MAXIMAL MONOTONE OPERATORS ON HILBERT SPACES

AND APPLICATIONS

BY

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Certification

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RECOMMENDED:

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Supervisor: PROF. K. EZZINBI.

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Head, Department of Pure and Applied Mathematics:

APPROVED:

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Date

ABSTRACT

Let H be a real Hilbert space and $A : D(A) \subset H \to H$ be an unbounded, linear, self-adjoint, and maximal monotone operator. The aim of this thesis is to solve $u'(t) + Au(t) = 0$, when A is linear but not bounded. The classical theory of differential linear systems cannot be applied here because the exponential formula $\exp(tA)$ does not make sense, since A is not continuous. Here we assume A is maximal monotone on a real Hilbert space, then we use the Yosida approximation to solve. Also, we provide many results on regularity of solutions. To illustrate the basic theory of the thesis, we propose to solve the heat equation in $L^2(\Omega)$. In order to do that, we use many important properties from Sobolev spaces, Green's formula and Lax-Milgram's theorem.

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DEDICATION

This work is dedicated to God Almighty and to my family, particularly my son Master, Suanu Kenule T. for his patience.

Contents

INTRODUCTION

This work exhibits existence and uniqueness results for a differential equation of first order with initial condition, governed by a maximal monotone operator on a Hilbert space. Firstly, we show the main proofs in maximal monotone operators needed for the existence and uniqueness theorem, the Hille-Yosida's Theorem on Hilbert spaces and its proof are also presented.

One very efficient way to describe some natural phenomena is through partial differential equation. This field of mathematics constantly explains and solves real world problems emanating from different fields of study. The concept of maximal monotonicity results is a very rich theory that has been developed in this field of mathematics since the early 1960s. Maximal monotone operators are considered due to the existence and uniqueness Theorem presented.

Definition 0.1. *Let $A : D(A) \subset H \to H$ be a linear operator. We define graph of A by*

$$Gr(A) = \{(x, y) \in H \times H : Ax = y\}.$$

Definition 0.2. *Let $A : D(A) \subset H \to H$ be an operator. A is said to be closed if for all sequence $(x_n) \in D(A)$ such that $x_n \to x$ in H and $Ax_n \to y$ in H. Then:*

(a) $x \in D(A)$,

(b) $y = Ax$.

The aim of this work is to solve the following differential equation

$$(P)\begin{cases} u' + Au = 0 \quad on \quad [0,\infty) \\ u(0) = u_0 \end{cases}$$

where H is a real Hilbert space, $u : [0,\infty) \to H$ is the unknown function, $A : D(A) \subset H \to H$ is maximal monotone, $u_0 \in H$ is a given initial state.
If A is linear and continuous, then existence and uniqueness is established by Theorem 2.7. However, if the operator is not continuous, the known Theorem 2.8 is applicable.
For $D(A) \neq H$, if $u_0 \in H \big| D(A)$, the solution of (P) may not belong to D(A). In this case, we then define a property for A such that if it is satisfied, we still have to obtain the existence and uniqueness of solution for the equation. We require A to be **self-adjoint**.
Given $A : D(A) \subset H \to H$, a linear operator with $\overline{D(A)} = H$. The adjoint operator $A^* : D(A^*) \subset H \to H$ is defined by

$$\langle u, Av\rangle = \langle A^*u, v\rangle \ for \ all \ v \in D(A)$$

The operator A is called self-adjoint if $A^* = A$.
A simplified version of the main Theorem in this thesis states as follows:
Given $A : D(A) \subset H \to H$ self-adjoint, linear, maximal monotone operator. Then for every $u_0 \in H$ there exists a unique solution u, say of (P).
Monotone operators on Hilbert spaces was first introduced by George Minty [9] in 1962 (see J.M Borwein,[2]), in order to study electrical networks, in his famous paper, although the operators were presented in the setting of partial differential equation by Browder in [5]. One of the most important and celebrated results was a characterization of maximal monotonicity given by Minty, which states that a monotone operator A is maximal monotone implies $I + A$ is surjective. By 1975, the main ideas about maximal monotonicity on Hilbert spaces were firmly established; an important and satisfactory monograph was published around that time by Brezis [3].
The first chapter is divided into three sections. The first section introduces Hilbert spaces, some examples and some of its properties. The second section briefly reviewed the main function spaces used in this thesis and the third section discussed the general Sobolev spaces.
Chapter two introduces the concept of the Yosida approximation of an operator and its main properties. We stated some existence and uniqueness Theorems which are

Theorem 2.7 and Theorem 2.8 in Hilbert spaces, here the operator is just maximal monotone. We further assumed the operator is self adjoint, hence we presented Theorem 2.15 which is the main interest of this thesis. We discussed self-adjoint operators in the second section. Finally, we apply Theorem 2.15 to heat (diffusion) problem.

For proper assimilation of this thesis, the reader is expected to be familiar with basic results and techniques of mathematical analysis. For example, good background material can be found in [3]. Other than this, the thesis is practically self-contained.

CHAPTER 1

Hilbert Spaces and Sobolev Spaces

The aim of this chapter is to recall some results on L^p spaces, distributions and Sobolev spaces that we use in the next chapter.

1.1 Hilbert spaces

A normed vector space is closed under vector addition and scalar multiplication. The norm defined on such a space generalises the elementary concept of the length of a vector. However, it is not always possible to obtain an analogue of the dot product, namely

$$a.b = a_1b_1 + a_2b_2 + a_3b_3$$

which yields

$$|a| = \sqrt{a.a}$$

which is an important tool in many applications. Hence, the question arises whether the dot product can be generalised to arbitrary vectors spaces. In fact, this can be done and leads to inner product spaces and complete inner product spaces, called Hilbert spaces.

Definition 1.1. *Let H be a linear space. An inner product on H is a function*

$$\langle .,. \rangle : H \times H \to \mathbb{R}$$

defined on $H \times H$ with values in $\mathbb{R}$ such that the following conditions are satisfied. For $x, y, z \in H, \lambda, \mu \in \mathbb{R}$

a) $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$

b) $\langle x, y\rangle = \langle y, x\rangle$

c) $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle y, z\rangle$

The pair $(H, \langle ., .\rangle)$ is called an inner product space. A Hilbert space, H is a complete inner product space (complete in the metric defined by the inner product).

1.1.1 Examples

1. **Euclidean space $\mathbb{R}^n$.**
   The space $\mathbb{R}^n$ is a Hilbert space with inner product defined by
   $$\langle x, y\rangle = \sum_{i=0}^{n} x_i y_i$$
   where,
   $$x = (x_1, x_2, ..., x_n) \text{ and } y = (y_1, y_2, ..., y_n)$$
   We obtain
   $$||x|| = \sqrt{\langle x, x\rangle} = (\sum_{i=0}^{n} x_i^2)^{\frac{1}{2}}$$

2. **Space $L^2(\Omega)$.**
   $L^2(\Omega) := \{f : \Omega \to \mathbb{R} : f \text{ is measurable and} \int_\Omega f^2 dx < \infty\}$, where Ω is an open set in $\mathbb{R}^n$, is a Hilbert space with the inner product defined
   $$\langle f, g\rangle = \int_\Omega f(x)g(x)dx$$
   and
   $$||f|| = (\int_\Omega |f(x)|dx)^{\frac{1}{2}}$$

3. **Hilbert sequence space l^2.**
   $l^2 := \{(x_n)_{n\geq 0} \subset \mathbb{R} : \sum_{i=0}^{\infty} |x_i|^2 < \infty\}$ is a Hilbert space with inner product defined by
   $$\langle x, y\rangle = \sum_{i=0}^{\infty} x_i y_i$$

Convergence of this series follows from Cauchy-Schwar'z inequality and the fact that $x, y \in l^2$, by assumption.
The norm is defined by

$$||x|| = (\sum_{i=0}^{\infty} |x_i|^2)^{\frac{1}{2}}$$

An inner product on H defines a norm on H given by

$$||x|| = \sqrt{\langle x, x\rangle}$$

and a metric on H given by

$$d(x, y) = ||x - y|| = \sqrt{\langle x - y, x - y\rangle}$$

Hence, inner products are normed spaces and Hilbert spaces are Banach space.
A norm on an inner product space satisfies the important parallelogram equality

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2) \; for \; all \; x, y \in H$$

Not all normed spaces are inner product spaces.

4. **Space** l^p.
   Let $1 \leq p < \infty$ be a fixed real number, we define l^p space as

$$l^p = \{(x_n)_{n\geq 0} \subset \mathbb{R} : \sum_{i=0}^{\infty} |x_i|^p < \infty\}.$$

   When $p \neq 2$, l^p is not a Hilbert space.

5. **Space** $C([a, b]; \mathbb{R})$.
   The space $C([a, b]; \mathbb{R})$ provided with supremum norm is not a Hilbert space.

Proposition 1.2. *Let $(H, \langle ., .\rangle)$ be an inner product space. Then, for all $x, y \in H$*

a. $|\langle x, y\rangle| \leq ||x||||y||$ *(Schwar'z inequality) where the equality holds if and only if x,y are linearly dependent.*

b. $||x + y|| \leq ||x|| + ||y||$ *(triangle inequality) where the equality holds if and only if x=cy $(c \geq 0)$*

Proposition 1.3. ***(Continuity of inner product).*** *Let $(x_n)_{n\geq 0}, (y_n)_{n\geq 0}$ be sequences in H, such that $x_n \to x$ and $y_n \to y$, then*

$$\langle x_n, y_n\rangle \to \langle x, y\rangle.$$

1.2 Function Spaces

Here, we recall the definitions of functions spaces used in this thesis.

1.2.1 L^p Spaces

Definition 1.4. *Let Ω be a nonempty open set in $\mathbb{R}^n$, for $1 \leq p < \infty$, we define*

$$L^p(\Omega) := \{f : \Omega \to \mathbb{R} : f \textit{ is measurable and } \int_\Omega |f(x)|^p dx < \infty\}$$

Remark 1.5. *We say two functions f and g are equivalent if $f = g$ almost everywhere. Then we define $L^p(\Omega)$ spaces as the equivalent classes for this relationship. The space $L^p(\Omega)$ can be seen as a space of functions. We do however, need to be careful sometimes. For example, saying that $f \in L^p(\Omega)$ is continuous means that f is equivalent to a continuous function. Now, for $f \in L^p(\Omega)$, we define*

$$||f||_p = (\int_\Omega |f(x)|^p dx)^{\frac{1}{p}}, \qquad 1 \leq p < \infty$$

The $L^p(\Omega)$ is a Banach space.

