Isoperimetric Variational Techniques and Applications.

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Epigraph

" Discoveries, small or great, are never born of spontaneous generation. They always suppose a soil seeded preliminary knowledge and well prepared by labor, both conscious and subconscious."

- Henri Poincaré -

Dedication

This project is dedicated to my Lord and Savior, my God and King, for inspiration and wisdom. He really makes all things beautiful in his time.

Secondly I dedicate this project to my beloved parents ***GABA A. Prudence, ADJAGBA Gilberte*** for their parental care, support and love towards me during my training at the African University of Science and Technology.

My dedication extends also to my brother ***Nathan*** and my sisters ***Charlène*** and ***Bénédicte*** who have supported me to achieve this goal.

Finally, I dedicate this work to ***AKADIRI Clémence***, my dear friend.

Preface

This project is at the interface between Nonlinear Functional Analysis, Convex Analysis and Differential Equations. It concerns one of the most powerful methods often used to solve optimization problems with constraints; namely the Variational Method involving Isoperimetric conditions. As applications the existence of infinetely many periodic solutions of some 2^{nd} order dynamical systems will be proven in the line of M.S. Berger[1].

Variational methods refer to proofs established by showing that a suitable auxilliary function attains a minimum or has a critical point (cf. Definition ...). In the former case, this can be viewed as a mathematical form of the principle of least action in Physics and justifies why so many results in Mathematics are somehow related to variational techniques as they have their origin in the physical sciences. Their applications cover numerous theoretic as well as applied areas including optimization, Banach space geometry, nonsmooth analysis, economics, control theory and Game theory. But we shall focus on a branch linking minimization and periodic differential equations.

My interest in this subject has been steadily fascinated by the successive lectures delivered at the African University of Sciences and Technology by Prof. C. Chidume (*Functional Analysis*)[2], Dr. N. Djitte (*Sobolev spaces and linear elliptic partial differential equations*)[3], Dr G. Degla (*Topics in Differential Analysis*)[4] and Prof. Thibault (*Convex Analysis*)[5].

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Contents

0 Introduction and Motivations

The exploitation of nature's propensity offers us ample opportunities to achieve or deal with an optimal objective concerning constrained shape, volume, time, velocity, energy or gain. This vivifies the need to study Optimization Theory and related topics.

In order to make the concepts clear, let us recall some keywords. Given a nonempty set X and a function $f : X \to \mathbb{R}$ which is bounded below, computing the number

$$\inf_X f \;:=\; \inf\{f(x) \,:\, x \in X\} \qquad (F_1)$$

represents a *minimization problem* posed in X: namely that of finding a minimizing sequence, i.e. $(x_k)_k \subset X$ such that

$$\lim_{k\to\infty} f(x_k) \;=\; \inf_X f\,.$$

The number $\inf_X f$ is often called the *infimal value* of f or more simply the *infimum* of f over X. The function f is usually called the *objective function* or also *infimand*. By analogy we have the concepts of *supremal value* (*supremum*) and *supremand*.
An optimal solution of (F_1) is an element $a \in X$ such that

$$f(a) \;\le\; f(x)\,, \quad \forall\, x \in X\,;$$

such an element a is usually called a *minimizer*, a *minimum point* or simply a *minimum* of f on X. We shall also speak of *global minimum*.
Let us emphasize that the notation

$$\min\{f(x) \,:\, x \in X\}$$

holds at the same time for a number (when there exists a solution to F_1) and a *problem to solve.*
Likewise one can meet *maximization problems* but they are all equivalent to minimization problems since for any real valued function g defined on a set X, one has

$$\sup\{g(x) \,:\, x \in X\} \;=\; -\inf\{-g(x) \,:\, x \in X\}\,.$$

When X has a topological structure, another problem related with (F_1), is to know whether a giving minimizing sequence (x_k) converges to an optimal solution when k tends to $+\infty$. Two conditions are essential to guarantee a positive answer to the above problem. A topological criterion on the structure of X (e.g., compactness) and a topological criterion on the behavior of the function f (e.g., continuity).

When X is an open set of a real normed linear space (respectively a manifold) and f is Fréchet differentiable or just Gâteaux differentiable (respectively differentiable in the geometric sense), a necessary condition for a point $a \in X$ to be a minimizer (according to Euler) is to be a *critical* (or *stationary*) *point* of f; this means that, $f'(a) \equiv 0$ on X (respectively $df(a) \equiv 0$ on T_aX, the tangent space of the manifold X at a). We say that a real number c is a *critical value* of f if there exists a critical point $a \in X$ such that $f(a) = c$. In the case of a Hilbert space $X = H$ endowed with a scalar product $\langle \cdot\,,\cdot \rangle$, and thanks to the Riesz representation theorem, the *gradient* ∇f of a Gâteaux differentiable is defined by setting

$$\langle h\,, \nabla f(x)\rangle \;=\; f'(x)(h)\,.$$

And so in this case, a critical point of f is just a solution of the equation

$$\nabla f(x) \;=\; 0\,.$$

The following surjectivity result illustrates well variational arguments.

Proposition 0.1 *Every continuous function $f : \mathbb{R} \to \mathbb{R}$ with a superlinear anti-derivative at infinities (i.e., $\lim\limits_{|x|\to\infty} \int_0^x f(t)dt/|x| \;=\; \infty$), is surjective.*

The proof follows immediately from the fact that for each arbitrary $r \in \mathbb{R}$ fixed, the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = \int_0^x f(t)dt - rx$ has a minimum point which is a critical point since this function is lower semi-continuous (in fact continuous), coercive (in the sense that its level sets $\{x \in \mathbb{R} :\; \varphi(x) \le t\}$ are relatively compact) and furthermore differentiable. $\square$

An interesting example which also illustrates well variational arguments (with differential analysis and ordinary differential equation tools) is the following:

Proposition 0.2 *Let X be the Banach space consisting of all continuously differentiable function u on $[0,1]$ satisfying the homogeneous Dirichlet boundary condition $u(0) = u(1) = 0$, that is,* $X = \{u \in \mathcal{C}^1[0,1]\,;\ u(0) = u(1) = 0\}$, *and equipped with the norm defined by*

$$\|u\|_{C^1} \;=\; \max_{x\in[0,1]} |u(x)| \,+\, \max_{x\in[0,1]} |u'(x)|\,.$$

Consider the functionals E and G defined on X respectively by:

$$E(u) \;=\; \int_0^1 |u'(x)|^2 dx \quad \text{and} \quad G(u) \;=\; \int_0^1 |u(x)|^2 dx\,.$$

Then the constrained minimization problem

$$\min\,\big\{E(u)\,;\ \ G(u) = 1\,,\ u \in X\big\}$$

is equivalent to the unconstrained minimization problem

$$\min\left\{\frac{E(u)}{G(u)}\,;\ \ u \in X\setminus\{0\}\right\}$$

and has an optimal solution $\varphi:\ [0,1] \to \mathbb{R}$ defined by $\varphi(x) = \sin(\pi x)$.

Note that in this Proposition 0.2, any critical point v of the functional J defined by $J(u) = E(u)/G(u)$ for $u \not\equiv 0$, is a solution to the eigenvalue problem

$$E'(v) \;=\; \lambda G'(v)$$

where λ is a Lagrange multiplier which in this case is explicitly $\lambda = J(v)$ since

$$J'(v) \;=\; \frac{1}{G(v)}\big(E'(v) \,-\, J(v)G'(v)\big) \;=\; 0\,.$$

Furthermore, a regularity argument from Distribution theory shows that the above eigenvalue problem is equivalent to

$$v'' \;=\; \lambda v\,, \qquad v \in X\setminus\{0\}\,.$$

Furthermore, based on elementary notions from differential analysis, distribution theory, and convex analysis (or simply, calculus of variations), it is not hard to get the following inspiring

Proposition 0.3 *Any critical point of the functional*

$$J: H^1\big((0,\pi),\mathbb{R}\big) \longrightarrow \mathbb{R}$$

defined by

$$J(u) = \int_0^\pi |\dot{u}(s)|^2\, ds$$

subject to the constraints

$$\int_0^\pi |u(s)|^2\, ds = 1 \quad \text{and} \quad \int_0^\pi u(s)\, ds = 0,$$

extends to a nonzero even periodic solution on $[-\pi,\pi]$ *of the ordinary differential equation (ODE)*

$$\ddot{u} + u = 0.$$

Abstractly, many boundary value problems are equivalent to

$$Au = 0 \tag{E}$$

where $A: U \subset X \to Y$ is a mapping from a nonempty open set U of a Banach space X into a Banach space Y. The problem is said to be *variational*, if there exists a differentiable functional $\varphi: U \subset X \to \mathbb{R}$ such that

$$A = \varphi', \qquad \text{(see Definition...).}$$

In this case, the space Y correspond to the dual X' of X and Equation (E) is equivalent to

$$\varphi'(u) = 0, \quad \text{i.e.,}$$
$$\langle h, \varphi'(u)\rangle = 0, \quad \forall\, h \in X \tag{2}$$

where $\langle\,,\,\rangle$ holds for the duality pairing of X and X'. This means that the critical points of φ are the solutions u of (2) and their images $\varphi(u)$ are the critical values of φ.

Besides given a $\mathcal{C}^1$ functional g defined from a Banach space X into a Banach space Y, if a point a is a minimizer of a $\mathcal{C}^1$ functional $f: X \to \mathbb{R}$ constrained to the condition $g(x) = 0_Y$, then a solves the problem

$$f'(a) = \lambda^* \circ g'(a),$$

where $\lambda^* \in Y^*$ is called a Lagrange parameter. (Cf. ...)

The aim of this dissertation is to stress the importance of the direct method of the **Theory of Calculus of Variations** -which deals with the existence and regularity of minimizers of functionals- through the study of the minimization of

$$\int_0^\pi |\dot{x}(s)|^2 ds$$

subject to the constraints :

$$\int_0^\pi U(x(s))ds \;=\; R\,, \quad \text{and} \quad \int_0^\pi \mathrm{grad}U(x(s))\,ds \;=\; 0\,;$$

which yields nonzero even periodic solutions of the dynamical system

$$\ddot{x}(t) \;+\; \mathrm{grad}U(x(t)) \;=\; 0 \tag{0.0.1}$$

where $x = x(t) = (x_1(t), \cdots, x_N(t)) \in \mathbb{R}^N$, $t \in [0, \pi]$, $\ddot{x}(t) = \frac{d^2x}{dt^2}(t)$ and U is a $\mathcal{C}^2(\mathbb{R}^N)$ real-valued function such that
(i) $0 \leq U(x)$ for $x \in \mathbb{R}^N$ *and* $U(0) = 0$,
(ii) U is convex and $U(x) \to \infty$ as $|x| \to \infty$.
(iii) $U \in C^2(\mathbb{R}^N)$ and the quadratic form $\Sigma U_{ij}(x)\zeta_i\zeta_j$ is positive definite for every $x \in \mathbb{R}^N$.

We divide this work into three chapters:

- In the first chapter, we present some preliminaries from Functionnal Analysis and the basic Optimization Theory.
- In the second chapter, we review minimization and variational principles and then consider some auxilliary constrained minimizations in the aim to solve our differential system .
- The third chapter is an application of the abstract.

We conclude this work by some important remarks, showing how the calculus of variations is related to equilibrium configuration of physical systems.

CHAPTER 1

Preliminaries:
Notations, Elementary notions and Important facts.

1.1 Banach Spaces

Definition 1.1.1 *Let X be a real linear space, and $\|.\|_X$ a norm on X and d_X the corresponding metric defined by $d_X(x,y) = \|x-y\|_X \quad \forall x, y \in X$.*
*The normed linear space $(X, \|.\|_X)$ is a **real Banach space** if the metric space (X, d_X) is complete, i.e., if any Cauchy sequence of elements of space $(X, \|.\|_X)$ converges in $(X, \|.\|_X)$. That is, every sequence satisfying the following Cauchy criterion:*

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : p, q \geq n_0 \Rightarrow d_X(x_p, x_q) \leq \varepsilon$$

converges in X.

Definition 1.1.2 *Given any vector space V over a field $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), the **topological dual space** (or simply) **dual space** of V is the linear space of all bounded linear functionals. We shall denote it by V^*.*

$$V^* := \{\varphi : \varphi : V \longrightarrow \mathbb{F}, \quad \varphi \text{ linear and bounded } \}$$

Remark 1.1.1
1)The topological dual space of V is sometimes denoted V'.

2)The dual space V^* has a canonical norm defined by

$$\|f\|_{V^*} = \sup_{x \in V, \|x\| \neq 0} \frac{|f(x)|}{\|x\|}, \quad \forall f \in V^*.$$

3)The dual of every real normed linear space, endowed with its canonical norm is a Banach space.

In order to define other useful topologies on dual spaces, we recall the following

Definition 1.1.3 ***(Initial topology)***
Let X be a nonempty set, $\{Y_i\}_{i \in I}$ a family of topological spaces (where I is an arbitrary index set) and $\phi_i : X \longrightarrow Y; i \in I$, a family of maps.
The ***smallest*** *toplogy on X such that the maps $\phi_i, i \in I$ are continous is called the* ***initial topology.***

Next, we define the ***weak topology*** of a normed vector space X and the ***weak star topology*** of its dual space X^* which are special initial topologies.

Definition 1.1.4 ***(weak topology)***
Let X be a real normed linear space, and let us associate to each $f \in X^*$ the map

$$\phi_f : X \longrightarrow \mathbb{R}$$

given by

$$\phi_f(x) = f(x) \qquad \forall x \in X.$$

The weak topology on X is the smallest topology on X for which all the ϕ_f are continous.
We write $\omega -$ topology for the weak topology.

Definition 1.1.5 ***(weak star topology)***
Let X be a real normed linear space and X^* its dual. Let us associate to each $x \in X$ the map

$$\phi_x : X^* \longrightarrow \mathbb{R}$$

given by

$$\phi_x(f) = f(x) \ \ \forall f \in X^*.$$

The weak star topology on X^* is the smallest topology on X^* for which all the ϕ_x are continous.
We write $\omega^* -$ topology for the weak star topology.

