second call today is a discounted value of $C^{''}u$ and $C^{''}d$ weighted by the probability P and $(1-P)$ i.e.

$$C^{''} = \frac{PC^{''}u + (1-P)C^{''}d}{1+r_f}$$

At this point we computed the value of the second call. Hence, we find the value of the first call option which is the **compound option**, since it is an option on the second call option.

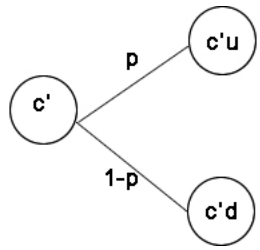


Fig: Tree for the first call option.

At year one, the maturity of the first call option, the call can only take two values as follows:

$$C^{'}u = max[C^{''}u - K_1, 0]$$

and

$$C^{'}u = max[C^{''}d - K_1, 0]$$

Note: At the maturity of the second option, the call can only take three values, one of which is $C^{''}ud = max[udS - K_2, 0]$, i.e. maximum between the value of the underlying asset after an "up down" movement minus the strike of the option, and 0. So the value of the call here depends on the value of the underlying asset at this up down movement (which is true in all the remaining two cases). Unlike the value of the first option at expiration, it depends on the value of the call option. i.e. $C^{''}u$ and $C^{''}d$.
Now, the value of $C^{'}u$ being the maximum value of 0, or the value of $C^{''}u$ from the second option minus the first option strike price K_1. The same calculation applied to $C^{'}d$ being the maximum value of 0, or the value of $C^{''}d$ from the second option minus the strike price K_1 of the first option.
The price of the first call today, which is obviously the value of the **compound option** is a discounted value of $C^{'}u$ and $C^{'}d$ weighted by the probability P and $(1 - P)$ i.e.

$$C^{'} = \frac{PC^{'}u + (1 - P)C^{'}d}{1 + r_f}$$

Generally, the method consist mainly of the following three steps:
First Step: Lattice of the underlying asset, based on the up and down

factors.
Second Step: Calculate the second long-term option, using risk neutral probabilities, and the backward induction technique.
Third Step: Calculate the option value lattice. The analysis depends on the lattice of the second long-term option.

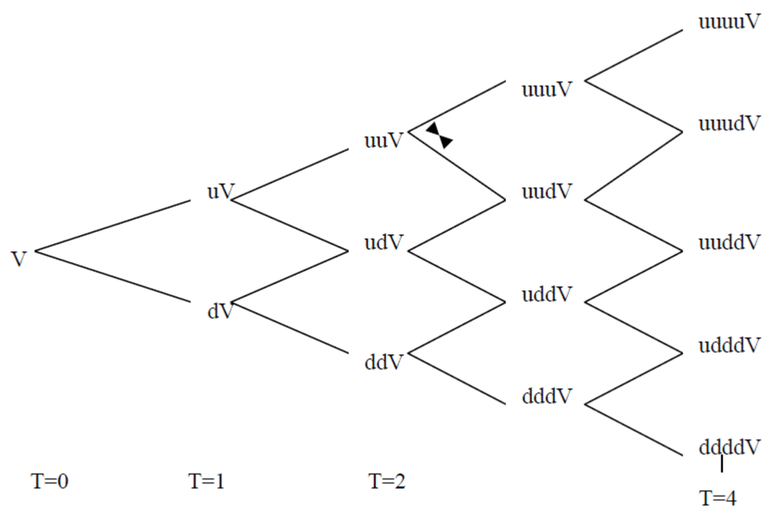
4.3.2 Four-Period Binomial lattice model

Four-period binomial lattice model is used to value sequential compound options. For sequential compound options, any type of phased investment fits this category. For instance, research and development (R&D) programs usually have four-phase investment scenario. When the firm invests in Phase 1 today, it acquires the option to investment in Phase 2 in year 1. If and when the firm invests in Phase 2, it then acquires the option to investment in Phase 3 in year 2. Similarly, investment in phase 3 will bring the option to invest in phase 4 in year 4. We will put the relevant variables into the table below.

Relevant Variables	Invest in Phase 1	Invest in Phase 2	Invest in Phase 3	Invest in Phase 4
	C_0	First Option C_1	Second Option C_2	Third Option C_3
Underlying Risky Asset		The Value Provided by the option to invest at Phase 3	The Value Provided by the Option to invest at Phase 4	The Value of the project
Exercise Price	Investment Outlay I_0	Investment Outlay I_1	Investment Outlay I_2	Investment Outlay I_3
Expire Date		T=1	T=2	T=4

From the table, it can be found that only the third option is the standard option, since its underlying asset is the total value of the project, V. Assuming that its value is uV with probability P or dV with probability $(1-P)$ after one year. The values of P, u and d are calculated using the same approach in the 2-period model above.

Following, we turn the project value into an event tree below, which is our first step.



With the sequential compound options, the order of economic priority is the opposite of the time sequence, just as in simultaneous compound option. Hence, we start first by valuing the third option which is a simple option on the value of the project. The values of the third option, second option, and first option at different time and various situations separately are given by C_3^0, C_2^0, and C_1^0 respectively.

At the end of the first time period, the first option expires. Therefore, it must be exercised at a cost of I_1, or left un-exercised. i.e. no cost. If it is exercised, the payouts are not directly dependent on the value of the underlying project, but dependent on the value provided by the option to investment at the next stage (i.e. second stage). Similarly, if the second option is exercised, the payouts will depend on the value provided by the option to investment at the third stage. But, only the third option is the standard European call option on the value of the project, thus we choose to firstly value the third option. At all end nodes, the intrinsic values of the third option expiring at $T = 4$ need to be solved, i.e. $C_3^{4,1}$, $C_3^{4,2}$, $C_3^{4,3}$, $C_3^{4,4}$, and $C_3^{4,5}$. The values of $C_3^{4,1}$, $C_3^{4,2}$, $C_3^{4,3}$, $C_3^{4,4}$, and $C_3^{4,5}$ are given below, just like in the 2-period:

