PRICING OF BASKET OPTIONS

BY

FAUZIYYA ADO YAKASAI

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Supervisor:

Professor G.O.S EKHAGUERE
PhD, DIC(london), FAAS

African University of Science and Technology
www.aust.edu.ng
P.M.B 681, Garki, Abuja F.C.T
Nigeria.

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BY

SUPERVISOR ______________________

Prof. G.O.S Ekhaguere

MEMBER ______________________

Prof. C.E Chidume

MEMBER ______________________

Dr. Guy Degla

DEDICATION

This work is dedicate to my beloved parents, Hajiya Binta Kilishi and Engr.Ado Umar Yakasai, a father like no other, who believes a daughter has equal right to education. They made me believe in myself and always feel I can do it. Thanks for your love and support.

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Contents

Chapter 1

General Introduction

In this chapter we give some definitions in probability theory needed for our thesis and provide some introduction to the work.

1.1 Preliminaries:

We begin by introducing a number of probabilistic concepts.

1.1.1 σ-algebra

Let Ω be a non-empty set and $\mathcal{B}$ a non-empty collection of subset of Ω, $\mathcal{B}$ is called a σ-algebra if the following properties hold:

i $\Omega \in \mathcal{B}$

ii $A \in \beta \Rightarrow A' \in \beta$

iii $\{A_j : j \in J\} \subset \mathcal{B} \Rightarrow \bigcup_{j \in J} A_j \in \mathcal{B}$ for any finite or infinite countable subset J of $\mathbb{N}$

1.1.2 Probability Space

1. Let Ω be a nonempty set and $\mathcal{B}$ a σ- algebra of subsets of Ω. Then the pair $(\Omega, \mathcal{B})$ is called a **measurable space** and a member of $\mathcal{B}$ is called a **measurable set**.

2. Let $(\Omega, \mathcal{B})$ be a measurable space and $\mu : \mathcal{B} \longrightarrow \mathbb{R}$ be a real valued map on β. Then μ is called a **probability Measure** if the following properties hold:

   i $\mu(A) \geq 0 \; \forall A \in \beta$

   ii $\mu(\Omega) = 1$

   iii For $\{A_n\}_{n\in\mathbb{N}} \subset \beta$, with $A_j \cap A_k = \emptyset$ for $j \neq k$

   $\mu(\bigcup\limits_{n\in\mathbb{N}} A_n) = \sum\limits_{n\in\mathbb{N}} \mu(A_n)$ i.e μ is σ-additive(or countably additive).

3. If (Ω, β) is a measurable Space and μ is a probability measure on (Ω, β),then the triple (Ω, β, μ) is called **Probability Space**.

1.1.3 Borel σ-algebra

If Υ is a collection of subsets of Ω,then the smallest σ-algebra of subsets of Ω which contains Υ, denoted by $\sigma(\Upsilon)$ is called the σ-algebra generated by Υ.

Let X be a nonempty set and τ a topology on X, i.e τ is the collection of all open subsets of X.Then $\sigma(\tau)$ ia called the Borel σ-algebra of the topological space (X, τ).

1.1.4 A random variable:

Let (Ω, β, μ) be an arbitrary probability space,$B(\mathbb{R}^d)$ be the Borel σ-algebra of $\mathbb{R}^d$ and $(\mathbb{R}^d, B(\mathbb{R}^d))$ the d-dimension Borel measurable space. Then, a measurable map $X : \Omega \longrightarrow \mathbb{R}^d$ is called a **random vector**. In the case d=1, X is called a **random variable**.

1.1.5 Probability distribution

Let (Ω, β, μ) be a probability space, $(\mathbb{R}^d, \beta(\mathbb{R}^d))$ be the d-dimensional Borel measurable space, and $X : \Omega \longrightarrow \mathbb{R}^d$ a random vector. Then the map $\mu_X : \beta(\mathbb{R}^d) \longrightarrow [0,1]$ defined by $\mu_X(A) = \mu(X^{-1}(A))$, $A \in \beta(\mathbb{R}^d)$ is called the probability distribution of X.

1.1.6 Normal distribution

A standard univariate normal distribution(i.e of mean zero and variance 1) has density $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$,$-\infty < x < \infty$ and cumulative distribution function
$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$

In general a normal distribution with mean μ and variance σ^2, $\sigma > 0$ has density $\phi_{\mu,\sigma(x)} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ and cumulative distribution function $\Phi_{\mu,\sigma(x)} = \Phi(\frac{x-\mu}{\sigma})$

The notation $X \sim N(\mu, \sigma^2)$ means the random variable X is normally distributed with mean μ and variance σ^2.

If $Y \sim N(0,1)$ (i.e Y has the standard normal distribution, then $\mu + \sigma Y \sim N(\mu, \sigma^2)$. Thus given a method for generating the samples $Y_1, Y_2, \cdots$ from the standard normal distribution, we can generate samples $X_1, X_2, \cdots$ from $N(\mu, \sigma^2)$. It therefore suffices to consider methods for sampling from N(0,1). [2]

1.1.7 A d-dimensional Normal distribution

This is characterised by a d-vector μ and a $d \times d$ covariance matrix Σ; and is abbreviated as $N(\mu, \Sigma)$. If Σ is positive definite (i.e $x^T \Sigma x > 0, \forall x \neq 0 \in \mathbb{R}^d$), then the normal distribution $N(\mu, \Sigma)$ has density

$$\phi_{\mu,\Sigma(x)} = \frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}} exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)),$$

$x \in \mathbb{R}^d$ with $|\Sigma|$ the determinant of Σ.

The standard d-dimensional normal distribution $N(0, I_d)$; with I_d the $d \times d$ identity matrix, is the special case

$$\frac{1}{(2\pi)^{\frac{d}{2}}} exp(-1/2 \; x^T x).$$

If $X \sim N(\mu, \Sigma)$ (i.e the random vector X has a multivariate normal distribution)then its ith component X_i has distribution $N(\mu_i, \Sigma_i^2)$ with $\sigma_i^2 = \Sigma_{ii}$.The ith and jth component have covariances $cov(X_i, X_j) = E[(X_i - \mu_i)(X_j - \mu_j)] = \Sigma_{ij}$ which justifies calling Σ the covariance matrix. The correlation between X_i and X_j is given by $\rho_{ij} = \frac{\Sigma_{ij}}{\sigma_i \sigma_j}$.

If a $d \times d$ symmetric matrix Σ is positive semi-definite but not positive definite then the rank of Σ is less than d, Σ fells to be invertible, and there is no normal density with covariance matrix Σ. In this case we can define the normal distribution $N(\mu, \Sigma)$ as the distribution of $X = \mu + AZ$ with $Y \sim N(0, I_d)$ for any $d \times d$ matrix A: $AA^T = \Sigma$. The resulting distribution is independent of which A is chosen. The random vector X does not have a density in $\mathbb{R}^d$, but if Σ has rank then one can find k component of X with multivariate normal density in $\mathbb{R}^k$.

Any linear transformation of a normal vector is again normal, $X \sim N(\mu, \Sigma) \Rightarrow AX \sim N(A\mu, A\Sigma A^T)$ for any d-vector μ and $d \times d$ matrix Σ and any $d \times k$ matrix A, for any k.[2]

1.1.8 Log-normal Distribution

In simple terms: A random variable X is said to have a lognormal distribution if its logarithm has a normal distribution. I.e $ln[X] \sim N(\mu, \sigma)$. An important property of this distribution is that it does not take values less than 0.

A lognormal distribution is very much what the name suggest "lognormal". Imagine that you have a function that is the exponent of some input variable X. The input variable itself is a normal distribution function . e.g. $y = k.e^X$

Now, if we take a natural log of this function gives a normal distribution.

1.1.9 Mathematical Expectation

Let (Ω,β,μ) be a probability space. If $X \in L^1(\Omega, \beta, \mu)$, then

$$\mathbb{E}(X) = \int_\Omega X(\omega) d\mu(\omega)$$

is called the mathematical expectation or expected value or mean of X.

The map $X \longmapsto \mathbb{E}(X)$,$X \in L^1(\Omega, \beta, \mu)$ has the following properties:

i $\mathbb{E}$ is linear: $\mathbb{E}(\alpha X + \beta Y) = \alpha \mathbb{E}(X) + \beta \mathbb{E}(Y)$, for all $X, Y \in L^1(\Omega, \beta, \mu$ and, $\alpha, \beta \in \mathbb{R}$

ii Markov's inequality holds, i.e let $X \in L^1(\Omega, \beta, \mu)$ be $\mathbb{R}$-valued. Then $(\{\omega \in \Omega : |X(\omega)| \geq \lambda\}) \leq \frac{\mathbb{E}(|X|)}{\lambda}) = \frac{\|X\|_1}{\lambda}$, where $\lambda > 0$.

iii $\mathbb{E}$ is positivity preserving i.e if X is real-valued and lies in $L^1(\Omega, \beta, \mu)$ and $X \geq 0$, then $\mathbb{E}(X) \geq 0$.

iv Chebychev's inequality holds: Let $X \in L^1(\Omega, \beta, \mu)$ be a $\mathbb{R}$-valued random variable with mean $\mathbb{E}(X) = \alpha$ and variance σ_X^2.Then for $\lambda > 0$ $(\{\omega \in \Omega : |X(\omega) - \alpha| \geq \lambda\}) \leq \frac{\sigma_X^2}{\lambda^2})$.

v Jensen's inequality holds i.e, if X is real-valued and lies in $L^1(\Omega, \beta, \mu)$. $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is convex and $\phi(X) \in L^1(\Omega, \beta, \mu)$, then $\mathbb{E}(\phi(X)) \geq \phi(\mathbb{E}(X))$.