Proposition 1.1.6 *Let X be a real normed linear space and X^* its dual space. Then, there exists on X^* three standard topologies, the strong topology given by the canonical norm $\|.\|_{X^*}$ on X^*, the weak topology ($\omega-$topology) and the weak star topology ω^*-topology such that :*

$$(X^*, \omega^*) \hookrightarrow (X^*, \omega) \hookrightarrow (X^*, \|.\|_{X^*}) \ .$$

The following part of this section is devoted to ***reflexive spaces.***
For any normed real linear space X, the space X^* of all bounded linear functionals on X is a real Banach space and as a linear space, it has its own corresponding dual space which we denote by $(X^*)^*$ or simply by X^{**} and often refer to as the *the second conjugate* of X or *double dual* or *the bidual* of X.
There exists a natural mapping $J : X \longrightarrow X^{**}$ defined , for each $x \in X$ by

$$J(x) = \phi_x$$

where

$$\phi_x : X^* \longrightarrow \mathbb{R}$$

is given by

$$\phi_x(f) = f(x)$$

for each $f \in X^*$.
Thus

$$\langle J(x), f \rangle \equiv f(x) \qquad \text{for each } f \in X^*.$$

J is linear and $\|Jx\| = \|x\|$ for all $x \in X$, (i.e.) J is an ***isometry embedding***. In general, the map J needs not to be onto. Since an isometry is injective,we always identify X to a subspace of X^{**}.
The mapping J is called ***canonical embedding.*** This leads to the following definition.

Definition 1.1.7 *Let X be a real Banach space and let J be the canonical embedding of X into X^{**}. If J is* **onto***, then X is said to be* ***reflexive****. Thus, a reflexive real Banach space is one for which the canonical embedding is onto.*

We now state the following important theorem.

Theorem 1.1.8 *(Eberlein-Smul'yan theorem)*
A real Banach space X is reflexive if and only if every (norm) bounded sequence in X has a subsequence which converges weakly to an element of X.

1.2 Hilbert Spaces

Definition 1.2.1
A map $\phi : E \times E \longrightarrow \mathbb{C}$ is ***sesquilinear*** *if:*

1) $\phi(x+y, z+w) = \phi(x,z) + \phi(x,w) + \phi(y,z) + \phi(y,w)$

2) $\phi(ax, by) = \bar{a}b\phi(x,y)$ *where the "bar" indicates the complex conjugation*

for all $x, y, z, w \in E$ and all $a, b \in \mathbb{C}$.

A ***Hermitian form*** *is a sesquilinear form $\phi : E \times E \longrightarrow \mathbb{C}$ such that*
3) $\phi(x,y) = \overline{\phi(y,x)}$;

A ***positive Hermitian form*** *is a Hermitian form such that*
4) $\phi(x,x) \geq 0$ *for all $x \in E$* ;

A ***definite Hermitian form*** *is a Hermitian form such that*
5) $\phi(x,x) = 0 \Longrightarrow x = 0$.

An ***inner product*** *on E is a* ***positive definite Hermitian form*** *and will be denoted $\langle . , . \rangle := \phi(. , .)$. The pair $(E, \langle . , . \rangle)$ is called an* ***inner product space.***

We shall simply write E for the inner product space $(E, \langle . , . \rangle)$ when the inner product $\langle . , . \rangle$ is known.
In the case where we are using more than one inner product spaces, specification will be made by writting $\langle . , . \rangle_E$ when talking about the inner product space $(E, \langle . , . \rangle)$.

Definition 1.2.2 *Two vectors x and y in an inner product space E are said to be* ***orthogonal*** *and we write $x \perp y$ if $\langle x, y \rangle = 0$. For a subset F of E, then we write $x \perp F$ if $x \perp y$ for every $y \in F$.*

Proposition 1.2.3 *Let E be an inner product space and $x, y \in E$.*
Then

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle . \langle y, y \rangle \quad .$$

For an inner product space $(E, \langle . \ , . \rangle)$, the function $\|.\|_{E} : E \longrightarrow \mathbb{R}$ defined by

$$\|x\|_{E} = \sqrt{\langle x, x \rangle_{E}}$$

is a norm on E.
Thus, $(E, \|.\|_{E})$ is a normed vector space, hence a metric space endowed with the distance $d_{E} : E \times E \longrightarrow \mathbb{R}$ defined by $d_{E}(x, y) = \|x - y\|_{E}$.

Definition 1.2.4 *(Hilbert Space)*
An inner product space $(E, \langle . \ , . \rangle)$ is called a ***Hilbert space*** *if the metric space (E, d_{E}) is complete.*

Remark 1.2.1
1)Hilbert spaces are thus a special class of Banach spaces.
2)Every finite dimension inner product space is complete and simply called ***Euclidian Space****.*

Proposition 1.2.5
Let H be a Hilbert space. Then, for all $u \in H$, $T_u(v) := \langle u, v \rangle$ defines a bounded linear functional, i.e. $T_u \in H^*$. Furthermore $\|u\|_{H} = \|T_u\|_{H^*}$.

Theorem 1.2.6 *(Riesz Representation theorem)*
Let H be a Hilbert space and let f be a bounded linear functional on H. Then,
(i) There exists a unique vector $y_0 \in H$ such that

$$f(x) = \langle x, y_0 \rangle \quad \textit{for each} \quad x \in H,$$

(ii) Moreover, $\|f\| = \|y_0\|$.

Remark 1.2.2 *The map $T : H \longrightarrow H^*$ defined by $T(u) = T_u$ is linear,(anti-linear in the complex case) and isometric. Therefore the canonical embedding is an isometry showing that* ***"any Hilbert space is reflexive"****.*

At the end of this part, we state this important proposition which is just a corollary of Eberlein-Smul'yan theorem.

Proposition 1.2.7 *Let H be a Hilbert space, then any bounded sequence in H has a subsequence which converges weakly to an element of H.*

1.3 Differential Calculus in Banach spaces

In this section, we define the derivative of a map defined between real Banach spaces.

Definition 1.3.1 *(Directional Differentiability)*
Let f be a function defined from a real linear space X into a real normed linear space Y and let $x_0 \in X$ and $v \in X \backslash \{0\}$.
The function f is said to be differentiable at x_0 in the direction v if the function $t \longmapsto f(x_0 + tv)$ is differentiable at $t = 0$. i.e.

$$t \longmapsto \frac{f(x_0 + tv) - f(x)}{t}; \qquad t \neq 0,$$

has a limit in Y when t tends to 0. This limit, when it exists is denoted $f'(x_0, v)$ or $\frac{\partial f}{\partial v}(x_0)$.

Definition 1.3.2 *(Gâteaux Differentiability)*
A function f defined from a real linear space X into a real normed linear space Y is Gâteaux Differentiable at a point $x_0 \in X$ if :
1) f is differentiable at x_0 in every direction $v \in X \backslash \{0\}$ and
2) there exists a bounded linear map $A : X \longrightarrow Y$ such that $f'(x_0, v) = A(v)$; in other words, the map

$$v \longmapsto f'(x_0, v)$$

is a bounded linear map from X into Y.
In this case the map $f'(x_0, .)$ is called the Gâteaux differential of f at x_0 and is denoted by $D_G f(x_0, .)$ or $f'_G(x_0)$.

Definition 1.3.3 *(Fréchet Differentiability)*
A map $f : U \subset X \longrightarrow Y$ whose domain U is an open set of a real Banach space X and whose range is a real Banach space Y is (Fréchet) differentiable at $x \in U$ if there is a bounded linear map $A : X \longrightarrow Y$ such that

$$\lim_{\|u\| \longrightarrow 0} \frac{\|f(x + u) - f(x) - Au\|}{\|u\|} = 0,$$

or equivalently

$$f(x + u) - f(x) - Au = o(\|u\|) .$$

Proposition 1.3.4 *If $f : U \subset X \longrightarrow Y$ is Fréchet Differentiable, then f is Gâteaux Differentiable.*

Proof. Indeed by taking $u = tv$, in the definition of Fréchet Differentiability we have

$$\frac{f(x+tv)-f(x)}{t} = \left(A(v) + \frac{o(\|tv\|)}{\|tv\|}\right)$$

by the Fréchet Differentiability. And since as $t \longrightarrow 0$, $u \longrightarrow 0$, so

$$\lim_{t\longrightarrow 0} \frac{f(x+tv)-f(x)}{t} = A(v)$$

and we are done.

Proposition 1.3.5 *Let X be a real Banach space and Y be a real normed linear space. Then*
1) The set of Gâteaux differentiable mappings from X into Y is a linear subspace of the linear space of all the mappings defined from X into Y space is contained in $\mathcal{B}(X,Y)$,
2) The set of Fréchet Differentiable mappings from X into Y is also a subspace of $\mathcal{B}(X,Y)$.

Theorem 1.3.6 *(Mean Value Theorem in Banach Spaces) Let X and Y be Banach spaces, $U \subset X$ be open and let $f : U \to Y$ be Gâteaux differentiable. Then for all $x_1, x_2 \in X$, we have*

$$\|f(x_1) - f(x_2)\| \le \sup_{t\in[0,1]} \|D_G f(x_1 + t(x_2 - x_1))\| \cdot \|x_1 - x_2\|$$

provided that $\sup_{t\in[0,1]} \|D_G f(x_1 + t(x_2 - x_1))\|$ *is finite.*

Proof. Suppose that the assumptions of Theorem 1.3.6 hold. Let $g^* \in Y^*$ (the dual of Y) such that $||g^*|| \le 1$. Then the real-valued function $\varphi : [0,1] \longrightarrow \mathbb{R}$ defined by

$$\varphi(t) = g^* \circ f(x_1 + th) \quad \text{where } h = x_2 - x_1$$

is differentiable on $[0,1]$ in the usual sense. Moreover we see that

$$\varphi'(t) = g^*\big(D_G f(x_1 + th)(h)\big), \quad \forall t \in (0,1).$$

It follows from the classical mean valued theorem that

$$|\varphi(1) - \varphi(0)| \le \sup_{0<t<1} |\varphi'(t)|,$$

that is

$$\|g^* \circ f(x_1) - g^* \circ f(x_2)\| \leq \sup_{0<t<1} |\varphi'(t)|.$$

Moreover for all $t \in (0,1)$, we have

$$\begin{aligned} |\varphi'(t)| &= \left|g^*\big(D_G f(x_1 + th)(h)\big)\right| \\ &\leq ||g^*|| \, \|D_G f(x_1 + th)\| \, \|h\| \\ &\leq \|D_G f(x_1 + th)\| \, \|h\| \end{aligned}$$

And so

$$\|g^*\big(f(x_1) - f(x_2)\big)\| = \|g^* o f(x_1) - g^* \circ f(x_2)\| \leq \left(\sup_{0<t<1} \|D_G f(x_1 + th)\|\right) \|h\|.$$

But it is well known as a consequence of the Hahn-Banach theorem that

$$\|y\| = \sup\{u^*(y), \ u^* \in Y^*, \ \|u^*\| \leq 1\}.$$

Therefore we finally have

$$\|f(x_1) - f(x_2)\| \leq \sup_{t\in[0,1]} \|D_G f(x_1 + t(x_2 - x_1)\| \cdot \|x_1 - x_2\|. \quad \square$$

Remark 1.3.1 : *The intereted reader is refered to [8] for another approach of the proof.*

Sufficient conditions for the Fréchet Differentiability is given by the following

Theorem 1.3.7 *Suppose that $f : U \subset X \longrightarrow Y$ is a Gâteaux Differentiable function defined from an open subset of a real Banach space X into a real Banach space Y. If the Gâteaux derivative $f'_G : U \subset X \longrightarrow \mathcal{B}(X, Y)$ is continous at $x \in U$, then f is Fréchet Differentiable at x and $f'(x) = f'_G(x)$.*

Proof. Let $x \in U$. Since U is open, there exixts $\delta > 0$ such that $B(x, \delta) \subset U$. Now for $h \in B(x, \delta)$, we define

$$r(h) = f(x + h) - f(x) - f'_G(x)h. \tag{1.3.1}$$

The Gâteaux Differentiability of f at x implies that r is also Gâteaux Differentiable, and

$$r'_G(h) = f'_G(x+h) - f'_G(x).$$

Applying theorem 1.3.6 on the segment line connecting 0 and h, we have that

$$\|r(h)\| \leq M(h)\|h\|,$$

where

$$M(h) = \sup_{0\leq t\leq 1} \|r'_G(th)\|.$$

The continuity of the Gâteaux Differential of f at x implies that $M(h) \to 0$ as $h \to 0$, so $r(h) = o(h)$. Relation 1.3.1 assures that f is Fréchet Differentiable at x, and so $f'(x) = f'_G(x)$.

1.4 Sobolev spaces and Embedding Theorems

We recall the following notations and basic results from Distridutions Theory.
Let $\Omega \subset \mathbb{R}^N$ be an open subset of $\mathbb{R}^N$.
A ***multi-index*** α is a vector $(\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N$. The ***length*** of α is $|\alpha| = \alpha_1 + \cdots + \alpha_N$.
Let $u \in L^1_{loc}(\Omega)$, where $L^1_{loc}(\Omega)$ is the set of functions which are integrable on every compact subset of Ω. If α is a multi-index, we set

$$D^\alpha := \frac{D^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$$

We also recall that we denote by $\mathcal{D}(\Omega)$ the set of $\mathcal{C}^\infty - functions$ defined on Ω with *compact support* in Ω.