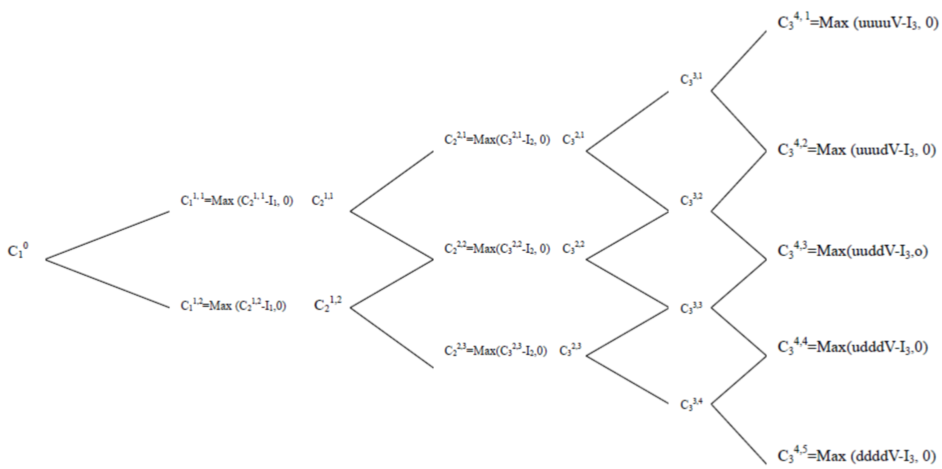
$$C_3^{4,1} = max(uuuuV - I_3, 0)$$

$$C_3^{4,2} = max(uuudV - I_3, 0)$$

$$C_3^{4,3} = max(uuddV - I_3, 0)$$

$$C_3^{4,4} = max(udddV - I_3, 0)$$

$$C_3^{4,5} = max(ddddV - I_3, 0)$$



Using the risk-neutral probability and risk-free rate, we can obtain the various values of the third option at $T = 3$:

$$C_3^{3,1} = \frac{PC_3^{4,1} + (1-P)C_3^{4,2}}{1+r_f}$$

$$C_3^{3,2} = \frac{PC_3^{4,2} + (1-P)C_3^{4,3}}{1+r_f}$$

$$C_3^{3,3} = \frac{PC_3^{4,3} + (1-P)C_3^{4,4}}{1+r_f}$$

$$C_3^{3,4} = \frac{PC_3^{4,4} + (1-P)C_3^{4,5}}{1+r_f}$$

We still use the same method to acquire the values of the third option at $T = 2$:

$$C_3^{2,1} = \frac{PC_3^{3,1} + (1-P)C_3^{3,2}}{1+r_f}$$

$$C_3^{2,2} = \frac{PC_3^{3,2} + (1-P)C_3^{3,3}}{1+r_f}$$

$$C_3^{2,3} = \frac{PC_3^{3,3} + (1-P)C_3^{3,4}}{1+r_f}$$

Secondly, we calculate the values of the second option. The second option is the compound option and its underlying asset is the third option. Intrinsic values of the second option expiring at $T = 2$ are:

$$C_2^{2,1} = max(C_3^{2,1} - I_2, 0)$$

$$C_2^{2,2} = max(C_3^{2,2} - I_2, 0)$$

$$C_2^{2,3} = max(C_3^{2,3} - I_2, 0)$$

After discounting these values multiplied by risk neutral probabilities, the values of the second option at $T = 1$ can be obtained.

$$C_2^{1,1} = \frac{PC_2^{2,1} + (1-P)C_2^{2,2}}{1+r_f}$$

$$C_2^{1,2} = \frac{PC_2^{2,2} + (1-P)C_2^{2,3}}{1+r_f}$$

Finally, standing at time zero, node C_1^0, we can estimate the present value of the compound option:

$$C_1^0 = \frac{PC_1^{1,1} + (1-P)C_1^{1,2}}{1+r_f}$$

In summary, the binomial lattice tree can be applied to any phased investment, depending on the number of phases. But, no matter the number of steps involved in the project, the approach of finding the value of the option will remain the same, only that the calculation will be long and time consuming.

4.4 THE FORWARD VALUATION OF COMPOUND OPTIONS

The forward valuation of compound options derives a solution for pricing compound options, when the underlying asset follows a diffusion process that is not a geometric Brownian motion. The solution is expressed as a forward integral of the price of the price surface of European vanilla options. The result can be applied to price defaultable corporate coupon-paying bonds when the value of the common stock is modelled as an option on the value of the firm. The forward solution is significantly more efficient than alternative numerical methods, to compute a cross-section of compound option prices. The majority of option pricing methods provide the price of options by a calculation that casts backward in time, starting from maturity date.

One exception is the forward partial differential equation of Dupire (1994), which provides the current prices of a panel of options as a function of their maturity dates, and strike prices, treating as a given constant the current price of the underlying asset. It has been believed that only European options could be priced by this method. So, we will show that compound options can also be priced by a forward method given the price surface of European options. Whence, we explain and show how to price a European (call) option on a European (call) option.

As we knew earlier, compound options are options written on an option, and it is characterised by two maturity dates: the intermediate maturity T_1, which is the date at which the buyer of the compound option may exercise the claimoption, and receive a European Option. And the final maturity T_2, at which the European option expires. At the intermediate date T_1, the underlying asset's price needs to be above a strike price say K_1, in order for the option to have positive value which makes this kind of option a path-dependent derivative.

Assume that the underlying asset follows a diffusion process different from a Brownian process. In this model we work with a generalised deterministic volatility i.e. a general diffusion process with deterministic volatility. We show how to obtain the price of a compound option on the basis of pure European option prices, starting with today as an intermediate maturity date for which an initial condition is applied, and continuing the calculation to any actual intermediate maturity date. This is a "forward" calculation of the option price in the spirit of Dupire's forward equation for European options. It gives a solution as a function of the strike price, and the time-to-maturity, just as any broker would quote an option.
We make the following assumptions, and assignments:

Let $t = 0$ be the initial date, T_1 be the intermediate maturity date, T_2 be the final maturity date. Let also K_1 be the intermediate strike price, and K_2 the strike price of the second stage-stage European option. Let S_0 be the underlying spot price at date 0, the dummy variables for the prices of the underlying asset at date t are denoted by x. We also assume that an option would require no payment at the intermediate date, and also we require that the underlying asset price fall some strike value inorder for the option to remain alive and continue in existence.

We denote by B_T the accumulated value of the cash account earning a continuously compounded interest rate equal to $r(t)$, $B_T = e^{\int_0^T r(u)du}$. Observe that the interest rates here is time varying. We do restrict interest rate to be deterministic, an assumption that would clearly be too restrictive if we were interested in pricing interest rate derivatives.

