A MODIFIED SUBGRADIENT EXTRAGRADIENT METHOD FOR SOLVING MONOTONE VARIATIONAL INEQUALITIES IN BANACH SPACES

A Thesis Presented to the Department of
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Master of Science

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CERTIFICATION

This is to certify that the thesis titled "A MODIFIED SUBGRADIENT EXTRAGRADIENT METHOD FOR SOLVING MONOTONE VARIATIONAL INEQUALITIES IN BANACH SPACES" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research work carried out by Agbo Ejike Cyprian in the department of Pure and Applied Mathematics.

APPROVAL

A MODIFIED SUBGRADIENT EXTRAGRADIENT METHOD FOR SOLVING MONOTONE VARIATIONAL INEQUALITIES IN BANACH SPACES

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ABSTRACT

The subgradient extragradient method is considered an improvement of the extragradient method for variational inequality problem for the class of monotone and Lipschitz continuous mappings in the setting of Hilbert spaces. In this Thesis, we proposed an improved subgradient extragradient method for variational inequality problem for the class of monotone and Lipschitz continuous mappings in the setting of real Banach spaces.

ACKNOWLEDGMENT

First and foremost, Praises and thanks to the God, the Almighty for His showers of blessing through out my research work and to the successful completion of my research. I am extremely grateful to my parents for there loves, prayers and sacrifices for educating and preparing me for my future. I would like to say a big thank you to my elder sister *Agbo Nkeiru Jennifer* for her constant encouragement through out my studies.

DEDICATION

This thesis is dedicated to God Almighty.

CONTENTS

CHAPTER 1

GENERAL INTRODUCTION AND LITERATURE REVIEW

In this chapter, we will give a general introduction on variational inequality problem and then a brief review of existing results on variational inequality.

1.1 Background of study

The contributions of this thesis falls within the general area of nonlinear functional analysis and applications, in particular, nonlinear operator theory. We are interested in finding or approximating solution(s) of a variational inequality problem for a monotone Lipschitz-continuous map on a level set of convex function in Banach spaces.

1.2 Variational Inequality Problems

The problem of finding $x^* \in C$ such that the variational inequality (VI)

$$\langle Ax^*, x - x^* \rangle \geq 0 \qquad \forall x \in C, \tag{1.2.1}$$

is called the variational inequality problem (VIP). Where C is a nonempty closed convex subset in a real Hilbert space H, $A : H \to H$ is a single valued mapping, $\langle ., . \rangle$ and $\|.\|$ are inner product and norm in H, respectively. Let $SOL(C, A)$ be the solution of the variational inequality (1.2.1). Finding a solution for VIP is a fundamental problem in optimization theory, partial differential equation, mathematical modeling, image recovery and Data processing.

It is well Known that the problem (1.2.1) is equivalent to solving the fixed point problem

$$x^* = P_C(x^* - \tau A(x^*)) \tag{1.2.2}$$

where τ is an arbitrary positive constant and P_C denotes the Euclidean least distant projection onto C.

We now show the equivalence between (1.2.1) and (1.2.2)

Proof

We know that,

$$x^* = P_C(x^* - \tau A(x^*))$$

is equivalent to

$$\langle \tau A(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C$$

which is also equivalent to

$$\langle A(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C \qquad (\tau > 0)$$

■

1.3 Statement of the Problem

Let $A : E \to E^*$ be a Lipschitz-continuous monotone map. In studying variational inequality problem in real Banach spaces more general than Hilbert spaces, several algorithms have been constructed for approximating solutions of variational inequality problem. Consequently, since most real life problems exist in spaces more general than Hilbert spaces, this intrigue mathematicians to ask if such results can be obtained for a Lipschitz-continuous monotone map defined on a level set of convex function in Banach spaces. In this thesis, we introduce an improved subgradient extragradient method

$$\begin{cases} x_0 \in E, k = 0 \\ C_k := \{w \in E : c(x_k) + \langle c'(x_k), w - x_k \rangle \leq 0\}; \\ y_k = \Pi_{C_k} J^{-1}(Jx_k - \beta_k f(x_k)); \\ T_k := \{w \in E : \langle Jx_k - \beta_k f(x_k) - Jy_k, w - y_k \rangle \leq 0\}; \\ x_{k+1} = \Pi_{T_k} J^{-1}(Jx_k - \beta_k f(y_k)). \end{cases} \tag{1.3.1}$$

for solving the Lipschitz-continuous monotone variational inequalities defined on a level set of convex function, *i.e.*; $C(x) := \{x \in E : c(x) \leq 0\}$ and $c : E \to R$ is a convex function defined on a uniformly smooth and 2-uniformly convex real Banach spaces E.

1.4 Motivation of Research and Objectives

Motivated by the result of *Censor et al* [2], Nadezhkina and Takahashi [14] and the works of He and Wu [26], we introduce an improved subgradient extragradient method for solving the Lipschitz-continuous monotone variational inequalities defined on a level set of convex function in uniformly smooth and 2-uniformly convex real Banach spaces which is a more general space than real Hilbert spaces.

1.5 Literature Review

We consider the following variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \qquad \forall x \in C, \tag{1.5.1}$$

Where C is a nonempty closed convex subset in a real Hilbert space H, $A : H \to H$ is a single valued mapping, $\langle ., . \rangle$ and $\|.\|$ are inner product and norm in H, respectively. Let $SOL(C, A)$ be the solution of the variational inequality (1.5.1). Finding a solution for $VIP's$ is a fundamental problem in optimization theory, partial differential equation, mathematical modeling, image recovery and data processing.

In the past years, numerous algorithm have been considered and proposed for solving $VIP's$. In this Thesis, we focus on the projection methods.

The simplest projection method for $VIP's$ is the gradient method in which only one projection onto C is performed.

$$\begin{cases} x_0 \in C, \\ x_{k+1} = P_C(x_k - \tau A(x_k)) \end{cases} \tag{1.5.2}$$

for each $k \geq 0$, where $\tau \in (0, \frac{2\eta}{L^2})$ and P_C denotes the Euclidean least distant projection onto C. However, the convergence of this method requires a slightly strong assumption that the operator $A : H \to H$ is $L-$ Lipschitz and $\tau-$ strongly monotone. To avoid these strong hypothesis above, Korpelevich's [12] proposed the extragradient (EG) method for a $L-$ Lipschitz continuous monotone operator $A : H \to H$, in which two projections onto C where calculated.

$$\begin{cases} x_0 \in C, \\ y_k = P_C(x_k - \tau Ax_k) \\ x_{k+1} = P_C(x_k - \tau A(y_k)) \end{cases} \tag{1.5.3}$$

for each $k \geq 0$, where $\tau \in (0, \frac{1}{L})$. If the the solution $SOL(C, A)$ is not empty, the two sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ generated by the algorithm converge weakly to the same element in $SOL(C, A)$.

In 2006, Nadezhkina and Takahashi [14] generalized the EG method to general Hilbert spaces (including infinite-dimensional spaces) and also established the weak convergence theorem.

However, in each iteration of the EG method, in order to get the next iterate x_{k+1}, two projections onto C need to be calculated. But projections onto a closed and convex subset are not easily executed and might greatly affect the EG method. In order to avoid this weakness, $Censor\ et\ al$ developed the subgradient extragradient method for solving $VIP's$ [4], in which the second projection in (1.5.3) onto C was replaced with a projection onto a specific constructible half-space.

$$\begin{cases} x_0 \in C, \\ y_k = P_C(x_k - \tau Ax_k) \\ T_k := \{w \in H : \langle x_k - \tau f(x_k) - y_k, w - y_k \rangle \leq 0\}, \\ x_{k+1} = P_{T_k}(x_k - \tau A(y_k)) \end{cases} \tag{1.5.4}$$

for each $k \geq 0$. Under appropriate conditions, the two sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ generated by the algorithm converges weakly to the same element in $SOL(C, A)$.