Definition 1.4.1
*We say that the function v is the α**-th weak partial derivative of** u if :*

1) $v \in L^1_{loc}(\Omega)$,

2) $v = D^\alpha u$ in the sens of distribution , i.e.

$$\int_\Omega u(x) D^\alpha \psi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x)\psi(x) dx, \quad \forall\ \psi \in \mathcal{D}(\Omega).$$

Definition 1.4.2 *Let $f, g \in L^1_{loc}(\mathbb{R}^N)$. We define the* ***convolution product*** *$f*g$ of f and g by*

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x-y)g(y)dy$$

Theorem 1.4.3 *Let $(\rho_n)_n$ be a sequence of functions such that :*

$$\rho_n \in \mathcal{D}(\mathbb{R}^N), \qquad supp\ \rho_n = \overline{B}(0, \frac{1}{n}), \qquad \int_{\mathbb{R}^N} \rho_n(x)dx = 1, \qquad \rho_n \geq 0 \quad on \quad \mathbb{R}^N.$$

(Such a sequence of smooth functions is called ***Friedrich mollifier*** *).*

If $f \in L^1_{loc}(\mathbb{R}^N)$ then the convolution product

$$f * \rho_n(x) = \int_{\mathbb{R}^N} f(x-y)\rho_n(y)dy$$

exists for each $x \in \mathbb{R}^N$.
Moreover
1. $f * \rho_n \in \mathcal{C}^\infty(\mathbb{R}^N)$,
2. If K is a compact set of points of continuity of f, then $f * \rho_n \longrightarrow f$ uniformly on K as $n \longrightarrow \infty$.

Proof. Since $supp\ \rho_n = \overline{B}(0, \frac{1}{n})$, (which is compact), and using $f \in L^1_{loc}(\mathbb{R}^N)$ we get

$$|f_n(x)| = |(f*\rho_n)(x)| = \left| \int_{\overline{B}(0,\frac{1}{n})} f(x-y)\rho_n(y)dy \right| = \int_{\overline{B}(0,\frac{1}{n})} |f(x-y)|\rho_n(y)dy < \infty.$$

Further, since

$$supp \left(\frac{\partial \rho_n}{\partial x_i} \right) \subset \overline{B}(0, \frac{1}{n}) \quad and \quad \frac{\partial}{\partial x_i}\left[f(y)\rho_n(x-y)\right] = \frac{\partial \rho_n(x-y)}{\partial x_i} f(y),$$

we get

$$\left| \frac{\partial \rho_n(x-y)}{\partial x_i} f(y) \right| \leq M_n |f(y)| \chi_{\overline{B}(0,\frac{1}{n})}$$

and using a corollary of Lebesgue dominated convergence theorem, we have :

$$\frac{\partial}{\partial x_i} \int_{\mathbb{R}^N} f(y)\rho_n(x-y)dy = \int_{\mathbb{R}^N} \frac{\partial \rho_n(x-y)}{\partial x_i} f(y)dy = \int_{\mathbb{R}^N} \frac{\partial \rho_n(y)}{\partial x_i} f(x-y)dy = f * \frac{\partial \rho_n}{\partial x_i}.$$

Let us prove now that $f_n \to f$ as $n \to \infty$, uniformly on compact subsets of $\mathbb{R}^N$.
Let K be a compact set of points of continuity of f_n. So, for any $\eta > 0$, there exists $\delta > 0$, such that for $x, z \in K$

$$\|x - z\| < \delta \Longrightarrow \|f(x) - f(z)\| < \eta.$$

Now,

$$f_n(x) - f(x) = \int_{\overline{B}(0,\frac{1}{n})} (f(x-y) - f(x))\rho_n(y)dy,$$

because

$$f(x) = f(x).1 = f(x)\int_{\mathbb{R}^N} \rho_n(y)dy = \int_{\mathbb{R}^N} f(x)\rho_n(y)dy \qquad and \qquad \int_{\mathbb{R}^N} \rho_n(y)dy = 1,$$

Hence, for $n \geq n_0$ with $n_0 = \left[\frac{1}{\delta}\right] + 1$,

$$\begin{aligned} |f_n(x) - f(x)| &\leq \int_{\overline{B}(0,\frac{1}{n})} |f(x-y) - f(x)|\rho_n(y)dy \\ &\leq \eta \int_{\mathbb{R}^N} \rho_n(y)dy = \eta \quad \text{for each } x \in K. \end{aligned}$$

Indeed

$$n \geq n_0 \Longrightarrow n \geq \frac{1}{\delta} \Longrightarrow \frac{1}{n} \leq \delta$$

so that

$$\|(x-y) - x\| = \|y\| \leq \frac{1}{n} \leq \delta$$

and the result follows from the uniform continuity of f_n.
We then conclude that $f * \rho_n \longrightarrow f$ uniformly on each compact.

Definition 1.4.4 *Let $1 \leq q \leq +\infty$, $m \in \mathbb{N}$. The **Sobolev space** $W^{m,p}(\Omega)$ is defined by*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega), |D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq m\}.$$

Clearly, $W^{m,p}(\Omega)$ is a real vector space .
The case $p = 2$ will play a special role. The Sobolev spaces $W^{m,2}(\Omega)$ are denoted by $H^m(\Omega)$, i.e.

$$H^m(\Omega) := W^{m,2}(\Omega).$$

The spaces $H^m(\Omega)$ have a natural inner-product defined by

$$\langle u, v\rangle_{H^m} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx, \qquad \forall u, v \in H^m(\Omega)$$

and are Hilbert spaces with the inner-product defined above. We will be more interested in our work by $H^1(\Omega)$.

Concerning Sobolev spaces, we will give here two important results, ***Rellich-Kondrachov compact embedding theorem*** (which is crucial in regularity analysis) and the ***Poincaré Inequality***.

Theorem 1.4.5 *(Rellich-Kondrachov)*
Let Ω be a $\mathcal{C}^1-$bounded open subset of $\mathbb{R}^N$, $1 \leq p < \infty$ and $p^* := \frac{Np}{N-p}$.
The followings embeddings are compact:

a. If $1 \leq p < N$ then $W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [1, p^*[$,

b. If $p = N$ then $W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [1, \infty[$,

c. If $p > N$ then $W^{1,p}(\Omega) \subset C(\bar{\Omega})$.

We have $\mathcal{D}(\Omega) \subset W^{m,p}(\Omega) \quad \forall m \in \mathbb{N}, \forall p \geq 1$, and we define $W_0^{m,p}(\Omega) := \overline{\mathcal{D}(\Omega)}$.

Proposition 1.4.6 *(Poincaré Inequality)*
Let $1 \leq p < \infty$ and Ω a bounded open subset of $\mathbb{R}^N$. Then there exists a constant $C = C(p, \Omega)$ such that

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega)$$

If Ω is connected and satisfies a $\mathcal{C}^1$ boundary condition, then there exists a constant $C = C(p, \Omega)$ such that

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega)$$

where

$$\bar{u} = \frac{1}{|\Omega|} \int_\Omega u(x) dx.$$

1.5 Basic notions of Convex analysis

Definition 1.5.1 *Let X be a real normed vector space, $x_0 \in X$ and $f : X \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ an extended real-valued function. One says that f is* ***lower semicontinuous (lsc)*** *at x_0 when for any real number r such that $r < f(x_0)$, there exists some neighborhood V of x_0 such that for all $x \in V$, $r < f(x)$.*

We next connect the lower semicontinuity to some geometric concept. For an extended real-valued function $f : X \longrightarrow \overline{\mathbb{R}}$, we define its ***epigraph epi f*** by

$$epi\ f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}.$$

We also introduce the concept of **lower level set** for $r \in \mathbb{R}$ by $\{f(.) \leq r\}$ where for $r \in \mathbb{R}$,

$$\{f(.) \leq r\} := \{x \in X : f(x) \leq r\}.$$

We therefore give the following characterisation ;

Theorem 1.5.2 *Let X be a real normed vector space and $f : X \longrightarrow \overline{\mathbb{R}}$ an extended real-valued function. The following assertions are equivalent*
a) f is lower semicontinous (lsc) ;
b) The epigraph epi f of f is closed in $X \times \mathbb{R}$;
c)For any $r \in \mathbb{R}$, the lower level set $\{f(.) \leq r\}$ is closed in X.

Definition 1.5.3 *Let C be a nonempty subset of a real normed vector space X. One says that the set C is* ***convex*** *provided that for $x, y \in C$, and $\lambda \in [0, 1]$, one has $\lambda x + (1 - \lambda)y \in C$.*

Through the epigraph of an extended real-valued function over a real vector space, one can define the concept of ***convex function*** as follow:

Definition 1.5.4 *Let $f : X \longrightarrow \overline{\mathbb{R}}$ an extended real valued function. Ones says that the function f is* ***convex*** *provided that its epigraph is a* ***convex*** *set in $X \times \mathbb{R}$.*

We also give the following important results.

Proposition 1.5.5 *Let X be a real normed vector space. If $f : X \longrightarrow \overline{\mathbb{R}}$ is lsc at $\bar{x} \in X$ and $\{x_n\}$ is a sequence in X which converges (strongly) to $\bar{x}$ then ,*

$$\liminf_{n\to\infty} f(x_n) \geq f(\bar{x}).$$

Proposition 1.5.6
Let $f : X \longrightarrow \overline{\mathbb{R}}$ be any map.
Then , f is convex and lsc $\Longleftrightarrow$ f is convex and weakly lsc.

And we obtain the following corollary

Corollary 1.5.7 *Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a convex and weakly lsc mapping. Suppose $\{x_n\}$ is a sequence in X which converges weakly to $\bar{x}$. Then,*

$$\liminf_{n\to\infty} f(x_n) \geq f(\bar{x}).$$

Definition 1.5.8 *Let X be a real normed vector space and C a nonempty convex subset of X. A function $f : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be convex relative to C, provided for all $\lambda \in]0, 1[$, $x, y \in C$*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

and f is said to be strictly convex relative to C if for $x, y \in C$ with $x \neq y$ and $f(x), f(y)$ finite, we have

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y).$$

Lemma 1.5.9 *(Slope inequality for convex functions)*
Let I be an unterval of $\mathbb{R}$ and $h : I \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Let $r_1, r_2, r_3 \in I$ such that $r_1 < r_2 < r_3$ and $h(r_1)$ and $h(r_2)$ are finite. Then

$$\frac{h(r_2) - h(r_1)}{r_2 - r_1} \leq \frac{h(r_3) - h(r_1)}{r_3 - r_1} \leq \frac{h(r_3) - h(r_2)}{r_3 - r_2}.$$

Furthermore, these inequalities for all such $r_1, r_2, r_3 \in I$ characterizes the convexity of f relative to I.
If we have

$$\frac{h(r_2) - h(r_1)}{r_2 - r_1} < \frac{h(r_3) - h(r_1)}{r_3 - r_1} < \frac{h(r_3) - h(r_2)}{r_3 - r_2}$$

for all $r_1, r_2, r_3 \in I$ such that $r_1 < r_2 < r_3$ and $h(r_1), h(r_2)$ and $h(r_3)$ are finite, we obtain a characterisation of the strict convexity of f relative to C.

Through the above lemma, we can characterize the convexity of differentiable functions of one real variable as follows.

Proposition 1.5.10 *Let I be an open interval of $\mathbb{R}$ and $h : I \longrightarrow \mathbb{R}$ be a real-valued differentiable function on I. The following assertions are equivalent :*
(a) h is convex on I;
(b) the derivative function h' is nondecreasing on I;
(c) $h'(r)(s-r) \leq h(s) - h(r)$ for all $r, s \in I$.
Similarly, the following are equivalent
(a') his strictly convex on I;
(b') the derivative function h' is increasing on I;
(c') $h'(r)(s-r) < h(s) - h(r)$ for all $r, s \in I$ with $r \neq s$.

Proof. $\underline{(a) \Rightarrow (b)}$ Let $r < t$ in I. According to the above lemma, we have

$$h'(r) = \lim_{s \downarrow r} \frac{h(s)-h(r)}{s-r} \leq \frac{h(t)-h(r)}{t-r} \leq \lim_{s \uparrow t} \frac{h(t)-h(s)}{t-s} = \lim_{s \uparrow t} \frac{h(s)-h(t)}{s-t} = h'(t),$$

which ensures the nondecreasing property of the derivative h' on I.

$\underline{(b) \Rightarrow (c)}$ Fix $r \in I$ and set $\varphi(s) := h(s) - h(r) - h'(r)(s-r)$ for all $s \in I$. The function φ is differentiable on I and $\varphi'(s) = h'(s) - h'(r)$. By the assumption (b), taking $s \in I$, we have that $\varphi'(s) \geq 0$ if $s \geq r$ and $\varphi'(s) \leq 0$ if $s \leq r$. We then deduce that $\varphi(s) \geq \varphi(r) = 0$ for all $s \in I$ and we are done.

$\underline{(c) \Rightarrow (a)}$ For s fixed in (c), we

$$h(s) \leq \sup_{r \in I} \left[h'(r)(s-r) + h(r)\right] \leq h(s)$$

that is

$$h(s) = \sup_{r \in I} \left[h'(r)(s-r) + h(r)\right]$$

Further, setting $H(s) = \left[h'(r)(s-r) + h(r)\right]$, for $s_1, s_2 \in I$ and $\alpha \in [0,1]$, we have that

$$\begin{aligned}
H(\alpha s_1 + (1-\alpha)s_2) &= h'(r)(\alpha s_1 + (1-\alpha)s_2 - r) + h(r) \\
&= h'(r)(\alpha s_1 + (1-\alpha)s_2 - \alpha r + (1-\alpha)r) + \alpha h(r) + (1-\alpha)h(r) \\
&= \alpha\left[h'(r)(s_1 - r) + h(r)\right] + (1-\alpha)\left[h'(r)(s_2 - r) + h(r)\right] \\
&= \alpha H(s_1) + (1-\alpha)H(s_2)
\end{aligned}$$

that is, H is convex and hence, h is convex on I as the pointwise supremum of a family of convex functions on I.
The case of the strict convexity of h follows the same arguments.

