AN ALGORITHM FOR SOLUTIONS OF HAMMERSTEIN INTEGRAL EQUATIONS WITH MONOTONE OPERATORS

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By

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Certification

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Abstract

Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let $F : X \to X^*$ and $K : X^* \to X$ be bounded monotone mappings such that the Hammerstein equation $u + KFu = 0$ has a solution in X. An explicit iteration sequence is constructed and proved to converge strongly to a solution of the equation. This is achieved by combining geometric properties of uniformly convex and uniformly smooth real Banach spaces recently introduced by Alber with our method of proof which is also of independent interest.

Acknowledgements

Let me start by giving my sincere thanks to GOD for making this work even possible. I have spent time searching for the words to express my gratitude to my supervisor, Professor C.E. Chidume, for all his efforts but the words weren't there. Being the man behind the scene, we worked tirelessly to make this work a success and so i want to say thank you. To my parents and my siblings, their endless support, patience and care will never be forgotten. Mmachuckwu, Sister Chinyerem, my classmates and all my friends who also supported me, I cannot find the words to say how grateful I am.

Dedication

To GOD.

Contents

Chapter 1

Introduction and literature review

The contents of this thesis fall within the general area of nonlinear operator theory, a flourishing area of research for numerous mathematicians. In this thesis, we concentrate on an important topic in this area-*approximation of solutions of nonlinear integral equations of Hammaerstein type involving monotone-type mappings.*

Let H be a real inner product space. A map $A : H \to 2^H$ is called *monotone* if for each $x, y \in H$,

$$\big\langle \eta - \nu, x - y \big\rangle \geq 0, \quad \forall \qquad \eta \in Ax, \qquad \nu \in Ay. \tag{1.1}$$

If A is single-valued, the map $A : H \to H$ is *monotone* if

$$\big\langle Ax - Ay, x - y \big\rangle \geq 0 \;\; \forall \;\; x, y \in H. \tag{1.2}$$

Monotone mappings were first studied in Hilbert spaces by Zarantonello [50], Minty [42], Kačurovskii [37] and a host of other authors. Interest in such mappings stems mainly from their usefulness in numerous applications. Consider, for example, the following:

Example 1. *Let $g : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The subdifferential of g at $x \in H$, $\partial g : H \to 2^H$, is defined by*

$$\partial g(x) = \big\{ x^* \in H : g(y) - g(x) \geq \big\langle y - x, x^* \big\rangle \;\; \forall \; y \in H \big\}.$$

It is easy to check that ∂g is a monotone operator on H, and that $0 \in \partial g(u)$ if and only if u is a minimizer of g. Setting $\partial g \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of g.

Example 2. *Again, let $A : H \to H$ be a monotone map. Consider the evolution equation*

$$\frac{du}{dt} + Au = 0. \tag{1.3}$$

At equilibrium state, $\frac{du}{dt} = 0$ so that

$$Au = 0. \tag{1.4}$$

Consequently, solving the equation $Au = 0$, in this case, corresponds to solving for the equilibrium state of the system described by (1.3).

Monotone maps also appear in Hammerstein equations. Since this thesis focuses on this class of equations, we give a brief review.

1.0.1 Hammerstein equations

Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of Hammerstein-type has the form

$$u(x) + \int_{\Omega} k(x,y) f(y, u(y)) dy = w(x), \tag{1.5}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions. If we define $F : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ and $K : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ by

$$Fu(y) = f(y, u(y)),\ y \in \Omega,$$

and

$$Kv(x) = \int_{\Omega} k(x,y) v(y) dy,\ x \in \Omega,$$

respectively, where $\mathcal{F}(\Omega, \mathbb{R})$ is a space of measurable real-valued functions defined from Ω to $\mathbb{R}$, then equation (1.5) can be put in the abstract form

$$u + KFu = 0. \tag{1.6}$$

where, without loss of generality, we have assumed that $w \equiv 0$. The operators K and F are generally of the monotone-type. A closer look at equation (1.6) reveals that it is a special case of (1.4), where

$$A := I + KF.$$

Interest in (1.6) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's function can, as a rule, be transformed into the form (1.5) (see e.g., Pascali and Sburlan [43], chapter IV, p. 164). Among these, we mention the problem of the forced ocsillation of finite amplitude of a pendulum.

Consider the problem of the pendulum:

$$\begin{cases} \frac{d^2 v(t)}{dt^2} + a^2 \sin v(t) &= z(t), \quad t \in [0,1], \\ v(0) = v(1) = 0, \end{cases} \tag{1.7}$$

where the driving force z is odd. The constant a $(a \neq 0)$ depends on the length of the pendulum and gravity. Since Green's function for the problem

$$v''(t) = 0, v(1) = v(0) = 0,$$

is the triangular function

$$K(t,x) = \begin{cases} t(1-x),\ if\ 0 \leq t \leq x \\ x(1-t),\ if\ x \leq t \leq 1, \end{cases} \tag{1.8}$$

it follows that problem (1.7) is equivalent to the nonlinear integral equation

$$v(t) = -\int_0^1 K(t,x)[z(x) - a^2 \sin v(x)]dx. \tag{1.9}$$

If $g(t) = -\int_0^1 K(t,x)z(x)dx$ and $v(t) + g(t) = u(t)$, then (1.9) can be written as the Hammerstein equation

$$u(t) = -\int_0^1 K(t,x)f(x,u(x))dx = 0, \tag{1.10}$$

where

$$f(x,u(x)) = a^2 sin[u(x) - g(x)],$$

(see e.g., [43])

Equations of Hammerstein-type also play a crucial role in the theory of optimal control systems and in automation and network theory (see e.g., Dolezale [33]).

Several existence results have been proved for equations of Hammerstein-type (see e.g., Brézis and Browder [4, 5, 6], Browder [7], Browder, De Figueiredo and Gupta [8]).

The concept of monotone maps has been extended to arbitrary real normed spaces. There are two well-studied extensions of Hilbert-space monotonicity to arbitrary normed spaces. We briefly explore the two.

The first is the class of *accretive operators.*

A map $A : E \to 2^E$ is called *accretive* if for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\big\langle \eta - \nu, j(x-y)\big\rangle \geq 0, \quad \forall \ \ \eta \in Ax, \ \ \nu \in Ay, \tag{1.11}$$

Where J is the normalized duality map on E. If A is single-valued, the map $A : E \to E$ is accretive if for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\big\langle Ax - Ay, j(x-y)\big\rangle \geq 0. \tag{1.12}$$

In a Hilbert space, the normalized duality map is the identity map, so that (1.11) and (1.12) reduce to (1.1) and (1.2) respectively, where $E = H$ and so, *accretivity is one extension of Hilbert space monotonicity to general normed spaces.*

A result of Kato [39] shows that (1.12) holds if and only if for all $x, y \in D(A)$, the following inequality holds

$$||x-y|| \leq ||x - y + s(Ax - Ay)|| \quad \ \forall s > 0. \tag{1.13}$$

The map A is called *generalized Φ-strongly accretive* if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \Phi(\|x-y\|). \tag{1.14}$$

It is called *ϕ-strongly accretive* if there exists a stritly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq ||x - y||\phi(\|x - y\|). \tag{1.15}$$

Finally, A is called *strongly accretive* if there exists $k \in (0, 1)$ such that for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq k\|x - y\|^2. \tag{1.16}$$

Clearly, the class of strongly accretive mappings is a sub-class of the class of ϕ-strongly accretive maps (one takes $\phi(t) = kt$); and the class of ϕ-strongly accretive maps is a sub-class of that of generalized Φ-strongly accretive (one takes $\Phi(t) = t\phi(t)$). It is well known that the inclusions are proper. For the equation $Au = 0$, when A is of accretive-type, existence theorems have been proved by various authors, in various Banach spaces and under various continuity conditions (see Browder [9], [10], [11], [12]). It is well known that the class of *generalized Φ-strongly accretive maps* is the largest, among the classes of accretive-type mappings, for which, if a solution exists, *it is necessarily unique.*

For approximating a solution of equation (1.4), where $A : E \to E$ is of accretive-type, Browder [13] defined an operator $T := I - A$, where I is the identity map on E. He called such an operator *pseudo-contractive*. It is trivial to observe that zeros of A correspond to fixed points of T.