In 2017, He and Wu [26] proposed a modified subgradient extragradient method for solving Lipschitz continuous monotone variational inequalities defined on a level set of convex function in real Hilbert spaces.

$$\begin{cases} x_0 \in H \\ C_k := \{w \in H : c(x_k) + \langle c'(x), w - x_k \rangle \leq 0\}, \\ y_k = P_{C_k}(x_k - \beta_k f(x_k)), \\ T_k := \{w \in H : \langle x_k - \beta_k f(x_k) - y_k, w - y_k \rangle \leq 0\}, \\ x_{k+1} = P_{T_k}(x_k - \beta_k f(y_k)). \end{cases} \tag{1.5.5}$$

where two projection onto two constructible half-spaces was computed and under appropriate conditions, the two sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ generated by the algorithm converge weakly to the same element in $SOL(C, A)$.

CHAPTER 2

PRELIMINARIES

In this chapter, we will give definition of some terms and results of interest used in the thesis.

2.1 Definition of terms

Through out this thesis, we will always let E be a real Banach space with dual space E^* and $\langle \cdot, \cdot \rangle$ denote the duality pairing of E and E^*. Whenever a sequence $\{x_n\}$ in E, converges strongly (weakly), we denote the convergence by $x_n \to x$ ($x_n \rightharpoonup x$) respectively.

Definition 2.1.1 *A normed space E is called smooth if for every $x \in E$, $\|x\| = 1$, there exists a unique $x^* \in E^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.*

Definition 2.1.2 *A normed space E is called uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$, such that for any $x, y \in E$, with $\|x\| = 1$, $\|y\| = 1$ and $\|x - y\| \geq \epsilon$ then $\|\frac{x+y}{2}\| \leq 1 - \delta$.*

Definition 2.1.3 *A normed space E is called strictly convex if for all $x, y \in E$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda)y\| < 1$, $\forall \lambda \in (0, 1)$.*

Definition 2.1.4 *The normalized duality mapping on a Banach space E:*

$$J : E \to 2^{E^*}$$

is defined as

$$Jx = \{jx \in E^* : \langle jx, x \rangle = ||x||^2 = ||jx||^2\}$$

Definition 2.1.5 *The Lyapunov fuctional $\phi : E \times E \to [0, \infty)$ is defined by*

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv \rangle + \|v\|^2 \ \forall \ u, v \in E, \tag{2.1.1}$$

where J is the normalized duality map from E to E^*.

Remark 2.1.6 *In a real Hilbert space H, equation (2.1.1) reduces to*

$$\phi(x, y) = \|x - y\|^2, \forall x, y \in H.$$

Proof *By Reiz representation theory, for any bounded linear functional f on H,*

- *There exist* $y \in H$ *such that*

$$f(x) = \langle x, y \rangle \quad \forall x \in H$$

- $\|f\| = \|y\|$

■

Furthermore, given $x, y, z \in E$*, we have the following properties:*

P1. $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,

P2. $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle z - x, Jy - Jz \rangle$,

Definition 2.1.7 *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H*. The map* $P_C : H \to C$ *defined by* $\tilde{x} := P_C(x) \in C$ *such that* $\|x - \tilde{x}\| = \inf_{y \in C} \|x - y\|$ *is called the metric projection of* x *onto* C*.*
It is well known that P_C *is characterized by the following inequalities:*

i. $||x - P_C(x)|| \leq ||x - y||$

ii. $\langle x - P_C, P_C - y \rangle \geq 0$

iii. $||x - y||^2 \geq ||x - P_C||^2 + ||y - P_C||^2$

for all $x \in H$ *and* $y \in C$*. [18] [7]*

Definition 2.1.8 *Let* E *be a smooth, strictly convex and reflexive real Banach space and let* C *be a nonempty, closed and convex subset of* E *. The map* $\Pi_C : E \to C$ *defined by* $\tilde{x} := \Pi_C(x) \in C$ *such that* $\phi(\tilde{x}, x) = \inf_{y \in C} \phi(y, x)$ *is called the generalized projection of* x *onto* C*.*

Remark 2.1.9 *Clearly, in a real Hilbert space, the generalized projection* Π_C *coincides with the metric projection* P_C *from* E *onto* C*.*

Definition 2.1.10 *A function* $c : E \to R$ *is said to be Gateaux differentiable at* $x \in E$*, if there exist an element, denoted by* $c'(x) \in E^*$*, such that*

$$\lim_{t \to 0} \frac{c(x + tv) - c(x)}{t} = \langle c'(x), v \rangle, \quad \forall v \in E.$$

where $c'(x)$ *is the Gateaux differential of* c *at* x*.*

Definition 2.1.11 *A function* $c : E \to R$ *is said to be weakly lower semi-continuous* $(w-lsc)$ *at* $x \in E$*, if* $x_k \rightharpoonup x$ *implies* $c(x) \leq \liminf_{k \to \infty} c(x_k)$*.*

Definition 2.1.12 *A function* $c : E \to R$ *is called convex, if*

$$c(tx + (1 - t)y) \leq tc(x) + (1 - t)c(y)$$

$\forall t \in [0, 1]$ *and* $\forall x, y \in E$

Definition 2.1.13 *A convex function* $c : E \to R$ *is said to be subdifferentiable at a point* $x \in E$ *if the set*

$$\partial c(x) := \{d \in E^* : c(y) \geq c(x) + \langle d, y - x \rangle, \forall y \in E\} \neq \emptyset$$

where each element in $\partial c(x)$ *is called a subdifferential of* c *at* x*.*

Remark 2.1.14 *It is well known that if c is Gateaux differentiable at x, c is subdifferentiable at x and $\partial c(x) = \{c'(x)\}$ [10].*

Definition 2.1.15 *The mapping $f : E \to E$ is said to be Lipschitz continuous [18], if there exist a positive constant k, such that*

$$||f(x) - f(y)|| \leq k||x - y|| \quad \forall x, y \in E$$

Definition 2.1.16 *A mapping $f : E \to E$ is said to be monotone on E, if*

$$\langle f(x) - f(y), x - y\rangle \geq 0 \quad \forall x, y \in E$$

Definition 2.1.17 *(Normal Cone) we denote the normal cone of c at $v \in C$, by $N_c(v)$ [17] i.e.,*

$$N_c(v) := \{w \in E^* : \langle w, y - v\rangle \leq 0 \quad \forall y \in C\}$$

Definition 2.1.18 *(Maximal monotone operator) Let $T : E \rightrightarrows 2^{E^*}$ be a point to set operator on E. T is called maximal monotone operator if T is monotone, i.e.,*

$$\langle u - v, x - y\rangle \geq 0, \qquad \forall u \in T(x) \text{ and } \forall v \in T(y)$$

and the graph $G(T)$ of T,

$$G(T) := \{(x, u) \in E \times E^* : u \in T(x)\},$$

is not properly contained in the graph of any other monotone operator.
Define

$$Tv = \begin{cases} f(v) + N_C(v), & if\ v \in C, \\ \emptyset & if\ v \notin C \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in SOL(C, f)$.