Proposition 1.5.11 *Let I be an open interval of $\mathbb{R}$ and $h : I \longrightarrow \mathbb{R}$ be a real-valued differentiable function on I.*
If the function h is twice differentiable on I, then h is convex on I if and only if $h''(r) \geq 0$ for all $r \in I$.
Similarly if h is twice differentiable on I and $h''(r) > 0$ for all $r \in I$, then h is strictly convex on I. The converse does not hold, that is, the strict convexity of a twice differentiable function h on I does not entail the positivity of h'' on I.

Proof. Since h is twice derivable, we have

$$h''(r) \geq 0 \qquad \forall r \in I \qquad \Longleftrightarrow \qquad h' \text{ is nondecreasing} \qquad \Longleftrightarrow \qquad h \text{ is convex}$$

and we are done.
The case of the strict convexity of h follows the same arguments.

We will consider now the more genaral case of differentiable functions on an open convex set of a normed vector space .

Theorem 1.5.12 *Let U be an open set of a real normed space $(X, \|.\|)$ and $f : U \longrightarrow \mathbb{R}$ be a function which is (Fréchet) differentiable on U. Then the following assertions are equivalent:*
(a) f is convex torelative U;
(b) $\langle f'(y) - f'(x), y - x\rangle \geq 0$ for all $x, y \in U$;
(c) $\langle f'(x), y - x\rangle \leq f(y) - f(x)$ for all $x, y \in U$.
Similarly, the following are equivalent :
(a') f is strictly convex relative to U;
(b') $\langle f'(y) - f'(x), y - x\rangle > 0$ for all $x, y \in U$ with $x \neq y$;
(c') $\langle f'(x), y - x\rangle < f(y) - f(x)$ for all $x, y \in U$ with $x \neq y$.

Proof. For fixed $x, y \in U$ with $x \neq y$, consider the open interval

$$I := \{s \in \mathbb{R} : x + s(y - x) \in U\}$$

and set $h(s) := f(x + s(y - x)$ for all $s \in I$. Observing that $0 \in I$ and $1 \in I$ with $h(0) = f(x)$ and $h(1) = f(y)$. we have
f is convex relative to U if and only if the function h is convex relative to I.

Indeed, since $0 \in I$ and $1 \in I$ and I is an interval, then $[0,1] \subset U$, so for all $\alpha \in [0,1] \subset U$

$$\begin{aligned} f(\alpha y + (1-\alpha)x) &= f(x + \alpha(y-x)) \\ &= h(\alpha) \\ &= h(\alpha.1 + (1-\alpha).0) \\ &\leq \alpha h(1) + (1-\alpha)h(0) \\ &= \alpha f(y) + (1-\alpha)f(x) \end{aligned}$$

We then apply proposition 1.5.10.

Theorem 1.5.13 *Let U be an open set of a real normed space $(X, \|.\|)$ and $f : U \longrightarrow \mathbb{R}$ be a function which is (Fréchet) differentiable on U.*
If f is twice differentiable on U, f is convex relative to U if and only if for each $x \in U$ the bilinear form associated with $f''(x)$ is positive semidefinite, i.e., $\langle f''(x).v, v\rangle \geq 0$ for all $v \in X$.
Similarly assuming the twice differentiabilty of f on U, a sufficient (but not necessary) condition for the strict convexity of f on U is for each $x \in U$ the positive definiteness of $f''(x)$, i.e., $\langle f''(x).v, v\rangle > 0$ for all $v \in X$ with $v \neq 0_X$

Proof. It follows the same arguments as in the proof of the above theorem, but in this case, we apply proposition 1.5.11

CHAPTER 2

Minimization and Variational methods

Here, we give some minimization and variational principles.
In fact, we prove the existence of minimum points (hence critical points) for a functional subject to some contraints and to this end, we will use the direct method of the calculus of variations.

Before we move further, let us recall the following:
The calculus of variations deals with functionals $\mathcal{J} : V \longrightarrow \mathbb{R}$, where V is some function space. The main interest of the subject is to find minimizers for such functionals, that is, functions $v \in V$ such that

$$\mathcal{J}(v) \leq \mathcal{J}(u) \text{ for every } u \in V.$$

But seeking a minimizer among the functions satisfying these may lead to false conclusions if the existence of a minimizer is not established beforehand.
The functional $\mathcal{J}$ must be bounded from below to have a minimizer. This means

$$\inf\{\mathcal{J}(u) : u \in V\} > -\infty.$$

It is not enough to know that a minimizer exists, but it shows the existence of a minimizing sequence, that is, a sequence (u_n) in V such that

$$\mathcal{J}(u_n) \longrightarrow \inf\{\mathcal{J}(u) : u \in V\}.$$

The direct method may be broken into the following steps :
1. Take a minimizing sequence $(u_n)_n$ for $\mathcal{J}$,

2. Show that $(u_n)_n$ admits some subsequence $(u_{n_k})_k$ that converges to an element $u^* \in V$ with respect to a topology τ on ,

3. Show that $\mathcal{J}$ is sequentially lower semi-continuous with respect to the topology τ.

To see that this shows the existence of a minimizer, consider the following definition of sequentially lower-semicontinuous functions.
The function J is sequentially lower-semicontinuous if

$$\liminf_{n\to\infty} \mathcal{J}(u_n) \geq \mathcal{J}(u^*) \text{ for any convergent sequence } u_n \xrightarrow{\tau} u^* \quad in \quad V.$$

The conclusion follows from ,

$$\inf\{\mathcal{J}(u) : u \in V\} = \lim_{n\to\infty} \mathcal{J}(u_n) = \lim_{k\to\infty} \mathcal{J}(u_{n_k}) \geq \mathcal{J}(u^*) \geq \inf\{\mathcal{J}(u) : u \in V\}$$

in other words

$$\mathcal{J}(u^*) = \inf\{\mathcal{J}(u) : u \in V\}.$$

We first give the following very important existence theorem.

Theorem 2.0.14 *Let K be a nonempty closed and convex subset of a reflexive real Banach space X. Let F be a convex and lower semicontinuous function on K.*
If K is bounded or F is coercive (i.e. $\lim_{\|x\|\to+\infty} F(x) = +\infty$),
then there exists a minimum of F over K.
Moreover, if F is strictly convex on K, then the minimum is unique.

Corollary 2.0.15 *(The projection Theorem)*
Let H be a Hilbert space and M a closed subspace of H. For arbitrary vector $x \in H$, there exists a vector $m^* \in M$ such that $\|x - m^*\| \leq \|x - m\|$ for all $m \in M$. Furthermore, $m^* \in M$ is unique if and only if $(x - m^*) \perp M$.

We also recall the following concepts.

Definition 2.0.16 *Let Ω be an open set of a real Banach space and F be a real-valued function defined on Ω and differentiable on Ω.*
$\bar{x} \in \Omega$ is a critical point of the function F without constraints if

$$F'(\bar{x}) = 0.$$

Definition 2.0.17 *Let X, Y be two real Banach spaces and $f : X \longrightarrow Y$ a differentiable map. Let $M \subset X$ be a manifold. One says that $\bar{x} \in M$ is a* ***critical point*** *of the function f* ***constrained to*** *M if*

$$T_{\bar{x}}M \subset ker \ [Df(\bar{x})]$$

where $T_{\bar{x}}M$ is the tangent plane of M at $\bar{x}$.

Remark 2.0.1 *We know that when the manifold M is described by $g(z) = 0$, i.e. $M = \{z \in X : g(z) = 0_Y\}$ where $g : X \longrightarrow Y$ is a* ***submersion****, then*

$$T_zM = ker \ [Dg(z)].$$

So that $\bar{x}$ is a critical point of the function f constrained to M if

$$ker \ [Dg(\bar{x})] \subset ker \ [Df(\bar{x})].$$

This inclusion between " kernels" implies (from Algebra) that there exists a linear form $\lambda : Y \longrightarrow \mathbb{R}$ (called ***Lagrange multiplier****) such that*

$$Df(\bar{x}) = \lambda \circ Dg(\bar{x}).$$

In the case now where $Y = \mathbb{R}^m$, the manifold (or constraints set) M takes the form $M = \{g_i(z) = 0, 1 \leq i \leq m\}$ and the linear form λ can be represented as a row-vector $\lambda := (\lambda_1, \cdots, \lambda_m)$ so that

$$Df(\bar{x}) = (\lambda_1, \cdots, \lambda_m).(Dg_1(\bar{x}), \cdots, Dg_m(\bar{x}))^t = \sum_{i=1}^{m} \lambda_i Dg_i(\bar{x}).$$

The reals $\lambda_i, 1 \leq i \leq m$ are by extension called the "Lagrange multipliers ".

This yields then to the following proposition

Proposition 2.0.18 *Let Ω be an open set of a real Banach space and F be a real-valued function defined on Ω and differentiable on Ω.*
Let $g_i : \Omega \longrightarrow \mathbb{R}^{m_i}$, $\quad i = 1, \cdots, n$ be differentiable functions on Ω.
The point $\bar{x} \in \Omega$ is a critical point of the function F ***subject to the constraints*** *($g_{m_i}(x) = 0,\ i = 1, \cdots, n$) if there exists a nonzero vector $(\lambda_0, \lambda_{m_1}, \cdots, \lambda_{m_n}) \in \mathbb{R} \times \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$, such that*

$$\lambda_0 \nabla F(\bar{x}) + \sum_{i=1}^{n} \lambda_{m_i} . \nabla g_{m_i}(\bar{x}) = 0$$

and

$$g_{m_i}(\bar{x}) = 0, \quad for \quad i = 1, \cdots, n \ .$$

Since we will be interested in minimization problem subject to constraints, we will add in the above proposition that $\lambda_0 \geq 0$.

It will be convenient for the rest of our work to prove the following useful results.

Consider now the space $H^1(]0, \pi[, \mathbb{R}^N)$ that we shall write simply H^1. We know that H^1 is also the set of absolutely continuous N-vector-valued functions x defined on $[0, \pi]$ such that $\dot{x} \in L_2 := L_2(]0, \pi[, \mathbb{R}^N)$ and is a Hilbert space with the norm induced by the inner product.

$$\langle x, y \rangle_{H^1} = \int_0^\pi (\dot{x}(s).\dot{y}(s) + x(s).y(s))ds \ .$$

Lemma 2.0.19 *Let $x \in H^1$ such that $\int_0^\pi x(s)\ ds = 0$, then there is a constant K independent of x such that*

$$max \left\{ \ \|x\|_{L_2}, \ \sup_{s \in [0,\pi]} |x(s)| \ \right\} \leq K \|\dot{x}\|_{L_2} \tag{2.0.1}$$

Furthermore

$$\text{if } x_n \longrightarrow x \text{ weakly in } H^1, \text{ then } x_n \longrightarrow x \quad \text{uniformly on } [0, \pi]. \tag{2.0.2}$$

Proof. We recall that for an absolutely continuous function, the fundamental theorem of analysis is true.
Therefore, for $x \in H^1$, and for any l fixed in $[0,\pi]$ we can write

$$x(s) - x(l) = \int_l^s \dot{x}(v)dv \quad \forall s \in [0,\pi]$$

and this implies that

$$\begin{aligned} |x(s) - x(l)| &= |\int_l^s \dot{x}(v)dv| \\ &\leq \int_0^\pi |\dot{x}(v)|dv, \qquad i.e. \\ &\leq \pi^{1/2}\left\{\int_0^\pi |\dot{x}(v)|^2 dv\right\}^{1/2} \qquad \text{for all } s \text{ in } [0,\pi] \quad . \end{aligned}$$

So

$$|x(s) - x(l)| \leq \pi^{1/2}\left\{\int_0^\pi |\dot{x}(v)|^2 dv\right\}^{1/2} \qquad \text{for all s in } [0,\pi]. \tag{2.0.3}$$

But we know (by the theorem of the mean) that there exists $s_0 \in [0,\pi]$ such that

$$x(s_0) = \frac{1}{\pi}\int_0^\pi x(v)dv = 0 \ ,$$

so for $l = s_0$ the inequality 2.0.3 becomes

$$|x(s)| \leq \pi^{1/2}\left\{\int_0^\pi |\dot{x}(v)|^2 \ dv\right\}^{1/2} \quad \forall s \in [0,\pi] \tag{2.0.4}$$

Now, we square the members of the inequality 2.0.4 and integrate them over $[0,\pi]$ to obtain

$$\left\{\int_0^\pi |x(s)|^2 ds\right\}^{1/2} \leq K_1 \left\{\int_0^\pi |\dot{x}(s)|^2 ds\right\}^{1/2} . \tag{2.0.5}$$

Using 2.0.4 again, by taking the supremum over $[0,\pi]$ of both sides ,we obtain

$$\sup_{[0,\pi]}|x(s)| \leq K_2 \left\{\int_0^\pi |\dot{x}(s)|^2 ds\right\}^{1/2} . \tag{2.0.6}$$

Combining 2.0.5 and 2.0.6 and taking $K = \max\{K_1, K_2\}$ we get the desired inequality 3.0.1

To prove the point 2.0.2, we will proceed by the way of contracdiction.
So, assume that we don't have the uniform convergence of $(x_n)_n$ to x, that is, there exists $\varepsilon_0 > 0$ and a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that

$$\|x_{n_k} - x\|_\infty > \varepsilon_0 \tag{2.0.7}$$

Since $(x_n)_n \subset W^{1,2} = H^1$, using the Rellich-Kondrachov compact embedding, there exists a subsequence $(x_{n_{k_j}})_j$ of $(x_{n_k})_k$ that converges uniformly (and therefore converges weakly) to some $x^* \in H^1$. The uniqueness of the weak limit assures us that $x = x^*$. The inclusion $(x_{n_{k_j}})_j \subset (x_{n_k})_k$ implies that $(x_{n_{k_j}})_j$ satisfies 2.0.7
So we have $\|x_{n_{k_j}} - x\|_\infty > \varepsilon_0$ and $x_{n_{k_j}} \longrightarrow x$ uniformly . Contradiction.

