# The Mountain Pass Theorem and Applications. 

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## MASTER DEGREE IN PURE AND APPLIED MATHEMATICS

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## Epigraph

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

John Louis von Neumann

## Dedication

Dedicated first of all to my Lord Jesus Christ for inspiration and wisdom.

Secondly I dedicate this project to my beloved parents - Djeungue Hubert and Fameni Jacqueline, my brothers and sisters; Serges, Huguette, Nina, Carlos, Fredy, and to my Fiancé, in appreciation of their genuine support wherever I am.

## Preface

This project lies at the interface between Nonlinear Functional Analysis,
unconstrained Optimization and Critical point theory. It concerns mainly the Ambrosetti-Rabinowitz's Mountain Pass Theorem which is a min-max theorem at the heart of deep mathematics and plays a crucial role in solving many variational problems. As application, a model of Lane-Emden equation is considered.

Minmax theorems characterize a critical value $c$ of a functonal $f$ defined on a Banach spaces as minmax over a suitable class $\mathcal{A}$ of subsets of $X$, that is :

$$
c=\inf _{A \in \mathcal{S}} \sup _{x \in A} f(x) .
$$

Variational methods refer to proofs established by showing that a suitable auxilliary function attains a minimum or has a critical point (see below). Minimum Variational principle can be viewed as a mathematical form of the principle of least action in Physics and justifies why so many results in Mathematics are related to variational techniques since they have their origin in the physical sciences.

The application of the Mountain Pass Theorem and more generally those of Variational Techniques cover numerous theoretic as well as applied areas of mathematical sciences such as Partial Differential Equations, Optimization, Banach space geometry, Control theory, Economics and Game theory.

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## Contents

Epigraph ..... ii
Dedication ..... iii
Preface ..... iv
Acknowledgement ..... v
Introduction ..... 1
1 Differentiability in Banach Spaces ..... 6
1.1 Gâteaux Derivative ..... 6
1.2 Fréchet Differentiability ..... 10
1.3 Second order derivative ..... 16
2 Nemytskii Operators ..... 20
2.1 Definition of a Nemytskii Operator ..... 20
2.2 Carathéodory condition ..... 21
2.3 Continuity and Differentiability of Nemytskii Operator ..... 23
3 Variational Principles and Minimization ..... 28
3.1 Lower Semicontinuous Functions ..... 28
3.2 Ekeland Theorem in Complete metric space ..... 31
3.3 Palais-Smale Conditions and Minimization ..... 35
3.4 Deformation Theorem and Palais-Smale Conditon ..... 37
3.5 Mountain-Pass Theorem ..... 41
4 Application: The lane Emden Equation ..... 44Bibliography54

## Introduction

Let us first introduce some keywords that will enable us to specify our principal objective.
Given a nonempty set $X$ and a funct ion $f: X \rightarrow \mathbb{R}$ which is bounded below, computing the number

$$
\begin{equation*}
\inf _{X} f:=\inf \{f(x): x \in X\} \tag{1}
\end{equation*}
$$

represents a minimization problem posed in $X$ : namely that of finding a minimizing sequence, i.e. $\left(x_{k}\right)_{k} \subset X$ such that

$$
\lim _{k \rightarrow \infty} f\left(x_{k}\right)=\inf _{X} f .
$$

The number $\inf _{X} f$ is often called the infimal value of $f$ or more simply the infimum of $f$ over $X$. The function $f$ is usually called the objective function or also infimand. By analogy we have the concepts of supremal value (supremum) and supremand.
An optimal solution of $\left(F_{1}\right)$ is an element $a \in X$ such that

$$
f(a) \leq f(x), \quad \forall x \in X ;
$$

such an element $a$ is usually called a minimizer, a minimum point or simply a minimum of $f$ on $X$. We shall also speak of global minimum.
Let us emphasize that the notation

$$
\min \{f(x): x \in X\}
$$

holds at the same time for a number (when there exists a solution to $F_{1}$ ) and a problem to solve.
Likewise one can meet maximization problems but they are all equivalent to minimization problems since for any real valued function $g$
defined on a set $X$, one has

$$
\sup \{g(x): x \in X\}=-\inf \{-g(x): x \in X\}
$$

When $X$ has a topological structure, another problem related with $\left(F_{1}\right)$, is to know
$\left(Q_{1}\right)$ whether a giving minimizing sequence $\left(x_{k}\right)$ converges to an optimal solution when $k$ tends to $+\infty$.

Two conditions are essential to guarantee a positive answer to $\left(Q_{1}\right)$. A topological criterion on the structure of $X$ (e.g., compactness) and a topological criterion on the behavior of the function $f$ (e.g., continuity).
When $X$ is an open set of a real normed linear space (respectively a manifold) and $f$ is Fréchet differentiable or just Gâteaux differentiable (respectively differentiable in the geometric sense), a necessary condition for a point $a \in X$ to be a minimizer (according to Euler) is to be a critical (or stationary) point of $f$; this means that, $f^{\prime}(a) \equiv 0$ on $X \quad$ (respectively $d f(a) \equiv 0$ on $T_{a} X$, the tangent space of the manifold $X$ at $a$ ). We say that a real number $c$ is a critical value of $f$ if there exists a critical point $a \in X$ such that $f(a)=c$. In the case of a Hilbert space $X$ endowed with a scalar product $\left\langle\cdot{ }^{\circ}\right\rangle$, and thanks to the Riesz representation theorem, the gradient $\nabla f$ of a Gâteaux differentiable is defined by setting

$$
\langle h, \nabla f(x)\rangle=f^{\prime}(x)(h) .
$$

And so in this case, a critical point of $f$ is just a solution of the equation

$$
\nabla f(x)=0
$$

For instance the following simple surjectivity result illustrates well the variational argument.

Proposition 0.1 The derivative of a differentiable function $f$ : $\mathbb{R} \rightarrow \mathbb{R}$ satisfying
$\lim _{|x| \rightarrow \infty} f(x) /|x|=\infty \quad$ is surjective.
The proof follows immediately from the fact that for each arbitrary $r \in \mathbb{R}$ fixed, the function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ defined by $\varphi(x)=$ $f(x)-r x$ has a minimum point which is a critical point since this function is lower semi-continuous (in fact continuous), coercive (in the sense that its level sets $\{x \in \mathbb{R}: \varphi(x) \leq t\}$ are compact) and
furthermore differentiable.
The interested reader can also see another interesting and illustrative example by checking (with differential analysis and ordinary differential equation tools) the following:

Proposition 0.2 Let $X$ be the normed linear space consisting of all continuously differentiable function $u$ on $[0,1]$ satisfying the homogeneous Dirichlet boundary condition $u(0)=u(1)=0$, that is, $X=\left\{u \in C^{1}[0,1] ; u(0)=u(1)=0\right\}$,
and equipped with the norm defined by

$$
\|u\|_{C^{1}}=\max _{x \in[0,1]}|u(x)|+\max _{x \in[0,1]}\left|u^{\prime}(x)\right| .
$$

Consider the functionals $E$ and $G$ defined on $X$ respectively by:

$$
E(u)=\int_{0}^{1}\left|u^{\prime}(x)\right|^{2} d x \quad \text { and } \quad G(u)=\int_{0}^{1}|u(x)|^{2} d x .
$$

Then the minimization problem

$$
\min \{E(u) ; G(u)=1, u \in X\}
$$

is equivalent to the minimization problem

$$
\min \left\{\frac{E(u)}{G(u)} ; u \in X \backslash\{0\}\right\}
$$

and has an optimal solution $\varphi:[0,1] \rightarrow \mathbb{R}$ defined by $\varphi(x)=$ $\sin (\pi x)$. See Appendix for the proof.

Many boundary value problems are equivalent to

$$
\begin{equation*}
A u=0 \tag{E}
\end{equation*}
$$

where $A: U \subset X \rightarrow Y$ is a mapping from a nonempty open set $U$ of a Banach space $X$ into a Banach space $Y$. The problem is said to be variational, if there exists a differentiable functional $\varphi: U \subset X \rightarrow \mathbb{R}$ such that

$$
A=\varphi^{\prime}, \quad(\text { see Definition1.2 })
$$

In this case, the space $Y$ correspond to the dual $X^{\prime}$ of $X$ and Equation $(E)$ is equivalent to

$$
\varphi^{\prime}(u)=0, \text { i.e., }
$$

$$
\begin{equation*}
\left\langle h, \varphi^{\prime}(u)\right\rangle=0, \quad \forall h \in X \tag{2}
\end{equation*}
$$

where $\langle$,$\rangle holds for the duality pairing of X$ and $X^{\prime}$. Hence the critical points of $\varphi$ are the solutions $u$ of (2) and their images $\varphi(u)$ are the critical values of $\varphi$. A critical point of $\varphi$ is a solution $u$ of (2) and the value of $\varphi$ at $u$ is a critical value.

Now how to find critical values in general?
When $\varphi: X \rightarrow Y$, defined between Banach spaces, is bounded from below, its infimum over $X ; \inf _{X} \varphi$ is a natural candidate according to Euler condition. In this case, Ekeland variational principle (Theorem 3.4) implies the existence of a sequence $\left(u_{n}\right)_{n}$ such that

$$
\varphi\left(u_{n}\right) \quad \longrightarrow \alpha:=\inf _{X} \varphi \quad \text { and } \quad \varphi^{\prime}\left(u_{n}\right) \longrightarrow 0
$$

Such a sequence is called Palais-Smale sequence at level $\alpha$. Given $c \in \mathbb{R}$, the functional $\varphi$ is said to satisfy (PS) ${ }_{c}$ condition, if any Palais-Smale sequence at level $c$ has a comvergent subsequence. If $\varphi$ is continuously differentiable, bounded below, and satisfies (PS) ${ }_{c}$ at level $c:=\inf _{X} \varphi$, then $c$ is a critical value of $\phi$.
There are many theorems regarding the existence of local extrema (minima or maxima) but very few concerning saddle points (i.e., critical points which are neither local minima nor local maxima).

The aim of this disertation is mainly to study the Mountain Pass Theorem which is an important tool from the Calculus of Variations that provides certain sufficient conditions on functionals to have saddle points. From a geometric point of view, one could say that the Mountain Pass lies along the path that passes at the Lowest elevation through the mountains. The Mountain Pass Theorem is extensively used to solve variational problems in Partial Differential Equations (for short PDEs). Here we shall focus our attention on the Lane-Emden equation

$$
\left\{\begin{array}{rccc}
-\Delta u & =u^{p} & \text { in } \Omega \\
u & >0 & \text { in } \Omega \\
u=0 & \text { on } \partial \Omega
\end{array}\right.
$$

where $\Omega$ is a nonempty bounded subset of $\mathbb{R}^{N}$ and $p>1$.
The organization of the project is as follows:

- In chapter 1 we review the notions of differentiable maps and critical points in Banach spaces.
- In the 2nd Chapter, we introduce the Nemytskii operators (defined between function spaces) and study some of their possible properties such as continuity and differentiability.
- In Chapter 3, we consider some variational principles and we state and prove the Mountain Pass Theorem.
- Finally we use the Mountain pass Theorem to show the existence of positive solutions to the Lane-Emden equation (mentioned above) in the case that the exponent $p$ satisfies
$1<p<\frac{N+2}{N-2}$, if $N \geq 3, \quad$ otherwise $1<p<\infty$ when $N=1,2$.


## ChAPTER 1

## Differentiability in Banach Spaces

We will define here two types of differentiability in Banach spaces as generalizations of the concept of differentiability in $\mathbb{R}$.

### 1.1 Gâteaux Derivative

Let us denote by $\mathcal{B}(X, Y)$ the space of all bounded linear maps from $X$ to $Y$ where $X, Y$ are Banach spaces.
Recall that a bounded linear map means a continuous linear map.

## Definition1.1

Let $f: U \mapsto Y$ be a mapping and $x \in U$; where $U \subset X$ open. We say that $f$ is Gâteaux differentiable at $x_{o}$ if there exists $A \in \mathcal{B}(X, Y)$, such that
$\forall h \in X \backslash\{0\}$, the map $t \mapsto \frac{f\left(x_{0}+t h\right)-f\left(x_{0}\right)}{t}$ has a limit as $t \rightarrow 0$ equal to $A(h)$; that is,

$$
\begin{equation*}
\lim _{t \rightarrow 0} \frac{f\left(x_{0}+t h\right)-f\left(x_{0}\right)}{t}=A(h) \tag{1.2}
\end{equation*}
$$

or equivalently

$$
f\left(x_{0}+t h\right)-f\left(x_{0}\right)=t A(h)+o(t) \quad \forall h \in X
$$

where $o(t)$ holds for the remainder $r(t)=f\left(x_{0}+t h\right)-f\left(x_{0}\right)-t A(h)$
satisfying

$$
\lim _{t \rightarrow 0} \frac{\|r(t)\|}{t}=0 .
$$

For simplicity we will write $A h$ instead of $A(h)$.
$A h$ is called the Gâteaux derivative of $f$ at $x_{0}$ in the direction of $h$ denoted $\frac{\partial f}{\partial h}\left(x_{0}\right)$.

The bounded linear operator $A$, depending on $x_{0}$, is denoted by $D_{G} f\left(x_{o}\right)$ or $f_{G}^{\prime}\left(x_{o}\right)$ and called the Gâteaux differential.

## Remarks.

- In Definition 1.1, one can simply require that $t \rightarrow 0^{+}$.
- Whenever $h \neq 0$ and the ratio $\frac{f\left(x_{0}+t h\right)-f\left(x_{0}\right)}{t}$ has a limit in $Y$ as $t \rightarrow 0$, we say that $f$ is differentiable in the direction of $h$ at $x_{o}$, and we call $\lim _{t \rightarrow 0} \frac{f\left(x_{0}+t h\right)-f\left(x_{0}\right)}{t}$ the directional dérivative of $f$ at $x_{o}$ in the direction $h$.


## Example 1.1:

The function $f: \mathbb{R}^{2} \mapsto \mathbb{R}$ defined by

$$
f(x, y)=x^{2}+y^{2}
$$

is Gateaux differentiable at every point $\left(x_{0}, y_{0}\right) \in \mathbb{R}^{2}$.
Indeed: Let $u_{0}=\left(x_{0}, y_{0}\right)$ and $h=\left(h_{1}, h_{2}\right)$. Then

$$
f\left(u_{0}+t h\right)-f\left(u_{0}\right)=2 t\left(x_{0} h_{1}+y_{0} h_{2}\right)+t^{2}\left(h_{1}^{2}+h_{2}^{2}\right), \quad \forall t \in \mathbb{R} .
$$

It follows that

$$
\lim _{t \rightarrow 0} \frac{f\left(u_{0}+t h\right)-f\left(u_{0}\right)}{t}=2\left(x_{0} h_{1}+y_{0} h_{2}\right)=2\left\langle u_{0}, h\right\rangle
$$

et since the map $h \mapsto 2\left\langle u_{0}, h\right\rangle$ is linear and continuous from $\mathbb{R}^{2}$ to $\mathbb{R}$, we conclude that $f$ is Gâteaux differentiable and

$$
D_{G} f\left(u_{0}\right)(h)=2\left\langle u_{0}, h\right\rangle \quad \forall h \mathbb{R}^{2} .
$$

Moreover by regarding $\mathbb{R}^{2}$ as a euclidean space, we can derive the gradient of $f$ at $u_{o}$ as

$$
\nabla f\left(u_{o}\right)=2 u_{o} .
$$

which is actually linear and bounded with respect to h as the inner product ( since $\mathbb{R}^{N}$ is an inner product space).

