A naive finite difference approximations for singularly perturbed parabolic reaction-diffusion problems

A Thesis presented to the Department of Pure and Applied Mathematics
African University of Science and Technology, Abuja
In partial fulfilment of the requirements for

MASTER DEGREE IN PURE AND APPLIED MATHEMATICS
By
Nnakwe Monday Ogudu

Supervised by
Prof. Jules Djoko Kamdem
University of Pretoria

African University of Science and Technology
www.aust.edu.ng
P.M.B 681, Garki, Abuja F.C.T
Nigeria.

June 7, 2016

Certification

A NAIVE FINITE DIFFERENCE APPROXIMATIONS FOR SINGULARLY PERTURBED PARABOLIC REACTION-DIFFUSION PROBLEMS

A THESIS APPROVED

BY

DEPARTMENT OF PURE AND APPLIED MATHEMATICS

RECOMMENDED:

....

Supervisor: PROF. J.K.DJOKO

....

H.O.D, Pure and Applied Mathematics:
PROF. C.E. CHIDUME (FTWAS, FAS, FNMS)

APPROVED:

....

Vice President (Academic)

....

Date

Abstract

A naive finite difference approximations for singularly perturbed parabolic reaction-diffusion problems

In this thesis, we treated a Standard Finite Difference Scheme for a singularly perturbed parabolic reaction-diffusion equation. We proved that the Standard Finite Difference Scheme is not a robust technique for solving such problems with singularities. First we discretized the continuous problem in time using the forward Euler method. We proved that the discrete problem satisfied a stability property in the $l_\infty - norm\ and\ l_2 - norm$ which is not uniform with respect to the perturbation parameter, as the solution is unbounded when the perturbation parameter goes to zero. Error analysis also showed that the solution of the SFDS is not uniformly convergent in the discrete $l_\infty - norm$ with respect to the perturbation parameter, (i.e., the convergence is very poor as the parameter becomes very small). Finally we presented numerical results that confirmed our theoretical findings.

Declaration

I, Nnakwe Monday Ogudu know the meaning of plagiarism and I declare that all of the work in the document, save for that which is properly acknowledged is my own.

Acknowledgement

First and foremost, I owe God a big thank you for the life I live today and making this program a huge success. For all He did, I can't thank Him enough.To my supervisor, **Professor Jules Djokokamdem**, I can't thank you enough for your constructive criticisms, guidance, patience, support and encouragement through out this thesis, Prof you are wonderful. I'm highly indebted to the management board of **Nelson Mandala Institution (NMI)**, for the scholarship she offered me which has greatly boosted my academic life.

I would like to express my sincere gratitude to the vice president Academic, African University of Science and Technology, Abuja and **HOD** institute of mathematics **Professor C.E. Chidume** for his leadership in the institute and for the intense/intensive knowledge he has impacted in me. *I wish to call him grandpa.* To all my lecturers; *Prof. Khalil Ezzinbi, Prof. Ngalla Djitte, Prof. Asha Wade, Prof. Gane Samb Lo, and a host of other Professors; discussing with you was very inspiring.* My humble gratitude goes to Prof A.U Osisiogu and Dr Jerry Ezeora whose reassuring presence in my life provided me with great strength to challenges and to strive higher. My special thanks to all the formal PHD students, *Mr Uche, Dr Maduabuchi, Dr Patricks and Dr Ushman for always devoting your time to help when the need arises.* A special thanks to the staff of AUST, especially Miss Udigwe Amaka for her administrative support. I call her **sis**. To all my classmates: Idu kennedy Obinna, Ken Nwigbo, Uba Markjoe, Uzorchukwu Mark, Aroh Innocent, Ogonna Michael, Onyekwere Gulfment and our little bride Nnyaba Ukamaka Victoria for all the everyday struggle we have been through. A special thank you to *Mrs Adedoyin Adegoke and Ndufor Korande* for your sincere humility and assistance during this adventure. To these special ones, *Ifebude Barnabas, Ojewale Mubarack, Abiola Saka and Ogonna Michael*, you guys are wonderful. To all my seniors and friends: Mr A. Friday Lawrence, Otubo E. Ezzaka, Ezea Chinedu, Izuchukwu Chinedu, Td, Ukwueze Kc, Eribo, Escober, Gbodo, N. Elshadai, N. Sunday, Jagaras, Ayokunle, Iyoke P., Ugwoke Blessing, Okpala Ifunanya,$\cdots$ God bless you. To *Engr. N. Christian Ukwueze and family,* ***Big Sis (Roseline)****, Big malle, Mama Jerry, Gold, Angel* etc.,God bless you greatly. A special thank you to my lovely parents, **Mr and Mrs Nnakwe Ogudu**, my siblings and the rest of my family members for standing by me in all this times. I will ever remain thankful for all you cares. Special gratitude to my mentor and friend his blessed memory, **Late Mr Okorchukwu Martins** who taught me how to burn my candle at nights, *"I wish the dead could see". Daddy I love you.* Thanks to all and sundry that I fail to mention here.

Nnakwe Monday Ogudu.

Dedication

This work with every humility of heart is dedicated to my lovely parents Mr and Mrs Nnakwe Ogudu and my siblings.

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CHAPTER 1

Introduction

This work falls within the general areas of numerical methods for partial differential equations (PDE), an area which prominent mathematicians have explored due to its diverse applications in numerous fields of sciences. This is evident since most D.Es can not be solved analytically, thus the method gives us useful insights into the solutions of the D.Es without necessarily solving them analytically.

1.1 Formulation of the problem

Standard Finite Difference Scheme is one of the most frequently used methods for solving differential equations numerically. To this end, we study a naive finite difference approximations for singularly perturbed parabolic reaction-diffusion problems. The governing equation of the problem is given by:

$$\begin{cases} u_t - \varepsilon u_{xx} + b(x,t)u = f(x,t) \quad (x,t) \in Q = \Omega \times (0,T] \\ u(x,0) = 0 \quad x \in \overline{\Omega} = [0,1] \\ u(0,t) = u(1,t) = 0 \ t \in (0,T], \end{cases} \tag{1.1}$$

where $b(x,t) \geq \beta > 0$ for all $(x,t) \in \overline{\Omega} \times [0,T]$, ε is the positive perturbation parameter and $f(x,t)$ is the external force. The diffusion term is u_{xx}, while the reaction term is $b(x,t)u$. The problem (1.1) is generally called singularly perturbed partial differential equation because of the small parameter ε in front of the second order derivative term in space u_{xx}. Thus (1.1) *is one in which a small positive per−turbation parameter* ε *is multiplied to the highest derivative term in the equa−tion of the problem.* Problems of these nature are well known in the literature of partial differential equations as they constitute an element of interest in the area of population dynamics and chemical reactors, and their numerical analysis is hard because of the presence of singularity when ε *goes to* 0. The existence and uniqueness result of (1.1) is well developed (see [4]). The objective of this thesis is to show that a naive numerical methods for (1.1) fails when ε *goes to* 0. To have an insight into the study, if one takes the stationary problem (as in [5])

$$\begin{cases} \varepsilon\psi'' + \psi' = \frac{1}{2} \quad 0 < x < 1 \ 0 < \epsilon << 1, \\ \psi(0) = 0, \\ \psi(1) = 1. \end{cases} \tag{1.2}$$

The exact solution of (1.2) is

$$\psi(x, \epsilon) = \frac{1 - \exp^{-\frac{x}{\epsilon}}}{2(1 - \exp^{-\frac{1}{\epsilon}})} + \frac{x}{2}.$$

Thus the solution as

$$\lim_{\epsilon \to 0} \psi(x, \epsilon) = \frac{1 + x}{2} = \psi_0(x)$$

does not live in $C^2[0, 1]$ since $\psi_0(x)$ does not satisfy the boundary condition at $x = 0$. So we infer that the solution is badly behaved.
In Chapter 2, we introduced the notion of the classical SFD approximation accompanied with some basic definitions and results. Then we formulated the classical SFD schemes for (1.1), an elegant proof of the existence and uniqueness of the solution of the discrete problem was presented.
In Chapter 3, we investigated the consistency and stability of the schemes of the continuous problem (1.1). It turned out that the stability was not uniform with respect to the perturbation parameters ε.
In Chapter 4, we studied the convergence of the schemes to our continuous problem (1.1). It turned out that the convergence was very poor as ε goes to zero. Basically this is why the classical SFDM failed to approximate (1.1), it had no control over ε and it found itself in damaging position.
In Chapter 5, computer programs were written and simulated for the several cases of interest and the numerical investigations corroborated with our theoretical findings.

CHAPTER 2

Numerical Schemes

The goal of this chapter is the following: first, to formulate the finite difference approximation associated with (1.1), we also state some preliminaries relating to the problem and finally discuss the existence and uniqueness of solution associated with (1.1).

2.1 Finite difference approximations of (1.1)

We first recall basic definitions and results for the definition of finite difference approximation.

The definitions and results we recall in this paragraph are standard and can be found in [1, 2, 3].
The fundamental idea of almost any numerical method for solving equations of the form (1.1) is to approximate the differential equation by a system of algebraic equations. The system of algebraic equations is set up in a way as to produce a "good solution" of the differential equation. The way to generating such a system is to replace the derivatives in the equation by finite differences. In fact for (1.1) we need to approximate u_t and u_{xx} departing from the classical definition of the derivative

$$u'(x) = \lim_{h\to 0} \frac{u(x+h) - u(x)}{h}.$$

This clearly indicates that in order to get a good approximation, h must be very small. Hence to compute good solutions, we need h approaching zero. This in

turn implies that we need to solve large system of algebraic equations.The solution is computed at the endpoints and determined by the differential equation in the interior solution domain.
The first step in deriving a finite difference approximation of (1.1) is to partition the unit domain $[0, 1]$ into a finite number of sub-intervals. We introduce the grid points $\{x_i\}_{i=0}^{N+1}$ given by $x_i = ih$, where $i \geq 1$ is an integer and the spacing h is the distance between two consecutive points, that is $h = x_{i+1} - x_i$. As we assume that the sub-division is uniform, then $h = 1/(N + 1)$. Typically N will be large so that h tends towards zero.
The second step, in the construction of finite difference scheme is the discrete operator. The following operators will be used throughout the work

$$D^+\phi(x) = \frac{1}{h}(\phi(x+h) - \phi(x)), \tag{2.1}$$

$$D^-\phi(x) = \frac{1}{h}(\phi(x) - \phi(x-h)). \tag{2.2}$$

The operator D^+ is called "forward", and D^- backward. From the definition of derivative, it is clear that if ϕ is twice differentiable, then $D^+\phi$ and $D^-\phi$ tend towards ϕ_x when $h \to 0$.

Definition 2.1. *An operator $\widetilde{A}_h$ is said to be a consistent approximation of A with respect to the discretization parameter h if*

$$\widetilde{A}_h\phi - A\phi \to 0 \quad \text{if} \quad h \to 0.$$

If furthermore the error $|\widetilde{A}_h\phi - A\phi|$ committed is bounded, up to a multiplicative constant, by h^p, then the approximation is said to be consistent of order p.