Lemma 2.0.20 *Let x_0 be a given element of H^1. Let $U \in \mathcal{C}^1(\mathbb{R}^N)$ a real-valued and strictly convex function.*
Then there exists a unique constant vector $a_0 \in \mathbb{R}^N$ sucht that

$$\int_0^\pi U(x_0(s) + a_0)ds = \min_{a \in \mathbb{R}^N} \int_0^\pi U(x_0(s) + a)ds.$$

Furthermore a_0 is characterised by

$$\int_0^\pi grad\ U(x_0(s) + a_0)ds = 0.$$

Proof. We want to find a minimum for the function $Q : \mathbb{R}^N \longrightarrow \mathbb{R}$, defined by

$$Q(a) = \int_0^\pi U(x_0(s) + a)ds.$$

Since U is $\mathcal{C}^1$ and strictly convex on $\mathbb{R}^N$, then Q is $\mathcal{C}^1$ and strictly convex on $\mathbb{R}^N$, hence strictly convex and lower semicontinuous on $\mathbb{R}^N$.
Moreover, the coercivity of U implies the coercivity of Q. So by theorem 2.0.14 the function Q has a unique minimum a_0.
The Euler condition gives us that $\nabla Q(a_0) = 0$, i.e.

$$\int_0^\pi grad\ U(x_0(s) + a_0)ds = 0.$$

Lemma 2.0.21 *Let U be a $\mathcal{C}^1(\mathbb{R}^N)$ real-valued and convex function such that :*
1) $0 = U(0) \leq U(x) \quad \forall x \in \mathbb{R}^N$,
2) $U(x) \longrightarrow \infty$ *as* $|x| \longrightarrow \infty$.
The set

$$\mathcal{S}_R = \left\{ x : x \in H^1, \int_0^\pi U(x(s))ds = R, \int_0^\pi grad\ U(x(s))ds = 0 \right\}$$

is nonempty for any $R > 0$, and is weakly closed in H^1.

Proof. Let $\{x_n\}$ be a sequence of elements of $\mathcal{S}_R$ such that $x_n \rightharpoonup x$ in H^1. We want to show that $x \in \mathcal{S}_R$.
Since $x_n \rightharpoonup x$ in H^1, then $x_n \longrightarrow x$ uniformly in H^1 by 2.0.2 . The sequence $(x_n)_n$ is a sequence of bounded functions that converges uniformly, so $(x_n)_n$ is uniformly bounded , i.e. there exits a constant M independent of n such that $|x_n(s)| \leq M$ for any $n \in \mathbb{N}$ and $s \in [0, \pi]$.
Let K be the closed ball $B'(0, M)$, K is compact and since U and ∇U are continous, there exists M_1 and M_2 two constants independent of n such that

$$|U(x_n(s))| \leq M_1 \quad and \quad |\nabla U(x_n(s))| \leq M_2 \quad \text{for all } n \in \mathbb{N} \quad and \quad s \in [0, \pi]$$

Hence, by Lebesgue's dominated convergence theorem

$$\int_0^\pi U(x_n(s))ds \underset{n \to \infty}{\longrightarrow} \int_0^\pi U(x(s))ds,$$

and

$$\int_0^\pi \nabla U(x_n(s))ds \underset{n \to \infty}{\longrightarrow} \int_0^\pi \nabla U(x(s))ds.$$

Thus, $x \in \mathcal{S}_R$ and $\mathcal{S}_R$ is weakly closed.

To show that $\mathcal{S}_R$ is nonempty for any $R > 0$, we consider the function $x(s) = (\sin 2s, 0, \cdots, 0)$ which obviously belongs to H^1.
For all $t > 0$, $tx(s) \in H^1$ and by the lemma 2.0.20, for each t, there exists a constant $N-$vector c_t such that $x_t(s) = tx(s) + c_t$ and $\int_\pi^0 grad\, U(x_t(s))ds = 0$.
Further, the fucntion

$$A : [0, +\infty) \longrightarrow \mathbb{R}^N \qquad \text{defined by} \qquad A(t) = tx(s) + c_t$$

is continous on $[0, +\infty)$ so that the function

$$B : [0, +\infty) \longrightarrow \mathbb{R}, \qquad \text{defined by} \qquad B(t) = \int_0^\pi U(x_t(s))ds = \int_0^\pi U(A(t))ds$$

is continuous and satisfies

$$B(0) = 0 \ , \quad \lim_{t \longrightarrow +\infty} \int_0^\pi U(x_t(s))ds = +\infty$$

So by the intermadiate value theorem, for any $R > 0$, there exists $t_0 \geq 0$ such that

$$\int_0^\pi U(x_{t_0}(s))ds = R \ .$$

This completes the proof $\blacksquare$.

We now state the following theorem that gives us the exitence of a minimum point for a constrained functional.

Theorem 2.0.22 *Let U be a $\mathcal{C}^1(\mathbb{R}^N)$ real-valued function such that*
(i) $\quad 0 = U(0) \leq U(x)$ *for* $x \in \mathbb{R}^N$,
(ii) $\quad U$ *is convex, and* $U(x) \to \infty$ *as* $|x| \to \infty$.
(*) $U \in \mathcal{C}^2(\mathbb{R}^N)$ *and the quadratic form* $\Sigma U_{ij}(x)\zeta_i\zeta_j$ *is positive definite, where* $U_{ij}(x) = \frac{\partial^2 U}{\partial x_i \partial x_j}(x)$.
Then for every number $R > 0$,

$$\beta = \inf_{x \in \mathcal{S}_R} \int_0^\pi |\dot{x}(s)|^2 ds > 0$$

and is attained on $\mathcal{S}_R$.

Proof. We begin by showing that

$$\beta = \inf_{x \in \mathcal{S}_R} \int_0^\pi |\dot{x}(s)|^2 ds \qquad \text{is attained .}$$

Set
$$T(x) = \int_0^{\pi} |\dot{x}(s)|^2 ds.$$
Since $\mathcal{S}_R \neq \varnothing$ then there exists a sequence $\{x_n\}$ of elements of $\mathcal{S}_R$ with
$$T(x_n) = \int_0^{\pi} |\dot{x_n}(s)|^2 ds \longrightarrow \beta \qquad \text{and} \qquad T(x_n) \leq \beta + 1.$$
Note that x_n can be written $x_n(s) = x_{0,n}(s) + x_{m,n}$ where $x_{0,n}$ has mean value zero on $[0,\pi]$ and $x_{m,n}$ is the mean value of x_n over $[0,\pi]$.
We want to show that
$$\|x_n\|^2_{H^1} \qquad \text{is uniformly bounded.}$$

Observe first that
$$x_{0,n}(s) = (x^1_{0,n}(s), \cdots, x^N_{0,n}(s)) \in \mathbb{R}^N, \qquad \text{and } x_{m,n}(s) = (x^1_{m,n}, \cdots, x^N_{m,n}) \in \mathbb{R}^N$$

so that
$$\begin{aligned} \|x_n\|^2_{H^1} &= \|x_{0,n}\|^2_{H^1} + \pi|x_{m,n}|^2 + 2\left[\int_0^{\pi} (\dot{x}_{0,n}(s).\dot{x}_{m,n} + x_{0,n}(s)x_{m,n})ds\right] \\ &= \|x_{0,n}\|^2_{H^1} + \pi|x_{m,n}|^2 , \end{aligned}$$

because $x_{0,n}$ has mean value 0 and $x_{m,n}$ is a constant vector.

By the relation 2.0.1 of lemma 2.0.19, we have that
$$\|x_{0,n}\|^2_{H^1} = \int_0^{\pi} (|\dot{x}_{0,n}|^2 + |x_{0,n}|^2)ds \leq (1+K)\int_0^{\pi} (|\dot{x}_{0,n}|^2 ds \leq (1+K)(\beta+1),$$
that is $\{\|x_{0,n}\|^2_{H^1}\}$ is uniformly bounded.

We next show that $|x_{m,n}|$ is uniformly bounded. Suppose by the way of contradiction that $x_{m,n} \longrightarrow \infty$.
Since $\left\{\|x_{0,n}\|^2_{H^1}\right\}$ is uniformly bounded; the relation 2.0.1 of lemma 2.0.19 implies that $\left\{\underset{s\in[0,\pi]}{sup} |x_{0,n}(s)|\right\}$ is uniformly bounded; so that if $|x_{m,n}| \longrightarrow \infty$, then

$\|x_n\| \longrightarrow \infty$ and

$$\int_0^\pi U(x_{0,n} + x_{m,n})ds \longrightarrow \infty,$$

which is a contradiction because

$$\int_0^\pi U(x_{0,n} + x_{m,n})ds = R \qquad (\text{ since } x_n \in \mathcal{S}_R)\ .$$

Hence $\|x_{m,n}\|_{H^1}$ is uniformly bounded so, by *Eberlein-Smul'yan Theorem* , has a weakly convergent subsequence $\{x_{n_k}\}$ with weak limit $\bar{x} \in H^1$. As the set $\mathcal{S}_\mathcal{R}$ is weakly closed by lemma 2.0.21,we conclude that $\bar{x} \in \mathcal{S}_R$.

We then show that $T(\bar{x}) = \beta$.
First observe that since T is convex and $\mathcal{C}^1$, T is convex and lsc, and hence convex and weakly lower semi-continuous in H^1.
Therefore we have

$$\liminf_{k\to\infty} T(x_{n_k}) \geq T(\bar{x}).$$

On the other hand,

$$\beta = \lim_{n\to\infty} T(x_n) = \lim_{k\to\infty} T(x_{n_k}) \geq \liminf_{k\to\infty} T(x_{n_k}) \geq T(\bar{x}) \geq \inf_{x\in\mathcal{S}_R} T(x) = \beta,$$

i.e.

$$T(\bar{x}) = \beta.$$

Hence, $\bar{x}$ is the desired minimal point .

Finally, we show that $\beta > 0$.
Since for all $x \in H^1 \quad T(x) \geq 0$, $\beta = \inf_{x\in\mathcal{S}_R} T(x) \geq 0$.
We write $\bar{x} = \bar{x}_0 + \bar{x}_m$ where $\bar{x}_0$ has mean value zero over $[0,\pi]$ and $\bar{x}_m$ is the mean value of $\bar{x}$ over $[0,\pi]$.
If

$$\beta = \int_0^\pi |\dot{\bar{x}}(s)|^2 ds = 0$$

then, $\int_0^\pi |\dot{\bar{x}}_0(s)|^2 ds = 0$, so $\bar{x}_0 = 0$ and by lemma 2.0.20, $\bar{x}_m = 0$ implying $\bar{x} = 0$, which is a contradiction to the fact that

$$\int_0^\pi U(\bar{x}(s))ds = R > 0$$

This completes the proof ■

CHAPTER 3

Existence Results of Periodic Solutions of some Dynamical Systems.

We now state the problem we want to solve, give its variational formulation, treat the obtained variational problem and show that its critical points are the desired solutions of the Dynamical System .

The Problem : to prove the following theorem,

Theorem 3.0.23 *Let U be a $\mathcal{C}^1(\mathbb{R}^N)$, real-valued function such that:*
(i) $0 \leq U(x)$ *for* $x \in \mathbb{R}^N$ *and* $U(0) = 0$,
(ii) U *is convex, and* $U(x) \to \infty$ *as* $|x| \to \infty$.
Then there exists a family of distinct periodic solutions for the system.

$$\ddot{x}(t) + grad\ U(x(t)) = 0 \tag{3.0.1}$$

where $x(t) = (x_1(t), \cdots, x_N(t)) \in \mathbb{R}^N, \quad t \in [0, \pi], \quad \ddot{x}(t) = \frac{d^2x(t)}{dt^2}$.

The Method : We minimize a measure of the kinetic energy of the N-vector-valued function x subject to appropriate constraints.

We begin by demonstrating that periodic solutions of 3.0.1 can be found as critical points of a certain isoperimetric problem.
To this end, we introduce the period λ of periodic solutions of 3.0.1 by making the change of variables $t = \lambda s$ in 3.0.1 and considering $2\pi-$period solutions of the resulting system

$$\ddot{y} + \lambda^2 grad\ U(y) = 0. \tag{3.0.2}$$

Because, if the function x is a solution of 3.0.1, then the function y defined by $y(s) = x(\lambda s)$ is a solution of 3.0.2 and inversely, if y is a solution of 3.0.2, then x defined by $x(t) = y(\frac{t}{\lambda})$ is a solution of 3.0.1.
Furthermore, for y given , $2\pi - periodic$ solution of 3.0.2, $x(t) = y(\frac{t}{\lambda})$ is a $2\pi\lambda - periodic$ solution of 3.0.1 and for x given, $2\pi\lambda - periodic$ solution of 3.0.1, $y(s) = x(\lambda s)$ is a $2\pi - periodic$ solution of 3.0.2.
Indeed

$$x(t + 2\pi\lambda) = x\left(\lambda\left(\frac{t}{\lambda}\right) + 2\pi\right) = y\left(\frac{t}{\lambda} + 2\pi\right) = y\left(\frac{t}{\lambda}\right) = x(t)$$

and

$$y(s + 2\pi) = y\left(\frac{1}{\lambda}(\lambda s + 2\pi\lambda)\right) = x(\lambda s + 2\pi\lambda) = x(\lambda t) = y(s)\ .$$

Lemma 3.0.24 *Even $2\pi-$periodic solutions of 3.0.2 may be obtained from the solutions of 3.0.2 together with the boundary conditions*

$$(2i) \qquad\qquad \dot{y}(0) = \dot{y}(\pi) = 0\ .$$

Proof. Given a solution y of 3.0.2 with the condition $(2i)$, we define an even extension of y to $[-\pi, \pi]$ by

$$Y(s) = \begin{cases} y(s) & \text{if } s \in [0, \pi]\ , \\ y(-s) & \text{if } s \in [-\pi, 0]. \end{cases}$$

The extension Y satisfies 3.0.2 on $[-\pi, 0]$. Indeed, for $s \in [-\pi, 0]$

$$\frac{d^2Y}{ds^2}(s) = \frac{d^2}{ds^2}Y(s) = \frac{d^2}{ds^2}y(-s) = \frac{d}{ds}\left((-1).\frac{dy}{ds}(-s)\right) = \frac{d^2y}{ds^2}(-s).$$

And since $\dot{y}(0) = 0$ the solution y is smooth across $s = 0$. Similarly, since $\dot{y}(\pi) = 0$, we may extend Y to a $2\pi-$periodic solution on $(-\infty, \infty)$ by setting $Y(s + 2k\pi) = Y(s), k \in \mathbb{Z}$.