Harrison and Kreps show that the absence of arbitrage opportunities is equivalent to the existence of a risk-neutral probability measure say Q such that the price $\pi_t(X)$ of any tradable contingent claim with pay off X that settles at T is equal to

$$\pi_t(X) = B_t E_Q(B_T^{-1} | \Omega_t)$$

where E_Q is the conditional expectation under the risk-neutral probability Q.

We assume that under the risk-neutral measure Q, the spot price of the underlying asset is:

$$dS(t) = S(t)r(t)dt + S(t)\sigma(S(t), t)dW(t) \tag{4.24}$$

where W is a d-dimensional standard Brownian motion under Q. Here the local volatility σ is time-varying, and restricted to be a bounded function of the price level of the underlying asset, and time. We express the values of tradable securities in terms of numeraire. Let the numeraire be the accumulated value of the cash account, and let us call $S^*(t) = B_T^{-1}S(t)$ the relative, or forward price of the underlying asset, and $X_T^* = B_T^{-1}X_T$ the discounted payoff.

With no loss of generality, when interest rates are deterministic, we can re-parametrize the local volatility as a function of the forward price, so that

$$\sigma^*(S_t^*, t) = \sigma(B_t S_t^*, t)$$

See[26]. The relative price $S^*(t)$ follows a local martingale under risk-neutral probability;

$$dS^*(t) = S^*(t)\sigma^*(S^*(t), t)dW(t) \tag{4.25}$$

Let $q(S_0^*; x, t)$ be the risk-neutral probability density of a transition of the relative price from S_0^* at date 0 to x at date t. The relative price $\pi_t^*(X_T^*)$ of any tradable security can be obtained as the risk-neutral expected value of its future cash flows, expressed in units of the numeraire $\pi_t^*(X_T^*) = E_Q(X_T^*|\Omega)$ In the case of compound option, its relative price Γ_0^* at the initial date is therefore equal to:

$$\Gamma_0^*(S_0^*, 0; K_1, K_2, T_1, T_2) = \int_{K_1^*}^{\infty} q(S_0^*; x, t)C^*(x, T_1; K_2, T_2)dx \tag{4.26}$$

where $C^*(x, T_1; K_2, T_2)$ is the relative price at date T_1 of a European call with srike price K_2, maturing on date T_2, conditional on the underlying asset being equal to x. However, the compound option is exercised when $S_{T_1} > K_1$.

To proceed, we need to use the Fokker-Planck (forward) partial differential equation under the risk neutral probability. Since the relative price follows a local martingale

$$dS^*(t) = S^*(t)\sigma^*(S^*(t), t)dW(t)$$

So, the drift term is zero here, and the diffusion coefficient is $\frac{1}{2}(S^{*^2}(t)\sigma^{*^2}(S^*(t),t))$. Then, the Fokker-Planck (forward) partial differential equation for the transition probabilities is given by equation 4.5:

$$\frac{\partial}{\partial t}q(S_0^*;x,t) = 0 + \frac{\partial^2}{\partial x^2}(\frac{1}{2}\sigma^{*^2}(x,t)x^2 q(S_0^*;x,t)) \tag{4.27}$$

$$-\frac{\partial}{\partial t}C^*(x,t;K_2,T_2) = \frac{1}{2}\sigma^{*^2}(x,t)x^2\frac{\partial^2}{\partial x^2}C^*(x,t;K_2,T_2) \tag{4.28}$$

$$\text{such that } C^*(x,T_1;K_2,T_2) = max(0, x - K_2 B_{T_2^{-1}}) \tag{4.29}$$

Equation 4.6 is the (backward) Black-Scholes partial differential equation for the relative price $C^*(t)$ of the European call, and 4.7 is the relative price of the European call. Both the Fokker-Planck (forward) PDE and backward Black-Scholes PDE are written at date t, $0 \leq t < T_1$.

Our aim is to obtain an ordinary differential equation for Γ_0^* as a function of the intermediate date of the compound option. Toward this end, we obtain the intermediate-maturity partial derivative:

$$\frac{\partial}{\partial t}\Gamma_0^*(S_0^*,K_1,K_2,t,T_2) = \int_{K_1^*}^{\infty}\frac{\partial}{\partial t}q(S_0^*;x,t)C^*(x,t;K_2,T_2)dx$$

Using the product rule, we have:

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma_0^*(S_0^*,K_1,K_2,t,T_2) &= \int_{K_1^*}^{\infty}\frac{\partial}{\partial t}q(S_0^*;x,t)C^*(x,t;K_2,T_2)dx \\ &+ \int_{K_1^*}^{\infty}q(S_0^*;x,t)\frac{\partial}{\partial t}C^*(x,t;K_2,T_2)dx\end{aligned} \tag{4.30}$$

We take the first term of the right hand side and integrate it by parts twice as follows: i.e.

$$\int_{K_1^*}^{\infty}\frac{\partial}{\partial t}q(S_0^*;x,t)C^*(x,t;K_2,T_2)dx$$

We substitute 4.5 in the integral:

$$\begin{aligned}\int_{K_1^*}^{\infty}\frac{\partial}{\partial t}q(S_0^*;x,t)C^*(x,t;K_2,T_2)dx = \int_{K_1^*}^{\infty}&\frac{\partial^2}{\partial x^2}(\frac{1}{2}\sigma^{*^2}(x,t)x^2) \\ &\times q(S_0^*;x,t)C^*(x,t;K_2,T_2)dx\end{aligned}$$

Let $u = C^*(x,t;K_2,T_2)$, $\quad dv = \frac{\partial^2}{\partial x^2}(\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_0^*;x,t))dx$.
Then $v = \frac{\partial}{\partial x}(\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_0^*;x,t))$
Now,

$$\int_{K_1^*}^{\infty} \frac{\partial^2}{\partial x^2}(\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_{0^*};x,t))C^*(x,t;K_2,T_2)dx$$
$$= lim_{x\to\infty}[C^*(x,t;K_2,T_2)\frac{\partial}{\partial x}(\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_0^*;x,t))]$$
$$- lim_{x\to K_1^*}[C^*(x,t;K_2,T_2)\frac{\partial}{\partial x}(\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_0^*;x,t))]$$
$$- \int_{K_{1^*}}^{\infty}[\frac{\partial}{\partial x}(\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_{0^*};x,t))]\frac{\partial}{\partial x}C^*(x,t;K_2,T_2)dx$$