1.1.10 Variance and covariance of random variables:

Let (Ω, β, μ) be a probability space and X an $\mathbb{R}$-valued random variable on Ω, such that $X \in L^2(\Omega, \beta, \mu)$. Then X is automatically in $L^1(\Omega, \beta, \mu)$ (because in general if $p \leq q$, then $L^q(\Omega, \beta, \mu) \subset L^p(\Omega, \beta, \mu)$ for all $p \in [1, \infty) \cup \{\infty\}$.) The **variance** of X is defined as

$$Var(X) = \mathbb{E}((X - \mathbb{E}(X))^2).$$

The number $\sigma_X = \sqrt{Var(X)}$ is called the **standard deviation/error** of X. Now let $X,\, Y \in L^2(\Omega, \beta, \mu)$. Then the **covariance** of X and Y is given by:

$$Cov(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y)))$$

And the correlation is given by:

$$corr(X, Y) = \rho(X, Y) = \frac{Cov(X)Cov(Y)}{\sqrt{Var(X)Var(Y)}}$$

Two random variables X,Y are called **uncorrelated** if $cov(X, Y) = 0$.

1.1.11 Characteristic function

Let (Ω, β, μ) be a probability space and $X \in L^0(\Omega, \mathbb{R})$. Define the $\mathbb{C}$-valued function on $\mathbb{R}$ by:

$$\phi(t) = \mathbb{E}(e^{itx}) = \mathbb{E}(costx) + i\mathbb{E}(sintx).$$

Then ϕ is called the characteristic function of X.

Note: $\frac{\phi'(0)}{i} = \mathbb{E}(X)$

1.1.12 Stochastic process

A stochastic process X indexed by a set J is a family $X = \{X(t) : t \in J\}$ of members of $L^{\circ}(\Omega, \mathbb{R}^d)$.The value of X(t) at $\omega \in \Omega$ is written as X(t,ω).

1.1.13 Sample Paths

If X is a stochastic process and $w \in \Omega$ then the map $t \longmapsto X(t, w) \in \mathbb{R}^d$ is called a sample path or trajectory of X.

1.1.14 Brownian Motion

Let $Z = \{Z(t) \in L^{\circ}(\Omega, \mathbb{R}^d) : t \in \triangle\}$,where $\triangle \subseteq \mathbb{R}_+ = [0, \infty]$ be an $\mathbb{R}^d$ Stochastic process on Ω with the following properties:

i $Z(0) = 0$, almost surely.

ii $Z(t) - Z(s)$ is an N(0,(t-s)I) random vector for all $t \geq s \geq 0$,where I is the $d \times d$ identity matrix.

iii Z has stochastically independent increments i.e for $0 < t_1 < t_2 < \cdots < t_n$, the random vectors $Z(t_1), Z(t_2) - Z(t_1), \cdots, Z(t_n) - Z(t_{n-1})$ are stochastically independent.

iv Z has continuous sample paths $t \longmapsto Z(t, w)$ for fixed $w \in \Omega$

Then Z is called the standard d-dimensional Brownian Motion or d-dimensional Weiner process.

For a d-dimensional Brownian motion $Z(t) = (Z_1(t), \cdots, Z_d(t))$ we have the following:

i $\mathbb{E}(Z_j(t)) = 0$, $j = 1, 2, \cdots, d$

ii $\mathbb{E}(Z_j(t)^2) = t$, $j = 1, 2, \cdots, d$

iii $\mathbb{E}(Z_j(t)Z_k(s)) = \delta_{jk} t \wedge k = \delta_{jk} min\{t, s\}$, for $t, s \in \triangle$, $j, k = 1, 2, \cdots, d$

1.1.15 Filtration:

Let (Ω, β, μ) be a probability space and consider $\mathbb{F}(\beta) = \{\beta_t : t \in \Delta\}$ a family of σ-subalgebras of β with the following properties:

i For each $t \in \Delta$, β_t contains all the μ-null members of β.

ii $\beta_s \subseteq \beta_t$ whenever $t \geq s$, $s, t \in \Delta$

Then $\mathbb{F}(\beta)$ is called a filtration of β and $(\Omega, \beta, \mathbb{F}(\beta), \mu)$ is called a filtered probability space or stochastic basis.
We interpret β_t as the information available at time t and $\mathbb{F}(\beta)$ describe the flow of information.

1.1.16 Adaptedness

A Stochastic process $X = \{X(t) \in L^{\circ}(\Omega, \mathbb{R}^n) : t \in T\}$ is said to be adapted to the filtration $\mathbb{F}(\beta) = \{\beta_t : t \in T\}$ if $X(t)$ is measurable with respect to β_t for each $t \in T$. It is plain that every stochastic process is adapted to its natural filtration.

1.1.17 Conditional expectation

Let (Ω, β, μ) be a probability space, X a real random variable in $L^1(\Omega, \beta, \mu)$ and ξ a σ-subalgebra of β. Then the conditional expectation of X given ξ

written $E(X \mid \xi)$ is defined as any random variable Y such that:
(i) Y is measurable with respect to ξ i.e. for any $A \in \beta(\mathbb{R})$, the set $Y^{-1}(A) \in \xi$.
(ii)$\int_B X(\omega)d\mu(\omega) = \int_B Y(\omega)d\mu(\omega)$ for arbitrary $B \in \xi$.
A random variable Y which satisfies (i) and (ii) is called a version of $E(X \mid \xi)$.

1.1.18 Martingale

Let $X = \{X(t) \in L^1(\Omega, \beta, \mu) : t \in \Delta\}$ be a real-valued stochastic process on a filtered probability space $(\Omega, \beta, \mathbb{F}(\beta), \mu)$. Then X is a

1. submartingale, if $E(X(t)/\beta_s) \geq X(s)$ a.s whenever $t \geq s$

2. Supermartingale, if $E(X(t)/\beta_s) \leq X(s)$ a.s whenever $t \geq s$

3. Martingale, if X is both a submartingale and supermartingale i.e $E(X(t)/\beta_s) = X(s)$ a.s whenever $t \geq s$

1.1.19 Quadratic variation

Let X be a stochastic process on a filtered probability space $(\Omega, \beta, \mathbb{F}(\beta), \mu)$. Then the quadratic variation of X on $[0, t]$, $t > 0$, is the stochastic process $\langle X \rangle$ defined by

$$\langle X \rangle(t) = lim_{|\mathbb{P}| \to 0} \sum_{j=0}^{n-1} |X(t_{j+1}) - X(t_j)|^2$$

where $\mathbb{P} = \{t_\circ, t_1, \cdot, t_n\}$ is any partition of $[0, t]$ i.e. $0 = t_1 < t_2 < \cdot < t_n = t$ and $|\mathbb{P}| = max_{0 \leq j \leq n-1} |t_{j+1} - t_j|$

Note: If X is a differentiable stochastic process, then ⟨X⟩=0.

1.1.20 Stochastic differential equations

These are differential equations in which one or more terms is a stochastic process, resulting in a solution which is itself a stochastic process. SDE are used to model diverse phenomena such as fluctuating stock prices or physical system subject to thermal fluctuations. They are of the form

$$dX(t) = g(t, X(t))dt + f(t, X(t))dW(t)$$

with initial condition $X(t_\circ) = x_\circ$, where W denotes a Wiener process (standard Brownian motion). These are equations of the form

$$dX(t) = g(t, X(t))dt + f(t, X(t))dW(t)$$

with initial condition $X(t_\circ) = x_\circ$

1.1.21 Ito formula and lemma

Let $(\Omega, \beta, \mathbb{F}(\beta), \mu)$ be a filtered probability space, X an adapted stochastic process on $(\Omega, \beta, \mathbb{F}(\beta), \mu)$ with quadratic variation $\langle X \rangle$ and $U \in C^{1,2}([0,1] \times \mathbb{R})$. Then

$$\begin{aligned} U(t, X(t)) = U(s, X(s)) &+ \int_s^t \frac{\partial U}{\partial t}(\tau, X(\tau))ds + \int_s^t \frac{\partial U}{\partial x}(\tau, X(\tau))dX(\tau) \\ &+ \frac{1}{2}\int_s^t \frac{\partial^2 U}{\partial x^2}(\tau, X(\tau))d\langle X\rangle(\tau) \end{aligned}$$

which may be written as

$$dU(t, x) = \frac{\partial U}{\partial t}(t, X(t))dt + \frac{\partial U}{\partial x}(t, X(t))dX(t) + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}(t, X(t))d\langle X\rangle(t)$$

The equation above is referred to as the **Ito formula.** If X satsifies the stochastic differential equation (SDE)

$$dX(t) = g(t, X(t))dt + f(t, X(t))dW(t)$$

$$X(t_\circ) = x_\circ;$$

then

$$dU(t, X(t)) = g_u(t, X(t))dt + f_u(t, X(t))dW(t)$$
$$U(t_\circ, X(t_\circ)) = U(t_\circ, x_\circ)$$

where

$$g_u(t,x) = \frac{\partial U}{\partial t}(t,x) + g(t,x)\frac{\partial U}{\partial x}(t,x) + \frac{1}{2}(f(t,x))^2\frac{\partial^2 U}{\partial x^2}(t,x);$$
$$f_u(t,x) = f(t,x)\frac{\partial U}{\partial x}(t,x)$$

We obtain a particular case of the Ito formula called the **Ito lemma**, if we take $X = Z$, by setting $g \equiv 0$ and $f \equiv 1$ on $\mathbb{T} \times \mathbb{R}$. Then

$$dU(t, Z(t)) = [\frac{\partial U}{\partial t}(t, Z(t)) + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}(t, Z(t))]dt + \frac{\partial U}{\partial x}(t, Z(t))dZ(t)$$

The equation above is referred to as the **Ito lemma**.