We easily verify that

Lemma 2.2. *1- If ϕ is twice differentiable on $[x, x+h]$, then $D^+\phi$ is a consistent approximation of order one of $\partial_x\phi$.*
2- If ϕ is twice differentiable on $[x-h, x]$, then $D^-\phi$ is a consistent approximation of order one of $\partial_x\phi$.

Next, we claim that

Lemma 2.3. *If ϕ is four times continuously differentiable on $[x-h, x+h]$, then*

$$D^-D^+\phi(x) = \frac{1}{h^2}\left[\phi(x+h) - 2\phi(x) + \phi(x-h)\right]$$

is a consistent, second order, approximation to ϕ_{xx}

Proof. . From Taylor's expansion,

$$\phi(x+h) = \phi(x) + h\phi'(x) + \frac{h}{2}\phi^{(2)}(x) + \frac{h^3}{6}\phi^{(3)}(x) + \frac{h^4}{24}\phi^{(4)}(x_1), \tag{2.3}$$

with $x_1 \in (x, x+h)$. Likewise

$$\phi(x-h) = \phi(x) - h\phi'(x) + \frac{h}{2}\phi^{(2)}(x) - \frac{h^3}{6}\phi^{(3)}(x) + \frac{h^4}{24}\phi^{(4)}(x_2), \tag{2.4}$$

with $x_2 \in (x-h, x)$ Adding (2.3) to (2.4) gives

$$\frac{\phi(x+h) - 2\phi(x) + \phi(x-h)}{h^2} - \phi^{(2)}(x) = \frac{h^2}{24}\left[\phi^{(4)}(x_1) + \phi^{(4)}(x_2)\right]. \tag{2.5}$$

To continue, we need the following result

Lemma 2.4. *(Mean Value for sum). Let g_i, be a function defined on* $(\min x_i, \max x_i)$ *such that*

(a) $g_i \geq 0$,

(b) $\sum_i g_i = 1$.

Let f be a function defined on $(\min x_i, \max x_i)$.
Then there is $z \in (\min x_i, \max x_i)$ such that for $y_i \in (\min x_i, \max x_i)$

$$\sum_i g_i(y_i) f(y_i) = f(z)$$

In (2.5), we take $g_1 = g_2 = 1/2$, and application of Lemma 2.4 leads to the existence of $z \in (x-h, x+h)$ such that

$$\frac{\phi(x+h) - 2\phi(x) + \phi(x-h)}{h^2} - \phi^{(2)}(x) = \frac{h^2}{12}\phi^{(4)}(z). \tag{2.6}$$

We also recall the following result

Lemma 2.5. *If $f : [0,1] \to \mathbb{R}$ is continuous,then*

(a) *f is bounded*

(b) *f attains its minimum at some point in* $[0, 1]$

(c) *f attains its maximum at some point in* $[0, 1]$

We then deduce that

$$|D^+D^-\phi(x) - \phi^{(2)}(x)| \leq \frac{h^2}{12} \sup_{x-h<z<x+h} \left|\phi^{(4)}(z)\right|.$$

Hence the approximation is consistent of order 2. It is noted that since $\phi^{(4)}$ is continuous, the maximum is well defined due to Lemma 2.5.

Remark 2.6. *By combining in different ways the operators D^+, D^-, it is possible to construct approximations to a partial derivative of any order; some are obviously better that others, meaning that their consistency order is higher.*

With the discrete operators D^+, D^-, and D^+D^-, we can define a finite difference approximation of (1.1). Since we have a space-time dependent problem, we have to approximate the time (here t) and the space (here x). After the partition of the space domain, we now partitioned the time domain $[0, T]$. We divide it into $M - 1$ sub-interval of equal length $k = t_{n+1} - t_n = T/M$. From now and the rest of this work we let

$$\begin{aligned} u_i^n &\approx u(x_i, t_n), \\ f_i^n &= f(x_i, t_n). \end{aligned} \tag{2.7}$$

Definition 2.7. *A scheme is said to be explicit if the solution at one time step can be computed directly from the solution at the previous time step. On the other hand, we call the scheme implicit if the solution at the next time level is obtained by solving a system of equations.*

Based on (2.7), and Lemma 2.3, we have the following approximations:

$$\begin{aligned} u_t(x_i, t_{n+1}) = u_t(x_i, t_n + k) &\approx \frac{u(x_i, t_n + k) - u(x_i, t_n)}{k} \approx \frac{u_i^{n+1} - u_i^n}{k}, \\ u_{xx}(x_i, t_n) \approx D^+D^-u(x_i, t_n) &\approx \frac{1}{k^2}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right], \\ & \text{or} \\ u_{xx}(x_i, t_{n+1}) \approx D^+D^-u(x_i, t_{n+1}) &\approx \frac{1}{k^2}\left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right]. \end{aligned} \tag{2.8}$$

With (2.8), the finite difference scheme approximating (1.1) is defined as follows:

$$\text{Explicit scheme}\quad\begin{cases}\text{Knowing } u_i^n,\ \text{find } u_i^{n+1} \text{ such that,}\\ \dfrac{u_i^{n+1}-u_i^n}{k} - \varepsilon D^+D^-u_i^n + b(x_i,t_n)u_i^n = f(x_i,t_n)\ ,\\ u_i^0 = 0 \ \text{ for all } i=0,1,2,\cdots N+1,\\ u_0^n = u_{N+1}^n = 0 \text{ for all } n=1,2,3,\cdots,M.\end{cases}\tag{2.9}$$

or

$$\text{Implicit scheme}\quad\begin{cases}\text{Knowing } u_i^n,\ \text{find } u_i^{n+1} \text{ such that,}\\ \dfrac{u_i^{n+1}-u_i^n}{k} - \varepsilon D^+D^-u_i^{n+1} + b(x_i,t_{n+1})u_i^{n+1} = f(x_i,t_{n+1})\ ,\\ u_i^0 = 0 \ \text{ for all } i=0,1,2,\cdots N+1,\\ u_0^n = u_{N+1}^n = 0 \text{ for all } n=1,2,3,\cdots,M.\end{cases}\tag{2.10}$$

According to definition 2.7, (2.9) is an explicit scheme since one can solve for u_i^{n+1} in term of u_i^n, while (2.10) is an implicit scheme.
Regarding the discrete schemes (2.9) and (2.10), two fundamental questions arise:

(a) does the discrete problem admit a unique solution?

(b) is the method convergent, that is does it hold that

$$\|u(x_i,t_n) - u_i^n\| \to 0, \quad when\ h \to 0\ ?$$

The first question, will be answered in the next paragraph while the second question will constitute the objective of the next Chapter, moreover, we want to see the influence of ε on the convergence.

2.2 Some preliminaries

We introduce some Mathematical facts without proofs of the relevance in the analysis in Chapter 3. The details can be found in [2].

For a vector $\boldsymbol{u} = (u_i)_{i=1}^n$, we define the following norms

$$\|\boldsymbol{u}\|^2 = \sum_{i=1}^{n} |u_i|^2, \qquad \|\boldsymbol{u}\|_\infty = \max_{1\le i\le n} |u_i|,$$

from which we deduce that

$$\frac{1}{n}\|\boldsymbol{u}\|^2 \le \|\boldsymbol{u}\|_\infty^2 \le \|\boldsymbol{u}\|^2. \tag{2.11}$$

- **discrete Holder inequality**

$$\sum_{i=1}^{N} |x_i y_i| \le \left[\sum_{i=1}^{N} |x_i|^2\right]^{1/2} \left[\sum_{i=1}^{N} |y_i|^2\right]^{1/2} \tag{2.12}$$

• **Young inequality**

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \text{ for all } a, b \in \mathbb{R} \text{ and } \varepsilon \in \mathbb{R}. \tag{2.13}$$

• **discrete Integration by parts**. Let $U = (u_i)_{0,\cdots,N+1}$ and $V = (v_i)_{0,\cdots,N+1}$ be two sequences. Then

$$\sum_{i=1}^{N} D^+ u_i v_i + \sum_{i=1}^{N+1} u_i D^- v_i = \frac{1}{h}(u_{N+1}v_{N+1} - u_1 v_0)$$

If in particular $v_0 = v_{N+1} = 0$, then

$$\sum_{i=1}^{N} D^+ u_i v_i = -\sum_{i=1}^{N+1} u_i D^- v_i \tag{2.14}$$

• **discrete Poincare's inequality**. Let $U = (u_i)_{i=0,\cdots,N+1}$, with $u_0 = u_{N+1} = 0$. Then there is a positive constant $\boldsymbol{c}$ independent of h such that

$$\boldsymbol{c}\sum_i |u_i|^2 \le \sum_i |D^+ u_i|^2 = \sum_i |D^- u_i|^2. \tag{2.15}$$

• **parallelogram identity**. Let $x = (x_i)$ and $y = (y_i)$ two sequences of vectors indexed for $i = 0, 1, 2, \cdots, N$. Then

$$2\sum_{i=0}^{N}(x_i - y_i)x_i \;\; = \;\; \|x\|^2 - \|y\|^2 + \|x - y\|^2. \tag{2.16}$$

2.3 Existence and Uniqueness of solution

In solving a mathematical problem by a numerical method, it is pertinent to check whether the solution is uniquely computable.

2.3.1 Existence and Uniqueness of solution of (2.9)

Solving (2.9) for u_i^{n+1}, one obtains

$$u_i^{n+1} = u_i^n + k\varepsilon D^+ D^- u_i^n - kb(x_i, t_n)u_i^n + kf(x_i, t_n) \ . \tag{2.17}$$

Thus (2.9) has a unique solution given at each time step by (2.17).

2.3.2 Existence and Uniqueness of solution of (2.10)

Solving (2.10), since at each time step, it is a linear system of equations of $N \times N$ unknowns. Hence, one can re-write it in the form:

$$\mathcal{A}\boldsymbol{u}^{n+1} = F, \tag{2.18}$$

where $\mathcal{A}$ is an $N \times N$ matrix, $\boldsymbol{u}^{n+1}$ is the unknown vector and F is a given vector taking into account the previous time step and the external force $f(x_i, t_n)$. With the reformation (2.10), it suffice to check one of the following conditions:

(a) the determinant of $\mathcal{A}$ is nonzero,

(b) $\mathcal{A}$ is symmetric and positive definite,

(c) $\mathcal{A}$ is diagonally dominant.

Checking of these conditions necessitate the knowledge of the operator $\mathcal{A}$ which is computationally intense. Also from the linear algebra, since we are in finite dimension, the following conditions are equivalent

(i) $\mathcal{A}$ is one to one (that is the kernel of $\mathcal{A}$ is the singleton 0),

(ii) $\mathcal{A}$ is surjective,

(iii) $\mathcal{A}$ is a bijection.