Consequently, solving the equation $Au = 0$ when A is an accretive-type operator is reduced to finding fixed points of pseudo-contractive-type mappings.

An important class of *pseudo-contractive* mappings is the class of *nonexpansive* maps, where a map $T : E \to E$ is called *nonexpansive* if for each $x, y \in E$, the following inequality holds; $||Tx - Ty|| \leq ||x - y||$.

Being an obvious generalization of contraction mappings(mappings $T : E \to E$ satisfying $||Tx-Ty|| \leq ||x-y||\forall x, y \in E$ and some $k \in (0, 1)$), is not all that makes them important. They are also important, as has been observed by Bruck [16], mainly for the following two reasons:

- Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960's and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.

- They appear in applications as the transition operators for initial value problems of differential inclusions of the form
$$0 \in \frac{du}{dt} + T(t)u,$$
where the operators $\{T(t)\}$ are in general set-valued and are accretive or dissipative and minimally continuous.

The following fixed point theorem has been proved for nonexpansive maps on uniformly convex spaces.

Theorem 1.0.1. *Let E be a reflexive Banach space and let K be a nonempty closed bounded and convex subset of E with normal structure. Let $T : K \to K$ be a nonexpansive mapping of K into itself. Then, T has a fixed point.*

While contractions guarantee existence and uniquness, nonexpansions do not. Trivial examples show that the sequence of successive approximations

$$x_{n+1} = Tx_n, \quad x_0 \in K, \quad n \geq 0 \tag{1.17}$$

(where K is a nonempty closed convex and bounded subset of a real Banach space E), for a nonexpansive mapping $T : K \to K$ even with a unique fixed point, may not converge to the fixed point. It is enough, for example, to take for T, a rotation of the unit ball in the plane around the origin of coordinates. Specifically, we have the following example.

Example 3. *Let $B := \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$ and let T denote an anticlockwise rotation of $\frac{\pi}{4}$ about the origin of coordinates. Then T is nonexpansive with the origin as the only fixed point. Moreover, the sequence defined by (1.17) where $B = K$ does not converge to zero.*

Krasnoselskii [40], however, showed that in this example, a convergent sequence of succesive approximations can be obtained if instead of T, the auxilliary nonexpansive mapping $\frac{1}{2}(I + T)$, is used, where I denotes the identity transformation of the plane, i.e., if the sequence of succesive approximations is definedd by

$$x_{n+1} = \frac{1}{2}\Big(x_n + Tx_n\Big) \quad n = 0, 1, .. \tag{1.18}$$

instead of by the usual so-called *Picard iterates*, $x_{n+1} = Tx_n \;\; x_0 \in K \;\; n \geq 0$. It is easy to see that the mappings T and $\frac{1}{2}(I + T)$ have the same set of fixed points, so that the limit of the convergent sequence defined by (1.18) is necessarily a fixed point of T.

Generally, if X is a normed linear space and K a convex subset of X, a generalization of equation (1.18) which has proved successful in the approximation of fixed points of nonexpansive mappings $T : K \to K$ (when they exist), is the following scheme: $x_0 \in K$,

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n = 0, 1, 2, ...\lambda \in (0, 1), \tag{1.19}$$

λ constant (see, e.g., Schaefer [47]). However, the most general *Mann-type* iterative scheme now studied is the following: $x_0 \in K$

$$x_{n+1} = (1 - c_n)x_n + c_nTx_n, \quad n = 0, 1, 2, ... \tag{1.20}$$

where $\{C_n\}_{n=1}^{\infty} \subset (0, 1)$ is a real sequence satisfying appropriate conditions (see, e.g., Chidume [21], Edelstein and O'Brian [34], Ishikawa [35]). Under the following additional assumptions

(i) $\lim C_n = 0$; and
(ii) $\sum_{n=1}^{\infty} C_n = \infty$,
the sequence $\{x_n\}$ generated by (1.20) is generally referred to as the *Mann* sequence [41]. The recurssion formula (1.19) is known as the *Krasnoselskii-Mann* (KM) formula for finding fixed points of nonexpansive mappings (when they exist).

Let K be a nonempty convex subset of a normed space E and $T : K \longrightarrow K$ be a nonexpansive map. Let the sequence $\{x_n\}_{n=0}^{\infty}$ in K be defined iteratively by $x_0 \in K$,

$$x_{n+1} = (1 - c_n)x_n + c_n T x_n, \quad n \geq 1, \tag{1.21}$$

where $\{c_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:
(i) $\displaystyle\sum_{n=0}^{\infty} c_n = \infty$,
(ii) $\displaystyle\lim_{n\longrightarrow\infty} c_n = 0$. Ishikawa [35] proved that If the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded, then it is an *approximate fixed point sequence* in the sense that

$$\lim_{n\longrightarrow\infty} ||x_n - Tx_n|| = 0. \tag{1.22}$$

Edelstein and O'Brian [34] considered the recursion formula

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \ x_0 \in K, \ n \in \mathbb{N}, \lambda \in (0, 1), \tag{1.23}$$

where T maps K into K and proved that *if K is bounded*, then the convergence in (1.22) is uniform.
Chidume [21] considered the recursion formular (1.21), introduced the concept of an admissible sequence and proved that *if K is bounded*, then the convergence in (1.22) is uniform.

Remark 1. *We therefore note that the best mode of convergence we can get from recursion formula (1.21) is weak convergence to a fixed point of T(see e.g, Reich [46]). It is always desirable to establish that the sequence is an approximate fixed point sequence i.e., that the sequence defined by (1.21) satisfies (1.22). In general, the iteration problem does not yield strong convergence of the sequence to a fixed point of T. To obtain convergence to a fixed point of T, some type of compactness condition must be imposed either on K or on the map T (e.g, T may be required to be demicompact, or $(I - T)$ may be required to map closed bounded subsets of E into closed subsets of E, etc, see e.g, Chidume [20]).*

For the more general class of Lipschitz pseudo-contractive maps, attempts to use the Mann formula, which has been successfully employed for nonexpansive mappings, to approximate a fixed point of a Lipschitz pseudo-contractive map even on a compact convex domain in a real Hilbert space proved abortive. In 1974, Ishikawa [36] proved the following theorem.

Theorem 1.0.2. *Let K be a nonempty compact convex subset of a real Hilbert space H and $T : K \longrightarrow K$ be a Lipschitz pseudo-contractive map. Let the sequence $\{x_n\}_{n=0}^{\infty}$ be defined by $x_0 \in K$,*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \tag{1.24}$$

$$y_n = (1-\beta_n)x_n + \beta_n T x_n, n \geq 1, \tag{1.25}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences satisfying the following conditions: (i)* $0 \leq \alpha_n \leq \beta_n < 1 \quad \forall n \geq 1$; *(ii)* $\sum \alpha_n \beta_n = \infty$; *(iii)* $\lim_{n \longrightarrow \infty} \beta_n = 0$. *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a fixed point of* T.