1.2.2 Test functions

Definition 1.6. *Let $f : \Omega \to \mathbb{R}$ be a continuous function. The* ***support*** *is*

$$supp(f) := \overline{\{x \in \Omega : f(x) \neq 0\}}$$

The function is said to be of compact support on Ω if the support is a compact set contained inside Ω.

Definition 1.7. *The space of test functions in Ω, denoted by $D(\Omega)$ is the space of all C^∞ functions defined on Ω which have compact supports in Ω.*

$C^\infty(\Omega)$ denotes the space of all real-valued functions on Ω of class C^∞.
$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n$ is called multi-index with length $|\alpha| = \sum_{i=1}^{n} \alpha_i$.
Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. We write $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}}$ and it acts on the space $C^\infty(\Omega)$. Thus, for $f \in C^\infty(\Omega)$, $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}}$ is it partial derivatives of order $|\alpha|$.

Definition 1.8. *Let $\{\psi_n\}_{n \geq 0}$ be a sequence in $D(\Omega)$ and $\psi \in D(\Omega)$.*
$\psi_n \to \psi$ in $D(\Omega)$ if
1. $\exists$ *a compact set* $K \subset \Omega : supp(\psi), supp(\psi_n) \subset K,\ for\ all\ n \geq 1$
2. $D^\alpha \psi_n \to D^\alpha \psi\ uniformly\ on\ K, \forall \alpha \in \mathbb{N}^n$.

1.2.3 Distributions

Definition 1.9. *A distribution on Ω is any continuous linear mapping $T : D(\Omega) \to \mathbb{R}$. The set of all distributions is denoted by $D'(\Omega)$.*

Remark 1.10. *By linearity, to show that T is continuous, it is enough to show that, if $\psi_n \to 0$ in $D(\Omega)$, then it is enough to show that $(T, \psi_n) \to 0$ in $\mathbb{R}$.*

Definition 1.11. *A function $f : \Omega \to \mathbb{R}$ is locally integrable if for any compact set, $K \subset \Omega$, we have that*

$$\int_K |f(x))|dx < \infty$$

The collection of all locally integrable functionals on Ω is denoted by $L^1_{loc}(\Omega)$
If $f \in C(\Omega)$, then $f \in L^1_{loc}(\Omega)$. For any $f \in L^1_{loc}(\Omega)$, f gives a distribution T_f defined by

$$(T_f, \psi) = \int_\Omega f(x)\psi(x)dx, \ \textit{for all } \psi \in D(\Omega)$$

Definition 1.12. *If $T \in D'(\Omega)$ is a distribution on an open set $\Omega \subset \mathbb{R}^n$, and if α is any multi-index, we define the distribution $D^\alpha T$ by*

$$(D^\alpha T, \psi) = (-1)^{|\alpha|}(T, D^\alpha \psi) \tag{1.1}$$

and it is the α^{th} partial derivative of T.
So, the map $D^\alpha : D'(\Omega) \to D'(\Omega)$ defined in (1.1) is linear and continuous.

1.3 Sobolev spaces

Sobolev spaces are based on the concept of weak (distributional) derivatives. It gives us a modern approach to the study of differential equations.

Definition 1.13. *Let $1 \leq p < \infty$ and k be a non-negative integer. Then, Sobolev space $W^{k,p}(\Omega)$ is defined by*

$$W^{k,p}(\Omega) := \{u \in L^P(\Omega) : D^\alpha u \in L^p(\Omega), \forall \ 0 \leq |\alpha| \leq k\}$$

The space is equipped with the norm

$$||u||_{W^{k,p}(\Omega)} := (\sum_{0 \leq |\alpha| \leq k} ||D^\alpha u||^p_{L^P(\Omega)})^{\frac{1}{p}}$$

$W_0^{K,p}(\Omega) = \overline{D(\Omega)}\big|_{W^{k,p}(\Omega)}$ *i.e.,* $W_0^{K,p}(\Omega)$ *is the closure of* $D(\Omega)$ *with respect to the norm* $||.||_{W^{k,p}(\Omega)}$.
When p=2, we write $H^k(\Omega) = W^{k,2}(\Omega)$ *and* $H_0^k(\Omega) = W_0^{k,2}(\Omega)$ *and these are real Hilbert spaces with the following inner product*

$$\langle u, v\rangle_{H^k(\Omega)} = \sum_{0\leq|\alpha|\leq k} \int_\Omega D^\alpha u D^\alpha v dx$$

and the norm

$$||u||_{H^k(\Omega)} = (\sum_{0\leq|\alpha|\leq k} ||D^\alpha u||^2_{L^2(\Omega)})^{\frac{1}{2}}$$

For, k=0,

$$W^{0,p}(\Omega) = L^P(\Omega).$$

$W^{k,p}(\Omega)$ *are Banach spaces.*
Given that Ω *is smooth, then:*

$$W_0^{k,p}(\Omega) := \{u \in W^{k,p}(\Omega) : u = Du = ... = D^{k-1}u = 0 \ on \ \partial\Omega\}.$$

For p=2, we have

$$W_0^{k,2}(\Omega) := \{u \in W^{k,2}(\Omega) : u = Du = ... = D^{k-1}u = 0 \ on \ \partial\Omega\}$$

For p=2,and k=1 , we have

$$W_0^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega) = H^1(\Omega) : u = 0 \ on \ \partial\Omega\}$$

and we denote it by $H_0^1(\Omega)$
For p=2, k=2, we write

$$W^{2,2}(\Omega) = H^2(\Omega).$$

Theorem 1.14. *Let* Ω *be smooth and* $u \in L^2(\Omega)$ *such that* $\Delta u \in L^2(\Omega)$. *Then* $u \in H^2(\Omega)$.

Green's Formula

Theorem 1.15. *Let* Ω *be bounded and smooth. Let* $u \in H^2(\Omega)$ *and* $v \in H^1(\Omega)$*, then*

$$\int_{\Omega} \nabla u . \nabla v dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} ds - \int_{\Omega} v \Delta u dx$$

where $\frac{\partial u}{\partial n}$ *denotes the normal derivative defined by* $\frac{\partial u}{\partial n} = \nabla u . \overrightarrow{n}$.
where $\overrightarrow{n}$ *denotes the normal vector.*

if $u = v$, then

$$\begin{aligned} \int_{\Omega} ||\nabla u||^2 dx &= \int_{\partial\Omega} u \frac{\partial u}{\partial n} ds - \int_{\Omega} u \Delta u ds \\ &= \int_{\partial\Omega} u \frac{\partial u}{\partial n} ds + \int_{\Omega} u(-\Delta u) ds \end{aligned}$$

Then,

$$\int_{\Omega} (-\Delta u) u dx = \int_{\Omega} ||\nabla u||^2 dx - \int_{\partial\Omega} u \frac{\partial u}{\partial n} ds$$

Theorem 1.16. ***(Lax-Milgram)****. Let* $a : V \times V \to \mathbb{R}$ *be a bilinear, continuous, and coercive functional. Then, for each* $f \in V^* \; \exists! \; u^* \in V$:

$$a(u^*, v) = (f, v), \; for \; all \; v \in V$$

Proposition 1.17. ***(Poincaré's inequality)****. Suppose* Ω *is a bounded set. Then there exists a constant* $C(\Omega) > 0$ *such that*

$$||u||_{L^2(\Omega)} \le C(\Omega) ||\nabla u||_{L^2(\Omega)}, \; for \; all \; u \in W_0^{1,2}(\Omega).$$

CHAPTER 2

Maximal Monotone Operators on Hilbert spaces

Some basic proofs about maximal monotone operators are presented in this chapter. These will be used in establishing our main existence and uniqueness theorem.
The study of maximal monotone operators started being considered around fifty years ago. Many differential equations that arise from different kind of applications are modelled using this kind of operators. The conventional technique for existence and uniqueness did not work to solve these problems because the operators are usually not continuous. Hence, the concept of maximal monotonicity is very useful.

Definition 2.1. *Let $A : D(A) \subset H \to H$ be a linear operator. A is monotone if*

$$\langle Av, v\rangle \geq 0, \quad for\ all\ v\ \in D(A).$$

It is called maximal monotone, if in addition R(I+A)=H, i.e.,
$for\ all\ f \in H,\ there\ exists\ u \in D(A)\ such\ that\ u + Au = f.$

Theorem 2.2. *Let A be a maximal monotone operator. Then*

1. *D(A) is dense in H*

2. *A is a closed operator*

3. *Let $\mathcal{L}(H)$ denote the space of all bounded linear operators on H, then $\forall\lambda > 0$, $(I + \lambda A)$ is bijective from D(A) onto H, $(I + \lambda A)^{-1}$ is bounded operator and $||(I + \lambda A)^{-1}||_{\mathcal{L}(H)} \leq 1$*

Proof:

First, observe that given any $f \in H$, there exists a unique $u \in D(A)$ such that $u + Au = f$, since if $\overline{u}$ is another solution, we have
$u - \overline{u} + A(u - \overline{u}) = 0$. Taking the scalar product with $(u - \overline{u})$ and using monotonicity, we see that $u - \overline{u} = 0$.

1. Suppose $\overline{D(A)} \neq H$, then $\exists\ f \in H, f \neq 0$ such that

$$\langle f, v \rangle = 0, \ for\ all\ v \in D(A).$$

But A is a maximal monotone operator, then

$$\exists\ v_0 \in D(A) : v_0 + Av_0 = f$$

Since, $\langle f, v \rangle = 0\ for\ all\ v \in D(A)$, thus $\langle f, v_0 \rangle = 0$
Then,

$$\begin{aligned} 0 &= \langle f, v_0 \rangle \\ &= \langle v_0 + Av_0, v_0 \rangle \\ &= \langle v_0, v_0 \rangle + \langle v_0, Av_0 \rangle \end{aligned}$$

Using the fact that A is linear and monotone, then

$$0 \geq ||v_0||^2$$

Then $v_0 = 0$, thus $f = 0$. This is a contradiction. Hence D(A) is dense in H.