The first equation in (2.10) can be re-written as follows

$$u_i^{n+1} - \varepsilon k D^+ D^- u_i^{n+1} + kb(x_i, t_{n+1})u_i^{n+1} = u_i^n + kf(x_i, t_{n+1}) \ . \tag{2.19}$$

It is then apparent that the operator $\mathcal{A}$ is defined by the left hand side of (2.19). So we want to check that;

$$\boldsymbol{\mathcal{A}u}^{n+1} = 0 \quad \text{implies that } \boldsymbol{u}_i^{n+1} = 0 \ \text{ for all } i. \tag{2.20}$$

Clearly, $\boldsymbol{\mathcal{A}u}^{n+1} = 0$ is

$$u_i^{n+1} - \varepsilon k D^+ D^- u_i^{n+1} + kb(x_i, t_{n+1})u_i^{n+1} = 0. \tag{2.21}$$

We multiply (2.21) by u_i^{n+1}, taking the summation for $i = 1, 2, \cdots, N$, apply (2.14). We find

$$\sum_{n=1}^{N} (u_i^{n+1})^2 + \varepsilon k \sum_{i=1}^{N+1} (D^- u_i^{n+1})^2 + k \sum_{i=1}^{N} b(x_i, t_{n+1})(u_i^{n+1})^2 = 0. \tag{2.22}$$

But we know that $b(x_i, t_n) \geq \beta \geq 0$, so that (2.22) implies that

$$(1 + k\beta)\|\boldsymbol{u}^{n+1}\|^2 + \varepsilon k \sum_{i=1}^{N+1} (D^- u_i^{n+1})^2 \leq 0. \tag{2.23}$$

$$(1 + k\beta + \varepsilon kc)\|\boldsymbol{u}^{n+1}\|^2 \leq 0 \tag{2.24}$$

Thus, from (2.24) we have that

$$\|\boldsymbol{u}^{n+1}\|^2 = 0.$$

Thus $u_i^{n+1} = 0$ for all i. Therefore $\mathcal{A}$ is injective, implying that the system of equation (2.10) has a unique solution.

CHAPTER 3

Consistency-Stability

3.1 Consistency Analysis

Our aim here is to discuss the consistency of the numerical scheme introduced in Chapter 2.
In general, any given partial differential equation, including its boundary/or initial conditions, can be written as an abstract operator equation

$$Au = f \tag{3.1}$$

with appropriately chosen function spaces U, V, a mapping $A : U \to V$, and $f \in V$. The related discrete problem can be stated analogously as

$$A_h u_h = f_h \tag{3.2}$$

with $A_h : U_h \to V_h$, $f_h \in V_h$, and discrete spaces U_h, V_h.

Definition 3.1. *Let $\widetilde{u}$ be the exact solution at the grid points, and $\|\cdot\|_{V_h}$ the norm on V_h. Then the value $\|A_h\widetilde{u} - f_h\|_{V_h}$ is called the consistency error relative to $u \in U$.*

It should be noted that the remainder term in Taylor's expansion gives a way of estimating the consistency error, provided that the solution u of (3.1) is smooth enough and f_h forms an appropriate discretization of f.

Definition 3.2. *A discretization of (3.1) is consistent if*

$$\|A_h\widetilde{u} - f_h\|_{V_h} \to 0 \ , \ h \to 0.$$

If in addition the consistency error satisfies the more precise estimate

$$\|A_h\widetilde{u} - f_h\|_{V_h} \leq ch^p$$

where c is a positive constant, then the discretization is said to be consistent of order p.

Remark 3.3. *The definition 2.1 is concern with an operator, while the definition 3.2 is concern with a discretization scheme.*

3.1.1 Consistency of finite difference scheme (2.9)

The formula used in the numerical schemes result from an approximation of the equation using a Taylor's expansion. To this end, we introduce a formal definition, what consistency means for problems, and then apply it to our problem at hand.

Theorem 3.4. *Let u be the solution of (1.1), assume that u is twice continuously differentiable in time and four times continuously differentiable in space. Then*

$$||\xi^n||_\infty \leq \boldsymbol{c}_1 k + \varepsilon \boldsymbol{c}_2 h^2, \tag{3.3}$$

where $\boldsymbol{c}_1, \boldsymbol{c}_2$ *are positive constants independent of* k, h *and* ε, *and*

$$\xi_i^n = \frac{1}{k}(u(x_i, t_{n+1}) - u(x_i, t_n)) - \varepsilon D^+ D^- u(x_i, t_n) + b(x_i, t_n)u(x_i, t_n) - f(x_i, t_n)$$

Proof. The proof is based on Taylor's expansion, Lemma 2.4 and Lemma 2.5. The result is based on Taylor expansions.
Since u is twice differentiable in time, there exist $\tau \in [t^n, t^{n+1}]$ such that:

$$\frac{1}{k}(u(x_i, t_{n+1}) - u(x_i, t_n)) = u_t(x_i, t_n) + \frac{k}{2} u_{tt}(x_i, \tau). \tag{3.4}$$

Also since u is four times continuously differentiable in space, there exist $y_1 \in [x_{i-1}, x_i]$, and $y_2 \in [x_i, x_{i+1}]$ such that

$$\varepsilon D^+ D^- u(x_i, t_n) = \varepsilon u_{xx}(x_i, t_n) + \frac{\varepsilon}{24} h^2 (u^{(4)}(y_1, t_n) + u^{(4)}(y_2, t_n)). \tag{3.5}$$

Using (3.4) and (3.5) in ξ_i^n, we have

$$\begin{aligned}
\xi_i^n &= \frac{1}{k}(u(x_i, t_{n+1}) - u(x_i, t_n)) - \varepsilon D^+ D^- u(x_i, t_n) + b(x_i, t_n)u(x_i, t_n) - f(x_i, t_n) \\
&= u_t(x_i, t_n) - \varepsilon u_{xx}(x_i, t_n) + b(x_i, t_n)u(x_i, t_n) - f(x_i, t_n) \\
&\quad + \frac{k}{2} u_{tt}(x_i, \tau) - \frac{\varepsilon}{24} h^2 (u^{(4)}(y_1, t_n) + u^{(4)}(y_2, t_n)).
\end{aligned} \tag{3.6}$$

From the continuous problem (1.1),

$$u_t(x_i,t_n) - \varepsilon u_{xx}(x_i,t_n) + b(x_i,t_n)u(x_i,t_n) - f(x_i,t_n) = 0.$$

Hence (3.6) is reduced to

$$\xi_i^n \;=\; \frac{k}{2}u_{tt}(x_i,\tau) - \frac{\varepsilon}{24}h^2(u^{(4)}(y_1,t_n) + u^{(4)}(y_2,t_n)). \tag{3.7}$$

We take the absolute value on both sides of (3.7), use the triangle inequality on the right hand side of the resulting equation.

$$|\xi_i^n| \;\leq\; \frac{k}{2}\left|u_{tt}(x_i,\tau)\right| + \frac{\varepsilon}{12}h^2\left(\frac{1}{2}|u^{(4)}(y_1,t_n)| + \frac{1}{2}|u^{(4)}(y_2,t_n)|\right). \tag{3.8}$$

Now we apply Lemma 2.4 on the second term of the right hand side of (3.8) and obtain $y \in (x_{i-1}, x_{i+1})$

$$|\xi_i^n| \;\leq\; \frac{k}{2}\left|u_{tt}(x_i,\tau)\right| + \frac{\varepsilon}{12}h^2\left|u^{(4)}(y,t_n)\right|. \tag{3.9}$$

Now, since u_{tt} is continuous on $[t_n, t_{n+1}]$ for all n, from lemma 2.5, u_{tt} is bounded and we let

$$2\boldsymbol{c}_1 = \max_{0\leq\tau\leq T} |u_{tt}(x_i,\tau)|.$$

With similar reasoning, we let

$$12\boldsymbol{c}_2 = \max_{0\leq y\leq 1}\left|u^{(4)}(y,t_n)\right|,$$

and (3.9) becomes

$$|\xi_i^n| \;\leq\; \boldsymbol{c}_1 k + \varepsilon\boldsymbol{c}_2 h^2. \tag{3.10}$$

Finally, taking the maximum over i, (3.10) implies that

$$\|\xi^n\|_\infty \;\leq\; \boldsymbol{c}_1 k + \varepsilon\boldsymbol{c}_2 h^2$$

which is the result announced.
It is evident from Theorem 3.4 that the order of consistency in space of the explicit scheme (2.9) depends on the value of ε. We then claim that

Corollary 3.5. *Let u be the solution of (1.1) assumed to be twice continuously differentiable in time and four times continuously differentiable in space. Then;*

• *If ε is independent of h, the numerical scheme (2.9) is consistent with respect to $\| \cdot \|_\infty$ norm, of order one in time and two in space in the sense that*

$$||\xi^n||_\infty \leq \boldsymbol{c}_1 k + \boldsymbol{c}_2 h^2.$$

• *If $\varepsilon = 0(h^{-1})$, the numerical scheme (2.9) is consistent with respect to $\| \cdot \|_\infty$ norm, of order one in time and one in space in the sense that*

$$||\xi^n||_\infty \leq \boldsymbol{c}_1 k + \boldsymbol{c}_2 h.$$

• *If $\varepsilon = 0(h^{\alpha})$ with $\alpha \geq 0$, the numerical scheme (2.9) is consistent with respect to $\| \cdot \|_\infty$ norm, of order one in time and $2 + \alpha$ in space in the sense that*

$$||\xi^n||_\infty \leq \boldsymbol{c}_1 k + \boldsymbol{c}_2 h^{2+\alpha}.$$

• *If $\varepsilon = 0(h^{-\alpha})$ with $\alpha \geq 2$, then*

$$||\xi^n||_\infty \leq \boldsymbol{c}_1 k + \boldsymbol{c}_2 h^{2-\alpha},$$

and $h^{2-\alpha}$ does not tends to zero when h tends to zero. Hence the numerical scheme (2.9) is not consistent with respect to $\| \cdot \|_\infty$ norm.