Theorem 1.1: (Euler necessary condition for extrema)
Let $X$ and $Y$ be real Banach spaces, $f: U \mapsto Y$ be a mapping and $x \in U$ where $U \subset X$ is open. If $f$ is Gâteaux differentiable at an extremum point $x_{0}$ (maximum or minimum point), then $D_{G} f\left(x_{0}\right)=0$

Proof: Under the hypothesis of this theorem, suppose without loss of generality that $x_{0}$ is a minimum point (otherwise consider the function $-f$ instead of $f$ ).
Since $x_{o} \in U$ and $U$ is open, there exists a positive real number $r$ such that the open ball $B_{r}\left(x_{o}\right)$ is contained in $U$. Now let $h \in X \backslash\{0\}$. Then for every $t$ such that $|t|<r /\|h\|$, we have, $f\left(x_{o}+t h\right) \geq f\left(x_{o}\right)$ and so by the Gâteaux differentiability of $f$ at $x_{o}$, we have

$$
D_{G} f\left(x_{o}\right)(h)=\lim _{t \rightarrow 0+} \frac{f\left(u_{0}+t h\right)-f\left(u_{0}\right)}{t} \geq 0
$$

It also follows that $D_{G} f\left(x_{o}\right)(-h) \geq 0$, i.e. $D_{G} f\left(x_{o}\right)(h) \leq 0$ by linearity of $D_{G} f\left(x_{o}\right)$. Therefore $D_{G} f\left(x_{o}\right)(h)=0$ for all $h \in X$ indeed. Thus $D_{G} f\left(x_{o}\right)=0$.

Theorem 1.2: (Mean Value Theorem in Banach Spaces)
Let $X$ and $Y$ be Banach spaces, $U \subset X$ be open and let $f: U \rightarrow Y$ be Gâteaux differentiable. Then for all $x_{1} x_{2} \in X$, we have

$$
\left\|f\left(x_{1}\right)-f\left(x_{2}\right)\right\| \leq \sup _{t \in[0,1]} \| D_{G} f\left(x_{1}+t\left(x_{2}-x_{1}\right)\|\cdot\| x_{1}-x_{2} \|\right.
$$

provided that the $\sup _{t \in[0,1]} \| D_{G} f\left(x_{1}+t\left(x_{2}-x_{1}\right) \|\right.$ is finite.
Proof. Suppose that the assumptions of Theorem 1.2 hold. Let $g^{*} \in Y^{*}$ (the dual of $Y$ ) such that $\left\|g^{*}\right\| \leq 1$. Then the real-valued function $\varphi:[0,1] \longrightarrow \mathbb{R}$ defined by

$$
\varphi(t)=g^{*} \circ f\left(x_{1}+t h\right) \quad \text { where } h=x_{2}-x_{1}
$$

is differentiable on $[0,1$ in the usual sense. Moreover we see that

$$
\varphi^{\prime}(t)=g^{*}\left(D_{G} f\left(x_{1}+t h\right)(h)\right), \quad \forall t \in(0,1) .
$$

It follows from the classical mean valued theorem that

$$
|\varphi(1)-\varphi(0)| \leq \sup _{0<t<1}\left|\varphi^{\prime}(t)\right|,
$$

that is

$$
\left\|g^{*} \circ f\left(x_{1}\right)-g^{*} \circ f\left(x_{2}\right)\right\| \leq \sup _{0<t<1}\left|\varphi^{\prime}(t)\right|
$$

Moreover for all $t \in(0,1)$, we have

$$
\begin{aligned}
\left|\varphi^{\prime}(t)\right| & =\left|g^{*}\left(D_{G} f\left(x_{1}+t h\right)(h)\right)\right| \\
& \leq\left\|g^{*}\right\|\left\|D_{G} f\left(x_{1}+t h\right)\right\|\|h\| \\
& \leq\left\|D_{G} f\left(x_{1}+t h\right)\right\|\|h\| .
\end{aligned}
$$

And so

$$
\left\|g^{*}\left(f\left(x_{1}\right)-f\left(x_{2}\right)\right)\right\|=\left\|g^{*} \circ f\left(x_{1}\right)-g^{*} \circ f\left(x_{2}\right)\right\|\left\|\leq\left(\sup _{0<t<1}\left\|D_{G} f\left(x_{1}+t h\right)\right\|\right)\right\| h \|
$$

But it is well known as a consequence of the Hahn-Banach theorem that

$$
\|y\|=\sup \left\{u^{*}(y), u^{*} \in Y^{*}, \quad\left\|u^{*}\right\| \leq 1\right\}
$$

Therefore we finally have
$\left\|f\left(x_{1}\right)-f\left(x_{2}\right)\right\| \leq \sup _{t \in[0,1]} \| D_{G} f\left(x_{1}+t\left(x_{2}-x_{1}\right)\|\cdot\| x_{1}-x_{2} \|\right.$.
Remark. If $f$ satisfies the assumptions of Theorem 1.2 and has a continuous Gâteaux differential, then one can prove the conclusion of Theorem 1.2 by using the notion of Riemann integration in Banach spaces following the next lemma.

Lemma. (Cf. ) Let $X$ be a Banach space and $\varphi:[a, b] \rightarrow X$ be continuous, where $-\infty<a<b<+\infty$. Then the sequence of partial sums

$$
\frac{b-a}{n} \sum_{k=0}^{n-1} \varphi\left(a+k \frac{b-a}{n}\right) \quad \text { converges as } n \rightarrow \infty
$$

and its limit is called the Riemann integral of $f$ over $[a, b]$ and is denoted by

$$
\int_{a}^{b} \varphi(t) d t
$$

That is

$$
\int_{a}^{b} \varphi(t) d t=\lim _{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=0}^{n-1} \varphi\left(a+k \frac{b-a}{n}\right) .
$$

It is easily seen that

$$
\left\|\int_{a}^{b} \varphi(t) d t\right\| \leq \int_{a}^{b}\|\varphi(t)\| d t
$$

Furthermore if $\varphi$ is continuously (Gâteaux) différentiable on $[a, b]$, then

$$
\varphi(b)=\varphi(a)+\int_{a}^{b} \varphi^{\prime}(t) d t
$$

Whenener the convergence in (1.2) is uniform for $h$, there arises an interesting stronger type of differentiability called the Fréchet differentiability.

### 1.2 Fréchet Differentiability

It is a refined notion differentiabilty of which concept is implicitly closer to that of the standard notion of differentiability known in $\mathbb{R}$

Recall: A function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is said to be differentiable at $x_{0} \in \mathbb{R}$ if and only if the mapping defined on $\mathbb{R} \backslash\{0\}$ as

$$
h \mapsto \frac{f\left(x_{0}+h\right)-f\left(x_{0}\right)}{h}
$$

has a limit $a \in \mathbb{R}$ as $h \rightarrow 0$; that is,

$$
\lim _{h \rightarrow 0} \frac{f\left(x_{0}+h\right)-f\left(x_{0}\right)}{h}=a \in \mathbb{R} .
$$

Obeserve that this condition is equivalent to "the existence of a real number $a \in \mathbb{R}$ such that

$$
f\left(x_{0}+h\right)=f\left(x_{0}\right)+a h+o(h) .
$$

Now, how can we extend this notion to operators defined between Banach spaces? The answer is in the following definition.

## Definition 1.2:

A function $f: U \longrightarrow Y$; where $X$ and $Y$ are Banach spaces and $U$ open in $X$, is said to be Fréchet differentiable at a point $x_{0} \in U$, if there exists a bounded linear map $A: X \rightarrow Y$ such that:

$$
\begin{equation*}
\lim _{\|h\| \rightarrow 0} \frac{\left\|f\left(x_{0}+h\right)-f\left(x_{0}\right)-A h\right\|}{\|h\|}=0, \tag{1.1}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
f\left(x_{0}+h\right)-f\left(x_{0}\right)=A h+o(\|h\|) ; \tag{1.2}
\end{equation*}
$$

where

$$
r(h):=f\left(x_{0}+h\right)-f\left(x_{0}\right)-A h=o(h)
$$

in the sense that

$$
\lim _{\|h\| \rightarrow 0} \frac{\|r(h)\|}{\|h\|}=0 .
$$

Such an operator $A$ is unique and called the Fréchet differential of $f$ at $x_{0}$ and is denoted by $D f\left(x_{0}\right)$ or $f^{\prime}\left(x_{0}\right)$ (somtimes it is also denoted by $d f\left(x_{o}\right)$ ).

The function $f$ is said to be Fréchet differentiable (or simply differentiable) on $U$, if it is Fréchet differentiable at every point of $U$. When there is no ambiguity about the domain of $f$, we just say that $f$ is differentiable.

## Definition 1.3:

Let $X$ and $Y$ be Banach spaces, $U$ open in $X$ and let $f: U \rightarrow Y$ be Fréchet differentiable on $U$. The Fréchet differential of $f$ on $U$ is the mapping

$$
\begin{aligned}
D f: U & \rightarrow \mathcal{B}(X, Y) \\
x & \mapsto D f(x) .
\end{aligned}
$$

We say that $f$ is continuously differentiable on $U$ or a mapping of class $C^{1}$ (or simply a $C^{1}$-mapping) if $D f$ is continuous as a mapping from $U$ into $\mathcal{B}(X, Y)$.

## Examples (Fréchet differentiable functions).

Let $H$ be a real Hilbert space. Then the function $F: H \longrightarrow \mathbb{R}$ defined by

$$
F(x)=\frac{1}{2}\|x\|^{2}
$$

is Fréchet differentiable on $H$ and its Fréchet differential is defined by:

$$
D F(x)(h)=\langle x, h\rangle=\langle h, x\rangle .
$$

Thanks to the Riesz representation we can write

$$
\nabla F(x)=x
$$

Indeed let us fix $x_{o} \in H$ arbitrarily. Then for every $h \in H$, we have

$$
\begin{aligned}
F\left(x_{o}+h\right)-F\left(x_{o}\right) & =\frac{1}{2}\left(\left\|x_{o}+h\right\|^{2}-\left\|x_{o}\right\|^{2}\right) \\
& =\frac{1}{2}\left(\left\langle x_{o}+h, x_{o}+h\right\rangle-\left\langle x_{o}, x_{o}\right\rangle\right) \\
& =\frac{1}{2}\left(\left\langle x_{o}, x_{o}\right\rangle+2\left\langle x_{o}, h\right\rangle+\langle h, h\rangle-\left\langle x_{o}, x_{o}\right\rangle\right) \\
& =\left\langle x_{o}, h\right\rangle+\frac{\langle h, h\rangle}{2} \\
& =\left\langle h, x_{o}\right\rangle+\frac{\|h\|^{2}}{2} .
\end{aligned}
$$

Now define the operator $A: H \rightarrow H$ by $A(h)=\left\langle h, x_{o}\right\rangle$. Then $A$ is linear (since the real inner product is bilinear) and bounded (according to Cauchy-Schwarz inequality). Moreover it is clear that

$$
\lim _{\|h\| \rightarrow 0} \frac{\left\|F\left(x_{0}+h\right)-F\left(x_{0}\right)-A(h)\right\|}{\|h\|}=\lim _{\|h\| \rightarrow 0}\|h\| / 2=0 .
$$

Next we present some properties of the Fréchet differential.
Proposition 1.1: Let $X$ and $Y$ be Banach spaces and $U \subset X$ open.

1. If $F: U \rightarrow Y$ is Fréchet differentiable at some point $x_{0} \in U$, then F is continuous at $x_{0}$.
2. If $F: U \rightarrow Y$ is Fréchet differentiable according to a norm in $X$,
then it is also Fréchet differentiable according to any norm equivalent to the first norm.
3. (linéarity)

If $F, G: U \rightarrow Y$ are Fréchet differentiable at some point $x_{o} \in$ $U$,
then for any $a, b \in \mathbb{R}, a F+b G$ is Fréchet differentiable at $x_{o}$ and

$$
D(a F+b G)\left(x_{o}\right)=a D F\left(x_{o}\right)+b D G\left(x_{o}\right) .
$$

4. (Chain rule).

Let also $V$ be an open set of a Banach space $Z$ and consider two mappings $F: U \longrightarrow Y$ and $G: V \longrightarrow Z$ such that $F(U) \subset V$. If F is Frechet differentiable at some point $x_{o} \in U$
and $G: V \longrightarrow Z$ is Frechet differentiable at $y_{o}=F\left(x_{o}\right) \in V$, then $G \circ F$ is Fréchet differentiable at $x_{o}$ and

$$
D(G \circ F)\left(x_{o}\right)=D G\left(y_{o}\right) \circ D F\left(x_{o}\right)
$$

## Proof:

1. Suppose that $f$ is Fréchet differentiable at $x_{0} \in U$. Then there exists a bounded linear map $A: X \rightarrow Y$ such that:

$$
f\left(x_{0}+h\right)-f\left(x_{0}\right)=A h+o(\|h\|) .
$$

It follows from the continuity of $A$ and the definition of $o(h)$ that

$$
\lim _{\|h\| \rightarrow 0}\left\|f\left(x_{0}+h\right)-f\left(x_{0}\right)\right\|=0
$$

that is

$$
\lim _{h \rightarrow 0}\left(f\left(x_{0}+h\right)-f\left(x_{0}\right)\right)=0 \text { in } Y
$$

or simply $\lim _{h \rightarrow 0} f\left(x_{0}+h\right)=f\left(x_{0}\right)$.
2. Let $\|\cdot\|_{1}$ and $\|\cdot\|_{2}$ be two equivalent norm in $X$. Then there exist constant $\alpha>0$ and $\beta>0$ such that

$$
\alpha\|x\|_{1} \leq\|x\|_{2} \leq \beta\|x\|_{1}, \quad \forall x \in X
$$

There a mapping $g$ is defined from an open neighbourhood of 0 in $\left(X,\|\cdot\|_{1}\right)$ into $Y$ if and only if is defined from an open neighbourhood of 0 in $\left(X,\|.\|_{2}\right)$ into $Y$. Moreover for any $h \neq 0$ in the domain of $g$, we have

$$
\frac{\|g(h)\|}{\beta\|h\|_{1}} \leq \frac{\|g(h)\|}{\|h\|_{2}} \leq \frac{\|g(h)\|}{\alpha\|h\|_{1}} \leq \frac{\beta\|g(h)\|}{\alpha\|h\|_{2}}
$$

which implies that

$$
\lim _{\|h\|_{1} \rightarrow 0} \frac{\|g(h)\|}{\|h\|_{1}}=0 \quad \Longleftrightarrow \quad \lim _{\|h\|_{2} \rightarrow 0} \frac{\|g(h)\|}{\|h\|_{2}}=0
$$

3. Let $\varepsilon>0$. Then by the Fréchet differentiability of the two functions $F$ and $G$ at $x_{o} \in U$, we get (indeed) the existence of $\delta>0$ such that for every $h \in X$ satisfying $\|h\|<\delta$, we have

$$
\left\|F\left(x_{o}+h\right)-F\left(x_{o}\right)-D F\left(x_{o}\right)(h)\right\| \leq \frac{\varepsilon}{2(|a|+1)}\|h\|
$$

and

$$
\left\|G\left(x_{o}+h\right)-G\left(x_{o}\right)-D G\left(x_{o}\right)(h)\right\| \leq \frac{\varepsilon}{2(|b|+1)}\|h\| .
$$

Thus we have

$$
\left\|(a F+b G)\left(x_{o}+h\right)-(a F+b G)\left(x_{o}\right)-a D F(x)(h)-b D G(x)(h)\right\| \leq \varepsilon\|h\| .
$$

4. We know that

$$
\begin{equation*}
F\left(x_{o}+h\right)-F\left(x_{o}\right)=D F\left(x_{o}\right)(h)+o(h) \tag{i}
\end{equation*}
$$

and

$$
\begin{equation*}
G\left(y_{o}+h\right)-G\left(y_{o}\right)=D G\left(y_{o}\right)(h)+o(h) \tag{ii}
\end{equation*}
$$

Therefore

$$
\begin{aligned}
G \circ F)\left(x_{o}+h\right)-(G \circ F)\left(x_{o}\right) & =G\left(F\left(x_{o}+h\right)\right)-G\left(F\left(x_{o}\right)\right) \\
& =G\left(F\left(x_{o}\right)+D F\left(x_{o}\right)(h)+o(\|h\|)\right)-G\left(F\left(x_{o}\right)\right) \\
& =G\left(F\left(x_{o}\right)\right)+D G\left(y_{o}\right)\left(\left(D F\left(x_{o}\right)(h)+o(h)-G\left(F\left(. x^{\prime}\right)\right.\right.\right. \\
& =D G\left(y_{o}\right) D F\left(x_{o}\right)(h)+D G\left(y_{o}\right)(o(h)) \\
& =D G\left(y_{o}\right) D F\left(x_{o}\right)(h)+\tilde{o}(h)
\end{aligned}
$$

which gives the result since $D G\left(y_{o}\right) D F\left(x_{o}\right)$ is a bounded linear map and

$$
\|\tilde{o}(h)\| \leq\left\|D G\left(y_{o}\right)\right\| \cdot\|o(h)\| .
$$

## Theorem 1.3:

Every Fréchet differentiable function is Gâteaux differentiable and the differentials coincide..

