On J-fixed points of J-pseudocontractions with applications

A Thesis
Submitted to the African University of Science and Technology
Abuja-Nigeria
in partial fulfilment of the requirements for

MASTER DEGREE IN PURE AND APPLIED MATHEMATICS

By

Kennedy Obinna Idu

Supervisor:

Prof. C.E. Chidume

African University of Science and Technology
www.aust.edu.ng
P.M.B 681, Garki, Abuja F.C.T
Nigeria.

June 2016

AFRICAN UNIVERSITY OF SCIENCE AND TECHNOLOGY (AUST), ABUJA, NIGERIA

CERTIFICATE OF APPROVAL

ON J-FIXED POINTS OF J-PSEUDOCONTRACTIONS WITH APPLICATIONS

M.Sc. THESIS

This is to certify that the M.Sc. thesis of

KENNEDY OBINNA IDU

has been approved by the Examining Committee for the thesis requirement for award of the degree of Master of Science degree in Mathematics.

RECOMMENDED: ______________________________

Supervisor: Prof. C. E. Chidume

____

HOD: Pure and Applied Mathematics

APPROVED: ______________________________

Chief Academic Officer

____

Date

DEDICATION

This work is in memory of my late Dad... I always loved him.

How I've always been a *son* to many.
This work is dedicated to *them*-
first my heavenly Father, my parents
Late Mr and Mrs Ephraim Idu, *et al*...

ACKNOWLEDGEMENTS

All glory to the God and Father of my Lord and Saviour Jesus Christ for His sure mercies and faithfulness wherein He has abounded towards me in the M.Sc. program; and for the vessels He used.

In this vein, I am grateful to my Supervisor, Prof. C.E. Chidume for a wonderful research experience. Prof. has been most generous in sharing his, over thirty years, research experience. He has given me taste and inculcated in me the rigorous culture and discipline of the researcher. It's been a whole wealth of learning for me: from mathematics to pedagogy and to calligraphy!

Prof. N. Djitte told me, "*if you want to do mathematics, you should not try to follow everybody...*" Prof. Ezzinbi would ask if I could "*touch these things*" and would go on to show me how to think as a mathematician. Prof. Gane Samb Lo showed me how its not all about the number of papers but the depth and richness of mathematical ingenuity and discovery that would last. He showed me the inspirations and motivation of the teacher. I would also love to appreciate Prof. A. Wade and Prof. J. Djoko for their lectures in differential geometry and numerical analysis respectively.

Dr. A. U. Bello was most kind as a Ph.D. teaching assistant. Thank you so much for your encouragements- they made a medalist and thank you so much for believing in me- it has made a mentee. I also would like to acknowledge Dr. P. Ndambovwe, Dr. M. Okpala and Mr Uche(Ph.D. students at the time).

Mr D. E. Mbonu will always be the gift of my mathematical career. Thank you so much Sir for your unwavering confidence and support. This work has also been dedicated to you.
I am also grateful to Prof. K.E. Osondu for a marvellous influence on my mathematical pursuits. Prof. E. N. Erumaka has always showed great concern and counsel to a *son*. Prof. A. Nduka has always been there when I needed him. To you all I have also dedicated this work.

Many thanks to the President and staff of the institution for making my program most memorable and successful. Also, thanks to Mr Bonaventure Mbida-Essama and to the Nelson Mandela Institute(NMI) for sponsorship.

I want to specially thank Mr Chukwuemeka Ani for the tremendous assistance and for being a source of inspiration throughout my M.Sc. program. Mr Gabriel Arusuaire was most benevolent and always stood by me in great difficulties and distress. I'll forever be grateful. Thank you Eric Mensah for helping *a perfect stranger*. To those who will never be forgotten: Sunday, Barnabas, Bih, Kate, Bayo, Precious, Henry, KD, Vivian, Sister, Japhet, Nurse Rose, Bolade, Mrs Buchi, Mrs Nsima, Jemila, Joseph, Stephan("My prof."), Chide, Mekiti, Gloria, Victoria, Almond, Bethel ... Thanks everyone for the gift of friendship.

Finally, I would love to thank you Miss Amaka Adigwe for your goodwill. Thank you Ogonna for being a brother and a friend; thank you Gulfment for your understanding and trust... and thanks to you my colleagues for *everything*.

... And to family, this is where you belong- more than the most deary and all that have now come along. I love you Mom... I love you Dee Prince, Aunts Chichi, Jane, Shirley, Ndidi, Chika- I love you Oby and Chukwuemeka.

Thanks so much to all my uncles, cousins, nephews and nieces. I would love to specially acknowledge my Uncle Joseph Iregbulem for the great support and care.

Contents

List of Figures

ABSTRACT

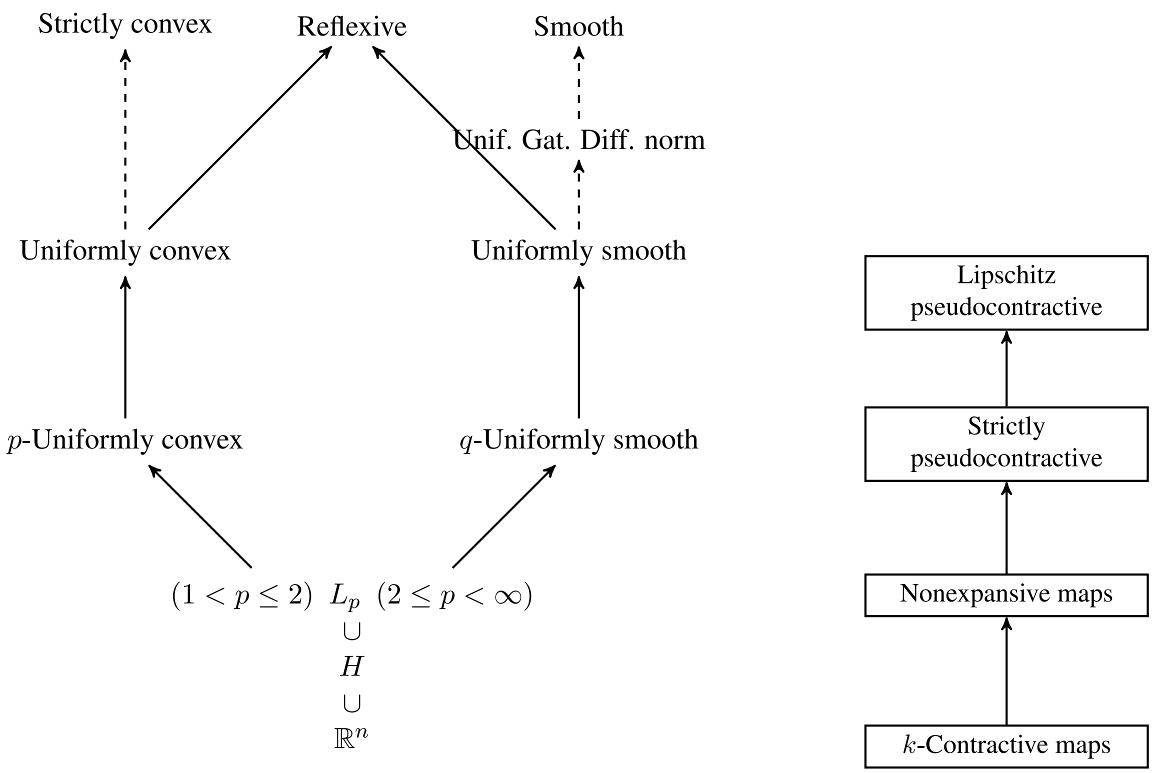
Let E be a real normed space with dual space E^* and let $A : E \to 2^{E^*}$ be any map. Let $J : E \to 2^{E^*}$ be the normalized duality map on E. A new class of mappings, *J-pseudocontractive maps*, is introduced and the notion of *J-fixed points* is used to prove that $T := (J - A)$ is J-pseudocontractive if and only if A is monotone. In the case that E is a uniformly convex and uniformly smooth real Banach space with dual E^*, $T : E \to 2^{E^*}$ is a bounded *J-pseudocontractive map* with a nonempty *J-fixed point* set, and $J - T : E \to 2^{E^*}$ is maximal monotone, a sequence is constructed which converges strongly to a J-fixed point of T. As an immediate consequence of this result, an analogue of a recent important result of Chidume for bounded m-accretive maps is obtained in the case that $A : E \to 2^{E^*}$ is bounded maximal monotone, a result which complements the *proximal point algorithm* of Martinet and Rockafellar. Furthermore, this analogue is applied to approximate solutions of Hammerstein integral equations and is also applied to convex optimization problems.

Chapter 1

INTRODUCTION

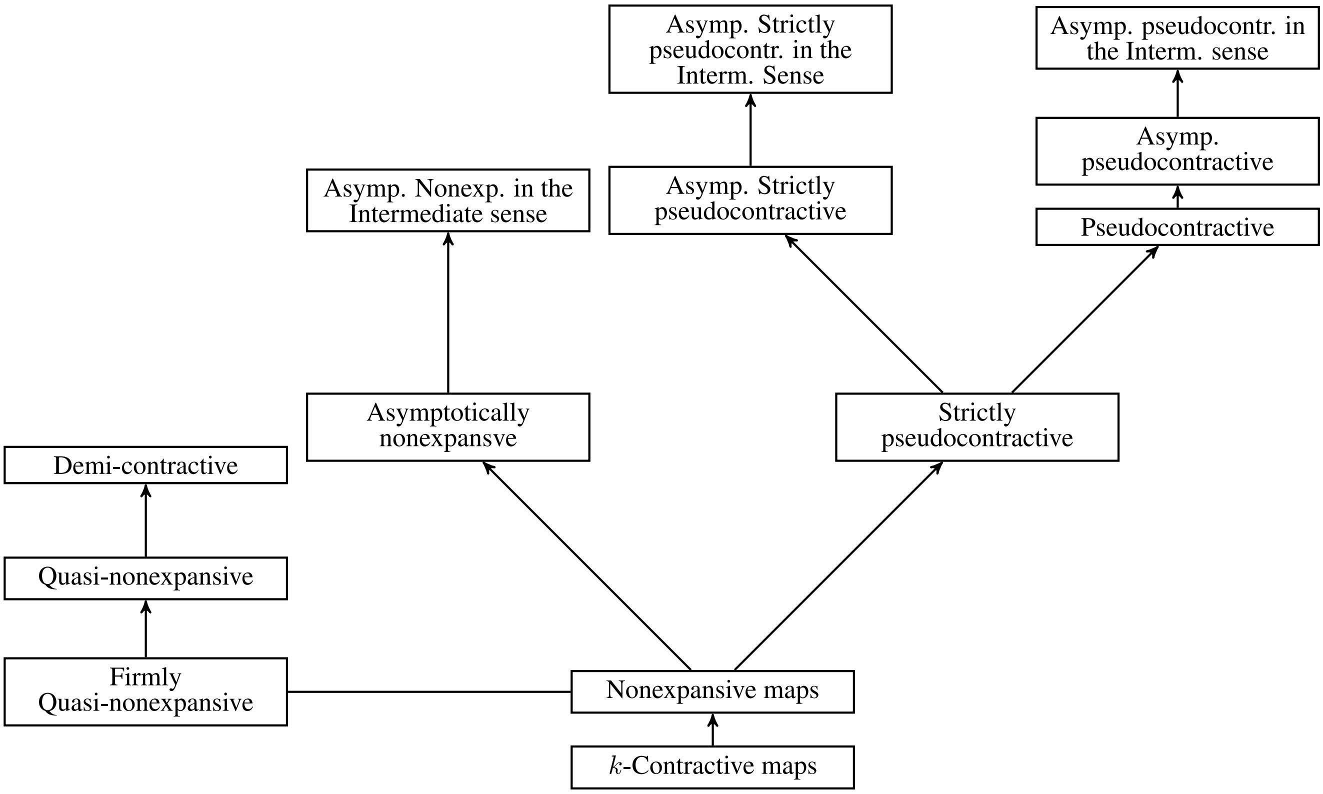
1.1 Background of study

The contributions of this thesis work fall within the general area of nonlinear functional analysis and applications, in particular, nonlinear operator theory. We are interested in the solution or approximation of solutions of nonlinear equations or inclusions (i.e., equations or inclusions defined by nonlinear operators) in Banach spaces.
Problems in the area involve methods of fixed point theory and application of iterative algorithms to approximate zeros or fixed points of nonlinear mappings. Research in the area is enormous due to varied classification of Banach spaces, operators and topological assumptions on them (e.g., continuity, boundedness, compactness, closedness e.t.c). The literature of the last four decades abounds with papers which establish fixed point theorems for selfmaps or nonselfmaps satisfying a variety of contractive type conditions on several ambient spaces. See figures 1.1, 1.2 and 3.1.



(a) Lattice of Banach spaces (b) Metric Fixed-point Operator lattice

Figure 1.1: Lattice of Spaces and Metric fixed-point Operator



(a) Contractive-type self map Operator lattice

Figure 1.2: Lattice of Operators

Let H be a real inner product space. A map $A : H \to 2^H$ is called *monotone* if for each $x, y \in H$,

$$\langle \eta - \nu, x - y \rangle \geq 0 \ \forall \quad \eta \in Ax, \quad \nu \in Ay. \tag{1.1.1}$$

Monotone mappings were first studied in Hilbert spaces by Zarantonello [120], Minty [84], Kačurovskii [64] and a host of other authors. Interest in such mappings stems mainly from their usefulness in applications.

1.1.1 Zeros of Monotone operators on Hilbert spaces

We consider the problem given by

$$Au \ni 0 \tag{1.1.2}$$

where $A : H \to 2^H$ is a monotone map on a Hilbert space. Problems of this kind find relevance in several areas of applications. In particular, we have the following examples:

Convex optimisation problems

Let $g : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *subdifferential* of g, $\partial g : H \to 2^H$, is defined for each $x \in H$ by

$$\partial g(x) = \{x^* \in H : g(y) - g(x) \geq \langle y - x, x^* \rangle \ \forall \, y \in H\}.$$

It is easy to check that ∂g is a *monotone operator* on H, and that $0 \in \partial g(u)$ *if and only if u is a minimizer of g*. Setting $\partial g \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of g.

In particular, as an example of the above, where $g(x) = |x|$, the subdifferential of g at zero, $\partial g(0) = [-1, 1]$, which trivially contains zero. Hence, zero is the minimizer of g.

Equilibrium problem of dynamical systems: Evolution equation

The equation $0 \in Au$ when A is a monotone map from a real Hilbert space to itself also appears in evolution systems. Consider the evolution equation for a single-valued operator,

$$\frac{du}{dt} + Au = 0$$

where A is a monotone map from a real Hilbert space to itself. At equilibrium state, $\frac{du}{dt} = 0$ so that $Au = 0$, whose solutions correspond to the equilibrium state of the dynamical system.

In particular, consider the following diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) & = \triangle u(t,x) + g(u(t,x)),\ t \geq 0,\ x \in \Omega, \\ u(t,x) & = 0,\ \ t \geq 0,\ x \in \partial\Omega, \\ u(0,x) & = u_0(x),\ u_0 \in L_2(\Omega), \end{cases} \tag{1.1.3}$$

where Ω is an open subset of $\mathbb{R}^n$.

By simple transformation i.e., by setting $v(t) = u(t,.)$, where $v : [0,\infty) \longrightarrow L_2(\Omega)$ is defined by $v(t)(x) = u(t,x)$ and $f(\varphi)(x) = g(\varphi(x))$, such that $f : L_2(\Omega) \longrightarrow L_2(\Omega)$, we see that equation (1.1.3) is equivalent to

$$\begin{cases} v'(t) & = Av(t) + f(v(t)),\ \ t \geq 0, \\ v(0) & = u_0, \end{cases} \tag{1.1.4}$$

where A is a nonlinear monotone-type mapping defined on $L_2(\Omega)$.
Setting f to be identically zero, at an equilibrium state (i.e., when the system becomes independent of time) we see that equation (1.1.4) reduces to

$$Au = 0. \tag{1.1.5}$$

Thus, approximating zeros of equation (1.1.5) is equivalent to the approximation of solutions of the diffusion equation (1.1.3) at equilibrium state.

Hammerstein integral equations

Definition 1.1.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of Hammerstein-type has the form*

$$u(x) + \int_\Omega k(x,y)f(y,u(y))dy = w(x), \tag{1.1.6}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions.

By simple transformation (1.1.6) can put in the abstract form

$$u + KFu = 0, \tag{1.1.7}$$

Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear part possesses Green's function can, as a rule, be transformed into the form (1.1.6) (see e.g., Pascali and Sburian [88], p. 164).

1.1.2 Extension of Hilbert space Monotonicity to arbitrary normed spaces

We recall that for a Hilbert space H, $\mathbf{H} = \mathbf{H}^*$. So, in the definition of a monotone operator in a Hilbert space, the map $A : H \to H$ could have been $A : H \to H^*$. Thus, the notion of monotone mappings has been extended to real normed spaces. We now briefly examine two well-studied extensions of Hilbert space monotonicity to arbitrary normed spaces, say , E.

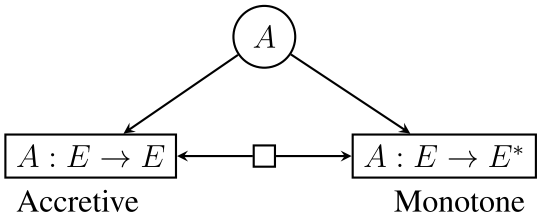


Figure 1.3: Extension of Hilbert space monotonicity

Accretive-type mappings

Let E be a real normed space with dual space E^*. A map
$J : E \to 2^{E^*}$ defined by

$$Jx := \left\{x^* \in E^* : \left\langle x, x^* \right\rangle = \|x\|.\|x^*\|,\ \|x\| = \|x^*\|\right\}$$

is called the *normalized duality map* on E. We denote J^{-1} by J_*

A map $A : D(A) \subseteq\to 2^E$ is called *accretive* if for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\left\langle \eta - \nu, j(x-y) \right\rangle \geq 0 \ \forall \quad \eta \in Ax, \quad \nu \in Ay. \tag{1.1.8}$$

Roughly speaking, accretive mappings acting in a space E are generalizations of non-decreasing real-valued functions. More precisely, A is said to be accretive if for all $x_1,\ x_2 \in D(A)$, $y_1 \in Ax_1$, $y_2 \in Ax_2$ and $\lambda \geq 0$,

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|.$$

A is called *maximal accretive* if, in addition, the graph of A is not properly contained in the graph of any other accretive operator. It is m-accretive if and only if A is accretive and $R(I + tA) = E$ for all $t > 0$. In a normed space, "m-accretive" implies "maximal accretive" . The converse assertion need not be true. The first counterexample was constructed in l_p by B.D. Calvert (1970). Moreover, A. Cernes (1974) showed that even if both E and E^* are uniformly convex, but E is not a Hilbert space, then there are maximal accretive mappings which are not m-accretive. However, it was proved by G. Minty (1962) that in Hilbert spaces, the notions of "m-accretive" and "maximal accretive" are equivalent (see e.g., [80]) In a Hilbert space, the normalized duality map is the identity map, and so, in this case, inequality (1.1.8) and inequality (1.1.1) coincide. Hence, *accretivity is one extension of Hilbert space monotonicity to general normed spaces*.

Monotone-type mappings in arbitrary normed spaces

Let E be a real normed space with dual E^*. A map $A : E \to 2^{E^*}$ is called *monotone* if for each $x, y \in E$, the following inequality holds:

$$\left\langle \eta - \nu, x - y \right\rangle \geq 0 \ \forall \quad \eta \in Ax, \quad \nu \in Ay. \tag{1.1.9}$$

It is called *maximal monotone* if, in addition, the graph of A is not properly contained in the graph of any other monotone operator. Also, A is m-monotone if and only if it is monotone and $R(J+tA)=E^*$ for all $t>0$. When E^* is a strictly convex Banach space with a Fréchet differentiable norm, a maximal monotone operator from E into E^* is m-monotone (see e.g., Kido [71]).

It is obvious that monotonicity of a map defined from a normed space *to its dual* is another extension of Hilbert space monotonicity to general normed spaces.

> The extension of the monotonicity condition from a Banach space into its dual has been the starting point for the development of nonlinear functional analysis.... The monotone mappings appear in a rather wide variety of contexts, since they can be found in many functional equations. Many of them appear also in calculus of variations, as subdifferential of convex functions (Pascali and Sburian [88], p. 101).

1.1.3 Application of Fixed Point Techniques

The theory of fixed point proves to be one of the most useful tools of modern mathematics. This comes from earlier development of the theory and the fact that most important nonlinear problems in applications can be transformed to fixed point problems.

Definition 1.1.2. *Let X be a non-empty set and f be a self-map on X. A fixed point of f is a point $x\in X$ such that $f(x)=x$. If f is a multivalued then a point p in X is called a fixed point of f if $p\in fp$.*

Theorems concerning the existence and properties of fixed points are known as fixed point theorems. Several fixed point theorems include the Banach contraction mapping principle, Brouwer fixed point theorem, Schauder fixed point theorem and a host of others (see e.g., Asati *et al.* [5], Khamsi [67], Smith [107], Lee [74]).
Let E be a real normed space and $A:E\to E$ be an accretive operator. Assume that $Au=0$ has a solution. Browder [14] introduced an operator $T:E\to E$ by $T=I-A$ and called the map T, **pseudo-contractive**. It is clear that zeros of A correspond to fixed points of T (i.e., $Au=0$ if and only if $Tu=u$). The class of pseudocontractive maps properly contains the class of *nonexpansive* maps which are a generalisation of contraction maps. A map $T:E\to E$ is called nonexpansive if for each $x,y\in E$, the inequality $\|Tx-Ty\|\leq\|x-y\|$ is true.

Several existence theorems have been proved for the equation $Au=0$, where A is of the monotone-type (or accretive-type) (see e.g., Brezis [11], Browder [14], Deimling [50], Pascali and Sburian [88], e.t.c.). Likewise, several results have appeared in the literature for approximating zeros of accretive-type (or fixed points of pseudo-contractive) mappings in certain Banach spaces (see e.g., Chidume *et al.*[20], Takahashi [113], Bruck [17], and host of other authors).

Let E be a real normed space and $T:=I-A:E\to E$ a pseudocontractive mapping. If K is a nonempty convex subset of E and $F(T):=\{x\in K:Tx=x\}\neq\emptyset$, the following recursion formula has been used to approximate fixed points of T, $x_0\in K$,

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_nTx_n,\ n\geq 0,$$

where $\{\alpha_n\}$ is a real sequence satisfying appropriate conditions. The most general iterative scheme for bounded pseudocontractive maps seems to be that obtained from the following

Theorem 1.1.3 (C. E. Chidume [23]). *Let E be a uniformly smooth real Banach space with modulus of smoothness ρ_E, and let $A:E\to 2^E$ be a multi-valued bounded $m-$accretive operator with $D(A)=E$ such that the inclusion $0\in Au$ has a solution. For arbitrary $x_1\in E$, define a sequence $\{x_n\}$ by,*

$$x_{n+1}=x_n-\lambda_nu_n-\lambda_n\theta_n(x_n-x_1),\ u_n\in Ax_n,\ n\geq 1,$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions: (i) $\lim_{n\to\infty}\theta_n = 0$, $\{\theta_n\}$ *is decreasing;* (ii) $\sum \lambda_n\theta_n = \infty$; $\sum \rho_E(\lambda_n M_1) < \infty$, *for some constant* $M_1 > 0$*;* (iii) $\lim_{n\to\infty} \frac{\left[\frac{\theta_{n-1}}{\theta_n}-1\right]}{\lambda_n\theta_n} = 0$*. There exists a constant* $\gamma_0 > 0$ *such that* $\frac{\rho_E(\lambda_n)}{\lambda_n} \leq \gamma_0\theta_n$*. Then, the sequence* $\{x_n\}$ *converges strongly to a zero of* A*.*

1.2 Statement of Problem

In studying the inclusion (1.1.2) on real Banach spaces more general than Hilbert spaces when A is of accretive-type mapping, several iterative algorithms have been constructed and results obtained for approximating solutions of problems of the equation (see e.g., the following monographs: Berinde [9], Browder [14], Chidume [22], Reich [90], and the references contained in them). Consequently, this has generated interests and the question asked if similar results for the case of *monotone-type mappings* in arbitrary Banach spaces can be obtained, where A maps a space into its dual.

Regrettably, the pursuit of analogous results has only been greeted with very little progress and seemingly unpropitious prospects as the success for the accretive-type case doesn't quite easily carry over to the case of monotone-type mappings. The difficulty, for the most part, seems to be that all efforts made to apply directly known geometric properties of Banach spaces proved abortive; also developing and understanding concepts with applying knowledge of the structure and geometry of the dual space, existence and uniqueness theorems for monotone-type mappings in arbitrary Banach spaces, weak topology and relevant tools of functional analysis, and other notions of operator theory were rather too slow for the ambitious researcher. Also, defining the iterative sequence to make sense posed a challenge.
Furthermore, the technique of converting the inclusion (1.1.2) into a fixed point problem of defining the map $T := I - A$ is not applicable since, in this case when A is monotone, A maps E into E^* and such T is never well-defined as the identity map does not make sense.

1.3 Aim and Objectives of Study

The aim of this work is to contribute to the efforts being made to approximate solutions of inclusion (1.1.2) where A is of monotone-type. We consider the problem of solving zeros of nonlinear equations of maximal monotone-type mappings with no continuity assumption. We proceed thus.

1. We introduce, as far as we know, a class of mappings called *J-Pseudocontractive mappings* and study the concept of *J-fixed points*. We establish the relationship between monotone mappings and J-pseudocontractive mappings and between J-fixed points and zeros of operators.

2. We construct an iterative algorithm which converges to a J-fixed point of a J-Pseudocontractive mappings and hence, by extension, to a zero of a monotone mapping.

3. We apply our results to:

    - Zeros of maximal monotone mappings. (This corresponds, as noted earlier, to the equilibrium state of some dynamical system)
    - Proximal point algorithm
    - Solutions Hammerstein integral equations
    - Convex minimization problems

Chapter 2

LITERATURE REVIEW

Let H be a real inner product space. A map $A : H \to 2^H$ is called *monotone* if for each $x, y \in H$,

$$\big\langle \eta - \nu, x - y \big\rangle \geq 0 \;\; \forall \quad \eta \in Ax, \quad \nu \in Ay. \tag{2.0.1}$$

Monotone mappings were first studied in Hilbert spaces by Zarantonello [120], Minty [84], Kačurovskii [64] and a host of other authors. Interest in such mappings stems mainly from their usefulness in applications. In particular, monotone mappings appear in convex optimization theory. Consider, for example, the following: Let $g : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *subdifferential* of g, $\partial g : H \to 2^H$, is defined for each $x \in H$ by

$$\partial g(x) = \big\{x^* \in H : g(y) - g(x) \geq \big\langle y - x, x^* \big\rangle \;\; \forall\, y \in H\big\}.$$

It is easy to check that ∂g is a *monotone operator* on H, and that $0 \in \partial g(u)$ *if and only if* u *is a minimizer of* g. Setting $\partial g \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of g.