3.1.2 Consistency of finite difference scheme (2.10)

Theorem 3.6. *Let u be the solution of (1.1), assume that u is twice continuously differentiable in time and four times continuously differentiable in space. Then the numerical scheme (2.10) is consistent of order one in time and two in space in the sense that*

$$||\xi^n||_\infty \leq \boldsymbol{c}_1 k + \varepsilon \boldsymbol{c}_2 h^2,$$

where $\boldsymbol{c}_1, \boldsymbol{c}_2$ are positive constants independent of k, h and ε, and

$$\xi_i^n = \frac{1}{k}(u(x_i, t_{n+1}) - u(x_i, t_n)) - \varepsilon D^+ D^- u(x_i, t_{n+1}) + b(x_i, t_{n+1}) u(x_i, t_{n+1}) - f(x_i, t_{n+1})$$

Proof. The proof is based on Taylor's expansion, Lemma 2.5, and Lemma 2.4.
Since u is twice differentiable on time, there exist $\tau \in [t^n, t^{n+1}]$ such that:

$$\frac{1}{k}(u(x_i, t_{n+1}) - u(x_i, t_n)) = u_t(x_i, t_{n+1}) - \frac{k}{2} u_{tt}(x_i, \tau). \tag{3.11}$$

Also since u is four times continuously differentiable in space, there exist $y_1 \in [x_{i-1}, x_i]$, and $y_2 \in [x_i, x_{i+1}]$ such that

$$\varepsilon D^+ D^- u(x_i, t_{n+1}) = \varepsilon u_{xx}(x_i, t_{n+1}) + \frac{\varepsilon}{24} h^2 (u^{(4)}(y_1, t_{n+1}) + u^{(4)}(y_2, t_{n+1})). \quad (3.12)$$

Using (3.11) and (3.12) in ξ_i^n, we have

$$\begin{aligned} \xi_i^n &= \frac{1}{k}(u(x_i, t_{n+1}) - u(x_i, t_n)) - \varepsilon D^+ D^- u(x_i, t_{n+1}) + b(x_i, t_{n+1}) u(x_i, t_{n+1}) \\ &\quad - f(x_i, t_{n+1}) \\ &= u_t(x_i, t_{n+1}) - \varepsilon u_{xx}(x_i, t_{n+1}) + b(x_i, t_{n+1}) u(x_i, t_{n+1}) - f(x_i, t_{n+1}) \\ &\quad - \frac{k}{2} u_{tt}(x_i, \tau) - \frac{\varepsilon}{24} h^2 (u^{(4)}(y_1, t_{n+1}) + u^{(4)}(y_2, t_{n+1})). \end{aligned} \quad (3.13)$$

From the continuous problem (1.1),

$$u_t(x_i, t_{n+1}) - \varepsilon u_{xx}(x_i, t_{n+1}) + b(x_i, t_{n+1}) u(x_i, t_{n+1}) - f(x_i, t_{n+1}) = 0.$$

Hence (3.13) is reduced to

$$\xi_i^n = -\frac{k}{2} u_{tt}(x_i, \tau) - \frac{\varepsilon}{24} h^2 (u^{(4)}(y_1, t_{n+1}) + u^{(4)}(y_2, t_{n+1})). \quad (3.14)$$

We take the absolute value on both sides of (3.14), use the triangle inequality on the right hand side of the resulting equation.

$$|\xi_i^n| \leq \frac{k}{2} |u_{tt}(x_i, \tau)| + \frac{\varepsilon}{12} h^2 \left(\frac{1}{2} |u^{(4)}(y_1, t_{n+1})| + \frac{1}{2} |u^{(4)}(y_2, t_{n+1})| \right). \quad (3.15)$$

Now we apply Lemma 2.4 on the second term of the right hand side of (3.15) and obtain $y \in (x_{i-1}, x_{i+1})$

$$|\xi_i^n| \leq \frac{k}{2} |u_{tt}(x_i, \tau)| + \frac{\varepsilon}{12} h^2 \left| u^{(4)}(y, t_{n+1}) \right|. \quad (3.16)$$

Now, since u_{tt} is continuous on $[t_n, t_{n+1}]$ for all n, from Lemma 2.5, u_{tt} is bounded and we let

$$2\boldsymbol{c}_1 = \max_{0 \leq \tau \leq T} |u_{tt}(x_i, \tau)|.$$

With similar reasoning, we let

$$12\boldsymbol{c}_2 = \max_{0 \leq y \leq 1} \left| u^{(4)}(y, t_{n+1}) \right|,$$

and (3.16) becomes

$$|\xi_i^n| \quad \leq \quad \boldsymbol{c}_1 k + \varepsilon \boldsymbol{c}_2 h^2. \tag{3.17}$$

Finally, taking the maximum over i, (3.17) implies that

$$\|\xi^n\|_\infty \quad \leq \quad \boldsymbol{c}_1 k + \varepsilon \boldsymbol{c}_2 h^2$$

which is the result announced.
It is evident from Theorem 3.6 that the order of consistency in space of the implicit scheme (2.10) depend of the value of ε. We then claim that

Corollary 3.7. *Let u be the solution of (1.1) assume to be twice continuously differentiable in time and four times continuously differentiable in space. Then;*
• *If ε is independent of h, the numerical scheme (2.10) is consistent with respect to $\|\cdot\|_\infty$ norm, of order one in time and two in space in the sense that*

$$||\xi^n||_\infty \leq \boldsymbol{c}_1 k + \boldsymbol{c}_2 h^2.$$

• *If $\varepsilon = 0(h^{-1})$, the numerical scheme (2.10) is consistent with respect to $\|\cdot\|_\infty$ norm, of order one in time and one in space in the sense that*

$$||\xi^n||_\infty \leq \boldsymbol{c}_1 k + \boldsymbol{c}_2 h.$$

• *If $\varepsilon = 0(h^\alpha)$ with $\alpha \geq 0$, the numerical scheme (2.10) is consistent with respect to $\|\cdot\|_\infty$ norm, of order one in time and $\alpha + 2$ in space in the sense that*

$$||\xi^n||_\infty \leq \boldsymbol{c}_1 k + \boldsymbol{c}_2 h^{2+\alpha}.$$

• *If $\varepsilon = 0(h^{-\alpha})$ with $\alpha \geq 2$, then*

$$||\xi^n||_\infty \leq \boldsymbol{c}_1 k + \boldsymbol{c}_2 h^{2-\alpha},$$

and $h^{2-\alpha}$ does not tends to zero when h tends to zero. Hence the numerical scheme (2.10) is not consistent with respect to $\|\cdot\|_\infty$ norm.

3.2 Stability analysis

Definition 3.8. *A discretization method (3.2) is stable if for some constant $\mathbf{c} > 0$ one has*

$$\|u_h - v_h\|_{U_h} \leq \mathbf{c}\|A_h u_h - A_h v_h\|_{V_h} \quad \text{for all} \;\; u_h, v_h \in U_h.$$

Hence stability ensures that rounding errors occurring in the problem will not have excessive effect on the final result.

Remark 3.9. *If the discrete operator A_h is linear, then stability of the discretization is equivalent to the existence of a constant $\boldsymbol{c} > 0$, independent of h, such that*

$$\|A_h^{-1}\| \leq \boldsymbol{c}.$$

The stability condition (see definition 3.8) is replaced by

$$\|u_h\|_{U_h} \leq \boldsymbol{c}\|A_h u_h\|_{V_h} \quad \textit{for all} \;\; u_h \in U_h.$$

But when A_h are nonlinear, the property mentioned in definition 3.8 is only needed in some neighborhood of the discrete solution u_h.

3.2.1 stability of the explicit scheme

Theorem 3.10. *Let $b(x,t)$ be a smooth and nonnegative function. The difference scheme (2.9) is conditionally stable. This is to say that if $1 - \dfrac{2k\varepsilon}{h^2} \geq k\beta$ with $\dfrac{k\varepsilon}{h^2} \leq \dfrac{1}{2}$ then*

$$\|u^n\|_\infty \leq ||u^0||_\infty + k\sum_{i=0}^{n} ||f^i||_\infty.$$

We recall that the solution vector is

$$u_i^{n+1} = (u_0^{n+1}, u_1^{n+1}, ..., u_{N+1}^{n+1}) \quad , \quad \text{and} \quad \|u^{n+1}\|_\infty = \max_{0\leq i\leq N+1} |u_i^{n+1}|\,.$$

proof. The finite difference scheme (2.8) is

$$\begin{cases} \text{for } i = 1, 2, ..., N, \\ u_i^{n+1} = u_i^n + \varepsilon k D^+ D^- u_i^n - kb(x_i, t_n)u_i^n + kf(x_i, t_n)\ , \\ u_i^0 = 0 \;\; \text{for all } i = 0, 1, 2, \cdots N+1, \\ u_0^n = u_{N+1}^n = 0 \text{ for all } n = 1, 2, 3, \cdots, M, \end{cases}$$

from which we deduce that

$$\begin{aligned} u_i^{n+1} &= \frac{k\varepsilon}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) - kb_i^n u_i^n + u_i^n + kf_i^n \\ &= \left(1 - \frac{2k\varepsilon}{h^2} - kb_i^n\right) u_i^n + \frac{k\varepsilon}{h^2}u_{i+1}^n + \frac{k\varepsilon}{h^2}u_{i-1}^n + kf_i^n. \end{aligned} \qquad (3.18)$$

We take the absolute value on both sides of (3.18), use the triangle inequality on the right hand side of the resulting equation. This gives

$$|u_i^{n+1}| \leq |1 - \frac{2k\varepsilon}{h^2} - kb_i^n||u_i^n| + \frac{k\varepsilon}{h^2}|u_{i+1}^n| + \frac{k\varepsilon}{h^2}|u_{i-1}^n| + k|f_i^n| \tag{3.19}$$

Now, taking the maximum over i, (3.19) implies that

$$\begin{aligned} ||u^{n+1}|| &\leq |1 - \frac{2k\varepsilon}{h^2} - kb_i^n|||u^n||_\infty + \frac{k\varepsilon}{h^2}||u^n||_\infty + \frac{k\varepsilon}{h^2}||u^n||_\infty + k||f^n||_\infty \\ &= |1 - \frac{2k\varepsilon}{h^2} - kb_i^n|||u^n||_\infty + 2\frac{k\varepsilon}{h^2}||u^n||_\infty + k||f^n||_\infty \end{aligned} \tag{3.20}$$

In order to continue with (3.20) we need to distinguish cases, and since $b(x,t)$ is smooth, then we let

$$\|b\| = \max_{0 \leq i \leq N+1} b(x_i, t_n).$$

Case 1. If

$$1 - \frac{2k\varepsilon}{h^2} \leq kb_i^n \leq k\|b\|, \tag{3.21}$$

then (3.20) is reduced to

$$\begin{aligned} \|u^{n+1}\|_\infty &\leq \left(\frac{4k\varepsilon}{h^2} + k\|b\| - 1\right)||u^n||_\infty + k||f^n||_\infty \\ &\leq \left(\frac{4k\varepsilon}{h^2} + k\|b\|\right)||u^n||_\infty + k||f^n||_\infty. \end{aligned} \tag{3.22}$$

So for stability we require that

$$\frac{2k\varepsilon}{h^2} + k\|b\| \leq 1 - \frac{2k\varepsilon}{h^2}. \tag{3.23}$$

But (3.21) is

$$1 \leq \frac{2k\varepsilon}{h^2} + k\|b\|,$$

which together with (3.23) implies

$$1 \leq \frac{2k\varepsilon}{h^2} + k\|b\| \leq 1 - \frac{2k\varepsilon}{h^2}$$

which is clearly a contradiction.