Proof: Let $f: U \longrightarrow Y$; where $X$ and $Y$ are Banach spaces and $U$ open in $X$, be Fréchet differentiable at a point $x_{0} \in U$. We show that $f$ is Gâteaux differentiable at $x_{o}$. Fix any $v \in X \backslash\{0\}$. Then we have $f^{\prime}\left(x_{o}\right) \in \mathcal{B}(X, Y)$ and

$$
\begin{aligned}
\lim _{t \rightarrow 0^{+}}\left(\frac{f\left(x_{o}+t v\right)-f\left(x_{o}\right)}{t}-f^{\prime}\left(x_{o}\right)(v)\right) & =\lim _{t \rightarrow 0^{+}}\|v\| \frac{f\left(x_{o}+t v\right)-f\left(x_{o}\right)-f^{\prime}\left(x_{o}\right)(t v)}{\|t v\|} \\
& =\lim _{\|h\| \rightarrow 0}\|v\| \frac{f\left(x_{o}+h\right)-f\left(x_{o}\right)-f^{\prime}\left(x_{o}\right)(h)}{\|h\|} \\
& =0 .
\end{aligned}
$$

Therefore $f$ is Gâteaux differentiable at $x_{o}$ and moreover $D_{G} f\left(x_{o}\right)=$ $f^{\prime}\left(x_{o}\right)$.

Remark: The converse of Theorem 1.2 is false as it can be seen by the example below.

## Example 1.2 :

The function $g: \mathbb{R}^{2} \longrightarrow \mathbb{R}$ defined by

$$
g(x, y)=\left\{\begin{array}{ccc}
\left(\frac{x^{2} y}{x^{4}+y^{2}}\right)^{4} & \text { if } & y \neq 0 \\
0 & \text { if } & y=0
\end{array}\right.
$$

is Gâteaux differentiable at $(0,0)$ but not Fréchet differentiable at this point.
Indeed, let $h=\left(h_{1}, h_{2}\right) \in \mathbb{R}^{2}$. Then for $t>0$, we have

$$
\frac{g(t h)-g(0,0)}{t}=\left\{\begin{array}{ccc}
\frac{t^{4} h_{1}^{4} h_{2}^{2}}{\left(t^{2} h_{1}^{4}+h_{2}^{2}\right)^{4}} & \text { if } & h_{2} \neq 0 \\
0 & \text { if } & h_{2}=0
\end{array}\right.
$$

and so

$$
\lim _{t \rightarrow 0^{+}} \frac{g(t h)-g(0,0)}{t}=0
$$

yielding the Gâteaux differentiability of $g$ at $(0,0)$ with $g^{\prime}(0,0) \equiv 0$. But it is seen that $g$ is not Fréchet differentiable at $(0,0)$, according to Theorem 1.3, by considering the perturbations $H=\left(h_{1}, h_{1}^{2}\right)$ as follows :

$$
\lim _{h_{1} \longrightarrow 0^{+}} \frac{g\left(h_{1}, h_{1}^{2}\right)-g(0,0)}{\left\|\left(h_{1}, h_{1}^{2}\right)\right\|}=\frac{1}{16} \neq 0
$$

The next theorem gives a useful sufficient condition under which Gâteaux differentiability implies Fréchet differrentiability.

## Theorem 1.4:

Let $X$ and $Y$ be Banach spaces, $U$ open and nonempty in $X$ and let $f: U \rightarrow Y$. If $f$ has a continuous Gâteaux differential, then $f$ is Fréchet differentiable and $f \in C^{1}(U, \mathbb{R})$.

## Proof:

Let $x \in U$ and choose $\epsilon>0$ such that $B(x, \epsilon) \subset U$. Let $h \in X$ such that $\|h\|<\epsilon$. Consider the function $\varphi:[0,1] \longrightarrow \mathbb{R}$ defined by :

$$
\varphi(t)=f(x+t h)-f(x)-t D_{G} f(x)(h)
$$

Since $f$ is Gâteaux differentiable, it follows that $\varphi$ is differentiable and

$$
\varphi^{\prime}(t)=D_{G} f(x+t h)(h)-D_{G} f(x)(h) .
$$

By applying the Mean Value Theorem to $\varphi$ we have :

$$
|\varphi(1)-\varphi(0)| \leq \sup _{0<t<1}\left|\varphi^{\prime}(t)\right|
$$

that is,

$$
\begin{aligned}
\left\|f(x+h)-f(x)-D_{G} f(x)(h)\right\| & \leq \sup _{t \in(0,1)}\left\|D_{G} f(x+t h)(h)-D_{G} F(x)(h)\right\| \\
& \leq \sup _{t \in(0,1)}\left\|D_{G} f(x+t h)-D_{G} F(x)\right\|\|h\| .
\end{aligned}
$$

By continuity of the mapping $D f: U \rightarrow \mathcal{B}(X, Y)$, we have

$$
\lim _{h \rightarrow 0}\left(\sup _{t \in(0,1)}\left\|D_{G} f(x+t h)-D_{G} f(x)\right\|\right)=0
$$

and so

$$
f(x+h)-f(x)-D_{G} f(x)(h)=o(h)
$$

with $D f(x) \in \mathcal{B}(X, Y)$.

## Definition 1.2:

Let $H$ be a Hilbert space equipped with inner product $\langle.$, . $\rangle$ and $f: X \longrightarrow \mathbb{R}$ be Fréchet differentiable. Then the mapping

$$
\begin{array}{rlc}
\nabla f: H & \longrightarrow & H \\
x & \mapsto & \nabla f(x),
\end{array}
$$

(where $\nabla f(x)$ is the gradient of $f$ at $x$ ) is called a potential operator with a potential $f: H \longrightarrow \mathbb{R}$.

### 1.3 Second order derivative

Let $X$ and $Y$ be real Banach spaces, $U$ open and nonempty in $X$, and let $f: U \rightarrow Y$ be differentiable. If

$$
f^{\prime}: U \longrightarrow \mathcal{B}(X, Y)
$$

is differentiable, then for every $x \in U$,

$$
\left(f^{\prime}\right)^{\prime}(x) \in L(X, \mathcal{B}(X, Y))
$$

and is simply denoted by $f^{\prime \prime}(x)$ or $D^{2} f(x)$. In this case we say that $f$ is twice differentiable at $x$ and $f^{\prime \prime}(x)$ is called the second order
differential of $f$ at $x$.
Observe that in fact

$$
f^{\prime \prime}(x): X \times X \longrightarrow Y
$$

is bilinear and bounded (i.e., continuous) .
We recall that a mapping $\Phi: X \times X \longrightarrow Y$ is a bounded bilinear map if:

1. $\forall\left(x_{1}, x_{2}\right) \in X \times X, \forall y \in X$ and $\forall \alpha, \beta \in \mathbb{R}$,

$$
\begin{aligned}
& \Phi\left(\alpha x_{1}+\beta x_{2}, y\right)=\alpha \Phi\left(x_{1}, y\right)+\beta \Phi\left(x_{2}, y\right) \\
& \Phi\left(y, \alpha x_{1}+\beta x_{2}\right)=\alpha \Phi\left(y, x_{1}\right)+\beta \Phi\left(y, x_{2}\right)
\end{aligned}
$$

2. $\exists K \in(0, \infty)$ such that

$$
\left\|\Phi\left(x_{1} \cdot x_{2}\right)\right\|_{Y} \leq K\left\|x_{1}\right\|_{X}\left\|x_{2}\right\|_{X}
$$

The norm of such a bounded bilinear map is given by:

$$
\|\Phi\|=\sup \left\{\left\|\Phi\left(x_{1}, x_{2}\right)\right\|_{Y},\left\|x_{1}\right\|_{X} \leq 1 \text { and }\left\|x_{2}\right\| \leq 1\right\}
$$

Note that, more generally, if $E_{1}, E_{2}$ and $E_{3}$ are given three normed linear spaces, we can define a bounded linear map from $E_{1} \times E_{2}$ into $E_{3}$.

The space of bounded bilinear maps from $X \times X$ into $Y$ is isometric to $\mathcal{B}(X, \mathcal{B}(X, Y))$. Indeed the map

$$
\begin{array}{clc}
\left.j: \mathcal{B}\left(X^{2}, Y\right)\right) & \longrightarrow & \mathcal{B}(X, \mathcal{B}(X, Y)) \\
A & \mapsto & j(A)
\end{array}
$$

where $j(A)$ is such that for all $x \in X$ and for all $y \in Y$,

$$
(j(A)(x))(y)=A(x, y) .
$$

Moreover $\|j(A)\|=\|A\|$.

Going back to the setting of the definition of the second order differential, if $f: U \rightarrow Y$ is twice differentiable, then $f^{\prime}: U \rightarrow$ $\mathcal{B}\left(X^{2}, Y\right)$ ). And if $f^{\prime \prime}$ is continuous, we say that $f$ is of class $\mathcal{C}^{2}$ and we write $f \in \mathcal{C}^{2}(U, Y)$.

Furthermore we have the following Taylor formula for $x \in U$ and $h$ sufficiently small :

$$
\begin{equation*}
f(x+h)=f(x)+f^{\prime}(x)(h)+\frac{1}{2} f^{\prime \prime}(x)(h, h)+o\left(\|h\|_{X}^{2}\right) \tag{1.5}
\end{equation*}
$$

that can be established by using a notion of Riemann integration in $Y$ such as

$$
f(x+h)=f(x)+f^{\prime}(x)(h)+\int_{0}^{1}(1-t) f^{\prime \prime}(x+t h)(h, h) d t .
$$

These Taylor expansions give the simplest sufficient conditions on a critical a $C^{2}$ functional to be a local extrema.

## Proposition 1.4

Let $X$ and $Y$ be Banach spaces, $U$ open in $X$ and let $f: U \rightarrow Y$ be twice continuously differentiable. Suppose that $x_{o}$ is a critical point of $f$.

1. If there exists a positive real number $\lambda$ such that

$$
D^{2} f\left(x_{o}\right)(h, h) \geq \lambda\|h\|^{2}, \quad \forall h \in X,
$$

Then $x_{o}$ is a local minimum point of $f$.
2. If for every $x$ in a neighbourhood of $x_{o}, D^{2} f(x)$ is positive semidefinite (in the sense that

$$
\left.D^{2} f(x)(h, h) \geq 0, \quad \forall h \in X\right)
$$

then $x_{o}$ is a local minimum point of $f$ over $U$.
3. If $U$ is convex and for every $x \in U, D^{2} f(x)$ is positive semidefinite (in the sense that

$$
\left.D^{2} f(x)(h, h) \geq 0, \quad \forall h \in X\right)
$$

then $x_{o}$ is a minimum point of $f$ over $U$. Observe in this case that $f$ is convex on $U$.

For instance if $H$ is a real Hilbert space and $b \in H$ is given then, the critical point $x_{o}$ of the the functional $\varphi$ defined on $H$ by

$$
\varphi(x)=\frac{\|x\|^{2}}{2}+\langle b, x\rangle,
$$

is the minimum point (i.e., $x_{o}=-b$ ).

## Note.

Let $H$ be a Hilbert space and $f: H \rightarrow \mathbb{R}$ be twice continuously differentiable. Then for every $x \in H$, there exists (according to the Riesz Representation Theorem) a bounded linear operator $A: H \rightarrow H$ which is symmetric and satisfies

$$
D^{2} f(x)\left(h_{1}, h_{2}\right)=\left\langle A h_{1}, h_{2}\right\rangle
$$

I

## chapter 2

## Nemytskii Operators

They take their name from the mathematician Viktor Vladimirovich Nemytskii.
Let $n$ and $m$ be positive integers and $\Omega$ be a nonempty subset of $\mathbb{R}^{n}$.

### 2.1 Definition of a Nemytskii Operator

Given a function $f$ defined from $\Omega \times \mathbb{R}^{m}$ into $\mathbb{R}$, denote by $\mathcal{F}\left(\Omega, \mathbb{R}^{m}\right)=$ $\left(\mathbb{R}^{m}\right)^{\Omega}$ the set of $m$-vector valued functions defined on $\Omega$.
Then the Nemytskii Operator, associated to $f$, is the operator,

$$
\begin{array}{rlc}
N_{f}: \mathcal{F}\left(\Omega, \mathbb{R}^{m}\right) & \longrightarrow \mathcal{F}(\Omega, \mathbb{R}) \\
u & \mapsto & N_{f} u
\end{array}
$$

where $N_{f} u$ is the function defined for $x \in \Omega$ by:

$$
\left(N_{f} u\right)(x):=f(x, u(x))
$$

. For simplicity we shall write $N_{f} u(x)$

## Example 2.1:

The Nemitskii operators associated to the maps

$$
\begin{array}{rlrl}
f: \mathbb{R} \times \mathbb{R} & \longrightarrow \mathbb{R} \quad \text { and } \quad g: \mathbb{R} \times \mathbb{R} & \longrightarrow \mathbb{R} \\
(x, s) & \mapsto|s| & & (x, s)
\end{array} \begin{array}{ll} 
& \mapsto e^{s},
\end{array}
$$

are respectively defined, for any function $u: \mathbb{R} \rightarrow \mathbb{R}$, by $N_{f} u$ and $N_{g} u$ with the following expressions:

$$
\left(N_{f} u\right)(x)=|u(x)| \quad \text { and } \quad\left(N_{g} u\right)(x)=x e^{u(x)},
$$

Observe, by continuity of $f$ and $g$, that both $N_{f}$ and $N_{g}$ map the set of real-valued continuous functions on $\Omega ; \mathcal{C}(\Omega)$, into itself. Moreover they map the set of real-valued measurable functions into itself.

## Example 2.2.

Let $\Omega \subset \mathbb{R}^{N}$ be a measurable set. Then the Nemitskii operator which assigns to each extended real-valued function on $\Omega$ its square, induces an operator

$$
\begin{array}{rlll}
L^{2}(\Omega) & \longrightarrow & L^{1}(\Omega) \\
u & \mapsto & u^{2}
\end{array}
$$

also called Nemitskii operator.
More generally, given any real number $p \in(1,+\infty)$ and setting $q:=\frac{p}{p-1} ;($ the conjugate of $p)$, the following map

$$
\begin{array}{rll}
L^{p}(\Omega) & \longrightarrow & L^{q}(\Omega) \\
u & \mapsto & |u|^{p-1}
\end{array}
$$

is a Nemitskii operator.
Because of the obvious usefulness of measurable functions, we introduce a notion called the caratheodory condition .