Even though the recursion formulas (1.18) and (1.19) of the Ishikawa scheme have been successfully used in approximating fixed points of Lipschitz pseudo-contractive mappings in real Hilbert spaces, when the domain of T is compact and convex the following question remained open.

OPEN QUESTION: Has the simpler and more efficient Mann sequence failed to converge strongly to some fixed point of a Lipschitz pseudo-contractive map in a Hilbert space, even when the domain of T is compact and convex?

This question remained open for many years until 2001, when Chidume and Mutangadura constructed a counter example to show a Lipschitz pseudo-contractive map defined on a compact convex subset of $\mathbb{R}^2$ with a unique fixed point for which no Mann sequence converges (see Chidume and Mutangadura, [19]).

Browder and Petryshyn [15], (1967) introduced a class of Lipschitz pseudo-contractive maps which contains the class nonexpansive maps called the class of strictly pseudo-contractive maps. Let H be a real Hilbert space. A map $T : H \to H$ is called *strictly pseudo-contractive* if for each $x, y \in H$, $||Tx-Ty||^2 \leq ||x-y||^2 + k||x-y-(Tx-Ty)||^2$, for some $k \in (0,1)$. For a real normed space E, $T : E \to E$ is called strictly pseudo-contractive if each $x, y \in H$, $\langle Tx - Ty, x - y\rangle \leq ||x-y||^2 - \lambda||x-y-(Tx-Ty)||^2$.

In 2004, a striking result was proved by Chidume *et.al.* That for this class of strictly pseudo-contractive maps, the sequence given by (1.21) is an approximate fixed point sequence for strictly pseudo-contractive maps. Furthermore, under an additional condition that T is demicompact, they proved that the sequence defined by (1.21) converges strongly to some fixed point T. Infact, they proved the following theorem.

Theorem 1.0.3. *Let E be a real Banach space. Let K be a nonempty closed and convex subset of E. Let $T : K \to K$ be a strictly pseudo-contractive map in the sense of Browder and Petryshyn with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. For a fixed $x_0 \in K$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a real sequence satisfying the following conditions: (i)$\sum_{n=1}^{\infty} \alpha_n = \infty$ and (ii)$\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. If T is demicompact, then $\{x_n\}$ converges strongly to some fixed point of T in K.

The following quotation further shows the importance of iterative methods for approximating fixed points of nonexpansive mappings.

> "Many well-known algorithms in signal processing and image reconstruction are iterative in nature.... A wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the KM iteration procedure, for particular choices of the operator...."(Charles Byrne [18]).

So far, we have seen the successes in approximating the solutions of (1.4) when the operator A is accretive. This is perhaps a result of numerous geometric properties of Banach spaces which play a crucial role.For example, Hilbert spaces have the nicest geometric properties. The availability of the inner product, the fact that the proximity map of a real Hilber space H onto a closed convex subset K of H is Lipschitzian with constant 1, and the following two identities:

$$||x+y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2, \tag{1.26}$$

$$||\lambda x + (1-\lambda)y||^2 = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||^2, \tag{1.27}$$

which hold for all $x, y \in H$, are some of the geometric properties that characterize Hilbert spaces and also make certain problems posed in ahilbert spaces more managable than those in general Banach spaces. However, as has been rightly observed by M. Hazewinkel,

> "...many, and probably most mathematical objects and models do not naturally live in Hilbert spaces".

Consequently, to extend some of the Hilbert space techniques to more general Banach spaces, analogues of the identities (1.26) and (1.27) have to be developed. For this development, the duality map which has become a most important tool in nonlineaar functional analysis plays a central role. In 1976, Bynum [17] obtained the following analogue of (1.26) for l_p spaces, $1 < p < \infty$:

$$||x+y||^2 \leq (p-1)||x||^2 + ||y||^2 + 2\langle x, j(y)\rangle, \quad 2 \leq p < \infty \tag{1.28}$$

$$(p-1)||x+y||^2 \leq ||x||^2 + ||y||^2 + 2\langle x, j(y)\rangle, \quad 1 < p < 2 \tag{1.29}$$

Analogues of (1.27) were also obtained by Bynum. In 1979, Reich [45] obtained an analogue of (1.26) in uniformly smooth Banach spaces. Other analogues of (1.26) and (1.27) obtained in 1991 and later can be found, for example, in Xu [48] and Xu and Roach [49].

For the past 30 years or so, the study of Krasnoselskii-Mann iterative procedures for the approximation of fixed points of nonexpansive maps and fixed points of some of their generalizations, and approximation of zeros of accretive-type operators have been flourishing areas of research for many mathematicians. Numerous applications of analogues (1.26) and (1.27) to nonlinear iterations involving various classes of nonlinear operators have since then been topics of intensive research. Today, substantial definitive results have been proved, some of the methods have reached their boundaries while others are still subjects of intensive research activity. However, it is apparent that the theory has now reached a level of maturity appropriate for an examination of its central themes.

Unfortunately, attempts to use these properties in approximating the solutions of (1.4) when A is of the monotone type have proved abortive and this is perhaps because of lack of the geometric properties in the spaces suitable for approaching such problems. Another reason is that only maps such as $A : E \to E$ are related to fixed points, also, the scheme is not well defined for $A : E \to E^*$. Fortunately, Alber [3] recently introduced the Lyapunov functional which has helped to develope new geometric properties suitable for monotone mappings. In this thesis, we use these properties to approximate solutions of Hammaerstein equations when the operators are maximal monotone. In the next section, we shall see how to approximate the solutions of Hammerstein equations (assuming existence).

1.0.2 Approximation of solutions of Hammerstein integral equations

In general, equations of Hammerstein-type are nonlinear and there is no known method to find close form solutions for them. Consequently, methods of approximating solutions of such equations, where solutions are known to exist, are of interest.
Let H be a real Hilbert space. A *nonlinear* operator $A : H \to H$ is said to be *angle-bounded* with angle $\beta > 0$ if

$$\langle Ax - Ay, z - y\rangle \leq \beta \langle Ax - Ay, x - y\rangle \tag{1.30}$$

for any triple elements $x, y, z \in H$. For $y = z$ inequality (1.30) implies the monotonicity of A. A monotone *linear* operator $A : H \to H$ is said to be *angle bounded* with angle $\alpha > 0$ if

$$|\langle Ax, y\rangle - \langle Ay, x\rangle| \leq 2\alpha \langle Ax, x\rangle^{\frac{1}{2}} \langle Ay, y\rangle^{\frac{1}{2}} \tag{1.31}$$

for all $x, y \in H$.
In the special case where *one of the operators is angle-bounded*, and the other is *bounded*, Brézis and Browder [4, 6] proved the strong convergence of a suitably defined *Galerkin approximation* to a solution of equation (1.6). In fact, they proved the following theorem.

Theorem 1.0.4 (Brézis and Browder [6]). *Let H be a separable Hilbert space and C be a closed subspace of H. Let $K : H \to C$ be a bounded continuous monotone operator and $F : C \to H$ be angle-bounded and weakly compact mapping. For a given $f \in C$, consider the Hammerstein equation*

$$(I + KF)u = f \tag{1.32}$$

and its nth Galerkin approximation given by

$$(I + K_n F_n)u_n = P^* f, \tag{1.33}$$

where $K_n = P_n^* K P_n : H \to C$ and $F_n = P_n F P_n^* : C_n \to H$, the symbols have their usual meanings (see [6]). Then, for each $n \in \mathbb{N}$, the Galerkin approximation (1.33) admits a unique solution u_n in C_n and $\{u_n\}$ converges strongly in H to the unique solution $u \in C$ of the equation (1.32).