2.2 Results of Interest

Condition 2.2.1 *The solution set of $VI(C, f)$, denoted by $SOL(C, f)$ is not empty.*

Condition 2.2.2 *The function $f : H \to R$ is monotone and Lipschitz-continuous on H.(but we have no need to extimate or know the Lipschitz constant of f)*

Condition 2.2.3 *The function c satisfies the following conditions:*

i. $c(x)$ is convex function.

ii. $c(x)$ is a weakly lower semicontinuous on H.

iii. $c(x)$ is Gateaux differentiable on H and $c'(x)$ is a M_1-Lipschitzian continuous map on H.

iv. There exist $M_2 \geq 0$ such that $||f(x)|| \leq M_2||c'(x)||$ for any $x \in \partial C$, where ∂C denotes the boundary of C.

Theorem 2.2.4 *Assume that the solution set $SOL(C, f)$ of $VI(C, f)$ is not empty. Given $x^* \in C$. Then $x^* \in SOL(C, f)$ iff we have either*

1. $f(x^*) = 0$, *or*

2. $x^* \in \partial C$ *and there exist a positive constant* β *such that* $f(x^*) = -\beta c'(x^*)$.

Lemma 2.2.5 *Let* C *be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive real Banach space* E. *Then,*

1. *if* $x \in E$ *and* $y \in C$, *then* $\tilde{x} = \Pi_C x$ *if and only if* $\langle \tilde{x} - y, Jx - J\tilde{x} \rangle \geq 0$, *for all* $y \in C$,

2. $\phi(y, \tilde{x}) + \phi(\tilde{x}, x) \leq \phi(y, x)$, *for all* $x \in E$, $y \in C$.

Lemma 2.2.6 *Let* E *be a* 2*-uniformly convex and smooth real Banach space. Then, there exists a positive constant* α *such that*

$$\alpha \|x - y\|^2 \leq \phi(x, y), \ \forall \ x, y \in E. \tag{2.2.1}$$

Remark 2.2.7 *Without loss of generality, we may assume* $\alpha \in (0, 1)$.

Lemma 2.2.8 *Let* E *be a real smooth and uniformly convex Banach space, and let* $\{u_n\}$ *and* $\{v_n\}$ *be two sequences of* E. *If either* $\{u_n\}$ *or* $\{v_n\}$ *is bounded, then* $\phi(u_n, v_n) \to 0 \Rightarrow \|u_n - v_n\| \to 0$.

Lemma 2.2.9 *(Opial condition) Let* E *be a real Banach space.* E *is said to have the opial property if [16], whenever any sequence* $\{x_k\}_{k=0}^{\infty} \subset E$ *converges weakly to a point* $x \in E$ *(*$x_k \rightharpoonup x \in E$*) and* $x \neq y$ *for any* $y \in E$ *it follows that*

$$\liminf_{k \to \infty} ||x_k - x|| < \liminf_{k \to \infty} ||x_k - y||$$

Lemma 2.2.10 *Let* $\{a_k\}$ *and* $\{t_k\}$ *be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{k+1} \leq a_k + t_k \quad \forall \quad n \geq 1$$

if $\sum_{k=1}^{\infty} t_k < \infty$, *then* $\lim_{k \to \infty} a_k$ *exist.*

Lemma 2.2.11 *Let* C *be a nonempty closed convex subset of a uniformly convex and smooth space* E. *Let* $\{x_k\}$ *be a bounded sequence in* E. *Supposed that, for all* $u \in C$

$$\phi(x_k, u) \leq (1 + \theta_k)\phi(x_k, u)$$

for every $k = 1, 2, 3, \ldots$ *and* $\sum_{k=1}^{\infty} \theta_k < \infty$. *then* $\{\Pi_C(x_k)\}$ *is a cauchy sequence.*

Remark 2.2.12 *Let* E *be a real Banach space and let* C *be a nonempty, closed and convex subset of* E. *Let the sequence* $\{x_k\}_{k=0}^{\infty} \in E$ *be fejer monotone with respect to* C*,i.e.; for any* $u \in C$

$$\phi(x_{k+1}, u) \leq \phi(x_k, u) \ \forall \ \ k \geq 0.$$

Then $\{\Pi_C(x_k)\}_{k=0}^{\infty}$ *converges strongly to some* $\bar{z} \in C$. *[13]*

CHAPTER 3

RESULTS OF HE AND WU

3.1 Introduction

In this chapter we will give a detailed prove of the results of Songnian He and Tao Wu which they proved in a real Hilbert space.

3.2 The Modified Subgradient Extragradient Algorithm

In this section, we give our algorithm for solving the $VI(C, f)$ in the setting of Hilbert spaces, where C is a level set given as follows:

$$C := \{x \in H : c(x) \leq 0\},$$

where $c : H \to R$ is a convex function.

Algorithm 3.2.1 *Select an initial guess $x_0 \in H$ arbitrarily, set $k = 0$ and construct the half-space*

$$C_k := \{w \in H : c(x_k) + \langle c'(x), w - x_k\rangle \leq 0\};$$

Given the current iteration x_k, compute

$$y_k = P_{C_k}(x_k - \beta_k f(x_k)),$$

where

$$\beta_k = \sigma\rho^{m_k},\ \sigma > 0,\ \rho \in (0, 1) \tag{3.2.1}$$

and m_k is the smallest nonnegative integer, such that

$$\beta_k^2||f(x_k) - f(y_k)||^2 + 2M\beta_k||x_k - y_k||^2 \leq v^2||x_k - y_k||^2, \tag{3.2.2}$$

where $M = M_1M_2$ and $v \in (0, 1)$
then calculate the next iterate,

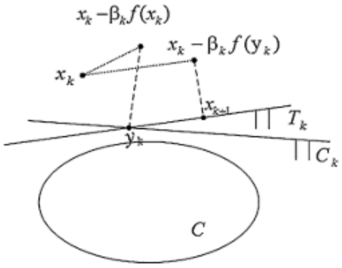
$$x_{k+1} = P_{T_k}(x_k - \beta_k f(y_k))$$

where

$$T_k := \{w \in H : \langle x_k - \beta_k f(x_k) - y_k, w - y_k\rangle \leq 0\},$$

T_k is the same as the half-space in Censor's method [3] and x_{k+1} is the metric projection of

Figure 3.1:



$x_k - \beta_k f(y_k)$ *onto the hyperplane* T_k.

$$\begin{cases} x_0 \in H \\ C_k := \{w \in H : c(x_k) + \langle c'(x), w - x_k\rangle \leq 0\}, \\ y_k = P_{C_k}(x_k - \beta_k f(x_k)), \\ T_k := \{w \in H : \langle x_k - \beta_k f(x_k) - y_k, w - y_k\rangle \leq 0\}, \\ x_{k+1} = P_{T_k}(x_k - \beta_k f(y_k)). \end{cases} \tag{3.2.3}$$

3.3 Convergence theorem of the algorithm

In this section, we prove the weak convergence theorem for Algorithm(3.2.3). First of all, we give the following lemma, which plays a crucial role in the proof of our main result.