Table 1.1: Ito Multiplication Table

x	dt	dZ(t)
dt	0	0
dZ(t)	0	dt

1.1.22 Gamma distribution

The probability density function $g\Gamma$ of a gamma distributed variable is given by $g\Gamma(x, \alpha, \beta) = \frac{e^{\frac{-x}{\beta}}(\frac{x}{\beta})^{\alpha-1}}{\beta\Gamma(\alpha)}$,$x \geq, \ \alpha, \beta \geq 0$

The corresponding cumulative distribution function $G\Gamma$ is define as:

$$\begin{aligned} G\Gamma(x, \alpha, \beta) &= \int_0^x g\Gamma(u, x, \beta)du \\ &= \frac{\int_0^x u^{\alpha-1}e^u du}{\Gamma(\alpha)} \\ &= \frac{\gamma(\alpha, \frac{x}{\beta})}{\Gamma(\alpha)}, \end{aligned}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

The ith moment of the gamma distribution is given by:

$$E[Y^i] = \frac{\beta^i \Gamma(i+\alpha)}{\Gamma(\alpha)}$$

The i^{th} moment of the inverse gamma distribution can be obtained for $-\alpha < i \leq 0$ for $i \leq -\alpha$ the moments are ∞
If Y is reciprocally gamma distributed then:

$$E[Y^i] = \frac{1}{\beta^i (\alpha-1)(\alpha-2)\cdots(\alpha-i)}.$$

Let gR be the inverse gamma probability distribution function. Then
$gR(x, \alpha, \beta) = \frac{g\Gamma(\frac{1}{x}, \alpha, \beta)}{x^2}$, $x \geq 0, \alpha, \beta > 0$

1.1.23 Risk-neutral Probabilities

These are probabilities for future outcomes adjusted for risk, which are then used to compute expected asset values. The benefit of this risk-neutral pricing approach is that once the risk-neutral probabilities are calculated, they can be used to price every asset based on its expected payoff. These theoretical risk-neutral probabilities differ from actual real world probabilities; if the latter were used, expected values of each security would need to be adjusted for its individual risk profile. A key assumption in computing risk-neutral probabilities is the absence of arbitrage. The concept of risk-neutral probabilities is widely used in pricing derivatives.

1.2 Overview

A financial derivative is a contract whose price is dependent upon or derived from one or more underlying assets. The underlying assets could be stocks, commodities, currencies e.t.c. An option is a financial derivative that gives the holder the right but not the obligation to buy or sell an underlying asset at a certain date and price. Options were first traded on the Chicago Board Options Exchange on April 26th, 1973. Basket option is a type of derivative security where the underlying asset is a group of commodities, securities or currencies. Since the early 1990s, basket options have been used as a tool for reducing risks (Hedging).

The pricing and hedging of basket options is difficult, due to the number of state variables. The usual methods employed in pricing options are not used to price Basket options, like Black and Scholes(1973) model. A single underlying asset is assumed to follow a geometric Brownian motion and therefore log-normally distributed, the problem arises from the fact that sum of correlated log-normally distributed random variables is not log-normal, thereby making it difficult to price the basket options and have a closed form pricing formula and hedging ratios. Some Practitioners sometimes take the basket itself also as a log-normal distribution. However, it leads to an inconsistency in the basic assumption "The distribution of a weighted average of correlated log-normals is anything but log-normal. Another difficulty that prevents the price of basket options from being exactly known the correlation structure involved in the basket. Correlation is observed to be volatile over time as is the volatility. A lot of research have been done to overcome this difficulty. Several methods has been proposed, comprising numerical methods and analytical approximation.

Instead of buying an option on each underlying asset, one may buy a single option on all the underlying assets "Basket options" as this will be cheaper, since there is only one option to monitor and exercise.

In the second chapter we give a literature review in pricing of basket options, highlighting some of the important contributions.

In the third chapter, we discuss financial derivatives and basket options, so as to have a clear idea of the financial market. We provide the: Definition of option, types of option, some examples of financial derivatives, traders, and some examples of basket options.

In the fourth chapter, we discuss the pricing of basket options and the methods used in the pricing, which is the main work of this thesis. The seller of a financial derivative, in particular options, requires a compensation for the risk he is bearing, by selling the option to the buyer. The buyer must pay a certain amount called a premium, in order to get the right to buy or sell the underlying asset and that is what is referred to as the **price of the option**. Several factors affect the pricing of basket option which include the initial prices, volatilities of the underlying asset, correlation e.t.c.

Various methods have been used in pricing of basket options, which include Monte-Carlo simulation (by assuming that the assets follow correlated geometric Brownian motion processes) first suggested by Boyle(1977). Monte-Carlo methods are suitable numerical methods used in pricing options that do not have an analytical closed form solution, especially basket options, Cox and Ross (1976) noted that if a riskless hedge can be formed, the option value is the risk-neutral and discounted expectation of its pay-off, that is the price can be represented by an integral, therefore making it possible to estimate the price of the option by Monte Carlo methods, which is done by simulating many independent paths of the underlying assets and taking the discounted mean of the generated pay-off's. We also have Tree based method (in the case of few state variables), analytical approximations such as Taylor approximation, Reciprocal gamma approximation, Log-normal approximation e.t.c.

In the last chapter, we give some applications of log-normal approximation by considering foreign exchange basket options, and give details on how they are priced.

Chapter 2

Literature Review

A lot of research has been done on the pricing of basket options, involving the use of different methods. Cox(1979) was the first to propose the Tree based approach, adopted in Wan(2002). Basket option pricing is done by approximating the underlying basket distribution. It employs the conditional expectation method first suggested by Curran(1994), Rogers and Shi(1995) and Nielsen and Sandmann (2003) for Asian options. Beisser(1999) estimated the price of the basket call from the weighted sum of (artificial) European call prices.

Gentle (1993) approximates the arithmetic average of the basket pay-off by a geometric average. Levy (1992) approximate the distribution of the basket by a log-normal distribution. Ju (2002) considers a Taylor expansion of the ratio of the characteristic function of the arithmetic average to that of the approximating log-normal random variable around zero volatility and provides a closed-form solution.[6]

The fact that the distribution of correlated log-normally distributed random variables converges to the reciprocal gamma distribution as the number of the underlying asset approaches infinity makes Milevsky and Posner(1998a) use the reciprocal gamma distribution as an approximation for the distribution of the basket,and provide an analytical solution.[4] The first two moments of both distributions where matched to obtain a closed-form solution. Later (1998b) they use distributions from the Johnson (1994) family as state-price densities to match higher moments of distribution of the arithmetic mean. Cox and Ross (1976) noted that if a risk-less hedge can be formed, the option value is the risk-neutral and discounted expectation of its pay-off.

Boyle(1977)[31] was the first to propose the Monte Carlo methods to option pricing, as an alternative to the previous methods.This draws the attention of researchers on the use of Monte Carlo method, and bed to the introduction of some variance reduction techniques. The price is estimated by simulating many independent paths of the underlying assets and taking the discounted mean of the generated play-offs. Boyle (1977), Kemna and Vorst (1990), Clewlow and Carverhill (1993), wrote some papers on variance reduction of the estimates of option prices. Ripley(1987) and Hammersley and Handscomb (1967) wrote on the use of control variates to reduce the variance of Monte Carlo estimates. P. Pellizzari (1997) presents two kinds of control variates to reduce variance of estimates, based on unconditional and conditional expectations of assets.

In a copula framework, an upper bound on a basket option is obtained by Rapuch and Roncalli (2001) and Cherubini and Luciano (2002). It is shown that this bound is equal to the so-called Frechet bound and corresponds to a particular case where the underlying assets are co-monotonic. Chen, Deelstra, Dhaene and Vanmaele (2006) use the related idea based on the theory of stochastic orders and on the theory of comonotonic risks, to derive the largest possible price that occurs when the components assets are comonotonic. Lars Oswald Dahl and Fred Espen Benth wrote on valuation of Asian Basket Options with Quasi-Monte Carlo Techniques and Singular Value Decomposition. Jinke Zhou1 and Xiaolu Wang (2008) provided a closed-form approximation formulae for pricing basket options. By approximating the distribution of the sum of correlated lognormals with some log-extended-skew-normal distribution.

Geoges Dionne, Genevieve Gauthier, Nadia Ouertani(2009) try to go beyond most research by treating basket options on heterogeneous underlying assets (different underlying assets)[15].

Chapter 3

Financial Derivatives

A financial derivative may be defined as a financial instrument whose value depends on (or derives from) the values of other, more basic underlying variables (assets, securities or indices) or is a financial contract whose value at an expiration date T, which is written into the contract, is determine by the price stochastic process of some financial asset called the underlying financial asset up to time T. Financial derivatives are traded in derivative exchanges (markets where individuals trade standardized contracts that have been defined by the exchange) and over the counter markets (a telephone and computer linked network of dealers,who do not physically meet). To every contract there are two positions: long position (the buyer of the contract)and short position (the seller/writer of the contract).[35]

Types of financial derivatives

3.0.1 Forward Contract

This is an agreement to buy or sell an asset at a certain future time for a certain price called the strike price. A forward contract is obligatory. The pay-off of an investor who holds the forward contract is:

$$S(T) - K$$

; where S(T) is the underlying asset price at time T and K is the strike price.

3.0.2 Future Contracts

This is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price. This kind of contract is marked to market daily i.e it is revalued daily to reflect the current values of relevant market variable. Unlike forward contracts, future contracts are normally traded on an exchange.

3.0.3 Options

This is a contract between two parties about trading an asset at a certain future time. One party is the writer often a bank, who fixes the terms of the contract and sells the option. The other party is the holder who purchases the option paying the market price which is the premium. They are two sides to every option contract: long position (buyer of the option) and short position (the seller or writer of the option).

Types of options

Call option

This gives the holder the right, without any obligation, to buy the underlying asset by a certain date for a certain price called the **strike price**.

Put option

This gives the holder the right, without any obligation to sell the underlying asset by a certain date for a certain price.