Case 2. If

$$1 - \frac{2k\varepsilon}{h^2} \geq kb_i^n \geq k\beta, \tag{3.24}$$

then (3.20) becomes

$$\begin{aligned} \|u^{n+1}\|_\infty &\leq (1-k\beta)||u^n||_\infty + k||f^n||_\infty \\ &\leq ||u^n||_\infty + k||f^n||_\infty. \end{aligned} \tag{3.25}$$

Finally summation for $i = 0, 1, ..., n$ gives the desired result. The proof is complete.

Remark 3.11. *Observe that the stability condition in Theorem 3.10 depends on ε. Hence we do not have a uniform stability with respect to ε.*

3.2.2 stability of the implicit scheme

Theorem 3.12. *Let b be a smooth and nonnegative function defined on $\Omega\times(0,T)$. The finite difference scheme (2.10) is unconditionally stable in the sense that*

$$||u^n||^2 \leq \frac{1}{(1+2k\beta)^n}||u^0||^2 + \frac{\boldsymbol{c}k}{\varepsilon}\sum_{i=0}^{n}\frac{1}{(1+2k\beta)^i}\|f^{n+1-i}\|^2.$$

Proof We recall that (2.10) is

$$\begin{cases} \text{Knowing } u_i^n, \text{ find } u_i^{n+1} \text{ such that,} \\ \dfrac{u_i^{n+1}-u_i^n}{k} - \varepsilon D^+D^-u_i^{n+1} + b(x_i,t_{n+1})u_i^{n+1} = f(x_i,t_{n+1}) \ , \\ u_i^0 = 0 \quad \text{for all } i = 0,1,2,\cdots N+1, \\ u_0^n = u_{N+1}^n = 0 \text{ for all } n = 1,2,3,\cdots, M. \end{cases}$$

We multiply the scheme by u_i^{n+1} and take summation to obtain

$$\frac{1}{k}\sum_{i=1}^{N}(u_i^{n+1}-u_i^n, u_i^{n+1}) - \varepsilon\sum_{i=1}^{N}D^+D^-u_i^{n+1}u_i^{n+1} + \sum_{i=1}^{N}b_i^{n+1}(u_i^{n+1})^2 = \sum_{i=1}^{N}f_i^{n+1}u_i^{n+1},$$

which after utilization of (2.14), and multiplication of the resulting equation by 2 ,and having in mind $b_i^{n+1} \geq \beta > 0$, we obtain

$$\begin{aligned} \frac{2}{k}\sum_{i=1}^{N+1}(u_i^{n+1}-u_i^n, u_i^{n+1}) &+ 2\varepsilon\sum_{i=1}^{N+1}(D^-u_i^{n+1})^2 + 2\beta\sum_{i=1}^{N+1}(u_i^{n+1})^2 \\ &\leq 2\sum_{i=1}^{N+1}f_i^{n+1}u_i^{n+1}. \end{aligned} \tag{3.26}$$

Application of (2.16) on (3.26) , and dropping some positive terms leads to

$$||u^{n+1}||^2 - ||u^n||^2 + 2k\varepsilon||D^- u^{n+1}||^2 + 2k\beta\|u^{n+1}\|^2 \leq 2k \sum_{i=1}^{N+1} |f_i^{n+1}|\,|u_i^{n+1}|. \quad (3.27)$$

Applying (2.12), (2.15) and (2.13) on (3.27), we obtain that

$$\begin{aligned} ||u^{n+1}||^2 - ||u^n||^2 + 2k\beta\|u^{n+1}\|^2 + 2k\varepsilon||D^- u^{n+1}||^2 &\leq 2k\|f^{n+1}\|\,\|u^{n+1}\| \\ &\leq 2\boldsymbol{c}k\|f^{n+1}\|\,\|D^- u^{n+1}\| \\ &\leq \frac{\boldsymbol{c}k}{\varepsilon}\|f^{n+1}\|^2 + \varepsilon k\,\|D^- u^{n+1}\|^2, \end{aligned}$$

which is

$$||u^{n+1}||^2 + 2k\beta\|u^{n+1}\|^2 + k\varepsilon||D^- u^{n+1}||^2 \leq ||u^n||^2 + \frac{\boldsymbol{c}k}{\varepsilon}\|f^{n+1}\|^2. \quad (3.28)$$

Now removing some positive terms, we obtain

$$(1 + 2k\beta)||u^{n+1}||^2 \leq ||u^n||^2 + \frac{\boldsymbol{c}k}{\varepsilon}\|f^{n+1}\|^2. \quad (3.29)$$

hence

$$||u^{n+1}||^2 \leq \frac{1}{(1 + 2k\beta)}||u^n||^2 + \frac{\boldsymbol{c}k}{\varepsilon(1 + 2k\beta)}\|f^{n+1}\|^2. \quad (3.30)$$

Hence by induction over n, one has

$$||u^n||^2 \leq \frac{1}{(1 + 2k\beta)^n}||u^0||^2 + \frac{\boldsymbol{c}k}{\varepsilon}\sum_{i=0}^{n} \frac{1}{(1 + 2k\beta)^i}\|f^{n+1-i}\|^2. \quad (3.31)$$

Remark 3.13. *It is manifest that as $\varepsilon \to 0$, the right hand side of (3.31) blow up.*

CHAPTER 4

Convergence analysis

The convergence of the method is defined similarly as the stability.

Definition 4.1. *A discretization method is convergent if the error satisfies*

$$\|\widetilde{u} - u_h\|_{U_h} \to 0 \quad h \to 0,$$

and convergent of order p if there exists a constant $\boldsymbol{c} > 0$ such that

$$\|\widetilde{u} - u_h\|_{U_h} \leq \boldsymbol{c} h^p.$$

Convergence is proved by demonstrating consistency and stability of the discretization.
Indeed definition 3.2 and definition 3.8 imply immediately the following abstract convergence theorem

Theorem 4.2. *Assume that both the continuous and the discrete problem have a unique solution. If the discretization method is consistent and stable then the method is also convergent. Furthermore, the order of convergence is at least as large as the order of consistency of the method.*

4.1 Convergence of the explicit scheme

Theorem 4.3. *Let $b(x,t)$ be a smooth and nonnegative function. The difference scheme (2.9) is conditionally convergent. This is to say that if*

$$1 - \frac{2k\varepsilon}{h^2} \geq k\beta \quad \textit{and} \quad \frac{k\varepsilon}{h^2} \leq \frac{1}{2}, \tag{4.1}$$

then

$$\|e^n\|_\infty \leq \boldsymbol{c}T(k+\varepsilon h^2). \tag{4.2}$$

Remark 4.4. *Using (4.1) in (4.2) gives*

$$\|e^n\|_\infty \leq \boldsymbol{c}T\left(\frac{h^2}{2\varepsilon}+\varepsilon h^2\right). \tag{4.3}$$

Proof We recall that the numerical solution at (x_i, t_n) is u_i^n, while the exact solution at the same point is $u(x_i, t_n)$. We define the error at the point (x_i, t_n) as

$$e_i^n = u(x_i, t_n) - u_i^n$$

from which we deduce that $u_i^n = u(x_i, t_n) - e_i^n$. Replacing that expression in (2.9), we find

$$\begin{aligned}&\frac{1}{k}[u(x_i, t_{n+1}) - e_i^{n+1} - u(x_i, t_n) + e_i^n] - \varepsilon D^+D^-[u(x_i, t_n) - e_i^n]\\ &+b_i^n[u(x_i, t_n) - e_i^n\ ] = f(x_i, t_n).\end{aligned} \tag{4.4}$$

Multiplying (4.4) by (-1), we obtain

$$\begin{aligned}\frac{1}{k}[\ e_i^{n+1} - e_i^n\ ] \quad &- \quad \varepsilon D^+D^-e_i^n + b_i^n e_i^n = \frac{1}{k}[u(x_i, t_{n+1}) - u(x_i, t_n)]\\ &- \quad \varepsilon D^+D^-u(x_i, t_n) + b(x_i, t_n)u(x_i, t_n) - f(x_i, t_n)\end{aligned} \tag{4.5}$$

Following the notation in Chapter 3 (consistency), we let

$$\xi_i^n = \frac{1}{k}[u(x_i, t_{n+1}) - u(x_i, t_n)] - \varepsilon D^+D^-u(x_i, t_n) + b(x_i, t_n)u(x_i, t_n) - f(x_i, t_n),$$

and re-write (4.5) as follows:

$$\begin{cases}\dfrac{e_i^{n+1} - e_i^n}{k} - \epsilon D^+D^-e_i^n + b_i^n e_i^n = \xi_i^n,\\ e_i^0 = 0,\\ e_0^n = e_{N+1}^n \qquad b(x,t) \gg \beta > 0.\end{cases}$$

From stability analysis (see Theorem 3.10), we deduce that if $1-\dfrac{2k\varepsilon}{h^2} \geq k\beta$ with $\dfrac{k\varepsilon}{h^2} \leq \dfrac{1}{2}$, then

$$||e^n||_\infty \leq ||e^0||_\infty + k\sum_{i=0}^{n} ||\xi^i||_\infty. \tag{4.6}$$

Next from Theorem 3.4

$$||\xi^n||_\infty \leq \boldsymbol{c}_1 k + \varepsilon \boldsymbol{c}_2 h^2$$

where $\boldsymbol{c}_1, \boldsymbol{c}_2$ are positive constants independent of k, h and ε. Thus (4.6) becomes

$$\begin{aligned} ||e^n||_\infty &\leq ||e^0||_\infty + k\sum_{i=0}^{n}(\boldsymbol{c}_1 k + \varepsilon \boldsymbol{c}_2 h^2) \\ &\leq ||e^0||_\infty + k(\boldsymbol{c}_1 k + \varepsilon \boldsymbol{c}_2 h^2)\sum_{i=0}^{n} 1 \\ &\leq ||e^0||_\infty + n\,k(\boldsymbol{c}_1 k + \varepsilon \boldsymbol{c}_2 h^2)\,. \end{aligned} \tag{4.7}$$

We observe that $e_i^0 = 0$ (initial condition), so that $||e^0||_\infty = 0$ and $n\,k \leq T$ (T is the total time of interest). Thus (4.7) becomes

$$||e^n||_\infty \leq T(\boldsymbol{c}_1 k + \varepsilon \boldsymbol{c}_2 h^2)\,. \tag{4.8}$$

We set $\boldsymbol{c} = \max(\boldsymbol{c}_1, \boldsymbol{c}_2)$ so that (4.8) gives

$$||e^n||_\infty \leq T\boldsymbol{c}(k + \varepsilon h^2)\,. \tag{4.9}$$

Hence proof is complete.
It is clear from Theorem 4.3 that the scheme (2.9) converges conditionally with respect to ε.
Observe that:
• If ε is independent of h , then one has

$$||e^n||_\infty \leq \boldsymbol{c}T(k + h^2) \tag{4.10}$$

• If ε goes to zero faster that h^2, then

$$\|e^n\|_\infty \leq \boldsymbol{c}T\left(\frac{h^2}{2\varepsilon} + \varepsilon h^2\right) \to \infty\,. \tag{4.11}$$

Thus we conclude that the convergence is not uniform with respect to ε.