Lemma 3.3.1 *Let* $\{x_k\}_{k=0}^{\infty}$ *and* $\{y_k\}_{k=0}^{\infty}$ *be the two sequences generated by (3.2.3). Let* $u \in SOL(C, f)$ *and let* β_k *be selected as (3.2.1). Then under conditions (2.2.1),(2.2.2) and (2.2.3), we have*

$$||x_{k+1} - u||^2 \leq ||x_k - u||^2 - (1 - v^2)||y_k - x_k||^2 \qquad \forall k \geq 0 \tag{3.3.1}$$

Proof Let $u \in SOL(C, f)$ arbitrarily, for all $k \geq 0$, using (2.1.7) and f being monotone, we have

$$||x_k - \beta_k f(y_k) - u||^2 \geq ||x_k - \beta_k f(y_k) - x_{k+1}||^2 + ||x_{k+1} - u||^2$$

$$\begin{aligned} \Leftrightarrow ||x_{k+1} - u||^2 &\leq ||x_k - \beta_k f(y_k) - x_{k+1}||^2 - ||x_k - \beta_k f(y_k) - u||^2 \\ &= ||x_k - u||^2 - ||x_k - x_{k+1}||^2 + 2\beta_k\langle f(y_k), u - x_{k+1}\rangle \\ &= ||x_k - u||^2 - ||x_k - x_{k+1}||^2 + 2\beta_k[\langle f(y_k) - f(u), u - y_k\rangle \\ &\quad +\langle f(u), y_k - x_{k+1}\rangle + \langle f(u), u - y_k\rangle + \langle f(y_k) - f(u), y_k - x_{k+1}\rangle] \\ &= ||x_k - u||^2 - ||x_k - x_{k+1}||^2 - 2\beta_k\langle f(y_k) - f(u), y_k - u\rangle \\ &\quad +2\beta_k\langle f(y_k), y_k - x_{k+1}\rangle - 2\beta_k\langle f(u), y_k - u\rangle \\ &\leq ||x_k - u||^2 - ||x_k - x_{k+1}||^2 + 2\beta_k\langle f(y_k), y_k - x_{k+1}\rangle + 2\beta_k\langle f(u), u - y_k\rangle \\ &= ||x_k - u||^2 - ||x_k - y_k||^2 - ||y_k - x_{k+1}||^2 - 2\langle x_k - y_k, y_k - x_{k+1}\rangle \\ &\quad +2\beta_k\langle f(y_k), y_k - x_{k+1}\rangle + 2\beta_k\langle f(u), u - y_k\rangle \\ &= ||x_k - u||^2 - ||x_k - y_k||^2 - ||y_k - x_{k+1}||^2 \\ &\quad +2\langle x_k - \beta_k f(y_k) - y_k, x_{k+1} - y_k\rangle + 2\beta_k\langle f(u), u - y_k\rangle \end{aligned} \tag{3.3.2}$$

By the definition of T_k, we have

$$\begin{aligned} \langle x_k - \beta_k f(y_k) - y_k, x_{k+1} - y_k\rangle &= \langle x_k - \beta_k f(x_k) - y_k, x_{k+1} - y_k\rangle + \beta_k\langle f(x_k) - f(y_k), x_{k+1} - y_k\rangle \\ &\leq \beta_k\langle f(x_k) - f(y_k), x_{k+1} - y_k\rangle \\ &\leq \beta_k||f(x_k) - f(y_k)||||x_{k+1} - y_k|| \end{aligned} \tag{3.3.3}$$

Substituting (3.3.3) into (3.3.2), we obtain

$$\begin{aligned} ||x_{k+1} - u||^2 &\leq ||x_k - u||^2 - ||x_k - y_k||^2 - ||y_k - x_{k+1}||^2 \\ &\quad +2\beta_k||f(x_k) - f(y_k)||||x_{k+1} - y_k|| + 2\beta_k\langle f(u), u - y_k\rangle \\ &\leq ||x_k - u||^2 - ||x_k - y_k||^2 + \beta_k||f(x_k) - f(y_k)||^2 + 2\beta_k\langle f(u), u - y_k\rangle \end{aligned}$$

The subsequent prove is divided into two cases;
Case 1: $f(u) \neq 0$
Using theorem (2.2.4), there exist $\beta_u > 0$ such that $f(u) = -\beta_u c'(u)$.By the subdifferentiablity of c and noting that $c(u) = 0$ since $u \in \partial C$, we have

$$\begin{aligned} c(u) + \langle c'(u), y_k - u\rangle &\leq c(y_k) \\ \langle -\beta_u c'(u), y_k - u\rangle &\geq -\beta_u c(y_k) \\ \langle f(u), u - y_k\rangle &\geq -\beta_u c(y_k) \\ \langle f(u), u - y_k\rangle &\leq \beta_u c(y_k) \end{aligned} \tag{3.3.4}$$

From the definition of C_k

$$c(x_k) + \langle c'(x_k), y_k - x_k\rangle \leq 0 \tag{3.3.5}$$

By sub-differentiability of c at y_k we have

$$c(y_k) + \langle c'(y_k), x_k - y_k\rangle \leq c(x_k) \tag{3.3.6}$$

Adding (3.3.5) and (3.3.6), we obtain

$$\begin{aligned} c(y_k) &+ \langle c'(y_k), x_k - y_k\rangle \leq \langle c'(x_k), x_k - y_k\rangle \\ &\Rightarrow c(y_k) \leq \langle c'(x_k) - c'(y_k), x_k - y_k\rangle \end{aligned} \tag{3.3.7}$$

Combining (3.3.4) and (3.3.7) and using condition (2.2.3), we have

$$\begin{aligned} \langle f(u), u - y_k\rangle &\leq \beta_u c(y_k) \\ &\leq \beta_u\langle c'(x_k) - c'(y_k), x_k - y_k\rangle \\ &\leq \beta_u||x_k - y_k||||c'(x_k) - c'(y_k)|| \\ &\leq M_2||x_k - y_k||||c'(x_k) - c'(y_k)|| \\ &\leq M_1M_2||x_k - y_k||^2 \\ &\leq M||x_k - y_k||^2 \end{aligned} \tag{3.3.8}$$

Substituting (3.3.8) in (3.3.4), we obtain

$$||x_{k+1} - u||^2 \leq ||x_k - u||^2 - ||x_k - y_k||^2 + \beta_k^2||f(x_k) - f(y_k)||^2 + 2M\beta_k^2||x_k - y_k||^2$$

Finaly, from the condition of β_k given in (3.2.1), we get

$$||x_{k+1} - u||^2 \leq ||x_k - u||^2 - (1 - v^2)||x_k - y_k||^2$$

Case 2: $f(u) = 0$
From (3.3.4), we have that

$$||x_{k+1} - u||^2 \leq ||x_k - u||^2 - ||x_k - y_k||^2 + \beta_k^2||f(x_k) - f(y_k)||^2$$

Obviously, (3.2.2) implies

$$\beta_k^2||f(x_k) - f(y_k)||^2 \leq v^2||x_k - y_k||^2$$

So,

$$||x_{k+1} - u||^2 \leq ||x_k - u||^2 - (1 - v^2)||x_k - y_k||^2 \tag{3.3.9}$$

■

Theorem 3.3.2 *Assume that conditions(2.2.1),(2.2.2) and (2.2.3) holds. Then the two sequences* $\{x_k\}_{k=0}^{\infty}$ *and* $\{y_k\}_{k=0}^{\infty}$ *generated by Algorithm (3.2.3) converges weakly to some point* $z \in SOL(C, f)$ *and further more*

$$z = \lim_{k\to\infty} \mathbb{P}_{SOL(C,f)} x_k$$

Proof By lemma (3.3.1),

$$||x_{k+1} - u||^2 \leq ||x_k - u||^2 \quad \forall k \geq 0$$

So there exist

$$a = \lim_{k\to\infty} ||x_k - u||^2$$

So from (3.3.9),

$$||x_k - y_k||^2 \leq \frac{1}{1-v^2}[||x_k - u||^2 - ||x_{k+1} - u||^2]$$

This implies that $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ are bounded and

$$x_k - y_k \to 0 \quad as \quad k \to \infty$$

Also, Since f is Lipschitz-continuous, we have

$$f(x_k) - f(y_k) \to 0 \quad as \quad k \to \infty$$

Let $w(x_k)$ be the set of weak limit points of $\{x_k\}_{k=0}^{\infty}$, *i.e.*;

$$w(x_k) = \{z \in E : \exists\{x_{k_j}\}_{k=0}^{\infty} \subset \{x_k\}_{k=0}^{\infty} \quad s.t \quad x_{k_j} \rightharpoonup z\}$$