European call option

This gives the holder the right, without any obligation, to buy the underlying asset on the maturity date at the specified price called the strike price. The

seller of the option promises to deliver the asset at the maturity date T to the buyer of the option. The buyer of the option decides whether to buy or not to buy the underlying asset.
The payoff (gain) of an investor holding the long position on a European call option is

$$max\{S(T) - K, 0\} = (S(T) - K)^+;$$

where S(T) is the price of the underlying asset at the maturity time T and K is the strike price. The option will be exercised by the holder if $S(T) > K$ as this will bring profit.

The payoff of an investor holding the short position on a European call option is

$$min\{K - S(T), 0\} = (K - S(T))^+.$$

European put option

This gives the holder the right, without any not the obligation, to sell the underlying asset on the maturity date at a specified price (strike price). The seller of the option promises to purchases the underlying asset at the maturity date T from the buyer of the option, the buyer of the option decides whether to sell or not to sell the underlying asset.

The payoff of an investor holding the long position on a European put option is

$$max\{K - S(T), 0\} = (K - S(T))^+.$$

The option will be exercised by the holder if $S(T) < K$ as this will give profit.
The payoff of an investor holding the short position on a European put option is

$$min\{S(T) - K, 0\} = (S(T) - K)^+.$$

American option

This gives the holder the right, without any obligation to buy(if call option) or sell (if put option) the underlying asset on or before the maturity date at the specified price(strike price). American options are more difficult to study than European options.

Exotic options

These are options more complicated than commonly traded options called the vanilla options (e.g call and put options).

Basket option

This is an exotic option whose underlying is a weighted sum or average of different assets. Its payoff depends on the value of a portfolio (or basket) of assets. Like other options a basket option gives the holder the right, without any obligation, to buy or sell the underlying assets on or before a certain date at a certain price. It often cost less than multiple single options.
Examples of basket options are:

i S&P 500; this is based on a portfolio stock of 500 different stocks which include 400 industrials, 40 utilities, 20 transportation companies and 40 financial institutions.

ii FTSE 100; this is based 100 UK stocks listed on the London stock exchange.

iii Equity index options which are traded on the exchange and usually based on 15 stocks.

iv Currency options written on more than one currency.

Suppose we have some d underlying assets, $S_i(T)$, $i = 1, 2, \cdots, d$. The payoff of an investor holding a long position on a European call basket option is:

$$max\{S_B(T) - K, 0\}$$

where $S_B(T) = \sum_{i=1}^{d} W_i S_i(T)$; W_i is the weight of asset i in the basket; $S_i(T)$ is the price of the underlying asset i at time T; and K is the strike price of the basket.

The payoff of an investor holding a short position on a European call basket option is:

$$min\{K - S_B(T), 0\}$$

The payoff of an investor holding a long position on a European put basket option is:

$$max\{K - S_B(T), 0\}$$

where $S_B(T) = \sum_{i=1}^{d} W_i S_i(T)$; W_i is the weight of asset i in the basket; $S_i(T)$ is the price of the underlying asset i at time T.

The payoff of an investor holding a short position on a European put basket option is:

$$min\{S_B(T) - K, 0\}$$

Types of traders

3.0.4 Hedgers

These are traders that use financial derivatives to reduce the risk that they face from potential future movements in a market variable. They take positions in a contract that will protect them against loss.

3.0.5 Speculators

These are traders that use financial derivatives to bet on the future direction of a market variable. They take a position in the market, either they are betting that the price will go up or that it will go down. They utilize their knowledge to forecast future price. They take high risk with the hope of getting high profit.

3.0.6 Arbitrageurs

These are traders that try to lock in a riskless profit by simultaneously entering into transactions in two or more markets. For example, if a stock has different values on different markets(exchange); say a price in exchange that is lower than the price in exchange B, An arbitrageur would buy the stock in exchange A and sell it in exchange B, thus making a risk-free profit.

In pricing option it is always assumed that there do not exist any arbitrage possibilities.

Put-Call Parity

In an arbitrage free market, a European call C and European put P with the same strike price K and maturity time T on an underlying asset S, paying no dividends, are related as follows:
$S(t) + P(t) - C(t) = Ke^{-r(T-t)}$, where r is the risk-less interest rate.

Chapter 4

Pricing of Basket option

In this chapter we discuss the pricing of basket options, factors that affect the pricing and various pricing techniques.

A lognormal distribution is the distribution of choice when pricing assets. There are three important observations that we have to recognize in order to understand this: the price of a stock at time t_1 is dependent on two variables: the stock price at t_0 and rate of return (r) for the interval $[t_0, t_1]$: What this means is that for one period of time $[t_0, t_1]$, we can get the new stock price by using the simple transformation:

$$past\ stock\ price \times rate\ of\ return = current\ stock\ price$$

The rate of return (r) follows a normal distribution: this is an important assumption. In order to completely understand this we should first think about the properties of a normal distribution, particularly: the shape of the distribution. Just by looking at a normal distribution graph, we can easily tell that the farther you move away from the mean the likelihood of getting a sample becomes less hence the likelihood of getting samples close to the mean is much higher. In other words: if we have a mean rate-of-return equal to r_0 for time t_1, we will see the new rate-of-return r_1 at time t_1 very "close to" r_0. This property of being "close to" gives us the idea of the rate-of-return conforming to a normal distribution. For the time interval $[t_0, t_1]$ we continuously compound the return: a very important concept in finance is compounding the return over an interval of time.

A continuously compounded rate of return is expressed using the mathematical concept of exponent e^x. So, if we have a rate of return = r, then the continuously compounded rate of return is: e^r.

Example

Suppose you are following a stock and you are interested in estimating the price of the stock after a time interval h, and this time interval could be anything from an hour to a year. Let's have the following variables:

$$initial\ stock\ price\ =\ S_0$$

$$end\ stock\ price\ = S_1$$

$$rate\ of\ return\ =\ r$$

We can have the following equation to get the stock end price:

$$S_1 = S_0 e^r$$

From the example equation above we can see that the rate of return is an input to the exponent function and S_0 is a constant. If we take the natural log of the expression then we are left with a normal distribution, because we know that the rate of return follows a normal distribution and also because taking the natural log of an exponent function leaves you with the input (i.e r in this example). So, this is why it is said that asset prices (stock price in this example) follow a lognormal distribution.

4.1 Definition

The price of an option is an amount of money known as the premium. The buyer of the option pays this premium to the seller of the option in an exchange for the right granted by the option.

Factors that affect the price of an option include: the initial price of the underlying assets at time $t = 0$,the maturity date(expiry date) of the option, the strike price, the volatility of the underlying assets, the interest rate r, dividend paid and the correlation of the underlying asset.

Call prices increase and put price decrease, if the underlying assets prices increase. Call prices decrease and put prices increase, if the underlying assets prices decrease. The longer the time until expiration,the higher the option price. The shorter the time until expiration, the lower the price. Option price increase as options become further in the money (when $s(T) > K$).The greater the expected volatility, the higher the option price. Call prices increase and put prices decrease, if the interest rate increase. Call prices decrease and put prices increase, if the interest rate decrease. Call prices decrease and put prices increase, if the dividend increase. Call prices increase and put prices decrease,if the dividend decrease. This is the main feature that distinguishes basket options from single underlying options.

4.2 Geometric Brownian Motion

An underlying asset S is said to be a geometric Brownian motion if

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ \tag{4.1}$$

where S(t) is the asset price at time t; μ is the instantaneous rate of return on the underlying asset; σ is the instantaneous volatility of the rate of return; and Z is a Brownian motion (Weiner process).

To get the solution of equation (4.1) we use the Ito lemma. We re-write the equation as follows:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dZ$$

Now let $u(t, S) = lnS$ then
$\dfrac{\delta u}{\delta t} = 0, \dfrac{\delta u}{\delta S} = \frac{1}{S}$

By Ito lemma. $du(t, S(t)) = \frac{dS(t)}{S(t)} - \frac{1}{2}\frac{d<S>}{S(t)^2}$
But,
$d < S >= (d(S(t))^2 = \sigma^2 S(t)^2 d(Z(t))^2 = \sigma^2 S(t)^2 dt$
(since $d(Z(t))^2 = dt$ by the Ito table).

Thus,

$$du(t, S(t)) = \frac{dS(t)}{S(t)} - \frac{\sigma^2}{2}dt$$

$$\Rightarrow du(t, S(t)) + \frac{1}{2}\sigma^2 dt = \mu dt + \sigma dZ(t)$$

$$\Rightarrow du(t, S(t)) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dZ(t)$$

Integrating both sides we have:

$$u(t, S(t)) - u(t_\circ, S(t_\circ)) = (\mu - \frac{1}{2}\sigma^2)(t - t_\circ) + \sigma(Z(t) - Z(t_\circ))$$

Next we substitute $U(t, S) = lnS$

$$ln(\frac{S(t)}{S(t_\circ)}) = lnS(t) - lnS(t_\circ) = (\mu - \frac{\sigma^2}{2})(t - t_\circ) + \sigma(Z(t) - Z(t_\circ))$$

taking exponent of both side we have;

$$\frac{S(t)}{S(t_\circ)} = exp((\mu - \frac{\sigma^2}{2})(t - t_\circ) + \sigma(Z(t) - Z(t_\circ)))$$

$$\Rightarrow S(t) = S(t_\circ)exp((\mu - \frac{\sigma^2}{2})(t - t_\circ) + \sigma(Z(t) - Z(t_\circ)))$$

When pricing basket options, each of the underlying asset is assumed to be a geometric Brownian motion and therefore log-normally distributed.

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)dZ_i \tag{4.2}$$

with solution;

$$S_i(t) = S_i(0)exp((\mu_i - \frac{\sigma_i^2}{2})t + \sigma_i Z_i(t)) \tag{4.3}$$

$$ln(\frac{S_i(t)}{S_i(0)}) \sim N(lnS_i(0) + (\mu_i - \frac{\sigma_i^2}{2})t, \sigma_i^2 t)$$

But the problem is that the sum of correlated log-normally distributed random variables is not log-normal. This makes it difficult to have an exact closedform pricing formula of the basket call and put.