It is obvious that if an *iterative algorithm* can be developed for the approximation of solutions of equation of Hammerstein-type (1.6), this will certainly be a welcome complement to the Galerkin approximation method. Attempts had been made to approximate solutions of equations of Hammerstein-type using *Mann-type* iteration scheme (see e.g., Mann [41]). However, the results obtained were not satisfactory (see [31]). The recurrence formulas used in these attempts, even in real Hilbert spaces, involved K^{-1} which is required to be strongly monotone when K is, and this, apart from limiting the class of mappings to which such iterative schemes are applicable, is also not convenient in any possible applications.

Part of the difficulties in establishing *iterative algorithms* for approximating solutions of Hammerstein equations is that the composition of two monotone maps need not be monotone. If A is linear, (1.2) reduces to

$$\langle Ax, x\rangle \geq 0,$$

for all $x \in \mathbb{R}^2$. Now let $E = \mathbb{R}^2$, take
$F = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $K = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, and $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Clearly, $KF = \begin{bmatrix} -1 & 3 \\ -3 & 1 \end{bmatrix}$.
Now,
$\langle Fx, x\rangle = 5$, $\langle Kx, x\rangle = 5$ but $\langle KFx, x\rangle = -3$. This shows that although F and K are monotone, KF is not.

The first satisfactory results on *iterative methods* for approximating solutions of Hammerstein equations *involving accretive-type mappings*, as far as we know, were obtained by Chidume and Zegeye [28, 29, 30].

Let X be a real Banach space and $F, K : X \to X$ be *accretive-type* mappings. Let $E := X \times X$. Then, Chidume and Zegeye (see [29, 30]) defined $A : E \to E$ by

$$A[u, v] = [Fu - v, Kv + u] \text{ for } [u, v] \in E.$$

We note that $A[u, v] = 0$ if and only if u solves (1.6) and $v = Fu$. The authors defined an iterative sequence and obtained strong convergence theorems in the Cartesian product space E, for solutions of Hammerstein equations under various continuity conditions on F and K, for special classes of real Banach spaces, X. It turns out that, in the case of a real Hilbert space, H, the operator A defined on $H \times H$ is monotone whenever F and K are. The method of proof used by Chidume and Zegeye provided the authors a clue for the establishment of the following coupled explicit iterative algorithm for computing a solution of the equation $u + KFu = 0$ *in the original space,* X. With initial vectors $u_0, v_0 \in X$, sequences $\{u_n\}$ and $\{v_n\}$ in X are defined iteratively as follows:

$$u_{n+1} = u_n - \alpha_n(Fu_n - v_n), \ n \geq 0, \tag{1.34}$$

$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n),\ n \geq 0, \tag{1.35}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying appropiate conditions. The recursion formulas (1.34) and (1.35) have been used successfully to approximate solutions of Hammerstein equations involving nonlinear *accretive-type mappings* (see e.g., Chidume and Djitte [25, 26], Chidume and Ofoedu [24], Chidume and Yekini [23], Chidume [20], and the references contained in them). In particular, the following theorems have been proved as generalizations of recent important results.

Theorem 1.0.5. *[Chidume, [27]] Let E be a uniformly smooth real Banach space with modulus of smoothness ρ_E, and let $A : E \to 2^E$ be a multi-valued bounded $m-$accretive operator with $D(A) = E$ such that the inclusion $0 \in Au$ has a solution. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}_{n=1}^{\infty}$ by,*

$$x_{n+1} = x_n - \lambda_n u_n - \lambda_n \theta_n (x_n - x_1),\ u_n \in Ax_n,\ n \geq 1,$$

where $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$ satisfying the folliowing conditions:
(i) $\lim_{n\to\infty} \theta_n = 0$, $\{\theta_n\}_{n=1}^{\infty}$ is decreasing;
(ii) $\sum \lambda_n\theta_n = \infty$; $\sum \rho_E(\lambda_n M_1) < \infty$, for some constant $M_1 > 0$;
(iii) $\lim_{n\to\infty} \frac{\left[\frac{\theta_{n-1}}{\theta_n} - 1\right]}{\lambda_n\theta_n} = 0$. There exists a constant $\gamma_0 > 0$ such that $\frac{\rho_E(\lambda_n)}{\lambda_n} \leq \gamma_0\theta_n$. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a zero of A.

It is our purpose in this paper to use an analogue of Theorem 1.0.5 and approximate a solution of (1.6) in the case where F and K are bounded maximal monotone mappings from E to E^* and E^* to E, respectively.

Chapter 2

PRELIMINARIES

In the sequel, we give some definitions of most of the terms and concepts we shall use.

2.1 Definition of some terms and concepts.

Definition 2.1.1. *Let E be a real normed space. A map $J : E \to 2^{E^*}$ defined by*

$$Jx := \left\{x^* \in E^* : \left\langle x, x^* \right\rangle = \|x\|.\|x^*\|,\ \|x\| = \|x^*\|\right\}$$

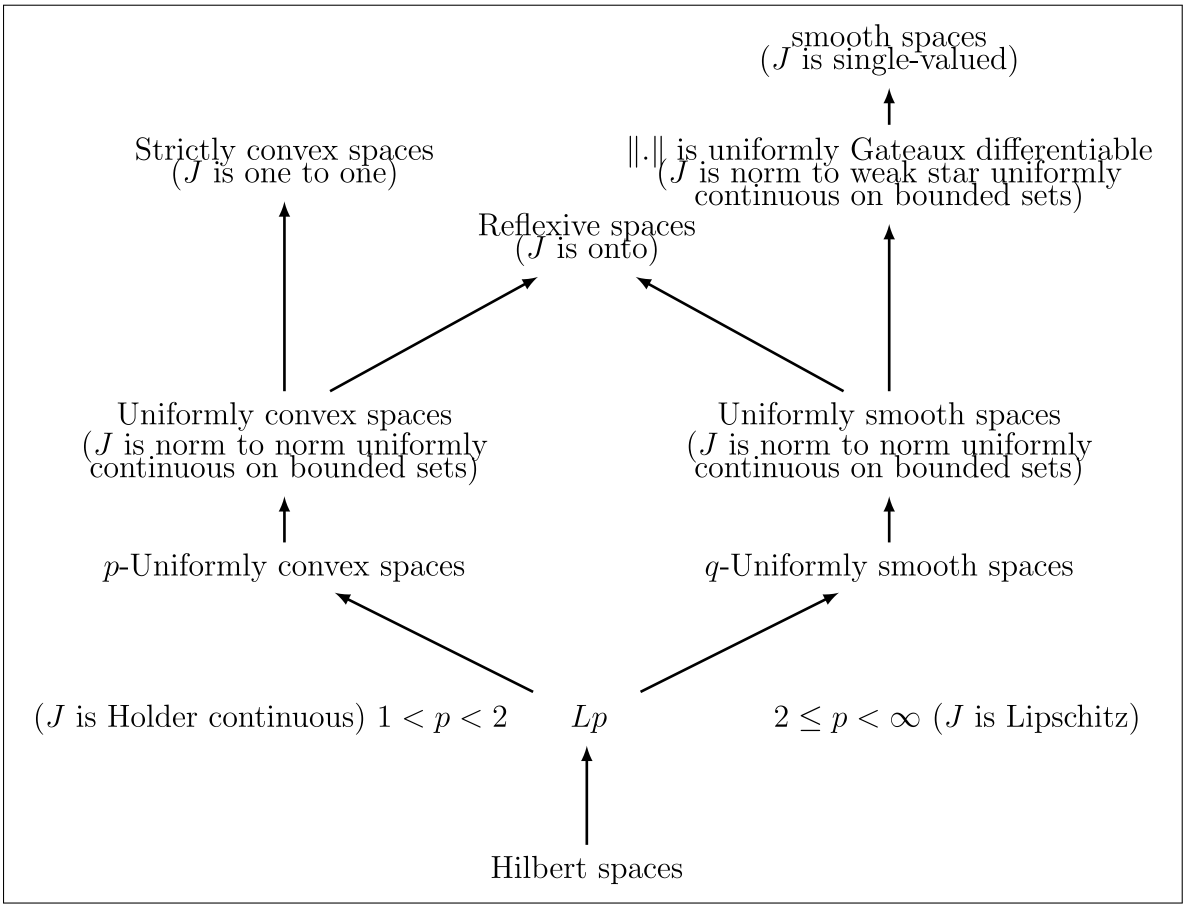
is called the normalized duality map on E.