Since the sequence $\{x_k\}_{k=0}^{\infty}$ is bounded, $w(x_k) \neq \emptyset$. Taking $z \in w(x_k)$ arbitrarily, Then $\exists\{x_{k_j}\}_{j=0}^{\infty} \subset \{x_k\}_{k=0}^{\infty} \quad s.t \quad x_{k_j} \rightharpoonup z \quad as \quad j \to \infty$
Also,

$$\exists\{y_{k_j}\}_{j=0}^{\infty} \subset \{y_k\}_{k=0}^{\infty} \quad s.t \quad y_{k_j} \rightharpoonup z \quad as \quad j \to \infty$$

Since $y_k \in C_k$,

$$c(x_k) + \langle y_k - x_k, c'(x_k)\rangle \leq 0$$

$\iff$

$$c(x_k) - \langle x_k - y_k, c'(x_k)\rangle \leq 0$$

$\Rightarrow$

$$\begin{aligned} c(x_k) &\leq ||c'(x_k)||||x_k - y_k|| \\ &\leq M'||x_k - y_k|| \end{aligned}$$

So,

$$c(x_{k_j}) \leq M'||x_{k_j} - y_{k_j}||$$

Since c is lower semi-continuous and $x_{k_j} \rightharpoonup z \quad as \quad j \to \infty$, then,

$$c(z) \leq \liminf_{j\to\infty} c(x_{k_j}) \leq 0$$

Hence, $z \in C$.
We now turn to showing that $z \in SOL(C, f)$.
Define

$$Tv = \begin{cases} f(v) + N_c(v), & if \ v \in C, \\ \emptyset, & if \ v \notin C \end{cases} \tag{3.3.10}$$

Clearly, T is maximal monotone, To show tha $0 \in Tz$, we proceed as follows. Let

$$(v, w) \in G(T),$$

This implies that

$$w \in Tv$$

But

$$Tv = f(v) + N_c(v)$$

So,

$$w - f(v) \in N_c(v)$$

By definition of $N_c(v)$, we have

$$\langle w - f(v), y - v\rangle \leq 0 \quad \forall y \in C$$

Since $z \in C$,

$$\langle w - f(v), z - v\rangle \leq 0 \tag{3.3.11}$$

Since $v \in C \subset T_k$, by definition of T_k,

$$\langle x_k - \beta_k f(x_k) - y_k, v - y_k\rangle \leq 0$$

$$\begin{aligned}
\Rightarrow \quad & \langle x_k - \beta_k f(x_k) - y_k, v - y_k\rangle \leq 0 \\
\Rightarrow \quad & \langle \frac{x_k - y_k}{\beta_k} - f(x_k), y_k - v\rangle \geq 0 \\
\Rightarrow \quad & \langle \frac{y_k - x_k}{\beta_k} + f(x_k), v - y_k\rangle \geq 0
\end{aligned} \tag{3.3.12}$$

From equation(3.3.11),

$$\langle w, z - v\rangle \leq \langle f(v), z - v\rangle$$

$$\begin{aligned}
\Rightarrow \quad & -\langle w, v - z\rangle \leq -\langle f(v), v - z\rangle \\
\Rightarrow \quad & \langle f(v), v - z\rangle \geq \langle f(v), v - z\rangle
\end{aligned} \tag{3.3.13}$$

From equation (3.3.12),

$$\langle \frac{y_{k_j} - x_{k_j}}{\beta} + f(x_{k_j}), v - y_{k_j}\rangle \geq 0 \ where\ \beta_k \leq \beta$$

So, equation (3.3.13) becomes,

$$\begin{aligned}
\langle w, v - z\rangle \geq\ & \langle f(v), v - z\rangle \\
\geq\ & \langle f(v), v - z\rangle - \langle \frac{Jy_{k_j} - Jx_{k_j}}{\beta} + f(x_{k_j}), v - y_{k_j}\rangle \\
=\ & \langle f(v), v - y_{k_j} + y_{k_j} - z\rangle - \langle \frac{Jy_{k_j} - Jx_{k_j}}{\beta} + f(x_{k_j}), v - y_{k_j}\rangle \\
=\ & \langle f(v), v - y_{k_j}\rangle + \langle f(v), y_{k_j} - z\rangle - \langle v - y_{k_j}, \frac{Jy_{k_j} - Jx_{k_j}}{\beta}\rangle \\
& -\langle f(x_{k_j}), v - y_{k_j}\rangle \\
=\ & \langle f(v) - f(y_{k_j}), v - y_{k_j}\rangle + \langle f(v), y_{k_j} - z\rangle - \langle \frac{Jy_{k_j} - Jx_{k_j}}{\beta}, v - y_{k_j}\rangle \\
& -\langle f(x_{k_j}) - f(y_{k_j}), v - y_{k_j}\rangle \\
=\ & \langle f(v) - f(y_{k_j}), v - y_{k_j}\rangle + \langle f(v), y_{k_j} - z\rangle - \langle \frac{Jy_{k_j} - Jx_{k_j}}{\beta}, v - y_{k_j}\rangle \\
& +\langle f(y_{k_j}) - f(x_{k_j}), v - y_{k_j}\rangle
\end{aligned} \tag{3.3.14}$$

So,

$$\begin{aligned} \langle w, v - z\rangle &\geq \langle f(v), y_{k_j} - z\rangle - \langle \frac{Jy_{k_j} - Jx_{k_j}}{\beta}, v - y_{k_j}\rangle && (3.3.15)\\ &+\langle f(y_{k_j}) - f(x_{k_j}), v - y_{k_j}\rangle && (3.3.16) \end{aligned}$$

Now, taking limit as $j \to \infty$ we have,

$$\langle w, v - z\rangle \geq 0$$

Hence

$$0 \in Tz$$

and consequently,

$$z \in T^{-1}0 = SOL(C, f).$$

We now show that $x_k \rightharpoonup z$ as $k \to \infty$.
Suppose there exist another $\{x_{k_i}\}_{k=0}^{\infty} \subset \{x_k\}_{k=0}^{\infty}$ *such that* $x_{k_i} \rightharpoonup \bar{z} \in SOL(C, f)$ as $i \to \infty$ but $z \neq \bar{z}$, noting that $\{\phi(u, x_k)\}_{k=0}^{\infty}$ and using the opial condition, we have

$$\begin{aligned} \lim_{k\to\infty} ||x_k - z|| = \lim_{j\to\infty} ||x_{k_j} - z|| &< \lim_{j\to\infty} ||x_{k_j} - \bar{z}|| \\ &= \lim_{k\to\infty} ||x_k - \bar{z}|| = \lim_{i\to\infty} ||x_{k_i} - \bar{z}|| \\ &< \lim_{i\to\infty} ||x_{k_i} - z|| = \lim_{k\to\infty} ||x_k - z|| \end{aligned} \tag{3.3.17}$$

This is a contradiction, so $\bar{z} = z$. Consequently, we have that $x_k \rightharpoonup z$ and $y_k \rightharpoonup z$ as $k \to \infty$
Finally we show that

$$z = \lim_{k\to\infty} \mathbb{P}_{SOL(C,f)} x_k.$$

Let

$$u_k = \mathbb{P}_{SOL(C,f)} x_k$$

By lemma(2.2.12) there exist $u^* \in SOL(C, f)$ such that $u_k \to u$ *as* $k \to \infty$
Since

$$u_k = \mathbb{P}_{SOL(C,f)} x_k$$

and

$$z \in SOL(C, f)$$

We have that

$$\langle z - u_k, x_k - u_k\rangle \leq 0$$

$\Rightarrow$

$$\langle z - u^*, z - u^*\rangle \leq 0$$

$$\begin{aligned} 0 &\geq \langle z - u^*, z - u^*\rangle \\ &= ||z||^2 - \langle z, u^*\rangle - \langle u^*, z\rangle + ||u^*||^2 \\ &\geq ||z||^2 - 2||z||||u^*|| + ||u^*||^2 \\ &= (||z|| - ||u^*||)^2 \geq 0 \end{aligned}$$

So,

$$\langle z - u^*, z - u^*\rangle = 0$$

Therefore $z = u^*$, This complete the proof. ■

CHAPTER 4

MAIN RESULTS

In this chapter we will present an extension of the results of Songnian He and Tao Wu [26], in a uniformly smooth and 2-uniformly convex real Banach spaces.