Furthermore, the equation $0 \in Au$ when A is a monotone map from a real Hilbert space to itself also appears in evolution systems. Consider the evolution equation $\frac{du}{dt} + Au = 0$ where A is a monotone map from a real Hilbert space to itself. At equilibrium state, $\frac{du}{dt} = 0$ so that $Au = 0$, whose solutions correspond to the equilibrium state of the dynamical system.

The notion of monotone mappings has been extended to real normed spaces. We now briefly examine two well-studied extensions of Hilbert space monotonicity to arbitrary normed spaces.

2.0.1 Accretive-type mappings

Let E be a real normed space with dual space E^*. A map $A : E \to 2^E$ is called *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\big\langle \eta - \nu, j(x - y) \big\rangle \geq 0 \;\; \forall \quad \eta \in Ax, \quad \nu \in Ay. \tag{2.0.2}$$

A is called *m-accretive* if, in addition, the graph of A is not properly contained in the graph of any other accretive operator. It is m-accretive if and only if A is accretive and $R(I + tA) = E$ for all $t > 0$.
In a Hilbert space, the normalized duality map is the identity map, and so, in this case, inequality (2.0.2) and inequality (2.0.3) coincide. Hence, *accretivity is one extension of Hilbert space monotonicity to general normed spaces*.

Accretive operators have been studied extensively by numerous mathematicians (see e.g., the following monographs: Berinde [9], Browder [14], Chidume [22], Reich [90], and the references contained in them).

2.0.2 Monotone-type mappings in arbitrary normed spaces

Let E be a real normed space with dual E^*. A map $A : E \to 2^{E^*}$ is called *monotone* if for each $x, y \in E$, the following inequality holds:

$$\big\langle \eta - \nu, x - y \big\rangle \geq 0 \;\; \forall \quad \eta \in Ax, \quad \nu \in Ay. \tag{2.0.3}$$

It is called *maximal monotone* if, in addition, the graph of A is not properly contained in the graph of any other monotone operator. Also, A is maximal monotone if and only if it is monotone and $R(J+tA) = E^*$ for all $t > 0$.
It is obvious that monotonicity of a map defined from a normed space *to its dual* is another extension of Hilbert space monotonicity to general normed spaces.

Accretive mappings were introduced independently in 1967 by Browder [14] and Kato [66]. Interest in such mappings stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in real Banach spaces. It is known (see e.g., Zeidler [122]) that many physically significant problems can be modeled in terms of an initial-value problem of the form

$$0 \in \frac{du}{dt} + Au, \quad u(0) = u_0, \tag{2.0.4}$$

where A is a multi-valued accretive map on an appropriate real Banach space. Typical examples of such evolution equations are found in models involving the heat, wave or Schrödinger equations (see e.g., Browder [15], Zeidler [122]). Observe that in the model (2.0.4), if the solution u is independent of time (i.e., at the equilibrium state of the system), then $\dfrac{du}{dt} = 0$ and (2.0.4) reduces to

$$0 \in Au \tag{2.0.5}$$

whose solutions then correspond to the equilibrium state of the system described by (2.0.4). Solutions of equation (2.0.5) can also represent solutions of partial differential equations (see e.g., Benilan, Crandall and Pazy [8], Khatibzadeh and Moroşanu [69], Khatibzadeh and Shokri [70], Showalter [106], Volpert [115] and so on).

In studying the equation $0 \in Au$, where A is a multi-valued accretive operator on a Hilbert space H, Browder introduced an operator T defined by $T := I - A$ where I is the identity map on H. He called such an operator *pseudo-contractive*. It is clear that solutions of $0 \in Au$, if they exist, correspond to fixed points of T.

Within the past 35 years or so, methods for approximating solutions of inclusion (2.0.5) when A is an accretive-type operator have become a flourishing area of research for numerous mathematicians. Numerous convergence theorems have been published for various Banach spaces and under various continuity assumptions. Many important results have been proved, thanks to geometric properties of Banach spaces developed from the mid 1980s to the early 1990s. The theory of approximation of solutions of the equation when A is of the accretive-type reached a level of maturity appropriate for an examination of its central themes. This resulted in the publication of several monographs which presented in-depth coverage of the main ideas, concepts and most important results on iterative algorithms for appropriation of fixed points of nonexpansive and pseudocontractive mappings and their generalisations, approximation of zeros of accretive-type operators; iterative algorithms for solutions of Hammerstein integral equations involving accretive-type mappings; iterative approximation of common fixed points (and common zeros) of families of these mappings; solutions of equilibrium problems; and so on (see e.g., Agarwal *et al.* [1]; Berinde [9]; Chidume [22]; Reich [92]; Censor and Reich [18]; William and Shahzad [116], and the references contained in them). Typical of the results proved for solutions of equation (2.0.5) is the following theorem:

Theorem 2.0.1 (Chidume, [23]). *Let E be a uniformly smooth real Banach space with modulus of smoothness ρ_E, and let $A : E \to 2^E$ be a multi-valued bounded $m-$accretive operator with $D(A) = E$ such that the inclusion $0 \in Au$ has a solution. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ by,*

$$x_{n+1} = x_n - \lambda_n u_n - \lambda_n \theta_n (x_n - x_1),\ u_n \in Ax_n,\ n \geq 1,$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \theta_n = 0$, $\{\theta_n\}$ is decreasing; (ii) $\sum \lambda_n \theta_n = \infty$; $\sum \rho_E(\lambda_n M_1) < \infty$, for some constant $M_1 > 0$;
(iii) $\lim_{n\to\infty} \frac{\left[\frac{\theta_{n-1}}{\theta_n} - 1\right]}{\lambda_n \theta_n} = 0$. There exists a constant $\gamma_0 > 0$ such that $\frac{\rho_E(\lambda_n)}{\lambda_n} \leq \gamma_0 \theta_n$. Then, the sequence $\{x_n\}$ converges strongly to a zero of A.

Unfortunately, developing algorithms for approximating solutions of inclusions of type (2.0.5) *when $A : E \to 2^{E^*}$ is of monotone-type* has not been very fruitful. Part of the difficulty seems to be that all efforts made to apply directly the geometric properties of Banach spaces developed from the mid 1980s to the early 1990s proved abortive. Furthermore, the technique of converting the inclusion (2.0.5) into a fixed point problem for $T := I - A : E \to E$ is not applicable since, in this case when A is monotone, A maps E into E^* and the identity map does not make sense.

Fortunately, Alber [2] (see also, Alber and Ryazantseva [4]) recently introduced a Lyapunov functional $\phi : E \times E \to \mathbb{R}$ which signalled the beginning of the development of new geometric properties of Banach spaces which are appropriate for studying iterative methods for approximating solutions of (2.0.5) when $A : E \to 2^{E^*}$ is of monotone-type. Geometric properties so far obtained have rekindled enormous research interest on iterative methods for approximating solutions of equation (2.0.5) where A is of the monotone-type, and other related problems (see e.g., Alber [2]; Alber and Guerre-Delabriere [3]; Chidume [23, 25]; Chidume *et al.* [34]; Diop *et al.*[51]; Moudafi [86], Moudafi and Tera [87]; Reich [91]; Sow *et al.*[111]; Takahashi [113]; Zegeye [121] and the references contained in them).

It is our purpose in this thesis to introduce, as far as we know, a new class of mappings called *J-pseudocontractive maps*; prove that $T := (J - A)$ is J-pseudocontractive if and only if A is monotone; and using the notion of *J-fixed points* (which has also been defined as *semi*-fixed point, *duality* fixed point, see e.g., H. Zegeye[121], B. Liu[78]) to prove that if E is a uniformly convex and uniformly smooth real Banach space with dual E^*, $T : E \to 2^{E^*}$ is a bounded *J-pseudocontractive map* with a nonempty *J-fixed point* set, and $J - T : E \to 2^{E^*}$ is maximal monotone, a sequence is constructed which converges strongly to a J-fixed point of T. As an immediate consequence of this result, an analogue of Theorem 2.0.1 for bounded maximal monotone maps is obtained which is also a complement of the *proximal point algorithm* of Martinet [82] and Rockafellar [100] which has also been studied by numerous authors (see e.g., Bruck [17]; Chidume [24]; Chidume [23]; Chidume and Djitte [19]; Reich and Sabach [96, 97], and the references contained in them). Furthermore, this analogue is applied to approximate solutions of Hammerstein integral equations and is also applied to convex optimization problems.

Chapter 3

PRELIMINARY CONCEPTS AND RESULTS

In this chapter we explore the geometric properties of abstract and classical Banach spaces as an attempt to recovering the rich properties of Hilbert spaces and their identities. We also study the characterization of Banach spaces using the duality mappings whose well-definedness is immediate from the Hahn-Banach theorem.

To understand our work in nonlinear operator theory and the study of nonlinear problems, we give an indepth study on nonlinear operators, their classes, characterizations, properties and applications. The notion of semi-groups of operators and the approximation of solutions of nonlinear equations are presented.
Finally, we study the areas of applications: convex analysis and Integral equations; with a gentle introduction to approximate fixed point techniques. Only a basic background in functional analysis is assumed.

3.1 Geometry of Some Banach spaces. Duality Mappings

Banach spaces are vector spaces with a norm (length) which generalises the euclidean structure on n dimensional real and complex space. The isomorphic theory of Banach spaces is the study of properties of Banach spaces which are invariant under renormings of the space. The study of properties of Banach spaces which depend on the specific choice of norm (or distance) on the space is known as the isometric theory of Banach spaces. These properties depend on the shape of the unit ball and so the isometric theory of Banach spaces is also known as the Geometry of Banach spaces.

An early interest of Banach space theory has been to relate analytic properties of a Banach space to various geometrical conditions on the norm of the Banach space. The simplest example of such a condition is that of strict convexity. Related to strict convexity is smoothness (by duality). The classical Banach spaces, L_p, $1 < p < \infty$, are strictly convex and smooth, while the spaces L_1 and $C(K)$ are neither strictly convex nor smooth except in the trivial case when they are one dimensional. Quantitative versions of strict convexity and smoothness are of particular importance in Banach space theory: the uniform smoothness and uniform convexity (see e.g., Lindenstrauss and Johnson [76], Megginson [83]). These are defined using *moduli* functions.

This leads us to the study of these Banach spaces (See e.g., Chidume [22], Bello [7] for the most part of this section).

3.1.1 Strictly Convex and Uniformly Convex Spaces

Let X be an arbitrary normed space and for fixed $x_0 \in X$, let $S_r(x_0)$ denote the *sphere* centred at x_0 with radius $r > 0$, that is,

$$S_r(x_0) := \{x \in X : \|x - x_0\| = r\}.$$

A point u of a convex set C in a vector space is called an *extreme point* of C if $u = tv + (1-t)w$ with $0 < t < 1$ and $v, w \in C$, implies that $u = v = w$.

Definition 3.1.1. *A normed linear space X is said to be* rotund or strictly convex *if for all $x, y \in X$ $x \neq y$, $\|x\| = \|y\| = 1$, we have*

$$\|\alpha x + (1-\alpha)y\| < 1 \text{ for all } \alpha \in (0,1).$$

The following are known characterizations of strictly convex spaces.

Theorem 3.1.2 (See e.g., Gudder and Strawther [59], Stringa [112])**.** *Let X be a normed space. The following conditions are equivalent:*

1. *X is strictly convex.*
2. *If $\|x + y\| = \|x\| + \|y\|$, $x \neq 0$, then $y = cx$ for some $c \geq 0$.*
3. *If $\langle x, y\rangle = \|x\| \cdot \|y\|$, $x \neq y$, then $y = \lambda x$ for some $\lambda \geq 0$.*
4. *Every nonzero continuous linear functional attains a maximum on at most one point of the unit sphere.*
5. *Every element of the unit sphere is an extreme point of the unit ball.*

Definition 3.1.3. *Let X be a normed linear space. Then X is said to be* uniformly rotund or uniformly convex *if for any $\varepsilon \in (0,2]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for each $x, y \in X$ with $\|x\| = 1$, $\|y\| = 1$, and $\|x - y\| \geq \varepsilon$, we have $\|\frac{1}{2}(x+y)\| \leq 1 - \delta$.*

In simple geometrical terms this property asserts that the mid-point of a variable chord of the unit sphere of the space cannot approach the surface of the sphere unless the length of the chord goes to zero.

Theorem 3.1.4. *Let X be a uniformly convex space. Then for any $d > 0, \varepsilon > 0$ and $x, y \in X$ with $\|x\| \leq d$, $\|y\| \leq d$, and $\|x - y\| \geq \varepsilon$, there exists a $\delta = \delta(\frac{\varepsilon}{d}) > 0$ such that $\|\frac{1}{2}(x+y)\| \leq (1-\delta)d$*

Proof. Let $x, y \in X$, set $k_1 = \frac{x}{d}, k_2 = \frac{y}{d}$ and $\overline{\varepsilon} = \frac{\varepsilon}{d}$. Applying the uniform convexity of X, the proof follows easily. ☐

Proposition 3.1.5. *Let X be a uniformly convex space, let $\alpha \in (0,1)$ and $\varepsilon > 0$, then for any $d > 0$, $x, y \in X$ such that $\|x\| \leq d$, $\|y\| \leq d$, and $\|x - y\| \geq \varepsilon$ there exists $\delta(\varepsilon) > 0$ independent of x and y such that*

$$\|\alpha x + (1-\alpha)y\| \leq [1 - 2\delta(\varepsilon)\, min\{\alpha, 1-\alpha\}]d.$$

Proof. Without loss of generality we shall assume that $\alpha \in (0, \frac{1}{2}]$, we also observe that

$$\|\alpha x + (1-\alpha)y\| = \|\alpha(x+y) + (1-2\alpha)y\| \leq 2\alpha\|\frac{1}{2}(x+y)\| + (1-2\alpha)\|y\|$$

Thus from the uniform convexity of X we have for some $\delta(\varepsilon) > 0$

$$\begin{aligned}\|\alpha x + (1-\alpha)y\| &\leq 2\alpha \left\| \frac{1}{2}(x+y) \right\| + (1-2\alpha)\|y\| \\ &\leq 2\alpha(1-\delta(\varepsilon))d + (1-2\alpha)d \\ &= (1-2\alpha\delta(\varepsilon))d \\ &\leq [1-2\delta(\overline{\varepsilon})min\{\alpha, 1-\alpha\}]d.\end{aligned}$$

Which completes the proof. □

Theorem 3.1.6. *Every uniformly convex space is strictly convex.*

Proof. Suppose X is uniformly convex, since $x \neq y$, set $\varepsilon = \|x-y\| > 0$ and $d = 1$. Then in view of proposition 3.1.5 we see that for each $\alpha \in (0,1)$, $\|\alpha x + (1-\alpha)y\| < 1$, which gives the desired result. □

The converse is not true. We give two counterexamples due to Goebel and Kirk [58].

Counterexample 1. *Let $\mu > 0$ and let $c_0 = c_0(\mathbb{N})$ be given the norm $\|\cdot\|_\mu$ defined for $x = \{x_n\} \in c_0$ by*

$$\|x\|_\mu := \|x\|_{c_0} + \mu\Big(\sum_{i=1}^{\infty} \Big(\frac{x_i}{i}\Big)^2\Big)^{\frac{1}{2}},$$

where $\|\cdot\|_{c_0}$ is the usual l_∞ norm. Then,

$$\|x\|_{c_0} \leq \|x\|_\mu \leq (1+\mu)\|x\|_{c_0}, \quad x \in c_0,$$

and the two norms are equivalent with $\|\cdot\|_\mu$ near $\|\cdot\|_{c_0}$ for small μ. However, the spaces $(c_0, \|\cdot\|_\mu)$ for $\mu > 0$ are strictly convex but not uniformly convex, while c_0 with its usual norm is not strictly convex.

Counterexample 2. *Fix $\mu > 0$ and let $C[0,1]$ be endowed the norm $\|\cdot\|_\mu$ defined as follows,*

$$\|x\|_\mu := \|x\|_0 + \mu\Big(\int_{i=1}^{\infty} \Big(\frac{x_i}{i}\Big)^2\Big)^{\frac{1}{2}},$$

where $\|\cdot\|_0$ is the usual supremum norm. As in the counterexample above, $(C[0,1], \|\cdot\|_\mu)$ for any $\mu > 0$ is strictly convex but not uniformly convex (for any $\varepsilon \in (0,2]$ there exist functions $x,\ y \in C[0,1]$ with $\|x\|_\mu = \|y\|_\mu = 1,\ \|x-y\| = \varepsilon$, and $\|\frac{x+y}{2}\|$ arbitrarily near), while c_0 with its usual norm is not strictly convex.

We now give further examples on uniformly and strictly convex spaces.

Example 1. *Every inner product space H is uniformly convex. In particular $\mathbb{R}^n$ with the euclidean norm is uniformly convex.*

To see this we shall apply parallelogram law which is valid in any inner product space. That is for all $x, y, \in H$, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Now, let $\varepsilon \in (0,2]$ be given, let $x, y \in H$ such that $\|x\| \leq 1, \|y\| \leq 1$, and $\|x-y\| \geq \varepsilon$, then from the above identity we have

$$\left\|\frac{1}{2}(x+y)\right\|^2 \leq \frac{1}{4}\left[2(2) - \|x-y\|^2\right] = 1 - \left\|\frac{1}{2}(x-y)\right\|^2 \leq 1 - \frac{1}{4}\varepsilon^2$$

So that

$$\left\|\frac{1}{2}(x+y)\right\| \le \sqrt{1-\frac{1}{4}\varepsilon^2}$$

To complete the proof we choose $\delta = \sqrt{1-\frac{1}{4}\varepsilon^2} > 0$.

Example 2. $\mathbb{R}^n$ *with* $\|\cdot\|_1$ *is not strictly convex. To see this we choose the canonical bases* e_1, e_2 *in* $\mathbb{R}^n$ *and take* $\lambda = \frac{1}{2}$. *Clearly* $\|e_1\| = \|e_2\| = 1$, $e_1 \neq e_2$ *and* $\left\|\frac{1}{2}e_1 + \frac{1}{2}e_2\right\| = \frac{1}{2}\|e_1+e_2\| = 1$. *Thus we have* $\mathbb{R}^n$ *with* $\|\cdot\|_1$ *is not strictly convex.*

Example 3. *The space* $C[a,b]$ *of all real valued continuous functions on the compact interval* $[a,b]$ *endowed with the "sup norm" is not strictly convex. To see this we choose two functions such that*

$$f(t) := 1 \text{ for all } t \in C[a,b], \;\; g(t) := \frac{b-t}{b-a} \text{ for all } t \in C[a,b].$$

Take $\varepsilon = \frac{1}{2}$. *Clearly,* $f, g \in C[a,b]$, $\|f\| = \|g\| = 1$ *and* $\|f-g\| = 1 > \varepsilon$. *But* $\|\frac{1}{2}(f+g)\| = 1$. *Thus,* $C[a,b]$ *is not strictly convex.*

Theorem 3.1.7. *Let* X *be a reflexive Banach space with norm* $\|\cdot\|$. *Then there exists an equivalent norm* $\|\cdot\|_0$ *such that* X *is strictly convex in this norm and* X^* *is strictly convex in the dual norm* $\|\cdot\|_0^*$.

3.1.2 Smooth and Uniformly smooth spaces

A smooth point of the unit ball of X is a point x in the unit sphere for which there is a unique linear functional with norm one which achieves its norm at x. The space X is called smooth provided every point in its unit sphere is a smooth point of the unit ball.

Definition 3.1.8. *A normed space* X *is called smooth if for every* $x \in X$, $\|x\| = 1$, *there exists a unique* x^* *in* X^* *such that* $x^* = 1$ *and* $\langle x, x^*\rangle = \|x\|$.

From an analytic point of view, smoothness of a space can be related to a notion of differentiability- *Gateaux differentiability*, of the space. Suppose x in the unit sphere of X is a smooth point. Consider arbitrary $y \neq x$ in X and a linear functional x^* on X with $\|x^*\| = \langle x, x^*\rangle = 1$. Then for any $t > 0$, $1 + t\langle y, x^*\rangle = \langle x+ty, x^*\rangle \le \|x+ty\|$. That is,

$$\langle y, x^*\rangle \le \frac{\|x+ty\| - \|x\|}{t}.$$

By using the triangle inequality one checks that the function $\frac{\|x+ty\|-\|x\|}{t}$ is an increasing function of t on $(0,\infty)$ and thus

$$\langle y, x^*\rangle \le \lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}.$$

Similarly,

$$\langle y, x^*\rangle \ge \lim_{t\to 0^+} \frac{\|x-ty\| - \|x\|}{-t}.$$

Thus if these two limits coincide, the value of $\langle y, x^*\rangle$ is uniquely determined. On the other hand, if these two limits differ, it follows from the Hahn-Banach theorem that for any λ satisfying

$$\lim_{t\to 0^+} \frac{\|x-ty\| - \|x\|}{-t} \le \lambda \le \lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

there is a linear functional x^* for which $\|x^*\| = \langle x, x^*\rangle = 1$ and $\langle y, x^*\rangle = \lambda$. Consequently, x is a smooth point if and only if $\|x+ty\| + \|x-ty\| - 2 = o(t)$ for every $y \in X$.

Definition 3.1.9. *A normed space X is said to be uniformly smooth whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = 1$ and $\|y\| \leq \delta$, then*

$$\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|.$$

Finally, in this survey of Banach spaces, we see the classical Banach spaces.

3.1.3 Classical Banach spaces: $L_p,\ 1 \leq p \leq \infty$

The *classical Banach spaces* are the earliest spaces ever studied (actually even before the formulation of the general theory). Roughly speaking, they refer to the class of spaces known to Banach and is composed of the Lebesgue spaces.

Definition 3.1.10 ($L_p(\Omega, \Sigma, \mu)$)**.** *Let μ be a positive measure on a σ-algebra Σ of subsets of a set Ω. For each p such that $1 \leq p \leq \infty$, the Lebesgue space $L_p(\Omega, \Sigma, \mu)$ is a normed space with the norm $\|\cdot\|_p$ given by letting*

$$\|f\|_p = \begin{cases} \Big(\int_\Omega |f|^p d\mu \Big)^{1/p} & \textit{if } 1 \leq p < \infty \\ inf\{t : t > 0,\ \mu(\{x : x \in \Omega,\ |f(x)|.t\}) = 0\} & \textit{if } p = \infty. \end{cases}$$

For each p, the elements of $L_p(\Omega, \Sigma, \mu)$ are equivalence classes (under equality almost everywhere [a.e.]) of Lebesgue measurable functions, $f : \Omega \to \mathbb{R}(\mathbb{C})$.
Let $p > 1$, then for $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, L_q is the conjugate space of L_p.

The $l_p (1 \leq p \leq \infty)$ spaces are the Lebesgue spaces $L_p(\Omega, \Sigma, \mu)$ where $\Omega = \mathbb{N}$ and μ is the counting measure on the σ-algebra Σ of all subsets of Ω. For $x = (x_i) \in l_p$, the norm is given by

$$\|x\|_p = \begin{cases} \Big(\sum_{i=1}^{\infty} |x_i|^p d\mu \Big)^{1/p} & \text{if } 1 \leq p < \infty \\ sup\{|x_i| : i \in \mathbb{N}\} & \text{if } p = \infty. \end{cases}$$

The $L_p (1 < p < \infty)$ possess the more nicer geometry properties after the Hilbert spaces. Apart from being both uniformly convex and uniformly smooth(this follows from Hanner's inequalities), we have explicit formula for computing most quantitative geometric parameters such as the *moduli of convexity and smoothness* and also the *duality mappings* and projection maps are well know. The pathological classes of L_1 and L_∞ can be used as checks in different areas of Banach space theory, e.g., in constructing counterexamples of spaces which are not strictly convex or smooth, among others.
Standard characteristic inequalities abound for work in L_p spaces, among them is:

Theorem 3.1.11 (see e.g., Clarkson [49])**.** *For space L_p, with $p \geq 2$, the following inequalities between the norms of two arbitrary elements x and y of the space are valid (here q is the conjugate index, $q = p/(p-1)$)*

$$2(\|x\|^p + \|y\|^p) \leq \|x + y\|^p + \|x - y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p);$$
$$2(\|x\|^p + \|y\|^p)^{q-1} \leq \|x + y\|^q + \|x - y\|^q;$$
$$\|x + y\|^p + \|x - y\|^p \leq 2(\|x\|^p + \|y\|^p)^{p-1}.$$

For $1 < p \leq 2$ these inequalities hold in the reverse sense.

Next, we consider geometric parameters for quantitative analysis of spaces.

3.1.4 Moduli. p-uniformly convex and q-uniformly smooth spaces

Often-times we are interested in quantitative descriptions of geometrical properties of Banach spaces. The most common way for creating these descriptions, is to define a real function (a "modulus") depending on the Banach space under consideration, and from this a suitable constant or coefficient closely related with this function. The moduli and/or the constants are attempts in order to get a better understanding about two facts:

- The shape of the unit ball of a space, and
- The hidden relations between weak and strong convergence of sequences

(see e.g., Llorens Fuster[79]). Thus, these moduli can be used in classifying and characterizing certain Banach spaces. We consider two well moduli in Banach space theory

The ***modulus of convexity*** of E is the function $\delta_X : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : \|x\| = \|y\| = 1;\ \epsilon = \|x-y\|\Big\}.$$

A related coefficient to the modulus of convexity is the *characteristic of convexity*,

$$\varepsilon_0(X) := sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\}.$$

The modulus of convexity and the characteristic of convexity are measures of "how convex" the unit ball in a Banach space is. It is also well known (see *e.g.*, Chidume [22] p. 34, Lindenstrauss and Tzafriri [77]) that δ_E is nondecreasing.
The following are geometric properties construed from the modulus of convexity (see e.g., Llorens Fuster[79]):

- We have the following definitions:

  Definition 3.1.12. *The space E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$, or equivalently if $\varepsilon_0(X) = 0$.*

  Definition 3.1.13 (**p-uniformly convex**)**.** *If there exists a constant $c > 0$ and a real number $p > 1$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, then E is said to be p-uniformly convex. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,*

  $$L_p \ (or\ l_p)\ or\ W_p^m\ is \left\{ \begin{array}{ll} p - uniformly\ convex & if\ \ 2 \leq p < \infty; \\ 2 - uniformly\ convex & if\ \ 1 < p < 2. \end{array} \right.$$

- The Banach space X is strictly convex (SC) if and only if $\delta_X(2) = 1$
- A classical result in metric fixed point theory is that the uniformly convex Banach spaces have the fixed point property for nonexpansive mappings (FPP). It was published in 1965. It is due (independently) to F. Browder, D. Gohde, and (in a more general form) to W.A. Kirk.
  If $\delta_X(1) > 0$ (that is, if $\varepsilon_0(X) < 1$), then X (is super-reflexive) and has (uniform) normal structure. In fact, see both X and X^* have the fixed point property for nonexpansive mappings.