2. Let $\{u_n\}_{n\geq 1} \subset D(A)$ be a sequence such that $u_n \to u$ and $Au_n \to f$
Now, $u_n + Au_n \to u + f$, so $(I + A)u_n \to u + f$.
Hence $u_n \to (I + A)^{-1}(u + f)$.
From uniqueness of limits, we have that $u = (I + A)^{-1}(u + f)$, then $u + Au = u + f$ implying that $Au = f$ and $u \in D(A)$
Hence, A is closed

3. We will prove that if $R(I + \lambda_0 A) = H$ for some $\lambda_0 > 0$, then $R(I + \lambda A) = H$ for every $\lambda > \frac{\lambda_0}{2}$. Note first as in part (1.) that for every $f \in H$ there is a unique $u \in D(A)$ such that $u + \lambda_0 Au = f$. Moreover, the map $f \mapsto u$, denoted by $(I + \lambda_0 A)^{-1}$, is a bounded linear operator with $||(I + \lambda_0 A)^{-1}||_{L(H)} \leq 1$. A is maximal

monotone , then for all $x \in H$

$$\begin{aligned}||(I+\lambda A)x||^2 &= ||x+\lambda Ax||^2\\ &= ||x||^2+\lambda^2||Ax||^2+2\lambda\langle x, Ax\rangle\\ &\geq ||x||^2\end{aligned}$$

Thus, we deduce that $\ker(I+\lambda A)=\{0\}$.
Hence, $I+\lambda A$ is 1-1.
Let $\lambda > \lambda_0$, We try to solve the equation

$$u+\lambda Au = f \; with \; \lambda > 0$$

which can be expressed as

$$u = (I+\lambda_0)^{-1}(\frac{\lambda_0}{\lambda}f + (1-\frac{\lambda_0}{\lambda})u) \tag{2.1}$$

But, $R(I+\lambda_0 A) = H$, which implies that $I+\lambda_0 A : D(A) \to H$, is bijective. On the other hand, $||(I+\lambda_0 A)x|| \geq ||x||$.
Taking, $x = (I+\lambda_0 A)^{-1}y$, we get

$$||(I+\lambda_0 A)^{-1}y|| \leq ||y||.$$

Hence, $||(I+\lambda_0 A)^{-1}|| \leq 1$
Let $\Psi : H \to H$ be defined by

$$\Psi(u) = (I+\lambda_0)^{-1}(\frac{\lambda_0}{\lambda}f + (1-\frac{\lambda_0}{\lambda})u)$$

Then,

$$\begin{aligned}||\Psi(u)-\Psi(v)|| &\leq ||(I+\lambda_0)^{-1}|||1-\frac{\lambda_0}{\lambda}|||u-v||\\ &\leq |1-\frac{\lambda_0}{\lambda}|||u-v||\end{aligned}$$

If $|1-\frac{\lambda_0}{\lambda}| < 1$, i.e., $\lambda > \frac{\lambda_0}{2}$, then Ψ is a strict contraction and thus has a unique fixed point, u in H
Since, $I+A$ is surjective, taking $\lambda_0 = 1$, we get that $I+\lambda A$ is surjective for every $\lambda > \frac{1}{2}$, repeating the arguments, we get that for every
$\lambda > \frac{1}{4}$, $\lambda > \frac{1}{8}$, ...
Hence, by induction, we conclude that $R(I+\lambda A) = H$ for $\lambda > 0$.

Example 2.3. *Let X be a real Hilbert space, $T: H \to H$ be a non-expansive map, i.e., $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in H$. Then the operator I - T is monotone. Infact, let $x, y \in H$, then*

$$\begin{aligned} \langle x - y, (I - T)x - (I - T)y \rangle &= \langle x - y, x - Tx - y + Ty \rangle \\ &= \langle x - y, x - y + Ty - Tx \rangle \\ &= ||x - y||^2 - (x - y, Tx - Ty) \\ &\geq ||x - y||^2 - ||x - y|| ||Tx - Ty|| \\ &= ||x - y||^2 - ||x - y||^2 \\ &= 0 \end{aligned}$$

Here, we have used Cauchy's inequality and the fact that T is non-expansive.Thus, we have I-T is monotone on H.

2.1 Examples of maximal monotone operators

1. Let $l^2 = \{(x_n)_{n \geq 0} \subset \mathbb{R} : \sum_{n=0}^{\infty} |x_n|^2 < \infty\}$. Let $(a_n)_{n \geq 0} \subset \mathbb{R}^+$, we define the operator $A : D(A) \subset l^2 \to l^2$ by $A((x_n)_{n \geq 0}) = (a_n x_n)_{n \geq 0}$.
   where $D(A) = \{(x_n)_{n \geq 0} \subset l^2 : (a_n x_n)_{n \geq 0} \in l^2\}$
   Then A is maximal monotone.

   Proof:
   The inner product on l^2 is defined by

   $$\langle x, y \rangle = \sum_{n=0}^{\infty} x_n y_n$$

   Then, $\langle Ax_n, x_n \rangle = \sum_{n=0}^{\infty} Ax_n x_n = \sum_{n=0}^{\infty} a_n {x_n}^2 \geq 0$
   Hence, A is monotone.
   Let $(f_n)_{n \geq 0} \in l^2$, we find $(u_n)_{n \geq 0} \in D(A)$, such that $u_n + Au_n = f_n$.
   Since $a_n \geq 0, \forall n \geq 0$, then $u_n = \frac{f_n}{1 + a_n} \leq f_n$.
   Thus,

   $$\sum_{n=0}^{\infty} {u_n}^2 = \langle u_n, u_n \rangle = ||u||^2 \leq ||f_n||^2 = \langle f_n, f_n \rangle = \sum_{n=0}^{\infty} {f_n}^2 < \infty$$

   Hence, A is maximal monotone.

2. Let Ω be open set in $\mathbb{R}^n$ and $m : \Omega \to \mathbb{R}^+$ be a measurable function. We define the operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ by $Af = mf$.
where $D(A) = \{f \in L^2(\Omega) : mf \in L^2(\Omega)\}$
Then A is maximal monotone on $L^2(\Omega)$.

Proof:
Since $m \geq 0$, then $\langle Af, f\rangle = \int\limits_{\Omega} mf^2 \geq 0$.
Hence A is monotone.
Let $f \in L^2(\Omega)$, we find $g \in D(A)$ such that $g + Ag = f$.
But $g = \frac{f}{1+m} \leq f$, so $u^2 \leq f^2$, implying that $u \in L^2(\Omega)$
Hence A is maximal monotone.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth. Consider the operator $-\Delta : H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega \to L^2(\Omega)$ defined by

$$-\Delta u := \quad -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}$$

$-\Delta$ is maximal monotone.

Proof:
$H_0^1(\Omega) := \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$, using Green's formula, we have

$$\begin{aligned}
\langle -\Delta u, u\rangle &= \int\limits_{\Omega} (-\Delta u)u \\
&= \int\limits_{\Omega} ||\nabla u||^2 - \int\limits_{\partial\Omega} \frac{\partial u}{\partial \sigma} u d\sigma \\
&= \int\limits_{\Omega} ||\nabla u||^2 \\
&\geq 0
\end{aligned}$$

Next, we show that $R(I - \Delta) = L^2(\Omega)$. i.e., for any $f \in L^2(\Omega)$ $\exists!$
$u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $u - \Delta u = f$.
Let $V = H_0^1(\Omega)$ with the following inner product

$$\langle u, v\rangle = \int\limits_{\Omega} \nabla u.\nabla v dx \ for \ all \ u, v \in H_0^1(\Omega)$$

We show that V is a Hilbert space.
$H_0^1(\Omega)$ is closed, since $H_0^1(\Omega) = \overline{D(\Omega)}\big|_{H^1(\Omega)}$ where Ω is bounded. Thus, $H_0^1(\Omega)$ is a

Hilbert space provided with the inner product of $H^1(\Omega)$

We now prove that $||u||_{H^1(\Omega)} \sim (\int_\Omega ||\nabla u||^2 dx)^{\frac{1}{2}}$.

Here, we use Poincare's inequality

$$\int_\Omega u^2 \leq c \int_\Omega ||\nabla u||^2$$

So, $\int_\Omega ||\nabla u||^2 dx + \int_\Omega u^2 dx \leq (1+c) \int_\Omega ||\nabla u||^2 dx$

But, $(1+c) \int_\Omega ||\nabla u||^2 dx \leq (1+c)||u||^2_{H^1\Omega}$

Then,

$$||u||_{H^1(\Omega)} \sim (\int_\Omega ||\nabla u||^2 dx)^{\frac{1}{2}}.$$

Therefore, $H^1_0(\Omega)$ provided with the new norm

$$||u||_{H^1_0(\Omega)} = (\int_\Omega ||\nabla u||^2)^{\frac{1}{2}}.$$

is complete.

But this norm comes from the inner product on $H^1_0(\Omega)$ which is

$$\begin{aligned} \langle u, v\rangle_{H^1_0(\Omega)} &= \int_\Omega \nabla u . \nabla v dx \\ &= \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}. \end{aligned}$$

Let $a(.,.) : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ be given by

$$a(u,v) = \int_\Omega \nabla u . \nabla v dx \quad \forall \ u, v \in H^1_0(\Omega)$$

$a(u,v) = \langle u, v\rangle_{H^1_0(\Omega)}$. Since $H^1_0(\Omega)$ provided with the inner product $\langle , \rangle_{H^1_0(\Omega)}$ ia a Hilbert space. Then $a(.,.)$ is continuous, bilnear and coercive.

By Theorem 1.16 , we deduce that $\exists! \ u \in H^1_0(\Omega)$ such that

$a(u,v) = \langle f, v\rangle$ for all $v \in H^1_0(\Omega)$.

Take $\psi \ in \ D(\Omega)$, it implies $a(u,v) = \langle f, v\rangle$. Hence -$\Delta u + u = f$ in distribution. Since $\Delta u = f - u \in L^2(\Omega)$ and from Theorem 1.14, we have that $u \in H^2(\Omega)$ and conclude thus, for $f \in L^2(\Omega)$ $\exists! \ u \in H^1_0(\Omega) \cap H^2(\Omega)$ such that -$\Delta u + u = f$. This means that $R(I - \Delta) = L^2(\Omega)$.

2.2 Yosida Approximation of a maximal monotone operator

Let A be a maximal monotone operator on a real Hilbert space H. Our main goal is to solve the following equation $u' + Au = 0$, where A is linear, maximal monotone but not necessarily continuous. This problem is easily solved by Theorem 2.7 if A is continuous on H. A natural step then is to find a sequence of Lipschitz functions that approximates A in some sense. This idea was introduced by the Japanese mathematician K. Yosida.