### 2.2 Carathéodory condition

Definition 2.2 (Caratheodory Condition):
A function $f: \Omega \times \mathbb{R}^{m} \longrightarrow \mathbb{R}$ is said to satisfy the Carathéodory conditions if
(i) $f(x, \cdot)$ is a continuous function for almost all $x \in \Omega$;
(ii) $f(\cdot, u)$ is a measurable function for all $u \in \mathbb{R}^{m}$.

## Theorem 2.1

Suppose that $\Omega$ is measurable. Let $f: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function and $u: \Omega \longrightarrow \mathbb{R}^{m}$ be measurable.
Then the function

$$
N_{f} u: x \mapsto f(x, u(x))
$$

is measurable.

## Proof:

It is well-known that every measurable function is the limit of a sequence of simple functions; i.e., functions having the form:

$$
\sum_{i=1}^{p} a_{i} 1_{A_{i}}
$$

where the coefficients $a_{i}$ are constant real numbers and the subsets $A_{i}$ are measurable.
By assumption, $u$ measurable, and so

$$
u=\lim _{k} u_{k}
$$

where every $u_{k}$ is a simple function. And since $f$ satisfies the Carathéodory condition, $f$ is continous with respect to its second argument for almost every first argument fixed. Therefore,

$$
\lim _{k} f\left(x, u_{k}(x)\right)=f(x, u(x)), \quad \text { for a. e. } x .
$$

For each $k$, set $f_{k}(x)=f\left(x, u_{k}(x)\right)$ for a.e. $x$. By the first Carathéodory property of $f$, each $f_{k}$ is measurable as a sum of products of measurable simple functions. In fact by writing $u_{k}$ on its canonical form we have

$$
u_{k}(x)=\sum_{i_{k}=1}^{n_{k}} a_{i_{k}} 1_{A_{i_{k}}}(x), \quad \text { for a. e. } x,
$$

and so

$$
f_{k}(x)=\sum_{i_{k}=1}^{n} f\left(x, a_{i_{k}}\right) 1_{A_{i}}(x), \quad \text { for a.e. } x .
$$

It follows that the function $x \mapsto f(x, u(x)$ is measurable as a pintwise limit of the sequence of measurable functions $\left(f_{k}\right)_{k}$.

### 2.3 Continuity and Differentiability of Nemytskii Operator

The following theorem gives us sufficient conditions under which a measurable Nemytskii operator is continous

## Theorem 2.2

Let $f: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Caratheodory function. If $\Omega$ is a bounded domain and $f$ satisfies the growth condition

$$
|f(x, s)| \leq a|s|^{p-1}+b(x), \quad \text { for a.e. } x \text { and for all } s,
$$

with $p>1, a>0$ and $b(\cdot) \in L^{q}(\Omega)$ nonnegative a.e.; where $1 / p+$ $1 / q=1$, then the Nemytskii operator $N_{f}: L^{p}(\Omega) \longrightarrow L^{q}(\Omega)$ is continuous.

## Proof:

- We show that $N_{f}$ maps $L^{p}(\Omega)$ into $L^{q}(\Omega)$.

$$
\begin{aligned}
\left|N_{f} u\right|^{q}=\mid f\left(x,\left.u(x)\right|^{q}\right. & \leq\left.|a| u(x)\right|^{p-1}+\left.b(x)\right|^{q} \\
& \leq 2^{q}\left(a^{q}|u(x)|^{q(p-1)}+|b(x)|^{q}\right) \\
& =2^{q}\left(a^{q}|u(x)|^{p}+|b(x)|^{q}\right)
\end{aligned}
$$

But we know that $b \in L^{q}$ and that $u \in L^{p}$ Hence :

$$
\int_{\Omega}\left|N_{f} u\right|^{q} \leq(2)^{q} m e s(\Omega)\left(a^{q}\|u\|_{L^{p}}^{p}+\|b\|_{L^{q}}^{q}\right)<\infty
$$

since $\Omega$ is a bounded domain. Therefore $N_{f}$ maps $L^{p}(\Omega)$ into $L^{q}(\Omega)$

- We show that $N_{f}$ is continuous.

Suppose on the contrary that $N_{f}$ is not continuous at some $u_{0}$. then there would exist $\varepsilon_{0}>0$ and a sequence $\left(u_{n}\right)_{n \geq 1} \subset L^{p}$ such that

$$
\begin{equation*}
u_{n} \xrightarrow{L^{p}} u_{0} \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|N_{f} u_{n}-N_{f} u_{0}\right\|_{L^{q}}>\varepsilon_{0} . \tag{2}
\end{equation*}
$$

But (1) implies that there exists a subsequence $u_{n_{k}}$ of $u_{n}$, and a function $g \in L^{p}(\Omega)$ such that

$$
\begin{equation*}
u_{n_{k}} \xrightarrow{\text { pointwise }} u_{0} \text { a.e. } \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|u_{n_{k}}\right| \leq g \quad \forall k \tag{4}
\end{equation*}
$$

It would follow from (3) and the continuity of $f(x, \cdot)$ for a.e. $x$, that

$$
\begin{equation*}
N_{f} u_{n_{k}} \quad \xrightarrow{\text { pointwise a.e. }} \quad N_{f} u_{0} . \tag{5}
\end{equation*}
$$

Moreover, (4) implies

$$
\begin{equation*}
\mid f\left(x,\left.u\left(n_{n_{k}}(x)\right)|\leq a| u_{n_{k}}\right|^{p-1}+b(x) \leq a|g(x)|^{p-1}+b(x)\right. \tag{6}
\end{equation*}
$$

with $a|g|^{p-1}+b \in L^{q}$ since $g \in L^{p}(\Omega)$ and $b \in L^{q}(\Omega)$.
Thus, (5), (6) and the Dominated Convergence Theorem would imply that

$$
f\left(x, u_{n_{k}}(x)\right) \xrightarrow{L^{q}} f\left(x, u_{0}(x)\right)
$$

that is,

$$
\left\|N_{f} u_{n_{k}}-N_{f} u_{0}\right\|_{L^{q}} \longrightarrow 0
$$

contradicting (2).

## Example 2.3

If $p \in(1,+\infty)$ and $q:=\frac{p}{p-1}$; (the conjugate of $p$ ), then the Nemitskii operator

$$
\begin{array}{rll}
L^{p}(\Omega) & \longrightarrow L^{q}(\Omega) \\
u & \mapsto & |u|^{p-1}
\end{array}
$$

is continuous. The proof follow readily from Theorem 3.2 by putting $a \equiv 1$ and $b \equiv 0$.

## Theorem 2.3

Let $\Omega$ a bounded domain and $f: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ a Carathéodory function satisfying the growth condition

$$
|f(x, s)| \leq a|s|^{p-1}+b(x), \quad \text { for a.e. } x \text { and for all } s,
$$

with $p>1, a>0$ and $b(\cdot) \in L^{q}(\Omega)$; where $1 / p+1 / q=1$.
Define

$$
F(x, s)=\int_{0}^{s} f(x, t) d t \quad \text { for a.e. } x \in \Omega \text { and all } s \in \mathbb{R} .
$$

Then $F: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function and the associated Nemitskii operator

$$
\begin{aligned}
N_{F}: L^{p}(\Omega) & \longrightarrow L^{1}(\Omega) \\
u & \mapsto \quad N_{F} u=F(\cdot, u(\cdot))
\end{aligned}
$$

is continuous.
Moreover the functional $\mathcal{F}: L^{p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$
\mathcal{F}(u)=\int_{\Omega} F(x, u(x)) d x
$$

is continuously Frechet differentiable and $\mathcal{F}^{\prime}(u)=N_{f}$.

## Proof:

Step 1. Show that $F$ is carathéodory.
The function $f(x,$.$) is continuous for almost all x \in \Omega$, Hence $F(x,$.$) is continuous for almost all x \in \Omega$ since it is the antiderivative of a continuous function over a bounded interval .
For every $s$ and for almost every $x \in \Omega, \quad F(x, s)$ is a Riemann integral. Without loss of generality, by supposing $s>0$, we have

$$
F(x, s)=\int_{0}^{s} f(x, t) d t=\lim _{n \rightarrow \infty} F_{n}(x, s)
$$

where

$$
F_{n}(x, s)=\frac{s}{n} \sum_{k=0}^{n-1} f\left(x, \frac{k s}{n}\right) .
$$

Clearly $F_{n}(\cdot, s)$ is measurable as a result of the Carathéodory properties de $f$, and it follows that $F(\cdot, s)$ is measurable as the pointwise limit of a sequence of measurable functions.
step 2. Continuity of $N_{F}$.

For all $u, v \in L^{p}(\Omega)$ and for a.e. $x \in \Omega$

$$
\begin{aligned}
\left|N_{F} u(x)-N_{F} v(x)\right| & =\mid F(x, u(x))-F(x, v(x) \mid \\
& \leq\left(2^{p-1} a\left(|u(x)|^{p-1}+|v(x)|^{p-1}\right)+b(x)\right)|u(x)-v(x)| .
\end{aligned}
$$

Thus
$\int_{\Omega}|F(x, u(x))-F(x, v(x))| \leq \int_{\Omega} a 2^{p-1}|u|^{p-1}|u-v|+\int_{\Omega} a 2^{p-1}|v|^{p-1}|u-v|+\int_{\Omega} b|u-v|$
Using Hölder's Inequality we have :
$\int_{\Omega}|F(x, u(x))-F(x, v(x))| \leq a 2^{p / q}\|u\|_{L^{p}}^{p / q}\|u-v\|_{L^{p}}+a 2^{p / q}\|v\|_{L^{p}}^{p / q} \| u-$
$v \|_{L^{p}}+$
$\|b\|_{L^{p}}\|u-v\|_{L^{p}}=\left(a 2^{p / q}\left(\|u\|_{L^{p}}^{p / q}+\|v\|_{L^{p}}^{p / q}\right)+\|b\|_{L^{q}}\right)\|u-v\|_{L^{p}}$
Since the the term $a 2^{p / q}\left(\|u\|_{L^{p}}^{p / q}+\|v\|_{L^{p}}^{p / q}\right)+\|b\|_{L^{q}}$ is bounded by the fact
that $u, v \in L^{p}$ and $b \in L^{q}$ we have that our function $N_{F}$ is locally Lipschitz and so continuous.
step 3 . Differentiability of $\mathcal{F}$.
We prove that $\mathcal{F}$ is Fréchet differentiable by showing that it is Gâteaux differentiable on $L^{p}(\Omega)$ and has a continuous Gâteaux differential.
Let us fix arbitrarily $u, h \in L^{p}(\Omega)$ and define

$$
\varphi(t):=\mathcal{F}(u+t h)=\int_{\Omega} F(x, u(x)+\operatorname{th}(x)) d x .
$$

Set

$$
G(x, t)=F(x, u(x)+t h(x)) .
$$

Then
(i) The function $G(\cdot, 0)$ is integrable on $\Omega$.
(ii) For a.e. $x \in \Omega, \quad G(x, \cdot)$ is differentiable according to the definition of $F$ and we have:

$$
\begin{aligned}
\frac{\partial G}{\partial t}(x, t) & =\frac{\partial}{\partial t}(F(x, u(x)+t h(x))) \\
& =\frac{\partial F}{\partial s}(x, u(x)+\operatorname{th}(x)) \frac{d}{d t}(u(x)+\operatorname{th}(x)) \\
& =f(x, u(x)+\operatorname{th}(x)) h(x) .
\end{aligned}
$$

Moreover, it follows from the assumptions on $f$ that

$$
\left|\frac{\partial G}{\partial t}(x, t)\right| \leq g(x) \quad \text { for a.e. } x \in \Omega \text { and all } t \in(-1,1)
$$

where

$$
g=\left(2^{p-1} a\left(|u|^{p-1}+|h|^{p-1}\right)+b\right)|h| \in L^{1}(\Omega) .
$$

Consequently $\varphi$ is differentiable on $(-1,1)$ and in particular at 0 and

$$
\varphi^{\prime}(0)=\int_{\Omega} \frac{\partial G}{\partial t}(x, 0) d x=\int_{\Omega} f(x, u(x)) h(x) d x
$$

Therefore $\mathcal{F}$ is differentiable at $u$ in every direction $h \neq 0$, with directional derivative

$$
\mathcal{F}^{\prime}(u ; h)=\int_{\Omega} f(x, u(x)) h(x) d x=\int_{\Omega} N_{f} u h .
$$

And since $N_{f} u \in L^{q}(\Omega)$ by Theorem 3.2, it is clear that $\mathcal{F}^{\prime}(u ; \cdot)$ is a bounded linear map from $L^{p}(\Omega)$ into $\mathbb{R}$. Thus $\mathcal{F}$ is Gâteaux differentiable at $u$,

$$
D_{\mathrm{G}} \mathcal{F}(u) \in\left(L^{p}(\Omega)\right)^{\prime} \equiv L^{q}(\Omega)
$$

and

$$
D_{\mathrm{G}} \mathcal{F}(u)=N_{f} u .
$$

It follows that

$$
D_{\mathrm{G}} \mathcal{F}=N_{f}
$$

Moreover $N_{f}$ is continuous (by Theorem2.2) which means that $D_{G} \mathcal{F}$ is continuous.

The Gâteaux derivative of $\mathcal{F}$ being continous, we conclude by Theorem 3.2 that $\mathcal{F}$ is Frechet differentiable with Frechet derivative

$$
\mathcal{F}^{\prime}=N_{f} .
$$

## Example 2.4

Let $\Omega$ be a non empty bounded measurable subset of $\mathbb{R}^{N}$. Then the functional

$$
\begin{array}{rlc}
T: L^{2}(\Omega) & \longrightarrow & \mathbb{R} \\
u & \mapsto & \int_{\Omega} u^{2}(x) d x
\end{array}
$$

is Frechet differentiable and

$$
T^{\prime}(u)=2 u \quad \forall u \in L^{2}(\Omega) ;
$$

in the sense that

$$
T^{\prime}(u)(h)=2 \int_{\Omega} u(x) h(x) d x \quad \forall h \in L^{2}(\Omega) .
$$

The proof follows from Theorem 2.3 by considering

$$
f(x, s)=s, \quad x \in \Omega, s \in \mathbb{R}
$$

$p=2, \quad a=1, b=0$

## CHAPTER 3

## Variational Principles and Minimization

### 3.1 Lower Semicontinuous Functions

Definition 3.1 Let $X$ be a Banach space and $f: X \rightarrow \mathbb{R}$ a functional bounded from below. A sequence $\left(a_{n}\right)_{n}$ of elements of $X$, is said to be a minimizing sequence if

$$
\lim _{n} f\left(a_{n}\right)=\inf _{x \in X} f(x)
$$

The functional $f: X \rightarrow \mathbb{R}$ is said to be semi-continuous (respectively weakly lower semi - continuous) if whenever $\lim _{n} x_{n}=x$ strongly (respectively weakly), it follows that

$$
\liminf _{n} f\left(x_{n}\right) \geq f(x)
$$

Or equivalently , $f$ is lower semi-continuous if its epigraf is closed. Where

$$
\operatorname{epigraf}(f)=\operatorname{epi}(f)=\{(x, y) \mid x \in X, y \in \mathbb{R} \text { and } y \geq f(x)\}
$$

## Some Properties of Lower Semicontinuous functions

## Proposition 3.1

1. The sum of two l.s.c (w.l.s.c.) functionals is a l.s.c (w.l.s.c.) functional.
2. The product of a positive constant and of a l.s.c (w.l.s.c.) functional is a l.s.c ( w.l.s.c.) functional.
3. If $\left(f_{j}\right)_{j}$ is a family of l.s.c (w.l.s.c.) functionals, then the function $\sup _{j} f_{j}$ is a l.s.c (w.l.s.c.) functional.