The ***modulus of smoothness*** of E is the function defined by:

$$\rho_X(\tau) := \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : x,\, y \in B_X\right\}.$$

A normed linear space E is called *uniformly smooth* if

$$\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau}=0.$$

It is well known (see *e.g.*, Chidume [22] p. 16, also Lindenstrauss and Tzafriri[77]) that ρ_E is nondecreasing.

If there exists a constant $c>0$ and a real number $q>1$ such that $\rho_E(\tau)\le c\tau^q$, then E is said to be q-***uniformly smooth***. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1<p<\infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} 2-\text{uniformly smooth} & \text{if } 2\le p<\infty;\\ p-\text{uniformly smooth} & \text{if } 1<p<2.\end{cases}$$

We obtain the following explicit formula of the moduli of convexity and smoothness, and the relation between them.

1. We have (see e.g., [4], p.47) for $p>1$, $q>1$, $X=L^p$, $X^*=L^q$, that

$$\delta_{X^*}(\epsilon)=1-\Big(1-\Big(\frac{\epsilon}{2}\Big)^q\Big)^{1/q},$$

   and, for every $\tau\ge 0$,

$$\rho_X(\tau)=\begin{cases}(1+\tau^p)^{\frac{1}{p}}-1), & \text{for } 1<p\le 2,\\ \Big(\frac{|1+t|^p+|1-t|^p}{2}\Big)^{\frac{1}{p}-1} & \text{for } p>2.\end{cases}$$

2. For Hilbert spaces H, take $p=q=2$ in the formula above. We remark here that the highest possible value of δ_X and the lowest value of ρ_X are attained in Hilbert spaces, that is, for any Banach space X,

$$\delta_X\le\delta_H \text{ and } \rho_X(\tau)\ge\rho_{l_2}(\tau).$$

3. The moduli of convexity and smoothness are related by the formulas:

$$\rho_{X^*}(\tau)=sup\Big\{\frac{1}{2}\varepsilon\tau-\delta_X(\varepsilon):\ 0\le\varepsilon\le 2\Big\}.$$
$$\rho_X(\tau)=sup\Big\{\frac{1}{2}\varepsilon\tau-\delta_{X^*}(\varepsilon):\ 0\le\varepsilon\le 2\Big\}.$$

   Consequently, X is uniformly convex if and only if its dual space is uniformly smooth and X is uniformly smooth if and only if its dual space is uniformly convex.

3.1.5 Duality Mapping of Banach spaces

From the concluding remarks in the previous subsection on the dual relationship between spaces, we here present the notion of duality mappings which will provide us with a pairing between elements of a normed space E and elements of its dual space X^*, which we shall also denote by $\langle\cdot,\cdot\rangle$ and will serve as a suitable analogue of the inner product in Hilbert spaces.

Define a map $J:X\longrightarrow 2^{X^*}$ by

$$Jx\ :=\ \Big\{x^*\in X^*:\langle x,x^*\rangle=\|x\|\|x^*\|;\ \|x\|=\|x^*\|\Big\}.$$

By Hahn Banach theorem for each $x \in X,\ x \neq 0$, there exists $y^* \in X^*$ such that $\|y^*\| = 1$, and $\langle x, y^*\rangle = \|x\|$. So if we set $x^* = \|x\|y^*$, then we see that for each $x \in X \ \exists\ x^* \in X^*$ such that $\|x^*\| = \|x\|$ and $\langle x, x^*\rangle = \|x\|^2$. So we see that for each $x \in X$, $Jx \neq \emptyset$. The mapping $J : X \longrightarrow 2^{X^*}$ is called the *duality mapping* of the space X. In general J is multivalued.

Remark. More generally, given an increasing continuous function $\varphi :\ [0, +\infty) \to [0, +\infty)$ such that $\varphi(0) = 0$ and $\lim\limits_{+\infty} \varphi = +\infty$, one defines the duality map J_φ corresponding to the (normalization) function φ, by

$$J_\varphi x \ := \ \Big\{x^* \in X^* : \langle x, x^*\rangle = \|x\|\|x^*\|;\ \|x^*\| = \varphi(\|x\|)\Big\}.$$

Proposition 3.1.14. *Let H be a real Hilbert space and identify H^* with H, then*

$$Jx = \{x\} \quad for\ all\ \ x \in H;$$

i.e The duality map J is the identity map.

Proof. Let $a \in H$. Define

$$\varphi_a(x)\ = \langle a,\, x\rangle \quad \forall x \in H.$$

Then $\varphi_a \in H^*$, $\|\varphi_a\| = \|a\|$ and $\varphi_a(a) = \|a\|^2$. Therefore $\varphi_a \in J(a)$and since φ_a is identified with a, via Riesz representation theorem, we can write $a \in J(a)$. Conversely, if $y \in Ja$ then $\langle a, y\rangle = \|a\|\|y\|$ and $\|a\| = \|y\|$ so that

$$\|a - y\|^2 = \langle a - y, a - y\rangle = \|a\|^2 + \|y\|^2 - 2\langle a, y\rangle = 2\|a\|^2 - 2\|y\|^2 = 0.$$

So we have $y = a$. Therefore $Ja = \{a\}$. □

Proposition 3.1.15. *Let X be a real Banach and J be the duality mapping on X, then*

$$J(\lambda x) = \lambda J(x)\ \forall\ \lambda \in \mathbb{R} \quad \forall\ x \in X.$$

Proof. Let $y^* \in J(x)$ and $\lambda \in \mathbb{R}$. For $\lambda = 0$ the result follows trivially. suppose $\lambda \neq 0$, then we have

$$\langle \lambda x, \lambda y^*\rangle = \lambda^2\langle x, y^*\rangle = \|\lambda x\|\|\lambda y^*\|, \text{we also have } \|\lambda x\| = \|\lambda y^*\|.$$

Thus we have $\lambda y^* \in J(\lambda x)$, which implies $\lambda J(x) \subset J(\lambda x)$. From the preceding inclusion we also obtained that $\frac{1}{\lambda}J(\lambda x) \subset J(x)$ which implies $J(\lambda x) \subset \lambda J(x)$. Therefore $J(\lambda x) = \lambda J(x)\ \forall \lambda \in \mathbb{R}$, $\forall x \in X$. □

Definition 3.1.16. *Let $f : X \longrightarrow Y$ be a map. Then f is said to be* demi-continuous *if it is norm to weak-star continuous, i.e f is continuous from X (endowed with the strong topology) to Y (endowed with the weak-star topology).*

Proposition 3.1.17. *Let X be a real norm space and J be the duality mapping on X. Then the following are true:*

(a) For each $x \in X$, Jx is a closed, convex subset of $B^*[0, \|x\|]$ in X^*. Where $B^*[0, \|x\|] = \{y^* \in X^* : \|y^*\| \leq \|x\|.\}$

(b) If X^* is strictly convex, then for each $x \in X$, Jx is single valued. Moreover the mapping J is demi-continuous, i.e J is continuous as a mapping from X with the strong topology to X^* with the weak-star topology.

(c) If X^* is uniformly convex, then for each $x \in X$, Jx is single valued and the mapping $x \longmapsto Jx$ is uniformly continuous on bounded subsets of X.

(d) If X^* is smooth, then J is one to one, and is onto if X is reflexive.

Proof. (a) Obviously we have $Jx \subset B^*[0, \|x\|]$. Let $\{y_n^*\}_{n\geq 1} \subset Jx$ such that $y_n^* \to y$, for each $n \geq 1$ we have $\langle x, y_n^* \rangle = \|x\|\|y_n^*\|$ and $\|x\| = \|y_n^*\|$. Letting $n \to +\infty$ we see that $\langle x, y \rangle = \|x\|\|y\|$ and $\|x\| = \|y\|$. Hence we have $y \in Jx$, which implies that Jx is closed.

For convexity, let $x^*, y^* \in Jx$ and $\lambda \in (0, 1)$, then

$$\begin{aligned}\langle x, \lambda x^* + (1-\lambda)y^* \rangle &= \lambda \langle x, x^* \rangle + (1-\lambda)\langle x, y^* \rangle \\ &= \lambda \|x\|\|x^*\| + (1-\lambda)\|y^*\| = \|x\|^2\end{aligned}$$

We also have from the triangular inequality that $\|\lambda x^* + (1-\lambda)y^*\| \leq \|x\|$, also,

$$\begin{aligned}\|x\|^2 &= \langle x, \lambda x^* + (1-\lambda)y^* \rangle \\ &\leq \|x\|\|\lambda x^* + (1-\lambda)y^*\|,\end{aligned}$$

which implies that $\|x\| \leq \|\lambda x^* + (1-\lambda)y^*\|$. Hence we have

$$\|x\| = \|\lambda x^* + (1-\lambda)y^*\|, \text{ which shows that } \lambda x^* + (1-\lambda)y^* \in Jx.$$

(b) Assume X^* is strictly convex, and suppose that there exit $x^*, y^* \in Jx$ such that $x^* \neq y^*$, then $\|x^*\| = \|y^*\| = \|x\|$ and by the strict convexity of X we have that for any $\lambda \in (0, 1)$, $\|\lambda x^* + (1-\lambda)y^*\| < \|x\|$. In particular taking $\lambda = \frac{1}{2}$, we have $\|\frac{1}{2}(x^* + y^*)\| < \|x\|$, which contradicts the fact that $\|\frac{1}{2}(x^* + y^*)\| = \|x\|$. (Since Jx is convex)

Let $\{x_n\}_{n\geq 1} \subset X$ such that $x_n \to x$. using the fact that $\|Jx_n\| = \|x_n\|$, i.e $\{Jx_n\}_{n\geq 1}$ is bounded and the fact that the unit ball is w^*-compact in X^* (Banach Alaoglo Theorem) we see that there exists a limit point y^* of $\{Jx_n\}_{n\geq 1}$. Now let $\{Jx_{n_k}\}_{k\geq 1} \subset X^*$ such that $w^* - lim Jx_{n_k} = y^*$, then we have $\lim\limits_{k\to\infty} \langle x_{n_k}, Jx_{n_k} \rangle = \langle x, y^* \rangle$. We also have that

$$\lim_{k\to\infty} \langle x_{n_k}, Jx_{n_k} \rangle = \lim_{k\to\infty} \|x_{n_k}\|^2 = \|x\|^2.$$

So we get $\langle x, y^* \rangle = \|x\|^2$, which implies $\|x\| \leq \|y^*\|$. To get the reverse inequality we use the fact that $w^* - lim Jx_{n_k} = y^*$ implies $\|y^*\| \leq \liminf \|Jx_{n_k}\| = \liminf \|x_{n_k}\| = \|x\|$. Thus we have $\|x\| = \|y^*\|$ and $\langle x, y^* \rangle = \|x\|^2$. i.e $y^* = Jx$. Therefore J is demicontinuous.

(c) Since a uniformly convex space is also strictly, then by part (b) above we see that J is single valued.

Assume J is not uniformly continuous on bounded subsets of X, then there exists a constant $M > 0$, $\alpha_0 > 0$, and subsequences $\{u_n\}, \{v_n\} \subset X$ such that

$$\begin{aligned}&\|u_n\| \leq M,\ \|v_n\| \leq M,\ n \geq 1, \\ &\|u_n - v_n\| \to 0 \; as \; n \to \infty, \\ &\|Ju_n - Jv_n\| \geq \alpha_0,\ n \geq 1\end{aligned} \tag{3.1.1}$$

Let $\beta > 0$ such that $\|u_n\| \geq \beta,\ \|v_n\| \geq \beta,$ for $n \geq 1$. Such β exist, for if there exists a subsequence $\{u_{n_k}\} \subset X$ such that $u_{n_k} \to 0$ as $n \to +\infty$, then we see that $v_{n_k} \to 0$. From the definition of duality map we obtained that $Ju_{n_k} \to 0$ and $Jv_{n_k} \to 0$, and this contradicts (3.1.1).

Now set

$$x_n = \frac{u_n}{\|u_n\|},\ y_n = \frac{v_n}{\|v_n\|}\ u_n, v_n\ \neq 0. \text{ Then we have,}$$

$$\begin{aligned}\|x_n - y_n\| &= \frac{1}{\|u_n\|\|v_n\|}\|\|u_n\|v_n\| - \|u_n\|v_n\| \\ &\leq \frac{1}{\beta^2}(\|v_n\|\|u_n - v_n\| + \|\|v_n\| - \|u_n\|\|\|v_n\|) \\ &\leq \frac{2M}{\beta^2}\|u_n - v_n\| \to 0\ as\ n \to +\infty\end{aligned}$$

We also have $2 \geq \|Jx_n + Jy_n\| \geq \langle x_n, Jx_n + Jy_n\rangle$ which together with

$$\begin{aligned}\langle x_n, Jx_n + Jy_n\rangle &= \|x_n\|^2 + \|y_n\|^2 + \langle x_n - y_n, Jy_n\rangle \\ &= 2 + \langle x_n - y_n, Jy_n\rangle \geq 2 - \|x_n - y_n\|\end{aligned}$$

implies

$$\lim_{n\to\infty}\|Jx_n + Jy_n\| = 2 \text{ i.e } \lim_{n\to\infty}\|\frac{1}{2}(Jx_n + Jy_n)\| = 1. \tag{3.1.2}$$

Now suppose there exists $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\},\ \{y_{n_k}\} \subset X$ such that $\|Jx_{n_k} - Jy_{n_k}\| \geq \varepsilon_0$, for $n \geq 1$. Observing that $\|Jx_{n_k}\| = \|Jy_{n_k}\| = 1$ and using the uniform convexity of X^* we see that there exists $\delta(\varepsilon_0) > 0$ such that $\|\frac{1}{2}(Jx_{n_k} + Jy_{n_k})\| \leq 1 - \delta(\varepsilon_0)$ which contradicts (3.1.2). Therefore we have $\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0$, which implies

$$\begin{aligned}\|Ju_n - Jv_n\| &= \|J(\|u_n\|x_n) - J(\|v_n\|y_n)\| \\ &= \|\|u_n\|Jx_n - \|v_n\|Jy_n\| \\ &\leq \|u_n\|\|Jx_n - Jy_n\| + \|v_n\|\|\|u_n\| - \|v_n\|\| \\ &\leq M\|Jx_n - Jy_n\| + \|u_n - v_n\| \to 0\ as\ n \to +\infty.\end{aligned}$$

This contradicts (3.1.1). Hence we have the result.

(d) *Left as a little exercise.* □

The analytical representations of duality mappings are known in a number of Banach spaces. For instance, in the spaces l^p, $L^p(G)$ and $W^p_m(G)$, $p \in (1, \infty)$ we have, respectively,

$$Jx = ||x||^{2-p}_{l^p}y \in l^q, \quad y = \{|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...\}, x = \{x_1, x_2, ...\},$$

$$Jx = ||x||^{2-p}_{L^p}|x(s)|^{p-2}x(s) \in L^q(G), s \in G,$$

and

$$Jx = ||x||^{2-p}_{W^p_m}\sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha(|D^\alpha x(s)|^{p-2}D^\alpha x(s)) \in W^q_{-m}(G), m > 0, s \in G,$$

where $p^{-1} + q^{-1} = 1$. [see e.g., Alber [4], p.36]

3.1.6 Important Banach space Identities and Characterizations

Beside the presence of inner product and the fact that the projection map is nonexpansive, duality map and *moduli* parameters are well-known and computable; the nice geometric properties construed in the following identities

(a) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$

(b) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 - \lambda(1-\lambda)\|x-y\|^2 + (1-\lambda)\|y\|^2$

make Hilbert spaces amiable but rather superficial for many real world applications. Most of the spaces where these application models are found are not Hilbert spaces, and so we lose these nice identities.

However, with the duality mapping, moduli parameter and other tools, we obtain characteristic analogous identities(inequalities) that suffice. In this subsection, we consider these identities.

We begin with the following identities which are valid in any normed space.

Theorem 3.1.18. *Let X be a real normed space, and $J_p : X \to 2^{X^*}$, $1 < p < \infty$, be the generalized duality map. Then, for any $x, y \in X$, the following inequality hold*

$$\|x+y\|^p \le \|x\|^p + p\langle y, j_p(x=y)\rangle$$

for all $j_p(x+y) \in J_p(x+y)$. In particular, if $p = 2$, then

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$.

Inequalities in uniformly convex spaces

Theorem 3.1.19. *Let $p > 1$ and $r > 0$ be two fixed real number. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing and convex function*

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that for all $x, y\, B_r$ and $0 \le \lambda \le 1$,

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|),$$

where $W_P := \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$ and $B_r := \{x \in X : \|x\| \le r\}$.

Corollary 3.1.20. *Let $p > 1$ and $r > 0$ be two fixed real numbers and X be a Banach space. Then the following are equivalent.*

(a) X is uniformly convex.

(b) There is a continuous, strictly increasing and convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that

$$\|x+y\|^p \ge \|x\|^p + p\langle y, j_p(x)\rangle + g(\|y\|)$$

for every $x, y \in B_r$ and $j_p(x) \in J_p(x)$.

(c) There is a continuous, strictly increasing and convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that

$$\langle x-y, j_p(x) - j_p(y)\rangle \ge g(\|x-y\|)$$

for every $x, y \in B_r$ and $j_p(x) \in J_p(x)$, $j_p(y) \in J_p(y)$.

Theorem 3.1.21 (p-uniformly convex). *Let $p > 1$ and X be a Banach space. Then the following are equivalent.*

(a) X is p-uniformly convex.

(b) There exists $c > 0$ such that

$$\frac{1}{2}(\|x+y\|^p + \|x-y\|^p) \geq \|x\|^p + c\|y\|^p \,\forall\, x, y \in X.$$

(c) $\|\cdot\|^p$ is uniformly convex at center in X.

(d) There is a constant $d > 0$ such that for every $x, y \in X$, $j_p(x) \in J_p(x)$, $j_p(y) \in J_p(y)$, the following inequality holds:

$$\langle x-y, j_p(x) - j_p(y)\rangle \geq d\|x-y\|^p.$$

Inequalities in uniformly smooth spaces

Theorem 3.1.22. *Let $q > 1$ and $r > 0$ be two fixed real number. Then a Banach space X is uniformly smooth if and only if there exists a continuous, strictly increasing and convex function*

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that for all $x, y\, B_r$ and $0 \leq \lambda \leq 1$,

$$\|\lambda x + (1-\lambda)y\|^p \geq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|),$$

where $W_P := \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$ and $B_r := \{x \in X : \|x\| \leq r\}$.

Corollary 3.1.23. *Let $q > 1$ and $r > 0$ be two fixed real numbers and X be a Banach space. Then the following are equivalent.*

(a) X is uniformly smooth.

(b) There is a continuous, strictly increasing and convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + g(\|y\|)$$

for every $x, y \in B_r$ and $j_q(x) \in J_q(x)$.

(c) There is a continuous, strictly increasing and convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that

$$\langle x-y, j_q(x) - j_q(y)\rangle \leq g(\|x-y\|)$$

for every $x, y \in B_r$ and $j_p(c) \in J_p(x)$, $j_p(y) \in J_p(y)$.

Theorem 3.1.24 (q-uniformly smooth). *Let $q > 1$ and X be a Banach space. Then the following are equivalent.*

(a) X is q-uniformly smooth.

(b) There exists $c > 0$ such that

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c\|y\|^q \,\forall\, x, y \in X.$$

(c) There is a constant $d > 0$ such that for every $x, y \in X$, the following inequality holds:

$$\langle x-y, J_q(x) - J_q(y)\rangle \leq d\|x-y\|^q.$$

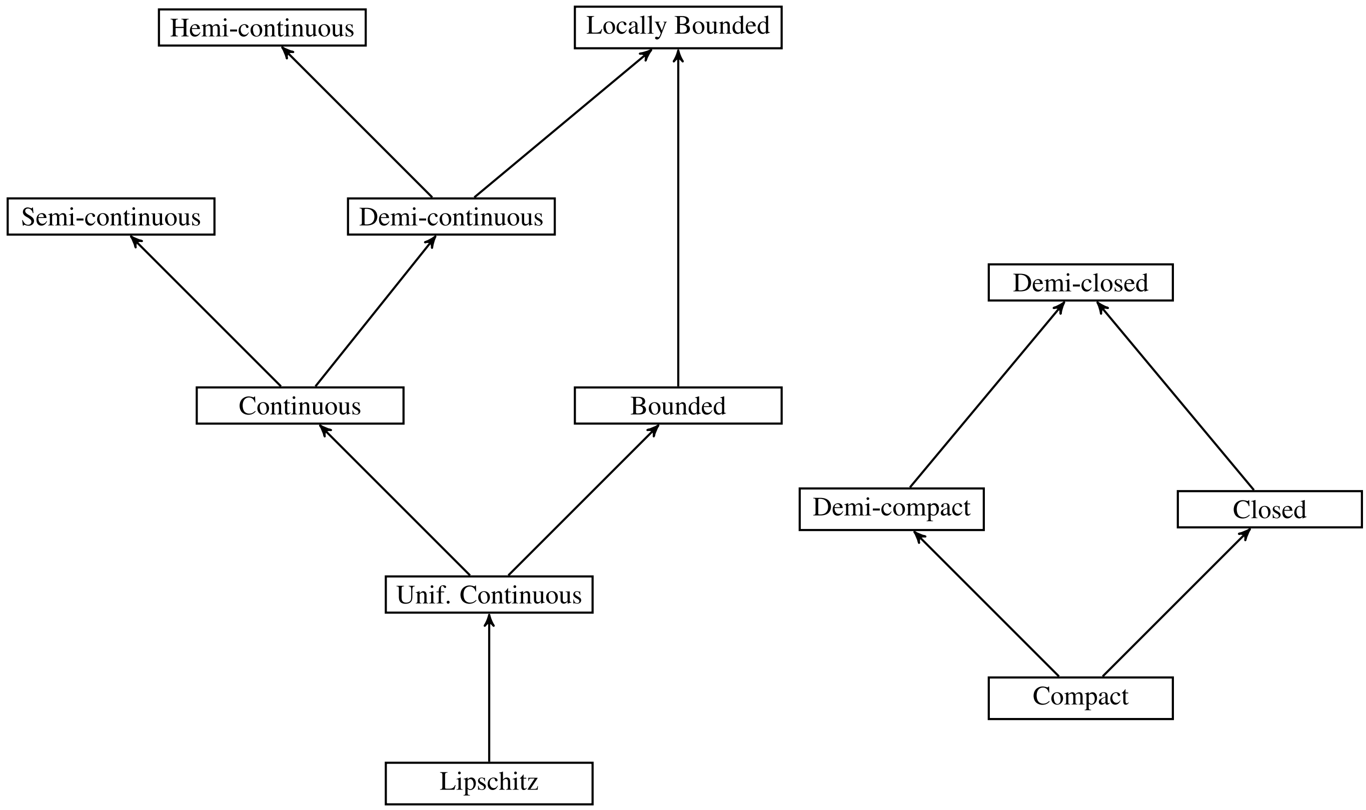
3.2 Nonlinear Operators. Maximal Monotone Mappings

In this section, we study certain classes of nonlinear operators that arise naturally in applications. A broader class producing a more general framework for the analysis of nonlinear problems, is provided by the so-called monotone operators from a Banach space X into its dual X^*. They are the natural generalization of increasing real functions on R and their definition does not require any order structure on the Banach space. They appear naturally in the calculus of variations and in the theory of nonlinear boundary value problems and provide an analytic framework broader than that of compact operators. Prominent among monotone operators are the so-called maximal monotone operators, which exhibit remarkable surjectivity properties.
We begin first, following the motivation in our background of study, by studying accretive operators that are the corresponding of monotone operators, when we deal with maps from X into itself (and not into X^*). The importance of *accretive* operators comes from the fact that they are the generators of linear and nonlinear *semigroups* which play a central role in the study of evolution equations.

We begin with some concepts and definitions of some topological properties of nonlinear operators.

3.2.1 Topological Properties of Nonlinear Operators

In what follows, we give some definitions of types of continuity and other topological properties of operators (see e.g., Cioranescu [48]).



(a) Continuity and Topological assumptions

(b) Topological Properties of Operators

Figure 3.1: Continuity and Topological Assumptions on Operators

Definition 3.2.1. *Suppose X be a normed linear space and $T : X \to X$ be a mapping. Then T is said* *to be Lipschitzian if there exists a constant $\beta \geq 0$ such that*

$$\|Tx - Ty\| \leq \beta\|x - y\| \textit{ for all} x,\ y \in X.$$

Here the smallest value of L for which the inequality holds is said to be Lipschitz constant for T and it is denoted by L.

For $L < 1$, T is contractive and for $L = 1$, T is non-expansive.