We now state the variational problem for periodic solutions of 3.0.1.

Theorem 3.0.25 *Let U be a $\mathcal{C}^2(\mathbb{R}^N)$, real-valued function such that:*
(i) $0 \leq U(x)$ *for* $x \in \mathbb{R}^N$ *and* $U(0) = 0$,
(ii) U *is convex, and* $U(x) \to \infty$ *as* $|x| \to \infty$, *and*
(*) the quadratic form $\Sigma U_{ij}(x)\zeta_i\zeta_j$ *is positive definite, where*
$U_{ij}(x) = \frac{\partial^2 U}{\partial x_i \partial x_j}(x)$.
Then, any critical point of the functionnal

$$T : H^1 \longrightarrow \mathbb{R}, \quad x \longmapsto T(x) = \int_0^\pi |\dot{x}(s)|^2 ds$$

subject to the constraints :

$$F(x) = \int_0^\pi U(x(s))ds = R \text{ (a positive constant), and } \int_0^\pi grad\, U(x(s))ds = 0$$

is a nonzero even periodic solution of 3.0.1.

Proof. Consider the following optimization problem

$$(\mathcal{P}) : \begin{cases} \text{Minimize } T(x) = \int_0^\pi |\dot{x}(s)|^2 ds \ , \\ \text{Subject to the constraints : } F(x) = \int_0^\pi U(x(s))ds = R, \quad G(x) = \int_0^\pi \nabla U(x(s))ds = 0 \ . \end{cases}$$

The functions T, F and G are $\mathcal{C}^1$ in H^1.
So we can write for a critical point x, that there exists a nonzero vector $(\lambda_0, \lambda_1, \beta) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N$ such that

$$\lambda_0 T'(x) + \lambda_1 f'(x) + \beta.G'(x) = 0 \ .$$

The computation of the Gateaûx derivatives of T, F, *and* G gives us :

$$\begin{aligned}
T'(x)(h) &= \nabla T(x).h = 2\int_0^\pi \dot{x}(s).\dot{h}(s)\ ds \qquad \text{for all } h \in H^1 \\
F'(x)(h) &= \nabla F(x).h = \int_0^\pi \nabla U(x(s)).h(s)\ ds \qquad \text{for all } h \in H^1 \\
\beta G'(x)(h) &= \beta \nabla G(x).h = \int_0^\pi \nabla(\beta.\nabla U(x(s))).h(s)\ ds \qquad \text{for all } h \in H^1.
\end{aligned}$$

Indeed,
1) For the function T, we have :

$$\begin{aligned}
\lim_{t\to 0}\frac{T(x+th)-T(x)}{t} &= \lim_{t\to 0}\int_0^{\pi}\frac{|\dot{x}(s)+t\dot{h}(s)|^2-|\dot{x}(s)|^2}{t}ds \\
&= \lim_{t\to 0}\left[\int_0^{\pi} 2\dot{x}(s)\dot{h}(s)\ ds + t\int_0^{\pi}|\dot{h}(s)|^2 ds\right] \\
&= 2\int_0^{\pi}\dot{x}(s).\dot{h}(s)ds
\end{aligned}$$

because $\displaystyle\int_0^{\pi}|\dot{h}(s)|^2 ds$ is finite (since $h\in H^1$),

$T'(x)$ is clearly linear and continuous. (since $|T'(x)(h)|\leq 2\|x\|_{H^1}.\|h\|_{H^1}$ for all $h\in H^1$).
T is therefore Gateaûx differentiable.
Further, the map

$$D_G : H^1 \longrightarrow (H^1)^* \text{ defined by } x \longmapsto T'(x).$$

is continous.
Indeed, for $x,y\in H^1, h\in H^1$, we have

$$\begin{aligned}
|(T'(x)-T'(y))(h)| &\leq \left|2\int_0^{\pi}(\dot{x}-\dot{y})(s).\dot{h}(s)ds\right| \\
&\leq 2\|\dot{x}-\dot{y}\|_{L_2}\ .\ \|h\|_{L_2} \\
&\leq 2\|\dot{x}-\dot{y}\|_{H^1}\ .\ \|h\|_{H^1} \qquad \text{for any } h\in H^1
\end{aligned}$$

i.e.

$$\|(T'(x)-T'(y))\|_{(H^1)^*}\leq \|\dot{x}-\dot{y}\|_{H^1}.$$

2) For the function F, we have :

$$\lim_{t\to 0}\frac{F(x+th)-F(x)}{t}=\lim_{t\to 0}\int_0^{\pi}\frac{U(x+th)-U(x)}{t}ds\ .$$

Firstly, we observe that, since U is differentiable

$$\lim_{t\to 0}\frac{U(x+th)-U(x)}{t}=U'(x)(h)\ .$$

Secondly, by the Mean value theorem, we can write

$$U(x+th)-U(x)=U'(c)(th)=tU'(c)(h)$$

for a vector $c=\alpha x+(1-\alpha)(x+th), \alpha\in(0,1)$ so that

$$\frac{U(x+th)-U(x)}{t}=U'(x)(h)$$

which is bounded on $[0,\pi]$ because h is continous and U is $\mathcal{C}^1$.
The Dominated convergence theorem gives us that

$$\lim_{t\to 0}\frac{F(x+th)-F(x)}{t}=\int_0^{\pi}\lim_{t\to 0}\frac{U(x+th)-U(x)}{t}ds$$

and like for the case of T, we have

$$\|(F'(x)-F'(y))\|_{(H^1)^*}\leq\|\nabla U(x)-\nabla U(y)\|_{H^1}.$$

The continuity of ∇U gives us the desired result .

3) For the function G, ,since it is a vectorial function we define first :

$$G_i(x)=\int_0^{\pi}\frac{\partial U}{\partial x_i}(x(s))\ ds$$

and for each G_i corresponds a $\beta_i\in\mathbb{R}$.
So using the same reasonning as in the case of F and since U is $\mathcal{C}^2$, we get that

$$G_i'(x)(h)=\int_0^{\pi}grad\left(\frac{\partial U}{\partial x_i}(x(s))\right).h(s)\ ds\ .$$

Therefore

$$\begin{aligned}
\beta.G'(x)h &= \sum_{i=1}^{N} \beta_i G'_i(x).h \\
&= \sum_{i=1}^{N} \beta_i \int_0^{\pi} grad\ \left(\frac{\partial U}{\partial x_i}(x)\right).h\ ds \\
&= \int_0^{\pi} \sum_{i=1}^{N} \beta_i\ grad\ \left(\frac{\partial U}{\partial x_i}(x)\right).h\ ds \\
&= \int_0^{\pi} grad\ \sum_{i=1}^{N} \left(\beta_i\ \frac{\partial U}{\partial x_i}(x)\right).h\ ds \\
&= \int_0^{\pi} grad\,(\beta.grad\ U(x))\,.h\ ds\ .
\end{aligned}$$

We apply at each G_i the same reasonning as in the case of F to get that G_i is $\mathcal{C}^1$ for each i so that G is $\mathcal{C}^1$.
So, for a critical point x, we have

$$2\lambda_0 \int_0^{\pi} \dot{x}(s)\dot{h}(s)\ ds + \lambda_1 \int_0^{\pi} \nabla U(x)h\ ds + \int_0^{\pi} grad\,(\beta.grad\ U(x))\,h\ ds = 0,\ \text{ for all } h \in H^1. \tag{3.0.3}$$

In particular, 3.0.3 is true for all $h \in \mathcal{D}(\Omega) \subset H^1$, i.e.

$$2\lambda_0 \int_0^{\pi} \dot{x}(s)\dot{h}(s)\ ds + \lambda_1 \int_0^{\pi} \nabla U(x)h\ ds + \int_0^{\pi} grad\,(\beta.grad\ U(x))\,h\ ds = 0,\ \text{ for all } h \in \mathcal{D}(\Omega). \tag{3.0.4}$$

But for $h \in \mathcal{D}(\Omega)$,

$$\int_0^{\pi} \dot{x}.\dot{h}\ ds = -\int_0^{\pi} \ddot{x}.h\ ds = -\langle \ddot{x}, h\rangle,$$

in the sense of distribution.

So that, 3.0.4 can be written as :

$$\langle -2\lambda_0 \ddot{x} + \lambda_1 \nabla U(x) + grad\,(\beta.grad\; U(x))\,, h\rangle = 0, \text{ for all } h \in \mathcal{D}(\Omega), \quad (3.0.5)$$

and this implies that

$$-2\lambda_0 \ddot{x} + \lambda_1 \nabla U(x) + grad\,(\beta.grad\; U(x)) = 0. \quad (3.0.6)$$

in the sense of distribution.

At this step, we will prove that $\lambda_0 \neq 0$.
If we assume by the way of contadiction that $\lambda_0 = 0$, the relation 3.0.6 becomes,

$$\lambda_1 \nabla U(x) + grad\,(\beta.grad\; U(x)) = 0. \qquad (\circledast)$$

Integrating $(\circledast)$ over $[0, \pi]$ and using $\int_0^\pi \nabla U(x(s))ds = 0$, we obtain

$$\int_0^\pi grad\,(\beta.grad\; U(x(s)))\,ds = 0,$$

i.e.

$$\sum_{i=1}^{N} \beta_i \int_0^\pi U_{ij}(x(s))ds = 0 \qquad (j = 1, \cdots, N).$$

since

$$\begin{aligned}
\int_0^\pi grad\,(\beta.grad\; U(x(s)))\,ds &= \int_0^\pi grad(\sum_{i=1}^{N} \beta_i \frac{\partial U}{\partial x_i}(x))\; ds \\
&= \int_0^\pi \sum_{i=1}^{N} \beta_i \left(grad(\frac{\partial U}{\partial x_i}(x)) \right) ds \\
&= \sum_{i=1}^{N} \beta_i \int_0^\pi \frac{\partial U}{\partial x_j} \left(\frac{\partial U}{\partial x_i}(x) \right) ds.
\end{aligned}$$

Since U satisfies (*), the quadratic form

$$\Sigma \left\{ \int_0^\pi U_{ij}(x(s))ds \right\} \zeta_i \zeta_j \qquad \text{is positive definite}$$

thus, $det| \int_0^\pi U_{ij}(x(s))ds| \neq 0$, implying that $\beta = 0$.
Hence $(\circledast)$ is reduced to $\lambda_1 \nabla U(x) = 0$.
We observe that $\lambda_1 \neq 0$, for otherwise $(\lambda_0, \lambda_1, \beta) = 0$ which is a contradiction.
Now, since $\lambda_1 \neq 0$, we have that $\nabla U(x) = 0$, i.e. U is constant, which is again a contradiction because U is coercive.

Conclusion: $\lambda_0 \neq 0$.

The fact that $\lambda_0 \neq 0$ and the relation 3.0.6 imply that

$$\ddot{x} = \frac{\lambda_1 \nabla U(x) + grad\,(\beta.grad\;U(x))}{2\lambda_0},$$

and the critical point x is a $\mathcal{C}^2-$function because U is $\mathcal{C}^2$.

Therefore the differential of the function T takes the following form

$$T'(x)(h) = 2\int_0^\pi \dot{x}(s).\dot{h}(s)\;ds = -2\int_0^\pi \ddot{x}.h\;ds + 2\left[\dot{x}(\pi)h(\pi) - \dot{x}(0)h(0)\right]$$

(using an integration by parts) and the relation 3.0.3 becomes

$$\begin{aligned}\lambda_0\left[-2\int_0^\pi \ddot{x}h\;ds + 2\left[\dot{x}(\pi)h(\pi) - \dot{x}(0)h(0)\right]\right] &+ \int_0^\pi \lambda_1 \nabla U(x) h ds \\ &+ \int_0^\pi grad\,(\beta.grad\;U(x))\,hds = 0,\;\text{ for all } h \in H^1,\end{aligned}$$

i.e.

$$\int_0^\pi \left[-2\lambda_0\ddot{x} + \lambda_1\nabla U(x) + grad\,(\beta.grad\;U(x))\right].hds + 2\lambda_0\left[\dot{x}(\pi)h(\pi) - \dot{x}(0)h(0)\right] = 0 \;\text{ for all } h \in H^1.$$

And by 3.0.6, the above relation is reduced to

$$\lambda_0\left[\dot{x}(\pi)h(\pi) - \dot{x}(0)h(0)\right] = 0,\;\text{ for all } h \in H^1.$$

By choosen h in H^1 to be respectively $h(x) = x$ and $h(x) = x - \pi$ we obtain ,

$$\lambda_0\dot{x}(\pi) = \lambda_0\dot{x}(0) = 0.$$

Hence, we find for a critical point x,

$$-2\lambda_0\ddot{x} + \lambda_1\nabla U(x) + grad\,(\beta.grad\;U(x)) = 0, \tag{3.0.7}$$

with the boundary conditions

$$\lambda_0\dot{x}(\pi) = \lambda_0\dot{x}(0) = 0\;. \tag{3.0.8}$$

We now show that the constants $\beta_i \quad (i = 1, \cdots, N)$ are in fact zero.
To this end, we integrate 3.0.7 over $[0, \pi]$. Using the boundary conditions and the fact that $\int_0^\pi \nabla U(x(s))ds = 0$, we find

$$\sum_{i=1}^{N} \beta_i \int_0^\pi U_{ij}(x(s))ds = 0 \qquad (j = 1, \cdots, N) \ ,$$

since

$$\begin{aligned}
\int_0^\pi grad\,(\beta.grad\; U(x)) &= \int_0^\pi grad(\sum_{i=1}^{N} \beta_i \frac{\partial U}{\partial x_i}(x))\; ds \\
&= \int_0^\pi \sum_{i=1}^{N} \beta_i \left(grad(\frac{\partial U}{\partial x_i}(x)) \right) ds \\
&= \sum_{i=1}^{N} \beta_i \int_0^\pi \frac{\partial U}{\partial x_j} \left(\frac{\partial U}{\partial x_i}(x) \right) ds.
\end{aligned}$$

Hence $\beta = 0$ provided that we guarantee

$$det|\int_0^\pi U_{ij}(x(s))ds| \neq 0.$$

Since U satisfies (*), the quadratic form

$$\Sigma \left\{ \int_0^\pi U_{ij}(x(s))ds \right\} \zeta_i \zeta_j \qquad \text{is positive definite .}$$

Thus, $det| \int_0^\pi U_{ij}(x(s))ds| \neq 0$ as required and $\beta = 0$.