## Proof.

Let $\left(x_{n}\right)_{n}$ be a sequence of elements of $X$ such that $x_{n} \rightarrow x$ strongly in $X$.

1. Let $f$ and $g$ be two l.s.c functionals defined on a Banach $X$.

$$
\begin{aligned}
\liminf _{n}(f+g)\left(x_{n}\right) & =\liminf _{n}\left(f\left(x_{n}\right)+g\left(x_{n}\right)\right) \\
& \geq \liminf _{n} f\left(x_{n}\right)+\liminf _{n} g\left(x_{n}\right) \quad \text { (by subadditivity of liminf), } \\
& \geq f(x)+g(x) \quad \text { by l.s.c. of } f \text { and } g) .
\end{aligned}
$$

2. Let $f$ be l.s.c and $c>0$. Then by the property of liminf we have

$$
\liminf _{n}(c f)\left(x_{n}\right)=c\left(\liminf _{n} f\left(x_{n}\right)\right) \geq c f(x)
$$

and so $c f$ is l.s.c.
3. Let $\left\{f_{j}\right\}_{j \in J}$ be a family of 1.s.c functions. Then

$$
\sup _{j \in J} f_{j}\left(x_{n}\right) \geq f_{i}\left(x_{n}\right) \text { for all } i \in J \text { and all } n .
$$

Thus, for each $i \in J$, we have

$$
\begin{aligned}
\liminf _{n}\left(\sup _{j} f_{j}\left(x_{n}\right)\right) & \geq \liminf _{n} f_{i}\left(x_{n}\right) \\
& \geq \liminf _{n} f_{i}(x) \quad\left(\text { by l.s.c. of } f_{i}\right) .
\end{aligned}
$$

And so

$$
\liminf _{n}\left(\sup _{j} f_{j}\left(x_{n}\right)\right) \geq \sup _{i \in J} f_{i}(x)
$$

## Theorem 3.1:

Let $f: X \rightarrow \mathbb{R}$ be a functional on a Banach $X$. If $f$ is convex and strongly lower semi-continuous, then $f$ is weakly lower semicontinuous.

## Theorem 3.2:

Let $f: X \rightarrow \mathbb{R}$ be a weakly lower semi-continuous functional on a reflexive Banach space $X$ with a bounded minimizing sequence. Then $f$ has a minimum on $X$.

## Proof:

Suppose that $f$ is lower semi-continuous and has a bounded minimizing sequence $\left(x_{n}\right)_{n}$. Then by the reflexivity of $X$ and according to Eberlein-Smulyan Theorem, $\left(x_{n}\right)_{n}$ has a weakly convergent subsequence $\left(x_{n_{k}}\right)_{k}$ which converges weakly to some $a \in X$. And by the lower semi-continuity of $f$, we have that

$$
\liminf _{k \rightarrow \infty} f\left(x_{n_{k}}\right) \geq f(a)
$$

Moreover, since $\left(x_{n}\right)_{n}$ is a minimizing sequence of $f$, we have

$$
\liminf _{n} f\left(x_{n}\right)=\inf _{x \in X} f(x) \leq f(a) .
$$

It follows that

$$
f(a)=\inf _{x \in X} f(x) .
$$

## Theorem 3.3:

Let $f: X \rightarrow \mathbb{R}$ be a weakly lower semi-continuous functional bounded from below on on the reflexive Banach space X . If $f$ is coercive, then $f$ has a minimum on $X$.

## Proof:

According to Theorem 3.2, it suffices now to prove that $f$ is bounded below and has a bounded minimizing sequence.
Since $f$ is coercive, we have

$$
\lim _{\|\mathbf{x}\| \rightarrow \infty} f(x)=\infty
$$

Besides $\inf _{x \in X} f(x) \in \mathbb{R}$ since $f$ is a real-valued functional which is bounded below. Moreover by property of the infimum, there exists a sequence $\left(x_{n}\right)_{n}$ of elements of $X$ such that $\left(f\left(x_{n}\right)\right)_{n}$ converges to $\inf f(x)$ as $n \rightarrow+\infty$, which means that $\left(x_{n}\right)_{n}$ is a minimizing
sequence.
We show that this minimizing sequence is bounded.
Suppose on the contrary that $\left(x_{n}\right)_{n}$ is not (norm) bounded. Then

$$
\forall k \in \mathbb{N}, \quad \exists n_{k} \text { such that }\left\|x_{n_{k}}\right\|>k .
$$

It would follow that

$$
\lim _{k}\left\|x_{n_{k}}\right\|=+\infty
$$

and so

$$
\lim _{k} f\left(x_{n_{k}}\right)=+\infty \quad(\text { by coercivity of } f)
$$

in contrast to the fact that

$$
\lim _{n} f\left(x_{n}\right)=\inf _{x \in X} f(x) \in \mathbb{R}
$$

### 3.2 Ekeland Theorem in Complete metric space

Let $M$ be a complete metric space and $\Phi: M \mapsto \mathbb{R}$ a lower semicontinuous functional, bounded below. If $\left(u_{j}\right)_{j}$ is a minimizing sequence, then

$$
\forall \varepsilon>0, \exists j_{0}: \quad \forall j>j_{0}, \quad \Phi\left(u_{j}\right)<\inf _{x \in \Phi} \Phi(x)+\varepsilon
$$

This fact motivates the definition of an $\varepsilon$-minimum point $u$ of $\Phi$, as a point satisfying :

$$
\Phi(u)<\inf _{x \in \Phi} \Phi(x)+\varepsilon .
$$

Theorem 3.4: ( Ekeland Principle, strong form, 1979).
Let M be a complete metric space and $\Phi: M \mapsto \mathbb{R}$ be a lower semi-continuous functional which is bounded from below. Let $k>$ $1, \varepsilon>0$ and $u \in M$ be an $\varepsilon$-minimum point of $\Phi$. Then there exists $v \in M$ such that:

$$
\begin{align*}
& \Phi(v) \leq \Phi(u)  \tag{1.7}\\
& d(u, v) \leq \frac{1}{k}  \tag{1.8}\\
& \Phi(v)<\Phi(w)+\varepsilon k d(w, v) \forall w \neq v . \tag{1.9}
\end{align*}
$$

## Proof:

Denote for simplicity $d_{k}(u, v):=k d(u, v)$ and define a partial ordering in M by :

$$
u \prec v \quad \Leftrightarrow \quad \phi(u) \leq \phi(v)-\varepsilon d_{k}(u, v) .
$$

Therefore we have

$$
\begin{array}{llll}
u \prec u, \quad \forall u \in M & & \text { (reflexivity) } \\
(u \prec v \text { and } v \prec u) \quad \Rightarrow & u=v & \text { (anti-symmetry) } \\
(u \prec v \text { and } v \prec w) \quad \Rightarrow \quad u \prec w & \text { (transitivity). }
\end{array}
$$

We prove only transitivity:
$(i) \Longrightarrow \phi(u)<\phi(v)-\varepsilon d_{k}(u, v)$
$\left(i^{\prime}\right) \Longrightarrow \phi(v)<\phi(w)-\varepsilon d_{k}(v, w)$
Substituting (i') in (i) we get

$$
\begin{aligned}
\phi(u) & \leq \phi(v)-\varepsilon d_{k}(u, v) \\
\leq \phi(w) & -\varepsilon\left(d_{k}(v, w)-d_{k}(v, u)\right) \\
\leq \phi(w) & -\varepsilon d_{k}(u, w) \text { by triangular inequality. }
\end{aligned}
$$

Now define a sequence of subsets $\left(S_{n}\right)_{n}$ s.t.:
Let $u_{1}=u$ and

$$
S_{1}:=\left\{w \in M: w \prec u_{1}\right\} .
$$

Construct inductively a sequence $\left(u_{n}\right)_{n}$ as follows :

$$
\begin{gathered}
u_{2} \in S_{1}, \quad \phi\left(u_{2}\right) \leq \inf _{S_{1}} \phi+\frac{\varepsilon}{2^{2}} \\
S_{2}=\left\{w \in M: w \prec u_{2}\right\}, \\
u_{n+1} \in S_{n}, \quad \phi\left(u_{n+1}\right) \leq \inf _{S_{n}} \phi+\frac{\varepsilon}{2^{n+1}} . \\
S_{n}=\left\{w \in M: w \prec u_{n}\right\} .
\end{gathered}
$$

Then we have :
$S_{1} \supset S_{2} \supset \ldots \supset S_{n} \supset \ldots$
$u_{1} \succ u_{2} \succ \ldots \succ u_{n} \succ \ldots$
$\left(\star_{1}\right)$ Each $S_{n}$ is closed: Indeed
let $v_{j} \in S_{n}$ and $\lim _{j} v_{j}=v \in M$ which means $\phi\left(v_{j}\right) \leq \phi\left(u_{n}\right)-\varepsilon d_{k}\left(v_{j}, u_{n}\right)$.
Letting $j \rightarrow \infty$, by the lower semicontinuity of $\phi$ and continuity of the distance $d_{k}$, we get
$\phi(v) \leq \phi\left(u_{n}\right)-\varepsilon d_{k}\left(v, u_{n}\right)$, which means that $v \in S_{n}$
$\left(\star_{2}\right) \lim _{n \rightarrow \infty} \operatorname{diam} S_{n}=0$.
Indeed, let $w \in S_{n}$ then $\phi(w) \leq \phi\left(u_{n}\right)-\varepsilon d_{k}\left(u_{n}, w\right)$
Also $w \in S_{n} \subset S_{n-1}$ so
$\phi\left(u_{n}\right) \leq \inf _{S_{n-1}} \phi+\frac{\varepsilon}{2^{n}} \leq \phi(w)+\frac{\varepsilon}{2^{n}}$ and
$\phi\left(u_{n}\right)-\frac{\varepsilon}{2^{n}} \leq \phi(w) \leq \phi\left(u_{n}\right)-\varepsilon d_{k}\left(u_{n}, w\right)$
which implies that $d_{k}\left(u_{n}, w\right) \leq \frac{1}{2^{n}} \forall w \in S_{n}$
For $w_{1}$ and $w_{2} \in S_{n}$ and triangular inequality we have :
$d_{k}\left(w_{1}, w_{2}\right) \leq d_{k}\left(w_{1}, u_{n}\right)+d_{k}\left(w_{2}, u_{n}\right) \leq \frac{1}{2^{n-1}} \rightarrow 0$ asn $\rightarrow \infty$
From $\left(\star_{1}\right) \operatorname{and}\left(\star_{2}\right)$ and the Nested Interval Property, $\exists$ a unique $v \in M$ s.t.
$\bigcap_{n=1}^{\infty} S_{n}=\{v\}$
We prove that v satisfies (1.7)-(1.9).
Since $v \in S_{1}$ and $v \prec u_{1}=u$ it follows that
$\phi(v) \leq \phi(u)-\varepsilon d_{k}(u, v) \leq \phi(u)$ which is)
Let $w \neq v$. If $w \prec v$ it follows $w \in \bigcap_{n=1}^{\infty} S_{n}$ and then $w=v$.
Therefore $w \nprec v$ that is
$\phi(w)>\phi(v)-\varepsilon d_{k}(w, v)$, which is (1.9).
Finally, by $\lim _{n} u_{n}=v$ and
$d_{k}\left(u_{n}, u\right) \leq \sum_{j=1}^{n-1} d_{k}\left(u_{j}, u_{j+1}\right) \leq \sum_{j=1}^{n-1} \frac{1}{2^{j}} \leq 1$
which implies that $d_{k}(u, v) \leq 1$

Corollary 3.1: (Ekeland principle, weak form).
Let $(M, d)$ be a complete metric space and $\Phi: M \mapsto \mathbb{R}$ be a lower semi-continuous functional bounded from below. Then $\forall \varepsilon>0$ there exists an $\varepsilon^{\iota}$ minimum point of $\Phi, v \in M$ such that:
$\Phi(v)<\Phi(w)+\varepsilon d(w, v), \quad \forall w \in M, \quad w \neq v$.

## Proof:

We show that there exists an $\varepsilon^{乞}$-minimum point $v$.
Let $\varepsilon>0$ and set

$$
\varepsilon_{o}=\min \left\{\frac{1}{2}, \varepsilon^{2}\right\}
$$

so that $0<\varepsilon_{o}<1$ and $\varepsilon_{o} \leq \varepsilon$. Then by the property of the infimum

$$
\exists u_{o} \in X \text { suchthat } f\left(u_{o}\right)<\inf _{X} \Phi+\varepsilon_{o}^{2} .
$$

And so $u_{o}$ is an $\varepsilon_{o}^{2}$-minimum point of $\Phi$. Let

$$
\kappa_{o}:=\frac{1}{\varepsilon_{o}}>1
$$

swe can apply Theorem 3.4 (1.9) to get the existence of some $v \in X$ such that

$$
\begin{gather*}
\Phi(v)<\Phi(w)+\varepsilon_{o}^{2} \kappa_{o} d(w, v), \quad \forall w \neq v .  \tag{1}\\
\Longrightarrow \Phi(v)<\Phi(w)+\varepsilon_{o} d(w, v), \quad \forall w \neq v, \text { since } \kappa_{o}:=\frac{1}{\varepsilon_{o}} \\
\Longrightarrow \Phi(v)<\Phi(w)+\varepsilon d(w, v), \quad \forall w \neq v, \text { since } \varepsilon_{0}<\varepsilon
\end{gather*}
$$

Also from Theorem3.4 we have that

$$
\Phi(v) \leq \Phi\left(u_{0}\right)<\inf _{x} \Phi(x)+\left(\varepsilon_{0}\right)^{2}<\inf _{x} \Phi(x)+\varepsilon
$$

which implies that v is an epsilon minimum point of $\Phi$

Corollary 3.2: Let $(M, d)$ be a complete metric space, $\Phi: M \mapsto \mathbb{R}$ be lower semi-continuous functional bounded from below. Let $\varepsilon>0$ and $u \in M$ be an $\varepsilon^{\breve{ }}$ minimum point of $\Phi$. Then there exists $v \in M$ such that:
$\Phi(v) \leq \Phi(u)$

```
\(d(v, u) \leq \sqrt{\varepsilon}\)
\(\Phi(v)<\Phi(w)+\sqrt{\varepsilon} d(v, w), \forall w \neq v\)
```


## Proof:

It follows from theorem 3.4 by taking $\kappa=\frac{1}{\sqrt{\varepsilon}}$ for $\varepsilon<1$

### 3.3 Palais-Smale Conditions and Minimization

Minimizing sequences for differentiable functionals are convergent under cer- tain compactness conditions. We shall use later the so called PalaisâSmale ( (PS) for short) conditions.
Let $X$ be a Banach space, $f: X \mapsto \mathbb{R}$ be a differentiable functional.
Definition 3.2: (3. Palais)
A $\mathcal{C}^{1}$-functional $f: X \longrightarrow \mathbb{R}$ satisfies the PalaisSmale $(P S)$ condition if every sequence $\left(x_{j}\right)_{j}$ in $X$ such that

$$
\left.f\left(x_{j}\right)_{j} \text { is }\right\} \text { bounded and } \lim _{\mathrm{j}} \mathrm{f}^{\prime}\left(\mathrm{x}_{\mathrm{j}}\right)=0 \text { in } \mathrm{X}
$$

has a convergent subsequence.
From (PS) condition, it follows that the set of critical points for bounded functional is compact. A variant of (PS) condition, noted as (PS)c, was introduced by Brezis, Coron and Nirenberg [BCN].

Definition 3.3: (Brezis, Coron, Nirenberg, 1980).
Let $c \in \mathbb{R}$. A $C^{1}$-functional and $f: X \longrightarrow \mathbb{R}$ satisfies the (PS)c condition if every sequence $\left(x_{j}\right)_{j}$ in X such that $\lim _{j} f\left(x_{j}\right)=c$ and $\lim _{j} f^{\prime}\left(x_{j}\right)=$ 0 in X has a convergent subsequence.