Definition 2.4. *Let A be a maximal monotone operator on a real Hilbert space H. The Yosida approximation of A corresponding to $\lambda > 0$ is defined by*

$$A_\lambda := \frac{1}{\lambda}(I - (I + \lambda A)^{-1})$$

where I denotes the identity on H.
Let J_λ be defined

$$J_\lambda := (I + \lambda A)^{-1} \ for \ \lambda > 0$$

This is called the Resolvent of A corresponding to $\lambda > 0$.

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$$

If A is maximal monotone, then

$$D(A_\lambda) = D(J_\lambda) = R(I + \lambda A) = H$$

.

Remark 2.5. *A_λ is a bounded linear operator on H.*

Theorem 2.6. *Let A be a maximal monotone operator. Then*

a) $A_\lambda v = A(J_\lambda v)$ *for all* $v \in H$ *and for all* $\lambda > 0$.

b) $A_\lambda v = J_\lambda(Av)$ *for all* $v \in D(A)$ *and for all* $\lambda > 0$.

c) $||A_\lambda v|| \leq ||Av||$ *for all* $v \in D(A)$ *and for all* $\lambda > 0$.

d) $\lim_{\lambda \to 0} J_\lambda v = v$ *for all* $v \in H$.

e) $\lim_{\lambda \to 0} A_\lambda v = Av$ *for all* $v \in D(A)$.

f) $\langle A_\lambda v, v\rangle \geq 0$ *for all* $v \in D(A)$ *and for all* $\lambda > 0$.

g) $||A_\lambda v|| \leq \frac{1}{\lambda}||v||$ *for all* $v \in H$ *and for all* $\lambda > 0$.

Proof:

a) Let $v \in H$, then $J_\lambda v = (I + \lambda A)^{-1}v$.

Thus,

$$J_\lambda v + \lambda A(J_\lambda v) = v \tag{2.2}$$

But, $A_\lambda v = (\frac{1}{\lambda}(I - J_\lambda))v$. Then, from equation (2.2) we deduce

$$\begin{aligned} A_\lambda v &= \frac{1}{\lambda}(v - (v - \lambda A(J_\lambda v)) \\ &= \frac{1}{\lambda}(\lambda A(J_\lambda v)) \\ &= A(J_\lambda v). \end{aligned}$$

b)

$$\begin{aligned} A_\lambda v &= \frac{1}{\lambda}(v - J_\lambda v) \\ \lambda A_\lambda v &= v - J_\lambda v \end{aligned}$$

From (a.), we have that $A_\lambda v = A(J_\lambda v)$, for all $v \in H$, $\lambda > 0$

Then,

$$\begin{aligned} A_\lambda v + Av - Av &= A(J_\lambda v) \\ A_\lambda v + Av - A(J_\lambda v) &= Av \\ A_\lambda v + A(v - J_\lambda v) &= Av \\ A_\lambda v + A(\lambda A_\lambda v) &= Av \\ A_\lambda v + \lambda A(A_\lambda v) &= Av \\ (I + \lambda A)(A_\lambda v) &= Av \\ A_\lambda v &= (I + \lambda A)^{-1}Av \\ A_\lambda v &= J_\lambda(Av) \end{aligned}$$

c) From b), we have $A_\lambda v = J_\lambda(Av)$, for all $v \in D(A)$, $\lambda > 0$

Then,

$$\begin{aligned} ||A_\lambda v|| &= ||J_\lambda(Av)|| \\ &\leq ||J_\lambda|| ||Av|| \\ &\leq ||Av|| \qquad [\text{since} \quad ||(I + \lambda A)^{-1}||_{L(H)} \leq 1] \\ ||A_\lambda v|| &\leq ||Av|| \end{aligned}$$

d) Assuming $v \in D(A)$. Then

$$||v - J_\lambda v|| = \lambda||A_\lambda v|| \leq \lambda||Av|| \quad \textit{by (b.)}$$

and thus $\lim_{\lambda\to 0} J_\lambda v = v$.

Suppose now that v is in H arbitrary. Given, $\epsilon > 0$ be given. Since $D(A)$ is dense in H, $\exists\, v_1 \in D(A) : ||v - v_1|| \leq \epsilon$. We have

$$\begin{aligned}
||J_\lambda v - v|| &= ||J_\lambda v - J_\lambda v_1 + J_\lambda v_1 - v_1 + v_1 - v|| \\
&\leq ||J_\lambda v - J_\lambda v_1|| + ||J_\lambda v_1 - v_1|| + ||v_1 - v|| \\
&\leq ||J_\lambda|| ||v - v_1|| + ||J_\lambda v_1 - v_1|| + ||v_1 - v|| \\
&\leq ||v - v_1|| + ||J_\lambda v_1 - v_1|| + ||v_1 - v|| \\
&\leq 2\epsilon + ||J_\lambda v_1 - v_1|| \\
&= 2\epsilon + ||\lambda A_\lambda v_1|| \\
&= 2\epsilon + \lambda||A_\lambda v_1|| \\
&\leq 2\epsilon + \lambda||Av_1|| \qquad \text{[from c)]} \\
||J_\lambda v - v|| &= 2\epsilon + \lambda||Av_1|| \\
\overline{\lim_{\lambda\to 0}}||J_\lambda v - v|| &\leq 2\epsilon, \quad \forall\, \epsilon > 0 \\
\text{So,} \quad \lim_{\lambda\to 0}||J_\lambda v - v|| &= 0
\end{aligned}$$

e)

$$\begin{aligned}
||A_\lambda v - Av|| &= ||J_\lambda(Av) - Av|| \\
\lim_{\lambda\to 0} J_\lambda(Av) &= Av \qquad \text{from (d)} \\
\Rightarrow \quad \lim_{\lambda\to 0} A_\lambda v &= Av \qquad [\text{since } A_\lambda v = J_\lambda(Av)]
\end{aligned}$$

f)

$$\begin{aligned}
\langle A_\lambda v, v\rangle &= \langle A_\lambda v, v - J_\lambda v + J_\lambda v\rangle \\
&= \langle A_\lambda v, v - J_\lambda v\rangle + \langle A_\lambda v, J_\lambda v\rangle \\
&= \langle A_\lambda v, \lambda A_\lambda v\rangle + \langle A(J_\lambda v), J_\lambda v\rangle \\
&= \lambda||A_\lambda v||^2 + \langle A(J_\lambda v), J_\lambda v\rangle \\
&\geq \lambda||A_\lambda v||^2 \qquad [\text{since } A \text{ is monotone}] \\
&\geq 0
\end{aligned}$$

So,

$$\langle A_\lambda v, v\rangle \geq 0$$

g)

$$\begin{aligned}
\langle A_\lambda v, v\rangle &= \langle A_\lambda v, v - J_\lambda v + J_\lambda v\rangle \\
&= \langle A_\lambda v, v - J_\lambda v\rangle + \langle A_\lambda v, J_\lambda v\rangle \\
&= \langle A_\lambda v, \lambda A_\lambda v\rangle + \langle A(J_\lambda v), J_\lambda v\rangle \\
&\geq \lambda ||A_\lambda v||^2
\end{aligned}$$

Then, $\lambda ||A_\lambda v||^2 \leq \langle A_\lambda v, v\rangle \leq ||A_\lambda v||||v||$
and

$$\begin{aligned}
\lambda ||A_\lambda v||^2 &\leq ||A_\lambda v||||v|| \\
\lambda ||A_\lambda v|| &\leq ||v|| \\
||A_\lambda v|| &\leq \frac{1}{\lambda}||v||
\end{aligned}$$

From Theorem 2.6, we get a very useful relation between the Yosida approximation and the resolvent which is constantly used in this thesis. We are interested in the Yosida approximation A_λ because it approximates the operator A. Thus, $(A_\lambda)_{\lambda>0}$ is a family of bounded operators that "approximates" the operator A as $\lambda \to 0$.
We want to discuss the existence and uniqueness of solutions of the problem.

$$\begin{cases} \frac{du}{dt} + Au = 0 \quad on \quad [0, \infty) \\ u(0) = u_0 \end{cases}$$

where A is linear but not necessarily continuous. Here, we assume that A is a maximal monotone operator on a real Hilbert space, H. We start with this very classical result.

Theorem 2.7. ***Cauchy-Lipschitz-Picard****: Let E be a Banach space and $F : E \to E$ be a lipschitz map i.e., there is a constant $L > 0$ such that*

$$||Fu - Fv|| \leq ||u - v|| \text{ for all } u, v \in E$$

Then given any $u_0 \in E$, there exists a unique solution $u \in C^1([0, \infty), E)$ of the following problem

$$(P_1) \quad \begin{cases} \frac{du}{dt}(t) = F(u(t)) \quad on \quad [0, \infty) \\ u(0) = u_0. \end{cases}$$

Proof: Existence:

Solving (P_1) amounts to finding some $u \in C([0,\infty);E)$ satisfying the integral equation

$$u(t) = u_0 + \int_0^t F(u(s))ds \tag{2.3}$$

Given $k > 0$, to be fixed, set

$$X = \{u \in C([0,\infty);E); \sup_{t\geq 0} e^{-kt}||u(t)|| < \infty)$$

X is a normed vector space, we claim that is is a Banach space for the norm $||u||_X = \sup_{t\geq 0} e^{-kt}||u(t)||$

Proof of claim:

Let $(u_n)_{n\geq 0}$ be a Cauchy sequence in X. Then for all $\epsilon > 0, \exists\, n_0 \in \mathbb{N} : \sup_{t\geq 0} e^{-kt}||u_n(t) - u_m(t)|| < \epsilon, \forall\, n,m \geq n_0$.

So, For all $t \geq 0$, $n,m \geq n_0$, we have that

$$e^{-kt}||u_n(t) - u_m(t)|| < \epsilon \tag{2.4}$$

Then, $(u_n(t))_{n\geq 0}$ is a Cauchy sequence in E. Since E is complete, then $u_n(t) \to v(t) \in E, \forall\, t \geq 0,\ as\ n \to \infty$.

We have to show that $v \in X$, $\sup_{t\geq 0} e^{-kt}||u_n(t) - v(t)|| \to 0,\ as\ n \to \infty$.