4.1 The Modified Subgradient Extragradient Method

In this section, we give our algorithm for solving the $VI(C, f)$ in the setting of uniformly smooth and 2-uniformly convex real Banach spaces, where C is a level set given as follows:

$$C := \{x \in E : c(x) \leq 0\},$$

where $c : E \to R$ is a convex function.

Algorithm 4.1.1 *Select an initial guess $x_0 \in E$ arbitrarily, set $k = 0$ and construct the half-space*

$$C_k := \{w \in E : c(x_k) + \langle c'(x), w - x_k\rangle \leq 0\};$$

Given the current iteration x_k, compute

$$y_k = \Pi_{C_k}(x_k - \beta_k f(x_k)),$$

where

$$\beta_k = \sigma\rho^{m_k},\ \sigma > 0,\ \rho \in (0, 1) \tag{4.1.1}$$

and m_k is the smallest nonnegative integer, such that

$$\beta_k^2||f(x_k) - f(y_k)||^2 + 2M\beta_k||x_k - y_k||^2 \leq v^2||x_k - y_k||^2, \tag{4.1.2}$$

where $M = M_1M_2$ and $v \in (0, 1)$
then calculate the next iterate,

$$x_{k+1} = \Pi_{T_k}(x_k - \beta_k f(y_k))$$

where

$$T_k := \{w \in E : \langle Jx_k - \beta_k f(x_k) - Jy_k, w - y_k\rangle \leq 0\}$$

which is the same as the half-space in Censor's method

$$\begin{cases} x_0 \in E, k = 0 \\ C_k := \{w \in E : c(x_k) + \langle c'(x_k), w - x_k\rangle \leq 0\}; \\ y_k = \Pi_{C_k}J^{-1}(Jx_k - \beta_k f(x_k)); \\ T_k := \{w \in E : \langle Jx_k - \beta_k f(x_k) - Jy_k, w - y_k\rangle \leq 0\}; \\ x_{k+1} = \Pi_{T_k}J^{-1}(Jx_k - \beta_k f(y_k)). \end{cases} \tag{4.1.3}$$

4.2 Convergence theorem of the algorithm

In this section, we prove the weak convergence theorem for Algorithm(4.1.3). First of all, we give the following lemma, which plays a crucial role in the proof of our main result.

Lemma 4.2.1 *Let $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ be the two sequences generated by (4.1.3). Let $u \in SOL(C,f)$ and let β_k be selected as (3.2.1). Then under the conditions (2.2.1),(2.2.2) and (2.2.3), we have*

$$\phi(x_{k+1},u) \leq \phi(x_k,u) - (1-v^2)\phi(x_k,y_k) \quad \forall k \geq 0 \tag{4.2.1}$$

Proof Let $u \in SOL(C,f)$ arbitrarily, for all $k \geq 0$ and using f being monotone, we have

$$\phi(J^{-1}(Jx_k - \beta_k f(y_k)), x_{k+1}) + \phi(x_{k+1},u) \leq \phi(J^{-1}(Jx_k - \beta_k f(y_k)), u)$$

$$\begin{aligned}
\Rightarrow \phi(x_{k+1},u) &\leq \phi(J^{-1}(Jx_k - \beta_k f(y_k)), u) - \phi(J^{-1}(Jx_k - \beta_k f(y_k)), x_{k+1}) \\
&= ||u||^2 - 2\langle Jx_k - \beta_k f(y_k), u\rangle + ||Jx_k - \beta_k f(y_k)||^2 \\
&\quad -||x_{k+1}||^2 + 2\langle Jx_k - \beta_k f(y_k), x_{k+1}\rangle - ||Jx_k - \beta_k f(y_k)||^2 \\
&= ||u||^2 - 2\langle Jx_k, u\rangle + 2\langle \beta_k f(y_k), u\rangle + ||x_{k+1}||^2 \\
&\quad +2\langle Jx_k, x_{k+1}\rangle - 2\langle \beta_k f(y_k), x_{k+1}\rangle \\
&= \phi(x_k,u) - \phi(x_k,x_{k+1}) + 2\beta_k\langle f(y_k), u - x_{k+1}\rangle \\
&= \phi(x_k,u) - \phi(x_k,x_{k+1}) + 2\beta_k\langle f(y_k) - f(u) + f(u), u + y_k - y_k - x_{k+1}\rangle \\
&= \phi(x_k,u) - \phi(x_k,x_{k+1}) + 2\beta_k[\langle f(y_k) - f(u), u - y_k\rangle \\
&\quad +\langle f(u), u - y_k\rangle + \langle f(y_k) - f(u), y_k - x_{k+1}\rangle + \langle f(u), y_k - x_{k+1}] \\
&= \phi(x_k,u) - \phi(x_k,x_{k+1}) + 2\beta_k[\langle f(y_k) - f(u), u - y_k\rangle \\
&\quad +\langle f(u), u - y_k\rangle + \langle f(y_k), y_k - x_{k+1}\rangle] \\
&\leq \phi(x_k,u) - \phi(x_k,x_{k+1}) + 2\beta_k[\langle f(u), u - y_k\rangle + \langle f(y_k), y_k - x_{k+1}\rangle] \\
&= \phi(x_k,u) - \phi(y_k,x_{k+1}) - \phi(x_k,y_k) \\
&\quad -2\langle Jx_k - Jy_k, y_k - x_{k+1}\rangle + 2\beta_k[\langle f(u), u - y_k\rangle + \langle f(y_k), y_k - x_{k+1}\rangle] \\
&= \phi(x_k,u) - \phi(y_k,x_{k+1}) - \phi(x_k,y_k) \\
&\quad -2\langle Jx_k - \beta_k f(y_k) - Jy_k, x_{k+1} - y_k\rangle + 2\beta_k\langle f(u), u - y_k\rangle
\end{aligned} \tag{4.2.2}$$

By the definition of T_k, we have

$$\begin{aligned}
\langle Jx_k - \beta_k f(y_k) - Jy_k, x_{k+1} - y_k\rangle &= \langle Jx_k - \beta_k f(x_k) - Jy_k, x_{k+1} - y_k\rangle + \beta_k\langle f(x_k) - f(y_k), x_{k+1} - \\
&\leq \beta_k\langle f(x_k) - f(y_k), x_{k+1} - y_k\rangle \\
&\leq \beta_k||f(x_k) - f(y_k)||||x_{k+1} - y_k|| \qquad (4.2
\end{aligned}$$