Definition 3.2.2 (Demi-continuity)**.** *Let $f : X \longrightarrow Y$ be a map. Then f is said to be* demi-continuous *if it is norm to weak-star continuous, i.e f is continuous from X (endowed with the strong topology) to Y (endowed with the weak-star topology)*

Definition 3.2.3 (Hemi-continuity)**.** *Let A be a single valued operator. Then A is said to be* hemi-continuous *if it is weakly continuous in every direction, i.e if for all $x_1, x_2, x \in X$, the function $\mathbb{R} \longrightarrow \mathbb{R}$ defined by $\lambda \longmapsto \langle x, A(x_1 + \lambda x_2)\rangle$ is continuous on $\mathbb{R}$.*

Definition 3.2.4 (Lower Semi-continuity)**.** *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Let $x_0 \in D(f)$, then f is* lower semicontinuous *at x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $f(x_0) - \varepsilon < f(x)$ for all $x \in B(x_0, \delta)$.*

Definition 3.2.5 (Demi-closed [89])**.** *Let $C \subseteq X$ be a nonempty unbounded closed convex subset of a Banach space, $h : C \to X$ is demi-closed if for any sequence $\{x_n\} \subseteq C$ weakly convergent to an element $x^* \in C$ with $\{h(x_n)\}$ norm-convergent to an element y^*, then $h(x^*) = y^*$*

Definition 3.2.6 (Demi-compactness)**.** *Let M be a subset of X, a normed linear space. Then a mapping $T : M \to X$ is said to be demi-compact if for any bounded sequence $\{x_n\}$ in M such that $(x_n - Tx_n) \to z \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ and a point $s \in M$ such that $x_{n_k} \to s$ as $k \to \infty$ and $(I - T)s = z$.*

Definition 3.2.7 (semi-compactness)**.** *Suppose X be a normed linear space and $T : X \to X$ be a mapping. Then T is said to be semi-compact if whenever there exists a sequence $\{x_n\}$ in X with $\lim \lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then the sequence $\{x_n\}$ has a convergent*

3.2.2 Accretive Operators and Pseudocontractive Mappings

In the study of nonlinear maps from a Banach space into itself, the corresponding notion of a monotone operator is that of an accretive operator. When X =H = a pivot Hilbert space, the two notions of monotone and accretive coincide.

Definition 3.2.8. *A map $A : E \to E$ is called accretive if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that*

$$\big\langle Ax - Ay, j(x - y)\big\rangle \geq 0. \tag{3.2.1}$$

A is called $generalized \Phi - strongly\ accretive$ if there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \Phi(\|x - y\|).$$

The map A is called ϕ-strongly accretive if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\big\langle Ax - Ay, j(x - y)\big\rangle \geq ||x - y||\phi(\|x - y\|)\ \ \forall\ x, y \in E. \tag{3.2.2}$$

Finally, A is called strongly accretive if there exists $k \in (0,1)$ such that

$$\big\langle Ax - Ay, j(x-y)\big\rangle \geq k\|x-y\|^2 \;\; \forall \;\, x, y \in E. \tag{3.2.3}$$

A is called maximal accretive if, in addition, the graph of A is not properly contained in the graph of any other accretive operator

It is easy to see that the class of strongly monotone mappings is a sub-class of the class of ϕ-strongly monotone maps (one takes $\phi(t) = kt$); and the class of ϕ-strongly monotone maps is a sub-class of that of generalized Φ-strongly monotone maps (one takes $\Phi(t) = t\phi(t)$). The inclusions are proper.
What follows is a characterization of accretive operators due to Kato.

Definition 3.2.9. *A mapping A with domain $D(A)$ and range $R(A)$ in E is called* accretive *if for all x , $y \in D(A)$, the following inequality is satisfied:*

$$\|x-y\| \leq \|x-y+s(Ax-Ay)\| \;\; \forall \, s > 0. \tag{3.2.4}$$

In His study of existence of solution of $Au = 0$, *where $A : E \to E$ is of accretive-type*, Browder defined an operator $T : E \to E$ by $T := I - A$, where I is the identity map on E. He called such an operator *pseudo-contractive*. Thus he establishes that existence of zeros of A corresponds to existence of fixed points of T. Furthermore, we have:

Definition 3.2.10. *A map $T : E \to E$ is called pseudocontractive if for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that*

$$\big\langle Tx - Ty, j(x-y)\big\rangle \leq \|x-y\|^2. \tag{3.2.5}$$

T is called $generalized \; \Phi - strongly \; pseudocontractive$ if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \Phi(\|x-y\|).$$

The map T is called ϕ-strongly pseudocontractive if there exists a strictly increasing function ϕ : $[0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\big\langle Tx - Ty, j(x-y)\big\rangle \geq ||x-y||\phi(\|x-y\|) \;\; \forall \;\, x, y \in E. \tag{3.2.6}$$

Finally, T is called strongly pseudocontractive if there exists $k \in (0,1)$ such that

$$\big\langle Tx - Ty, j(x-y)\big\rangle \leq k\|x-y\|^2 \;\; \forall \;\, x, y \in E. \tag{3.2.7}$$

3.2.3 Monotone and Maximal monotone Operators

Let X be a real normed space with dual X^*. A map $A : X \to 2^{X^*}$ is called *monotone* if for each $x, y \in E$, the following inequality holds:

$$\big\langle \eta - \nu, x - y\big\rangle \geq 0 \;\; \forall \quad \eta \in Ax, \quad \nu \in Ay. \tag{3.2.8}$$

The map A is called *generalized Φ-strongly monotone* if there exists a strictly increasing function Φ : $[0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\big\langle Ax - Ay, x - y\big\rangle \geq \Phi(\|x-y\|) \;\; \forall \;\, x, y \in E. \tag{3.2.9}$$

The map A is called *ϕ-strongly monotone* if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\big\langle Ax - Ay, x - y\big\rangle \geq ||x - y||\phi(\|x - y\|) \;\; \forall \;\; x, y \in E. \tag{3.2.10}$$

Finally, A is called *strongly monotone* if there exists $k \in (0, 1)$ such that

$$\big\langle Ax - Ay, x - y\big\rangle \geq k\|x - y\|^2 \;\; \forall \;\; x, y \in E. \tag{3.2.11}$$

It is easy to see that the class of strongly monotone mappings is a sub-class of the class of ϕ-strongly monotone maps (one takes $\phi(t) = kt$); and the class of ϕ-strongly monotone maps is a sub-class of that of generalized Φ-strongly monotone maps (one takes $\Phi(t) = t\phi(t)$). It is well known that the inclusions are proper.

EXAMPLES: The following are examples of monotone operators:

1. Every non decreasing function on $\mathbb{R}$ is monotone.

2. The duality map, J, defined earlier is monotone.

3. Let A be $n \times n$ matrix with real entries. Consider the function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $g(x) = Ax$. Then g is monotone if and only if A is positive semi definite.

4. Let $g : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *subdifferential* of g, $\partial g : H \to 2^H$, is defined for each $x \in H$ by

   $$\partial g(x) = \big\{x^* \in H : g(y) - g(x) \geq \big\langle y - x, x^*\big\rangle \;\; \forall\, y \in H\big\}.$$

   It is easy to check that the subdifferential is a monotone operator.

5. Let H be a real Hilbert space and $T : H \longrightarrow H$ be a non expansive map (i.e, $\|Tx - Ty\| \leq \|x - y\|$). Then the operator $I - T$ is monotone. In particular, it follows that, every orthogonal projection of a Hilbert space is monotone.

Definition 3.2.11. *A monotone operator $A : X \to X^*$ is called maximal monotone if its graph, $G(A) \subseteq X \times X^*$ is not properly contained in the graph of any other monotone operator.*

Maximal monotone operators have possess nice surjectivity attributes and are of particular interest because they are crucial in the solvability of evolution equations in Hilbert spaces as they generate semi-group of bounded linear operators.
A related notion is *m-monotone* operators defined in terms of the resolvent operator.

Definition 3.2.12. *A monotone operator $A : X \to X^*$ satisfying the condition that for all $\lambda > 0$, $R(J + \lambda A) = X^*$.*

When X^* is a strictly convex Banach space with a Fréchet differentiable norm, a maximal monotone operator from X into X^* is m-monotone.

3.2.4 Some Characterizations and Properties of Maximal Operators

We have the following results for characterizing maximal or m-monotone operators.

Theorem 3.2.13. *Assume X and X^* are reflexive and strictly convex. Let J denote the duality mapping on X and assume that $A \subset X \times X^*$ is monotone, then A is maximal monotone if and only if*

$$R(\lambda J + A) = X^*$$

for all $\lambda > 0$ (equivalently for some $\lambda > 0$).

Proof. Left as an exercise. □

Theorem 3.2.14. *Let X be a reflexive Banach space and $A : X \longrightarrow X^*$ be monotone and hemi-continuous, then A is maximal monotone.*

Proof. Suppose A is not maximal monotone, then there exists $x_0 \in X$ and $y_0 \in X^*$ such that $y_0 \neq x_0$ and

$$\langle x - x_0, Ax - y_0 \rangle \geq 0 \text{ for all } x \in X.$$

Set $x_t = tx_0 + (1-t)x$ for $t \in (0,1)$ and $x \in X$. Then $x_t - x_0 = (1-t)(x - x_0)$. Putting this in (3.2.4) we have

$$0 \leq (1-t)\langle x - x_0, Ax_t - y_0 \rangle \text{ for ll } t \in [0,1].$$

i.e

$$0 \leq \langle x - x_0, A(tx_0 + (1-t)x) - y_0 \rangle \text{ for all } t \in [0,1).$$

Hemi-continuous of A implies that

$$\langle x - x_0, Ax_0 - y_0 \rangle \geq 0 \text{ for all } x \in X.$$

Thus we have $y_0 = Ax_0$ which contradicts our assumption. Therefore A is maximal monotone. □

Theorem 3.2.15 (Rockafellar)**.** *Let X be a reflexive Banach space. Let $A \subset X \times X^*$ be monotone. Then A is maximal monotone if and only if*

$$grA + gr(-J_X) = X \times X^*.$$

Where J_X is the duality mapping on X.

Theorem 3.2.16. *Let X be a reflexive Banach space. Let $A \subset X \times X^*$ be monotone. If A is maximal monotone, then $A + J_X$ is onto. Conversely, if $A + J_X$ is onto and both J_X and J_X^{-1} are single valued, then A is maximal monotone.*

Proof. Assume A is maximal monotone. Let $y^* \in X^*$ then we see that $(0, y) \in X \times X^*$. According to Theorem 3.2.15,

$$(0, y^*) \in grA + gr(-J_X).$$

So there exists $(a, a^*) \in A$, $(u, u^*) \in -J_X$ such that

$$(a + u, a^* + u^*) = (0, y^*),$$

which implies that $a + u = 0$ and $a^* + u^* = y^*$. So we have $a = -u$ and $y^* \in A(a) + J_X(a)$ (here we have used the fact that $u^* \in -J_X(u)$). Thus we have $y^* \in (A + J)(a)$ which implies $X^* \subset A + J_X$. Hence

$$R(A + J_X) = X^*.$$

For the converse of the proof, one can follow the same process as in theorem (2.6). □

For any monotone mapping $T : X \to X^*$, we associate the Fitzpatrick functional defined as follows:

$$\mathcal{F}_T : X \times X^* \to \mathbb{R},$$

is defined by

$$\mathcal{F}_T(x, x^*) := \mathit{sup}\{\langle x^*, y\rangle + \langle y^*, x - y\rangle : (y, y^*) \in G(T)\}.$$

A monotone mapping T is called *almost maximal* (see e.g., Borwein [12]) if

$$\mathcal{F}_T(x, x^*) \geq \langle x, x^*\rangle,$$

for all $(x, x^*) \in X \times X^*$. We have the follwoing important results for maximal monotone mappings.

Lemma 3.2.17 (Borwein, [12])**.** *Suppose that S is monotone in Banach space and that either S is surjective or has full domain. Then S is almost maximal.*

Lemma 3.2.18 (Fitzpatrick, [57])**.** *If T is monotone mapping on a Banach space E and $(x, x^*) \in G(T)$, then $\mathcal{F}_T(x, x^*) = \langle x^*, x\rangle$ and $(x, x^*) \in \partial\mathcal{F}_T(x, x^*)$.*

Lemma 3.2.19 (Fitzpatrick, [57])**.** *If T is a monotone mapping on a Banach space E, then T is maximal monotone if and only if $\mathcal{F}_T(x, x^*) > \langle x^*, x\rangle$ whenever $x \in E$ and $x^* \in E^* \setminus T(x)$.*

A problem of great interest because of its application to the existence theory for perturbed partial differential equations is to know whether the ***sum of two maximal monotone*** operators is again maximal monotone. The following are prominent and crucial results in this regard.

Theorem 3.2.20 (Rockafellar, [103])**.** *Let X be reflexive, and let T_1 and T_2 be maximal monotone mappings from X to X^*. Suppose that either one of the following conditions is satisfied:*

(a.) $D(T_1) \cap \mathit{int}D(T_2) \neq \emptyset$ *, or*

(b.) there exists an $x \in \mathit{cl}D(T_1) \cap \mathit{cl}D(T_2)$ such that T_2 is locally bounded at x.

Then $T_1 + T_2$ is a maximal monotone mapping.

Theorem 3.2.21 (Browder, [16])**.** *Let X be a strictly convex reflexive Banach space with a strictly convex conjugate space X^*, T_1 a maximal monotone mapping from X to X^*, T_2 a hemicontinuous monotone mapping of all of X into X^* which carries bounded subsets of X into bounded subsets of X^*. Then, the mapping $T = T_1 + T_2$ is a maximal monotone map of X into X^*.*

3.2.5 Semigroup of Operators. Resolvents

Semigroup theory has been applied in the study of initial value and boundary valued problems of differential equations to determine *well-posedness* and to obtain solutions. In this subsection, we shall study the semigroup of nonlinear operators, in particular, we shall consider the class of *resolvents* of a nonlinear operator.

Definition 3.2.22. *Let X be a Banach space. A one parameter family of operators $T(t)$, $0 \leq t < \infty$, defined and acting on a closed subset $G \subseteq X$, with the following properties:*

(a) $T(t + s)x = T(t)(T(s)x)$ *for* $x \in G$, s, $t > 0$.

(b) $T(0)x = x$ *for any* $x \in G$.

(c) for any $x \in G$, the function $T(t) : X \to X$ is continuous with respect to t on $[0, \infty)$].

An operator $A : G \to X$ is called an *infinitesimal generator* or generating operator of the semigroup $T(t)$ if :

$$Ax = \lim_{h \to 0} \frac{T(h)x - x}{h}$$

exists for all $x \in G$. Accretive operators are infinitesimal generators of semigroups.

Two very important operators are given in:

Definition 3.2.23 (Resolvent)**.** *Let X be a Banach space and $A : X \to 2^X$ a nonlinear operator. For every $\lambda > 0$ we define $J_\lambda = (I + \lambda A)^{-1}$, the resolvent of A.*

The following are some properties of the resolvents of a maximal accretive operator $A : X \to 2^X$.

1. J_λ is nonexpansive for all $\lambda > 0$.
2. $\overline{D(A)}$ is convex and $\lim\limits_{\lambda \to 0} J_\lambda(x) = proj(x; \overline{D(A)})$ for all $x \in X$.

3.2.6 Approximation of the Nonlinear Equation $Au = 0$

In the existence of solution of $Au = 0$, it has been proved that the class of *generalised* Φ-strongly accretive(monotone) is the largest class for which we have uniqueness. Beyond this we lose uniqueness and say the problem is ***ill-posed***.

For ill-posed problems we need some form of a ***path*** to our solution. In the construction of a path we employ resolvents and a closely related operator called the *Yosida approximation of A.*

Definition 3.2.24. *Let X be a Banach space and $A : X \to 2^X$ a nonlinear operator. For every $\lambda > 0$ we define $A_\lambda = (1/\lambda)(I - J_\lambda)$, the Yosida approximation of A.*

Following are some properties of A_λ.

1. $A_\lambda(x) \in A(J_\lambda(x))$ for all $x \in X$ and all $\lambda > 0$.
2. A_λ is monotone and Lipschitz continuous with constant $1/\lambda$ for all $\lambda > 0$.
3. For all $x \in D(A)$, $\|A_\lambda(x)\| \leq \|A^0(x)\| = min\{\|x^*\| : x^* \in A(x)\}$ for all $\lambda > 0$.
4. $\lim\limits_{\lambda \to 0} A_\lambda(x) = A^0(x) := \{u \in A(x) : \|u\| = \|A(x)\|\}$ for all $x \in D(A)$.

What follows below are important theorems which give an application of the resolvent operator in the approximating the solution of the equation $Au = 0$.

The first theorem below gives a result on the generation of nonlinear semigroups. In chapter four, we shall see the celebrated path convergence result of Reich which applies the resolvent operator and the *proximal point algorithm* of Martinet and Rockafellar which employs the Yosida approximation of an accretive(monotone) operator.

Theorem 3.2.25. *If $A : D(A) \subseteq X \to 2^X$ is an m-accretive operator, then for every $x \in \overline{D(A)}$ $T(t)x = \lim\limits_{n \to \infty} (I + (t/n)A)^{-n}x$ exists for each $x \in \overline{D(A)}$ and uniformly in t in compact subsets of $\mathbb{R}^+$. The family of maps $T(t) : \overline{D(A)} \to \overline{D(A)}$, $t \geq 0$, is a semigroup of nonexpansive maps and for each $x \in D(A)$ and $t > 0$, we have $\|T(t)x - x\| \leq t|A(x)| := inf\{\|y\| : y \in A(x)\}$.*

Theorem 3.2.26 (see e.g., Reich and Shoikhet [98]). *Let f be the generator of a one parameter semigroup $S = \{F_t : t \in (0, \infty)\}$ of ρ-nonexpansive self-mappings of D and let f have no null point in $\mathbb{B}$. Then there is a unique boundary point $a \in \partial\mathbb{B}$ such that for each $x \in D$ the net of resolvents $\{J_r(x)(= (I + rf)^{-1}(x)) : r > 0\}$ strongly converges to a.*

3.3 Convex Analysis: Subdifferential and Optimization

In this section we present the basic properties of convex functions which are the fundamental objects of convex analysis. This is followed by the subdifferential and its connection to optimization problems.

3.3.1 Basic Definitions and Results in Convex Analysis

Definition 3.3.1. *Let C be a non empty subset of a real norm linear space X. The set C is said to be* convex *if for each $x, y \in C$ and for each $t \in (0, 1)$ we have $tx + (1-t)y \in C$.*

Definition 3.3.2. *Let C be a non empty convex subset of X. Then* the convex hull *of C denoted by coC is the intersection of all convex sets containing C. (Equivalently, convex hull of C is the set of all convex combinations of finite subsets of points of C).*

Definition 3.3.3. *Let C be a non empty convex subset of X. Let $f : C \longrightarrow \mathbb{R} \cup \{+\infty\}$. Then f is said to be* convex *if for each $t \in (0, 1)$ and for all $x, y \in C$ we have*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Moreover f is said to be proper if f is not identically $+\infty$ (i.e $\exists\, x_0 \in C$ such that $f(x_0) \in \mathbb{R}$)

Definition 3.3.4. *Let $f : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. The effective domain of f is defined by*

$$D(f) = \{x \in X : f(x) < +\infty\}.$$

The set

$$epi(f) = \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$$

is called the epigraph of f, while

$$S_\alpha = \{x \in X : f(x) \leq \alpha\}$$

is called the section of f.

Proposition 3.3.5. *A mapping $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is convex if and only if its epigraph is convex.*

Proposition 3.3.6 (Slope Inequality). *Let I be an interval of $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a convex function. Assume $r_1 < r_2 < r_3$ with $r_i \in I$ for $i = 1, 2, 3.$ and $f(r_1), f(r_2)$ are finite. Then*

$$\frac{f(r_2) - f(r_1)}{r_2 - r_1} \leq \frac{f(r_3) - f(r_1)}{r_3 - r_1} \leq \frac{f(r_3) - f(r_2)}{r_3 - r_2}.$$

Proposition 3.3.7. *Suppose $f : I \longrightarrow \mathbb{R}$ is convex and derivable on I. Then f' is increasing.*

Proof. Let $r < t$ we show that $f'(r) \leq f'(t)$. Now

$$f'(r) = \lim_{s \to r^+} \frac{f(s) - f(r)}{s - r} \leq \frac{f(t) - f(r)}{t - r} \leq \lim_{s \to r^-} \frac{f(s) - f(t)}{s - t} = f'(t)$$

$\square$

Definition 3.3.8. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Let $x_0 \in D(f)$, then f is* lower semicontinuous *at x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $f(x_0) - \varepsilon < f(x)$ for all $x \in B(x_0, \delta)$.*

Proposition 3.3.9. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Let $x_0 \in D(f)$, then f is lower semicontinuous at x_0 if and only if*

$$\liminf f(x_n) \geq f(x_0)$$

for all $\{x_n\} \subset X$ such that $x_n \to x_0$.

Proposition 3.3.10. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Then the following are equivalent.*

(a) f is lower semicontinuous,

(b) $epi(f)$ is closed,

(c) S_α is closed for each $\alpha \in \mathbb{R}$.

Definition 3.3.11. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Then f is said to be* coercive *if*

$$\lim_{\|x\| \to \infty} f(x) = +\infty.$$

Proposition 3.3.12. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Then f is convex and l.s.c if and only if f is convex and weakly l.s.c.*

Proof.

$$\begin{aligned} \text{f is convex and l.s.c} &\Leftrightarrow epi(f) \text{ is convex and closed} \\ &\Leftrightarrow epi(f) \text{ is convex and weakly closed} \\ &\Leftrightarrow f \text{ is convex and weakly l.s.c.} \end{aligned}$$

□

Theorem 3.3.13. *Assume X is reflexive. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be proper, convex, coercive and l.s.c function on X. Then f has a minimum on X. That is there exists $x_0 \in X$ such that*

$$f(x_0) = \inf_{x \in X} f(x).$$

Proof. Let $\eta = \inf_{x \in X} f(x)$. Since f is proper we see that $\eta < +\infty$. Let $\{x_n\} \subset X$ such that $f(x_n) \to \eta < +\infty$, then from the coercivity condition of f we see that $\{x_n\}$ is bounded. Since X is reflexive, then there exists $x_0 \in X$ and a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup x_0$. In view of proposition 3.3.12 f is weakly lower semi continuous. So we have

$$\eta \leq f(x_0) \leq \liminf f(x_{n_k}) = \lim_{k \to \infty} f(x_{n_k}) = \eta.$$

Therefore

$$f(x_0) = \eta = \inf_{x \in X} f(x).$$

□

Definition 3.3.14. *Let X be Banach space. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function. Let $x \in D(f)$ and $v \in X$, then we say that f has a* directional derivative *at x in the direction of $v \neq 0$ if the limit*

$$\lim_{t \to 0^+} \frac{f(x+tv) - f(x)}{t} \text{ exist.}$$

We denote by $f^{'}(x, v)$ the directional derivative of f at x in the direction of v, and we write

$$f^{'}(x, v) = \lim_{t \to 0^+} \frac{f(x+tv) - f(x)}{t}.$$

The function $f : X \longrightarrow \mathbb{R}$ is said to be Gâteaux differentiable at $x \in X$ if for all $v \in X$ $f^{'}(x, v)$ exists in $\mathbb{R}$ and the function $v \mapsto f^{'}(x, v)$ is linear and continuous. We denote by $D_G f(x)$ the Gâteaux differential of f at x and

$$\langle D_G f(x), v \rangle := f^{'}(x, v) \text{ for all } v \in X.$$

3.3.2 Subdifferential and Optimization

Definition 3.3.15. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper and convex function. Let $x \in D(f)$, then the* subdifferential *$\partial f(x)$ of f at x is the set*

$$\partial f(x) = \{x^* \in X^* : \langle y - x, x^* \rangle \leq f(y) - f(x) \; \forall y \in X\}.$$

We remarked that if x is not in $D(f)$ then $\partial f(x) = \emptyset$.

Proposition 3.3.16. *Let X be proper and convex function which is Gâteaux differentiable at $x \in D(f)$ then*

$$\partial f(x) = \{D_G f(x)\}.$$

Proof. To see this we pick $y \in X$. Convexity of f implies that

$$\frac{f(x + t(y-x)) - f(x)}{t} \leq f(y) - f(x), \quad 0 < t < 1.$$

Since f is Gâteaux differentiable at x we obtained that

$$\langle y - x, D_G f(x) \rangle = \lim_{t \to 0^+} \frac{f(x + t(y-x)) - f(x)}{t} \leq f(y) - f(x).$$

Thus $D_G f(x) \in \partial f(x)$.
Conversely, let $w^* \in \partial f(x)$, then for any $y \in X$ and $t > 0$

$$\frac{f(x+ty) - f(x)}{t} \geq \langle y, w^* \rangle.$$

Using the Gâteaux differentiability of f at x we obtained that

$$\langle y, D_G f(x) \rangle \geq \langle y, w^* \rangle \quad \forall y \in X$$

which implies $D_G f(x) = w^* \in \partial f(x)$. Therefore $\partial f(x) = \{D_G f(x)\}$. $\square$

Proposition 3.3.17. *Define a function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ by*

$$f(x) = \frac{1}{2}\|x\|^2 \quad \forall x \in X.$$

Then f is proper, convex and continuous. Moreover $\partial f(x) = J(x)$ for each $x \in X$ where J is the duality map on X.

Proof. Indeed choose first $x^* \in Jx$. Then for any $y \in x$ we have

$$\begin{aligned}\langle y-x, x^*\rangle = \langle y, x\rangle - \|x\|^2 &\leq \|y\|\|x\| - \|x\|^2 \\ &\leq \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \\ &= f(y) - f(x).\end{aligned}$$

Thus we have $x^* \in \partial f(x)$. Conversely, for $x^* \in \partial f(x)$ we have

$$\langle y-x, x^*\rangle \leq f(y) - f(x)\ \forall y \in X.$$

So considering $x+ty$, $t \in (0,1)$ we get

$$\langle x^*, y\rangle \leq \frac{1}{2t}(\|x+ty\|^2 - \|x\|^2) \leq \|x\|\|y\| + \frac{t}{2}\|y\|.$$

As $t \to 0^+$ we have $\langle x^*, y\rangle \leq \|x\|\|y\|$, which implies $\|x^*\| \leq \|x\|$. Also using the fact that $x^* \in \partial f(x)$ and considering $x + tx \in X$ we have

$$2t\langle -x, x^*\rangle \leq \|x - tx\|^2 - \|x\|^2 = (t^2 - 2t)\|x\|^2, \quad t > 0.$$

So we have $(2-t)\|x\|^2 \leq 2\langle x, x^*\rangle$. Now as $t \to 0^+$ we obtained

$$\|x\|^2 \leq \langle x, x^*\rangle \leq \|x\|\|x^*\| \text{ which implies } \|x\| \leq \|x^*\|.$$

Therefore we have $\|x\| = \|x^*\|$ and $\langle x, x^*\rangle = \|x\|^2$. Thus $x^* \in J(x)$. □

Example 1. *Let K be a closed, convex subset of X. Define a map I_k on X by*

$$I_k(x) = \begin{cases} 0 & \text{for } x \in K, \\ +\infty & \text{for } x \notin K. \end{cases}$$

It is easy to see that I_k is convex and lower semi-continuous (since K is convex and closed). Furthermore for any $x \in K$ we get

$$\partial I_k(x) = \{x^* \in X^* : \langle y-x, x^*\rangle \leq 0,\ \forall y \in X\}.$$

As we introduce the application of the subdifferential to **optimisation**, we consider the following optimisation problem:

$$\text{Find } x^* \in X \text{ such that } f(x^*) \leq f(x)\ \ \forall\, x \in X \tag{3.3.1}$$

where $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ convex and proper (i.e., f is not identically ∞).