From equation (2.6), letting $m \to \infty$, we get for all $t \geq 0$,

$$e^{-kt}||u_n(t) - v(t)|| < \epsilon,\ for\ all\ n \geq n_0$$

This implies that $\sup_{t\geq 0} e^{-kt}||u_n(t) - v(t)|| < \epsilon$, $\forall n \geq n_0$. Thus, $e^{-kt}u_n(t) \to e^{-kt}v(t)$ uniformly on $\mathbb{R}^+$. Since, $t \mapsto e^{-kt}u_n(t)$ is continuous, then $t \mapsto e^{-kt}v(t)$. But $v(t) = e^{kt}(e^{-kt}v(t))$, therefore $t \mapsto v(t)$. Since,$e^{-kt}u_n(t) \to e^{-kt}v(t)$ uniformly on $\mathbb{R}^+$, taking $\epsilon = 1$, we get that for some $n \in \mathbb{N}$,

$$\begin{aligned}
e^{-kt}v(t) &= e^{-kt}v(t) - e^{-kt}u_n(t) + e^{-kt}u_n(t) \\
e^{-kt}||v(t)|| &\leq ||e^{-kt}(v(t) - u_n(t))|| + e^{-kt}||u_n(t)|| \\
\sup_{t\geq 0} e^{-kt}||v(t)|| &\leq \sup_{t\geq 0} e^{-kt}||u_n(t) - v(t)|| + \sup_{t\geq 0} e^{-kt}||u_n(t)|| \\
\sup_{t\geq 0} e^{-kt}||v(t)|| &\leq 1 + \sup_{t\geq 0} e^{-kt}||u_n(t)||
\end{aligned}$$

So, $\sup_{t\geq 0} e^{-kt}||v(t)|| < \infty$, hence $v \in X$, and $||u_n - v||_X \to 0$.

Thus, X is complete.

For every $u \in X$, the function Φu defined by

$$(\Phi u)(t) = u_0 + \int_0^t F(u(s))ds$$

Then

a) Φu belongs to X

b) $||\Phi u - \Phi v||_X \leq \frac{L}{k}||u - v||_X$, for all $u, v \in X$

The prove of those is as follows:

$$\begin{aligned} (\Phi u)(t) &= u_0 + \int_0^t F(u(s))ds \\ e^{-kt}||\Phi u(t)|| &\leq e^{-kt}||u_0|| + e^{-kt}\int_0^t ||F(u(s))||ds \end{aligned}$$

But, $||Fu - Fv|| \leq L||u - v||$, for all $u, v \in E$.
So, $||F(u(s)) - F(0)|| \leq L||u(s)||$, for all $s \geq 0$.
Therefore, $||F(u(s))|| \leq ||F(0)|| + L||u(s)||$.
Then,

$$\begin{aligned} e^{-kt}||\Phi u(t)|| &\leq e^{-kt}||u_0|| + e^{-kt}\int_0^t (||F(0)|| + L||u(s)||)ds \\ &\leq e^{-kt}||u_0|| + e^{-kt}L\int_0^t ||u(s)||ds + e^{-kt}\int_0^t ||F(0)||ds \\ &\leq e^{-kt}||u_0|| + Le^{-kt}\int_0^t e^{ks}e^{-ks}||u(s)||ds + e^{-kt}t||F(0)|| \\ &\leq e^{-kt}||u_0|| + Le^{-kt}\int_0^t e^{ks}\sup_{t\geq 0} e^{-ks}||u(s)||ds + e^{-kt}t||F(0)|| \\ &\leq e^{-kt}||u_0|| + \frac{L}{k}||u||_X e^{-kt}(e^{-kt} - 1) + e^{-kt}t||F(0)|| \\ &\leq e^{-kt}||u_0|| + \frac{L}{k}||u||_X + e^{-kt}t||F(0)|| \end{aligned}$$

Taking the supremum over all $t \geq 0$, we get

$$\sup_{t\geq 0} e^{-kt}||\Phi u(t)|| \leq \sup_{t\geq 0} e^{-kt}||u_0|| + \frac{L}{k}||u||_X + \sup_{t\geq 0} e^{-kt}t||F(0)||$$

Thus, $\sup_{t\geq 0} e^{-kt}||\Phi u(t)|| < \infty$.
Hence, $\Phi u \in X$.
Also,

$$\begin{aligned}
||\Phi u(t) - \Phi v(t)|| &= ||\int_0^t (F(u(s)) - F(v(s)))ds|| \\
&\leq L \int_0^t ||u(s) - v(s)||ds \\
&= L \int_0^t e^{-ks}||u(s) - v(s)||e^{ks}ds \\
&= L \int_0^t \sup_{s\geq 0} e^{-ks}||u(s) - v(s)||e^{ks}ds \\
&= \frac{L}{k}||u - v||_X(e^{kt} - 1) \\
&\leq \frac{L}{k}||u - v||_X e^{kt}
\end{aligned}$$

So, $e^{-kt}||\Phi u(t) - \Phi v(t)|| \leq \frac{L}{k}||u - v||_X$
Taking the supremum over all $t \geq 0$, we get

$$||\Phi u - \Phi v||_X \leq \frac{L}{k}||u - v||_X, \ for\ all\ u, v \in X$$

Since X is a complete metric space, fixing $k > L$, we get that Φ has a unique fixed point u in X, which is a solution of (2.3). Uniqueness:
Let u, v be solutions of (P_1), then

$$\begin{aligned}
\varphi(t) = ||u(t) - v(t)|| &= ||\int_0^t F(u(s)) - F(v(s))ds|| \\
&\leq L \int_0^t ||u(s) - v(s)||ds \\
&= L \int_0^t \varphi(s)ds
\end{aligned}$$

So, $\varphi(t) \leq L \int_0^t \varphi(s)ds$, for all $t \geq 0$. By Gronwall's lemma, we deduce that $\varphi \equiv 0$, that means $u = v$.

This Theorem is extremely useful in ordinary differential equations. However, it is of little use in partial differential equation. We introduce the Hille-Yosida theorem in Hilbert spaces which is a very powerful tool in solving evolution partial differential equation.

Theorem 2.8. ***(Hille-Yosida)****: Let $A : D(A) \subset H \to H$ be a maximal monotone operator on a real Hilbert space H. Then given any $u_0 \in D(A)$ there exists a unique solution*

$$u \in C^1([0,\infty), H) \cap C([0,\infty), D(A))$$

satisfying

$$(P_2) \begin{cases} \frac{du}{dt}(t) + Au(t) = 0 \quad on \quad [0,\infty) \\ u(0) = u_0 \end{cases}$$

Moreover, $||u(t)|| \leq ||u_0||$ *and* $||\frac{du}{dt}(t)|| = ||Au(t)|| \leq ||Au_0||$*, for all* $t \geq 0$

Proof:
We shall divide the proof into different steps
Step 1. Uniqueness.
Let u and $\overline{u}$ be two solutions of (P_2). Then we have

$$\frac{d}{dt}(u - \overline{u}) = -A(u - \overline{u})$$

and

$$\frac{1}{2}\frac{d}{dt}||u - \overline{u}||^2 = \langle \frac{d}{dt}(u - \overline{u}), (u - \overline{u})\rangle = -\langle A(u - \overline{u}), (u - \overline{u})\rangle \leq 0$$

Thus,the function $t \mapsto ||u(t) - \overline{u}(t)||$ is non-increasing on $[0,\infty)$.
Since $||u(0) - \overline{u}(0)|| = ||u_0 - u_0|| = 0$, it follows that

$$||u(t) - \overline{u}(t)|| \leq ||u(0) - \overline{u}(0)|| = 0, for\ all\ t \geq 0.$$

So, $u = \overline{u}$

Step 2. We consider the approximate solution u_λ to

$$(P_3) \begin{cases} \frac{du_\lambda}{dt}(t) + A_\lambda u_\lambda(t) = 0 \quad on \quad [0,\infty) \\ u_\lambda(0) = u_0 \in D(A), \end{cases}$$

where A_λ is the Yosida approximation of A. Since A_λ is a Lipschitzian map from H to H, (P_3) has a unique continuously differentiable solution u_λ defined on $[0,\infty)$.

We shall prove that u_λ is a Cauchy sequence in $C^1([0,\infty),H)$ and check that u_λ converges to the solution of the differential equation (P_1). We have the following estimates from Lemma 2.9

(a) $||u_\lambda(t)|| \leq ||u(0)||, \quad for\ all\ t \geq 0,\ \lambda > 0$

(b) $||\frac{du_\lambda}{dt}(t)|| = ||A_\lambda u_\lambda(t)|| \leq ||Au_0|| \quad for\ all\ \lambda > 0$

Lemma 2.9. *Let $w_\lambda \in C^1([0,\infty),H)$ be a function satisfying*

$$\frac{dw_\lambda}{dt} + A_\lambda w_\lambda = 0 \quad on\ [0,\infty). \tag{2.5}$$

Then the functions $t \mapsto ||w_\lambda(t)||$ and $t \mapsto ||\frac{dw_\lambda}{dt}(t)||$ are nonincreasing on $[0,\infty)$.

Proof of Lemma 2.9: Infact, we have

$$\langle \frac{dw_\lambda}{dt}, w_\lambda\rangle + \langle A_\lambda w_\lambda, w_\lambda\rangle = 0 \tag{2.6}$$

By Theorem 2.6 f., we know that $\langle A_\lambda w_\lambda, w_\lambda\rangle \geq 0$ and thus $\frac{1}{2}\frac{d}{dt}||w_\lambda||^2 \leq 0$, so that $t \to ||w_\lambda(t)||$ is nonincreasing. On the other hand, since A_λ is a linear bounded operator, we deduce (by induction) from equation (2.5) that $w_\lambda \in C^\infty([0,\infty);H)$ and also that

$$\frac{d}{dt}(\frac{dw_\lambda}{dt}) + A_\lambda(\frac{dw_\lambda}{dt}) = 0$$

Applying the preceding fact to $\frac{dw_\lambda}{dt}$, we see that the function
$t \to ||\frac{dw_\lambda}{dt}(t)||$ is nonincreasing. In fact, at any order k, the function
$t \to ||\frac{d^k w_\lambda}{dt^k}||$ is nonincreasing.

