APPROXIMATION OF SOLUTION OF GENERALIZED EQUILIBRIUM PROBLEMS AND COMMON FIXED POINT OF FINITE FAMILY OF STRICT PSEUDOCONTRACTIONS WITH APPLICATION

BY

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CERTIFICATION

This is to certify that the thesis titled "Approximation of Solution of Generalized Equilibrium Problems and Common Fixed Point of Finite Family of Strict Pseudocontractions with Application" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research carried out by Nwokpoku, Ikechukwu Ugbanu in Department of Pure and Applied Mathematics.

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ABSTRACT

In this thesis, we consider the problem of approximating solution of generalized equilibrium problems and common fixed point of finite family of strict pseudocontractions. The result obtained is applied in approximation of solution of generalized mixed equilibrium problems and common fixed point of finite family of strict pseudocontractions. Our theorems improve and unify some existing results that were recently announced by several authors. Corollaries obtained and our method of proof are of independent interest.

keywords and Phrases: Equilibrium problems, Fixed point problems, Generalized equilibrium problems, Generalized mixed equilibrium problem,Variational inequality problems, Halpern-type algorithm, Strictly pseudocontractive mappings, firmly nonexpansive maps.

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DEDICATION

To our Lord Jesus Christ, for His grace and infinite mercy.

CONTENTS

CHAPTER 1

INTRODUCTION

The content of this thesis falls within the area of nonlinear operator theory. This area has attracted attention of several researchers due to its wide range of application in different areas of pure and applied sciences. The research documented in this thesis concentrated on the following topic: *Approximation of solution of generalized equilibrium problems and common fixed point of finite family of strict pseudocontractions.*

1.1 Background of Study

In sciences, engineering, economics and in some other areas where there is a quantitative analysis, we are greatly interested in describing how systems evolve in time, that is, in describing system's dynamics. We will restrict ourselves to one dimensional case for the purpose of illustration. We will always write $u = u(t)$, which is the state of the system. We think of the dependent variable u as the state variable of a system that is varying with time t, which is the independent variable. Thus, knowing u is virtually the same as knowing what state the system is, at time t. For example, $u(t)$ could be the number of patience admitted in a hospital, the quantity of data processed by CPU, the concentration of a chemical substance such as sugar in the body, the number of immigrants into a country, the current in an electrical circuit, the speed of a spacecraft, or the monthly sales of an advertised item. Knowledge of $u(t)$ for a given system tells us how the system changes with respect to time. Often, we relate the state $u(t)$ to its rates of change, as expressed by its derivatives $u^{'}(t)$, $u^{''}(t), \cdots$, and so on. It is important to note that some of the dynamical system can be described by the following model,

$$\frac{\mathrm{du}}{\mathrm{dt}} + Au = f(t, u(t)). \tag{1.1}$$

Where A is an operator defined on some appropriate spaces. Equation (1.1) is called nonhomogeneous first order ordinary differential equation if $f(t, u(t)) \neq 0$, otherwise it is homogeneous first order ordinary differential equation. Assuming that $u(t)$ is a solution to equation (1.1) and suppose that t_0 is the initial reference time that we want to start studying the above model, we can always use $u(t)$ to make comparative analysis of the behaviour of the dynamical system between the time t_0 and t. If $f(t, u(t)) = 0$, then equation (1.1) becomes

$$\frac{\mathrm{du}}{\mathrm{dt}} + Au = 0. \tag{1.2}$$

If we put $A \equiv 0$ in equation (1.1), then, equation (1.1) reduces to

$$\frac{\mathrm{du}}{\mathrm{dt}} = f(t, u(t)). \tag{1.3}$$

Picard proved that under some certain assumptions on f, its domain and co-domain, that problem (1.3) is equivalent to problem of finding fixed point of an operator T defined by

$$\begin{aligned}(Tu)(t) &= \phi(t) \\ &= u_0 + \int_{t_0}^{t} f(s, u(s))\mathrm{d}s,\end{aligned} \tag{1.4}$$

where T is a self map defined on some appropriate infinite dimensional function space and $u_0 = u(t_0)$. Though equation (1.3) looks simple, it happens that most times, we do not have exact solution of equation (1.3) rather the numerical solution. This numerical solution corresponds to the approximated fixed point of some nonlinear operators. Furthermore, it is well known that at equilibrium state, $\frac{\mathrm{du}}{\mathrm{dt}} = 0$, hence at equilibrium state, equation (1.2) becomes

$$Au = 0. \tag{1.5}$$

Consequently, equation (1.2) reduces to problem of finding zero (zeros) of A which corresponds (correspond) to problem of finding fixed point of some operator T by defining $A \equiv I - T$.