Next we demonstrate that the constants $-\lambda_0$ and λ_1 are both nonzero and of the same sign. Indeed, since $\beta = 0$, the relation 3.0.7 can be written

$$-2\lambda_0 \ddot{x} + \lambda_1 \nabla U(x) = 0. \tag{3.0.9}$$

Multiplying 3.0.9 by x, integrating over $[0, \pi]$ and using the conditions 3.0.8, we find

$$-\lambda_0 \int_0^\pi |\dot{x}(s)|^2 ds = \lambda_1 \int_0^\pi \nabla U(x(s)).x(s)ds. \tag{3.0.10}$$

The strict convexity of U allows us to write that
$U(0) > U(x) + \nabla U(x(s))(0 - x)$ for all $x(s) \neq 0$ and integrating this inequality over $[0, \pi]$ we obtain

$$0 < R < \int_0^\pi \nabla U(x).x(s)ds.$$

Also,

$$\int_0^\pi |\dot{x}(s)|^2 ds \neq 0,$$

for otherwise $x(s) = c$ is a constant,
but

$$\begin{aligned} x(s) = c \quad &\Longrightarrow \quad \int_0^\pi grad\ U(c) = 0 \text{ by } \int_0^\pi grad\ U(x(s))ds = 0 \\ &\Longrightarrow \quad \pi.grad\ U(c) = 0, \end{aligned}$$

i.e. c is a minimum for U. Since U is strictly convex and has a minimum at 0,we conclude that $c = 0$, and this contradicts

$$\int_0^\pi U(x(s))ds = R > 0.$$

Thus 3.0.10 implies that $-\lambda_0$ and λ_1 are both of the same sign and both nonzero. Hence,without lost of generality, we may write 3.0.7 and 3.0.8

$$\ddot{x} + \lambda^2 \nabla U(x) = 0 \qquad (\lambda^2 \neq 0) \tag{3.0.11}$$

$$\dot{x}(\pi) = \dot{x}(0) = 0\ . \tag{3.0.12}$$

Now the lemma 3.0.24 implies that the critical points $x(s)$ may be extended to an even $2\pi-$periodic solution of 3.0.2, and thus correspond to an even $2\pi\lambda-$periodic solution of 3.0.1 after reparametrisation .
This completes the proof.

Corollary 3.0.26 *Let U be a $\mathcal{C}^2(\mathbb{R}^N)$, real-valued function such that:*
(i) $\alpha = U(a) \leq U(x)$ *for* $x \in \mathbb{R}^N$,
(ii) U *is convex, and* $U(x) \to \infty$ *as* $|x| \to \infty$, *and*
(*) the quadratic form $\Sigma U_{ij}(x)\zeta_i\zeta_j$ *is positive definite, where*
$U_{ij} = \frac{\partial^2 U}{\partial x_i \partial x_j}$.
Then, any critical point of the functionnal

$$T : H^1 \longrightarrow \mathbb{R}, \quad z \longmapsto T(z) = \int_0^\pi |\dot{z}(s)|^2 ds$$

subject to the constraints :

$$F(z) = \int_0^\pi U(z(s))ds = r = R + \alpha\pi \ (where\ r > \alpha\pi), \quad \int_0^\pi grad\ U(z(s))ds = 0,$$

is a nonzero even periodic solution of 3.0.1.

Proof. Since the function U is convex, $\mathcal{C}^1$ and coercive,it has a minimizer, i.e. there exists $a \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$ such that

$$\alpha = U(a) \leq U(x) \quad for \quad x \in \mathbb{R}^N.$$

We define now the function V by $V(x) = U(x+a) - \alpha$. Then, the function V satisfies :
1) V is a $\mathcal{C}^2(\mathbb{R}^N)$ real-valued function ,
2) $0 \leq V(x)$ for $x \in \mathbb{R}^N$ *and* $V(0) = 0$,
3) V is convex, and $V(x) \to \infty$ as $|x| \to \infty$, and
(*) the quadratic form $\Sigma \frac{\partial^2 V}{\partial x_i \partial x_j}(x)\zeta_i\zeta_j$ is positive definite.
We then apply theorem 3.0.25 with the function V.
So, if x_V is a critical point obtained, we know that x_V is a $2\pi\lambda-$periodic solution of 3.0.1, i.e. $\ddot{x}_V + grad\ V(x_V) = 0$.
Now, setting $z_V = x_V + a$, it is obvious that z is a $2\pi\lambda-$periodic and satisfies

$$\ddot{z}_V + grad\ U(z_V) = 0,$$

i.e. z_V is a solution of 3.0.1.

In this part, we give the last step in the resolution or our problem.
We start by the following proposition.

Proposition 3.0.27 *Let V be a $\mathcal{C}^1(\mathbb{R}^N)$ real-valued and convex function such that :*

⋇ $\lim\limits_{|x|\to+\infty} V(x) = +\infty$,
Then there exists a sequence $\{V_k\}$ of functions such that :
1) *V_k is a $\mathcal{C}^2(\mathbb{R}^N)$ real-valued functions and the Hessian of V_k is positive definite for all $x \in \mathbb{R}^N$,*
2) *$V_k \to V$ and grad $V_k \to$ grad V as $k \to \infty$, uniformly on any compact subset of $\mathbb{R}^N$,*
3) $\lim\limits_{|x|\to+\infty} V_k(x) = +\infty$ *uniformly in k.*

Proof.

Let

$$V_k(x) = (V * \rho_k)(x) + \frac{|x|^2}{k}.$$

By theorem 1.4.3 the function V_k is well defined, $\mathcal{C}^\infty$, and $V * \rho_k \longrightarrow V$ uniformly on each compact subset of $\mathbb{R}^N$. Moreover the Hessian of V_k is positive definite.

Furthermore, for any compact K, there exists $M > 0$ such that
$|x|^2 \leq M \quad \forall x \in K$,
so that

$$\sup_K \frac{|x|^2}{k} \leq \frac{M}{k}, \quad i.e. \quad \frac{|x|^2}{k} \longrightarrow 0 \quad \text{uniformly on} \quad K.$$

So, given $\varepsilon > 0$, there exists :

$$N_1 : \forall k \geq N_1 \quad \forall x \in K \qquad |V * \rho_k(x) - V(x)| \leq \frac{\varepsilon}{2}$$

$$N_2 : \forall k \geq N_2 \quad \forall x \in K \qquad \frac{|x|^2}{k} \leq \frac{\varepsilon}{2}$$

Taking $N = \max\{N_1, N_2\}$, we have that for $k \geq N$

$$\left| V * \rho_k(x) - V(x) + \frac{|x|^2}{k} \right| \leq |V * \rho_k(x) - V(x)| + \frac{|x|^2}{k} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

$$i.e. \qquad V_k \longrightarrow V \text{ uniformly on } K.$$

The same reasonning applies for $grad\ V_k$.

We finally show that $\lim_{|x|\to+\infty} V_k(x) = +\infty$ uniformly in k.
V is coercive, so

$$\forall B > 0, \quad \exists C > 0 : \quad \|z\| > C \Longrightarrow |U(z)| > B.$$

But

$$(V * \rho_k)(x) = \int_{\mathbb{R}^N} V(x-y)\rho_k(y)dy = \int_{B'(0,1)} V(x-y)\rho_k(y)dy$$

since $supp\ \rho_k = B'(0, \frac{1}{k}) \subset B'(0,1)$.
For $\|x\| \geq C+1$, we have $\|x-y\| \geq \|x\| - \|y\| \geq C+1-1 = C$ (because $\|y\| \leq 1$), and we obtain

$$(V*\rho_k)(x) = \int_{\mathbb{R}^N} V(x-y)\rho_k(y)dy = \int_{B'(0,1)} V(x-y)\rho_k(y)dy \geq B \int_{B'(0,1)} \rho_k(y) = B.$$

The proof is then complete.

Theorem 3.0.28 *Let U be a $\mathcal{C}^1(\mathbb{R}^N)$ real-valued function such that*
(i) $0 = U(0) \leq U(x)$ *for* $x \in \mathbb{R}^N$,
(ii) U *is convex, and* $U(x) \to \infty$ *as* $|x| \to \infty$.
For every number $R > 0$, there exists some

$$\bar{x} \in \mathcal{S}_R = \left\{ x \in H^1, \int_0^\pi U(x(s)) = R, \int_0^\pi \nabla U(x(s)) = 0 \right\}$$

which gives a solution to the differential system 3.0.1.

Proof. For the function U, by using the above proposition, we get a sequence $\{U_k\}$ of functions such that
$\circledast_1)$ $U_k \in \mathcal{C}^2(\mathbb{R}^N)$ and the Hessian of U_k is positive definite for all $x \in \mathbb{R}^N$,
$\circledast_2)$ $U_k \to U$ and $grad\ U_k \to grad\ U$ as $k \to \infty$, uniformly on compact subsets of $\mathbb{R}^N$,
$\circledast_3)$ $\lim_{|x|\to+\infty} U_k(x) = +\infty$ uniformly in k.
Applying theorem 2.0.22 and theorem 3.0.25, we find a sequence $\{x_k\}$ of even $2\pi - periodic$ elements of H^1 and a sequence $\{\lambda_k\}$ of real numbers with

$$\int_0^\pi \dot{x}_k(s).\dot{h}(s)\ ds = \lambda_k^2 \int_0^\pi grad\ U_k(x_k(s) + a_k).h(s)\ ds \text{ for all } h \in H^1 \quad (3.0.13)$$

and such that

$$T(x_k) = \int_0^\pi |\dot{x}_k(s)|^2 ds = \inf \int_0^\pi |\dot{x}(s)|^2 ds$$

over the set

$$\mathcal{S}_k = \left\{ x : x \in H^1, \int_0^\pi U_k(x(s))ds = R + \alpha_k \pi, \int_0^\pi grad\ U_k(x(s))ds = 0 \right\}.$$

where

$$\alpha_k = U_k(a_k) = \min_{x\in\mathbb{R}^N} U_k(x)$$

We know that by setting $V_k(x) = U_k(x + a_k) - \alpha_k$, we obtain a sequence $\{z_k\}$ of solutions of 3.0.1 satisfying

$$\int_0^\pi \dot{z}_k(s).\dot{h}(s)\ ds = \lambda_k^2 \int_0^\pi grad\ U_k(z_k).h\ ds \text{ for all } h \in H^1$$

and such that

$$T(z_k) = \int_0^\pi |\dot{z}_k(s)|^2 ds = \inf \int_0^\pi |\dot{z}_k(s)|^2 ds$$

over the set

$$\mathcal{S}_{k,R} = \left\{ x : x \in H^1, \int_0^\pi V_k(x(s))ds = R, \quad \int_0^\pi grad\ V_k(x(s))ds = 0 \right\}.$$

Claim:

Lemma 3.0.29 *The sequence*

$$\left\{ \int_0^\pi |\dot{z}_k(s)|^2 ds \right\}$$

is uniformly bounded .