It is well known if the function f is differentiable and x^* exists, then $f'(x^*) = 0$. The absolute value function $x \mapsto |x|$, however, has a minimizer, which, in fact, is 0. But, it is not differentiable at 0. Thus, the above fact is not applicable. However, with the more general notion of differentiability, using the *subdifferential*, we obtain analogous result for optimisation analysis.

Proposition 3.3.18. *Let $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ be a convex and proper function. Then, the* **subdifferential** *of f at $x \in X$ is the map $\partial f : X \to 2^{X^*}$ defined by*

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y-x\rangle \leq f(y) - f(x)\ \ \forall\, y \in X\}. \tag{3.3.2}$$

It is easy to see that $0 \in \partial f(a)$ if and only if a is a minimizer of f. Setting $\partial f \equiv A$, it follows that in this case, solutions of the inclusion $0 \in Au$ correspond to minimizers of f.

In particular, in the above example, where $f(x) = |x|$, the subdifferential of f at zero, $\partial f(0) = [-1, 1]$, which trivially contains zero. Hence, zero is the minimizer of f.

3.4 Fixed Point Theory: Approximate Fixed Points

The theory of fixed point is one of the most useful tools of modern mathematics. This comes from the fact that most important nonlinear problems in applications (such as existence and stability of periodic and quasiperiodic orbits in celestial mechanics; determining material constants or material laws (e.g., coefficients of partial differential equations) from experimental data (inverse problems); deformation of rods, plates, and shells; nonlinear oscillation in physics, chemistry, and biology; spectra of molecules, considering quantum mechanical electron interaction; game-theoretic models in economics; e.t.c) reduce to solving a given equation which in turn may be reduced to finding the fixed points of some operator (see e.g., Zeidler [123]).

Definition 3.4.1 (Fixed point)**.** *Let X be a non-empty set and f be a function which maps X to X. A fixed point of f is a point $x \in X$ such that $f(x) = x$. If f is a multivalued map, i.e., from X to the collection of nonempty subsets of, X then a point p in X is called a fixed point of f if $p \in fp$.*

Definition 3.4.2 (Fixed point problem)**.** *Let X be a set, A and B two nonempty subsets of X such that $A \cap B \neq \emptyset$, and $f : A \to B$ be a map. When does a point $x \in A$ such that $f(x) = x$.*

Theorems concerning the existence and properties of fixed points are known as fixed point theorems. Several fixed point theorems include the Banach contraction mapping principle, Brouwer fixed point theorem, Schauder fixed point theorem and a host of others.

Let E be a real normed space and $A : E \to E$ be an accretive operator. Assume that $Au = 0$ has a solution. Browder [14] introduced an operator $T : E \to E$ by $T = I - A$ and called the map T, **pseudo-contractive**. It is clear that zeros of A correspond to fixed points of T (i.e., $Au = 0$ if and only if $Tu = u$). The class of pseudocontractive maps properly contains the class of *nonexpansive* maps which are a generalisation of contraction maps. A map $T : E \to E$ is called nonexpansive if for each $x, y \in E$, the inequality $\|Tx - Ty\| \leq \|x - y\|$ is true.

3.4.1 Approximation and Iterative Algorithm

Methods of *approximation* (including classical approximation, abstract approximation, constructive approximation, multivariate approximation e.t.c) entail representing non-arithmetic quantities by arithmetic quantities so that the accuracy can be ascertained to a desired degree. Nonlinear equations of nonlinear from physical sciences and other areas of applied mathematics are always of an approximate character, since it is not possible to take full account of all influences (see e.g., Zeidler [123]).

The beginning of fixed point theory is from the application of the method of successive approximations in proving existence of differential equations. In many cases of application, the mapping under consideration may not have an exact fixed point due to some strong restriction on the space or the map; or that an approximate fixed point is more than enough: an approximate solution plays an important role in such situations (see e.g., Khamsi and Kozlowski [68]).

Definition 3.4.3 (Approximate fixed point)**.** *Let (X, d) be a metric space and $T : X \to X$ any mapping. If for all $\epsilon > 0$ there exists a $z \in X$ such that $d(z, Tz) \leq \epsilon$, then z is called an approximate fixed point of T.*
A sequence $\{x_n\}$ is called an approximated fixed point sequence for T if $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

An *iterative algorithm* or method is a mathematical procedure that generates a sequence of improving approximate solutions for a class of problems. In the classical problems of finding the root of an equation (or solution of a system of equations), an iterative method uses an initial guess to generate successive approximations to a solution. Iterative methods are often the only choice for nonlinear equations. Techniques of functional analysis are used to derive analytical relationships between approximation methods and convergence properties for general classes of algorithms.

Iterative Algorithms of Fixed Points and Zeros of Operators

In this section, we refer to the paper of Chidume [47]. We have the well known and celebrated Contraction Mapping Principle.

Theorem 3.4.4. *(Contraction Mapping Principle) Let (X, ρ) be a complete metric space and $T : X \to X$ be a contraction map of X into itself. Then,*

(a) *T has a unique fixed point, say x^* in X;*

(b) *the sequence $\{x_n\}_{n=0}^{\infty}$ in X defined by $x_0 \in X$,*

$$x_{n+1} = Tx_n, \ n = 0, \ 1, \ 2, \ 3, \cdots \tag{3.4.1}$$

converges to x^*.

The sequence of the recursion formula $(3.4.1)$ is called the *Picard sequence*. One important class of nonlinear mappings generalizing the class of contraction mappings is the class of nonexpansive mappings (see e.g., [22], p.57).

If K is a nonempty *compact convex* subset of $\mathbb{R}^2$ and $T : K \longrightarrow K$ is *a nonexpansive map*, even with a unique fixed point, the Picard sequence defined by $(3.4.1)$ may fail to converge to the fixed point. It suffices to take $K = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$ and let $T : K \longrightarrow K$ be a rotation of K about the origin of coordinates through a fixed angle θ, $0 < \theta < \frac{\pi}{2}$ (say). It is easy to check that T is nonexpansive, zero is the unique fixed point of T and that the Picard sequence $(3.4.1)$ with $x_0 = (1, 0)$ fails to converge to zero.

Following research efforts by Mann [81], Krasnoselkii [73], Schaefer [104], Ishikawa [61], Edelstein $\big($[54], [55], [56]$\big)$, Reinermann [99], Edelstein and O'Brian [56], Chidume [21], and a host of other authors, the following recursion formula was developed and found to be effective for approximating fixed points of *nonexpansive mappings:*

Let K be a nonempty convex subset of a normed space E and $T : K \longrightarrow K$ be a nonexpansive map. Let the sequence $\{x_n\}_{n=0}^{\infty}$ in K be defined by

$$x_{n+1} = (1 - c_n)x_n + c_nTx_n, \ x_0 \in K, \ n \in \mathbb{N}, \tag{3.4.2}$$

where $\{c_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions: $(i) \displaystyle\sum_{n=0}^{\infty} c_n = \infty, \ \ (ii) \lim_{n \longrightarrow \infty} c_n = 0$. *If the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded*, Ishikawa [61] proved that the sequence is an *approximate fixed point sequence* in the sense that

$$\lim_{n \longrightarrow \infty} ||x_n - Tx_n|| = 0. \tag{3.4.3}$$

Edelstein and O'Brian [56] considered the recursion formula

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \ x_0 \in K, \ n \in \mathbb{N}, \lambda \in (0, 1), \tag{3.4.4}$$

where T maps K into K and proved that *if K is bounded*, then the convergence in (3.4.3) is uniform.

Chidume [21] considered the recursion formular (3.4.2), introduced the concept of admissible sequence and proved that *if K is bounded*, then the convergence in (3.4.3) is uniform *for the sequence defined by* (10).

Remark 1 We note here that the recursion formula (3.4.2) which is certainly cumbersome when compared with Picard iteration was developed for the class of nonexpansive maps because the simpler Picard sequence will not always converge for nonexpansive maps. Furthermore, the recursion formula (3.4.2) can only yield that the sequence defined by (3.4.2) satisfies (3.4.3). In general, it does not yield convergence of the sequence to a fixed point of T. To obtain convergence to a fixed point of T, *some type of compactness condition* must be imposed either on K or on the map T (e.g, T may be required to be *demicompact at zero*, or $(I-T)$ may be required to *map closed bounded subsets of E into closed subsets of E*, etc, see e.g, Chidume [22]). The recursion formula (3.4.2) is now generally referred to as *Mann formula* in the light of Mann [81].

An important class of mappings generalizing the class of nonexpansive mappings is the class of *Lipschitz pseudo-contractive maps*. It is not difficult to check that every nonexpansive map is a Lipschitz pseudo-contraction. We have already given an example of a pseudo-contractive map which is not even continuous. All attempts to use the Mann formula, which has been successfully employed for nonexpansive mappings, to approximate a fixed point of a Lipschitz pseudo-contractive map even on a compact convex domain in a real Hilbert space proved abortive. In 1974, Ishikawa [62] proved the following theorem.

Theorem IS. *Let K be a nonempty compact convex subset of a real Hilbert space H and $T : K \longrightarrow K$ be a Lipschitz pseudo-contractive map. Let the sequence $\{x_n\}_{n=0}^{\infty}$ be defined by $x_0 \in K$,*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \tag{3.4.5}$$

$$y_n = (1-\beta_n)x_n + \beta_n T x_n, n \geq 1, \tag{3.4.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the following conditions: (i) $0 \leqslant \alpha_n \leqslant \beta_n < 1 \, \forall n \geqslant 1$; (ii) $\sum \alpha_n \beta_n = \infty$; (iii) $\lim\limits_{n \longrightarrow \infty} \beta_n = 0$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.

Remark 2. It is clear that the recursion formulas (3.4.5) and (3.4.6) of the Ishikawa scheme are more cumbersome than the Mann formula (3.4.2). However, since it was not known whether or not the simpler Mann sequence would always converge to fixed points of Lipschitz pseudo-contractive maps, the cumbersome Ishikawa scheme was applied for this class of maps. The question of whether or not the simpler Mann sequence had actually failed for this class of maps remained open for many years. This was resolved in 2001 by Chidume and Mutangadura [27] who produced an example of a Lipschitz pseudo-contractive map defined on a compact convex subset of $\mathbb{R}^2$ with a unique fixed point for which no Mann sequence converges.

Other iteration methods have been introduced and have successfully been employed to approximate fixed points of Lipschitz pseudo-ontractive mappings even in *Banach spaces more general than Hilbert spaces* (see e.g., Schu [105]; Chidume [26]; Bruck [17]; Reich [93])

Motivated by the papers of Reich $\big($[93], [95], [94]$\big)$, Chidume and Zegeye [28] studied the following perturbation of the Mann recurrence relation to approximate fixed points of Lipschitz pseudo-contractive mappings in real Banach spaces much more general than Hilbert spaces. They proved the following theorem.

Theorem CZ ([28] *Let K be a nonempty closed convex subset of a real Banach space E. Let $T: K \to K$ be a Lipschitz pseudo-contractive map with constant $L > 0$ and $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \tag{3.4.7}$$

for all positive integers n, where λ_n and θ_n are real sequences in $(0,1)$ satisfying appropriate conditions. Then, $\|x_n - Tx_n\| \to 0 \quad \text{as n} \to \infty$.

3.4.2 Important Recurrent Inequalities

We present in this section some standard recurrent inequalities often employed in proving several convergence theorems (see e.g., Berinde [9]).

Lemma 3.4.5. *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers and $0 \leq q < 1$, so that*

$$a_{n+1} \leq q a_n + b_n, \text{ for all } n \geq 0.$$

(i) If $\lim\limits_{n\to\infty} b_n = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

(ii) If $\sum\limits_{n=0}^{\infty} b_n < \infty$, then $\sum\limits_{n=0}^{\infty} a_n < \infty$.

Remark. We obtain the weaker form of the lemma above when $q = 1$. This is given in the next lemma.

Lemma 3.4.6. *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying*

$$a_{n+1} \leq a_n + b_n, \text{ for all } n \geq 0.$$

(ii) If $\sum\limits_{n=0}^{\infty} b_n < \infty$, then $\lim\limits_{n\to\infty} a_n$ exists.

(ii) If $\sum\limits_{n=0}^{\infty} b_n < \infty$ and $\{a_n\}_{n=0}^{\infty}$ has a subsequence converging to zero, then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 3.4.7. *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers and let $\{a_n\}_{n=0}^{\infty}$ be a real sequence in $[0,1]$ such that $\sum\limits_{n=0}^{\infty} a_n = \infty$.*

(i) If for a given $\epsilon > 0$ there exists a positive integer n_0 such that

$$x_{n+1} \leq (1-a_n)x_n + \epsilon \cdot a_n, \text{ for all } n \geq n_0,$$

then we have $0 \leq \limsup\limits_{n\to\infty} x_n \leq \epsilon$.

(ii) If there exists a positive integer n_1 such that

$$x_{n+1} \leq (1-a_n)x_n + a_n b_n, \text{ for all } n \geq n_1,$$

where $b_n \geq 0$ for all $n \geq 0$ and $\lim\limits_{n\to\infty} b_n = 0$, then we have $\lim\limits_{n\to\infty} x_n = 0$.

Lemma 3.4.8. *Let* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers satisfying*

$$a_{n+1} \leq (1-\omega_n)a_n + b_n + c_n, \textit{ for all } n \geq 0,$$

where $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$. *If (i)* $\sum\limits_{n=0}^{\infty} \omega_n = \infty$, *(ii)* $b_n = o(\omega_n)$ *and (iii)* $\sum\limits_{n=0}^{\infty} < \infty$, *then*

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 3.4.9. *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of nonnegative numbers satisfying*

$$a_{n+1} \leq (1+\delta_n)a_n - \lambda_n \frac{\Phi(a_{n+1})}{1+\Phi(a_{n+1})+a_{n+1}} \cdot a_n, \textit{ for all } n \geq 0,$$

where $\Phi : [0,\infty) \to [0,\infty)$ *is a strictly increasing function with* $\Phi(0) = 0$, *and* $\{\lambda_n\}_{n=0}^{\infty}$, $\{\delta_n\}_{n=0}^{\infty}$ *are sequences of nonnegative numbers satisfying*
(i) $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$; *(ii)* $\sum\limits_{n=0}^{\infty} \delta_n < \infty$.
Then

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 3.4.10. *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of nonnegative numbers satisfying*

$$a_{n+1} \leq (1+\delta_n)a_n - \lambda_n \frac{\Phi(a_{n+1})}{1+\Phi(a_{n+1})+a_{n+1}} \cdot a_n + \theta_n, \textit{ for all } n \geq 0,$$

where $\Phi : [0,\infty) \to [0,\infty)$ *is a strictly increasing function with* $\Phi(0) = 0$, *and* $\{\lambda_n\}_{n=0}^{\infty}$, $\{\delta_n\}_{n=0}^{\infty}$ *are sequences of nonnegative numbers satisfying*
(i) $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$; *(ii)* $\sum\limits_{n=0}^{\infty} \delta_n < \infty$. *(iii)* $\sum\limits_{n=0}^{\infty} \theta_n < \infty$ *Then*

$$\lim_{n\to\infty} a_n = 0.$$

Remarks.

1. It is easy to see that Lemma 3.4.10 follows by Lemma 3.4.8 for

$$\omega_n = -\delta_n + \lambda_n \frac{\Phi(a_{n+1})}{1+\Phi(a_{n+11})+a_{n+1}}, \; n \geq 0,$$

   while Lemma 3.4.9 is obtained from Lemma 3.4.8 for

$$\omega_n = -\delta_n + \lambda_n \frac{\Phi(a_{n+1})}{1+\Phi(a_{n+11})+a_{n+1}} \text{ and } c_n = 0, \; n \geq 0.$$

2. in the case $\omega_n = 1-q$, for all $n \geq 0$, with $0 \leq q < 1$ and $c_n = 0$, $n \geq 0$, we can obtain from Lemma 3.4.8 a stronger result given by Lemma 3.4.5

Lemma 3.4.11. *Let* $\{\lambda_n\}_{n=0}^{\infty}$ *and* $\{\gamma_n\}_{n=0}^{\infty}$ *be sequences of non-negative real numbers and* $\{\alpha_n\}$ *be a sequence of positive numbers satisfying the conditions* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\frac{\gamma_n}{\alpha_n} \to 0$ *as* $n \to 0$. *Let the recursive inequality*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \psi(\lambda_{n+1}) + \gamma_n, \quad n \geq 0, \tag{3.4.8}$$

be given, where $\psi : [0,\infty) \to [0,\infty)$ *is a strictly increasing continuous function and* $\psi(0) = 0$. *Then,* $\lim_{n\to\infty} \lambda_n = 0$.

Chapter 4

MAIN RESULTS AND APPLICATIONS

In the sequel, we shall need the following definitions and results.

Let E be a smooth real Banach space with dual E^*. The Lyapounov functional $\phi : E \times E \to \mathbb{R}$, defined by:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \text{ for } x, y \in E, \tag{4.0.1}$$

where J is the normalized duality mapping from E into E^* will play a central role in the sequel. It was introduced by Alber and has been studied by Alber [2], Alber and Guerre-Delabriere [3], Kamimura and Takahashi [65], Reich [92] and a host of other authors. If $E = H$, a real Hilbert space, then equation (4.0.1) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \text{ for } x, y \in E. \tag{4.0.2}$$

Define a map $V : X \times X^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \quad \text{for } x \in X,\ x^* \in X^*. \tag{4.0.3}$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}(x^*))\ \forall x \in X,\ x^* \in X^*. \tag{4.0.4}$$

Lemma 4.0.12. *([Alber, [4]]) Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*) \tag{4.0.5}$$

for all $x \in X$ and $x^*, y^* \in X^*$.

Lemma 4.0.13. *([Alber, [4], p.50]) Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Let $W : X \times X \to \mathbb{R}^{\not{k}}$ be defined by $W(x, y) = \frac{1}{2}\phi(y, x)$. Then,*

$$W(x, y) - W(z, y) \geq \langle Jx - Jz, z - y\rangle,$$

i.e.,

$$\phi(y, x) - \phi(y, z) \geq 2\langle Jx - Jz, z - y\rangle,$$

and also

$$W(x, y) \leq \langle Jx - Jy, x - y\rangle,$$

for all $x,\ y,\ z \in X$.

Lemma 4.0.14 (Alber, [4], p.45). *Let X be a uniformly convex Banach space. Then, for any $R > 0$ and any $x,\, y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$, the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_X(c_2^{-1}\|x - y\|),$$

where $c_2 = 2max\{1, R\}$, $1 < L < 1.7$.
Define

$$K := 4RLsup\{\|Jx - Jy\| : \|x\| \leq R,\, \|y\| \leq R\} + 1. \tag{4.0.6}$$

Lemma 4.0.15 (Alber, [4], p.46). *Let X be a uniformly smooth and strictly convex Banach space. Then for any $R > 0$ and any $x,\, y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$ the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_{X^*}(c_2^{-1}\|Jx - Jy\|),$$

where $c_2 = 2max\{1, R\}$, $1 < L < 1.7$.

Let E^* be a real strictly convex dual Banach space with a Fréchet differentiable norm. Let $A : E \to 2^{E^*}$ be a maximal monotone operator with no monotone extension. Let $z \in E^*$ be fixed. Then for every $\lambda > 0$, there exists a unique $x_\lambda \in E$ such that $Jx_\lambda + \lambda Ax_\lambda \ni z$ (see Reich [90], p. 342). Setting $J_\lambda z = x_\lambda$, we have the *resolvent* $J_\lambda := (J + \lambda A)^{-1} : E^* \to E$ of A for every $\lambda > 0$. The following is a celebrated result of Reich.

Lemma 4.0.16 (Reich, [90]; see also, Kido, [71]). *Let E^* be a strictly convex dual Banach space with a Fréchet differentiable norm, and let A be a maximal monotone operator from E to E^* such that $A^{-1}0 \neq \emptyset$. Let $z \in E^*$ be arbitrary but fixed. For each $\lambda > 0$ there exists a unique $x_\lambda \in E$ such that $Jx_\lambda + \lambda Ax_\lambda \ni z$. Furthermore, x_λ converges strongly to a unique $p \in A^{-1}0$.*

Lemma 4.0.17. *From Lemma 4.0.16, setting $\lambda_n := \frac{1}{\theta_n}$ where $\theta_n \to 0$ as $n \to \infty$ and $\frac{1}{2}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big) \in (0,1)$ for all $n \geq 1$ and some K, $z = Jv$ for some $v \in E$, and $y_n := \Big(J + \frac{1}{\theta_n}A\Big)^{-1} z$, we obtain that:*

$$Ay_n = \theta_n(Jv - Jy_n), \tag{4.0.7}$$

$$y_n \to y^* \in A^{-1}0,$$

where $A : E \to E^*$ is maximal monotone.

Remark 1. *Let $R > 0$ such that $\|v\| \leq R$, $\|y_n\| \leq R$ for all $n \geq 1$. We observe that equation (4.0.7) yields*

$$Jy_{n-1} - Jy_n + \frac{1}{\theta_n}\Big(Ay_{n-1} - Ay_n\Big) = \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big(Jv - Jy_{n-1}\Big). \tag{4.0.8}$$

Taking the duality pairing of the LHS of this equation with $y_{n-1} - y_n$, applying Cauchy-Schwarz and using (4.0.8)*, we obtain that,*

$$\langle Jy_{n-1} - Jy_n, y_{n-1} - y_n\rangle \leq \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big\|Jv - Jy_{n-1}\Big\|\|y_{n-1} - y_n\|.$$

It follows that if E is uniformly convex and uniformly smooth, using lemma 4.0.14 we obtain that,

$$\begin{aligned}(2L)^{-1}\delta_E(c_2^{-1}\|y_{n-1} - y_n\|) &\leq \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big\|Jv - Jy_{n-1}\Big\|\|y_{n-1} - y_n\|, \\ &\leq 2Rsup\Big\{\Big\|Jv - Jy_{n-1}\Big\|\Big\}\frac{\theta_{n-1} - \theta_n}{\theta_n}\end{aligned} \tag{4.0.9}$$

which gives, using equation (4.0.6)

$$\|y_{n-1} - y_n\| \le c_2 \delta_E^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n} K\right). \tag{4.0.10}$$

Similarly, using lemma 4.0.15, we obtain that,

$$\|Jy_{n-1} - Jy_n\| \le c_2 \delta_{E^*}^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n} K\right). \tag{4.0.11}$$

Remark 2. *In p-uniformly convex spaces, we have (see e.g., Chidume [22], p.34,) that, for some constant $c > 0$,*

$$\delta_E(\epsilon) \ge c\epsilon^p, \qquad \textit{for } 0 < \epsilon \le 2. \tag{4.0.12}$$

From inequality (4.0.9), using inequality (4.0.12), we obtain that,

$$\frac{c}{2Lc_2^p}\|y_{n-1} - y_n\|^p \le \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)\Big\|Jv - Jy_{n-1}\Big\|\|y_{n-1} - y_n\|,$$

which gives that:

$$\|y_{n-1} - y_n\| \le \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)^{1/p} K_1, \quad \textit{for some } K_1 > 0. \tag{4.0.13}$$

Also, we have from lemma 4.0.15 that

$$(2L)^{-1}\delta_{X^*}(c_2^{-1}\|Jx - Jy\|) \le \langle Jx - Jy, x - y\rangle.$$

Again, using inequality (4.0.12), we obtain that

$$\frac{c}{2Lc_2^p}\|Jy_{n-1} - Jy_n\|^p \le \langle Jy_{n-1} - Jy_n, y_{n-1} - y_n\rangle \le \|Jy_{n-1} - Jy_n\|\|y_{n-1} - y_n\|,$$

which gives that:

$$\|Jy_{n-1} - Jy_n\| \le \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)^{1/p} K_2, \quad \textit{for some } K_2 > 0. \tag{4.0.14}$$

Lemma 4.0.18 (Kamimura and Takahashi, [65])**.** *Let X be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 4.0.19 (Xu, [117])**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the following relation*

$$a_{n+1} \le (1 - \sigma_n)\, a_n + \sigma_n b_n + c_n, \quad n \ge 0, \tag{4.0.15}$$

where $\{\sigma_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ satisfy the conditions:

(i) $\{\sigma_n\}_{n=1}^{\infty} \subset [0, 1]$, $\sum_{n=1}^{\infty} \sigma_n = \infty$, *or equivalently,* $\prod_{n=1}^{\infty}(1 - \sigma_n) = 0$;

(ii) $\limsup_{n\to\infty} b_n \le 0$;

(iii) $c_n \ge 0\, (n \ge 0)$, $\sum_{n=1}^{\infty} c_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

In the definition below, we consider the fixed point notion which has also been defined as *semi*-fixed point(see e.g., H. Zegeye[121]), *duality* fixed point(see e.g., B. Liu[78]). However, for the purpose of this thesis, we choose to uphold the name *J-fixed points*.

Definition 4.0.20 (*J-fixed point*)**.** *Let E be an arbitrary normed space and E^* be its dual. Let $T : E \to 2^{E^*}$ be any mapping. A point $x \in E$ will be called a J-fixed point of T if and only if there exists $\eta \in Tx$ such that $\eta \in Jx$*

We introduce the following definition:

Definition 4.0.21 (*J-pseudocontractive mappings*)**.** *Let E be a normed space. A mapping $T : E \to 2^{E^*}$ is called J-pseudocontractive if for every $x,\, y \in E$,*

$$\langle \tau - \zeta, x - y\rangle \leq \langle \eta - \nu, x - y\rangle \quad \textit{for all } \tau \in Tx,\ \zeta \in Ty,\ \eta \in Jx,\ \nu \in Jy.$$

Example 4. *If $E = H$, a real Hilbert space, then J is the identity map on H. Consequently, every pseudocontractive map on H is J-pseudocontractive.*

For our next example, we need the following characterization of the normalized duality map on l_p, $1 < p < \infty$.