Remark 2. *Many properties of the normalized duality map have been established. We list some of them here.*

- *If E is smooth, then J is single-valued.*
- *If E is strictly convex, then J is one-to-one, i.e., if $x \neq y$, then $Jx \cap Jy = \emptyset$.*
- *If E is reflexive, then J is onto.*
- *In particular, if a Banach space E is uniformly smooth and uniformly convex, the dual space is also uniformly smooth and uniformly convex. Hence, the normalized duality map on E and the normalized duality map on its dual space E^*, are both uniformly continuous on bounded sets.*

Remark 3. *If E is smooth, strictly convex and reflexive and $J^* : E^* \to E$ is the normalized duality mapping on E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$, where I_E and I_{E^*} are the identity mappings on E and E^*, respectively.*

The lattice below, shows the properties of J on different normed linear spaces.



Definition 2.1.2. *Let E be a real normed linear space with dual E^*. A mapping T with $D(T)$ and $R(T)$ in E is called accretive if and only if for all $x, y \in D(T)$, the following inequality is satisfied:*

$$\|x - y\| \leq \|x - y + s(Tx - Ty)\| \; \forall \; s > 0.$$

Equivalently, by using the normalized duality map which we define as $J : E \to 2^{E^*}$ given by

$$J(x) \colon = \{x^* \in X^* \colon \langle x, x^* \rangle = \|x\|\|x^*\|; \|x^*\| = \|x\|\}$$

we can give the above definition as:
T is accretive if and only if for all $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq 0,$$

where J is the normalized duality map on E.

Definition 2.1.3. *A map $T : D(T) \subset E \to E^*$ is called monotone if for each $x, y \in D(T)$,*

$$\langle Tx - Ty, x - y\rangle \geq 0.$$

We observe that since in a Hilbert space, the normalized duality map is the identity, consequently, in Hilbert spaces, monotonicity and accretivity coincide.

Definition 2.1.4. *Let E be a real normed linear space. A real-valued function f defined on a convex subset of E is said to be convex if for all $x, y \in E$ and for every $\lambda \in [0, 1]$;*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Definition 2.1.5. *Let E be a real normed linear space and f a proper and convex function defined on E. The subdifferential operator $\partial f : E \to 2^{E^*}$ is defined by:*

$$\partial f(x) := \{x^* \in E^* : \langle x - y, x^* \rangle \leq f(x) - f(y) \forall y \in E\}.$$

Definition 2.1.6. *A Banach space E is said to be uniformly convex if for each $\epsilon \in (0,2]$ there exists a $\delta(\epsilon) > 0$ such that, for all $x, y \in E$ with $\|x\| = 1 = \|y\|$ and $\|x - y\| \geq \epsilon$, we have $\|\frac{x+y}{2}\|) \leq 1 - \delta(\epsilon)$.*

Definition 2.1.7. *Let E be a real normed space with dim $E \geq 2$; the modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$, defined by*

$$\delta_E(\epsilon) := \inf_{x,y \in E} \Big\{1 - \|\frac{x+y}{2}\| : \|x\|, \|y\| \leq 1, \|x - y\| \geq \epsilon\Big\}.$$

Definition 2.1.8. *A normed linear space E is said to be strictly convex if for all $x, y \in E$, with $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda)y\| < 1 \, \forall \, \lambda \in (0,1)$.*

Definition 2.1.9. *Let E be a real Banach space and $S(E) = \{x \in E : \|x\| = 1\}$; then the norm of E is said to be Gâteaux differentiable if*

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. In this case E is called smooth.

We give a characterization of smooth Banach spaces which is also used as an equivalent definition by some authors.

A normed space E is called smooth if for every $x \in E$, with $\|x\| = 1$ there exists a unique $x^* \in X^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.

Definition 2.1.10. *Let E be a normed linear space with dim $E \geq 2$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by*

$$\begin{aligned} \rho_E(t) &:= sup\Big\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = t\Big\} \\ &= sup\Big\{\frac{\|x + ty\| + \|x - ty\|}{2} - 1 : \|x\| = 1 = \|y\|\Big\}. \end{aligned}$$

Definition 2.1.11. *A normed space E is said to be uniformly smooth whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = 1$ and $\|y\| \leq \delta$, we have,*

$$\|x + y\| + \|x - y\| < 2\varepsilon\|y\|$$

A useful characterization of Uniformly smooth spaces by the modulus of smoothness is given in the following theorem which we state without proof as follows:

Theorem 2.1.12. *(see for example, Chidume [?]) A normed linear space E is uniformly smooth if and only if*

$$\lim_{t \to 0^+} \frac{\rho_E(t)}{t} = 0.$$

Remark

(i) Every Uniformly smooth space is smooth.

(ii) Every Uniformly smooth space is reflexive.

(iii) $\rho_E : [0, \infty) \to [0, \infty)$ is a convex and continuous function.

(iv) $\frac{\rho_E(t)}{t}$ is a nondecreasing function on $[0, \infty)$

(v) $\rho_E(t) \leq t$ for all $t \geq 0$.

There is a very interesting relationship between uniformly smooth and uniformly convex spaces which we present in the following theorem.

Theorem 2.1.13. *(see, for example Chidume [20] Let E be a Banach space.*

(i) E is uniformly convex if and only if E^* is uniformly smooth.

(ii) E is uniformly smooth if and only if E^* is uniformly convex.

Definition 2.1.14. *Let $p > 1$ be a real number. A normed linear space E is p-uniformly convex if there exists a constant $c > 0$ such that*

$$\delta_E(\epsilon) \geq c\epsilon^p.$$

Examples: L_p (or l^p), spaces $(1 < p < \infty)$, are p-uniformly convex; where,

(a) $\delta_E(\epsilon) \geq \epsilon^p$ if $2 \leq p < \infty$.

(b) $\delta_E(\epsilon) \geq \frac{1}{2^{p+1}}\epsilon^2$, if $1 < p < 2$.