Step 3. We will prove that for every t ≥ 0, $u_\lambda(t)$ converges,as
$\lambda \to 0$, to some limit, denoted by $u(t)$. Moreover, the convergence is uniform on every bounded interval [0,T].
Let $\lambda,\ p \in \mathbb{R}^+$. Define

$$a(t) = \frac{1}{2}||u_\lambda(t) - u_p(t)||^2$$

Then,

$$\begin{aligned} a(t) &= \int_0^t \frac{1}{2}\frac{d}{dt}||u_\lambda(s) - u_p(s)||^2 ds \\ &= \int_0^t -\langle A_\lambda u_\lambda(t) - A_p u_p(t), u_\lambda(t) - u_p(t)\rangle ds \end{aligned}$$

since $\lambda A_\lambda = I - J_\lambda$, and $A_\lambda x = A(J_\lambda(x))$ and the monotonicity of A.
Dropping t for simplicity, we write

$$\begin{aligned}
\langle A_\lambda u_\lambda - A_p u_p, u_\lambda - u_p\rangle &= \langle A_\lambda u_\lambda - A_p u_p, u_\lambda - J_\lambda u_\lambda + J_\lambda u_\lambda - J_p u_p + J_p u_p - u_p\rangle \\
&= \langle A_\lambda u_\lambda - A_p u_p, \lambda A_\lambda u_\lambda - pA_p u_p + J_\lambda u_\lambda - J_p u_p\rangle \\
&= \langle A_\lambda u_\lambda - A_p u_p, \lambda A_\lambda u_\lambda - pA_p u_p\rangle \\
&+ \langle A_\lambda u_\lambda - A_p u_p, J_\lambda u_\lambda - J_p u_p\rangle \\
&= \langle A_\lambda u_\lambda - A_p u_p, \lambda A_\lambda u_\lambda - pA_p u_p\rangle \\
&+ \langle A(J_\lambda u_\lambda) - A(J_p u_p), J_\lambda u_\lambda - J_p u_p\rangle \\
&\geq \langle A_\lambda u_\lambda - A_p u_p, \lambda A_\lambda u_\lambda - pA_p u_p\rangle \\
&= \langle A_\lambda u_\lambda, \lambda A_\lambda u_\lambda\rangle - \langle A_\lambda u_\lambda, pA_p u_p\rangle \\
&- \lambda\langle A_p u_p, A_\lambda u_\lambda\rangle + \langle A_p u_p, pA_p u_p\rangle \\
&= \lambda||A_\lambda u_\lambda||^2 + p||A_p u_p||^2 - p\langle A_\lambda u_\lambda, A_p u_p\rangle \\
&- \lambda\langle A_p u_p, A_\lambda u_\lambda\rangle \\
&= \lambda||A_\lambda u_\lambda||^2 + p||A_p u_p||^2 - p\langle A_\lambda u_\lambda, A_p u_p\rangle \\
&- \lambda\langle A_\lambda u_\lambda, A_p u_p\rangle \\
&= \lambda||A_\lambda u_\lambda||^2 + p||A_p u_p||^2 - (p+\lambda)\langle A_\lambda u_\lambda, A_p u_p\rangle
\end{aligned}$$

Then,

$$\begin{aligned}
-\langle A_\lambda u_\lambda - A_p u_p, u_\lambda - u_p\rangle &\leq (p+\lambda)\langle A_\lambda u_\lambda, A_p u_p\rangle - (\lambda||A_\lambda u_\lambda||^2 + p||A_p u_p||^2) \\
&\leq (p+\lambda)\langle A_\lambda u_\lambda, A_p u_p\rangle \\
&\leq (p+\lambda)||A_\lambda u_\lambda||||A_p u_p|| \\
&\leq (p+\lambda)||A_\lambda u_0||^2
\end{aligned}$$

Thus,

$$\begin{aligned}
a(t) &\leq \int_0^t (\lambda+p)||A_\lambda u_0||^2 ds \\
&= (\lambda+p)t||A_\lambda u_0||^2
\end{aligned}$$

That is, $\frac{1}{2}||u_\lambda(t) - u_p(t)||^2 \leq (\lambda+p)t||A_\lambda u_0||^2$
Then,

$$||u_\lambda(t) - u_p(t)|| \leq \sqrt{2(\lambda+p)t}||A_\lambda u_0|| \tag{2.7}$$

It follows that for every fixed $t \geq 0,\ u_\lambda(t)$ is a Cauchy sequence as $\lambda \to 0$ and thus converges to a limit, denoted by $u(t)$. passing to the limit in (2.7) as $p \to 0$, we have

$$||u_\lambda(t) - u(t)|| \leq \sqrt{2\lambda t}||Au_0||$$

Therefore, the convergence is uniform in t on every bounded interval [0,T] and so $u \in C([0,\infty), H)$

Step 4: Assuming $u_0 \in D(A)$ *and* $Au_0 \in D(A)$. We prove here that $\frac{du_\lambda}{dt}(t)$ convergence,as $\lambda \to 0$ to some limit and that the convergence is uniform on every bounded interval [0,T].

Set $v_\lambda = \frac{du_\lambda}{dt}$, so that $\frac{du_\lambda}{dt} = \frac{d}{dt}(\frac{du_\lambda}{dt})$

Now, $\frac{d}{dt}(\frac{du_\lambda}{dt} + A_\lambda u_\lambda) = 0$, therefore $\frac{dv_\lambda}{dt} + A_\lambda v_\lambda = 0$

For $\lambda,\ p \in \mathbb{R}^+$ following the same argument as in **step 3,** we see that:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||v_\lambda - v_p||^2 &= \langle A_p v_p - A_\lambda v_\lambda, \lambda A_\lambda v_\lambda - pA_p v_p\rangle \\ &\leq ||A_p v_p - A_\lambda v_\lambda||||\lambda A_\lambda v_\lambda - pA_p v_p|| \\ &\leq (||A_p v_p|| + ||A_\lambda v_\lambda||)(\lambda||A_\lambda v_\lambda|| + p||A_p v_p||). \end{aligned}$$

It follows that,

$$\frac{1}{2}\frac{d}{dt}||v_\lambda - v_p||^2 \leq (||A_p v_p|| + ||A_\lambda v_\lambda||)(\lambda||A_\lambda v_\lambda|| + p||A_p v_p||) \tag{2.8}$$

From Lemma 2.9, we have

$||A_\lambda v_\lambda(t)|| \leq ||A_\lambda v_\lambda(0)|| = ||A_\lambda \frac{du_\lambda}{dt}(0)|| = ||A_\lambda A_\lambda u_\lambda(0)|| = ||A_\lambda A_\lambda u_0||$.

Similarly,

$$||A_p v_p(t)|| \leq ||A_p v_p(0)|| = ||A_p A_p u_0||. \tag{2.9}$$

From Theorem 2.6 (a.) and (b.), we have that,

$$A(J_\lambda v) = A_\lambda v = J_\lambda(Av),\ \textit{for all } v \in D(A) \textit{ and } \lambda > 0.$$

Since $Au_0 \in D(A)$, we obtain

$A_\lambda A_\lambda u_0 = A_\lambda(J_\lambda Au_0) = J_\lambda(A(J_\lambda Au_0)) = J_\lambda(J_\lambda(AAu_0)) = J_\lambda^2 A^2 u_0$.

Consequently,

$$||A_\lambda A_\lambda u_0|| = ||J_\lambda^2 A^2 u_0|| \leq ||J_\lambda^2||||A^2 u_0|| = ||A^2 u_0|| \tag{2.10}$$

and

$$||A_p A_p u_0|| \leq ||A^2 u_0|| \tag{2.11}$$

Thus from (2.8), (2.9), (2.10), (2.11), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||v_p - v_p||^2 &\leq (||A_p v_p|| + ||A_\lambda v_\lambda||)(\lambda||A_\lambda v_\lambda|| + p||A_p v_p||) \\
&\leq (||A_p A_p u_0|| + ||A_\lambda A_\lambda u_0||)(\lambda||A_\lambda A_\lambda u_0|| + p||A_p v_p u_0||) \\
&\leq (||A^2 u_0|| + ||A^2 u_0||)(\lambda||A^2 u_0|| + p||A^2 u_0||) \\
&= (2||A^2 u_0||)((\lambda + p)||A^2 u_0||) \\
&= 2(\lambda + p)||A^2 u_0||^2
\end{aligned}$$

Thus, $\frac{d}{dt}||v_\lambda - v_p||^2 \leq 4(\lambda + p)||A^2 u_0||^2$ and on integrating this inequality, we get

$$||v_\lambda - v_p|| \leq 2\sqrt{(\lambda + p)t}||A^2 u_0||$$

We conclude as in **step 3,** that $v_\lambda(t) = \frac{du_\lambda}{dt}(t)$ converges, as $\lambda \to 0$, to some limit and that the convergence is uniform on every bounded interval $[0, T]$.

Step 5. Assuming that $u_0 \in D(A)$ and $Au_0 \in D(A)$, we prove here that u is a solution of (P_2)

By **steps 3 and 4**, we know that for all $T < \infty$.

$$\begin{cases} u_\lambda(t) \to u(t), as\ \lambda \to 0,\ uniformly\ on\ [0, T] \\ \frac{du_\lambda}{dt}(t)\ converges,\ as\ \lambda \to 0,\ uniformly\ on\ [0, T] \end{cases}$$

It follows that $u \in C^1([0, \infty); H)$ and that $\frac{du_\lambda}{dt}(t) \to \frac{du}{dt}(t)$ as $\lambda \to 0$ uniformly on [0,T]. Rewrite (P_2) as

$$\begin{aligned}
0 &= \frac{du_\lambda}{dt}(t) + A_\lambda u_\lambda(t) \\
&= \frac{du_\lambda}{dt}(t) + A(J_\lambda u_\lambda(t))
\end{aligned}$$

We observe that $J_\lambda u_\lambda(t) \to u(t)\ as\ \lambda \to 0$ since,

$$\begin{aligned}
||J_\lambda u_\lambda(t) - u(t)|| &= ||J_\lambda u_\lambda(t) - J_\lambda u(t) + J_\lambda u(t) - u(t)|| \\
&\leq ||J_\lambda u_\lambda(t) - J_\lambda u(t)|| + ||J_\lambda u(t) - u(t)|| \\
&= ||J_\lambda||||(u_\lambda(t) - u(t))|| + ||J_\lambda u(t) - u(t)|| \to 0\ as\ \lambda \to 0
\end{aligned}$$

Applying the fact that A is closed, we deduce that
$u(t) \in D(A), \forall t \geq 0,$ and that

$$\frac{du}{dt}(t) + Au(t) = 0$$

Finally, since $u \in C^1([0,\infty); H)$, the function $t \mapsto Au(t)$ is continuous from $[0,\infty)$ to H and thus $u \in C([0,\infty); D(A))$. Hence we have obtained a solution of (P_2) satisfying , in addition,

$$||u(t)|| \leq ||u_0|| \; and \; ||\frac{du}{dt}(t)|| = ||Au(t)|| \leq ||Au_0||, \forall\, t \geq 0$$

Step 6. We now conclude the Theorem with the following Lemma.