Note: PS condition implies (PS)c condition since in $\mathbb{R}$ every convergent sequence is bounded.

Theorem 3.5: Let $f: X \longrightarrow \mathbb{R}$ be a $\mathcal{C}^{1}$ functional bounded below. Then for each $\varepsilon>0$ and $x \in X$ such that $f(x) \leq \inf _{X} f+\varepsilon$ $\exists y \in X$ such that
(1) $f(y) \leq f(x)$
(2) $\|x-y\| \leq \sqrt{\varepsilon}$
(3) $\left\|f^{\prime}(y)\right\| \leq \sqrt{\varepsilon}$

Proof:
Apply Corollary 3.2 with $M=X$ and $f=\Phi$. Then we get the
existence of $y$ such that

$$
f(z)>f(y)-\sqrt{\varepsilon}\|y-z\|, \forall z \neq y
$$

let $z=y+$ th and $h \in X$ with $\|h\|=1, t>0$. Then we have $f(y+t h)-f(y) \geq-\sqrt{\varepsilon} t$.
Letting $t \rightarrow 0$ we get $f^{\prime}(y)(h) \geq-\sqrt{\varepsilon}$.
Applying the above inequality to $-h$, we have $f^{\prime}(y)(h) \leq \sqrt{\varepsilon}$. Combining this with the other inequality, we have $\left|f^{\prime}(y)(h)\right| \leq \sqrt{\varepsilon}$. Thus

$$
\left\|f^{\prime}(y)\right\|=\sup \left\{\left|f^{\prime}(y)(h)\right|,\|y\| \leq 1\right\} \leq \sqrt{\varepsilon}
$$

Corollary 3.3: Let $f: X \mapsto \mathbb{R}$ be a $C^{1}$ functional bounded from below and $\left(x_{j}\right)_{j}$ be a minimizing sequence. Then there exists another minimizing sequence $\left(y_{j}\right)_{j}$ such that

$$
\begin{aligned}
& f\left(y_{j}\right) \leq f\left(x_{j}\right), \\
& \lim _{j}\left\|x_{j}-y_{j}\right\|=0 \\
& \lim _{j}\left\|f^{\prime}\left(y_{j}\right)\right\|=0 .
\end{aligned}
$$

## Proof:

$f$ bounded below implies that $\inf f=c \in R . \lim _{j} f\left(x_{j}\right)=c$ implies by property of infimum and definition of limits that, $\exists j_{0}$ such that. $j>$ $j_{0} \Longrightarrow f\left(u_{j}\right)<\inf f+\frac{1}{j^{2}}$
So by Theorem $3.4 \exists v_{j}$ s.t.

$$
\begin{align*}
& f\left(v_{j}\right) \leq f\left(u_{j}\right) \\
& \left\|u_{j}-v_{j}\right\| \leq \frac{1}{j}  \tag{1}\\
& \left\|f^{\prime}\left(v_{j}\right)\right\| \leq \frac{1}{j}
\end{align*}
$$

Taking limit as j tends to infinity on both sides of (1) and (2) we get the result.

## Theorem 3.6:

Let $f: X \rightarrow \mathbb{R}$ be $C^{1}$ bounded below, X a Banach space, and let $c=\inf f$.Assume that $f$ satisfies $(P S)_{c}$ then c is achieved at some $x_{0} \in X$ and $f^{\prime}\left(x_{0}\right)=0$

## Proof:

f bounded below implies that $c=\inf f \in \mathbb{R}$ and by definition of infimum $\exists$ a minimizing sequence.
By corollary $3.3 \exists$ another minimizing sequence $\left(y_{n}\right)_{n}$ s.t.
(1) $f\left(y_{n}\right)<f\left(x_{n}\right)$
(2) $\lim _{n}\left\|x_{n}-y_{n}\right\|=0$
(3) $\lim _{n}\left\|f^{\prime}\left(y_{n}\right)\right\|=0$

Thus $\lim _{n} f\left(y_{n}\right)=c \ldots(*)$ and $\lim \left\|f^{\prime}\left(y_{n}\right)\right\|=0$
So by (PS)c condition we have that $\left(y_{n}\right)_{n}$ has a convergent subsequence $y_{n_{k}}$ which converges to say $x_{0} \in X$
Now by (3) and continuity of $f^{\prime}$ and of $\|$.$\| we have that$
$\left\|f^{\prime}\left(x_{0}\right)\right\|=0 \Longleftrightarrow f^{\prime}\left(x_{0}\right)=0$ by properties of norm.
By $(*)$ and the continuity of f we have that $f\left(x_{0}\right)=c$

### 3.4 Deformation Theorem and Palais-Smale Conditon

Let $f \in C^{1}(X, \mathbb{R})$ be a functional defined on the open subset X in the Banach space E . We introduce the following notations
$K=\left\{x \in E: f^{\prime}(x)=0\right\}, K_{c}=\{x \in K(f): f(x)=c\}$,
$f^{c}=\{x \in E: f(x) \leq c\}, f_{c}=\{x \in E: f(x) \geq c\}$,
$f_{a}^{b}=\{x \in E: a \leq f(x) \leq b\}=f_{a} \cap f^{b}$,
$B_{\rho}=\{x \in E:\|x\| \leq \rho\}, S_{\rho}=\{x \in E:\|x\|=\rho\}$.and
$d(x, F)=\inf \{\|x-y\|, y \in F\}, F_{\delta}=\{x \in E: d(x, F)<\delta\}$, where F is a closed set in E .

## Definition 3.4

A continuous mapping $\eta(t, x):[0,1] \times X \longrightarrow X$ is said to be a homotopy of homeomorphisms if for all $t \in[0,1]$
(1) $\eta(t, \cdot): X \mapsto X$ is a homeomorphism,
(2) $\eta(0, x)=x, \forall x \in X$

The homotopy $\eta$ is $f$-decreasing (respectively $f$-increasing) if whenever $0 \leq t_{1} \leq t_{2} \leq 1$ then
$f\left(\eta\left(t_{1}, x\right)\right) \geq f\left(\eta\left(t_{2}, x\right)\right),\left(f\left(\eta\left(t_{1}, x\right)\right) \leq f\left(\eta\left(t_{2}, x\right)\right)\right) \forall x \in X$.

## Theorem 3.7

Let $f \in C^{1}(X, \mathbb{R})$ and $F$ and $G$ be closed disjoint subsets of $X$.
Let $c \in \mathbb{R}, \varepsilon$ and $\delta>0$ be numbers such that $F_{2 \delta} \cap G=\varnothing$ and $x \in f^{-1}[c-\varepsilon, c+\varepsilon] \cap F_{2 \delta} \Longrightarrow\|f(x)\| \geq 4 \frac{\varepsilon}{\delta}$

Then there exists a $f$-decreasing homotopy of homeomorphisms $\eta:[0,1] \times X \rightarrow X$ such that:
(1) $\eta(t, x)=x$ if either $x \in G$ or $|f(x)-c| \geq 2 \varepsilon$,
(2) $\eta\left(1, f^{c+\varepsilon} \cap F\right) \subset f^{c-\varepsilon} \cap F_{2 \delta}$
(3) $\|\eta(t, x)-x\| \leq 2 \delta t$.

Proof: Consider the sets :
$A:=\{x:|f(x)-c| \geq 2 \varepsilon\} \cup\left\{x: f(x) \leq \frac{2 \varepsilon}{\delta}\right\} \cup G$,
$B:=f^{-1}[c-\varepsilon, c+\varepsilon] \cap F_{2 \delta}$,
Define the function

$$
\chi(x)=\frac{d(x, A)}{d(x, A)+d(x, B)} .
$$

Claim. There exists a locally Lipschitz mapping (called a pseudogradient vector field of $f$ ) $V: X \backslash K \mapsto E$ with the property:
(i) $\|V(x)\| \leq 2\left\|f^{\prime}(x)\right\|$,
(ii) $\left\|f^{\prime}(x)\right\|^{2} \leq\left\langle f^{\prime}(x), V(x)\right\rangle$
. Proof of the claim.
Given $x \in X \backslash K$, there exists, by definition of the canonical norm in $E, w_{x} \in E$ such that $\left\|w_{x}\right\|=1$ and

$$
\left\langle f^{\prime}(x), w_{x}\right\rangle>\frac{2}{3}\left\|f^{\prime}(x)\right\| .
$$

Let $V_{x}=\frac{3}{2}\left\|f^{\prime}(x)\right\| \omega_{x}$ Then:

$$
\begin{gathered}
\left\|V_{x}\right\|<\beta\left\|f^{\prime}(x)\right\|, \\
\left\langle f^{\prime}(x), V_{x}\right\rangle>\left\|f^{\prime}(x)\right\|^{2} .
\end{gathered}
$$

Since $f^{\prime}$ is continuous there exists an open neighborhood $U_{x}$ of $x$ such that for every $y \in U_{x}$,

$$
\begin{align*}
& \left\|V_{x}\right\| \leq 2\left\|f^{\prime}(y)\right\|  \tag{1}\\
& \left\langle f^{\prime}(y), V_{x}\right\rangle \geq\left\|f^{\prime}(x)\right\|^{2} \tag{2}
\end{align*}
$$

The family $\left\{U_{x}: x \in X \backslash K\right\}$ is an open covering of $X \backslash K$ and let $\left\{W_{i}\right\}_{i}$ be a locally finite refinement of $\left\{U_{x}\right\}_{x}$ and $p_{i}(x)=d(x, X \backslash$ $\left.W_{i}\right)$. For each i, let $x_{i}$ be such that $W_{i} \subset U_{x_{i}}$ and put $V_{i}=V_{x_{i}}$. The function $p_{i}(x)$ is Lipschitz continuous and $p_{i}(x)=0$ if $x \in W_{i}$. The sum $\sum_{i} p_{i}(x)$ is locally finite and $\sum_{i} p_{i}(x)>0$ for all $x \in X \backslash K$. Define for $x \in X \backslash K$

$$
V(x)=\frac{1}{\sum_{i} p_{i}} \sum_{j} p_{j} V_{j}
$$

The mapping $V: X \backslash K \mapsto E$ is a locally Lipschitz continuous and since $V(x)$ is a convex combination of vectors satisfying (1) and (2) we have

$$
\begin{aligned}
\|V(x)\| & \leq \frac{1}{\sum_{i} p_{i}} \sum_{i} p_{i}\left\|V_{i}\right\| \leq \beta\left\|f^{\prime}(x)\right\| \\
\left\langle f^{\prime}(x), V(x)\right\rangle & =\frac{1}{\sum_{i} p_{i}} \sum_{j} p_{j}\left\langle f^{\prime}(x), V_{j}\right\rangle \geq \alpha\left\|f^{\prime}(x)\right\|^{2}
\end{aligned}
$$

Now we are ready to prove Theorem 3.7.
Let $g(x)=\chi(x) V(x)$ and consider the Cauchy problem:

$$
\begin{equation*}
\sigma^{\prime}(t)=g(\sigma(t)), \quad \sigma(0)=x \tag{3.7.1}
\end{equation*}
$$

for every $x \in X$. We have $g(x)=0$ if $x \in A$. If $x \notin A$ and $x \in X \backslash K$, then

$$
\begin{equation*}
\|g(x)\| \leq\|V(x)\| \leq \frac{2}{\left\|f^{\prime}(x)\right\|} \leq \frac{\delta}{\varepsilon} \tag{3.7.2}
\end{equation*}
$$

By the fundamental existence-uniqueness theorem for ordinary differential equations in Banach spaces the problem (3.7.1) has a unique solution $\sigma(., x): \mathbb{R}^{+} \times X \mapsto E$ and $\sigma(t,):. X \mapsto X$ is a homeomorphism.

The homotopy $\sigma(\cdot, \cdot)$ is f -decreasing because:

$$
\begin{aligned}
\frac{d}{d t} f(\sigma(t, x)) & =<f^{\prime}(\sigma(t, x)), \sigma^{\prime}(t, x)> \\
& =-\chi(\sigma(t, x))<f^{\prime}(\sigma(t, x)), V(\sigma(t, x))>
\end{aligned}
$$

$$
\leq-\chi(\sigma(t, x)) \leq 0
$$

Let

$$
\eta(t, x)=\sigma(2 \varepsilon t, x)
$$

If $x \in A$, we have $\chi(x)=0$ and so we have $g(x)=0$. Thus the Cauchy problem has solution $\sigma(t, x) \equiv x$ and it follows that $\eta(t, x)=x$ if $x \in G$ or $|f(x)-c| \geq 2 \varepsilon$. Hence (1) is proved.
We get (3) by (3.7.2)

$$
\begin{aligned}
\|\eta(t, x)-x\| & =\|\sigma(2 \varepsilon t, x)-\sigma(0, x)\| \\
& \leq \int_{0}^{2 \varepsilon t}\left\|\sigma^{\prime}(s)\right\| d s=\int_{0}^{2 \varepsilon t}\|g(\sigma(s))\| d s \\
& \leq \frac{\delta}{\varepsilon} 2 \varepsilon t=2 \delta t
\end{aligned}
$$

Next we prove (2).
From(3) it follows that $\eta(t, F) \subset F_{2 \delta}$ for all $t \in[0,1]$. Let $x \in$ $f^{c+\varepsilon} \cap F$. If there exists $t_{0} \in[0,1]$ such that $f\left(\sigma\left(2 \varepsilon t_{0}, x\right)\right) \leq c-\varepsilon$, then

$$
f(\sigma(2 \varepsilon, x)) \leq f\left(\sigma\left(2 \varepsilon t_{0}, x\right)\right) \leq c-\varepsilon
$$

since $\sigma$ is $f$-decreasing, and the assertion follows.
If otherwise $f(\sigma(2 \varepsilon t, x))>c-\varepsilon$ for every $t \in[0,1]$, then by using the fact that $c+\varepsilon \geq f(x)=f(\sigma(0, x)) \geq f(\sigma(2 \varepsilon, x))>c-\varepsilon$, we get $\sigma(2 \varepsilon, x) \in f^{-1}[c-\varepsilon, c+\varepsilon] \cap F_{2 \delta}=B$. Then

$$
\begin{aligned}
f(\sigma(2 \varepsilon, x))-f(x) & =\int_{0}^{2 \varepsilon} \frac{d}{d s} f(\sigma(s, x)) d s \\
& =\int_{0}^{2 \varepsilon}<f^{\prime}(\sigma(s, x)), \sigma^{\prime}(s, x)>d s \\
& =-\int_{0}^{2 \varepsilon}<f^{\prime}(\sigma(s, x)), V(\sigma(s, x))>d s \leq-2 \varepsilon
\end{aligned}
$$

and so $c-\varepsilon<f(\sigma(2 \varepsilon, x)) \leq f(x)-2 \varepsilon \leq c+\varepsilon-2 \varepsilon=c-\varepsilon$, which is a contradiction.

## Corollary 3.4:

Let $f$ satisfies (P S)c condition and $K_{c}=\varnothing$ Then there exist $\varepsilon>0$ and
an $f$-decreasing homotopy of homeomorphisms $\eta:[0,1] \times X \rightarrow X$ such that:
(1) $\eta(t, x)=x$ if $|f(x)-c| \geq 2 \varepsilon$,
(2) $\eta\left(1, f^{c+\varepsilon}\right) \subset f^{c-\varepsilon}$.