The third term on the right hand side can further be integrated by parts again:

$$\int_{K_{1^*}}^{\infty} \frac{\partial^2}{\partial x^2}(\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_0^*;x,t))C^*(x,t;K_2,T_2)dx$$
$$= lim_{x\to\infty}[C^*(x,t;K_2,T_2)\frac{\partial}{\partial x}(\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_0^*;x,t))]$$
$$- lim_{x\to K_1^*}[C^*(x,t;K_2,T_2)\frac{\partial}{\partial x}(\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_0^*;x,t))]$$
$$- lim_{x\to\infty}\frac{\partial}{\partial x}[C^*(x,t;K_2,T_2)\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_{0^*};x,t))]$$
$$+ lim_{x\to K_{1^*}}\frac{\partial}{\partial x}[C^*(x,t;K_2,T_2)]\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_{0^*};x,t))$$
$$+ \int_{K_1^*}^{\infty}\frac{1}{2}{\sigma^*}^2(x,t)x^2 q(S_0^*;x,t))\frac{\partial^2}{\partial x^2}[C^*(x,t;K_2,T_2)]dx \tag{4.31}$$

Then,

$$\int_{K_1^*}^{\infty} \frac{\partial^2}{\partial x^2}(\frac{1}{2}\sigma^{*^2}(x,t)x^2 q(S_0^*;x,t))C^*(x,t;K_2,T_2)dx$$
$$= -\frac{\partial}{\partial K_1^*}[\frac{1}{2}\sigma^{*^2}(K_1^*,t)K_1^{*^2}q(S_0^*;K_1^*,t))]C^*(K_1^*,t;K_2,T_2)$$
$$+ \frac{\partial}{\partial K_1^*}C^*(K_1^*,t;K_2,T_2)[\frac{1}{2}\sigma^{*^2}(K_1^*,t)K_1^{*^2}q(S_0^*;K_1^*,t))]$$
$$+ \int_{K_1^*}^{\infty} \frac{1}{2}\sigma^{*^2}(x,t)x^2 q(S_0^*;x,t))\frac{\partial^2}{\partial x^2}C^*(x,t;K_2,T_2)dx$$

If we observe, we will see that some terms in equation 4.9 vanishes, this is due to the assumption that the probability density goes to zero fast enough as x goes to infinity, so that all the boundary contributions at infinity are equal to zero. i.e.

$$lim_{x\to\infty}\frac{\partial}{\partial x}[\frac{1}{2}\sigma^{*^2}(x,t)x^2 q(S_0^*;x,t))]C^*(x,t;K_2,T_2) = 0$$

and

$$lim_{x\to\infty}[\frac{1}{2}\sigma^{*^2}(x,t)x^2 q(S_0^*;x,t))]\frac{\partial}{\partial x}[C^*(x,t;K_2,T_2)] = 0$$

Thus, we have

$$\int_{K_1^*}^{\infty} \frac{\partial}{\partial t}q(S_0^*;x,t)C^*(x,t;K_2,T_2)dx$$
$$= \int_{K_1^*}^{\infty} \frac{\partial^2}{\partial x^2}(\frac{1}{2}\sigma^{*^2}(x,t)x^2 q(S_0^*;x,t))C^*(x,t;K_2,T_2)dx$$
$$= -\frac{\partial}{\partial K_1^*}[\frac{1}{2}\sigma^{*^2}(K_1^*,t)K_1^{*^2}q(S_0^*;K_1^*,t))]C^*(K_1^*,t;K_2,T_2)$$
$$+ \frac{\partial}{\partial K_1^*}C^*(K_1^*,t;K_2,T_2)[\frac{1}{2}\sigma^{*^2}(K_1^*,t)K_1^{*^2}q(S_0^*;K_1^*,t))]$$
$$+ \int_{K_1^*}^{\infty} \frac{1}{2}\sigma^{*^2}(x,t)x^2 q(S_0^*;x,t))\frac{\partial^2}{\partial x^2}C^*(x,t;K_2,T_2)dx \tag{4.32}$$

Now we add the other term in equation 4.8 to equation 4.10, therefore, we have:

$$\frac{\partial}{\partial t}\Gamma_0^*(S_0^*, K_1, K_2, t, T_2) =$$
$$- \frac{\partial}{\partial K_1^*}[\frac{1}{2}\sigma^{*^2}(K_1^*, t)K_1^{*^2} q(S_0^*; K_1^*, t))]C^*(K_1^*, t; K_2, T_2)$$
$$+ \int_{K_1^*}^{\infty} \frac{1}{2}\sigma^{*^2}(x,t)x^2 q(S_0^*; x, t))\frac{\partial^2}{\partial x^2}C^*(x, t; K_2, T_2)dx$$
$$+ \frac{\partial}{\partial K_1^*}C^*(K_1^*, t; K_2, T_2)[\frac{1}{2}\sigma^{*^2}(K_1^*, t)K_1^{*^2} q(S_0^*; K_1^*, t))]$$
$$+ \int_{K_1^*}^{\infty} q(S_0^*; x, t)\frac{\partial}{\partial t}C^*(x, t; K_2, T_2)dx$$

Substituting 4.6 we get:

$$\frac{\partial}{\partial t}\Gamma_0^*(S_0^*, K_1, K_2, t, T_2) =$$
$$- \frac{\partial}{\partial K_1^*}[\frac{1}{2}\sigma^{*^2}(K_1^*, t)K_1^{*^2} q(S_0^*; K_1^*, t))]C^*(K_1^*, t; K_2, T_2)$$
$$+ \frac{\partial}{\partial K_1^*}C^*(K_1^*, t; K_2, T_2)[\frac{1}{2}\sigma^{*^2}(K_1^*, t)K_1^{*^2} q(S_0^*; K_1^*, t))]$$
$$+ \int_{K_1^*}^{\infty} \frac{1}{2}\sigma^{*^2}(x,t)x^2 q(S_0^*; x, t))\frac{\partial^2}{\partial x^2}C^*(x, t; K_2, T_2)dx$$
$$- \int_{K_1^*}^{\infty} \frac{1}{2}\sigma^{*^2}(x,t)x^2 q(S_0^*; x, t))\frac{\partial^2}{\partial x^2}C^*(x, t; K_2, T_2)dx$$