Definition 2.1.15. *Let $q > 1$ be a real number. A normed linear space E is q-uniformly smooth if there exists a constant $c > 0$ such that for all $t > 0$*

$$\rho_E(t) \leq ct^q.$$

As a consequence of theorem (2.1.13) we have the following theorem:

Theorem 2.1.16. *Let E be a real Banach space.*

(i) E is p-uniformly convex if and only if E^* is q-uniformly smooth.

(ii) E is q-uniformly smooth if and only if E^* is p-uniformly convex, where $\frac{1}{p} + \frac{1}{q} = 1$.

2.2 Results of Interest

Definition 2.2.1. *Let E be a normed space with $dim E \geq 2$. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$, defined by*

$$\delta_E(\epsilon) := inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\Big\}.$$

In the sequel, we shall need the following definitions and results. Let E be a smooth real Banach space with dual E^*. The function $\phi : E \times E \to \mathbb{R}$, defined by,

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for } x, y \in E, \tag{2.1}$$

where J is the normalized duality mapping from E into 2^{E^*} will play a central role. It was introduced by Alber and has been studied by Alber [1], Alber and Guerre-Delabriere [2], Kamimura and Takahashi [38], Reich [44] and a host of other authors. If $E = H$, a real Hilbert space, then equation (2.1) reduces to $\phi(x,y) = \|x-y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2 \ \text{ for } x, y \in E. \tag{2.2}$$

Define a map $V : X \times X^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2. \tag{2.3}$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}(x^*)) \ \forall x \in X, \ x^* \in X^*. \tag{2.4}$$

Lemma 2.2.2. *([Alber, [1]]) Let X be a reflexive striclty convex and smooth Banach space with X^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*) \tag{2.5}$$

for all $x \in X$ and $x^*, y^* \in X^*$.

Lemma 2.2.3 (Kamimura and Takahashi, [38])**.** *Let X be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 2.2.4. *(Xu [?]) Let ρ_n be a sequence of non-negative real numbers satisfying the relation:*

$$\rho_{n+1} \leq (1-\beta_n)\rho_n + \beta_n\zeta_n + \gamma_n, \ n \geq 0, \tag{2.6}$$

where,
(i) $\beta_n \subset [0,1], \quad \sum \beta_n = \infty$;
(ii) $\limsup \zeta_n \leq 0$;
(iii) $\gamma_n \geq 0; \quad (n \geq 0), \quad \sum \gamma_n < \infty$.
Then, $\rho_n \to 0$ as $n \to \infty$.

Remark 4. *Let E^* be a strictly convex dual Banach space with a Frechet differentiable norm and $A : E \to 2^{E^*}$, be a maximal monotone map with no monotone extension. Let $z \in E^*$ be fixed. Then for every $\lambda > 0$, $\exists!\ x_\lambda \in E$ such that $z \in Jx_\lambda + \lambda Ax_\lambda$ (see Reich [45], p. 342). Setting $J_\lambda z = x_\lambda$, we have the resolvent $J_\lambda := (J + \lambda A)^{-1} : E^* \to E$ of A, for every $\lambda > 0$. A celebrated result of Reich follows.*

Lemma 2.2.5. *(Reich, [45]). Let E^* be a strictly convex dual Banach space with a Frechet differentiable norm and let $A : E \to E^*$ be maximal monotone such that $A^{-1}0 \neq \emptyset$. Let $z \in E^*$ be an arbitrary bu fixed vector. For each $\lambda > 0, \exists! x_\lambda \in E$ such that $z \in Jx_\lambda + \lambda Ax_\lambda$. Furthermore, x_λ converges strongly to a unique $v \in A^{-1}0$.*

Lemma 2.2.6 (Alber, [1], p45). *Let X be a uniformly convex Banach space. Then for any $R > 0$ and any $x, y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$ the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_X(c_2^{-1}\|x - y\|),$$

where $c_2 = 2\max\{1, R\}$, $1 < L < 1.7$.

Lemma 2.2.7 (Alber, [1], p46). *Let X be a uniformly smooth and strictly convex Banach space. Then for any $R > 0$ and any $x, y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$ the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_{X^*}(c_2^{-1}\|Jx - Jy\|),$$

where $c_2 = 2\max\{1, R\}$, $1 < L < 1.7$.

Lemma 2.2.8 (Alber, [1], p.50). *Let X be a reflexive strictly convex and smooth Banach space with dual X^*. Let $W : X \times X \to \mathbb{R}$ be defined by $W(x, y) = \frac{1}{2}\phi(y, x)$. Then,*

$$\phi(y, x) - \phi(y, z) \geq 2\langle Jx - Jz, z - y\rangle, \tag{2.7}$$

and

$$W(x, y) \leq \langle Jx - Jy, x - y\rangle, \tag{2.8}$$

for all $x, y, z \in X$

Lemma 2.2.9. *From Lemma 2.2.5 above, setting $\lambda_n := \frac{1}{\theta_n}$ where $\theta_n \to 0$ as $n \to \infty$, $u = z$, $y_n := \left(J + \frac{1}{\theta_n}\right)^{-1} u$, we obtain that:*

$$Ay_n = \theta_n(Ju - Jy_n), \tag{2.9}$$

$$y_n \to y^* \in A^{-1}0,$$

where $A : E \to E^*$ is maximal monotone. We observe equation (2.9) yields

$$Jy_{n-1} - Jy_n + \frac{1}{\theta_n}\Big(Ay_{n-1} - Ay_n\Big) = \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big(Ju - Jy_{n-1}\Big). \tag{2.10}$$

Taking the duality pairing of this with $y_{n-1} - y_n$, we obtain that

$$\begin{aligned}
&\langle Jy_{n-1} - Jy_n, y_{n-1} - y_n\rangle \\
&\quad\le \Big\langle Jy_{n-1} - Jy_n + \frac{1}{\theta_n}(Ay_{n-1} - Ay_n), y_{n-1} - y_n\Big\rangle \\
&\quad\le \Big\|Jy_{n-1} - Jy_n + \frac{1}{\theta_n}\Big(Ay_{n-1} - Ay_n\Big)\Big\|\|y_{n-1} - y_n\| \\
&\quad= \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big\|Ju - Jy_{n-1}\Big\|\|y_{n-1} - y_n\|.
\end{aligned}$$

It follows that if E is uniformly convex and uniformly smooth, using lemma 2.2.6 we obtain that,

$$(2L)^{-1}\delta_E(c_2^{-1}\|y_{n-1} - y_n\|) \le \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big\|Ju - Jy_{n-1}\Big\|\|y_{n-1} - y_n\|, \tag{2.11}$$

which gives that

$$\|y_{n-1} - y_n\| \le c_2\delta_E^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\right), \text{ for some } K > 0. \tag{2.12}$$

Similarly, using lemmam 2.2.7, we obtain that,

$$\|Jy_{n-1} - Jy_n\| \le c_2\delta_{E^*}^{-1}\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}K\right), \text{for some } K > 0. \tag{2.13}$$

2.3 Some interesting properties of Normalized Duality map

Definition 2.3.1. *Let E be a real Banach space, and $p > 1$, the mapping $J_p\colon E \to 2^{E^*}$ defined by;*

$$J_p(x)\colon = \{x^* \in E^*\colon \langle x, x^*\rangle = \|x\|\|x^*\|; \|x^*\| = \|x\|^{p-1}\}$$

is called the generalized duality map. In particular, if $p = 2$, then the duality map J_2 is denoted by J and called the normalized duality map.

The following propositions which are known outlines some interesting properties of the normalized duality mapping in different Banach spaces.