4.2 Convergence of the implicit scheme

Theorem 4.5. *Let b be a non-negative and smooth function defined on $[0,T]\times[0,1]$. The numerical solution $\boldsymbol{u}_i^n$ of (2.10) converges unconditionally to $u(x_i,t_n)$ of (1.1) in the sense that there exists generic positive constant* $\mathbf{c}$ *independent of h, ε and k such that*

$$\|u(x_i,t_n)-u_i^n\|_\infty^2 \leq \boldsymbol{c}\frac{T}{\varepsilon}(k+\varepsilon h^2)^2\,. \tag{4.12}$$

Proof We recall that (2.10) approximating (1.1) is

$$\begin{cases} \text{Knowing } u_i^n, \text{ find } u_i^{n+1} \text{ such that,} \\ \dfrac{u_i^{n+1}-u_i^n}{k} - \varepsilon D^+D^-u_i^{n+1} + b(x_i,t_{n+1})u_i^{n+1} = f(x_i,t_{n+1})\ , \\ u_i^0 = 0 \ \text{ for all } i = 0,1,2,\cdots N+1, \\ u_0^n = u_{N+1}^n = 0 \text{ for all } n = 1,2,3,\cdots,M. \end{cases}$$

The error is $e_i^n = u(x_i,t_n)-u_i^n$. Replacing it in (2.10) leads to

$$\begin{aligned} \frac{1}{k}[\ u(x_i,t_{n+1}) &- e_i^{n+1} - u(x_i,t_n) + e_i^n] - \varepsilon D^+D^-[u(x_i,t_{n+1})-e_i^{n+1}] \\ &+ b_i^{n+1}[u(x_i,t_{n+1})-e_i^{n+1}] = f(x_i,t_{n+1})\,. \end{aligned} \tag{4.13}$$

Multiplying (4.13) by (-1), we obtain

$$\begin{aligned} \frac{1}{k}[e_i^{n+1} &- e_i^n] - \varepsilon D^+D^-e_i^{n+1} + b_i^{n+1}e_i^{n+1} = \frac{1}{k}[u(x_i,t_{n+1})-u(x_i,t_n)] \\ &- \varepsilon D^+D^-u(x_i,t_{n+1}) + b(x_i,t_{n+})u(x_i,t_{n+1}) - f(x_i,t_{n+})\,. \end{aligned} \tag{4.14}$$

We let

$$\xi_i^n = \frac{1}{k}[u(x_i,t_{n+1})-u(x_i,t_n)]-\epsilon D^+D^-u(x_i,t_{n+1})+b(x_i,t_{n+1})u(x_i,t_{n+1})-f(x_i,t_{n+1}).$$

Thus (4.14) is

$$\begin{cases} \dfrac{e_i^{n+1}-e_i^n}{k} - \varepsilon D^+D^-e_i^{n+1} + b_i^{n+1}e_i^{n+1} = \xi_i^n, \\ e_i^0 = 0, \\ e_0^n = e_{N+1}^n \qquad b(x,t) \gg \beta > 0. \end{cases}$$

From stability analysis (see Theorem 3.12), we have

$$\begin{aligned}
\|e^n\|_\infty^2 \leq ||e^n||^2 &\leq \frac{1}{(1+2k\beta)^n}||e^0||^2 + \frac{\boldsymbol{c}k}{\varepsilon}\sum_{i=0}^{n}\frac{1}{(1+2k\beta)^i}||\xi^{n+1-i}||^2 \\
&\leq \frac{\boldsymbol{c}k}{\varepsilon}\sum_{i=0}^{n}\frac{1}{(1+2k\beta)^i}||\xi^{n+1-i}||_\infty^2 \\
&\leq \max_{1\leq m\leq n}\|\xi^m\|_\infty^2 \frac{\boldsymbol{c}k}{\varepsilon}\sum_{i=0}^{n}\frac{1}{(1+2k\beta)^i}. \qquad (4.15)
\end{aligned}$$

From Theorem 3.6

$$||\xi^m||_\infty^2 \leq \boldsymbol{c}_1 k + \varepsilon \boldsymbol{c}_2 h^2,$$

where$\boldsymbol{c}_1, \boldsymbol{c}_2$ are positive constants independent of k, h and ε. Next

$$\frac{1}{1+2k\beta} < 1\,.$$

Thus (4.15) implies that

$$||e^n||^2 \;\; \leq \;\; nk\boldsymbol{c}(k+\varepsilon h^2)^2\frac{1}{\varepsilon}\,. \qquad (4.16)$$

Observe that $nk \leq T$, (T is the total time of interest). Thus (4.16) becomes

$$\|e^n\|_\infty^2 \leq ||e^n||^2 \;\; \leq \;\; \frac{\boldsymbol{c}T}{\varepsilon}(k+\varepsilon h^2)^2\,. \qquad (4.17)$$

Hence proof is complete.
It is clear from Theorem 4.5 that the convergence of the scheme depends on ε.
Observe that:

• If ε is independent of h, then (4.12) becomes

$$\|u(x_i, t_n) - u_i^n\|_\infty^2 \leq \boldsymbol{c}T(k+h^2)^2\,. \qquad (4.18)$$

• If ε goes faster to zero, than k and h^2, then the right hand side of (4.18) becomes unbounded.

Hence the convergence estimates obtained in Theorem 4.5 is not uniform with respect to ε.

CHAPTER 5

Numerical simulations and future works

In this Chapter, we demonstrate computationally the theoretical results obtained in Chapter 4. More particularly, we show the poor convergence display in Theorem 4.3 and Theorem 4.5 when ε is very small compare to the mesh size h. We also show that if the mesh size goes to zero faster than ε then good behavior is obtained. We conclude this Chapter by reviewing our results, and indicating some future research to correct the bad behavior of the numerical solution for (2.9) and (2.10). The examples discussed are from all the examples discussed in [6].The exact solutions to our test examples are not known. We therefore use a variant of the double mesh principle to estimate the errors of the computed approximations. The maximum errors at all mesh points are calculated using the formula

$$e^{\varepsilon,N,\tau} = \max_{0\leq i\leq N;0\leq k\leq M} |U_{i,k}^{\varepsilon,N,\tau} - U_{i,k}^{\varepsilon,2N,\frac{\tau}{4}}|$$

where $U_{i,k}^{\varepsilon,N,\tau}$ and $U_{i,k}^{\varepsilon,2N,\frac{\tau}{4}}$ are the approximate solution obtained using a constant
time step τ, space step h and constant time step $\tau/4$, space step $h/2$ respectively. Furthermore, we compute

$$e_{N,\tau} = \max_{0<\varepsilon\leq 1} e^{\varepsilon,N,\tau}$$

The numerical rate of convergence is computed using the formula

$$r_{N,\tau} = log_2(e^{\varepsilon,N,\tau}/e^{\varepsilon,2N,\frac{\tau}{4}})$$

with the mesh size

$$h = \frac{1}{N+1}.$$

5.1 Numerical examples for (2.9)

The focus in this section is to show that the parameters ε is responsible for the poor/good convergence of the finite difference (2.9). Since (2.9) is explicit, we solve it directly, and step by step we compute the solution.

We consider the problems:

5.1.1 Example 1

$$\begin{cases} u_t - \varepsilon u_{xx} + \dfrac{2+x}{3} u = 1 + t^2, \text{for } (x,t) \in (0,1) \times (0,1] \\ u(x,0) = 0, \ \text{for } x \in [0,1] \\ u(0,t) = u(1,t) = 0, \ \text{for } t \in (0,1]. \end{cases}$$

5.1.2 Example 2

$$\begin{cases} u_t - \varepsilon u_{xx} + \dfrac{1+x^2}{2} u = \exp(x) - 1 + \sin(\pi x) \text{ for } (x,t) \in (0,1) \times (0,1] \\ u(x,0) = 0, \ \text{for } x \in [0,1] \\ u(0,t) = u(1,t) = 0, \ \text{for } t \in (0,1]. \end{cases}$$

ε	$N = 32$ $\tau=0.1$	$N = 64$ $\tau=0.1/4$	$N = 128$ $\tau=0.1/4^2$	$N = 256$ $\tau=0.1/4^3$	$N = 512$ $\tau=0.1/4^4$
2^{-8}	0.2277	0.1409	0.0799	0.0427	0.0221
2^{-10}	0.2747	0.2161	0.1404	0.0802	0.0429
2^{-16}	0.0970	0.0797	0.1648	0.2427	0.2144
2^{-18}	0.0862	0.0375	0.0659	0.1622	0.2426
2^{-20}	0.0834	0.0251	0.0223	0.0624	0.1616
2^{-50}	0.0825	0.0207	0.0050	0.0012	$2.9964E^{-4}$
$\vdots$					
2^{-100}	0.0825	0.0207	0.0050	0.0012	$2.9964E^{-4}$
$e^{\varepsilon,N,\tau}$	0.2747	0.2161	0.1648	0.2427	0.2426

Table 5.1: Maximum Error for example 1

ε	r_1	r_2	r_3	r4
2^{-8}	0.6928	0.8177	0.9034	0.9506
2^{-10}	0.3434	0.6225	0.8069	0.9033
2^{-16}	0.2831	-1.0475	-0.5585	0.1789
2^{-18}	1.2018	-0.8127	-1.3005	-0.5806
2^{-20}	1.7349	0.1705	-1.4868	-1.3723
2^{-50}	1.9923	2.0604	2.0526	1.9998
$\vdots$				
2^{-100}	1.9923	2.0604	2.0526	1.9998
$r_{N,\tau}$	1.9923	2.0604	2.0526	1.9998

Table 5.2: Rate of convergence for example 1

ε	$N = 32$ $\tau=0.1$	$N = 64$ $\tau=0.1/4$	$N = 128$ $\tau=0.1/4^2$	$N = 256$ $\tau=0.1/4^3$	$N = 512$ $\tau=0.1/4^4$
2^{-8}	0.2903	0.1699	0.0935	0.0492	0.0252
2^{-10}	0.3797	0.2757	0.1690	0.0936	0.0493
2^{-16}	0.1301	0.1206	0.2494	0.3383	0.2726
2^{-18}	0.1301	0.0541	0.1032	0.2457	0.3379
2^{-20}	0.1301	0.0343	0.0350	0.0985	0.2446
2^{-50}	0.1301	0.0324	0.0081	0.0022	$7.0831E^{-4}$
$\vdots$					
2^{-100}	0.1301	0.0324	0.0081	0.0022	$7.0831E^{-4}$
$e^{\varepsilon,N,\tau}$	0.3797	0.2757	0.2494	0.3383	0.3379