Substituting (4.2.3) into (4.2.2) and using the fact that E is uniformly smooth and 2-uniformly convex we obtain

$$\begin{aligned}
\phi(x_{k+1},u) &\leq \phi(x_k,u) - \phi(y_k,x_{k+1}) - \phi(x_k,y_k) + 2\beta_k||f(x_k) - f(y_k)||||x_{k+1} - y_k|| + 2\beta_k\langle f(u), u - \\
&\leq \phi(x_k,u) - \phi(y_k,x_{k+1}) - \phi(x_k,y_k) + ||f(x_k) - f(y_k)||^2 + {\beta_k}^2||x_{k+1} - y_k||^2 \\
&\quad +2\beta_k\langle f(u), u - y_k\rangle \\
&\leq \phi(x_k,u) - \phi(y_k,x_{k+1}) - \phi(x_k,y_k) + ||f(x_k) - f(y_k)||^2 + \frac{{\beta_k}^2}{\alpha}\phi(y_k,x_{k+1}) \\
&\quad +2\beta_k\langle f(u), u - y_k\rangle \\
&\leq \phi(x_k,u) - \phi(y_k,x_{k+1}) - \phi(x_k,y_k) + ||f(x_k) - f(y_k)||^2 + \phi(y_k,x_{k+1}) \\
&\quad +2\beta_k\langle f(u), u - y_k\rangle \\
&= \phi(x_k,u) - \phi(x_k,y_k) + ||f(x_k) - f(y_k)||^2 + 2\beta_k\langle f(u), u - y_k\rangle \qquad (4.2
\end{aligned}$$

The subsequent proves is divided into two cases;
Case 1: $f(u) \neq 0$
Using theorem (3.5), there exist $\beta_u > 0$ such that $f(u) = -\beta_u c'(u)$.By the subdifferentiablity of c and noting that $c(u) = 0$ since $u \in \partial C$, we have

$$\begin{aligned} c(u) + \langle c'(u), y_k - u \rangle &\leq c(y_k) \\ \langle -\beta_u c'(u), y_k - u \rangle &\geq -\beta_u c(y_k) \\ \langle f(u), u - y_k \rangle &\geq -\beta_u c(y_k) \\ \langle f(u), u - y_k \rangle &\leq \beta_u c(y_k) \end{aligned} \tag{4.2.5}$$

From the definition of C_k

$$c(x_k) + \langle c'(x_k), y_k - x_k \rangle \leq 0 \tag{4.2.6}$$

By subdifferentiability of c at y_k we have

$$c(y_k) + \langle c'(y_k), x_k - y_k \rangle \leq c(x_k) \tag{4.2.7}$$

Adding (4.2.6) and (4.2.7), we obtain

$$\begin{aligned} c(y_k) \quad + \quad & \langle c'(y_k), x_k - y_k \rangle \leq \langle c'(x_k), x_k - y_k \rangle \\ & \Rightarrow c(y_k) \leq \langle c'(x_k) - c'(y_k), x_k - y_k \rangle \end{aligned} \tag{4.2.8}$$

Combining (4.2.5) and (4.2.8) and using condition 3.3, we have

$$\begin{aligned} \langle f(u), u - y_k \rangle &\leq \beta_u c(y_k) \\ &\leq \beta_u \langle c'(x_k) - c'(y_k), x_k - y_k \rangle \\ &\leq \beta_u ||x_k - y_k|| ||c'(x_k) - c'(y_k)|| \\ &\leq M_2 ||x_k - y_k|| ||c'(x_k) - c'(y_k)|| \\ &\leq M_1 M_2 ||x_k - y_k||^2 \\ &\leq M ||x_k - y_k||^2 \\ &\leq \frac{M}{\alpha} \phi(x_k, y_k) \end{aligned} \tag{4.2.9}$$

Substituting (4.2.9) in (4.2.4), we obtain

$$\begin{aligned} \phi(x_{k+1}, u) &\leq \phi(x_k, u) - \phi(x_k, y_k) + ||f(x_k) - f(y_k)||^2 + 2\frac{\beta_k M}{\alpha} \phi(x_k, y_k) \\ &\leq \phi(x_k, u) - \phi(x_k, y_k) + ||f(x_k) - f(y_k)||^2 + 2M\phi(y_k, x_k) \end{aligned}$$

where m_k is the smallest non-negative integer such that

$$\beta_k = \sigma \rho_{m_k} \alpha$$

and

$$||f(x_k) - f(y_k)||^2 + 2M\phi(x_k, y_k) \leq v^2 \phi(x_k, y_k)$$

So,

$$\phi(x_{k+1}, u) \leq \phi(x_k, u) - (1 - v^2)\phi(x_k, y_k) \tag{4.2.10}$$

Case 2: $f(u) = 0$
From (4.2.4), we have

$$\phi(x_{k+1}, u) \leq \phi(x_k, u) - \phi(x_k, y_k) + ||f(x_k) - f(y_k)||^2$$

Clearly,

$$||f(x_k) - f(y_k)||^2 \leq v^2\phi(x_k, y_k)$$

We now have

$$\phi(x_{k+1}, u) \leq \phi(x_k, u) - (1 - v^2)\phi(x_k, y_k) \qquad where \quad v \in (0, 1)$$

■

Theorem 4.2.2 *Assume that conditions* 3.1 − 3.2 *holds. Then the two sequences* $\{x_k\}_{k=0}^{\infty}$ *and* $\{y_k\}_{k=0}^{\infty}$ *generated by Algorithm (4.1.3) converges weakly to some point* $z \in SOL(C, f)$ *and further more*

$$z = \lim_{k\to\infty} \Pi_{SOL(C,f)}x_k$$

Proof *By lemma (4.2.1),*

$$\phi(x_{k+1}, u) \leq \phi(x_k, u) \quad \forall k \geq 0$$

So there exist

$$a = \lim_{k\to\infty} \phi(u, x_k)$$

So from equation (4.2.10),

$$\phi(x_k, y_k) \leq \frac{1}{1 - v^2}(\phi(x_k, u) - \phi(x_{k+1}, u))$$

This implies that

$$\phi(x_k, y_k) \to 0 \quad as \quad k \to \infty$$

Thus $\{x_k\}_{k=0}^{\infty}$ *and* $\{y_k\}_{k=0}^{\infty}$ *are bounded and*

$$x_k - y_k \to 0 \quad as \quad k \to \infty$$

Also, Since f *is Lipschitz-continuous, we have*

$$f(x_k) - f(y_k) \to 0 \quad as \quad k \to \infty$$

Let $w(x_k)$ *be the set of weak limit points of* $\{x_k\}_{k=0}^{\infty}$*, i.e.;*

$$w(x_k) = \{z \in E : \exists\{x_{k_j}\}_{k=0}^{\infty} \subset \{x_k\}_{k=0}^{\infty} \quad s.t \quad x_{k_j} \rightharpoonup z\}$$

Since the sequence $\{x_k\}_{k=0}^{\infty}$ *is bounded,* $w(x_k) \neq \emptyset$*. Taking* $z \in w(x_k)$ *arbitrarily, Then* $\exists\{x_{k_j}\}_{j=0}^{\infty} \subset \{x_k\}_{k=0}^{\infty} \quad s.t \quad x_{k_j} \rightharpoonup z \quad as \quad j \to \infty$
Also,

$$\exists\{y_{k_j}\}_{j=0}^{\infty} \subset \{y_k\}_{k=0}^{\infty} \quad s.t \quad y_{k_j} \rightharpoonup z \quad as \quad j \to \infty$$

Since $y_k \in C_k$*,*

$$c(x_k) + \langle c'(x_k), y_k - x_k\rangle \leq 0$$

$\Longleftrightarrow$

$$c(x_k) - \langle c'(x_k), x_k - y_k\rangle \leq 0$$

$\Rightarrow$

$$\begin{aligned} c(x_k) &\leq ||c'(x_k)||||x_k - y_k|| \\ &\leq M'||x_k - y_k|| \end{aligned}$$