Proposition 2.3.2. *In any normed linear space E for any $x \in E$ $J(x) \neq \phi$, convex, closed and bounded subset of E^**

Remark Since $J(x)$ is closed and convex, it is weakly closed; and being bounded, in a reflexive space, it is weakly compact.

Proposition 2.3.3. *The normalized duality mapping is Homogenous in the sense that, for all $x \in X$, $J(\lambda x) = \lambda J(x)$ $\forall$ $\lambda \in \mathbb{R}$.*

Other properties of the normalized duality map are embodied in the following theorems.

Theorem 2.3.4. *Let E be a real uniformly smooth Banach space, with dual E^*, then the normalized duality map $J : E \to E^*$ is norm-to-norm uniformly continuous on bounded subsets of E.*

Theorem 2.3.5. *Let E be a real Banach space with a uniformly Gateaux differentiable norm. Then the normalized duality mapping $J : E \to E^*$ is norm-to-weak* uniformly continuous on bounded subsets of E.*

Chapter 3

A Strong convergence theorem

3.1 Main Result

Let X be a real reflexive space with dual space X^*. Let $F : X \to X^*$, $K : X^* \to X$ be monotone maps. Consider the Hammerstein equation

$$u + KFu = 0,\ u \in X. \tag{3.1}$$

In this chapter, we construct an iterative sequence which converges strongly to a solution of Hammaerstein equation (3.1) when F and K are maximal monotone and bounded maps. We shall obtain this as an application of the following theorem.

Theorem 3.1.1. *(Chidume et al. [32]) Let X be a uniformly convex and uniformly smooth real Banach space and X^* be its dual. Let $A : X \to 2^{X^*}$ be a maximal monotone and bounded map with $A^{-1}(0) \neq \emptyset$. For arbitrary $u_1 \in X$, define a sequence $\{u_n\}$ iteratively by:*

$$u_{n+1} = J^{-1}(Ju_n - \alpha_n\xi_n - \alpha_n\theta_n(Ju_n - Ju_1)), \quad \xi_n \in Au_n, n \geq 1. \tag{3.2}$$

Then, the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to a solution of $0 \in Au$, where $\alpha_n \in (0,1)$ and $\theta_n \in (0,1)$ satisfy the following conditions:
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $M^*\alpha_n \leq \gamma_0\theta_n, \quad \delta_E^{-1}(\alpha_n M^*) \leq \gamma_0\theta_n$;

(iii) $\dfrac{\delta_E^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\alpha_n\theta_n} \to 0, \quad \dfrac{\delta_{E^*}^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\alpha_n\theta_n} \to 0$ *as* $n \to \infty$;

(iv) $\frac{1}{2}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right) \leq 1$;

for some constants $M^* > 0,\ K > 0$ and $\gamma_0 > 0$; where δ_X is the modulus of convexity of X and δ_{X^*} is the modulus of convexity of X^*.

The following theorem shall be used in the sequel.

Theorem 3.1.2. *(Browder [14]) Let X be a strictly convex and reflexive Banach space with a strictly convex conjugate space X^*, T_1 a maximal monotone mapping* *from X to X^*, T_2 a hemicontinuous monotone mapping of all of X into X^* which carries bounded subsets of X into bounded subsets of X^*. Then, the mapping $T = T_1 + T_2$ is a maximal monotone map of X into X^*.*

We shall also need the following corollaries.

Corollary 3.1.3. *([20], pg. 40) Let $p > 1$ and $r > 0$ be two fixed real number and X be a Banach space. Then the following are equivalent.*
(i) X is uniformly convex.
(ii) There is a continuous, strictly increasing convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that

$$||x + y||^p \geq ||x||^p + p\langle y, f_x\rangle + g(||y||) \tag{3.3}$$

for every $x, y \in B_r(0)$ and $f_x \in J_p(x)$.
(iii) There is a continuous, strictly increasing convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that

$$\langle x - y, f_x - f_y\rangle \geq g(||x - y||) \tag{3.4}$$

for every $x, y \in B_r(0)$ and $f_x \in J_p(x),\ f_y \in J_p(y)$, where $B_r(0) := \{u \in X : ||u|| \leq r\}$.

Corollary 3.1.4. *([20], pg. 50) Let $q > 1$ and $r > 0$ be two fixed real number and X be a smooth Banach space. Then the following are equivalent.*
(i) X is uniformly smooth.
(ii) There is a continuous, strictly increasing convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that for every $x, y \in B_r(0)$, we have

$$||x + y||^q \leq ||x||^q + q\langle y, J_q(x)\rangle + g(||y||) \tag{3.5}$$

(iii) There is a continuous, strictly increasing convex function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ g(0) = 0$$

such that for all $x, y \in B_r(0)$, we have

$$\langle x - y, J_q(x) - J_q(y)\rangle \leq g(||x - y||), \tag{3.6}$$

where $B_r(0) := \{u \in X : ||u|| \leq r\}$.

Let X be a real normed space with dual X^*. Let J_X and J_{X^*} denote the normalized duality maps on X and X^*, respectively. Define $E = X \times X^*$. Then $E^* = X^* \times X$.

It has been proved (see e.g, Chidume [20]) that if $J_{X\times X^*}$ denotes the duality map on $X \times X^*$, then, for any $[u, v] \in X \times X^*$,

$$J_{X\times X^*}[u, v] = [J_X(u), J_{X^*}(v)],$$

so that for arbitrary $z_1 = [u_1, v_1]$, $z_2 = [u_2, v_2]$ in E, the duality pairing $\langle \cdot, \cdot \rangle$ is given by

$$\langle z_1, J_E(z_2)\rangle := \langle u_1, J_X(u_2)\rangle + \langle v_1, J_{X^*}(v_2)\rangle.$$

Lemma 3.1.5. *Let X be a uniformly convex and uniformly smooth real Banach space with dual X^*. Let $E := X \times X^*$, $F : X \to X^*$ and $K : X^* \to X$ be maximal monotone and bounded maps. Define $A : E \to E^*$ by*

$$A[u, v] = [Fu - v, Kv + u], \tag{3.7}$$

for all $[u, v] \in E$. Then,
(i) E is uniformly convex and uniformly smooth,
(ii) A is maximal monotone,
(iii) A is bounded,
(iv) $A[u^*, v^*] = [0, 0]$ if and only if u^* solves (3.1) and $v^* = Fu^*$.

Proof. (*i*) Let $x = [x_1, x_2]$, $y = [y_1, y_2]$ be arbitrary elements of E. Then,

$$\begin{aligned}
&\langle x - y, J_E(x) - J_E(y)\rangle \\
&\quad = \Big\langle [x_1 - y_1, x_2 - y_2], \Big[J_X(x_1) - J_X(y_1), J_{X^*}(x_2) - J_{X^*}(y_2)\Big]\Big\rangle \\
&\quad = \Big\langle x_1 - y_1, J_X(x_1) - J_X(y_1)\Big\rangle + \Big\langle x_2 - y_2, J_{X^*}(x_2) - J_{X^*}(y_2)\Big\rangle \\
&\quad \le g_1^*(\|x_1 - y_1\|) + g_2^*(\|x_2 - y_2\|),
\end{aligned}$$

where g_1^*, g_2^* are strictly increasing continuous and convex functions on $\mathbb{R}^+$ and $g_1^*(0) = g_2^*(0) = 0$. It follows that:

$$\Big\langle x - y, J_E(x) - J_E(y)\Big\rangle \le g^*(\|x - y\|),$$

where $g^*(\|x - y\|) := g_1^*(\|x_1 - y_1\|) + g_2^*(\|x_2 - y_2\|)$. Hence the result follows from inequality (3.6) of corollary 3.1.4 that E is uniformly smooth.