So,

$$\frac{\partial}{\partial t}\Gamma_0^*(S_0^*, K_1, K_2, t, T_2) =$$
$$- \frac{\partial}{\partial K_1^*}[\frac{1}{2}\sigma^{*^2}(K_1^*, t)K_1^{*^2} q(S_0^*; K_1^*, t))]C^*(K_1^*, t; K_2, T_2)$$
$$+ \frac{\partial}{\partial K_1^*}C^*(K_1^*, t; K_2, T_2)[\frac{1}{2}\sigma^{*^2}(K_1^*, t)K_1^{*^2} q(S_0^*; K_1^*, t))] \tag{4.33}$$

Whence we obtain an ordinary differential equation for Γ_0^* as a function of the intermediate date of the compound option, which may be trivially solved by integration for any given S_0^*, K_1, K_2, T_2, with an initial condition

$$\Gamma_0^*(S_0^*, K_1, K_2, 0, T_2) = C^*(S_0^*, 0; K_2, T_2) \times 1(S_0^* > K_1)$$

where 1 is the indicator function. Integrating both sides of 4.11 we get:

$$\begin{aligned}\Gamma_0(S_0, &K_1, K_2, T_1, T_2) - \Gamma_0^*(S_0, K_1, K_2, 0, T_2)\\ &= \int_0^{T_1} [-\frac{\partial}{\partial K_1^*}[\frac{1}{2}\sigma^{*^2}(K_1^*, t) K_1^{*^2} q(S_0^*; K_1^*, t))] C^*(K_1^*, t; K_2, T_2)\\ &+ \frac{\partial}{\partial K_1^*} C^*(K_1^*, t; K_2, T_2)[\frac{1}{2}\sigma^{*^2}(K_1^*, t) K_1^{*^2} q(S_0^*; K_1^*, t))] dt\end{aligned}$$

Now,

$$\begin{aligned}\Gamma_0(S_0, K_1, &K_2, T_1, T_2) = C(S_0, 0; K_2, T_2) \times 1(S_0 > K_1)\\ &+ \int_0^{T_1} [-\frac{\partial}{\partial K_1^*}(\frac{1}{2}\sigma^{*^2}(K_1^*, t) K_1^{*^2} q(S_0^*; K_1^*, t))) C^*(K_1^*, t; K_2, T_2)\\ &+ \frac{\partial}{\partial K_1^*} C^*(K_1^*, t; K_2, T_2)[\frac{1}{2}\sigma^{*^2}(K_1^*, t) K_1^{*^2} q(S_0^*; K_1^*, t))] dt.\end{aligned} \tag{4.34}$$

Observing that $\Gamma_0^* = \Gamma_0$, $S_0^* = S_0$, and $C_0^* = C_0$.

The price of compound option is given by equation 4.12, which is equal to the sum of three terms. The first term is the value at time 0 of a plain vanilla European option with strike equal to the final strike price, and maturity equal to the final date, truncated at the intermediate strike price of the compound option. The second term is a weighted average over the time dimension $0 \leq t \leq T_1$ of the prices of plain vanilla European option with strikes K_2 considered only for a value of the underlying asset equal to K_1^*. The third term is a weighted average of the strike sensitivities of these same plain vanilla European options.

Chapter 5

APPLICATIONS

5.1 Black-Scholes-Merton Model

Consider a put-on-call option that gives the option holder the right to sell a call option for \$50, three months from today. The strike on the underlying call option is \$520, the time to maturity on the call is six months from today, the price on the underlying stock index is \$500, the risk-free-rate is 8%, and the stock index pays dividends at a rate of 3% annually and, has a volatility of 35%. Calculate the value of the option (Put-on-call) using the BSM model.
Solution:
S=500, K_1=520, K_2=50, T_1=0.25, T_2=0.5, r=0.08, b=0.08-0.03=0.05, and σ=0.35.
We use the formula to value the put-on-call option. The formula is as follows:

$$P_{call} = K_1 e^{-rT_2} M(z_2, -y_2; -\rho) - S e^{(b-r)T_2} M(z_1, -y_1; -\rho) + K_2 e^{-rT_1} N(-y_2)$$

where

$$y_1 = \frac{ln(\frac{S}{I}) + (b + \frac{\sigma^2}{2})T_1}{\sigma\sqrt{T_1}}$$

$$y_2 = y_1 - \sigma\sqrt{T_1}$$

$$z_1 = \frac{ln(\frac{S}{K_1}) + (b + \frac{\sigma^2}{2})T_2}{\sigma\sqrt{T_2}}$$

$$z_2 = z_1 - \sigma\sqrt{T_2}$$

and

$$\rho = \sqrt{T_1/T_2}$$

So to obtain the value of the option we have to first evaluate the value of the critical value I, and we get the value of I by solving the equation

$$C_{BSM}(I, K_1, T_2 - T_1) = K_2$$

Substituting the values K_1, T_2, T_1 and K_2 we have:
$C_{BSM}(I, K_1, T_2 - T_1) = K_2$
$\Longrightarrow$
$C_{BSM}(I, 520, 0.5 - 0.25) = 50$
$\Longrightarrow$
$C_{BSM}(I, 520, 0.25) = 50.$
Using the generalised BSM formula we get:

$$C_{BSM}(I, 520, 0.25) = Ie^{(b-r)0.25}N(d_1) - 520e^{-r(0.52)}N(d_2)$$

We substitute the values of r and b as follows:

$$C_{BSM}(I, 520, 0.25) = Ie^{(0.05-0.08)0.25}N(d_1) - 520e^{-(0.08)(0.52)}N(d_2)$$

Now,

$$Ie^{(0.05-0.08)0.25}N(d_1) - 520e^{-(0.08)(0.52)}N(d_2) = 50$$

$\Longrightarrow$

$$Ie^{-0.0075}N(d_1) - 520e^{-0.0416}N(d_2) = 50$$

To find the value of $N(d_1)$ and $N(d_2)$ we first evaluate d_1 and d_2. Therefore,
$d_1 = \frac{ln(\frac{S}{K_1})+(b+\frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}} = \frac{ln(\frac{500}{520})+(0.05+\frac{(0.35)^2}{2})0.25}{0.35\sqrt{0.25}} = \frac{ln(0.9615)+0.0278}{0.175}$