Lemma 2.10. *Let $u_0 \in D(A)$.Then, $\forall \varepsilon > 0 \exists\ \overline{u_0} \in D(A)$ such that $||u_0 - \overline{u_0}|| < \epsilon$ and $||Au_0 - A\overline{u_0}|| < \epsilon$. In otherwords, $D(A^2)$is dense in $D(A)$ (for the graph norm).*

Proof:: We want to prove that $D(A^2)$ is dense in $D(A)$ using the graph norm, which is given by

$$||x||_A = ||x||_H + ||Ax||_H, \forall\, x \in D(A)$$

Since A is maximal monotone, for $\lambda > 0$, $J_\lambda = (I + \lambda A)^{-1}$ is well-defined and is a bijective map from H to D(A).
Then, $(I + \lambda A)J_\lambda x = x, \forall x \in H$ and $J_\lambda(I + \lambda A)x = x, \forall x \in D(A)$. We know that $J_\lambda x \to x, \forall x \in H$. Now, let $u_0 \in D(A) \subset H$. Set $\overline{u_0} = J_\lambda u_0,\ for\ some\ \lambda > 0$.
Since $ImJ_\lambda = D(A)$, it implies that $\overline{u_0} \in D(A)$, then $(I + \lambda A)\overline{u_0} = u_0$ and hence we have $A\overline{u_0} = \frac{u_0 - \overline{u_0}}{\lambda}$.
Since D(A) is a vector space and $u_0,\ \overline{u_0} \in D(A)$,then $A\overline{u_0} \in D(A)$.
Hence, $\overline{u_0} \in D(A^2)$.
Using the graph norm, we have

$$\begin{aligned} ||J_\lambda u_0 - u_0||_A &= ||J_\lambda u_0 - u_0||_H + ||A(J_\lambda u_0 - u_0)||_H \\ &= ||J_\lambda u_0 - u_0||_H + ||AJ_\lambda u_0 - Au_0||_H \\ &= ||J_\lambda u_0 - u_0||_H + ||J_\lambda Au_0 - Au_0||_H \end{aligned}$$

But $J_\lambda u_0 \to u_0$ and $J_\lambda Au_0 \to Au_0$. Hence,$||J_\lambda u_0 - u_0||_A \to 0\ as\ \lambda \to 0$
But $u_0 \in D(A)$ and $J_\lambda u_0 = \overline{u_0} \in D(A)$. It implies that $D(A^2)$ is dense in D(A) using the graph norm.
Thus, given $u_0 \in D(A)$, we construct (using lemma 2.10) a sequence $u_{0n} \in D(A^2)$ such that $u_{0n} \to u_0$ and $Au_{0n} \to Au_0$
By step 5, We know that there is a solution u_n of the problem

$$(P_4)\ \begin{cases} \frac{du_n}{dt}(t) + Au_n(t) = 0 \quad on \quad [0,\infty) \\ u_n(0) = u_{0n} \end{cases}$$

We have,for all $t \geq 0$

$$||u_n(t) - u_m(t)|| \leq ||u_{0n} - u_{0m}|| \rightarrow 0, \ as \ m, n \ \rightarrow \infty$$

and

$$||\frac{du_n}{dt}(t) - \frac{du_m}{dt}(t)|| \leq ||Au_{0n} - Au_{0m}|| \rightarrow 0, \ as \ m, n \ \rightarrow \infty$$

Therefore,

$$u_n(t) \rightarrow u(t) \ uniformly \ on \ [0, \infty)$$

and

$$\frac{du_n}{dt}(t) \rightarrow \frac{du}{dt}(t) \ uniformly \ on \ [0, \infty)$$

with $u \in C^1([0, \infty); H)$, passing limit in (P_4) and using the fact that A is a closed operator, we see that $u(t) \in D(A)$ and u satisfies (P_2). From (P_2)we deduce that $u \in C([0, \infty); D(A).)$

Remark 2.11. *Let u_λ be the solution of (P_2)*

a. *Assume $u_0 \in D(A)$.*
   We have proved that $u_\lambda \rightarrow u \in C^1([0, \infty); H) \cap C([0, \infty); D(A))$ as $\lambda \rightarrow 0$.

b. *Assume $u_0 \in H \big| D(A)$*
   *One can still prove that as $\lambda \rightarrow 0$, u_λ converges to some limit u. However, this limit u may not belong to D(A) or may be nowhere differentiable on $[0, \infty)$. Hence u may not be a "classical" solution of (P_2). Nevertheless, we may view u(t) as a "generalised" solution of (P_2). We shall see in Theorem 2.20 that this does not happen when A is **self-adjoint**; in this case u is a classical solution of (P_2), for any $u_0 \in H$, even when $u_0 \notin D(A)$*

2.3 Self adjoint Operators

Let $A : D(A) \subset H \rightarrow H$ be an unbounded operator with $\overline{D(A)} = H$. The adjoint $A^* : D(A^*) \subset H \rightarrow H$ is defined by the requirements

$$\langle u, Av \rangle = \langle A^* u, v \rangle, \forall \ v \in D(A)$$

Define $u \in D(A^*)$ if and only if $u \in H$ and $\exists \ f \in H$ such that

$$\langle u, Av \rangle = \langle f, v \rangle, \forall \ v \in D(A) \tag{2.12}$$

For $u \in D(A^*)$, set

$$A^*u = f$$

A^* makes sense. In fact, if (2.12) holds for any $g \in H$, then $g = f$.

Definition 2.12. *A linear operator $A : D(A) \subset H \to H$ is called symmetric if $\overline{D(A)} = H$ and*

$$\langle Au, v\rangle = \langle u, Av\rangle, \forall\ u, v \in D(A)$$

A is self adjoint if and only if $A = A^*$.
Note that for bounded operators, the notions of symmetric and self adjoint operators coincide. However if A is unbounded, there is a difference between symmetric and self adjoint operators.

Proposition 2.13. *If A is self-adjoint, then it is symmetric.*

. The adjoint of any densely defined operator is closed. This implies that a self adjoint operator is closed.

Theorem 2.14. *Let A be a maximal monotone, symmetric operator. Then A is self adjoint.*

Proof: Let $J_1 = (I + A)^{-1}$. We will first show that J_1 is self adjoint. Since $J_1 \in L(H)$, it suffices to check that

$$\langle J_1u, v\rangle = \langle u, J_1v\rangle\ for\ all\ u, v \in H$$

Set $u_1 = J_1u\ and\ v_1 = J_1v, such\ that$

$$\begin{aligned} u_1 + Au_1 &= u. \\ v_1 + Av_1 &= v. \end{aligned}$$

Since A is symmetric, it follows that

$$\langle u_1, v\rangle = \langle u, v_1\rangle$$

Then, $\langle J_1u, v\rangle = \langle u, J_1v\rangle$
Now, let $u \in D(A^*)$ and set $f = u + A^*u$. We have

$$\begin{aligned} \langle f, v\rangle &= \langle u + A^*u, v\rangle,\ \forall v \in D(A) \\ &= \langle u, v\rangle + \langle A^*u, v\rangle \\ &= \langle u, v\rangle + \langle u, Av\rangle \\ &= \langle u, (I + A)v\rangle \end{aligned}$$

For $w \in H$ arbitrary, since A is maximal monotone, then

$$\exists\ v \in D(A)\ such\ that\ (I + A)v = w$$

Then, $\langle f, J_1 w\rangle = \langle u, w\rangle$
But J_1 symmetric,i.e., $\langle J_1 f, w\rangle = \langle f, J_1 w\rangle$
Therefore, $J_1 = u$ and thus $u \in D(A)$.
Hence, $D(A^*) = D(A)$.

Theorem 2.15. *Let A be a self-adjoint, maximal monotone operator. Then for every $u_0 \in H$, there exists a unique function*

$$u \in C([0, \infty); H) \cap C^1((0, \infty); H) \cap C([0, \infty); D(A))$$

such that

$$(P_5)\ \begin{cases} \frac{du}{dt}(t) + Au(t) = 0 \quad on \quad [0, \infty) \\ u(0) = u(0) \end{cases}$$

Moreover, we have

$$||u(t)|| \leq ||u_0||\ and\ ||\frac{du}{dt}(t)|| = ||Au(t)|| \leq \frac{1}{t}||u_0||, \forall\ t > 0$$

Proof: uniqueness:
Let $u\ and\ v$ be two solutions. By the monotonicity of A,we see that $\psi(t) = ||u(t) - v(t)||^2$ is non-increasing on $(0, \infty)$.On the other hand, ψ is continuous on $[0, \infty)$ and $\psi(0) = 0$. Thus $\psi \equiv 0$.