Also,

$$\begin{aligned}
&\langle x - y, J_E(x) - J_E(y)\rangle \\
&\quad = \Big\langle [x_1 - y_1, x_2 - y_2], \Big[J_X(x_1) - J_X(y_1), J_{X^*}(x_2) - J_{X^*}(y_2)\Big]\Big\rangle \\
&\quad = \Big\langle x_1 - y_1, J_X(x_1) - J_X(y_1)\Big\rangle + \Big\langle x_2 - y_2, J_{X^*}(x_2) - J_{X^*}(y_2)\Big\rangle \\
&\quad \ge g_1(\|x_1 - y_1\|) + g_2(\|x_2 - y_2\|),
\end{aligned}$$

where g_1, g_2 are strictly increasing continuous and convex functions on $\mathbb{R}^+$ and $g_1(0) = g_2(0) = 0$. It follows that:

$$\Big\langle x - y, J_E(x) - J_E(y) \Big\rangle \geq g(\|x - y\|),$$

where $g(\|x-y\|) := g_1(\|x_1 - y_1\|) + g_2(\|x_2 - y_2\|)$. Hence the result follows from inequality (3.4) corollary 3.1.3 that E is uniformly convex.

(ii) Let $S, T : E \times E^* \to E^* \times E$ be defined by

$$S[u, v] = [Fu, Kv], \quad T[u, v] = [-v, u].$$

Then, $A = S + T$. It suffices to show that S and T are maximal monotone. Observe that S is monotone. Let $h = [h_1, h_2] \in E^* \times E$. Since F, K are maximal monotone, take $u = (J + \lambda F)^{-1} h_1$ and $v = (J_* + \lambda K)^{-1} h_2$, where J_* is the normalized duality map on E^*. Then, $(J + \lambda S)w = h$, where $w = [u, v]$. Hence, S is maximal monotone.

Clearly, T is bounded and monotone. Furthermore it is continuous. Hence, it is hemi-continuous. Therefore, by theorem 3.1.2, $A = S + T$ is maximal monotone.

(iii) Let V be a bounded subset of $E \times E^*$. Then, $\exists\, R > 0$ such that $||[u, v]||_{E \times E^*} \leq R$, for all $[u, v] \in V$. This implies that $||u||_E \leq R$ and $||v||_{E^*} \leq R$. Now, since F and K are bounded, we have that

$$\begin{aligned}
||A[u, v]||^2_{E^* \times E} &= ||[Fu - v, Kv + u]||^2_{E^* \times E} \\
&= ||Fu - v||^2_{E^*} + ||Kv + u||^2_E \\
&\leq \big(||Fu||_{E^*} + ||v||_{E^*}\big)^2 + \big(||Kv||_E + ||u||_E\big)^2 \\
&\leq \big(M_1 + R\big)^2 + \big(M_2 + R\big)^2 \\
&=: M^2,
\end{aligned}$$

for some positive constants, M_1 and M_2. Hence, $||A[u, v]||_{E^* \times E} \leq M, \forall\ [u, v] \in V$ and so A is bounded.

(iv) Let $z^* = (u^*, v^*) \in E \times E^*$. Now,

$$A[u^*, v^*] = [Fu^* - v^*, Kv^* + u^*] \quad = \quad [0, 0]$$

$\Leftrightarrow$

$$Fu^* - v^* = 0 \text{ and } Kv^* + u^* = 0$$

$\Leftrightarrow$

$$v^* = Fu^* \text{ and } u^* + KFu^* = 0.$$

Hence, z^* is a zero of A if and only if u^* solves (3.1), where $v^* = Fu^*$. □

Let the sequences $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{\theta_n\}_{n=1}^{\infty} \subset (0,1)$ satisfy the conditions in theorem 3.1.1 above.

We prove the following theorem.

Theorem 3.1.6. *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^*. Let $F : X \to X^*$ and $K : X^* \to X$ be maximal monotone and bounded maps. For arbitrary $u_1 \in X$ and $v_1 \in X^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in X and X^*, respectively by*

$$u_{n+1} = J^{-1}(Ju_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n(Ju_n - Ju_1)),\ n \geq 1, \tag{3.8}$$

$$v_{n+1} = J_*^{-1}(J_*v_n - \alpha_n(Kv_n + u_n) - \alpha_n\theta_n(J_*v_n - J_*v_1)),\ n \geq 1. \tag{3.9}$$

Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of $u + KFu = 0$ with $v^* = Fu^*$.

Proof. Denote J and J_* by J_X and J_{X^*}, respectively, then, we have from equations (3.8) and (3.9), that

$$\begin{aligned}
&[u_{n+1}, v_{n+1}] \\
&= \Big[J_X^{-1}(J_Xu_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n(J_Xu_n - J_Xu_1)), J_{X^*}^{-1}(J_{X^*}v_n - \alpha_n(Kv_n + u_n) \\
&\quad -\alpha_n\theta_n(J_{X^*}v_n - J_{X^*}v_1))\Big] \\
&= J_{X\times X^*}^{-1}\Big[J_Xu_n - \alpha_n(Fu_n - v_n) - \alpha_n\theta_n(J_Xu_n - J_Xu_1), J_{X^*}v_n - \alpha_n(Kv_n + u_n) \\
&\quad -\alpha_n\theta_n(J_{X^*}v_n - J_{X^*}v_1)\Big]. \\
&= J_{X\times X^*}^{-1}\Big[(J_Xu_n, J_{X^*}v_n) - \alpha_n(Fu_n - v_n, Kv_n + u_n) \\
&\quad -\alpha_n\theta_n[(J_Xu_n, J_{X^*}v_n) - (J_Xu_1, J_{X^*}v_1)]\Big] \\
&= J_{X\times X^*}^{-1}\Big[J_{X\times X^*}[u_n, v_n] - \alpha_n[Fu_n - v_n, Kv_n + u_n] \\
&\quad -\alpha_n\theta_n(J_{X\times X^*}[u_n, v_n] - J_{X\times X^*}[u_1, v_1])\Big] \\
&= J_{X\times X^*}^{-1}\Big[J_{X\times X^*}[u_n, v_n] - \alpha_nA[v_n, u_n] - \alpha_n\theta_n(J_{X\times X^*}[u_n, v_n] - J_{X\times X^*}[u_1, v_1])\Big],
\end{aligned}$$

i.e.,

$$[u_{n+1}, v_{n+1}] = J_{X\times X^*}^{-1}\Big[J_{X\times X^*}[u_n, v_n] - \alpha_nA[v_n, u_n] - \alpha_n\theta_n(J_{X\times X^*}[u_n, v_n] - J_{X\times X^*}[u_1, v_1])\Big] \tag{3.10}$$

Setting $z_n = [u_n, v_n]$ and $J_E = J_{X\times X^*}$, equation (3.10) becomes

$$z_{n+1} = J_E^{-1}(J_Ez_n - \alpha_nAz_n - \alpha_n\theta_n(J_Ez_n - J_Ez_1)), n \geq 1. \tag{3.11}$$

It follows by theorem 3.2, that the sequence $\{z_n\}_{n=1}^{\infty}$ converges strongly to a zero of A and by Lemma 3.1.5, (iv), $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge to u^* and v^*, respectively, where u^* is a solution of the Hammerstein equation $u + KFu = 0$, with $v^* = Fu^*$. $\square$

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