Table 5.3: Maximum Error for example 2

ε	r_1	r_2	r_3	r4
2^{-8}	0.7730	0.8623	0.9268	0.9625
2^{-10}	0.4619	0.7062	0.8522	0.9263
2^{-16}	0.1089	-1.0479	-0.4397	0.3112
2^{-18}	1.2654	-0.9315	-1.2512	-0.4598
2^{-20}	1.9252	-0.0300	-1.4933	-1.3127
2^{-50}	2.0069	2.0034	1.8865	1.6241
$\vdots$				
2^{-100}	2.0069	2.0034	1.8865	1.6241
$r_{N,\tau}$	2.0069	2.0034	1.8865	1.6241

Table 5.4: Rate of convergence for example 2

Discussion: table 5.1,$\cdots$,table 5.4 represent the numerical results, where we computed the maximum errors $e^{\varepsilon,N,\tau}$ and the corresponding rate of convergence $r_{N,\tau}$ respectively for the above examples. The numerical analysis are as follows:
• the last row of table 5.1 and 5.3 give the maximum errors $e^{\varepsilon,N,\tau}$
• the numerical rate of convergence $r_{N,\tau}$ are displaced in the last row of table 5.2 and 5.4.
We observed from table (5.2 and 5.4) that the rate of convergence is:
• not uniform (column 1,4 and 3,4) respectively as the parameter ε becomes smaller and smaller which shows a poor convergence as mentioned in theorem 4.3.
• uniform (column 2,3 and 1,2) with respect to the parameter ε which shows a good convergence as mentioned in theorem 4.3 (i.e. convergence is conditional). To further corroborate our analysis, graphs are plotted.
• figure 5.1 represents the graph of the numerical solution when ε ranges from 2^{-6} *to* 1. This shows how severe the parameter ε affects the numerical solution (rapid oscillation) for the scheme presented.
• figure (5.2, 5.3 and 5.4) represent the graph of the numerical solution when $\varepsilon = (2^{-8}\ , 2^{-50}\ \textit{and}\ 2^{-100})$ respectively. Clearly, figure (5.3 and 5.4) show that the numerical solution is unbounded as ε *becomes very small* as mentioned in theorem 4.3. figure 5.2 shows a better result than figure (5.3 and 5.4), at least the solutions at the boundaries are known.

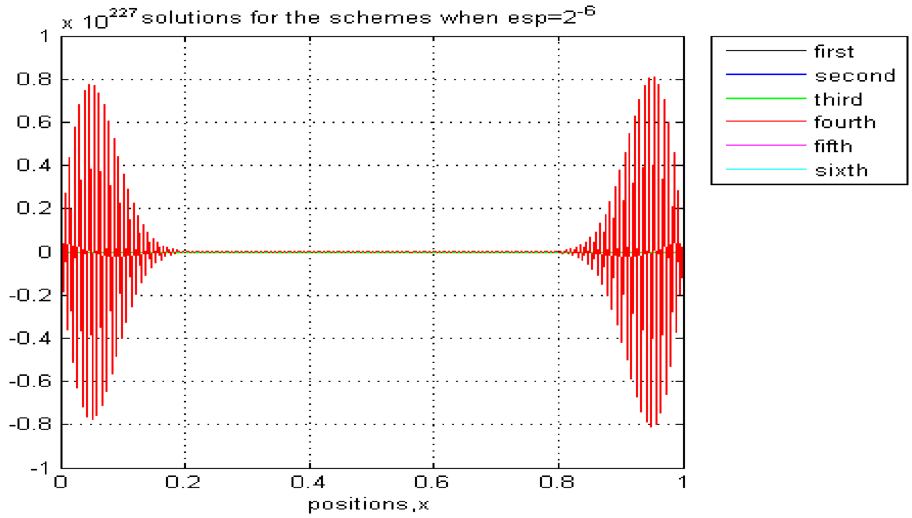

Figure 5.1: solution profile of (2.9)

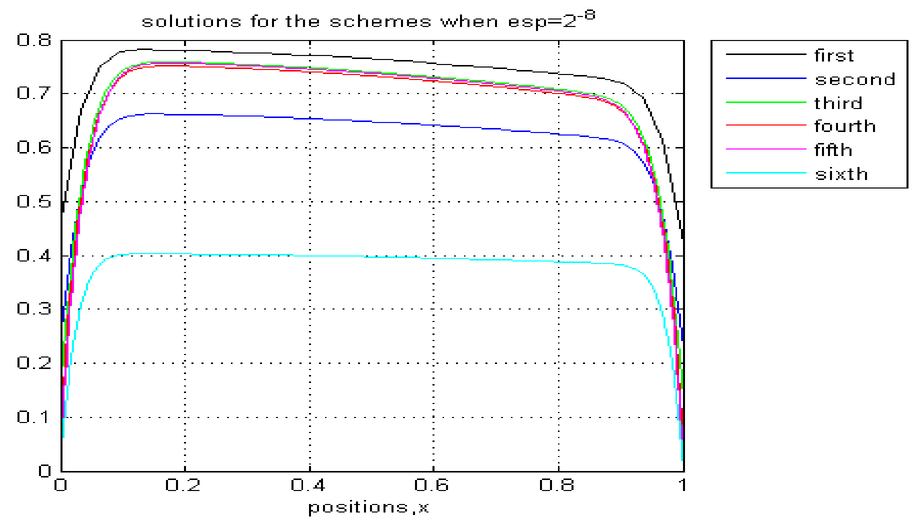

Figure 5.2: solution profile of (2.9)

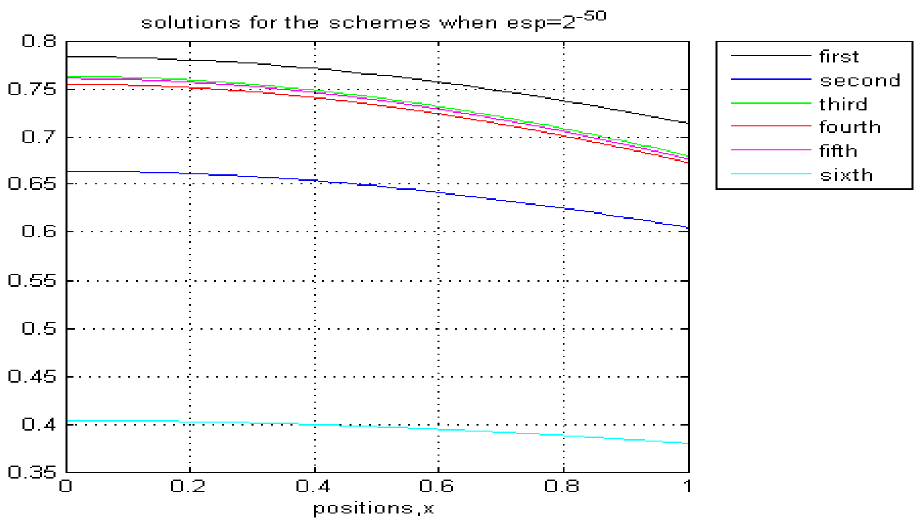

Figure 5.3: solution profile of (2.9)

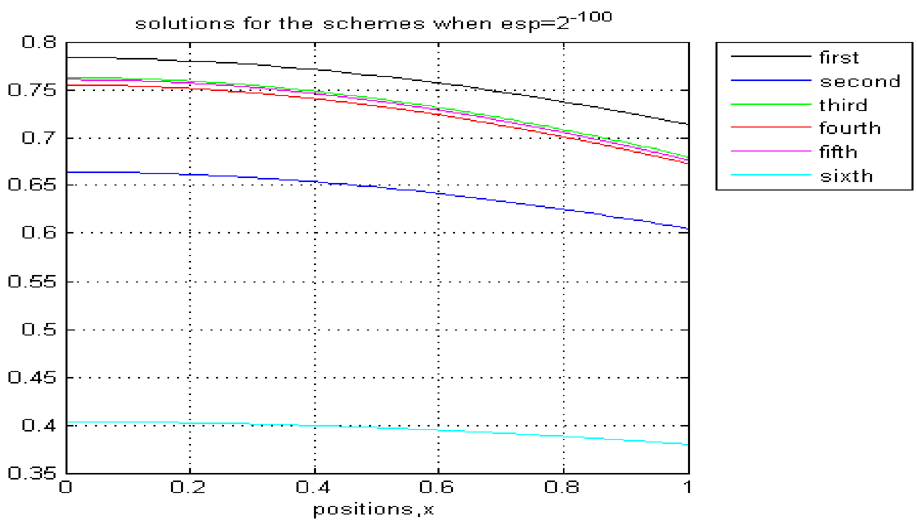

Figure 5.4: solution profile of (2.9)

5.2 Numerical examples for (2.10)

In this section, we first write down the system of equations (2.10) in the form

$$\mathcal{A}\boldsymbol{\psi} = F, \tag{5.1}$$

with $N+1$, the number of grid points in $[0,1]$, $\mathcal{A}$ is a $N \times N$ matrix, $\boldsymbol{\psi}^T = (u_1^{n+1}, u_2^{n+1}, \cdots, u_N^{n+1})$ is the unknown vector at each time step, $F = (f_1^{n+1}, ..., f_N^{n+1})$ is a given vector of size $N \times 1$.

5.2.1 Example 1

$$\begin{cases} u_t - \varepsilon u_{xx} + \dfrac{1+x^2}{2} u = t^3, \text{for } (x,t) \in (0,1) \times (0,1] \\ u(x,0) = 0, \ \text{for } x \in [0,1] \\ u(0,t) = u(1,t) = 0, \ \text{for } t \in (0,1]. \end{cases}$$

5.2.2 Example 2

$$\begin{cases} u_t - \varepsilon u_{xx} + \dfrac{1+x^2}{2} u = (1+t)^3 \text{ for } (x,t) \in (0,1) \times (0,1] \\ u(x,0) = 0, \ \text{for } x \in [0,1] \\ u(0,t) = u(1,t) = 0, \ \text{for } t \in (0,1]. \end{cases}$$

ε	$N=32$ $\tau=0.1$	$N=64$ $\tau=0.1/4$	$N=128$ $\tau=0.1/4^2$	$N=256$ $\tau=0.1/4^3$	$N=512$ $\tau=0.1/4^4$
2^{-6}	0.0550	0.0282	0.0087	0.0023	$5.828E^{-4}$
2^{-8}	0.0562	0.0286	0.0087	0.0023	$5.833E^{-4}$
2^{-10}	0.0567	0.0288	0.0088	0.0023	$5.8345E^{-4}$
2^{-16}	0.0569	0.0289	0.0088	0.0023	$5.8352E^{-4}$
2^{-18}	0.0570	0.0289	0.0088	0.0023	$5.8353E^{-4}$
2^{-20}	0.0570	0.0289	0.0088	0.0023	$5.8353E^{-4}$
$\vdots$					
2^{-50}	0.0570	0.0289	0.0088	0.0023	$5.8353E^{-4}$
$e^{\varepsilon,N,\tau}$	0.0570	0.0289	0.0088	0.0023	$5.8353E^{-4}$