In l_p spaces, $1 < p < \infty$, for arbitrary $x \in l_p$, $x = (x_1, x_2, x_3, \cdots)$,

$$Jx = \|x\|^{2-p}(|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, |x_3|^{p-2}x_3, \cdots),$$

(see e.g., Alber [4], p.36).

Example 5. *Let $1 < q < p < \infty$ and let $\lambda \in \mathbb{R}$ be arbitrary. Define $T : l_p \to l_q$ by*

$$Tx = (\lambda, x_2, x_3, \cdots).$$

Then, $x_\lambda := (\lambda, 0, 0, \cdots)$ is a J-fixed point of T.

Remark 3. *We observe that, assuming existence, zeros of a monotone mapping $A : E \to 2^{E^*}$ correspond to J-fixed points of a J-pseudocontractive mapping T.*

The following lemma asserts that $A : E \to 2^{E^*}$ is monotone if and only if $T := (J - A) : E \to 2^{E^*}$ is J-pseudocontractive.

Lemma 4.0.22. *Let E be an arbitrary real normed space and E^* be its dual space. Let $A : E \to 2^{E^*}$ be any mapping. Then A is monotone if and only if $T := (J - A) : E \to 2^{E^*}$ is J-pseudocontractive.*

Proof. Let $x,\, y \in E$ be arbitrary. Suppose A is monotone. Then, for every $\mu_x \in Ax,\quad \mu_y \in Ay$, $jx \in Jx,\ jy \in Jy$, $\tau_x \in Tx$, $\tau_y \in Ty$, such that $\tau_x = jx - \mu_x$, $\tau_y = jy - \mu_y$, we have that:

$$\begin{aligned}\langle \tau_x - \tau_y, x - y\rangle &= \langle jx - jy, x - y\rangle - \langle \mu_x - \mu_y, x - y\rangle \\ &\leq \langle jx - jy, x - y\rangle.\end{aligned}$$

Hence, T is J-pseudocontractive.

Conversely, suppose $T := (J - A)$ is J-pseudocontractive, we prove $A := J - T$ is monotone. For all $x,\, y \in E$, let $\mu_x \in Ax$, $\mu_y \in Ay$. Then, $\mu_x = jx - \zeta_x$ and $\mu_y = jy - \zeta_y$ for some $\zeta_x \in Tx$, $\zeta_y \in Ty$, $jx \in Jx$ and $jy \in Jy$. We have that:

$$\begin{aligned} \langle \mu_x - \mu_y, x - y \rangle &= \langle jx - \zeta_x - jy + \zeta_y, x - y \rangle \\ &= \langle jx - jy, x - y \rangle - \langle \zeta_x - \zeta_y, x - y \rangle \\ &\geq 0. \end{aligned}$$

Hence, A is monotone. □

Influenced by the work of Kato [63], we give the proposition below which can easily be verified:

Proposition 4.0.23. *Let X be a Banach space and X^* be its dual. Let $A : D \subseteq X \to 2^{X^*}$ be a map. Then A is monotone if and only if for each $u, v \in D$ and $\lambda \geq 0$,*

$$\langle \nu_u - \nu_v, u - v \rangle \leq \langle \nu_u - \nu_v + \lambda(\tau_u - \tau_v), u - v \rangle,\ \forall \nu_u \in Ju,\ \nu_v \in Jv,\ \tau_u \in Au \text{ and } \tau_v \in Av.$$

Thus a mapping $A : d \subseteq X \to 2^{E^*}$ is monotone if and only if the mapping $J_\lambda := (J + \lambda A)^{-1}$ (called the resolvent of A) is J-pseudocontractive on its domain for each positive λ.

We now prove the following lemma which will be crucial in the sequel.

Lemma 4.0.24. *Let E be a smooth real Banach space with dual E^*. Let $\phi : E \times E \to \mathbb{R}$ be the Lyapounov functional. Then,*

$$\phi(y, x) = \phi(x, y) - 2\langle x + y, Jx - Jy \rangle + 2(\|x\|^2 - \|y\|^2), \text{ for all } x, y \in E.$$

Proof. Let $x,\, y \in E$, we have:

$$\begin{aligned} \phi(y, x) &= \|x\|^2 - 2\langle y, Jx \rangle + \|y\|^2 \\ &= \phi(x, y) - 2\Big(\langle y, Jx \rangle - \langle x, Jy \rangle\Big). \end{aligned} \tag{4.0.16}$$

But,

$$\langle x + y, Jx - Jy \rangle = \|x\|^2 - \langle x, Jy \rangle + \langle y, Jx \rangle - \|y\|^2,$$

so that,

$$\langle y, Jx \rangle - \langle x, Jy \rangle = \langle x + y, Jx - Jy \rangle + \|y\|^2 - \|x\|^2;$$

and substituting in (4.0.16), the result follows. □

In theorem 4.0.25 below, the sequence $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$ satisfies the following conditions:

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(ii) $\lambda_n M_0^* \leq \gamma_0 \theta_n$; $\delta_E^{-1}(\lambda_n M_0^*) \leq \gamma_0 \theta_n$,

for all $n \geq 1$ and for some constants $M_0^* > 0$, $\gamma_0 > 0$.

Theorem 4.0.25. *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $T : E \to 2^{E^*}$ be a multi-valued J-pseudocontractive and bounded map. Suppose $F_E^J(T) := \{v \in E : Jv \in Tx\} \neq \emptyset$. For arbitrary $u \in E$, define a sequence $\{x_n\}$ iteratively by: $x_1 \in E$,*

$$x_{n+1} = J^{-1}\Big((1-\lambda_n)Jx_n + \lambda_n\eta_n - \lambda_n\theta_n(Jx_n - Ju)\Big), \;\; n \geq 1, \text{ where } \eta_n \in Tx_n. \tag{4.0.17}$$

Then, the sequence $\{x_n\}$ is bounded.

Proof. Since $F_E^J(T) \neq \emptyset$, let $x^* \in F_E^J(T)$. Then, there exists $r > 0$ such that $max\{\phi(x^*, u), \phi(x^*, x_1)\} \leq \frac{r}{8}$. Let $B := \{x \in E : \phi(x^*, x) \leq r\}$, and since T is bounded, we define:

$M_0 := \sup\{||Jx - \eta + \theta(Jx - Ju)|| : \;\; \theta \in (0,1),\, x \in B,\, \eta \in Tx\} + 1$

$M_1 := \sup\{||Jx - Ju|| : \;\; x \in B\} + 1$

$M_2 := \sup\{||J^{-1}\left[Jx - \lambda\left(Jx - \eta + \theta(Jx - Ju)\right)\right] - x|| : \;\; \lambda,\, \theta \in (0,1),\, x \in B,\, \eta \in Tx\} + 1$

Let $M := \max\{M_2M_0, c_2M_0, c_2M_1\}$, and

$$\gamma_0 := min\Big\{1, \frac{r}{16M}\Big\},$$

where c_2 is the constant in Lemma 4.0.14. We show that $\phi(x^*, x_n) \leq r$ for all $n \geq 1$. We proceed by induction. Clearly, $\phi(x^*, x_1) \leq r$. Suppose $\phi(x^*, x_n) \leq r$ for some $n \geq 1$. We show $\phi(x^*, x_{n+1}) \leq r$. Suppose this is not the case, then $\phi(x^*, x_{n+1}) > r$. Observe that

$$\|x_{n+1} - x_n\| = \|J^{-1}\left[Jx_n - \lambda_n\left(Jx_n - \eta_n + \theta_n(Jx_n - Ju)\right)\right] - J^{-1}Jx_n\|.$$

From lemma 4.0.14 and the recurrrence relation (4.0.17), we have that

$$\begin{aligned} (2L)^{-1}\delta_E(c_2^{-1}\|x_{n+1} - x_n\|) &\leq \langle Jx_{n+1} - Jx_n, x_{n+1} - x_n\rangle \\ &\leq \|Jx_{n+1} - Jx_n\|\|x_{n+1} - x_n\| \\ &\leq \lambda_n M_0\|x_{n+1} - x_n\|. \end{aligned} \tag{4.0.18}$$

We hence obtain that

$$\|x_{n+1} - x_n\| \leq c_2\delta_E^{-1}(\lambda_n M_0^*), \text{ for some } M_0^* > 0. \tag{4.0.19}$$

Using inequality (4.0.5) with $y^* = \lambda_n\left[Jx_n - \eta_n + \theta_n(Jx_n - Ju)\right]$, we obtain using also inequality (4.0.19) that:

$$\phi(x^*, x_{n+1}) = V(x^*, Jx_n - \lambda_n\left[Jx_n - \eta_n + \theta_n(Jx_n - Ju)\right])$$

$$\begin{aligned} &\leq V(x^*, Jx_n) - 2\lambda_n\langle x_n - x^*, Jx_n - \eta_n + \theta_n(Jx_n - Ju)\rangle \\ &\quad -2\lambda_n\langle x_{n+1} - x_n, Jx_n - \eta_n + \theta_n(Jx_n - Ju)\rangle \\ &\leq V(x^*, Jx_n) - 2\lambda_n\langle x_n - x^*, Jx_n - \eta_n + \theta_n(Jx_n - Ju)\rangle \\ &\quad +2\lambda_n\|x_{n+1} - x_n\|\|Jx_n - \eta_n + \theta_n(Jx_n - Ju)|| \\ &\leq V(x^*, Jx_n) - 2\lambda_n\langle x_n - x^*, Jx_n - \eta_n\rangle - 2\lambda_n\theta_n\langle x_n - x^*, Jx_n - Ju\rangle + 2\lambda_n M_0 c_2\delta_E^{-1}(\lambda_n M_0^*). \end{aligned}$$

Since T is J-pseudocontractive, so that $(J - T)$ is monotone, and using the recursion formula, we have that:

$$\begin{aligned} \phi(x^*, x_{n+1}) &\leq V(x^*, Jx_n) - 2\lambda_n\theta_n\langle x_n - x^*, Jx_n - Ju\rangle + 2\lambda_n M_0 c_2\delta_E^{-1}(\lambda_n M_0^*) \\ &= \phi(x^*, x_n) - 2\lambda_n\theta_n\langle x_n - x_{n+1}, Jx_n - Ju\rangle - 2\lambda_n\theta_n\langle x_{n+1} - x^*, Jx_n - Jx_{n+1}\rangle \\ &\quad -2\lambda_n\theta_n\langle x_{n+1} - x^*, Jx_{n+1} - Ju\rangle + 2\lambda_n M_0 c_2\delta_E^{-1}(\lambda_n M_0^*). \end{aligned} \tag{4.0.20}$$

We have from lemma 4.0.13 that,

$$-2\lambda_n\theta_n\langle x_{n+1}-x^*, Jx_{n+1}-Ju\rangle \leq \lambda_n\theta_n\phi(x^*,u)-\lambda_n\theta_n\phi(x^*,x_{n+1}).$$

Substituting this in inequality (4.0.20), we obtain that:

$$\begin{aligned}
& r < \phi(x^*,x_{n+1}) \\
\leq\ & \phi(x^*,x_n)-\lambda_n\theta_n\phi(x^*,x_{n+1})+\lambda_n\theta_n\phi(x^*,u)+2\lambda_n\theta_n M_1c_2\delta_E^{-1}(\lambda_n M_0^*) \\
& +2\lambda_n\theta_n M_2(\lambda_n M_0)+2\lambda_n M_0c_2\delta_E^{-1}(\lambda_n M_0^*) \\
\leq\ & \phi(x^*,x_n)-\lambda_n\theta_n\phi(x^*,x_{n+1})+\lambda_n\theta_n\phi(x^*,u)+2\lambda_n\theta_n\gamma_0 M_1c_2+2\lambda_n\theta_n\gamma_0 M_2M_0+2\lambda_n\theta_n\gamma_0 M_0c_2 \\
\leq\ & \phi(x^*,x_n)-\lambda_n\theta_n\phi(x^*,x_{n+1})+4\lambda_n\theta_n\frac{r}{8} \\
\leq\ & r-\lambda_n\theta_n r+\frac{\lambda_n\theta_n r}{2}=r-\frac{\lambda_n\theta_n r}{2}<r.
\end{aligned}$$

This is a contradiction. Hence, $\{x_n\}_{n=1}^{\infty}$ is bounded. $\square$

In theorem 4.0.26 below, λ_n and θ_n are real sequences in $(0,1)$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty}\lambda_n\theta_n=\infty$;

(ii) $\lambda_n M_0^*\leq\gamma_0\theta_n$; $\delta_E^{-1}(\lambda_n M_0^*)\leq\gamma_0\theta_n$,

(iii) $\dfrac{\delta_E^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\lambda_n\theta_n}\to 0,\ \ \dfrac{\delta_{E^*}^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\lambda_n\theta_n}\to 0,$ as $n\to\infty$,

(iv) $\dfrac{1}{2}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)\in(0,1)$,

for some constants $M_0^*>0$, and $\gamma_0>0$; where $\delta_E:(0,\infty)\to(0,\infty)$ is the modulus of convexity of E and $K>0$ is as defined in lemma 4.0.14.

Theorem 4.0.26. *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $T:E\to 2^{E^*}$ be a J-pseudocontractive and bounded map such that $(J-T)$ is maximal monotone. Suppose $F_E^J(T)=\{v\in E: Jv\in Tv\}\neq\emptyset$. For arbitrary $x_1,\,u\in E$, define a sequence $\{x_n\}$ iteratively by:*

$$x_{n+1}=J^{-1}\left[(1-\lambda_n)Jx_n+\lambda_n\eta_n-\lambda_n\theta_n(Jx_n-Ju)\right],\ \ \eta_n\in Tx_n,\ \ n\geq 1, \tag{4.0.21}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying conditions $(i)-(iv)$ above. Then, the sequence $\{x_n\}$ converges strongly to a J-fixed point of T.

Proof. Setting $y^*=\lambda_n\left[Jx_n-\eta_n+\theta_n(Jx_n-Ju)\right]\in E^*$, applying inequality (4.0.5) and using Lemma 4.0.24, we compute as follows:

$$\begin{aligned}
&\phi(y_n,x_{n+1}) \\
&\quad= V(y_n, Jx_n-\lambda_n\left(Jx_n-\eta_n+\theta_n(Jx_n-Ju)\right)) \\
&\quad\leq V(y_n,Jx_n)-2\langle x_{n+1}-y_n,\lambda_n\left(Jx_n-\eta_n+\theta_n(Jx_n-Ju)\right)\rangle \\
&\quad= \phi(y_n,x_n)-2\lambda_n\langle x_{n+1}-y_n,Jx_n-\eta_n\rangle-2\lambda_n\theta_n\langle x_{n+1}-y_n,Jx_n-Ju\rangle \\
&\quad= \phi(x_n,y_n)-2\langle x_n+y_n,Jx_n-Jy_n\rangle+2\left(\|x_n\|^2-\|y_n\|^2\right)-2\lambda_n\langle x_{n+1}-y_n,Jx_n-\eta_n\rangle \\
&\qquad -2\lambda_n\theta_n\langle x_{n+1}-y_n,Jx_n-Ju\rangle.
\end{aligned} \tag{4.0.22}$$

But we have from Lemma 4.0.17 that $y_n = J^{-1}\left[\tau_n - \theta_n(Jy_n - Ju)\right]$ for some $\tau_n \in Ty_n$ and thus obtain:

$$\phi(x_n, y_n) = V(x_n, Jy_n) = V(x_n, Jy_{n-1} + Jy_n - Jy_{n-1}) \leq V(x_n, Jy_{n-1}) - 2\langle y_n - x_n, Jy_{n-1} - Jy_n\rangle.$$

Hence, substituting this in inequality (4.0.22) and using Lemma 4.0.24, we obtain:

$$\begin{aligned}
&\phi(y_n, x_{n+1})\\
&\quad\leq V(x_n, Jy_{n-1}) - 2\langle y_n - x_n, Jy_{n-1} - Jy_n\rangle + 2\left(\|x_n\|^2 - \|y_n\|^2\right) - 2\langle x_n + y_n, Jx_n - Jy_n\rangle\\
&\qquad -2\lambda_n\langle x_{n+1} - y_n, Jx_n - \eta_n\rangle - 2\lambda_n\theta_n\langle x_{n+1} - y_n, Jx_n - Ju\rangle\\
&\quad= \phi(x_n, y_{n-1}) - 2\langle y_n - x_n, Jy_{n-1} - Jy_n\rangle + 2\left(\|x_n\|^2 - \|y_n\|^2\right) - 2\langle x_n + y_n, Jx_n - Jy_n\rangle\\
&\qquad -2\lambda_n\langle x_{n+1} - y_n, Jx_n - \eta_n\rangle - 2\lambda_n\theta_n\langle x_{n+1} - y_n, Jx_n - Ju\rangle\\
&\quad= \phi(y_{n-1}, x_n) + 2\left(\|y_{n-1}\|^2 - \|x_n\|^2\right) + 2\langle y_{n-1} + x_n, Jx_n - Jy_{n-1}\rangle - 2\langle y_n - x_n, Jy_{n-1} - Jy_n\rangle\\
&\qquad +2\left(\|x_n\|^2 - \|y_n\|^2\right) - 2\langle x_n + y_n, Jx_n - Jy_n\rangle - 2\lambda_n\langle x_{n+1} - y_n, Jx_n - \eta_n\rangle\\
&\qquad -2\lambda_n\theta_n\langle x_{n+1} - y_n, Jx_n - Ju\rangle\\
&\quad= \phi(y_{n-1}, x_n) + 2\left(\|y_{n-1}\|^2 - \|y_n\|^2\right) + 2\langle y_{n-1} + x_n, Jx_n - Jy_{n-1}\rangle - 2\langle y_n - x_n, Jy_{n-1} - Jy_n\rangle\\
&\qquad -2\langle x_n + y_n, Jx_n - Jy_n\rangle - 2\lambda_n\langle x_{n+1} - y_n, Jx_n - \eta_n\rangle - 2\lambda_n\theta_n\langle x_{n+1} - y_n, Jx_n - Ju\rangle. \qquad (4.0.23)
\end{aligned}$$

Furthermore, using Lemma 4.0.13, we obtain that:

$$\begin{aligned}
&-2\lambda_n\theta_n\langle x_{n+1} - y_n, Jx_n - Ju\rangle\\
&\qquad= -2\lambda_n\theta_n\langle x_{n+1} - x_n, Jx_n - Ju\rangle - 2\lambda_n\theta_n\langle x_n - y_{n-1}, Jx_n - Jy_{n-1}\rangle\\
&\qquad\quad -2\lambda_n\theta_n\langle x_n - y_{n-1}, Jy_{n-1} - Ju\rangle - 2\lambda_n\theta_n\langle y_{n-1} - y_n, Jx_n - Ju\rangle\\
&\qquad\leq -2\lambda_n\theta_n\langle x_{n+1} - x_n, Jx_n - Ju\rangle - \lambda_n\theta_n\phi(y_{n-1}, x_n)\\
&\qquad\quad -2\lambda_n\theta_n\langle x_n - y_{n-1}, Jy_{n-1} - Ju\rangle - 2\lambda_n\theta_n\langle y_{n-1} - y_n, Jx_n - Ju\rangle.
\end{aligned}$$

Substituting this inequality in inequality (4.0.23), we thus have:

$$\begin{aligned}
&\phi(y_n, x_{n+1})\\
&\quad\leq \phi(y_{n-1}, x_n) + 2\left(\|y_{n-1}\|^2 - \|y_n\|^2\right) + 2\langle y_{n-1} + x_n, Jx_n - Jy_{n-1}\rangle - 2\langle y_n - x_n, Jy_{n-1} - Jy_n\rangle\\
&\qquad -2\langle x_n + y_n, Jx_n - Jy_n\rangle - 2\lambda_n\langle x_{n+1} - y_n, Jx_n - \eta_n\rangle - 2\lambda_n\theta_n\langle x_{n+1} - x_n, Jx_n - Ju\rangle\\
&\qquad -\lambda_n\theta_n\phi(y_{n-1}, x_n) - 2\lambda_n\theta_n\langle x_n - y_{n-1}, Jy_{n-1} - Ju\rangle - 2\lambda_n\theta_n\langle y_{n-1} - y_n, Jx_n - Ju\rangle\\
&\quad\leq \phi(y_{n-1}, x_n) - \lambda_n\theta_n\phi(y_{n-1}, x_n) + 2\left(\|y_{n-1}\|^2 - \|y_n\|^2\right) + 2\langle y_{n-1} - y_n, Jx_n - Jy_{n-1}\rangle\\
&\qquad -2\langle y_n - x_n, Jy_{n-1} - Jy_n\rangle - 2\langle x_n + y_n, Jy_n - Jy_{n-1}\rangle - \underline{2\lambda_n\langle x_{n+1} - y_n, Jx_n - \eta_n\rangle}\\
&\qquad -2\lambda_n\theta_n\langle x_{n+1} - x_n, Jx_n - Ju\rangle - \underline{2\lambda_n\theta_n\langle x_n - y_{n-1}, Jy_{n-1} - Ju\rangle} - 2\lambda_n\theta_n\langle y_{n-1} - y_n, Jx_n - Ju\rangle.
\end{aligned}$$

Estimating the underlined terms, we obtain :

$$\begin{aligned}
&-2\lambda_n\langle x_{n+1} - y_n, Jx_n - \eta_n\rangle - 2\lambda_n\theta_n\langle x_n - y_{n-1}, Jy_{n-1} - Ju\rangle\\
&\quad= -2\lambda_n\langle x_{n+1} - x_n, Jx_n - \eta_n\rangle - 2\lambda_n\langle x_n - y_n, Jx_n - \eta_n\rangle - 2\lambda_n\theta_n\langle x_n - y_n, Jy_{n-1} - Ju\rangle\\
&\qquad -2\lambda_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1} - Ju\rangle\\
&\quad= -2\lambda_n\langle x_{n+1} - x_n, Jx_n - \eta_n\rangle - 2\lambda_n\langle x_n - y_n, Jx_n - \eta_n\rangle - 2\lambda_n\theta_n\langle x_n - y_n, Jy_{n-1} - Jy_n\rangle\\
&\qquad -2\lambda_n\langle x_n - y_n, \theta_n(Jy_n - Ju)\rangle - 2\lambda_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1} - Ju\rangle\\
&\quad= -2\lambda_n\langle x_{n+1} - x_n, Jx_n - \eta_n\rangle\underline{-2\lambda_n\langle x_n - y_n, Jx_n - \eta_n\rangle} - 2\lambda_n\theta_n\langle x_n - y_n, Jy_{n-1} - Jy_n\rangle\\
&\qquad -\underline{2\lambda_n\langle x_n - y_n, -(Jy_n - \tau_n)\rangle} - 2\lambda_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1} - Ju\rangle\\
&\quad\leq -2\lambda_n\langle x_{n+1} - x_n, Jx_n - \eta_n\rangle - 2\lambda_n\theta_n\langle x_n - y_n, Jy_{n-1} - Jy_n\rangle - 2\lambda_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1} - Ju\rangle.
\end{aligned}$$

We thus have:

$$
\begin{aligned}
&\phi(y_n, x_{n+1})\\
&\le \phi(y_{n-1}, x_n) - \lambda_n\theta_n\phi(y_{n-1}, x_n) + 2\|y_{n-1} - y_n\|\left(\|y_{n-1}\| + \|y_n\|\right) + 2\langle y_{n-1} - y_n, Jx_n - Jy_{n-1}\rangle\\
&\quad -2\langle x_n + y_n, Jy_n - Jy_{n-1}\rangle - 2\langle y_n - x_n, Jy_{n-1} - Jy_n\rangle - 2\lambda_n\theta_n\langle x_{n+1} - x_n, Jx_n - Ju\rangle\\
&\quad -2\lambda_n\theta_n\langle y_{n-1} - y_n, Jx_n - Ju\rangle - 2\lambda_n\langle x_{n+1} - x_n, Jx_n - \eta_n\rangle - 2\lambda_n\theta_n\langle x_n - y_n, Jy_{n-1} - Jy_n\rangle\\
&\quad -2\lambda_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1} - Ju\rangle\\
&\le \phi(y_{n-1}, x_n) - \lambda_n\theta_n\phi(y_{n-1}, x_n) + 2\|y_{n-1} - y_n\|\left(\|y_{n-1}\| + \|y_n\|\right) + 2\langle y_{n-1} - y_n, Jx_n - Jy_n\rangle\\
&\quad -2\langle y_{n-1} + x_n, Jy_n - Jy_{n-1}\rangle - 2\langle y_n - x_n, Jy_{n-1} - Jy_n\rangle - 2\lambda_n\theta_n\langle x_{n+1} - x_n, Jx_n - Ju\rangle\\
&\quad -2\lambda_n\theta_n\langle y_{n-1} - y_n, Jx_n - Ju\rangle - 2\lambda_n\langle x_{n+1} - x_n, Jx_n - \eta_n\rangle - 2\lambda_n\theta_n\langle x_n - y_n, Jy_{n-1} - Jy_n\rangle\\
&\quad -2\lambda_n\theta_n\langle y_n - y_{n-1}, Jy_{n-1} - Ju\rangle\\
&\le (1 - \lambda_n\theta_n)\phi(y_{n-1}, x_n) + 2\lambda_n\theta_n M_a\left(\|x_{n+1} - x_n\| + \|y_{n-1} - y_n\| + \|Jy_{n-1} - Jy_n\|\right)\\
&\quad +M_b\left(\lambda_n\|x_{n+1} - x_n\| + \|y_{n-1} - y_n\| + \|Jy_{n-1} - Jy_n\|\right), \text{ for some } M_a > 0,\ M_b > 0 \qquad (4.0.24)\\
&\le (1 - \lambda_n\theta_n)\phi(y_{n-1}, x_n) + 2\lambda_n\theta_n M_a\left(c_2\delta_E^{-1}(\lambda_n M_0^*) + \delta_E^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\right) + \delta_{E^*}^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\right)\right)\\
&\quad +M_b\left(c_2\lambda_n\delta_E^{-1}(\lambda_n M_0^*) + \delta_E^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\right) + \delta_{E^*}^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\right)\right) \qquad (4.0.25)\\
&\le (1 - \lambda_n\theta_n)\phi(y_{n-1}, x_n) + \lambda_n\theta_n M_a^*\Bigg(c_2\delta_E^{-1}(\lambda_n M_0^*) + \delta_E^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big) + \delta_{E^*}^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big)\\
&\quad +\frac{\delta_E^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)}{\lambda_n\theta_n} + \frac{\delta_{E^*}^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)}{\lambda_n\theta_n} + \frac{c_2\delta_E^{-1}(\lambda_n M_0^*)}{\theta_n}\Bigg), \quad \text{where } M_a^* = 2max\{M_a, M_b\}. (4.0.26)
\end{aligned}
$$

Now, setting

$$a_n := \phi(y_{n-1}, x_n); \quad \sigma_n := \lambda_n\theta_n; \quad c_n \equiv 0$$

and

$$
\begin{aligned}
b_n := \Bigg[M_a^*\Bigg(&c_2\delta_E^{-1}(\lambda_n M_0^*) + \delta_E^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big) + \delta_{E^*}^{-1}\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\Big) + \frac{\delta_E^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)}{\lambda_n\theta_n}\\
&+\frac{\delta_{E^*}^{-1}\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\Big)}{\lambda_n\theta_n} + \frac{c_2\delta_E^{-1}(\lambda_n M_0^*)}{\theta_n}\Bigg)\Bigg],
\end{aligned}
$$

inequality (4.0.26) becomes

$$a_{n+1} \le (1 - \sigma_n)\,a_n + \sigma_n b_n + c_n, \quad n \ge 0,$$

It now follows from Lemma 4.0.19 that $\phi(y_{n-1}, x_n) \to 0$ as $n \to \infty$. From Lemma 4.0.18, we have that $\|x_n - y_{n-1}\| \to 0$ and since $y_n \to y^* \in (J - T)^{-1}0$, we obtain that $x_n \to y^* \in (J - T)^{-1}0$. This completes the proof. □

Example 6. *We have (see e.g., [4], p.47) for $p > 1,\ q > 1,\ X = L^p,\ X^* = L^q$, that*

$$\delta_{X^*}(\epsilon) = 1 - \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{1/q},$$

and so obtain that:

$$\delta_{X^*}^{-1}(\epsilon) = 2\left[1-(1-\epsilon)^q\right]^{1/q} \le 2q^{1/q}\epsilon^{1/q}, \text{ since } (1-\epsilon)^q > 1-q\epsilon, \text{ for } q>1.$$

The prototypes for our theorems are the following:

$$\lambda_n = \frac{1}{(n+1)^a}, \quad \theta_n = \frac{1}{(n+1)^b}.$$

$0 < b < \frac{1}{r}\cdot a, \quad a+b < 1/r, \quad b < 1/K;$ *where* $K>0$ *is as defined in lemma 4.0.14,* $r = max\{p,q\}$.