## Proof:

By (PS)c condition there exist $\varepsilon_{0}, \beta>0$ such that $|f(x)-c| \leq$ $\varepsilon_{0} \Longrightarrow\left\|f^{\prime}(x)\right\| \geq \beta$. Otherwise, there exists a sequence $\left(x_{j}\right)_{j}$ such that

$$
\left|f\left(x_{j}\right)-c\right| \leq \frac{1}{j} \text { and }\left\|f^{\prime}\left(x_{j}\right)\right\| \leq \frac{1}{j}
$$

By (PS)c condition it will follow that c is a critical value, which contradicts to $K_{c}=\varnothing$.
Let $\delta>0$ and $\varepsilon \in\left(0, \min \left(\varepsilon_{0}, \frac{\beta \delta}{4}\right)\right)$.
So,$|f(x)-c| \leq \varepsilon \Longrightarrow\left\|f^{\prime}(x)\right\| \geq \beta>\frac{4 \varepsilon}{\delta}$.
The assertion follows from Theorem 4.6 taking $G=\varnothing$ and $\mathrm{F}=$ X.

### 3.5 Mountain-Pass Theorem

In critical point theory, minimax theorems characterize a critical value $c$ of a functional $f: X \mapsto \mathbb{R}$ as a minimax over a suitable class of sets $\mathcal{A}$

$$
c=\inf _{A \in \mathcal{A}} \max _{x \in A} f(x)
$$

Theorem 3.8: (Ambrosetti and Rabinowitz, 1973).
Let X be a real Banach space and $f \in C^{1}(X, \mathbb{R})$. Suppose that $f$ satisfies (PS) condition, $f(0)=0$ and
(i) there exist constants $\rho>0$ and $\alpha>0$ such that $f(x) \geq \alpha$ if $\|x\|=\rho$,
(ii) there is $e \in X,\|e\|>\rho$, such that $f(e) \leq 0$.

Then f has a critical value $c \geq \alpha$ which can be characterized as

$$
\begin{equation*}
c=\inf _{\gamma \in \Gamma} \max _{t \in[0,1]} f(\gamma(t)) \tag{3.8.1}
\end{equation*}
$$

Where

$$
\begin{equation*}
\Gamma=\{\gamma \in C([0,1], X): \gamma(0)=0, \gamma(1)=e\} \tag{3.8.2}
\end{equation*}
$$

Geometrically, when $X=\mathbb{R}^{2}$ the assumptions (i) and (ii) mean that the origin lies in a valley surrounded by a âmountainâ

$$
\Gamma_{f}=\left\{(x, f(x)) \in \mathbb{R}^{3}: x \in \mathbb{R}^{2}\right\}
$$

So, there must exist a mountain pass joining $(0,0)$ and $(e, f(e))$ that contains a critical value.
Note that (PS) condition is essential in Theorem 3.8 as the following example shows.

Example 4.1 The function $h(x, y)=x^{2}+(x+1)^{3} y^{2}$ satisfies assumptions (i) and (ii) of Theorem 3.8 but does not satisfy (PS) condition and its unique critical point is $(0,0)$.

## Proof.

The point $(0,0)$ is a strict local minima and the unique critical point. If (PS) condition is satisfied then (PS)c, with $c>0$ defined by (3.8.1), is also satisfied. Let $\left(x_{j}, y_{j}\right)_{j}$ be a sequence such that:

$$
\begin{align*}
& \lim _{j} x_{j}^{2}+\left(x_{j}+1\right)^{3} y_{j}^{2}=c>0  \tag{1.40}\\
& \lim _{j} 2 x_{j}+3\left(x_{j}+1\right)^{2} y_{j}^{2}=0 \\
& \lim _{j} 2\left(x_{j}+1\right)^{3} y_{j}=0 .
\end{align*}
$$

Suppose that $\lim _{j}\left(x_{j}, y_{j}\right)=\left(x_{0}, y_{0}\right)=(0,0)$. Passing to the limit in (1) we obtain:

$$
\begin{gathered}
x_{0}^{2}+\left(x_{0}+1\right)^{3} y_{0}^{2}=c>0, \\
2 x_{0}+3\left(x_{0}+1\right)^{2} y_{0}^{2}=0, \\
2\left(x_{0}+1\right)^{3} y_{0}=0,
\end{gathered}
$$

which is a contradiction.

## Proof of Theorem 3.8

Suppose by contradiction that $K_{c}=$. Take $\varepsilon$ such that $0<\varepsilon<\frac{\alpha}{2}$. From (i) and (ii) we have $c \geq \alpha>2 \varepsilon$ and let $\gamma \in \Gamma$ be such that

$$
\begin{equation*}
\max _{t \in[0,1]} f(\gamma(t))<c+\varepsilon \tag{2}
\end{equation*}
$$

By (PS), the condition (PS)c with c defined by (3.8.1), holds.

Let $\eta:[0,1] \times X \mapsto X$ be a f -decreasing homotopy according to Corollary 4.4 and $\gamma_{1}=\eta(1, \gamma)$.
Then 0 and e belong to $\{x:|f(x)-c| \geq 2 \varepsilon\}$ because $f(0)=0$, $f(e) \leq 0$ and $c>2 \varepsilon$.
By Corollary 4.4 (1) it follows that
$\gamma_{1}(0)=\eta(1, \gamma(0))=\eta(1,0)=0$,
$\gamma_{1}(1)=\eta(1, \gamma(1))=\eta(1, e)=e$,
which means that $\gamma_{1} \in \Gamma$. By Corollary 4.4 (2) and (2) we obtain

$$
\max _{t \in[0,1]} f\left(\gamma_{1}(t)\right) \leq c-\varepsilon<c
$$

which is a contradiction to the definition of c .

## CHAPTER 4

## Application: The lane Emden Equation

Let $N$ be a positive integer. We shall focus our attention on the exponents $1<p<\frac{(N+2)}{(N-2)}$, if $N \geq 3$, and $1<p<\infty$ provided that $\mathrm{N}=1,2$.

Consider the nonlinear elliptic boundary value problem

$$
\begin{align*}
&-\Delta u=u^{p} \text { in } \quad \Omega \\
& \mathrm{u}>0  \tag{P}\\
& u=0 \\
& \text { in } \Omega \\
& \text { on } \partial \Omega,
\end{align*}
$$

where $\Omega$ is a nonempty subset of $\mathbb{R}^{N}$.

This equation is called the Lane-Emden equation.It is basic model of nonlinear elliptic boundary problem .First formulated by Lane, an astrophysicist, in the mid 19th. century, the role of this equation and related elliptic PDEs is very broad outside and inside mathematics. The most interesting thing about solving this equation is that the existence and structure of the solution set of this problem is surprisingly complex, depending not only on the different values taken by $p$
but also on the geometry of $\Omega$. For instance the following facts hold:

Proposition 4.1 The problem (P) has no solution if $p \geq(N+$ $2) /(N-2)$ and if $\Omega$ is a star-shaped domain with respect to a certain point, e.g. the open unit ball.
(the proof of this nonexistence result follows the Pohozaev identity, which is obtained after multiplication of the first equation of (P) with $x \nabla u$ and integration by parts).

But in the case of an annulus $\Omega$ (a non star-shaped domain) Kazdan-Warner [7] proved the result below that also follows from a work by Degla

Proposition 4.2 The problem (P) has a solution for any $p>1$.

Proposition 4.3 If $p=(N+2) /(N-2)$ then the energy functional associated to problem $(P)$ does not have the Palais-Smale property.

If we let $p$ take the value 1 , then we would have a linear problem, and the existence of a solution depends on the geometry of the domain: clearly if 1 is not an eigenvalue of the opposite of the Laplacian operator; $\Delta$, in $H_{0}^{1}(\Omega)$, then there is no solution to our problem $(P)$.

Proposition 4.4 If $0<p<1$, then there would exist a unique solution (since the mapping $u \mapsto f(u) / u=u^{p-1}$ is decreasing) and, moreover, this solution would be stable. The arguments could be done in this case by using the method of sub- and super-solutions.

For the proofs or details of these propositions, the reader is refered to Willem Michel...

We use the Mountain Pass theorem to prove the existence of a solution (in the case $p>1$ is appropriately chosen as specified at the beginning of the introduction).
consider the functional

$$
\begin{equation*}
F(u)=\frac{1}{2} \int_{\Omega}|\nabla u|^{2}-\frac{1}{p+1} \int_{\Omega}\left(u^{+}\right)^{p+1}, u \in H_{0}^{1}(\Omega) \tag{3.165}
\end{equation*}
$$

Firstly we show that $F(u)$ is a functional on $H_{0}^{1}(\Omega)$.

Given that $1 \leq 2<N$ and that $p+1<2 N / N-2($ since $p<$ $(N+1) / N-2)$ ),
using the Rellich Kondradov Theorem, we have that $H^{1}(\Omega) \subset \subset \rightarrow$ $L^{p+1}$ i.e. $H^{1}(\Omega)$ is Compactly Embedded in $L^{p+1}$.
Hence $u \in H_{0}^{1} \Rightarrow u \in L^{p+1}$
Thus

$$
|F(u)| \leq \frac{1}{2}\|\nabla u\|_{L^{2}}^{2}+\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}<\infty
$$

1. We show that $F \in \mathcal{C}^{1}\left(H_{0}^{1}(\Omega), \mathbb{R}\right)$ and $F^{\prime}(u)=0$ if and only if $u$ is a solution of (P)

1: We show that $F$ is Gâteaux differentiable.
Recall that $H_{0}^{1}$ is a Hilbert space endowed with the norm defined by

$$
\|u\|_{H_{0}^{1}}=\|\nabla u\|_{L^{2}}^{2}+\|u\|_{L^{2}}^{2}
$$

a) It is not hard to check that the map $\psi: s \mapsto\left(s^{+}\right)^{p+1}$ is differentiable at every point of $\mathbb{R}$ and its derivative is defined by $\psi(s)=(p+1)\left(s^{+}\right)^{p}$. Indeed we have

$$
s^{+}=\max \{s, 0\}=\left\{\begin{array}{lll}
s & \text { if } & s>0 \\
0 & \text { if } & s \leq 0
\end{array}\right.
$$

which shows clearly that $\psi$ is differentiable on $\mathbb{R} \backslash\{0\}$ and

$$
\psi^{\prime}(s)=\left\{\begin{array}{lll}
(p+1) s^{p} & \text { if } & s>0 \\
0 & \text { if } & s<0
\end{array}\right.
$$

Furthermore, on the one hand we have

$$
\lim _{s \rightarrow 0^{-}} \frac{\psi(s)-\psi(0)}{s}=0
$$

and on the other hand

$$
\lim _{s \rightarrow 0^{+}} \frac{\psi(s)-\psi(0)}{s}=\lim _{s \rightarrow 0^{+}} s^{p}=0 \text { since } p>0 .
$$

Thus $\psi$ is also differentiable at 0 and $\psi^{\prime}(0)=0$. It follows that

$$
\psi^{\prime}(s)=(p+1)\left(s^{+}\right)^{p}, \quad \forall s \in \mathbb{R} .
$$

b) Given $u, h \in H_{0}^{1}(\Omega)$, let us consider the map $\varphi$ defined from $\mathbb{R}$ into $H_{0}^{1}(\Omega)$ by $\varphi(t)=F(u+t h)$; that is,

$$
\begin{aligned}
\varphi(t) & =\frac{1}{2} \int_{\Omega}|\nabla(u(x)+t h(x))|^{2} d x-\frac{1}{p+1} \int_{\Omega}\left((u(x)+t h(x))^{+}\right)^{p+1} d x \\
& =\frac{1}{2} \int_{\Omega}|\nabla u(x)+t \nabla h(x)|^{2} d x-\frac{1}{p+1} \int_{\Omega}\left((u(x)+t h(x))^{+}\right)^{p+1} d x
\end{aligned}
$$

We show that $\varphi$ is differentiable at 0 . To this end, let us set

$$
\begin{aligned}
G(x, t) & =\frac{1}{2}|\nabla u(x)+t \nabla h(x)|^{2}-\frac{1}{p+1}\left((u(x)+t h(x))^{+}\right)^{p+1} \\
& =\frac{1}{2}|\nabla u(x)+t \nabla h(x)|^{2}-\frac{1}{p+1} \psi(u(x)+t h(x)) \\
& =\frac{1}{2}(\nabla u(x)+t \nabla h(x))^{2}-\frac{1}{p+1} \psi(u(x)+t h(x))
\end{aligned}
$$

where the power 2 denotes an inner square in $\mathbb{R}^{n}$. Then we see that $G(\cdot, 0) \in L^{1}(\Omega)$ and that $G$ is differentiable with respect to $t$ with

$$
\begin{aligned}
\frac{\partial G}{\partial t}(x, t) & =\langle\nabla u(x)+t \nabla h(x), \nabla h(x)\rangle+\frac{1}{p+1} \psi^{\prime}(u(x)+t h(x)) h(x) \\
& =\langle\nabla u(x)+t \nabla h(x), \nabla h(x)\rangle+\left((u(x)+t h(x))^{+}\right)^{p} h(x)
\end{aligned}
$$

and

$$
\left|\frac{\partial G}{\partial t}(x, t)\right| \leq g(x), \quad \forall|t| \leq 1 \text {.and for a.e. } x \in \Omega
$$

where
$g(x)=|\nabla u(x)||\nabla h(x)|+|\nabla h(x)|^{2}+2^{p}|h(x)|\left(|u(x)|^{p}+|h(x)|^{p}\right)$.

Using the Cauchy Schwarz Inequality ,the inequality $(a+b)^{p} \leq$ $2^{p}\left(|a|^{p}+|b|^{p}\right)$, and the Triangle Inequality.

Also, using Hölder's inequality and the fact that $H^{1} \subset \subset \rightarrow L^{p+1}$ we have :

$$
\int_{\Omega}|g(x)| d x \leq\|\nabla u\|_{L^{2}}\|\nabla h\|_{L^{2}}+\|\nabla h\|_{L^{2}}^{2}+2^{p}\|h\|_{L^{p+1}}\|u\|_{L^{p+1}}+2^{p}\|h\|_{L^{p+1}}^{p+1}<\infty
$$

Hence $g \in L^{1}$, and by using the Theorem of differentiation under the integral symbol, we have

$$
\begin{aligned}
\varphi^{\prime}(0) & =\int_{\Omega} \frac{\partial G}{\partial t}(x, 0) d x \\
& =\int_{\Omega}\left(\nabla u \cdot \nabla h+\left(u^{+}\right)^{p} h\right)
\end{aligned}
$$

For $u$ fixed in $H_{0}^{1}(\Omega)$, the mapping

$$
H: H_{0}^{1} \rightarrow H_{0}^{1}, h \mapsto \int_{\Omega}\left(\nabla u \cdot \nabla h+\left(u^{+}\right)^{p} h\right)
$$

is linear and continuous from $H_{0}^{1}(\Omega)$ into itself.
Indeed: Linearity follows from the fact that integration and laplacian are linear.
To prove continuity, we prove that the function $H$ is bounded in $H_{0}^{1}$ :

$$
\begin{aligned}
|H(h)| & \leq\|\nabla u\|_{L^{2}}\|\nabla h\|_{L^{2}}+\|h\|_{L^{p+1}}\|u\|_{L^{p+1}}^{p} \\
& \leq\|\nabla u\|_{L^{2}}\|h\|_{H^{1}}+C_{1}\|u\|_{L^{p+1}}^{p}\|h\|_{H^{1}} \\
& \leq C\|h\|_{H^{1}}
\end{aligned}
$$

according to the compact embedding $H^{1}(\Omega) \subset \subset \rightarrow L^{p+1}(\Omega)$ and the continuous inclusion $L^{p+1}(\Omega) \subset \rightarrow L^{2}(\Omega)$. Using now the fact that $\Omega$ is bounded, we have that differentiable at $u$ and $\forall h \in H_{0}^{1}(\Omega)$, we have

$$
D_{G} F(u)(h)=\int_{\Omega}\left(\nabla u \cdot \nabla h+\left(u^{+}\right)^{p} h\right)
$$