$\Longrightarrow d_1 = \frac{-(0.0393)+0.0278}{0.175} = -0.0657$. Therefore, $d_1 = -0.0657$. So,

$$d_2 = d_1 - \sigma\sqrt{T} = -0.0657 - 0.175 = -0.2407.$$

Hence, $d_1 = -0.0657$, and $d_2 = -0.2407$.
Now,

$$N(d_1) = N(-0.0657) = 0.4761 \ \ and \ \ N(d_2) = N(-0.2407) = 0.4052$$

We obtain the value of $N(d_1)$ and $N(d_2)$ using the cumulative normal distribution table.

$$Ie^{-0.0075}N(d_1) - 520e^{-0.0416}N(d_2) = I \times 0.4761e^{-0.0075}$$

$$-520 \times 0.4052e^{-0.0416} = 50$$

$\Longrightarrow$

$$0.4725I - 202.1284 = 50$$

$\Longrightarrow$

$$0.4725I = 252.1284$$

$\Longrightarrow$

$$I = 533.6051$$

Next we compute the values of y_1, y_2, z_1, z_2, and ρ.

$$y_1 = \frac{ln(\frac{S}{I})+(b+\frac{\sigma^2}{2})T_1}{\sigma\sqrt{T_1}} = \frac{ln(\frac{500}{533.6051})+(0.05+\frac{(0.35)^2}{2})0.25}{0.35\sqrt{0.25}} = \frac{ln(0.9370)+0.00278}{0.175}$$

$\Longrightarrow$

$$y_1 = -0.2131$$

Therefore, $y_2 = y_1 - \sigma\sqrt{T_1} = -0.2131 - 0.175 = -0.3881$

Next we compute z_1 and z_2 as follows:

$$z_1 = \frac{ln(\frac{S}{K_1})+(b+\frac{\sigma^2}{2})T_2}{\sigma\sqrt{T_2}} = \frac{ln(\frac{500}{520})+0.0556}{0.2475} = \frac{-0.0393+0.0556}{0.2475} = 0.0659$$

Then, $z_2 = z_1 - \sigma\sqrt{T_2} = 0.659 - 0.2475 = -0.1816$, and

$$\rho = \sqrt{\frac{T_1}{T_2}} = 0.7071.$$

Having known the values of y_1, y_2, z_1, z_2 and ρ, the value of the option after substitution is given by:

$$P_{call} = 520e^{-0.08\times0.5}M(-0.1816, 0.3881, -0.7071) - 500e^{(0.05-0.08)0.5}$$

$$\times M(0.0659, 0.2131, -07071) + 50e^{-0.08\times0.25}N(0.3881).$$

To find the value of the option we have to again compute the value of the bivariate cumulative distribution function i.e. $M(-0.1816, 0.3881, -0.7071)$

and $M(0.0659, 0.2131, -07071)$
To calculate the value for the bivariate cumulative normal distribution function we use the Drezner and Wesolowsky Algorithm. The algorithm is as follows:
Drezner and Wesolowsky (1990) suggest two algorithms for calculating the bivariate cumulative normal distribution function. Their first algorithm is much simpler, and four to five times as fast as the Drezner (1978) algorithm. The simplest version of the Drezner and Wesolowsky (1990) algorithm is

$$M(a,b,\rho) = N(a)N(b) + \rho \sum_{i=1}^{5} \frac{x_i e^{\frac{2aby_i\rho - a^2 - b^2}{2(1-y_i^2\rho^2)}}}{\sqrt{1-y_i^2\rho^2}}$$

where $N(\cdot)$ is the cumulative normal distribution function and

$$\begin{aligned} x_1 &= 0.01885404 & y_1 &= 0.04691008 \\ x_2 &= 0.038088059 & y_2 &= 0.23076534 \\ x_3 &= 0.0452707394 & y_3 &= 0.50000000 \\ x_4 &= 0.038088059 & y_4 &= 0.76923466 \\ x_5 &= 0.018854042 & y_5 &= 0.95308992 \end{aligned}$$

We apply the following algorithm to find the value of the two bivariate cumulative normal distributions. Let's first consider $M(0.0659, 0.2131, -0.7071)$. Here, our $a = 0.0659$, $b = 0.2131$, and $\rho = -0.7071$.

$$M(a,b,\rho) = N(a)N(b) + \rho\{\frac{x_1 e^{\frac{2aby_1\rho - a^2 - b^2}{2(1-y_1^2\rho^2)}}}{\sqrt{1-y_1^2\rho^2}} + \frac{x_2 e^{\frac{2aby_2\rho - a^2 - b^2}{2(1-y_2^2\rho^2)}}}{\sqrt{1-y_2^2\rho^2}} + \frac{x_3 e^{\frac{2aby_3\rho - a^2 - b^2}{2(1-y_3^2\rho^2)}}}{\sqrt{1-y_3^2\rho^2}}$$
$$+ \frac{x_4 e^{\frac{2aby_4\rho - a^2 - b^2}{2(1-y_4^2\rho^2)}}}{\sqrt{1-y_4^2\rho^2}} + \frac{x_5 e^{\frac{2aby_5\rho - a^2 - b^2}{2(1-y_5^2\rho^2)}}}{\sqrt{1-y_5^2\rho^2}}\}$$