In particular, without loss of generality, let $r = p$. *Then, one can choose* $a := \frac{1}{(p+1)}$ *and* $b := min\{\frac{1}{2K}, \frac{1}{2p(p+1)}\}$.

We now verify that, with these prototypes, the conditions $(i)-(iii)$ *of Theorem 4.0.26 are satisfied. Clearly* (i) *and the first part of* (ii) *are easily verified.*
For the second part of condition (ii)*, we have that:*

$$\begin{aligned}\frac{\delta_E^{-1}(\lambda_n M_0^*)}{\theta_n} &= \frac{2\left[1-(1-\lambda_n M_0^*)^p\right]^{1/p}}{\theta_n}\\ &\le \frac{2(pM_0^*)^{1/p}\lambda_n^{1/p}}{\theta_n} = 2(pM_0^*)^{1/p}\cdot(n+1)^{b-(a/p)} \longrightarrow 0.\end{aligned}$$

For condition (iii)*, we have that:*

$$\frac{\delta_{E^*}^{-1}\left(\frac{\theta_{n-1}}{\theta_n}-1\right)}{\lambda_n\theta_n} = \frac{2\left[1-\left(2-\frac{\theta_{n-1}}{\theta_n}\right)^q\right]^{1/q}}{\lambda_n\theta_n}.$$

$$\begin{aligned}&= \frac{2\left[1-\left(2-\left(\frac{n+1}{n}\right)^b\right)^q\right]^{1/q}}{1/(n+1)^{a+b}} = 2\left[1-\left(2-\left(1+\frac{1}{n}\right)^b\right)^q\right]^{1/q}\cdot(n+1)^{a+b}\\ &\le 2\left[1-\left(2-1-\frac{b}{n}\right)^q\right]^{1/q}\cdot(n+1)^{a+b} \le 2\left[\frac{bq}{n}\right]^{1/q}\cdot(n+1)^{a+b}\\ &= 2(bq)^{1/q}\cdot\frac{1}{n^{1/q}}\cdot(n+1)^{a+b} \le 2^{a+b+1}(bq)^{1/q}\cdot n^{a+b-(1/q)} \longrightarrow 0.\end{aligned}$$

Similarly, we obtain that

$$\frac{\delta_E^{-1}\left(\frac{\theta_{n-1}}{\theta_n}-1\right)}{\lambda_n\theta_n} = \frac{2\left[1-\left(2-\frac{\theta_{n-1}}{\theta_n}\right)^p\right]^{1/p}}{\lambda_n\theta_n} \longrightarrow 0.$$

Finally, for condition (iv)*, we have that:*

$$\frac{1}{2}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right) = \frac{1}{2}\left[\left(1+\frac{1}{n}\right)^b - 1\right]\cdot K \le \frac{bK}{2n} < 1.$$

This completes the verification.

Remark 4. *We remark, following Lindenstrauss and Tzafriri [77], that in applications, we do not often use the precise value of the modulus of convexity but only a power type estimate from below.*

A uniformly convex space X has modulus of convexity of power type p if, for some $0 < K < \infty$, $\delta_X(\epsilon) \geq K\epsilon^p$. For instance, L_p spaces have modulus of convexity of power type 2, for $1 < p \leq 2$, and of power type p, for $p > 2$ (see e.g., [77],p.63). We observe that the condition for modulus of convexity of power type p corresponds to that of p-uniformly convex spaces. However, we have that L_p spaces are p-uniformly convex, for $1 < p < 2$, and are 2-uniformly convex, for $p \geq 2$.

These lead us to prove the following corollary of Theorem 4.0.25, which will be crucial in several applications.

Corollary 4.0.27. *For $p > 1$, $q > 1$, let E be a p-uniformly convex and q-uniformly smooth real Banach space and let E^* be its dual. Let $T : E \to E^*$ be a J-pseudocontractive and bounded map. Suppose $F_E^J(T) := \{u^* \in E : Tu^* = Ju^*\} \neq \emptyset$. For arbitrary x_1, $u \in E$, define a sequence $\{x_n\}$ iteratively by:*

$$x_{n+1} = J^{-1}\left[(1-\lambda_n)Jx_n + \lambda_n\eta_n - \lambda_n\theta_n(Jx_n - Ju)\right], \ \ n \geq 1 \textit{ where } \eta_n \in Tx_n. \tag{4.0.27}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying conditions $(i)-(iii)$ of theorem (4.0.25). Then, the sequence $\{x_n\}$ converges strongly to a J-fixed point of T.

Proof. We observe, for p-uniformly convex space, using Remark 2, that conditions $(i)-(iv)$ of Theorem 4.0.26 reduce to:

$(i)^*$ $\lambda_n \leq \gamma_0\theta_n$

$(ii)^*$ $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$;

$(iii)^*$ $\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)^{1/p} \to 0$, $\frac{M^*\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)^{1/p}}{\lambda_n\theta_n} \to 0$, $\frac{\left(\lambda_n^{(1/p)}M_0^{**}\right)}{\theta_n} \to 0$, as $n \to \infty$, for some M_0^{**}, $M^* > 0$,

and for p-uniformly convex spaces, we have from (4.0.18), using equation (4.0.12), that

$$\begin{aligned} c_2^{-1}\|x_{n+1}-x_n\|^p &\leq 2LM_0\lambda_n\|x_{n+1}-x_n\| \\ \|x_{n+1}-x_n\| &\leq \lambda_n^{1/p}M_0^{**} \quad \text{for some } M_0^{**} > 0. \end{aligned} \tag{4.0.28}$$

Following the proof of theorem 4.0.26, we have from inequality (4.0.24), using (4.0.28), that:

$$\begin{aligned} &\phi(y_n, x_{n+1}) \\ &\leq (1-\lambda_n\theta_n)\phi(y_{n-1},x_n) + 2\lambda_n\theta_n M_a\left(\lambda_n^{1/p}M_0^{**} + K_1\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)^{1/p} + K_2\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)^{1/p}\right) \\ &\quad + M_b\left(\lambda_n^{1+(1/p)}M_0^{**} + K_1\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)^{1/p} + K_2\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)^{1/p}\right) \\ &\leq (1-\lambda_n\theta_n)\phi(y_{n-1},x_n) + \lambda_n\theta_n M_a^*\left(\lambda_n^{1/p}M_0^{**} + M^*\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)^{1/p} + \frac{M^*\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)^{1/p}}{\lambda_n\theta_n}\right. \\ &\quad \left. + \frac{\left(\lambda_n^{(1/p)}M_0^{**}\right)}{\theta_n}\right), \qquad \text{where } M^* = max\{K_1, K_2\},\ M_a^* = 2max\{M_a, M_b\}. \end{aligned} \tag{4.0.29}$$

Now, setting

$$a_n := \phi(y_{n-1}, x_n); \quad \sigma_n := \lambda_n\theta_n; \quad c_n \equiv 0$$

and

$$b_n := \left[M_a^* \left(\lambda_n^{1/p} M_0^{**} + M^* \left(\frac{\theta_{n-1} - \theta_n}{\theta_n} \right)^{1/p} + \frac{M^* \left(\frac{\theta_{n-1}-\theta_n}{\theta_n} \right)^{1/p}}{\lambda_n\theta_n} + \frac{\left(\lambda_n^{(1/p)} M_0^{**} \right)}{\theta_n} \right) \right],$$

$$a_{n+1} \leq (1 - \sigma_n)\, a_n + \sigma_n b_n + c_n, \quad n \geq 0,$$

It now follows from Lemma 4.0.19 that $\phi(y_{n-1}, x_n) \to 0$ as $n \to \infty$. From Lemma 4.0.18, we have that $\|x_n - y_{n-1}\| \to 0$ and since $y_n \to y^* \in (J - T)^{-1}0$, this completes the proof.

□

Example 7. *Real sequences that satisfy the conditions $(i)^* - (iv)^*$ in corollary 4.0.27 are the folliowing:*

$$\lambda_n = (n+1)^{-a} \textit{ and } \theta_n = (n+1)^{-b}, \;\; n \geq 1.$$
$$0 < b < \frac{1}{p} \cdot a, \quad a + b < 1/p.$$

For example, one can choose $a := \frac{1}{(p+1)}$ and $b := \frac{1}{2p(p+1)}$. We now check these prototypes.

Clearly conditions $(i)^*$ - $(ii)^*$ are satisfied. We verify condition $(iii)^*$. Using the fact that $(1+x)^s \leq 1 + sx$, for $x > -1$ and $0 < s < 1$, we have that

$$\begin{aligned} 0 \;\; &\leq \;\; \frac{M^* \left(\frac{\theta_{n-1}}{\theta_n} - 1 \right)^{1/p}}{\lambda_n\theta_n} = M^* \Big[\left(1 + \frac{1}{n} \right)^b - 1 \Big]^{1/p} \cdot (n+1)^{a+b} \\ &\leq \;\; M^* b^{1/p} \cdot \frac{(n+1)^{a+b}}{n^{1/p}} = 2^{a+b} M^* b^{1/p} \cdot n^{a+b-(1/p)} \longrightarrow 0. \end{aligned}$$

Also,

$$0 \;\; \leq \;\; \left(\frac{\theta_{n-1}}{\theta_n} - 1 \right)^{1/p} = \Big[\left(1 + \frac{1}{n} \right)^b - 1 \Big]^{1/p} \leq \frac{b^{1/p}}{n^{1/p}} \longrightarrow 0,$$

and

$$0 \;\; \leq \;\; \frac{\lambda_n^{(1/p)} M_0^{**}}{\theta_n} = M_0^{**} (n+1)^{b-(a/p)} \longrightarrow 0. \tag{4.0.30}$$

4.1 Application to zeros of maximal monotone maps

Corollary 4.1.1. *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $A : E \to 2^{E^*}$ be a multi-valued maximal monotone and bounded map such that $A^{-1}0 \neq \emptyset$. For fixed u, $x_1 \in E$, let a sequence $\{x_n\}$ be iteratively defined by:*

$$x_{n+1} = J^{-1} \left[Jx_n - \lambda_n \mu_n - \lambda_n \theta_n (Jx_n - Ju) \right], \;\; n \geq 1, \quad \mu_n \in Ax_n. \tag{4.1.1}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1)$. Then, the sequence $\{x_n\}$ converges strongly to a zero of A.

Proof. Recall that A is monotone if and only if $T = (J - A)$ is J-pseudocontractive and that zeros of A correspond to J-fixed points of T.. Now, if we replace A by $J - T$ in equation (4.1.1), the equation reduces to (4.0.21) and hence the proof follows. □

4.2 Complement to proximal point algorithm

The *proximal point algorithm* of Martinet [82] and Rockafellar [100] was introduced to approximate a solution of $0 \in Au$ where A is the subdifferential of some convex functional defined on a real Hilbert space. A solution of this inclusion gives the minimizers of the convex functional. Let E be a real normed space with dual space, E^* and $f : E \to \mathbb{R}$ be a convex functional. The subdifferential of f, $\partial f : E \to 2^{E^*}$ at $u \in E$ is defined as follows:

$$(\partial f)(u) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle \ \ \forall y \in E\}.$$

It is well known that ∂f is a maximal monotone map on E and that $0 \in (\partial f)(u)$ if and only if u is a minimizer of f. Following this, the proximal point algorithm has been studied for minimizers of f in real Banach spaces more general than Hilbert spaces.

Rockafellar [100] proved that the *proximal point algorithm* defined as follows:

$$x_{k+1} = \Big(I + \frac{1}{\lambda_k}A\Big)^{-1}(x_k) + e_k, \quad x_1 \in H, \tag{4.2.1}$$

where $\lambda_k > 0$ is a regularizing parameter; converges *weakly* to a solution of $0 \in Au$ where A is the subdifferential of a convex functional on a Hilbert space *provided a solution exists*. He then asked if the proximal point algorithm always converge strongly.

This was resolved in the negative by Güler [60] who produced a proper closed convex function g in the infinite dimensional Hilbert space l_2 for which the proximal point algorithm converges *weakly* but *not strongly*, (see also Bauschke *et al. [6]*). Several authors modified the proximal point algorithm to obtain *strong* convergence (see e.g., Bruck [17]; Kamimura and Takahashi [72]; Lehdili and Moudafi [75]; Reich [95]; Solodov and Svaiter [108]; Xu [118]). We remark that in every one of these modifications, the recursion formula developed involved either the computation of $(I + \lambda_k A)^{-1}(x_k)$ at each point of the iteration process or the construction, at each iteration, of two subsets of the space, intersecting them and projecting the initial vector onto the intersection. As far as we know, the first iteration process to approximate a solution of $0 \in Au$ in real Banach spaces more general than Hilbert spaces and which does not involve either of these setbacks was given by Chidume and Djitte [19] who proved a special case of Theorem 2.0.1 in which the space E is a 2-uniformly smooth real Banach space. These spaces include L_p spaces, $2 \leq p < \infty$, but do not include L_p spaces, $1 < p < 2$. This result of Chidume and Djitte has recently been proved in uniformly convex and uniformly smooth real Banach spaces (which include L_p spaces, $1 < p < \infty$) (Chidume, (Theorem 2.0.1) above).

Corollary 4.1.1 of this paper is an analogue of Theorem 2.0.1 for maximal monotone maps when $A : E \to 2^{E^*}$ is a maximal monotone and bounded map, a result which complements the proximal point algorithm, under this setting, in the sense that it yields strong convergence to a solution of $0 \in Au$ and without requiring either the computation of $(J + \lambda A)^{-1}(z_n)$ at each iteration process or the construction of two subsets of E, and projection of the initial vector onto their intersection, *at each stage of the iteration process*.

4.3 Application to solutions of Hammerstein integral equations

Definition 4.3.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of Hammerstein-type has the form*

$$u(x) + \int_\Omega k(x,y) f(y, u(y)) dy = w(x), \tag{4.3.1}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions.

By simple transformation (4.3.1) can put in the form

$$u + KFu = w. \tag{4.3.2}$$

which, without loss of generality can be written as

$$u + KFu = 0. \tag{4.3.3}$$

Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear part posses Green's function can, as a rule, be transformed into the form (4.3.1) (see e.g., Pascali and Sburian [88], chapter p. 164).

Among the first early results on the approximation of solution of Hammerstein equations is the following result of Brézis and Browder.

Theorem 4.3.2 (Brézis and Browder [13]). *Let H be a seprable Hilbert space and C be a closed subspace of H. Let $K : H \to C$ be a bounded continuous monotone operator and $F : C \to H$ be angle-bounded and weakly compact mapping. For a giving $f \in C$, consider the Hammerstein equation*

$$(I + KF)u = f \tag{4.3.4}$$

and its nth Galerkin approximation given by

$$(I + K_nF_n)u_n = P^*f, \tag{4.3.5}$$

where $K_n = P_n^*KP_n : H \to C$ and $F_n = P_nFP_n^* : C_n \to H$, where the symbols have their usual meanings (see [88]). Then, for each $n \in \mathbb{N}$, the Galerkin approximation (4.3.5) admits a unique solution u_n in C_n and $\{u_n\}$ converges strongly in H to the unique solution $u \in C$ of the equation (4.3.4) where $K_n = P_n^*KP_n : H \to C$ and $F_n = P_nFP_n^* : C_n \to H$, where the symbols have their usual meanings (see [13]). Then, for each $n \in \mathbb{N}$, the Galerkin approximation (4.3.5) admits a unique solution u_n in C_n and $\{u_n\}$ converges strongly in H to the unique solution $u \in C$ of the equation (4.3.4).

It is obvious that if an iterative algorithm can be developed for the approximation of solutions of equation of Hammerstein-type (4.3.3), this will certainly be preferred.
Attempts have been made to approximate solutions of equations of Hammerstein-type using Mann-type iteration scheme. However, the results obtained were not satisfactory (see e.g., [30]). The recurrance formulas used in early attempts involved K^{-1} which is also required to be strongly monotone, and this, apart from limiting the class of mappings to which such iterative schemes are applicable, it is also not convenient in applications. Part of the difficulty is the fact that the composition of two monotone operators need not to be monotone.
The first satisfactory results on *iterative methods* for approximating solutions of Hammerstein equations in real Banach spaces more general Hilbert spaces, as far as we know, were obtained by Chidume and Zegeye [31, 32, 33]. For the case of real Hilbert space H, for $F, K : H \to H$, they defined an auxillary map on the Cartesian product $E := H \times H$, $T : E \to E$ by

$$T[u, v] = [Fu - v, Kv + u].$$

We note that

$$T[u, v] = 0 \Longleftrightarrow u \text{ solves } (4.3.3) \text{ and } v = Fu.$$

With this, they were able to obtain strong convergence of an iterative scheme defined in the Cartesian product space E to a solution of Hammerstein equation (4.3.3). The method of proof used by Chidume and Zegeye provided the clue to the establishement of the following couple explicit algorithm for computing a solution of the equation $u + KFu = 0$ in the original space X. With initial vectors $u_0, v_0 \in X$, sequences $\{u_n\}$ and $\{v_n\}$ in X are defined iteratively as follows:

$$u_{n+1} = u_n - \alpha_n(Fu_n - v_n),\ n \geq 0, \tag{4.3.6}$$

$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n),\ n \geq 0, \tag{4.3.7}$$

where α_n is a sequence in $(0,1)$ satisfying appropriate conditions.
Some typical results obtained using the recursion formulas described above in approximating solutions of nonlinear Hammerstein equations involving monotone maps in Hilbert spaces can be found in the following references ([36], [33]).

In real Banach space X more general than Hilbert spaces, where $F, K : X \to X$ are of *accretive-type*, Chidume and Zegeye considered an operator $A : E \to E$ where $E := X \times X$ and were able to successfully approximate solutions of Hammerstein equations using recursion formulas described above. These schemes have now been employed by Chidume and other authors to approximate solutions of Hammerstein equations in various Banach spaces under various continuity assumptions (see e.g., [31, 32, 33, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35], [52, 53], [51], [111], [109, 110]). This success has not carried over to the case of *monotone-type mappings* in Banach spaces where K and F map a space into its dual. In this section, we introduce a new iterative scheme and prove that a sequence of our scheme converges strongly to a solution of a Hammerstein equation under this setting. For this, we begin with the following preliminaries and lemmas.

We now prove the following lemmas.

Lemma 4.3.3. *Let X, Y be real uniformly convex and uniformly smooth spaces. Let $E = X \times Y$ with the norm $\|z\|_E = (\|u\|_X^q + \|v\|_Y^q)^{\frac{1}{q}}$, for arbitrary $z = [u, v] \in E$. Let $E^* = X^* \times Y^*$ denote the dual space of E. For arbitrary $x = [x_1, x_2] \in E$, define the map $j_q^E : E \to E^*$ by*

$$j_q^E(x) = j_q^E[x_1, x_2] := [j_q^X(x_1), j_q^Y(x_2)],$$

so that for arbitrary $z_1 = [u_1, v_1]$, $z_2 = [u_2, v_2]$ in E, the duality pairing $\langle \cdot, \cdot \rangle$ is given by

$$\langle z_1, j_q^E \rangle := \langle u_1, j_q^X(u_2) \rangle + \langle v_1, j_q^Y(v_2) \rangle.$$

Then,

$(a.)$ *E is uniformly smooth and uniformly convex,*

$(b.)$ *j_q^E is single-valued duality mapping on E.*

Proof. $(a.)$ Let $p > 1$, $q > 1$. Let $x = [x_1, x_2]$, $y = [y_1, y_2]$ be arbitrary elements of E. Using condition $(iii)'$ of corollary 2^r in [119], We have that:

$$\begin{aligned}
&\langle x - y, j_q(x) - j_q(y) \rangle \\
&\quad = \Big\langle [x_1 - y_1, x_2 - y_2], \Big[j_q^X(x_1) - j_q^X(y_1), j_q^Y(x_2) - j_q^Y(y_2)\Big] \Big\rangle \\
&\quad = \Big\langle x_1 - y_1, j_q^X(x_1) - j_q^X(x_2) \Big\rangle + \Big\langle x_2 - y_2, j_q^Y(x_2) - j_q^Y(y_2) \Big\rangle \\
&\quad \leq g_1^*(\|x_1 - y_1\|) + g_2^*(\|x_2 - y_2\|).
\end{aligned}$$

Where $g_1^*,\ g_2^*$ are strictly increasing continuous and convex functions on $\mathbb{R}^+$ and $g_1^*(0) = g_2^*(0) = 0$. It follows that:

$$\left\langle x - y, j_q^E(x) - j_q^E(y) \right\rangle \leq g^*(\|x - y\|),$$

where $g^*(\|x-y\|) = g_1^*(\|x_1 - y_1\|) + g_2^*(\|x_2 - y_2\|)$. Hence the result follows from corollary $2'$ that E is uniformly smooth.
Also, using condition (iii) of corollary 3 in [119], we have that:

$$\begin{aligned}
&\langle x - y, j_p(x) - j_p(y) \rangle \\
&\quad = \left\langle [x_1 - y_1, x_2 - y_2], \left[j_p^X(x_1) - j_p^X(y_1), j_p^Y(x_2) - j_p^Y(y_2) \right] \right\rangle \\
&\quad = \left\langle x_1 - y_1, j_p^X(x_1) - j_p^X(x_2) \right\rangle + \left\langle x_2 - y_2, j_p^Y(x_2) - j_p^Y(y_2) \right\rangle \\
&\quad \geq g_1(\|x_1 - y_1\|) + g_2(\|x_2 - y_2\|).
\end{aligned}$$

Where $g_1,\ g_2$ are strictly increasing continuous and convex functions on $\mathbb{R}^+$ and $g_1(0) = g_2(0) = 0$. It follows that:

$$\left\langle x - y, j_p^E(x) - j_p^E(y) \right\rangle \geq g(\|x - y\|),$$

where $g(\|x-y\|) = g_1(\|x_1 - y_1\|) + g_2(\|x_2 - y_2\|)$. Hence the result follows from corollary 3 that E is uniformly convex. Since E is uniformly smooth, it is smooth and hence any duality mapping on E is single-valued.

$(b.)$ For arbitrary $x = [x_1, x_2] \in E$, let $j_q^E(x) = j_q^E[x_1, x_1] = \psi_q$. Then $\psi_q = [j_q^X(x_1), j_q^Y(x_2)] \in E^*$. We have that, for $p > 1$ such that $1/p + 1/q = 1$,

$$\begin{aligned}
\|\psi_q\|_{E^*} &= \left(\left\| \left[j_q^X(x_1), j_q^Y(x_2) \right] \right\| \right)^{1/p} = (\|j_q(x_1)\|_{X^*}^p + \|j_q(x_2)\|_{Y^*}^p)^{1/p} \\
&= (\|x_1\|_X^{(q-1)p} + \|x_2\|_Y^{(q-1)p})^{1/p} = (\|x_1\|_X^q + \|x_2\|_Y^q)^{(q-1)p} \\
&= \|x\|_E^{q-1}.
\end{aligned}$$

Hence, $\|\psi\|_{E^*} = \|x\|_E^{q-1}$. Furthermore,

$$\begin{aligned}
\langle x, \psi_q \rangle &= \left\langle [x_1, x_2], \left[j_q^X(x_1), j_q^Y(x_2) \right] \right\rangle = \left\langle x_1, j_q^X(x_1) \right\rangle + \left\langle x_2, j_q^Y(x_2) \right\rangle \\
&= \|x_1\|_X^q + \|x_2\|_Y^q = \left(\|x_1\|_X^q + \|x_2\|_Y^q \right)^{1/q} \left(\|x_1\|_X^q + \|x_2\|_Y^q \right)^{(q-1)/q} \\
&= \|x\|_E \cdot \|\psi\|_{E^*}^{q-1}.
\end{aligned}$$

Hence, j_q^E is a single-valued normalized duality mapping on E. □

We recall the following lemma which will be needed in what follows.

Lemma 4.3.4 (Browder, [16]). *Let X be a strictly convex reflexive Banach space with a strictly convex conjugate space X^*, T_1 a maximal monotone mapping from X to X^*, T_2 a hemicontinuous monotone mapping of all of X into X^* which carries bounded subsets of X into bounded subsets of X^*. Then, the mapping $T = T_1 + T_2$ is a maximal monotone map of X into X^*.*

Using lemma 4.3.4, we prove the following important lemma which will be used in the sequel.