So,

$$c(x_{k_j}) \leq M'||x_{k_j} - y_{k_j}||$$

Since c is lower semi-continuous and $x_{k_j} \rightharpoonup z \quad as \quad j \to \infty$, then,

$$c(z) \leq \liminf_{j\to\infty} c(x_{k_j}) \leq 0$$

Hence, $z \in C$.
We now turn to showing that $z \in SOL(C, f)$.
Define

$$Tv = \begin{cases} f(v) + N_c(v), & if\ v \in C, \\ \emptyset, & if\ v \notin C \end{cases} \tag{4.2.11}$$

Clearly, T is maximal monotone, To show tha $0 \in Tz$, we proceed as follows. Let

$$(v, w) \in G(T),$$

This implies that

$$w \in Tv$$

But

$$Tv = f(v) + N_c(v)$$

So,

$$w - f(v) \in N_c(v)$$

By definition of $N_c(v)$, we have

$$\langle w - f(v), y - v\rangle \leq 0 \quad \forall y \in C$$

Since $z \in C$,

$$\langle w - f(v), z - v\rangle \leq 0 \tag{4.2.12}$$

Since $v \in C \subset T_k$, by definition of T_k,

$$\langle Jx_k - \beta_k f(x_k) - Jy_k, v - y_k\rangle \leq 0$$

$$\begin{aligned} \Rightarrow \quad & \langle Jx_k - \beta_k f(x_k) - Jy_k, v - y_k\rangle \leq 0 \\ \Rightarrow \quad & \langle \frac{Jx_k - Jy_k}{\beta_k} - f(x_k), y_k - v\rangle \geq 0 \\ \Rightarrow \quad & \langle \frac{Jy_k - Jx_k}{\beta_k} + f(x_k), v - y_k\rangle \geq 0 \end{aligned} \tag{4.2.13}$$

From equation(4.2.12),

$$\langle w, z - v\rangle \leq \langle f(v), z - v\rangle$$

$$\begin{aligned} \Rightarrow \quad & -\langle w, v - z\rangle \leq -\langle f(v), v - z\rangle \\ \Rightarrow \quad & \langle w, v - z\rangle \geq \langle f(v), v - z\rangle \end{aligned} \tag{4.2.14}$$

From equation (4.2.13),

$$\langle \frac{Jy_{k_j} - Jx_{k_j}}{\beta} + f(x_{k_j}), v - y_{k_j}\rangle \geq 0 \ where\ \beta_k \leq \beta$$

So, equation (4.2.14) becomes,

$$\begin{aligned}
\langle w, v-z\rangle &\geq \langle f(v), v-z\rangle \\
&\geq \langle f(v), v-z\rangle - \langle \frac{Jy_{k_j}-Jx_{k_j}}{\beta} + f(x_{k_j}), v-y_{k_j}\rangle \\
&= \langle f(v), v-y_{k_j}+y_{k_j}-z\rangle - \langle \frac{Jy_{k_j}-Jx_{k_j}}{\beta} + f(x_{k_j}), v-y_{k_j}\rangle \\
&= \langle f(v), v-y_{k_j}\rangle + \langle f(v), y_{k_j}-z\rangle - \langle \frac{Jy_{k_j}-Jx_{k_j}}{\beta}, v-y_{k_j}\rangle \\
&\quad -\langle f(x_{k_j}), v-y_{k_j}\rangle \\
&= \langle f(v)-f(y_{k_j}), v-y_{k_j}\rangle + \langle f(v), y_{k_j}-z\rangle - \langle \frac{Jy_{k_j}-Jx_{k_j}}{\beta}, v-y_{k_j}\rangle \\
&\quad -\langle f(x_{k_j})-f(y_{k_j}), v-y_{k_j}\rangle \\
&= \langle f(v)-f(y_{k_j}), v-y_{k_j}\rangle + \langle f(v), y_{k_j}-z\rangle - \langle \frac{Jy_{k_j}-Jx_{k_j}}{\beta}, v-y_{k_j}\rangle \\
&\quad +\langle f(y_{k_j})-f(x_{k_j}), v-y_{k_j}\rangle \qquad (4.2.15)
\end{aligned}$$

So,

$$\begin{aligned}
\langle w, v-z\rangle &\geq \langle f(v), y_{k_j}-z\rangle - \langle \frac{Jy_{k_j}-Jx_{k_j}}{\beta}, v-y_{k_j}\rangle \qquad (4.2.16)\\
&\quad +\langle f(y_{k_j})-f(x_{k_j}), v-y_{k_j}\rangle \qquad (4.2.17)
\end{aligned}$$

Now, taking limit as $j\to\infty$ we have,

$$\langle w, v-z\rangle \geq 0$$

Hence

$$0\in Tz$$

and consequently,

$$z\in T^{-1}0 = SOL(C,f).$$

We now show that $x_k \rightharpoonup z$ as $k\to\infty$.
Suppose there exist another $\{x_{k_i}\}_{k=0}^{\infty}\subset\{x_k\}_{k=0}^{\infty}$ such that $x_{k_i}\rightharpoonup \bar{z}\in SOL(C,f)$ as $i\to\infty$ but $z\neq\bar{z}$, noting that $\{\phi(x_k,u)\}_{k=0}^{\infty}$ and using the opial condition, we have

$$\begin{aligned}
\lim_{k\to\infty}||x_k-z|| = \lim_{j\to\infty}||x_{k_j}-z|| &< \lim_{j\to\infty}||x_{k_j}-\bar{z}|| \\
&= \lim_{k\to\infty}||x_k-\bar{z}|| = \lim_{i\to\infty}||x_{k_i}-\bar{z}|| \\
&< \lim_{i\to\infty}||x_{k_i}-z|| = \lim_{k\to\infty}||x_k-z||
\end{aligned}$$

This is a contadiction, so $\bar{z}=z$. Consequently, we have that $x_k\rightharpoonup z$ and $y_k\rightharpoonup z$ as $k\to\infty$
Finally we show that

$$z=\lim_{k\to\infty}\Pi_{SOL(C,f)}x_k.$$

Let

$$u_k = \Pi_{SOL(C,f)}x_k$$

By lemma (2.2.12) there exist $u^* \in SOL(C, f)$ *such that* $u_k \to u$ *as* $k \to \infty$
Since J *is weakly sequentially continuous on bounded sets and* $x_k \rightharpoonup z$ *as* $k \to \infty$*, we have that*

$$Jx_k \rightharpoonup Jz \text{ as } k \to \infty$$

Since

$$u_k = \Pi_{SOL(C,f)} x_k$$

and

$$z \in SOL(C, f)$$

We have that

$$\langle Jx_k - Ju_k, z - u_k \rangle \leq 0$$

$\Rightarrow$

$$\langle Jz - Ju^*, z - u^* \rangle \leq 0$$

$$\begin{aligned} 0 &\geq \langle Jz - Ju^*, z - u^* \rangle \\ &= ||z||^2 - \langle Ju^*, z \rangle - \langle Jz, u^* \rangle + ||u^*||^2 \\ &\geq ||z||^2 - 2||z||||u^*|| + ||u^*||^2 \\ &= (||z|| - ||u^*||)^2 \geq 0 \end{aligned}$$

So,

$$\langle Jz - Ju^*, z - u^* \rangle = 0$$

Therefore $z = u^*$*, This complete the proof.* ■

4.2.1 Conclusion

Although the extragradient method and the subgradient methods has been widely studied, the existing algorithms all face the problem that the projection operator is hard to calculate. The problem can be solved effectively by using the modified subgradient extragradient method proposed in this paper, since two projections onto the original domain are all replaced with projections onto some half-spaces, which is very easily calculated. Besides, the step size can be selected in some adaptive ways, which means that we have no need to know the Lipschitz constant of the operator. Furthermore, our result was proved in uniformly smooth and 2-uniformly convex real Banach spaces.

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