(See [1]) Substituting the values of a, b, and ρ we can compute the values separately as follows:

$$\begin{aligned} \frac{x_1 e^{\frac{2aby_1\rho - a^2 - b^2}{2(1-y_1^2\rho^2)}}}{\sqrt{1-y_1^2\rho^2}} &= \frac{0.018854042 e^{\frac{2\times 0.0659\times 0.2131\times 0.04691008\times(-0.7071) - 0.0043 - 0.0454}{2(1-(0.04691008)^2(-0.7071)^2)}}}{\sqrt{1-(0.04691008)^2(-0.7071)^2}} \\ &= \frac{0.018854042 \times 0.975}{0.9994} \\ &= 0.0179 \end{aligned}$$

$$\frac{x_2 e^{\frac{2aby_2\rho - a^2 - b^2}{2(1-y_2^2\rho^2)}}}{\sqrt{1-y_2^2\rho^2}} = \frac{0.038088059 e^{\frac{-0.0199\times 0.23076534 - 0.0043 - 0.0454}{2(1-(0.23076534)^2 0.499)}}}{\sqrt{1-(0.23076534)^2 0.499}}$$
$$= \frac{0.038088059 \times 0.9725}{0.9866}$$
$$= 0.0375$$

$$\frac{x_3 e^{\frac{2aby_3\rho - a^2 - b^2}{2(1-y_3^2\rho^2)}}}{\sqrt{1-y_3^2\rho^2}} = \frac{0.0452707394 e^{\frac{-0.0199\times 0.5 - 0.0043 - 0.0454}{2(1-(0.5)^2 0.499)}}}{\sqrt{1-(0.5)^2 0.499}}$$
$$= \frac{0.0452707394 \times 0.9666}{0.9355}$$
$$= 0.0582$$

$$\frac{x_4 e^{\frac{2aby_4\rho - a^2 - b^2}{2(1-y_4^2\rho^2)}}}{\sqrt{1-y_4^2\rho^2}} = \frac{0.038088059 e^{\frac{-0.0199\times 0.76923466 - 0.0043 - 0.0454}{2(1-(0.76923466)^2 0.499)}}}{\sqrt{1-(0.76923466)^2 0.499}}$$
$$= \frac{0.038088059 \times 0.954}{0.8411}$$
$$= 0.0432$$

$$\frac{x_5 e^{\frac{2aby_5\rho - a^2 - b^2}{2(1-y_5^2\rho^2)}}}{\sqrt{1-y_5^2\rho^2}} = \frac{0.018854042 e^{\frac{-0.0199\times 0.95308992 - 0.0043 - 0.0454}{2(1-(0.95308992)^2 0.499)}}}{\sqrt{1-(0.95308992)^2 0.499}}$$
$$= \frac{0.018854042 \times 0.9392}{0.7394}$$
$$= 0.0239$$

Therefore,

$$\rho \sum_{i=1}^{5} \frac{x_i e^{\frac{2aby_i\rho - a^2 - b^2}{2(1-y_i^2\rho^2)}}}{\sqrt{1-y_i^2\rho^2}} = (-0.7071)\{0.0179 + 0.0375 + 0.0582 + 0.0432 + 0.0239\}$$
$$= -0.7071 \times 0.1807$$
$$= -0.1278$$

Now we find the value of $N(0.0659)$ and $N(0.2131)$ using the cumulative normal distribution table. So, $N(0.0659) = 0.5239$ and $N(0.2131) = 0.5832$. Then

$$N(0.0659) \times N(0.2131) = 0.3055$$

Now

$$M(0.0659, 0.2131, -0.7071) = 0.3055 + (-0.1278) = 0.1777$$

We also compute the value of $M(-0.1816, 0.381, -0.7071)$ in a similar way, and it happens to be 0.1596. Now, we can compute the value of the option as follows:

$$P_{call} = 520e^{-0.08\times 0.5} \times 0.1596 - 500e^{(0.05-0.08)0.5} \times 0.1777$$
$$+50e^{-0.08\times 0.25} N(0.3881)$$

$\Longrightarrow$

$$P_{call} = 520 \times 0.9608 \times 0.1596 - 500 \times 0.9851 \times 0.1777$$
$$+50 \times 0.9802 \times 0.6480$$

$\Longrightarrow$

$$P_{call} = 79.7387 - 87.5261 + 31.7585 = 23.97$$

Hence, the value of the put-on-call option is $23.97.

5.2 Binomial Lattice Model

Suppose a project has two phases, of which the first cost $500 million and takes a year to complete. The second phase's expiration is one year also and cost $700 million. Suppose that the volatility of the logarithmic returns on the projected future cash flows is calculated to be 20%. The risk-free rate on a risk-less asset is found to be yielding 7.7%. The static valuation of future profitability using a discounted cash flow model, in other words the present value of the future cash flows discounted at an appropriate market risk-adjusted discount rate is found to be $1000 million; and could increase by 50% or decrease by 33.3% in a year. What is the present value of this project?

Solution:

$S = 1000$, $r_f = 0.077$, $\sigma = 0.2$, $K_1 = 500$, $K_2 = 700$.

First, we calculate the value of u, d, and P.

$u = 1.5$, $d = \frac{1}{u} = \frac{1}{1.5} = 0.67$ and

$$P = \frac{e^{r_f T} - d}{u - d} = \frac{e^{0.077\times 1} - 0.67}{1.5 - 0.67} = \frac{1.08 - 0.67}{0.83} = 0.49$$

Hence, $1 - P = 1 - 0.49 = 0.51$.

First Step: Since we obtain the values of u, d, and P, we directly go to the first step. i.e. lattice of the underlying asset based on the up, and down factors (here our underlying asset is the total value of the project). The figure below gives the value of the project at different nodes. That is, either by multiplying S by u for an up-movement, or multiplying S by d for a down-movement.

So, at period one we got the value of the project by multiplying the value S with the variable u (uS) for up-movement, and multiplying S by d (dS) for the down-movement. Likewise at period two, we have uS, and dS, where uS can take two more values: uuS for "up up-movement" (i.e. uS multiplied by the variable u), and udS for "up down-movement", (i.e. uS multiplied by the variable d). Also, dS can take two values: duS for "down up-movement", and ddS for "down down-movement. We turn the value of the project into an event tree below:

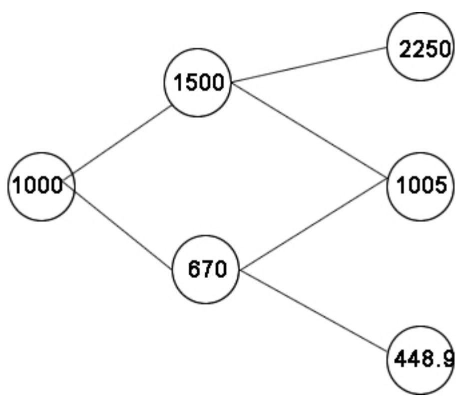


Fig: Tree for the value of the project.