Lemma 4.3.5. *Let E be a Banach space. Let $F : E \to E^*$ and $K : E^* \to E$ be bounded and maximal monotone mappings with $D(F) = D(K) = E$. Let $T : E \times E^* \to E^* \times E$ be defined by*

$$T[u, v] = [Ju - Fu + v, J_*v - Kv - u] \text{ for all } (u, v) \in E \times E^*,$$

then, the mapping $A := (J - T)$ is maximal monotone.

Proof. We show that the mapping $A = (J - T) : E \times E^* \to E^* \times E$ defined as

$$A[u, v] = [Fu - v, Kv + u]$$

is bounded maximal monotone. Let $S, T : E \times E^* \to E^* \times E$ be defined as

$$S[u, v] = [Fu, Kv], \quad T[u, v] = [-v, u].$$

Then, $A = S + T$. It suffices to show S, T are maximal monotone.

Observe that S is monotone and bounded (from the boundedness of F, K). Let $h = [h_1, h_2] \in E^* \times E$. Since F, K are maximal monotone, take $u = (J + \lambda F)^{-1}h_1$ and $v = (J_* + \lambda K)^{-1}h_2$. Then, $(J + \lambda S)w = h$, where $w = [u, v]$. Hence, S is maximal monotone.

Clearly, T is bounded and monotone. Furthermore it is continuous. Hence, it is hemi-continuous. Therefore by lemma 4.3.4, $A = S + T$ is maximal monotone and bounded. □

Lemma 4.3.6. *Let E be a uniformly convex and uniformly smooth real Banach space. Let $F : E \to E^*$ and $K : E^* \to E$ be monotone mappings with $D(F) = D(K) = E$. Let $T : E \times E^* \to E^* \times E$ be defined by $T[u, v] = [Ju - Fu + v, J_*v - Kv - u]$ for all $(u, v) \in E \times E^*$, then T is J-pseudocontractive. Moreover, if the Hammerstein equation $u + KFu = 0$ has a solution in E, then u^* is a solution of $u + KFu = 0$ if and only if $(u^*, v^*) \in F_E^J(T)$, where $v^* = Fu^*$.*

Proof. Using the monotonicity of F and K, we easily obtain that $\langle Tw_1 - Tw_2, w_1 - w_2\rangle \leq \langle Jw_1 - Jw_2, w_1 - w_2\rangle$ for all $w_1 = [u_1, v_1],\ w_2 = [u_2, v_2] \in E \times E^*$.

Moreover, we observe that

$$\begin{aligned}
& T(u^*, v^*) = J(u^*, v^*) \\
\iff\ & [Ju^* - Fu^* + v^*, J_*v^* - Kv^* - u^*] = [Ju^*, J_*v^*] \\
\iff\ & Ju^* - Fu^* + v^* = Ju^* \text{ and } J_*v^* - Kv^* - u^* = J_*v^* \\
\iff\ & v^* = Fu^* \text{ and } u^* + Kv^* = 0 \iff u^* + KFu^* = 0.
\end{aligned}$$

□

We now prove the following theorem.

Theorem 4.3.7. *Let E be a uniformly smooth and uniformly convex real Banach space and $F : E \to E^*$, $K : E^* \to E$ be maximal monotone and bounded maps, respectively. For $(x_1, y_1), (u_1, v_1) \in E \times E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^* respectively, by*

$$u_{n+1} = J^{-1}\left[Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju)\right], \ \ n \geq 1, \tag{4.3.8}$$

$$v_{n+1} = J_*^{-1}\left[Jv_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(J_*v_n - J_*y_1)\right], \ \ n \geq 1. \tag{4.3.9}$$

Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

Proof. From Lemma 4.3.6 we have that $T : E \times E^* \to E^* \times E$ defined by $T[u, v] = [Ju - Fu + v, J_*v - Kv - u]$ for all $(u, v) \in E \times E^*$ is J-pseudocontractive, and from Lemma 4.3.5, $A := (J - T)$ is bounded maximal monotone.

Applying theorem 4.0.25 where $X = E \times E^*$, from lemma 4.3.3, X is uniformly convex and uniformly smooth. We obtain (4.3.8) and (4.3.9) and the proof follows □

4.4 Application to convex optimization problem

The following lemma is well known (see e.g., [114], p.23, for similar proof in the Hilbert space case).

Lemma 4.4.1. *Let X be a normed space. Let $f : X \to \mathbb{R}$ be a convex function that is bounded on bounded subsets of X. Then, the subdifferential, $\partial f : X \to 2^{X^*}$ is bounded on bounded subsets of E.*

We now prove the following strong convergence theorem.

Theorem 4.4.2. *Let E be a uniformly convex and uniformly smooth real Banach space with dual E^*. Let $f : E \to (-\infty, \infty]$ be a lower semi-continuously Frèchet differentiable convex and bounded functional such that $(\partial f)^{-1}0 \neq \emptyset$. For given $u, x_1 \in E$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = J^{-1}\left[Jx_n - \lambda_n(\partial f)x_n - \lambda_n\theta_n(Jx_n - Ju)\right], \;\; n \geq 1. \tag{4.4.1}$$

Then, $\{x_n\}$ converges strongly to some $x^* \in (\partial f)^{-1}0$.

Proof. Since f is convex and bounded, we obtain that ∂f is bounded. By Rockafellar [101, 102] (see also, e.g., Minty [84], Moreau [85]), we have that (∂f) is maximal monotone mapping from E^* into E and $0 \in (\partial f)^{-1}v$ if and only if $f(v) = \min_{x\in E} f(x)$. Since f is convex and bounded, from lemma 4.4.1 we have that ∂f is bounded, hence, the conclusion follows from corollary 4.1.1 $\square$

REFERENCES

[1] R.P. Agarwal, M. Meehan and D. O'Regan, *Fixed point theory and applications*, vol. 141, Cambridge university press, 2001

[2] Y. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*. In *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type* (A. G. Kartsatos, Ed.), Marcel Dekker, New York (1996), pp. 15-50.

[3] Y. Alber and S. Guerre-Delabriere, *On the projection methods for fixed point problems*, Analysis (Munich), vol. 21 (2001), no. 1, pp. 17-39.

[4] Y. Alber and I. Ryazantseva, *Nonlinear Ill Posed Problems of Monotone Type*, Springer, London, UK, 2006.

[5] A. Asati, A. Singh and C.L. Parihar, *127 Years of fixed point theory-"A brief Survey of development of fixed point theory"*, International Journal Publications of Problems and Applications in Engineering Research (IJPAPER), Vol 04, Issue 01(2013), pp. 34-39.

[6] H. H. Bauschke Bruck, E. Matouskov and S. Reich; *Projection and Proximal Point Methods: convergence results and counterexeamples*, Nonlinear Anal. 56 (2004), 715-738.

[7] A. U. Bello, *Monotone operators and applications* , M.Sc. Thesis, African Univ. of Sci. and Tech.,(AUST), Abuja, 2011.

[8] P. Benilan, M.G. Crandall and A. Pazy, *Nonlinear evolution equations in Banach spaces [preprint]*, Besançon 1994.

[9] V. Berinde, *Iterative Approximation of Fixed points*, Lecture Notes in Mathematics, Springer, London, UK, 2007.

[10] V. Berinde, St. Maruster, I.A. Rus, *An abstract point of view on iterative approximation of fixed points of nonself operators,* J. Nonlinear Convex Anal. 15 (2014), no. 5, 851-865.

[11] H. Brezis, *Monotone Operators, Nonlinear Semigroups and applications, Proc.. of the Int. Cong. of Math.* Vancouver, (1974), 249-255.

[12] J.M. Borwein, *Maximality of sums of two maximal monotone operators in general Banach space*, Proc. Amer. Math. Soc., vol. 135 (2007), pp. 3917-3924.

[13] H. Brézis and F. E. Browder; *Nonlinear integral equations and systems of Hammerstein type*, Bull. Amer. Math. Soc. 82 (1976), 115-147.

[14] F. E. Browder, *Nonlinear mappings of nonexpansive and accretive-type in Banach spaces*, Bull. Amer. Math. Soc. vol. 73 (1967), 875-882.

[15] F. E. Browder, *Nonlinear equations of evolution and nonlinear accretive operators in Banach spaces*, Bull. Amer. Math. Soc. vol. 73 (1967), 875-882.

[16] F. E. Browder, *Existence and perturbation theorems for nonlinear maximal monotone operators in Banach spaces*, Bull. Amer. Math. Soc. Volume 73, Number 3 (1967), 322-327.

[17] R. E. Bruck Jr, *A strongly convergent iterative solution of* $0 \in U(x)$ *for a maximal monotone operator* U *in Hilbert spaces*, J. Math. Anal. Appl., vol. 48 (1974), 114-126.

[18] Y. Censor and S. Riech, *Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization*, Optimization, vol. 37, no. 4 (1996), pp. 323-339.

[19] C.E. Chidume and N. Djitte, *Strong convergence theorems for zeros of bounded maximal monotone nonlinear operators*, Abstract and Applied Analysis, Volume 2012, Article ID 681348, 19 pages, doi:10.1155/2012/681348.

[20] C.E. Chidume, J.N. Ezeora, *Krasnoselskii-type Algorithm for family of multi-valued strictly pseudo-conttractive mappings,* Fixed Point Theory Appl. 2014, 111(2014).

[21] C.E. Chidume, *On the approximation of fixed points of nonexpansive mapping*, Houston J. math. 7 (1981), 345-554.

[22] C. E. Chidume. *Geometric Properties of Banach Spaces and Nonlinear iterations*, vol. 1965 of Lectures Notes in Mathematics, Springer, London, UK, 2009.

[23] C. E. Chidume. *Strong convergence theorems for bounded accretive operators in uniformly smooth Banach spaces*, Contemporary Mathematics,vol. 659, Nonlinear Analysis and Optimization,(B. S. Mordukhovich, S. Reich, A. J. Zaslavski), AMS, Providence, RI, 2016.

[24] C.E. Chidume, *The iterative solution of the equation* $f \in x + Tx$ *for a monotone operator* T *in* L^p *spaces*, J. Math. Anal. vol 116 (1986), no. 2, 531-537.

[25] C. E. Chidume, *An approximation method for monotone Lipschitzian operators in Hilbert-spaces*, Journal of the Australian Mathematical Society, series A-pure mathematics and statistics, vol. 41 (1986), pp. 59-63.

[26] C.E. Chidume, *Iterative approximation of fixed points of Lipschitz pseudocontractive maps*, Pro. Amer. math. Soc. 129 (2001), no. 8, 2245-2251.

[27] C.E. Chidume and S.A. Mutangadura, *An example of Mann iteration method for Lipschitz pseudocontractions*, Pro. Amer. math. Soc. 129 (2001), no. 8, 2359-2363.

[28] C.E. Chidume and H. Zegeye, *Approximate fixed point sequences and convergence theorems for Lipschitz pseudocontractive maps*, Pro. Amer. Math. Soc. 132 (2004), no. 3, 831-840.

[29] C. E. Chidume and M. O. Osilike, *Iterative solutions of nonlinear accretive operator equations in arbitrary Banach spaces*, Nonlinear Analysis-Theory Methods & Applications, vol. 36 (1999), pp. 863-872.

[30] C.E. Chidume, M.O. Osilike; Iterative solution of nonlinear integral equations of Hammerstein-type, J. Nigerian Math. Soc. 11 (1992) 9-18 (MR96c:65207).

[31] C.E. Chidume, H. Zegeye; *Iterative approximation of solutions of nonlinear equation of Hammerstein-type*, Abstr. Appl. Anal. 6 (2003) 353-367.

[32] C.E. Chidume, H. Zegeye; *Approximation of solutions of nonlinear equations of monotone and Hammerstein-type,* Appl. Anal. 82 (8) (2003) 747-758.

[33] C.E. Chidume, H. Zegeye; *Approximation of solutions of nonlinear equations of Hammerstein-type in Hilbert space,* Proc. Amer. Math. Soc. 133 (3) (2005) 851-858.

[34] C.E.Chidume, C.O.Chidume and A.U.Bello, *An algorithm for computing zeros of generalized phi-strongly monotone and bounded maps in classical Banach spaces*, Optimization, vol. 65 (2016), no.4, 827-839, DOI:10.1080/02331934.2015.1074686.

[35] C. E. Chidume, A.U. Bello, *An iterative algorithm for approximating solutions of Hammerstein equations with monotone maps in Banach spaces*, Appl. Math. Comp., 2015, to appear.

[36] C.E. Chidume and Y. Shehu, *Approximation of solutions of equations of Hammerstein type in Hilbert spaces*, Fixed Point Theory, Vol. 16, no. 1, (2015), pp. 91-102.

[37] C.E. Chidume and Y. Shehu, *Iterative approximation of solutions of generalized equations of Hammerstein type*, Fixed Point Theory 15 (2014), no. 2, 427-440.

[38] C. E. Chidume, N. Djitte, *Iterative method for solving nonlinear integral equations of Hammerstein type*, Applied Mathematics and Computation, (Elsevier), 219 (2013). 5613-5621.

[39] C.E. Chidume and Y. Shehu, *Iterative approximation of solution of equations of Ham- merstein type in certain Banach spaces*, Applied Mathematics and Computation (Elsevier), 219 (2013) 5657-5667.

[40] C.E. Chidume and Y. Shehu, *Strong convergence theorem for approximation of solutions of equations of Hammerstein type*, Nonlinear Analysis (Elsevier), 75 (2012), 5664-5671.

[41] C.E. Chidume and Y. Shehu, *Approximation of solutions of generalized equations of Hammerstein type*, Computer Math. Appl. (Elsevier), 63 (2012) 966-974.

[42] C.E. Chidume and N.Djitte, *Convergence theorems for solutions Hammerstein equations with accretive type nonlinear operators*, PanAmer. Math. J. Vol. 22 (2012), no. 2, 19 -29.

[43] C.E. Chidume and N. Djitte, *Approximation of solutions of nonlinear integral equations of Hammerstein type*, International Scholarly Research Network: ISRN Mathematical Analysis, volume 2012, Article ID 169751, 12 pages, doi:10.1155/2012/16975.

[44] C.E. Chidume and E.U. Ofoedu, *Solution of nonlinear integral equations of Hammerstein type*, Nonlinear Analysis, (Elsevier), Vol. 74(13), (2011) 4293-4299.

[45] C.E. Chidume, N. Djitte, *Approximation of solutions of Hammerstein equations with bounded strongly accretive nonlinear operators*, Nonlinear Analysis (Elsevier), Vol.70, no. 11(2009) 4071-4078.

[46] C.E. Chidume, N. Djitte, *Iterative approximation of solutions of nonlinear equations of Hammerstein type*, Nonlinear Analysis (Elsevier), Vol.70, no. 11(2009), 4086-4092.

[47] C.E. Chidume, *Strong convergence and stability of Picard iteration sequences for a general class of contractive-type mappings*, Fixed Point Theory and Applications 2014, 2014:233.wel

[48] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62, Kluwer Academic Publishers, 1990.

[49] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. 40 (1936), 396-414.

[50] K. Deimling, *Zeros of accretive operators*, Manuscripta Math. 13 (1974), 365-374.

[51] C. Diop, T.M.M. Sow, N. Djitte and C.E. Chidume, *Constructive techniques for zeros of monotone mappings in certain Banach spaces*, SpringerPlus, vol. 4(2015), No. 1.

[52] N. Djitte and M. Sene, *An iterative algorithm for approximating solutions of Hammerstein integral equations*, Numerical Functional Analysis and Optimisation, vol. 34(2012): 1299-1316.

[53] N. Djitte and M. Sene, *Iterative solution of nonlinear integral equations of Hammerstein type with Lipschitz and accretive operators*, ISRN Appl. Math., (2012), Article ID 963802, 15 pages, DOI:5402/2012/963802.

[54] M. Edelstein, *A remark on a theorem of Krasnoselkii*, Amer. Math. Monthly, 13(1966), 507-510.

[55] M. Edelstein, *On nonexpansive mapping*, Pro. Amer. math. Soc. 15 (1964), 689-695.

[56] . Edelstein and R.C. O'Brian, *Nonexpansive mappings, asymptotic regularity and successive approximation*, J. London math. Soc. 17(1978), no.3, 547-554.

[57] S. Fitzpatrick, *Representing monotone operators by convex functions*, Workshop/Miniconference on Functional Analysis and Optimization (Canberra, 1988), 59–65, Proc. Centre Math. Anal. Austral. Nat. Univ., 20, Austral. Nat. Univ., Canberra, 1988.

[58] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics 28, CUP, Cambridge, New York, Port Chester, Melbourne, Sydney, 1990.

[59] S. Gudder and D. Strawther, *Strictly convex normed linear spaces*, Proceedings of the American Mathematical Society, Vol. 59, No. 2 (Sep., 1976), pp. 263 267.

[60] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim. **29**(1991) 403-419.

[61] S. Ishikawa, *Fixed points and iteration of nonexpansive mapping in a Banach space*, Pro. Amer. math. Soc. 73 (1976), 61-71.

[62] S. Ishikawa, *Fixed points by a new iteration method*, Pro. Amer. math. Soc. 44 (1974), no.1, 147-150.

[63] T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan 19 (1967), 508-520.

[64] R. I. Kačurovskii, *On monotone operators and convex functionals*, Uspekhi Mathematicheskikh Nauk, vol. 15 (1960), no. 4, pp. 213-215.

[65] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAMJ. Optim., vol. 13 (2002), no. 3, pp. 938-945.

[66] T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan, vol. 19 (1967), pp. 508-520.

[67] M.A. Khamsi, *Introduction to metric fixed point theory*, International Workshop on Nonlinear Functional Analysis and its Applications, Shahid Beheshti University, Iran, January 20-24, 2002.

[68] M. A. Khamsi and W. M. Kozlowski, *Fixed point theory in modular function spaces*, Birkhäuser Basel, 2015.

[69] H. Khatibzadeh and G. Moroşanu, *Strong and weak solutions to second order differential inclusions governed by monotone Operators*, Set-Valued and Variational Analysis, Vol. 22 (2014),Issue 2, pp 521-531.

[70] H. Khatibzadeh and A. Shokri, *On the first- and second-order strongly monotone dynamical systems and minimization problems*, Optimization Methods and Software Vol. 30 (2015) Issue 6, p. 1303-1309.

[71] K. Kido, *Strong convergence of resolvents of monotone operators in Banach spaces*, Proc. Amer. Math. Soc., vol. 103 (1988), no. 3,pp. 755-7588.

[72] S. Kamimura and W. Takahashi; *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optimization **13**(3) (2003), 938-945.

[73] M.A. Krasnoselkii, *Two observations about the method of successive approximations*, Uspehi Math. Nauk 10 (1957), Abt. 1, 131-140.

[74] , J. B. Lee, *Topological fixed point theory*, Asia Pacific Mathematics Newsletter Vol. 3 No. 3(2013).

[75] N. Lehdili and A. Moudafi; *Combining the proximal algorithm and Tikhonov regularization*, Optimization **37**(1996), 239-252.

[76] J. Lindenstrauss and W. B. Johnson, *Handbook of the geometry of Banach spaces* Vol 1, Elsevier (2001)

[77] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II: Function Spaces*, Ergebnisse Math. Grenzgebiete Bd. **97**, Springer-Verlag, Berlin, 1979.

[78] B. Liu, *Fixed point of strong duality pseudocontractive mappings and applications*, Abstract and Applied Analysis Vol 2012, Article ID 623625, 7 pages, doi:10.1155/2012/623625

[79] E. Llorens Fuster *Some moduli and constants related to Metric Fixed Point Theory*, in Handbook of Metric Fixed Point Theory, W.A. Kirk and B. Sims eds. Kluwer, 2002. 133–175. (MR.2003j:46019). (Zbl. 1021. 47030).

[80] M-accretive-operator. A.G. Kartsatos (originator), Encyclopedia of Mathematics. URL: http://www.encyclopediaofmath.org/index.php?title=M-accretive-operator&oldid=17391.

[81] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., vol. 4 (1953), pp. 506-510.

[82] B. Martinet.; *Régularisation ďinéquations variationnelles par approximations successives*, Revue française ďinformatique et de Recherche Opérationnelle, vol. 4 (1970), pp. 154-158.

[83] R. E. Megginson, *An Introduction to Banach space theory*, Springer-Verlag New york 1998.

[84] G. J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J, vol. 29 (1962), no. 4, pp. 341-346.

[85] J.J. Moreau, *Proximité et dualité dans un espace Hilbertien*, Bull. Soc. Math., France, 93(1965), pp. 273-299.

[86] A. Moudafi, *Proximal methods for a class of bilevel monotone equilibrium problems,* J. Global Optim. 47 (2010), no. 2, 45-52.

[87] A. Moudafi and M. Thera, *Finding a zero of the sum of two maximal monotone operators,* J. Optim. Theory Appl. 94 (1997), no. 2, 425-448.

[88] D. Pascali and S. Sburian, *Nonlinear mappings of monotone type*, Editura Academia Bucuresti, Romania, 1978.

[89] D. O'Regan,N. Sahzad and R.P. Agarwal, *Random fixed point theory in spaces with two metrics*, Journal of Applied Mathematics and Stochastic Analysis,16:2 (2003), 171-176

[90] S. Reich, *Constructive techniques for accretive and monotone operators*, Applied non-linear analysis, Academic Press, New York (1979), pp. 335-345.

[91] S. Reich, *The range of sums of accretive and monotone operators*,J. Math. Anal. Appl. 68 (1979), no. 1, 310–317.

[92] S. Reich, *A weak convergence theorem for the alternating methods with Bergman distance*, in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in Lecture notes in pure and Appl. Math., vol. 178 (1996), Dekker, New York. pp. 313-318.

[93] S. Reich, *Iterative methods for accretive sets in Banach Spaces,* Academic Press, New York, 1978, 317-326.

[94] S. Reich, *Product of formulas, nonlinear semigroups and accretive operators,* J. Funct. Anal. 36 (1980), 147-168.

[95] S. Reich.; *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, Journal od Mathematical Analysis and Applications, (1980) vol. 75, no. 1, pp. 287-292.

[96] S. Reich and S. Sabach, *A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces*, Journal of Nonlinear and Convex Analysis, vol. 10 (2009), no. 3, pp. 471-485.

[97] S. Reich and S. Sabach, *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numerical Funtional Analysis and Optimization, vol. 31 (2010), no. 1-3, pp. 22-44.

[98] S. Reich and D. Shoikhet, *Nonlinear semigroups, fixed points, and geometry of domains*, Imperial College Press, London, 2005

[99] J. Reinermann, *Uber Fipunkte kontrahierender Abbildungen und schwach konvergente Toeplitz Verfahren,* Arch. Math. 20 (1969), 59-64.

[100] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM Journal on Control and Optimization, vol. 14, (1976), no. 5, pp. 877-898.

[101] R. T. Rockafellar, *Characterization of subdifferentials of convex functions*, Pacific J. Math., vol. 17, (1966), pp. 497-510.

[102] R. T. Rockafellar, *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math., vol. 33, (1970), pp. 209-216.

[103] R.T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., vol. 149(1970), 75–88.

[104] H. Schaefer, *Ber die Methode Sukzessiver Approximation,* (German) Jber. Deutsch Math. Verein 59 (1957), Abt. 1, 131-140.

[105] Schu. *Iterative construction of fixed points of asymptotically nonexpansive mapping*, J. Math. Anal. Appl. 158 (1991), 407-413.

[106] R.E. Showalter, *Monotone operators in Banach spaces and nonlinear partial differential equations*, Mathematical Surveys and Monographs, vol. 49 (1997), AMS.

[107] Z. Smith, *Fixed point methods in nonlinear in nonlinear analysis*, in Directions for Mathematics Research Experience for Undergraduates, M. A. Peterson, Y. A. Rubinstein eds. World Scientific, 2015.

[108] M.V. Solodov and B.F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilber space*, Math. Program., Ser. A 87 (5000) 189-202.

[109] T.M.M. Sow and N. Djitte, *Nonlinear integral equations of Hammerstein type with phi-monotone mappings in certain Banach spaces*, Journal of Convex Analysis, August, 2015.

[110] T.M.M. Sow and N. Djitte, *Hammerstein equations involving monotone operators in classical Banach spaces: On Chidume's open problem*, SpringerPlus, 2015.

[111] T.M.M. Sow, C. Diop and N. Djitte, *Algorithm for Hammerstein equations with monotone mappings in certain Banach spaces*, Creat. Math. Inform., vol. 25(2016), No. 1, 101-114.

[112] A. Stringa, *On strictly convex and strictly convex according to an index semi-normed vector spaces*, Gen. Math. Notes, Vol. 4, No. 2(2011), pp.10-22.

[113] W. Takahashi, *Proximal point algorithms and four resolvents of nonlinear operators of monotone type in Banach spaces*, Taiwanese J. of Math., vol. 12 (2008), no. 8, 1883-1910.

[114] V. V. Vasin and I. I. Eremin, *Operators and Iterative Processes of Fejer Type: Theory and Applications (Inverse and III-Posed Problems Series)[1 ed.]*, Walter de Gruyter, Berlin, 2009.

[115] V. Volpert, *Elliptic Partial Differential Equations: Volume 2: Reaction-Diffusion Equations*, Volume 104 of Monographs in Mathematics, Springer, 2014.

[116] K. William and N. Shahzad, *Fixed point theory in distance spaces*, Springer Verlag, 2014.

[117] H.K. Xu.; *Iterative algorithms for nonlinear operators*, Journal of the London Mathematical Society *II*, (2002) vol. 66, no. 1, pp.240-256.

[118] H.K. Xu, *A regularization method for the proximal point algorithm*, Journal of Global Optimization, vol. 36 (2006), no. 1, pp. 115-125

[119] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., Vol. 16 (1991), no. 12, pp. 1127-1138. 1991.

[120] E. H. Zarantonello, *Solving functional equations by contractive averaging*,Tech. Rep. 160, U. S. Army Math. Research Center, Madison, Wisconsin, 1960.

[121] H. Zegeye, *Strong convergence theorems for maximal monotone mappings in Banach spaces*, J. Math. Anal. Appl. 343 (2008) 663–671.

[122] E. Zeidler, *Nonlinear Functional Analysis and its Applications Part II: Monotone Operators*, Springer-Verlag, Berlin, 1985.

[123] E. Zeidler, *Nonlinear Functional Analysis and its Applications Part I: Fixed-Point Theorems*, Springer-Verlag, Berlin, 1986.