Cox and Ross, (1976) noted that if a risk-less hedge can be formed, the option value is the risk-neutral and discounted expectation of its pay-off. Therefore the price of the basket is given by:

$V_{call} = e^{-rt}E_Q(max\{S_B(t)-K,0\})$:for a call; where r is the risk free interest rate and Q is the risk neutral probability or equivalent martingale measure. The value of the put can be obtained from the put-call parity.

To evaluate the above expectation, several numerical methods are used. These will be described in the next chapter.

4.3 Methods used in pricing Basket options

4.3.1 Numerical Methods

Tree, Monte-Carlo, and finite difference methods are some of the numerical methods used in pricing basket options. However, tree and finite difference methods are not flexible, as dimensions increase. The Monte Carlo method is highly flexible as it deals with random factors such as stochastic volatility and it is able to incorporate more realistic price processes such as jumps in asset prices. The major drawback of standard Monte Carlo is the speed of convergence and the apparent inapplicability to path dependent options. Also large numbers of simulations are required for convergence to the exact price of the option if standard Monte Carlo is used.

Monte-Carlo Method

This involves the use of random numbers with the structure of a stochastic process. It is often used to price options whose value are difficult to find and where the value of an option can be expressed as an expectation of some random variables.

The following are some of the principle underlying the application of the Monte-Carlo method for option pricing:

1. If a derivative security can be perfectly hedged through trading other assets, then the price of the derivative security is the cost of the hedging strategy.

2. Discounted asset prices are martingales under a probability measure associated with the choice of discount factor. Prices are **expected values of discounted pay-offs** under such a martingale measure.

3. In a market where risk can be perfectly hedged(complete market) any pay-off can be perfectly synthesized through a trading strategy, and the martingale measure associated with the discount factor is unique.

statement 2 above gives us a clue to calculating option prices:To express it as an expectation of pay-offs and to describe the dynamics of asset prices as they will be under a risk adjusted probability measure. Since the price of a basket option can be expressed as an expectation, we can therefore use the Monte-Carlo method to evaluate the expectation.

Monte-Carlo Methodology

To get the price of an option, the following steps are followed:

Step 1: Generate the sample paths of the underlying assets over the given time horizon
Step 2: Evaluate the discounted pay-offs of the option for each path.
Step 3: Average the pay-offs over all paths.[1]

In order to estimate a quantity ϑ using Monte-Carlo method, an estimator $\bar{f_n}$ is used and independent sample paths $\bar{\vartheta_k}$, $k = 1, 2, ..., n$ are generated in a sample space Ω, where $E(\bar{\vartheta_k}) = \vartheta$ and $\bar{\vartheta_k} = \sigma^2$ for each $\bar{\vartheta_k}$

Next, set the estimator $\bar{f_n} = \frac{1}{n}\sum_{k=1}^{n} \bar{\vartheta_k}$

Therefore,

$$E(\bar{f_n}) = E(\frac{1}{n}\sum_{k=1}^{n} \bar{\vartheta_k}) = \frac{1}{n}\sum_{k=1}^{n} E(\bar{\vartheta_k}) = \vartheta$$

and

$$var(\bar{f_n}) = var(\frac{1}{n}\sum_{k=1}^{n} \bar{\vartheta_k}) = \frac{1}{n^2}(\sum_{k=1}^{n} var(\bar{\vartheta_k})) = \frac{\sigma^2}{n}$$

Then $\lim \bar{f_n} = \vartheta$ as $n \to \infty$

Now, to estimate the value of the basket call given by the expectation above,we let

$$\vartheta = e^{-rT} E_Q(\max\{S_B(t) - K, 0\}).$$

We sample n independently generated prices $S_i^k(t)$ of S_i, $k = 1, ...n$; $i =$

$1, ...d$. Then we use this sample average to estimate the option price to be

$$\vartheta = \frac{e^{-rt}}{n} \sum_{k=1}^{n} (\max\{S_B^k(t) - K, 0\})$$

To generate the sample $S_i^k(t)$ of $S_i(t)$, we have to simulate the paths of the stochastic process

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)dZ_i$$

where Z_i are standard Brownian motion and Z_i, Z_j have correlation ρ_{ij}.[1]

Since we are dealing with correlated underlying asset,we need to simulate correlated normal random variables. Because in a model based on Brownian motion simulating correlated asset prices means simulating correlated normal random variables [3]. We use the following steps:

1. Determine a suitable correlation matrix ρ with ρ_{ij} entries and a covariance matrix Σ, for the underlying asset with the following properties:

   i Symmetric $\Sigma^T = \Sigma$

   ii The diagonal element satisfies $\Sigma_{i,i} \geq 0$

   iii positive semi-definite so that $x^T \Sigma x \geq 0$, $\forall x \in \mathbb{R}^n$

2. Generate normal random variables with mean of 0 and variance 1.

3. Standardize the variables.

4. Decompose the correlation matrix Σ by applying the Cholesky decomposition approach.

5. For each path, loop the simulation from 0 to d and 0 to k, where d is the number of the underlying asset and k the number of time periods.

Note: When referring to the correlation of underlying asset returns,is the same as referring to the correlation of the standard Brownian motions driving the underlying asset prices.[20]

To the above steps we define the $d \times d$ matrix Σ by setting $\Sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$. Our goal is to generate $(Z_1, \cdots, Z_d)$ where,

$$(\sigma_1 Z_1, \cdots, \sigma_d Z_d) \sim N(0, \Sigma) \tag{4.4}$$

Motivation Let $y_i \sim N(0,1)$ for $i = 1, \cdots, d$ be independently identically distributed random variables. Since a linear combination of normal random variables is normal, then $c_1 y_1 + \cdots + c_n y_n \sim N(0, \sigma^2)$, where $\sigma^2 = c_1^2 + \cdots + c_n^2$. In general if A is an $(n \times m)$ matrix and $Y = (y_1, \cdots, y_n)^T$.
Then, $A^T Y \sim N(0, A^T A)$.

So we can apply this motivation to $(\sigma_1 Z_1, \cdots, \sigma_d Z_d)$. Therefore our goal reduces to finding a matrix A such that $A^T A = \Sigma$. To do this we find the Cholesky decompostion of Σ.

Note: Any symmetric positive-definite matrix M, can be written as $M = U^T D U$, where U is an upper triangular matrix and D is a diagonal matrix with positive diagonal elements.

Now,since Σ is positive-definite matrix, we have $\Sigma = U^T D U = (U^T \sqrt{D})(\sqrt{D} U) = (\sqrt{D} U)^T (\sqrt{D} U)$. So we can let $A = \sqrt{D} U$ which is called the Cholesky decomposition of Σ [3].

Therefore we re-write equation (4.2) as follows:

$$dS_i(t) = \mu_i S_i(t)dt + a_i S_i(t)dY(t),$$

where a_i is the i^{th} row of A^T [2].
Explicitly

$$dS_i(t) = \mu_i S_i(t)dt + S_i(t)\sum_{j=1}^{d} A_{ij}^T dy_j(t)$$

Finally, we simulate the paths at times $0 = t_0 < t_1 < \cdots < t_n$ with the following algorithms:

Generate $Y \sim N(0, I)$
compute A such that $A^T A = \Sigma$
for i=0 to d, k=0 to n-1
we set $S_i(t_{k+1}) = S_i(t_k)exp(\mu_i - \frac{1}{2}\sigma^2(t_{k+1} - t_k) + \sqrt{t_{k+1} - t_k}\sum_{j=1}^{d} A_{ij}^T y_{k+1,j})$ [3].

Tree or Lattice Method:

This involves the representation of the underlying asset processes (4.3) in form of a tree.The instantaneous rate of return μ_i is assumed to be constant and $cov(dZ_i(t), dZ_j(t)) = \rho_{ij}dts \neq t$ where all the ρ_{ij}^s are assumed to be constant.so the process becomes

$$S_i(t) = rS_i(t)dt + \sigma_i S_i(t)dZ_i, i = 1, \cdots, n \quad (4.5)$$

Tree or Lattice Methodology

The following steps are followed when pricing basket options using the lattice approach:

Step1: Transform equation (4.2) into Ito process with constant drifts,standard deviations and correlation coefficients.

Step2: Transform the process in step1 into a system of uncorrelated processes. The system of uncorrelated processes is approximated with a multivariate binomial lattice.

Step3: Then the price of the basket option is a function of the prices of the underlying assets and time. Let us denote the price of the basket option by $V(S_1, \cdots S_n, t)$.

A multivariate binomial tree, with d underlying assets, can be described as an $(d+1)-$dimensional tree, one time dimension and one dimension for each underlying asset. Each node in the tree, has 2^d branches. The reasoning behind the binomial approximation is as follows: First, divide the time to maturity T for the basket option into N equally long time periods. For one time period, the continuous joint end of period distribution of the transformed state underlying assets (conditional on the price of the transformed underlying assets at the beginning of the time period) is then approximated with a 2^d-jump distribution. To derive the 2^d-jump distribution, all jump probabilities are first set to be equal. The jump sizes are then determined in a fashion that ensures convergence between the 2^d-jump distribution and the continuous conditional joint end of period distribution as N approaches infinity (i.e as the length of the time period approaches 0).

After the tree has been constructed the price of the basket option is obtained by using risk-neutral pricing and working recursively backward through time. That is, the terminal condition gives the price of the Basket option at each of the $(N+1)^d$ nodes at the expiration date. The price of the basket option at each of the N^d nodes one time period earlier can be calculated by discounting the expected price of the basket option at the riskless rate.