Second Step: Next, we use the tree, and move to the second step. i.e. We calculate the value of the second option using the risk-neutral probabilities, and the backward induction technique. The following step is as follows:

At year two, maturity of the second option (second phase), the value of the call can take three values as follows:

$$C''uu = max[uuS - K_2, 0] = max[2250 - 700, 0] = max[1550, 0] = 1550$$

$$C''ud = max[udS - K_2, 0] = max[1005 - 700, 0] = max[305, 0] = 305$$

$$C''dd = max[ddS - K_2, 0] = max[448.89 - 700, 0] = max[-251.11, 0] = 0$$

At year one, value of the second call $C^{''}u$ is a discounted value of $C^{''}uu$ and $C^{''}ud$ weighted by the probability P and $(1-P)$. Also price $C^{''}d$ is a discounted value of $C^{''}ud$ and $C^{''}dd$ weighted by the probability P and $(1-P)$. i.e.

$$
\begin{aligned}
C^{''}u = \frac{PC^{''}uu + (1-P)C^{''}ud}{1+r_f} &= \frac{0.49 \times 1550 + 0.51 \times 305}{1+0.077} \\
&= \frac{759.5 + 155.55}{1.077} \\
&= 849.63
\end{aligned}
$$

$$
C^{''}d = \frac{PC^{''}ud + (1-P)C^{''}dd}{1+r_f} = \frac{0.49 \times 305 + 0.51 \times 0}{1.077} = \frac{149.45}{1.077} = 138.77
$$

The price of the second call today is a discounted value of $C^{''}u$ and $C^{''}d$ weighted by the probability P and $(1-P)$ i.e.

$$
\begin{aligned}
C^{''} = \frac{PC^{''}u + (1-P)C^{''}d}{1+r_f} &= \frac{0.49 \times 849.63 + 0.51 \times 138.77}{1.077} \\
&= \frac{495.32}{1.077} = 459.91
\end{aligned}
$$

Therefore the value of the second call today is \$459.91 million. Hence, we evaluated the value of the second call, we then move to the third and final step.

Third Step: The next step is the final step. i.e. We find the option value lattice, whereby its analysis depends on the second option.

At year one, the maturity of the first call option (first phase), the call can take two values:

$$
C^{'}u = max[C^{''}u - K_1, 0] = max[849.63 - 500, 0] = max[349.63, 0] = 349.63
$$

$$
C^{'}u = max[C^{''}d - K_1, 0] = max[13877 - 500, 0] = max[-361.23, 0] = 0
$$

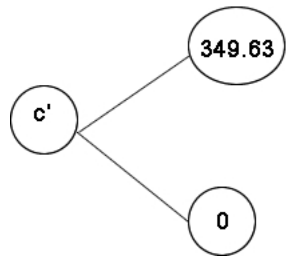


Fig: Tree for the value of the first call option (compound option).
Finally, the price of the first call today, is a discounted value of $C'u$ and $C'd$ weighted by the probability P and $(1-P)$ i.e.

$$C' = \frac{PC'u + (1-P)C'd}{1+r_f} = \frac{0.49 \times 349.63 + 0}{1.077} = \frac{171.32}{1.077} = 159.02$$

Hence, the present value of the project is \$159.02 million.

Bibliography

[1] Espen Gaarder Haug: *The complete guide to option pricing formulas*

[2] Andrea Buraschi and Bernard Dumas: *The forward valuation of compound options*. The Journal of Derivatives

[3] Jamilu Auwalu Adamu: *The role of option pricing and valuation models in Nigerian financial markets*. School of Mathematical Studies, National Mathematical Centre, Abuja, Nigeria.

[4] Black F., and M. Scholes (1973): *The pricing of options and corporate liabilities*. Journal of political economy.

[5] Brach Marion A. (2003): *Real options in practice*. John Wiley and Sons, Inc.

[6] Copeland T; Antikarov, V(2000): *Real options: A practitioner's guide.* New York, Texere.

[7] Cox J.C. Ross M., (1979): *Option Pricing: A Simplified Approach*. The Journal of financial Economics.

[8] Geske Robert (1979): *The valuation of compound options*. Journal of financial Economics.

[9] Hull, John (2000): *Options, Futures and Other Derivatives.*

[10] Tian, Yisong (1993): *A modified lattice approach to option pricing*. The Journal of futures markets.

[11] Ling Chen, Tze Leung Lai, and Tiong Wee Lin: *Option prices and pricing theory: Combining financial mathematics with statistical modelling.*

[12] Liesbeth Thomassen, and Martine Van Wouwe: *The n-fold compound option*. University of Antwerp, Belgium.

[13] Carl Charella, and Boda Kang: *The evaluation of American Compound option prices under stochastic volatility, and Stochastic interest rates*. Quantitative Finance Research Centre.

[14] Jefrey owen Katz, Donna L. Mc Commick: *Advanced option pricing models: An Empirical Approach to valuing options*

[15] Meng Yu Lee, Fang-Bo Yeh, An-Pin Chen: *The generalised sequential compound options pricing, and sensitivity analysis*. Journal of mathematical social sciences.

[16] Sheldon Natenberg: *Option volatility and pricing*.

[17] Yue-Kuen Kwok: *Mathematical models of financial derivatives*

[18] Gaurav Dhingra: *An understanding of financial derivatives*.

[19] Ronald H.W., Hoppe: *Numerical methods for option pricing in finance*.

[20] Robert Geske: *The valuation of corporate liabilities as compound options*. Journal of Financial Economics.

[21] Victor L. H. Sheen: *Real Options Theory and the Pricing of Residential Land-Use Rights in Shanghai*. Masters thesis, Massachusetts Institute of Technology.

[22] Christoph Kaserer: *Multiperiod Capital Budgeting under Uncertainty: Real Options Analysis II*. Department of Financial Management and Capital Markets.

[23] Shao Rong Song (2006): *Real Option Approach to R&D Project Valuation*. Masters thesis.

[24] Shilling James D., C. F. Sirmans, John D. Benjamin: *On Option-pricing Models in Real Estate: A Critique*. 1987, AREUEA Journal, 15, 1, 742-752.

[25] Wen-li Tang, Liang-rong Song: *The Analysis of Black-Scholes Option Pricing*. Advances in Applied Economics and Finance (AAEF), Vol. 1, No. 3, 2012, ISSN 2167-6348.

[26] Mark H.A. Davis: *The Dupire Formula*

[27] Fernando Antonio Lucena Aiube: *The Solution of Black and Scholes Partial Differential Equation*. Pontificia Universidade Catolica do Rio de Janeiro