Table 5.5: Maximum Error for example 1

ε	r_1	r_2	r_3	r4
2^{-6}	0.9641	1.6990	1.9194	1.9787
2^{-8}	0.9725	1.7114	1.9256	1.9810
2^{-10}	0.9762	1.7161	1.9276	1.9818
2^{-16}	0.9779	1.7185	1.9289	1.9822
2^{-18}	0.9795	1.7185	1.9286	1.9822
2^{-20}	0.9792	1.7192	1.9289	1.9822
$\vdots$				
2^{-50}	0.9791	1.7192	1.9291	1.9822
$r_{N,\tau}$	0.9791	1.7192	1.9291	1.9822

Table 5.6: Rate of convergence for example 1

ε	$N = 32$ $\tau=0.1$	$N = 64$ $\tau=0.1/4$	$N = 128$ $\tau=0.1/4^2$	$N = 256$ $\tau=0.1/4^3$	$N = 512$ $\tau=0.1/4^4$
2^{-6}	0.6817	0.2560	0.0718	0.0185	0.0047
2^{-8}	0.7038	0.2605	0.0723	0.0186	0.0047
2^{-10}	0.7122	0.2621	0.0725	0.0186	0.0047
2^{-16}	0.7165	0.2630	0.0726	0.0186	0.0047
2^{-18}	0.7176	0.02630	0.0726	0.0186	0.0047
2^{-20}	0.7179	0.2631	0.0726	0.0186	0.0047
$\vdots$					
2^{-50}	0.7180	0.2632	0.0726	0.00186	0.0047
$e^{\varepsilon,N,\tau}$	0.7180	0.2632	0.0726	0.00186	0.0047

Table 5.7: Maximum Error for example 2

ε	r_1	r_2	r_3	r4
2^{-6}	1.4130	1.8341	1.9543	1.9875
2^{-8}	1.4339	1.8491	1.9608	1.9899
2^{-10}	1.4422	1.8547	1.9629	1.9906
2^{-16}	1.4461	1.8576	1.9642	1.9911
2^{-18}	1.4484	1.8576	1.9642	1.9911
2^{-20}	1.4482	1.8584	1.9642	1.9911
$\vdots$				
2^{-50}	1.4482	1.8584	1.9644	1.9911
$r_{N,\tau}$	1.4482	1.8584	1.9644	1.9911

Table 5.8: Rate of convergence for example 2

Discussion: table 5.5,$\cdots$,table 5.8 represent the numerical results, where we computed the maximum errors $e^{\varepsilon,N,\tau}$ and the corresponding rate of convergence $r_{N,\tau}$ respectively for the above examples. The numerical analysis are as follows:
• the last row of table (5.5 and 5.7) give the maximum errors $e^{\varepsilon,N,\tau}$
• the numerical rate of convergence $r_{N,\tau}$ are displaced in the last row of table 5.6 and 5.8.
We observed from table (5.6 and 5.8) that the numerical rate of convergence is not uniform as the parameter ε becomes smaller and smaller which shows a poor convergence as mentioned in theorem 4.5.
To further corroborate our analysis, graphs have been plotted.
• figure (5.5 and 5.6) represent the graph of the numerical solution when $\varepsilon = (2^{-6}\ and\ 2^{-10})$ respectively.
• figure (5.7 and 5.8) represent the graph of the numerical solution when $\varepsilon = (2^{-20}\ and\ 2^{-50})$ respectively. Clearly, figure (5.7 and 5.8) show that the numerical solution is unbounded as ε *becomes very small* as mentioned in theorem 4.5. Figure (5.5 and 5.6) show a better result than figure (5.7 and 5.8), at least the solutions at the boundaries are almost known.

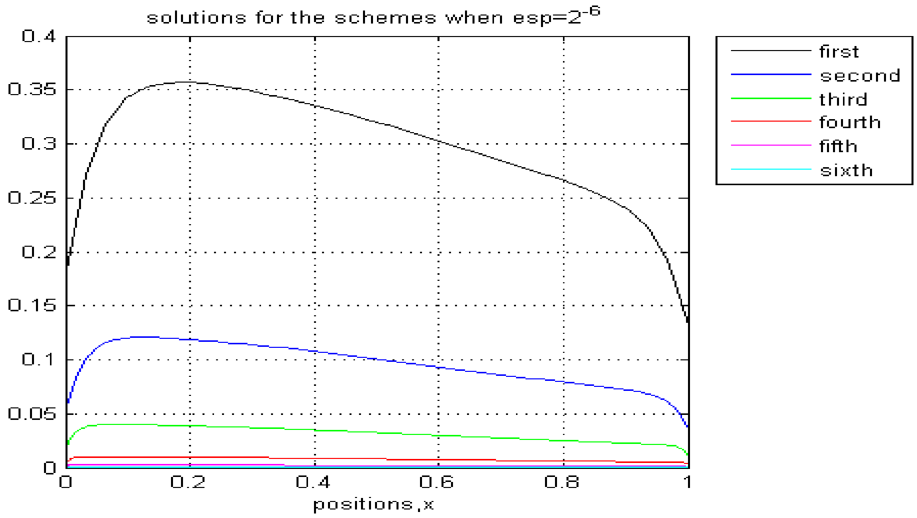

Figure 5.5: solution profile of (2.10)

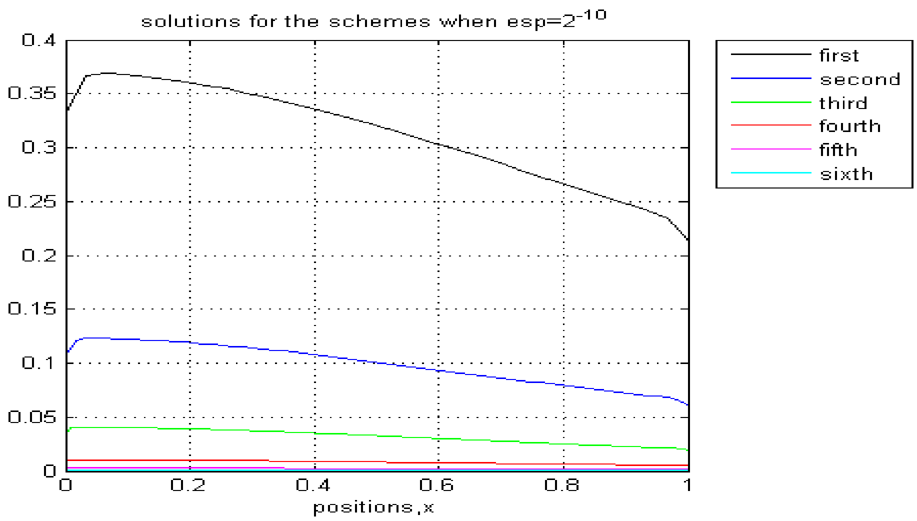

Figure 5.6: solution profile of (2.10)

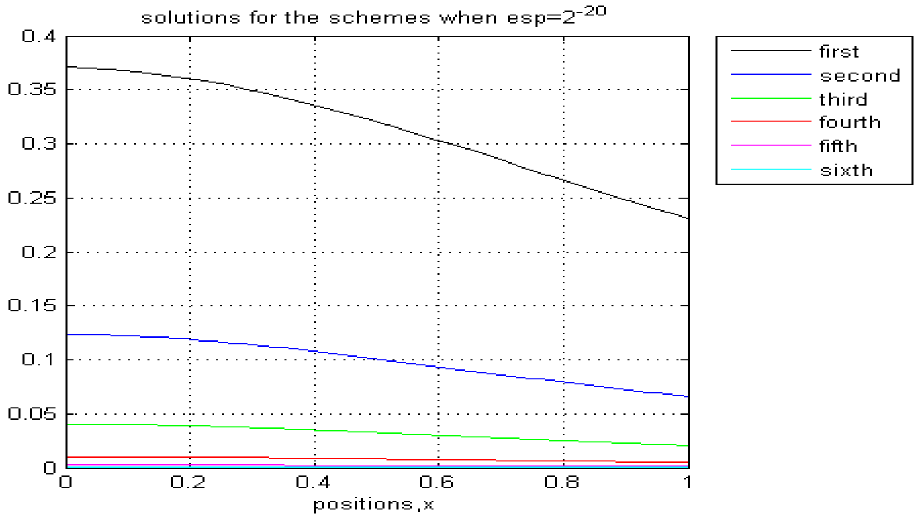

Figure 5.7: solution profile of (2.10)

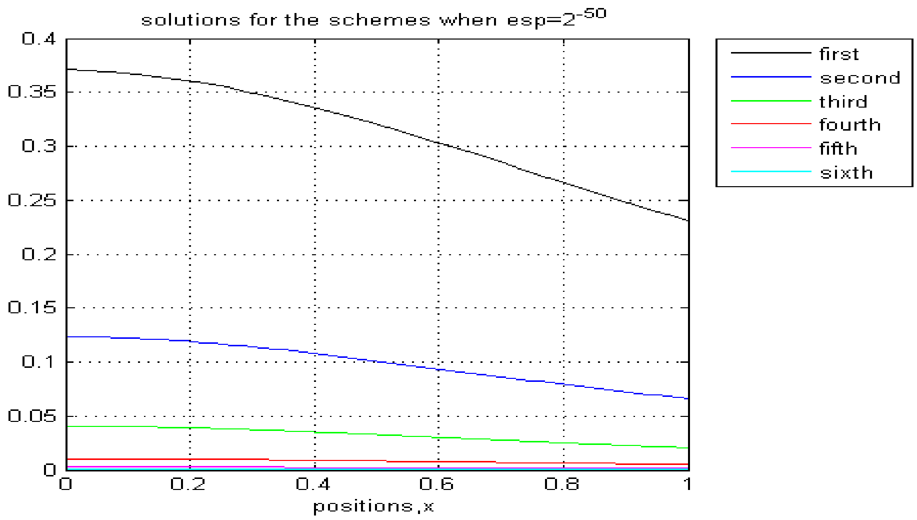

Figure 5.8: solution profile of (2.10)

5.3 Concluding remarks and Future works

In this thesis, we have studied a naive finite difference approximations for singularly perturbed parabolic reaction-diffusion problems. We have shown that the underlying schemes are not good approximations for such problems. First we proved the existence and uniqueness of solution of the discrete problems, we also demonstrated the consistency of our schemes to the continuous problem and we further analysed the stability in $l_\infty - norm\ and\ l_2 - norm$. It turned out that the solution of the discrete problems are not uniformly stable with respect to ε as the solution are unbounded when ε tends to zero. Error analysis also showed that the solution of the discrete problems do not converge uniformly to the solution of the continuous problem with respect to ε . Furthermore, a variant of the double mesh principle was introduced to simulate and investigate the errors of the discrete problems, as several cases were considered and found that the error increases as ε gets smaller and smaller with respect to the step size h . Finally we observed that the schemes displayed numerically a qualitative behaviour that coincided with our theoretical findings.
Subsequently, we intend to investigate other numerical schemes such as the Non Standard Finite Difference Scheme (NSFDS) or even the Finite Element Method (FEM) where we feel the method could be more effective and efficient than the method used.

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