We prove that $F$ is Fréchet differentiable.
Let $h \in H_{0}^{1}$ and $u_{n}$ a sequence which converges to $u$ in $H_{0}^{1}$. We have that:
$\nabla u_{n}$ converges to $\nabla u$ in $H_{0}^{1}(\Omega) \subset L^{2}$ (by definition of convergence in $\left.H^{1}(\Omega)\right)$ and $u_{n}$ converges to $u$ in $L^{p+1}$ by compact embedding of $H_{0}^{1}$ into $L^{p+1}$.
Hence $D_{G} F\left(u_{n}\right)$ converges to $D_{G} F(u)$, that is $D_{G} F$ is continuous at $u$,therefore $F$ is Frechet differentiable and is therefore $C^{1}$ We have that

$$
F^{\prime}(u)(h)=\int_{\Omega}\left(-\Delta u-\left(u^{+}\right)^{p}\right) h
$$

Since $\mathcal{D}(\Omega)$ is dense in $H_{0}^{1}(\Omega), F^{\prime}(u)$ is completely defined by knowing just its action on $\mathcal{D}(\Omega)$. But for every $\phi \in \mathcal{D}(\Omega)$, we have

$$
\int_{\Omega} \nabla u \cdot \nabla \phi=-\int_{\Omega} u \Delta \phi(\text { by Green formula) }
$$

and so

$$
\begin{aligned}
F^{\prime}(u)(\phi) & =-\int_{\Omega} u \Delta \phi-\int_{\Omega}\left(u^{+}\right)^{p} \phi \\
& =-\langle\Delta u, \phi\rangle-\left\langle\left(u^{+}\right)^{p}, \phi\right\rangle \text { (in the distributional sense) } \\
& =-\left\langle\Delta u+\left(u^{+}\right)^{p}, \phi\right\rangle .
\end{aligned}
$$

That is

$$
F^{\prime}(u)=-\Delta u-\left(u^{+}\right)^{p} \quad \text { (in the distributional sense). }
$$

So $F^{\prime}(u)=0$ if and only if $u$ is a weak solution of the

$$
-\Delta u=\left(u^{+}\right)^{p} .
$$

Moreover $u \in H_{0}^{1}$ also implies that $u=0$ on the boundary of $\Omega$.
Suppose that $u \leq 0$ in $\Omega$ then $u^{+}=0$ so that the equation is reduced to :

$$
\left\{\begin{array}{rll}
-\Delta u & =0 & \text { in } \Omega \\
u= & 0 & \text { on } \partial \Omega .
\end{array}\right.
$$

which has as only solution the zero solution. Hence $u>0$ on $\Omega$.
Therefore we have

$$
\left\{\begin{array}{rll}
-\Delta u & =u^{p} & \text { in } \Omega \\
u & =0 & \text { on } \partial \Omega .
\end{array}\right.
$$

(which comes from the fact that $u \in H_{0}^{1}$ )
2. $F(0)=0$ follows directly from the definition of $F$.

Let us show that there exists $\rho>0$ and $R$ such that

$$
F(u)>\rho \text { for } \quad\|u\|_{H_{0}^{1}}=R .
$$

$$
\int_{\Omega}\left(u^{+}\right)^{p+1} \leq \int_{\Omega}|u|^{p+1}=\|u\|_{L^{p+1}}^{p+1} \leq C\|u\|_{H_{0}^{1}}^{p+1}
$$

Using Poincare inequality :

$$
\|u\|_{L^{2}} \leq C\|\nabla u\|_{L^{2}}
$$

We get that

$$
\begin{aligned}
F(u) & \geq \frac{1}{2}\|u\|_{H_{0}^{1}}^{2}-\frac{C}{p+1}\|u\|_{H_{0}^{1}}^{p+1} \\
& \geq \frac{1}{2}\|u\|_{H_{0}^{1}}^{2}\left(1-\frac{2 C}{p+1}\|u\|_{H_{0}^{1}}^{p-1}\right) \\
& =\|u\|_{H_{0}^{1}}^{2}\left(\frac{1}{2}-\frac{2 C}{p+1}\|u\|_{H_{0}^{1}}^{p-1}\right) \\
& =R^{2}\left(\frac{1}{2}-\frac{2 C R^{p-1}}{p+1}\right)
\end{aligned}
$$

with

$$
\lim _{R \rightarrow 0^{+}}\left(\frac{1}{2}-\frac{2 C R^{p-1}}{p+1}\right)=\frac{1}{2}>0
$$

So for $R>0$ sufficiently small, we have

$$
\left(\frac{1}{2}-\frac{2 C R^{p-1}}{p+1}\right)>\frac{1}{4}
$$

which gives

$$
F(u)>\frac{R^{2}}{4} .
$$

Therefore taking

$$
\rho=\frac{R^{2}}{4}>0
$$

we get for $\|u\|_{H_{0}^{1}}=R, F(u)>\rho$ provided $\|u\|_{H_{0}^{1}}=R$ is small enough.
3. Now we prove that there exists $v_{0} \in H_{0}^{1}(\Omega)$ such that

$$
\left\|v_{0}\right\|_{H_{0}^{1}}>R \text { and } F\left(v_{0}\right) \leq 0 .
$$

To this end, choose $w_{0} \in H_{0}^{1}(\Omega) \backslash\{0\}$ such that $w_{0} \geq 0$. Thus

$$
\begin{aligned}
F\left(t w_{0}\right) & =\frac{t^{2}}{2} \int_{\Omega}\left|\nabla w_{0}\right|^{2}-\frac{t^{p+1}}{p+1} \int_{\Omega} w_{0}^{p+1} \\
& =t^{p+1}\left(\frac{1}{t^{p-1}} \int_{\Omega}\left|\nabla w_{0}\right|^{2}-\frac{1}{p+1} \int_{\Omega} w_{0}^{p+1}\right),
\end{aligned}
$$

with

$$
\lim _{t \rightarrow+\infty}\left(\frac{1}{t^{p-1}} \int_{\Omega}\left|\nabla w_{0}\right|^{2}-\frac{1}{p+1} \int_{\Omega} w_{0}^{p+1}\right)=-\frac{\left\|w_{0}\right\|_{p+1}^{p+1}}{p+1}<0 .
$$

Hence for $t>0$ sufficiently large,

$$
F\left(t w_{0}\right)<-\frac{t^{p+1}}{2} \frac{\left\|w_{0}\right\|_{p+1}^{p+1}}{p+1} \ll 0 .
$$

So it suffices to consider such a $t>0$ large enough and to take $v_{o}=t w_{0}$ in order to have

$$
F\left(v_{o}\right) \leq 0 \text { with }\left\|v_{o}\right\|_{H_{0}^{1}}>R .
$$

4. $F$ satisfies the (PS) condition.

Let $\left(u_{n}\right)_{n}$ be a sequence of elements of $H_{0}^{1}(\Omega)$ such that $\left(F\left(u_{n}\right)\right)_{n}$ is bounded and $F^{\prime}\left(u_{n}\right) \rightarrow 0$.
We show that $\left(u_{n}\right)_{n}$ has a convergent subsequence.
The idea is to make use of the computation of $F^{\prime}\left(u_{n}\right)\left(u_{n}\right)$.

$$
\begin{aligned}
F^{\prime}\left(u_{n}\right)\left(u_{n}\right) & =-\int_{\Omega}\left(\nabla u_{n} \cdot \nabla u_{n}-\left(u_{n}^{+}\right)^{p} u_{n}\right) \\
& =\int_{\Omega}\left|\nabla u_{n}\right|^{2}-\int_{\Omega} u_{n}\left(u_{n}^{+}\right)^{p} \\
& =\int_{\Omega}\left|\nabla u_{n}\right|^{2}-\int_{\Omega}\left(u_{n}^{+}\right)^{p+1}
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{1}{p+1} F^{\prime}\left(u_{n}\right)\left(u_{n}\right) & =\frac{1}{p+1} \int_{\Omega}\left|\nabla u_{n}\right|^{2}-\frac{1}{p+1} \int_{\Omega}\left(u_{n}^{+}\right)^{p+1} \\
& =\frac{1}{p+1} \int_{\Omega}\left|\nabla u_{n}\right|^{2}+F\left(u_{n}\right)-\frac{1}{2} \int_{\Omega}\left|\nabla u_{n}\right|^{2} \\
& =F\left(u_{n}\right)-\frac{p-1}{p+1} \int_{\Omega}\left|\nabla u_{n}\right|^{2} .
\end{aligned}
$$

Thus,

$$
\left\|\nabla u_{n}\right\|_{L^{2}}^{2}=\frac{2(p+1)}{p-1} F\left(u_{n}\right)-\frac{2}{p-1} F^{\prime}\left(u_{n}\right) u_{n}
$$

which implies that

$$
\left\|\nabla u_{n}\right\|_{L^{2}}^{2} \leq \left\lvert\, \frac{2(p+1)}{p-1}\left\|F\left(u_{n}\right)\right\|+\frac{2}{p-1}\left\|F^{\prime}\left(u_{n}\right)\right\|\left\|u_{n}\right\|_{L^{2}}\right.
$$

and using Poincaré inequality we have,

$$
\left\|\nabla u_{n}\right\|_{L^{2}}^{2} \leq \frac{2(p+1)}{p-1}\left\|F\left(u_{n}\right) \left\lvert\,+\frac{2 C}{p-1}\right.\right\| F^{\prime}\left(u_{n}\right)\| \| \nabla u_{n} \|_{L^{2}}
$$

But from the hypothesis, $F\left(u_{n}\right)$ is bounded and $F^{\prime}\left(u_{n}\right) \rightarrow 0$. Then it follows from the above inequality that $\left(\left\|\nabla u_{n}\right\|_{L^{2}}\right)_{n}$ is bounded, and so $\left(u_{n}\right)_{n}$ is bounded in $H_{0}^{1}(\Omega)$ by Poincaré inequality. To see this, suppose on the contrary that it is not bounded. Therefore there would exist a subsequence $\left(\left\|\nabla u_{n_{j}}\right\|_{L^{2}}\right)_{j}$ that tends to $+\infty$. We would then get a contrast with the following inequality

$$
\left\|\nabla u_{n_{j}}\right\|_{L^{2}} \leq \left\lvert\, \frac{2(p+1)}{(p-1)\left\|\nabla u_{n_{j}}\right\|_{L^{2}}}\left\|F\left(u_{n_{j}}\right)\right\|+\frac{2}{p-1)}\left\|F^{\prime}\left(u_{n_{j}}\right)\right\|\right.
$$

while letting $j$ tend to $+\infty$. Since $H_{0}^{1}(\Omega)$ is a Hilbert space which is compactly embedded in $L^{p+1}(\Omega)$ according to Rellich compactness theorem, it follows from the boundedness of $\left(u_{n}\right)_{n}$ in $H_{0}^{1}(\Omega)$ that there exists a subsequence $\left(u_{n_{k}}\right)_{k}$ of $\left(u_{n}\right)_{n}$ and an element $u \in H_{0}^{1}(\Omega)$ such that
(i) $\left(u_{n_{k}}\right)_{k}$ converges weakly to $u$ in $H_{0}^{1}(\Omega)$.
(ii) $\left(u_{n_{k}}\right)_{k}$ converges strongly to $u$ in $L^{p+1}(\Omega)$.
(iii) $\left(u_{n_{k}}\right)_{k}$ converges almost everywhere to $u$ on $\Omega$ and there exists some nonnegative $g \in L^{p+1}(\Omega)$ such that $\left|u_{n_{k}}\right| \leq g$.

From the convergences $F^{\prime}\left(u_{n_{k}}\right) \rightarrow 0$ (by assumption)

$$
\int u_{n_{k}}^{+} u_{n_{k}}=\int\left(u_{n_{k}}^{+}\right)^{2} \longrightarrow \int\left(u^{+}\right)^{2} \text { by (iii) }
$$

and the continuous embedding of $L^{p+1}(\Omega)$ into $L^{2}(\Omega)$. Thus it follows from the convergence $F^{\prime}\left(u_{n_{k}}\right) \rightarrow 0$ and the boundedness of $\left(u_{n_{k}}\right)$ that $F^{\prime}\left(u_{n_{k}}\right)\left(u_{n_{k}}\right) \rightarrow .0$ Thus

$$
\begin{equation*}
\int\left(\nabla u_{n_{k}}\right)^{2}=F^{\prime}\left(u_{n_{k}}\right)\left(u_{n_{k}}\right)+\int\left(u^{+}\right)^{2} \rightarrow \int\left(\nabla u^{+}\right)^{2} \tag{C1}
\end{equation*}
$$

and

$$
\begin{equation*}
\int\left(u_{n_{k}}^{+}\right)^{p+1} \longrightarrow \int\left(u^{+}\right)^{p+1}, \text { as well. } \tag{C2}
\end{equation*}
$$

Besides since $u_{n_{k}} u$ in $H_{0}^{1}$ we have

$$
\int_{\Omega} \nabla u_{n_{k}} \nabla u=\left\langle u_{n_{k}}, u\right\rangle-\int u_{n_{k}} u \longrightarrow\langle u, u\rangle-\int u^{2},
$$

that is,

$$
\int_{\Omega} \nabla u_{n_{k}} \nabla u \longrightarrow \int|\nabla u|^{2} .
$$

It follows that

$$
\nabla u_{n_{k}} \longrightarrow \nabla u \quad \in \quad L^{2}(\Omega)
$$

because

$$
\left|\nabla u_{n_{k}}-\nabla u\right|^{2}=\left|\nabla u_{n_{k}}\right|^{2}-2\left\langle\nabla u_{n_{k}}, \nabla u\right\rangle+|\nabla u|^{2} .
$$

Consequently, $u_{n_{k}} \rightarrow u$ in $H_{0}^{1}$ and so $F$ satisfies the PS condition.

Concusion:
From the steps 1, 2, 3, and 4, we have that our function satisfies the hypothesis of the Mountain Pass Theorem due to Ambrosseti and Ravinowitz, Hence F has a critical value $c>\rho$ characterised by

$$
c=\inf _{\gamma \in \Gamma} \max _{t \in[0,1]} f(\gamma(t))
$$

Where

$$
\Gamma=\left\{\gamma \in C([0,1], X): \gamma(0)=0, \gamma(1)=v_{0}\right\}
$$

But $F$ having a critical value $c$ implies that it has a critical point say $u$ such that $F(u)=c$ and $F^{\prime}(u)=0$, which is the solution of our equation as shown above.

## Bibliography

[1] Maria do Rosario Grossinho and Stephan Agop Tersian An Introduction to Minimax Theorems and their applications to differential equations, 2001
[2] E.A.Sheina Mountain Pass Theorem in the Problem on a Nontrivial Solution of a Quasilinear Equation with a Parameter, Moscow State University, Moscow, Russia,2008
[3] Louis Jean Jean and Kazunaga Tanaka A remark onleast energy solution in $\mathbb{R}^{N}$
[4] Vicentiu D. Radulescu Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations: Monotonicity, Analytic, and Variational Methods
[5] Klaus schimtt Revisiting the method of Sub and Supersolutions for non linear elliptic problems
[6] Willem Michel Minimax Theorems
[7] Massimo Grossi Asymptotic behaviour of the Kazdan-Warner solution in the annulus
[8] C.E. Chidume; Applicable functional analysis, International Centre for Theoretical Physics Trieste, Italy, July 2006.
[9] Thibault Lecture notes in convex analysis,African University of Science and Technology
[10] Guy Degla Lecture notes on Differentiability in Banach spaces, African University of science and technology,2010
[11] Khalil Ezzinbi; Lecture Notes on Differential Equations in Banach Spaces, African University of Science and Technology, 2009.
[12] NGalla Djitte Lecture notes on distribution theory and sobolev spaces, African University of Science and Technology, 2010
[13] Obi Lecture notes on Real Analysis,African University of Sciences and Technology,2009