Existence:
The proof is divided into two steps.
Step 1. Assume first that $u_0 \in D(A)$, $Au_0 \in D(A)$ and let u be the solution of (P_2) given by theorem 2.14 .We claim that

$$||\frac{du}{dt}(t)|| \leq \frac{1}{t}||u_0||,\ for\ all\ t > 0.$$

As in the proof of proposition 2.18, we have that

$$J_\lambda^* = J_\lambda\ and\ A_\lambda^* = A_\lambda,\ for\ all\ t > 0.$$

We go back to the approximate problem introduced in the proof of theorem 2.8

$$\frac{du_\lambda}{dt} + A_\lambda u_\lambda = 0\ on\ [0, \infty), u_\lambda(0) = u_0 \tag{2.13}$$

Taking the scalar product of (2.13) with u_λ and integrating on [0,T], we obtain

$$\frac{1}{2}||u_\lambda(T)||^2 + \int_0^T \langle A_\lambda u_\lambda, u_\lambda\rangle = \frac{1}{2}||u_\lambda(0)||^2 \tag{2.14}$$

Taking the scalar product of (2.13) with $t\frac{du_\lambda}{dt}$ and integrating over [0.T], we obtain

$$\int_0^T ||\frac{du_\lambda}{dt}(t)||^2 tdt + \int_0^T \langle A_\lambda u_\lambda(t), \frac{du_\lambda}{dt}(t)\rangle tdt = 0 \tag{2.15}$$

But, setting $v_\lambda = \frac{du_\lambda}{dt}$

$$\begin{aligned} \frac{d}{dt}\langle A_\lambda u_\lambda, u_\lambda\rangle &= \langle A_\lambda \frac{du_\lambda}{dt}, u_\lambda\rangle + \langle A_\lambda u_\lambda, \frac{du_\lambda}{dt}\rangle \\ &= \langle A_\lambda v_\lambda, u_\lambda\rangle + \langle A_\lambda u_\lambda, v_\lambda\rangle \\ &= \langle v_\lambda, A_\lambda^* u_\lambda\rangle + \langle A_\lambda u_\lambda, v_\lambda\rangle \\ &= \langle v_\lambda, A_\lambda u_\lambda\rangle + \langle A_\lambda u_\lambda, v_\lambda\rangle \\ &= 2\langle A_\lambda u_\lambda, v_\lambda\rangle \end{aligned}$$

Then,

$$\frac{d}{dt}\langle A_\lambda u_\lambda, u_\lambda\rangle = 2\langle A_\lambda u_\lambda, \frac{du_\lambda}{dt}\rangle \tag{2.16}$$

From equation (2.16) and integrating by parts, we get

$$\begin{aligned} \int_0^T \langle A_\lambda u_\lambda, \frac{du_\lambda}{dt}\rangle tdt &= \frac{1}{2}\int_0^T \frac{d}{dt}\langle A_\lambda u_\lambda, u_\lambda\rangle tdt \\ &= \frac{1}{2}\{t\langle A_\lambda u_\lambda, u_\lambda\rangle\Big|_0^T - \int_0^T \langle A_\lambda u_\lambda, u_\lambda\rangle dt\} \\ &= \frac{1}{2}\{\langle A_\lambda u_\lambda(T), u_\lambda(T)\rangle T\} - \frac{1}{2}\{\int_0^T \langle A_\lambda u_\lambda, u_\lambda\rangle dt\} \end{aligned}$$

Then,

$$\int_0^T \langle A_\lambda u_\lambda, u_\lambda\rangle dt = \langle A_\lambda u_\lambda(T), u_\lambda(T)\rangle T - 2\int_0^T \langle A_\lambda u_\lambda, \frac{du_\lambda}{dt}\rangle tdt \tag{2.17}$$

From lemma 2.9, since the function $t \mapsto ||\frac{du_\lambda}{dt}(t)||$ is nonincreasing, we have for $0 < t \leq T$

$$\begin{aligned}\int_0^T ||\frac{du_\lambda}{dt}(t)||^2 t dt &\geq \int_0^T ||\frac{du_\lambda}{dt}(T)||^2 t dt \\ &= ||\frac{du_\lambda}{dt}(T)||^2 \frac{T^2}{2}\end{aligned}$$

So,

$$\int_0^T ||\frac{du_\lambda}{dt}(t)||^2 t dt \geq ||\frac{du_\lambda}{dt}(T)||^2 \frac{T^2}{2} \tag{2.18}$$

From (2.14),(2.15),(2.17),and (2.18) we get

$$\begin{aligned}\frac{1}{2}||u_\lambda(0)||^2 &= \frac{1}{2}||u_\lambda(T)||^2 + \int_0^T \langle A_\lambda u_\lambda, u_\lambda \rangle \\ &= \frac{1}{2}||u_\lambda(T)||^2 + \{\langle A_\lambda u_\lambda(T), u_\lambda(T)\rangle T - 2\int_0^T \langle A_\lambda u_\lambda, \frac{du_\lambda}{dt}\rangle t dt\} \\ &= \frac{1}{2}||u_\lambda(T)||^2 + \{\langle A_\lambda u_\lambda(T), u_\lambda(T)\rangle T + 2\int_0^T ||\frac{du_\lambda}{dt}(t)||^2 t dt\} \\ &\geq \frac{1}{2}||u_\lambda(T)||^2 + \langle A_\lambda u_\lambda(T), u_\lambda(T)\rangle T + T^2||\frac{du_\lambda}{dt}(T)||^2\} \\ &\geq \frac{1}{2}||u_\lambda(T)||^2 + T^2||\frac{du_\lambda}{dt}(T)||^2 \\ &\geq T^2||\frac{du_\lambda}{dt}(T)||^2\end{aligned}$$

It follows in particular that

$$||\frac{du_\lambda}{dt}(T)|| \leq \frac{1}{T}||u_0||, T > 0$$

Since $||.||$ is continuous, on passing limit as $\lambda \to 0$, we get

$$||\frac{du_\lambda}{dt}(t)|| \leq \frac{1}{t}||u_0||, t > 0$$

since $\frac{du_\lambda}{dt} \to \frac{du}{dt}$ (see step 5 in the proof of theorem 2.8).

Step 2. Assume now that $u_0 \in H$. Let (u_{0n}) be a sequence in $D(A^2)$ such that $u_{0n} \to u_0$ (recall that$D(A^2)$) is dense in $D(A)$ and $D(A)$ is dense in H; thus $D(A^2)$

is dense in H).

Let u_n be the solution of

$$(P_5) \begin{cases} \frac{du_n}{dt} + Au_n = 0 \quad on \quad [0, \infty) \\ u_n(0) = u_{0n} \end{cases}$$

We know by (theorem 2.8) that

$$||u_n(t) - u_m(t)|| \leq ||u_{0n} - u_{0m}|| \forall\, m, n\ \forall t > 0$$

and (by step 1) that

$$||\frac{du_n}{dt}(t) - \frac{du_m}{dt}(t)|| \leq \frac{1}{t}||u_{0n} - u_{0m}|| \forall\, m, n\ \forall t > 0$$

It follows that u_n converges uniformly on $[0, \infty)$ to some limit u(t)and that $\frac{du_n}{dt}(t)$ converges to $\frac{du}{dt}(t)$ uniformly on every interval $[\delta, \infty), \delta > 0$ The limiting function u satisfies

$$u \in C([0, \infty); H) \cap C^1((0, \infty); H).$$

Since A is a closed operator, then $u(t) \in D(A)\ \forall\, t > 0$ and $\frac{du}{dt}(t) + Au(t) = 0, \forall\, t > 0$.

2.4 Application

Now, we solve the heat equation by using the approach of maximal monotone operators on Hilbert Spaces.

Firstly, we show that $-\Delta$ with $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, when Ω is smooth is maximal monotone on $L^2(\Omega)$.

Secondly, we apply Theorem 2.15 to prove that the heat equation has solution. Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and open with boundary Γ. Set

$$\begin{aligned} \Upsilon &= \Omega \times (0, \infty) \\ \Sigma &= \Gamma \times (0, \infty) \end{aligned}$$

Consider the following problem:find a function

$u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$ such that

$$\begin{cases} (1.) \quad \frac{\partial u}{\partial t} - \Delta u = 0, \text{ on } \Upsilon \\ (2.) \quad u = 0 \ on\ \Sigma \quad (Dirichlet\ Boundary\ condition) \\ (3.) \quad u(x, 0) = u_0(x) \text{ on } \Omega, u_0 \in L^2(\Omega) \end{cases}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian in the space variables x, t is the time variable and $u_0(x)$ is called the initial (or Cauchy) data.
Equation (1) is called the heat equation because it models the temperature distribution u in the domain Ω at time t.
Equation (2) is the (homogeneous) Dirichlet boundary condition. It corresponds to the assumption that the boundary Γ is kept at zero temperature. We solve problem (1),(2),(3) by viewing $u(x,t)$ as a function defined on $[0,\infty)$ with values in $H = L^2(\Omega)$, or $H = H_0^1(\Omega)$. When we write just u(t), we mean that $u(t)$ is an element in H, namely the function $x \to u(x,t)$. This viewpoint allows us to easily solve problem (1),(2),(3) by combining theorem 2.8 with some known results.
Let $H = L^2(\Omega)$, where Ω is assumed C^∞ and Γ is bounded. Consider the unbounded operator $A : D(A) \subset H \to H$ defined by

$$\begin{cases} D(A) = H^2(\Omega) \cap H_0^1(\Omega) \\ Au = -\Delta u \end{cases}$$

Observe that the boundary condition (2) has been incorporated in the definition of the domain of A.

Theorem 2.16. *Assume $u_0 \in L^2(\Omega)$. Then there exists a unique function u satisfying (1),(2),(3)and*

(4) $u \in C([0,\infty); L^2(\Omega)) \cap C((0,\infty); H^2(\Omega) \cap H_0^1(\Omega))$.

(5) $u \in C^1((0,\infty); L^2(\Omega))$.

Proof:
We claim that A is a self-adjoint maximal monotone operator. We may then apply theorem 2.20 and deduce the existence of a unique of (1),(2),(3) satisfying (4) and (5)

i) A is monotone (see example 2.1 3.)

ii) A is maximal monotone (see example 2.1 3.)

iii) A is self adjoint.
In view of Theorem 2.14, it suffices to verify that A is symmetric.

Now,$u, v \in D(A)$. Then,

$$\begin{aligned}
\langle Au, v\rangle_{L^2} &= \int_{\Omega} (-\Delta u)v \\
&= \int_{\Omega} \nabla u.\nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial n} v dr
\end{aligned}$$

But $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$
and $H_0^1(\Omega) = \{u \in H^1(\Omega) : u\big|_{\partial\Omega} = 0\}$.
Since, $v \in H_0^1(\Omega)$, we have that

$$\langle Au, v\rangle_{L^2} = \int_{\Omega} (-\Delta u)v = \int_{\Omega} \nabla u.\nabla v$$

. Also,

$$\langle u, Av\rangle_{L^2} = \int_{\Omega} u(-\Delta v) = \int_{\Omega} (-\Delta v)u = \int_{\Omega} \nabla v.\nabla u = \int_{\Omega} \nabla u.\nabla v$$

So, $\langle Au, v\rangle = \langle u, Av\rangle$.
Hence, A is self-adjoint.
From Theorem 2.15, there exists a unique function

$$u \in C([0, \infty); L^2(\Omega)) \cap C^1((0, \infty); L^2(\Omega)) \cap C((0, \infty); H^2(\Omega) \cap H_0^1(\Omega))$$

satisfying (1),(2),(3).

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