The expected price is a conditional expectation. It is conditional on the information available at the specific node in question.With these prices, the basket option prices at each of the $(N-1)^d$ nodes two time periods before the expiration date can be calculated, and so on. Thus, the basket option price at the current time, which is the desired price, can be obtained by working recursively backward through time with the following formula:

$$V^s_{u_1,u_2,\cdots,u_d} = e^{r\triangle t}\frac{1}{2^d}(V^{s+1}_{u_1+1,u_2+1,\cdots,u_d+1} + V^{s+1}_{u_1+1,u_2+1,\cdots,u_d} + \cdots + V^{s+1}_{u_1,u_2,\cdots,u_d}) \tag{4.6}$$

$0 \le s < N$; $0 \le u_1 \le s$; $0 \le u_2 \le s$; $\cdots$; $0 \le u_d \le s$;

where,

$V^s_{u_1,u_2,\cdots,u_d}$: is the basket option price at s time steps from current time when the underlying asset price S_1 has jumped upward u_1 times, the underlying asset price S_2 has jumped upward u_2 times, ..., the underlying asset price S_d has jumped upward u_d times. The number of down jumps for a given underlying asset is given by s minus its number of up jumps.[5]

T: is the time to maturity from $t = t_0$.The time to maturity from an arbitrary $t(t_0 \le t \le t_0 + T)$ is equal to $T - (t - t_0)$

N: is the number of time steps that T is divided into.

$\triangle t = \frac{T}{N}$: is the length of a time step.

2^n: is the number of branches from each node of the tree.

$\frac{1}{2^n}$: is the equal probability for each branch of a node.

4.3.2 Approximation Methods

Several approximation Methods had been used in pricing basket option. These include Levy Log-normal approximation, Ju-Taylor approximation, Milevsky and Posner Reciprocal Gamma Distribution, e.t.c. Before we describe some of the approximation Methods, let us define the moments of a basket option.

Definition: Let $F_i = S_i(0)e^{\mu_i T}$ and $F = \sum_{i=1}^{d} W_i F_i$, The non-normalized momentsM_i, $i = 1, 2, \cdots$, of the basket option is defined in the following way:

$$
\begin{aligned}
M_1 &= \sum_i W_i F_i \\
M_2 &= \sum_{ij} W_i W_j F_i F_j e^{\rho_{i,j}\sigma_i\sigma_j T} \\
M_3 &= \sum_{i,j,k} W_i W_j W_k F_i F_j F_k e^{\rho_{i,j}\sigma_i\sigma_j T + \rho_{i,k}\sigma_i\sigma_k T + \rho_{j,k}\sigma_j\sigma_k T} \\
M_4 &= \sum_{i,j,k,l} W_i W_j W_k W_l F_i F_j F_k F_l e^{\rho_{i,j}\sigma_i\sigma_j T + \rho_{i,k}\sigma_i\sigma_k T + \rho_{i,l}\sigma_i\sigma_l T + \rho_{j,k}\sigma_j\sigma_k T + \rho_{j,l}\sigma_j\sigma_l T + \rho_{k,l}\sigma_k\sigma_l T}
\end{aligned}
$$

The log-normal approximation

Levy (1992) approximated the basket option by assuming that the sum of correlated assets is still log-normally distributed. The approximation is done by matching the first two normalized moments, this means that the first moment $M_1 = 1$ and the second moment $M_2 = \frac{1}{F^2}\sum_{ij} W_i W_j F_i F_j e^{\rho_{i,j}\sigma_i\sigma_j T}$ and thus can the variance be matched according to $\sigma^2 = ln(M_2)$, which results in:

$$V_{call}(T) \simeq e^{-rT}(FN(\frac{ln(\frac{F}{K}) + \frac{1}{2}\sigma^2}{\sigma}) - KN(\frac{ln(\frac{F}{K}) - \frac{1}{2}\sigma^2}{\sigma}))$$

where N(.) is the cumulative distribution of a standard normal random variable, $F = \sum_{i=1}^{d} W_i F_i$, and K is the basket strike price.[4]

The reciprocal gamma approximation

Milevsky and Posner (1998a) showed that a summation of correlated log-normally distributed stochastic variables will converge in distribution towards a reciprocal gamma distribution when $d \longrightarrow \infty$. We will therefore approximate our finite summation log-normally distributed variables as a reciprocal gamma distribution. A random variable is reciprocal gamma distributed if the inverse is gamma distributed. The valuing of a basket option with correlated underlying assets is done by a moment matching technique. By matching the first two normalised moments $M_1 = 1$ and $M_2 = \frac{1}{F^2} \sum_{ij} W_i W_j F_i F_j e^{\rho_{i,j}\sigma_i \sigma_j T}$, one gets the following system of equations:

$$\beta = \frac{1}{\alpha - 1}$$

$$M_2 = \frac{1}{\beta^2(\alpha-1)(\alpha-2)}$$

Solving the system for α and β, one gets

$$\alpha = \frac{2M_2 - 1}{M_2 - 1}$$

$$\beta = \frac{M_2 - 1}{M_2}$$

Let gR be the Reciprocal gamma distribution The goal is to evaluate the following integral

$$\begin{aligned}
V_{call}(T) &= e^{-rT} \int_0^{\infty} (S_B(T) - K)^+ dp(S_B(T), S_B(0)) \\
&= e^{-rT} \int_{\frac{K}{F}}^{\infty} (\frac{S_B(T)}{T} - \frac{K}{F}) gR(\frac{S_B(T)}{F}, \alpha, \beta) dS_B(\frac{S_B(T)}{FS_B(0)}) \\
&= e^{-rT} [\int_{\frac{K}{F}}^{\infty} x gR(x, \alpha, \beta) dx - K \int_{\frac{K}{F}}^{\infty} gR(x, \alpha, \beta) dx] \\
&= e^{-rT} [\int_{\frac{K}{F}}^{\infty} \frac{G\Gamma(\frac{1}{x, \alpha, \beta})}{x} - K \int_{\frac{K}{F}}^{\infty} \frac{g\Gamma(\frac{1}{x, \alpha, \beta})}{x^2} dx] \\
&= e^{-rT} [\int_0^{\frac{F}{K}} g\Gamma(\frac{u, \alpha, \beta}{u}) du - K \int_0^{\frac{F}{K}} g\Gamma(u, \alpha, \beta) du]
\end{aligned}$$

$$
\begin{aligned}
&= e^{-rT}[\int_0^{\frac{F}{K}} \frac{e^{\frac{-u}{\beta}}(\frac{u}{\beta})^{\alpha-1}}{u\beta\Gamma(\alpha)} - K\int_0^{\frac{F}{K}} g\Gamma(u,\alpha,\beta)du] \\
&= e^{-rT}[\int_0^{\frac{F}{K}} \frac{e^{\frac{-u}{\beta}}(\frac{u}{\beta})^{\alpha-2}}{\beta^2\Gamma(\alpha)} - K\int_0^{\frac{F}{K}} g\Gamma(u,\alpha,\beta)du] \\
&= e^{-rT}[\int_0^{\frac{F}{K}} \frac{e^{\frac{-u}{\beta}}(\frac{u}{\beta})^{\alpha-2}}{\beta\Gamma(\alpha)/(\alpha-1)} - K\int_0^{\frac{F}{K}} g\Gamma(u,\alpha,\beta)du] \\
&= e^{-rT}[\int_0^{\frac{F}{K}} \frac{e^{\frac{-u}{\beta}}(\frac{u}{\beta})^{\alpha}-2}{\beta\Gamma(\alpha-1)} - K\int_0^{\frac{F}{K}} g\Gamma(u,\alpha,\beta)du] \\
&= e^{rT}[\int_0^{\frac{F}{K}} g\Gamma(u,\alpha-1,\beta)du - K\int_0^{\frac{F}{K}} g\Gamma(u,\alpha,\beta)du] \\
&= e^{-rT}[F.G\Gamma(\frac{F}{K},\alpha-1,\beta) - KG\Gamma(\frac{F}{K},\alpha,\beta)].
\end{aligned}
$$

where p is the risk risk neutral probability density function, $G\Gamma$ the gamma cumulative distribution function.[4]

The taylor approximation

This is based on the observation that though the weighted average of log-normal variables is no longer log-normal, it can be approximated by a log-normal random variable if the first two moments match the true first moments. Ju(2002) derived the basket value by considering the Taylor expansion of the ratio of the characteristic function of the average to that of the approximating log-normal random variable around Zero volatility. He included terms up to σ^6 in the expansion. If we study the process it does not seem reasonable to perform a Taylor expansion around zero volatility, since the volatility is different for each underlying asset. We overcome this difficulty by introducing a scale parameter z. The process (4.3) becomes,

$$
S_i(t,z) = S_i(0)exp((\mu_i - \frac{\sigma_i^2}{2})t + z\sigma_i Z_i(t) \tag{4.7}
$$

for z=1 we are back to the standard process (4.3). [4]

Let $B(z) = \sum_{i=1}^{d} W_i S_i(z,t)$ and let M_1 and $M_2(z^2)$ represents the first two moments of B(z),let Y(z) be a normal random variable with mean $m(z^2)$

and variance $v(z^2)$, and then match the first two moments of $e^{Y(z)}$ to those of B(z) in the following way:

$$m(z^2) = 2logM_1 - 0.5logM_2(z^2)$$

$$v(z^2) = logM_2(z^2) - 2logM_1$$

Let X(z)=log(B(z)). The goal is to find the probability density function of X(z) by considering its characteristic function given by:

$$E[e^{i\phi X(z)}] = E[e^{i\phi Y(z)}]\frac{E[e^{i\phi X(z)}]}{E[e^{i\phi Y(z)}]} = E[e^{i\phi Y(z)}]f(z)$$

where

$E[e^{i\phi Y(z)}] = e^{i\phi m(z^2) - \phi^2 \frac{v(z^2)}{2}}$, is the characteristic function of the normal random variable Y(z)

$f(z) = E[e^{i\phi X(z)}]e^{-i\phi m(z^2) + \phi^2 \frac{v(z^2)}{2}}$, is the ratio of the characteristic function of X(z) to that of Y(z).