Proof. Let $x(s) = (\sin 2s, 0, \cdots, 0)$.Then, using the result on the nonvacuity of $\mathcal{S}_{k,R}$ for each V_k, (cf the proof of Lemma 2.0.21)there exists a number $t_k > 0$ and a vector $c_k \in \mathbb{R}^N$ such that

$$y_k(s) = t_k x(s) + c_k \in \mathcal{S}_{k,R},$$

that is

$$\int_0^\pi grad\ V_k(y_k(s))ds = 0, \qquad \int_0^\pi V_k(y_k(s))ds = R;$$

The condition

$$\lim_{|x|\to+\infty} U_k(x) = +\infty \quad \text{uniformly in } k$$

implies that

$$\lim_{|x|\to+\infty} V_k(x) = +\infty \quad \text{uniformly in } k,$$

so that there exists $C > 0$ such that

$$|x| > C \Longrightarrow V_k(x) \geq \frac{2R}{\pi} \qquad for \qquad k = 1, 2, \cdots .$$

By setting

$$E_k := \{s : 0 \leq s \leq \pi, |y_k(s)| > C\} \ ,$$

we have that $E_k \subset [0, \pi]$ and so

$$\int_{E_k} \frac{2R}{\pi}\, ds \leq \int_{E_k} V_k(y_k(s))ds \leq \int_0^\pi V_k(y_k(s))ds$$

i.e.

$$\frac{2R}{\pi}\, mes(E_k) \leq R$$

so that

$$mes(E_k^c) = mes\,\{s : 0 \leq s \leq \pi, |y_k(s)| \leq C\} \geq \frac{\pi}{2},$$

where $E_k^c = [0, \pi] \setminus E_k$.
Thus, there is an interval $[a, b] \subset [0, \pi]$ on which $|y_k(s)| \leq C$, $b - a \geq \frac{\pi}{4}$ and $\theta = |\sin 2b - \sin 2a| > 0$.
And since,

$$t_k\theta = t_k|\sin 2b - \sin 2a| = |y_k(b) - y_k(a)| \leq 2C,$$

it follows that

$$t_k \leq \frac{2C}{\theta}.$$

But

$$\dot{y}_k(s) = t_k\dot{x}(s) = t_k(2\cos 2s, 0, \cdots, 0) \Longrightarrow \int_0^\pi |\dot{y}_k(s)|^2 ds = t_k^2 \int_0^\pi |\dot{x}(s)|^2 ds,$$

and since

$$\int_0^\pi |\dot{x}(s)|^2 ds = 4\int_0^\pi (\cos 2s)^2 ds \leq 4\pi,$$

it follows that

$$\int_0^\pi |\dot{y}(s)|^2 ds \leq \left(\frac{2C}{\theta}\right)^2 4\pi.$$

Consequently, the sequence

$$\left\{\int_0^\pi |\dot{z}_k(s)|^2 ds\right\} \quad \text{is also bounded by} \quad \left(\frac{2C}{\theta}\right)^2 4\pi.$$

Consider now the sequence $\{z_k\}$ of even $2\pi - periodic$ elements generated by the sequence $\{V_k\}$ given above.
As in the proof of the theorem 2.0.22, we can write z_k as the sum of its mean value and a function having mean value 0, i.e

$$z_k = z_{0,k} + z_{m,k}.$$

Using the fact that

$$\int_0^\pi V_k(z_k(s))ds = R$$

and the above lemma, the property $\circledast_3$) of $\{U_k\}$ implies that the sequences $\left\{\sup_{[0,\pi]}|z_{0,k}(s)|\right\}$ and $\{\|z_k\|\}$ are uniformly bounded.
In fact $z_k = x_k + a_k$ where $U_k(a_k) = \alpha_k$.
We show that $\{a_k\}$ is bounded.
Assume by the way of contradiction that there exists a subsequence $a_{k_j} \longrightarrow \infty$.
Let $C > \min\limits_{\mathbb{R}^N} U$ and $M = \left\{x \in \mathbb{R}^N : U(x) \leq C\right\}$.
M is compact, so $U_k \longrightarrow U$ uniformly and $\min\limits_M U_k \longrightarrow \min\limits_M U$.
The uniform coercivity of the sequence $\{U_k\}$ implies that

$$\forall B, \exists k_0 \in \mathbb{N},\ \exists A : |x| > A \Longrightarrow U_k(x) > B \quad \forall k \geq k_0.$$

So that $a_{k_j} \longrightarrow \infty \Longrightarrow \exists j_0 :\ |a_{k_j}| > A \quad \forall j \geq j_0$.
Hence, for j large enough, $U_{k_j}(a_{k_j}) > B$ so that $\limsup U_k \geq B$.

One the other hand $\limsup \min\limits_{\mathbb{R}^N} U_k \leq \limsup \min\limits_M U_k \leq C$.
So for $B = C + 1$, we have $C + 1 \leq \limsup U_k \leq C$, contradiction.

Therefore, the sequence $\{a_k\}$ is bounded.
Hence $\{x_k\}$ has a weakly convergent subsequence $\{x_{k_j}\}$ such that $x_{k_j} \rightharpoonup \bar{x}$.

Furthermore, since $x_{k_j} \longrightarrow \bar{x}$ uniformly and each x_k is $\mathcal{C}^2 - function$ (by construction), then $\bar{x}$ is continuous .
We now show that

$$\int_0^\pi U(\bar{x}(s))ds = R > 0, \quad and \quad \int_0^\pi \nabla U(\bar{x}(s))ds = 0.$$

Observe first that the implication :
there exists $q > 0$ and $k_0 \in \mathbb{N}$ such that

$$|x| > q \quad \Longrightarrow \quad U_k(x) > C \quad \forall k \geq k_0$$

gives us that $\alpha_k \in B'(0, C) \quad \forall k \geq k_0$ and since $B'(0, C)$ is compact, the uniform convergence of U_k to U on $B'(0, C)$ implies that $\alpha_k \longrightarrow 0$.
(Because the uniform convergence on a compact K implies that $\min\limits_K U_k \longrightarrow \min\limits_K U$ on the compact K).

Let $\quad K_1 = \{\bar{x}(s) : s \in [0, \pi]\}, \qquad K_2 = \bigcup\limits_{j\in\mathbb{N}} \left\{x_{k_j}(s) : s \in [0, \pi]\right\} \qquad and \quad K = K_1 \cup K_2.$

Since $\bar{x}$ is continous, then K_1 is a compact of $\mathbb{R}^N$, and due to the uniform convergence of $\{x_{k_j}\}$, K_2 is also a compact of R^N, so that K is compact. We then have

$$\begin{aligned}
\left|\int_0^\pi [U_{k_j}(x_{k_j}(s)) - U(\bar{x}(s))]ds\right| &\leq \int_0^\pi \left|U_{k_j}(x_{k_j}(s)) - U(x_{k_j}(s))\right| \, ds \\
&+ \int_0^\pi \left|U(x_{k_j}(s)) - U(\bar{x}(s))\right| \, ds \\
&\leq \int_0^\pi \sup_{\zeta\in K}|U_{k_j}(\zeta) - U(\zeta)| \, ds + \int_0^\pi \left|U(x_{k_j}(s)) - U(\bar{x}(s))\right| \, ds.
\end{aligned}$$

The uniform convergence of U_k to U on each compact gives that:

$$\sup_{\zeta\in K}|U_{k_j}(\zeta) - U(\zeta)| \longrightarrow 0.$$

Moreover the continuity of U and the Lebesgue dominated convergence theorem imply that

$$\int_0^\pi \left|U(x_{k_j}(s)) - U(\bar{x}(s))\right| \, ds \longrightarrow 0,$$

and this gives us the desired result,i.e.

$$\int_0^\pi U(\bar{x}(s))ds = R.$$

Similarly, we obtain
$$\int_0^\pi grad\ U(\bar{x}(s))ds = 0.$$
We show next, that $\bar{x}$ is not constant.
Assume by the way of contradiction that $\bar{x}(s) = c$, then by $\int_0^\pi \nabla U(\bar{x}(s))ds = 0$ we have $grad\ U(c) = 0$.
Since U is convex, we can write $U(0) \geq U(c) + \nabla U(c)(-c)$, i.e. $U(0) \geq U(c)$. It follows that $U(c) = 0$ (since $U(0) \leq U(c)$) which is a contradiction because $\int_0^\pi U(\bar{x}(s))ds = R > 0$. Thus $\bar{x}(s) \neq constant$.

Furthermore, if we set $h = x_{_{k_j}} + a_{_{k_j}}$ in the equation 3.0.13 , we get
$$\int_0^\pi |\dot{x}_{_{k_j}}(s)|^2 ds = \lambda^{^2}_{_{k_j}} \int_0^\pi grad\ U_{k_j}(x_{_{k_j}}(s) + a_{_{k_j}}).(x_{_{k_j}}(s) + a_{_{k_j}})ds$$
The condition $\int_0^\pi V_k(z_{_k}(s))ds = R > 0$ ensures us that $V_k(z_{_k}(s)) > 0$ and then that the inner product $\langle grad\ U_{k_j}(x_{_{k_j}}(s) + a_{_{k_j}}), x_{_{k_j}}(s) + a_{_{k_j}}\rangle > 0$ (cf proof of Theorem 3.0.25) so that
$$\lambda^{^2}_{_{k_j}} = \frac{\int_0^\pi |\dot{x}_{_{k_j}}(s)|^2 ds}{\int_0^\pi grad\ U_{k_j}(x_{_{k_j}}(s) + a_{_{k_j}}).(x_{_{k_j}}(s) + a_{_{k_j}})ds}.$$
The sequence $\{a_{_k}\}$ being bounded,we have up to a convergent subsequence that we relabel $(a_{_{k_j}})$, that
$$\int_0^\pi \langle grad\ U_{k_j}(x_{_{k_j}}(s)+a_{_{k_j}}), (x_{_{k_j}}(s)+a_{_{k_j}})\rangle ds \longrightarrow \int_0^\pi \langle \nabla U(\bar{x}(s)+\bar{a}), (\bar{x}(s)+\bar{a})\rangle ds,$$
where $a_{_{k_j}} \longrightarrow \bar{a}$ uniformly.
Since
$$\int_0^\pi |\dot{x}_{_{k_j}}(s)|^2 ds \longrightarrow T(\bar{x})$$
and
$$\int_0^\pi \langle grad\ U_{k_j}(x_{_{k_j}}(s)+a_{_{k_j}}), (x_{_{k_j}}(s)+a_{_{kj}})\rangle ds \longrightarrow \int_0^\pi \langle \nabla U(\bar{x}(s)+\bar{a}).(\bar{x}(s)+\bar{a})\rangle ds,$$
we obtain that the sequence $\{\lambda^{^2}_{_{k_j}}\}$ is convergent with limit $\lambda^2 > 0$.

If we rewrite the equation 3.0.13 for the convergent sequence $\lambda^2_{_{k_j}}$, we obtain

$$\int_0^\pi \dot{x}_{k_j}(s).h(s)\ ds = \lambda_{k_j}^2 \int_0^\pi \langle grad\ U(x_{k_j}(s)+a_{k_j}), h(s)\rangle\ ds \quad \text{for all} \quad h \in H^1.$$

In particular, for $h \in \mathcal{D}(0,\pi)$

$$\int_0^\pi \dot{x}_{k_j}(s).\dot{h}(s)\ ds = -\int_0^\pi x_{k_j}(s).\ddot{h}(s)\ ds$$

and

$$-\int_0^\pi x_{k_j}(s).\ddot{h}(s)\ ds \underset{k\to\infty}{\longrightarrow} -\int_0^\pi \bar{x}(s).\ddot{h}(s)\ ds.$$

In addition, for $h \in \mathcal{D}(0,\pi)$,

$$-\int_0^\pi x_{k_j}(s).\ddot{h}(s)\ ds = -\langle \ddot{\bar{x}}, h\rangle \qquad \text{in the sens of distribution .}$$

and

$$\lambda_{k_j}^2 \int_0^\pi \langle grad\ U_{k_j}(x_{k_j}(s)+a_{k_j}), h(s)\rangle ds \underset{k\to\infty}{\longrightarrow} \lambda^2 \int_0^\pi grad\ U(\bar{x}(s)+\bar{a}).h(s)ds = \lambda^2\langle grad\ U(\bar{x}+\bar{a}), h\rangle$$

so that

$$-\langle \ddot{\bar{x}}, h\rangle = \lambda^2\langle grad\ U(\bar{x}+\bar{a}), h\rangle \quad \text{for all} \quad h \in \mathcal{D}(0,\pi).$$

Consequently, $\bar{x}$ satisfies

$$\ddot{\bar{x}} + \lambda^2 grad\ U(\bar{x}+\bar{a}) = 0,$$

and $\bar{z} = \bar{x} + \bar{a}$ is a solution for the problem.

Theorem 3.0.30 *The system 3.0.1 possesses a one parameter family of distinct solutions $\tilde{x}_R(t)$ where the parameter R varies over the positive real numbers. Furthermore the average potential energy of $\tilde{x}_R(t)$,*

$$\frac{1}{T_R}\int_0^{T_R} U(\tilde{x}_R(t))dt,$$

over a period T_R is proportional to R.

Proof. By virtue of the threorem 3.0.28, we have the existence of critical points $x_R(s)$ of the variational problem and they correspond to even $2\pi\lambda - periodic$ solutions $\tilde{x}_R(t) = x_R\left(\frac{t}{\lambda}\right)$ of 3.0.1.
For two positive real numbers R_1 and R_2 such tha $R_1 \neq R_2$, because of the

equality condition in $\mathcal{S}_R$ we obviously get that $x_{_{R_1}}(s) \neq x_{_{R_2}}(s)$.
In

$$\int_0^{\pi} U(x_{_R}(s))ds = R,$$

we make the change of variables $t = \lambda s$ to get

$$\frac{1}{\lambda}\int_0^{\lambda\pi} U(\tilde{x}_{_R}(t))dt = R = \frac{1}{2\lambda}\int_0^{2\lambda\pi} U(\tilde{x}_{_R}(t))dt$$

so that

$$2\lambda R = \int_0^{2\lambda\pi} U(\tilde{x}_{_R}(t))dt$$

It means that the mean value of $U(\tilde{x}_{_R}(t))$ over the period $2\pi\lambda$ is proportional to the parameter R.

Remark 3.0.2

1• *A general model for the function* U *is given , for* $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, *by*

$$U(x) = \|x\|^2 = \sum_{i=1}^{N} x_i^2.$$

(Cf. also Propostion 0.2*).*

2• *We proved in lemma 3.0.24 that a solution of 3.0.2 can be extended to an "even" function, which is still solution of 3.0.2. We now observe that for somme additional conditions on the function* U*, for instance that* U *is even, we can also obtain, for our solutions, "odd" extensions which are still solutions.*

3• *We can give a physical interpretation to the problem through the second law of Newton.*
If $\ddot{x}$ *designes the body's acceleration,* m *the mass of the body, we have*

$$m\ddot{x} = F$$

where F *is the net force on the body.*
When F *derives from a potential* U*, it has the form* $F = grad\ U$*, or* $F = -grad\ U$*.*
So, for a given potential, we look for the possible orbits of the body.

4• *The functional* T *involved in the variational formulation is just the kinectic energy of the system and since the solution are parametrized by the average potential energy* $\frac{1}{2\pi\lambda}\int_0^{2\lambda\pi} U(\tilde{x}_R(t))dt$*, by setting*
$E_c(x) = T(x) = kinetic\ energy$ *and* $E_p(x) = \int_0^{2\lambda\pi} U(\tilde{x}_R(t))dt = potential\ energy$*,*
we can define another optimization problem by :

$$\inf_{x \in K} \mathcal{F}(x)$$

where

$$\mathcal{F}(x) = E_c(x) + E_p(x) \text{ denotes the total energy of the system and}$$

$$K \text{ a suitable constraints set .}$$

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