Ju performs a Taylor expansion of the two factors of f(z) up to z^6, leading to:

$$f(z) \backsimeq 1 - i\phi d_1(z) - \phi^2 d_2(z) + i\phi^3 d_3(z) + \phi^4 d_4(z).$$

where d_i are polynomial of z and terms of order higher than z^6 are ignored.

For details see[6]

$$
\begin{aligned}
d_1(z) &= \frac{1}{2}(6a_1^2(z) + a_2(z) - 4b_1(z) + 2b_2(z)) - \frac{1}{6}(1206a_1^3(z) - a_3(z) \\
&\quad + 6(24c_1(z) - 6c_2(z) + 2C_3(z) - c_4(z))) \\
d_2(z) &= \frac{1}{2}(10a_1^2(z) + a_2(z) - 6b_1(z) + 2b_2(z)) - (128a_1^3(z)/3 - a_3(z)/6 \\
&\quad + 2a_1(z)b_1(z) - a_1(z)b_2(z) + 50c_1(z) - 11c_2(z) + 3c_2(z) + 3c_3(z) - c_4(z)) \\
d_3(z) &= (2a_1^2(z) - b_1(z)) - \frac{1}{3}(88a_1^3(z) + 3a_1(z)(5b_1(z) - 2b_2(z)) \\
&\quad + 3(25c_1(z) - 6c_2(z) + c_3(z))) \\
d_4(z) &= (-20a_1^3(z)/3 + a_1(z)(-4b_1(z) + b_2(z) - 10c_1(z) + c_2(z)) \\
a_1(z) &= -\frac{z^2 \sum_{ij}^d W_i W_j F_i F_j (\rho_{ij}\sigma_i\sigma_j T)}{2\sum_{i,j}^d W_i W_j F_i F_j} \\
a_2(z) &= 2a_1^2 - \frac{z^4 \sum_{ij}^d W_i W_j F_i F_j (\rho_{ij}\sigma_i\sigma_j T)^2}{2\sum_{ij}^d W_i W_j F_i F_j} \\
a_3(z) &= 6a_1a_2 - 4a_1^3 - \frac{z^6 \sum_{i,j}^d W_i W_j F_i F_j (\rho_{ij}\sigma_i\sigma_j T)^3}{2\sum_{ij}^d W_i W_j F_i F_j} \\
b_1(z) &= \frac{z^4}{4B^3(0)}2) \sum_{ijk}^d W_i W_j W_k F_i F_j F_k (\rho_{ik}\sigma_i\sigma_k T)(\rho_{jk}\sigma_j\sigma_k T) \\
b_2(z) &= a_1^2(z) - \frac{1}{2}a_2(z) \\
c_1(z) &= -a_1(z)b_1(z) \\
c_2(z) &= \frac{z^6}{144B^4(0)}(9.8 \sum_{ijkl}^d W_i W_j W_k W_l F_i F_j F_k F_l (\rho_{il}\sigma_i\sigma_l T)(\rho_{jk}\sigma_j\sigma_k T)(\rho_{kl}\sigma_k\sigma_l T) \\
&\quad + 2\sum_{ij}^d W_i W_j F_i F_j (\rho_{ij}\sigma_i\sigma_j T). \sum_{ij}^d W_i W_j F_i F_j (\rho_{ij}\sigma_i\sigma_j T)^2 \\
&\quad + 4\sum_{i,j}^d W_i W_j F_i F_j (\rho_{ij}\sigma_i\sigma_j T) \sum_{i,j}^d W_i W_j F_i F_j (\rho_{ij}\sigma_i\sigma_j T)^2
\end{aligned}
$$

$$
\begin{aligned}
c_3(z) &= \frac{z^6}{48B^6(0)}(4.6\sum_{ijk}^{d} W_i W_j W_k F_i F_j F_k (\rho_{ik}\sigma_i\sigma_k T)(\rho_{jk}\sigma_j\sigma_k T)^2 \\
&\quad + 8\sum_{ijk}^{d} W_i W_j W_k F_i F_j F_k (\rho_{ik}\sigma_i\sigma_k T)(\rho_{jk}\sigma_j\sigma_k T) \\
c_4(z) &= a_1(z)a_2(z) - \frac{2}{3}a_1^3(z) - \frac{1}{6}a_3(z)
\end{aligned}
$$

Finally $E[e^{i\phi X(z)}]$ is approximated by:

$$E[e^{i\phi X(z)}] \backsimeq e^{i\phi m(z^2) - \phi^2 \frac{v(z^2}{2}}(1 - i\phi d_1(z) - \phi^2 d_2(z) + i\phi^3 d_3(z) + \phi^4 d_4(z))$$

and thus the probability density function h(x) of X(1) is derived as

$$h(x) = p(x) + (\frac{d}{dx}d_1(1) + \frac{d^3}{dx^3}d_3(1) + \frac{d^4}{dx^4}d_4(1))p(x)$$

where $p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi x + i\phi m(1) - \phi^2 \frac{v(1)}{2}} d\phi = \frac{1}{\sqrt{2\phi(1)}} e^{-\frac{(x-m(1))^2}{2v(1)}}$ is the normal density with mean of m(1) and variance v(1).

The approximate price of the basket call is given by:

$$
\begin{aligned}
P_{call}(T) = e^{-rT}E[(e^{X(1)} - K)^+] = e^{-rT}[B(0)N(y_1) - KN(y_2) + K(z_1 p(y) \\
+ z_2 \frac{d}{dy}p(y) + z_3 \frac{d^2}{dy^2}p(y))]
\end{aligned}
$$

where $y = logK$, $y_1 = \frac{m(1)-y}{\sqrt{v(1)}} + \sqrt{v(1)}$, $y_2 = y_1 - \sqrt{v(1)}$ and
$z_1 = d_2(1) - d_3(1) + d_4(1)$, $z_2 = d_3(1) - d_4(1)$, $z_3 = d_4(1)$.

NOTE: If we remove the last term in the basket price we get exactly the same price as for the levy log-normal approximation.

Chapter 5

APPLICATION

In this chapter we will give some applications of the methods discussed in the previous chapter by considering the foreign exchange basket Option.

5.1 Foreign Exchange Market

The foreign exchange market is today one of the biggest and most liquid (maturity of about 2 years) markets. The bank for International Settlements reported in their final year report of 2007 that the daily turnover on the foreign exchange market was approximately \$3.2 trillion. It can be seen as a market that never sleeps, it is open 24 hours from UTC 22:00 on Sunday until 22:00 UTC Friday, compared to, for instance, the stock market, which closes for the day when the time hits the closing hour. The spot exchange rate, is the exchange between two currencies, i.e. the amount that one has to pay in one currency to receive units in another currency. All these exchanges are accomplished through market makers, that is why we have the bid-ask spread, the difference between the rate at which the currency is purchased from and the rate that it is sold to these market makers.

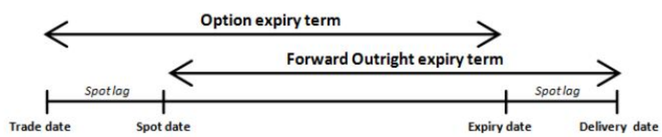
The simplest foreign exchange transaction one can imagine is going to a bureau de change, such as one might find in an airport, and exchanging a certain number of banknotes or coins of one currency for a certain amount of notes and coins of another realm. For example, if an exchange was quoting a GBPNGN rate of 1.63935 NG naira per pound sterling (or conversely, 0.6100 pounds sterling per NG naira). Thus, neglecting two-sided bidoffer pricing and commissions, a holidaymaker at Heathrow seeking to buy ₦100.00 for his/her holiday in lagos should expect to pay 61.00 pounds.

This transaction is, to within a minute or two, immediate. Suppose instead that the transaction is larger in notional by a factor of 1000. Suppose that a traveller is seeking to transfer pounds sterling into a Nigerian account as a deposit on the purchase of a condo in lagos. A trade of this size will be executed in the foreign exchange spot market, and instead of the NG naira funds being available in a minute or two (and the UK pound funds being transferred away from the client), the exchange of funds happens at the **spot date**, which is generally in a couple of days. This lag is largely for historical reasons. On that day, the NG naira funds appear in the clients Nigerian account and the UK pound sterling funds are transferred out of the clients UK account. This process is referred to as **settlement**. The risk that one of these payments goes through but the other does not is referred to as **foreign exchange settlement risk** or Herstatt risk (after the famous example of Herstatt bank defaulting on dollar payments on 26 June 1974).

Another possibility is that perhaps the traveller is flying to lagos to see a new apartment building being built, and he/she knows that the ₦100 000.00 will be needed in six months time. To lock in the currency rate today and protect against currency risk, he/she could enter into a foreign exchange forward, which fixes the rate today and requires the funds to be transferred in six months. The date in the future when the settlement must take place is called the **delivery date**.

A third possibility is that the traveller, being structurally long in pounds sterling, could buy an option to protect against depreciation of the sterling amount over the six month interval in other words, an option to buy NGN (and, equivalently, sell GBP) i.e. a put on the GBPNGN exchange rate, at a prearranged strike price. This removes any downside risk, at the cost of the option premium. Since the transaction is deferred into the future, we have the **delivery date** (just as for the forward) but also an **expiry date** when the option holder must decide whether or not to exercise the option.[7]

There are, therefore, as many as four dates of importance: today (sometimes called the horizon date), spot, expiry and delivery. The period between the trade date and the expiry date is the expiry term for options and the period between spot date and delivery date is the forward outright expiry term (deposit), illustrated in the following figure:



The forward outright contract allows anyone to buy/sell a currency at a specified date for a specified rate in the future, the expiry date for this type of contracts will be referred to as the delivery date. The spot date for a currency cross is the first common day that is the second good **business day**. For an option which expires over the night or week, the expiry term is found as the first open day after the trade date. For an option which expires during the month or year, the expiry date are determined from the forward outright contract by identifying the spot date, and the expiry date from the forward outright contract and finally by stepping back the length of the spot lag to determine the option expiry date for this contract. The option expiry day is the first open day which is the second previous open day from the delivery date. [4]

Note: A day is a good business day for a currency ccy if it is not a weekend (Saturday and Sunday for most currencies, but not always for Islamic countries). For currency pairs ccy1ccy2, a day is only a good business day if it is a good business day for both ccy1 (the foreign currency or sometimes the base currency) and ccy2 (also known as the domestic currency, the terms currency or the quote currency). [7]

5.1.1 Quotation Style

Suppose we have a domestic currency (NG naira, ₦) with the domestic interest rate rd and a foreign currency (US dollars,$) with interest rate rf , and these interest rates will be assumed to be deterministic. The **quotation** or **the exchange rate** is defined how much one need to pay on the domestic currency to buy on unit of the foreign currency. The choice of which way around they are quoted is purely market convention is according to FOR-DOM(foreign domestic). Some uses USD/NGN, the problem with this notation is one could read USD/NGN as USD per NGN, which is not what the market quotes. For US dollars against the NG naira,the market standard quote could be USDNGN (the price of 1 USD in NGN) or NGNUSD (the price of 1 NGN in USD). The USDNGN quote is for NG naira per US dollar. [7]

Table 5.1: Currency pair quotation conventions and market terminology

Currency pair	Common trading floor jargon
EURUSD	Euro-dollar
EURGBP	Euro-Sterlin
GBPUSD	Cable (from the late 1800s transatlantic telegraph cables)
EURJPY	Euro-yen
USDJPY	Dollar-yen
USDNGN	Dollar-naira
NGNUSD	Naira-dollar
GBPNGN	pound-naira

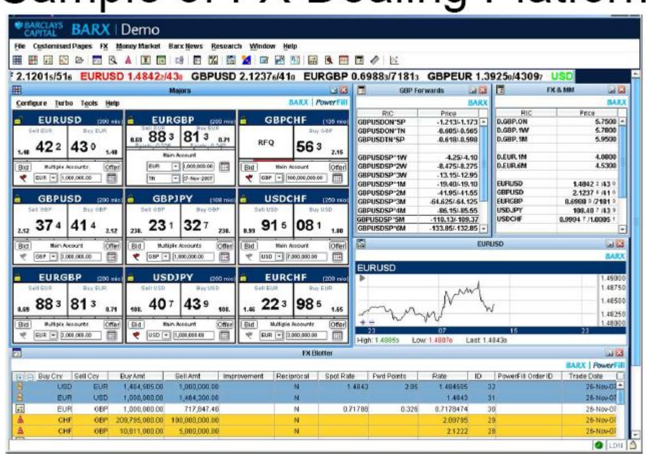
Let S_t define the current spot exchange rate at time t, S_t is then define as

$$S_t = \frac{units\ of\ domestic\ currency}{units\ of\ foreign\ currency}$$

For example, if USDNGN is 157.60, then one US dollar can be bought for ₦157.63 in the spot market. It is a NGN per USD price. It is the cost of one dollar, in nairas.

On foreign exchange trading floors, the spot rate S_t is invariably read aloud with the word '**spot**' used to indicate the decimal place, e.g. one fifty seven **spot** six three in the USDNGN example above. The exception is where there are no digits trailing the decimal place; e.g. if the spot rate for USDNGN was 157.00, we would read this out as 157 the **figure**.

Sample of FX Dealing Platform



5.2 Foreign Exchange Basket Option

Foreign exchange basket options are derivatives based on a common base currency ccy1 and several other risky currencies(ccy2,...,ccyn).The risky currencies have different weights in the basket.A basket option protects against drop in currencies at the same time.

In pricing foreign exchange basket currency pair is assumed to satisfy the stochastic differential equation

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)dZ_i$$

Here, μ_i is the difference between foreign and domestic interest rates of the i^{th} currency pair, S_i is the spot exchange rate of the i^{th} currency pair, σ_i is the volatility of the i^{th} currency pair.

The covariance matrix is also defined as $\Sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, where ρ_{ij} is the correlation coefficient between the ith and jth currency pair.

5.2.1 Correlation in foreign exchange

Obtaining the correlations of market instruments is not a trivial task, It can either be done by observing historical data or by implied calibration.[32] However in foreign exchange transaction the cross currency pairs e.g ccy2ccy3 spot and option are traded, and the correlation can be determined from this contract.

Theorem 1 *Assume that in a market with three spot exchange rates, the correlation coefficient between two currencies are determined by*

$$\rho_{ij} = \frac{\sigma_i^2 + \sigma_j^2 - \sigma_{ij}^2}{2\sigma_i\sigma_j}$$

for $i \neq j$,where σ_i and σ_j are the volatility of the ith and jth currency pair respectively,and σ_{ij}^2 is the covariance between two currency pair(ith and jth). The following property must hold

$$|\sigma_i - \sigma_j| < \sigma_{ij} < \sigma_i + \sigma_j$$

with $\sigma_{ij}^2 = \sigma_i^2 + \sigma_j^2 - 2\sigma_i\sigma_j \Rightarrow \rho_{ij} \in [0,1]$ and when $\rho_{ij} = 1$, we have $\sigma_i = \sigma_j$.

For the proof, see [4]

5.3 Numerical Example

Let NGN be the base currency and consider a basket of four risky(domestic) currencies(GBP,USD,EURO,YEN) with the following setup:
initial foreign exchange spot values $S_o = [10\ 10\ 10\ 10]$ (i.e $S_i(0) = 10$ for each i), $K = 10$, $\mu_i = 0$, $T = 1$, the volatilities $\sigma = [0.3\ 0.3\ 0.3\ 0.3]$ (i.e $\sigma_i = 0.3$ for each i), the weight $W = [0.25\ 0.25\ 0.25\ 0.25]$ (i.e $W_i = 0.25$, for each i). Then correlation matrix is given as follows:

$$\rho = \begin{bmatrix} 1 & 0.2 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$$

Note: In a real foreign exchange market our correlation setup is not possible: The correlation coefficients are determine by the previous theorem.

We price the basket using the log-normal approximation by varying the time.

Now, $F_i = 10$ $F = 10$, for $T = 0.1$:

$$
\begin{aligned}
M_2 &= \frac{1}{10^2}\sum_{ij}^{4} W_i W_j F_i F_j e^{\rho_{i,j}\sigma_i\sigma_j(0.1)} \\
&= \frac{1}{10^2}[W_1W_1F_1F_1e^{\rho_{1,1}\sigma_1\sigma_1(0.1)} + W_1W_2F_1F_2e^{\rho_{1,2}\sigma_1\sigma_2(0.1)} \\
&\quad + W_1W_3F_1F_3e^{\rho_{1,3}\sigma_1\sigma_3(0.1)} + W_1W_4F_1F_4e^{\rho_{1,4}\sigma_1\sigma_4(0.1)} \\
&\quad + W_2W_1F_2F_1e^{\rho_{2,1}\sigma_2\sigma_1(0.1)} + W_2W_2F_2F_2e^{\rho_{2,2}\sigma_2\sigma_2(0.1)} \\
&\quad + W_2W_3F_2F_1e^{\rho_{2,3}\sigma_2\sigma_3(0.1)} + W_2W_4F_2F_4e^{\rho_{2,4}\sigma_2\sigma_4(0.1)} \\
&\quad + W_3W_1F_3F_1e^{\rho_{3,1}\sigma_3\sigma_1(0.1)} + W_3W_2F_3F_2e^{\rho_{3,2}\sigma_3\sigma_2(0.1)} \\
&\quad + W_3W_3F_3F_3e^{\rho_{3,3}\sigma_3\sigma_3(0.1)} + W_3W_4F_3F_4e^{\rho_{3,4}\sigma_3\sigma_4(0.1)} \\
&\quad + W_4W_1F_4F_1e^{\rho_{4,1}\sigma_4\sigma_1(0.1)} + W_4W_2F_4F_2e^{\rho_{4,2}\sigma_4\sigma_2(0.1)} \\
&\quad + W_4W_3F_4F_3e^{\rho_{4,3}\sigma_4\sigma_3(0.1)} + W_4W_4F_4F_4e^{\rho_{4,4}\sigma_4\sigma_4(0.1)} \\
&= 1.013
\end{aligned}
$$

so, $\sigma^2 = ln(M_2) = ln(1.013) \ \Rightarrow \sigma = 0.1203$
substituting in the log-normal approximation formula:

$$V_{call}(T) \simeq e^{-rT}(FN(\frac{ln(\frac{F}{K}) + \frac{1}{2}\sigma^2}{\sigma}) - KN(\frac{ln(\frac{F}{K}) - \frac{1}{2}\sigma^2}{\sigma}))$$

We have, $V_{call}(0.1) \simeq 10(N(0.03005) - N(-0.03005)) \simeq 0.24$,
Therefore, $V_{call}(0.1) \simeq 0.24$.

The cumulative normal distribution function value table can be obtained from the table provided at the end of the chapter [34].
If we repeat this for the rest of the time interval, we obtain the following results:

Table 5.2: Result obtain by varying the time

T	V_{call}
0.1	0.24
0.2	0.32
0.3	0.40
0.4	0.48
0.5	0.56
0.6	0.56
0.7	0.64
0.8	0.72
0.9	0.72
1.0	0.80

5.4 Conclusion

As the time increases the option value increases.

We have successfully described some of the methods used in pricing basket options. We considered the European type, which is easier to price than an American type, which allows the buyer to exercise the option on or before expiration. Further research can focus on basket options of American type.

The Cumulative Normal Distribution $N(x)$ when $x \leq 0$

d	0.00	−0.01	−0.02	−0.03	−0.04	−0.05	−0.06	−0.07	−0.08	−0.09
−4.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−4.4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−4.3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−4.2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−4.1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−4.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−3.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.7	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.6	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

The Cumulative Normal Distribution $N(x)$ when $x \geq 0$

d	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

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