We recall that if a function f is twice differentiable at a point x^* i.e $f''(x^*)$ exists and $f''(x^*) \neq 0$ and $f'(x^*) = 0$ then, x^* is an extremum point. This leads us to the following question: How do we get the optimizer of a function whenever it exist without necessarily differentiating f in the usual sense? We have to note that some of the important operators involved in optimization problems are not differentiable in the usual sense. We give an example to illustrate our point. Consider the map $f : H \to \mathbb{R}$ defined by

$$f(x) = \|x\|,$$

where H is a real Hilbert space. It is well known that f is not differentiable at zero. However, it is easy to see that zero is the minimizer. From the foregoing analysis, it is worthy to study optimization problems.

Let us consider the problem of finding $u^* \in K$ such that

$$f(u^*, y) \geq 0, \ \forall\, y \in K, \tag{1.6}$$

where K is a nonempty, closed and convex subset of real Hilbert space H and $f : K \times K \to \mathbb{R}$, a bifunction. We observe that it includes fixed point problems and optimization problems as special cases. Furthermore, if we consider a nonlinear operator $A : K \longrightarrow H$ and a problem of finding $x^* \in K$ such that

$$f(u^*, y) = \langle Au^*, y - u^* \rangle \geq 0, \ \forall\, y \in K. \tag{1.7}$$

We obtain another special case of equation (1.6)

If however, we consider the problem of finding $u^* \in K$ such that

$$f(u^*, y) + \langle Au^*, y - u^* \rangle \geq 0, \ \forall\, y \in K, \tag{1.8}$$

then, we have a new problem which include problems (1.6) and (1.7) as special cases, we are going to study problem (1.8) extensively in this thesis.

Problem (1.6) was introduced by Blum and Oettli (1994) and Noor and Oettli (1994). It has a great impact and influence in the development of several branches of Pure and Applied Sciences.

Motivated by the above example and forgoing analysis, We are interested in studying some iterative algorithm for approximating the solution of equation (1.8) and common fixed point of finite family of strict pseudocontractions.

We present some preliminary results, definitions and some well known facts in Hilbert spaces, understanding them plays a crucial role in comprehending the entire work. We shall therefore, immediately turn to the preliminary section where most of the necessary definitions and explanation of terms are displayed.

1.2 Preliminary

In this section, we give definitions of some crucial concepts that shall be needed in sequel.

Definition 1.1. Let $T : D(T) \subseteq H \to H$ be a map. then, T is said to be

(i) Asymptotically k-strictly pseudocontraction in the intermediate sense (Sahu, *et ai.*, 2008) with sequence $\{\gamma_n\}$ if there exists a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that for all $x, y \in K$ and for all $n \in \mathbb{N}$,

$$\limsup_{n\to\infty} \sup_{x,y\in K} (\|T^n x - T^n y\| - (1 + \gamma_n)\|x - y\|^2 - k\|(I - T^n)x - (I - T^n)y\|^2) \leq 0. \quad (1.9)$$

(ii) k-Lipschitz if there exists $k \geq 0$ such that for all $x, y \in D(T)$,

$$\|Tx - Ty\| \leq k\|x - y\|.$$

If $k \in [0, 1)$ in (ii), then T is called *contraction* and if $k \in [0, 1]$, then the mapping T is called *nonexpansive.*

(iii) k-strictly pseudocontractive mapping if there exists a constant $k \in [0, 1)$ such that for all $x, y \in D(T)$.

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2.$$

(iv) *firmly nonexpansive* if for all $x, y \in D(T)$,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle .$$

(v) *monotone* if for all $x, y \in D(T)$, $\langle Tx - Ty, x - y\rangle \geq 0$.

(vi) *α-inverse strongly monotone* if there exists $\alpha > 0$ such that for all $x, y \in D(T)$,

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2.$$

Furthermore, a point $x^* \in D(T)$ is called fixed of T if $Tx^* = x^*$.

Remark 1.2. (i) It has been shown by Marino and Xu (2007) that the class of strict pseudocontractions are Lipschitz with Lipschitz constant $\frac{1+k}{1-k}$. Therefore, the class of strict Pseudocontractions is a subclass of uniformly continuous mappings, as well as a subclass of Lipschitz pseudocontractive mappings.

(ii) It is easy to see that every nonexpansive map is 0-strictly pseudocontraction. Hence, the class of strict pseudocontractions contains the class of nonexpansive maps. We, however, emphasize that the converse is false. In fact, we have the following example.

Example 1.3. Let H be a real Hilbert space and let $T : H \to H$ be defined by

$$T(x) = -2x$$

It is not difficult to see that T is not nonexpansive map. We argue as follow to show that T is strictly pseudocontraction. First, we observe that for any $x, y \in H$,

$$\begin{aligned} \|Tx - Ty\|^2 = 4\|x - y\|^2 &= (1 + 3)\|x - y\|^2 \\ &= \left(1 + \left(\frac{3}{9}\right)(9)\right)\|x - y\|^2 \\ &= \|x - y\|^2 + \frac{3}{9}\|3(x - y)\|^2 \\ &= \|x - y\|^2 + \frac{1}{3}\|(1 + 2)x - (1 + 2)y)\|^2 \end{aligned}$$

$$\begin{aligned} &= \|x-y\|^2 + \frac{1}{3}\|(1-(-2))x-(1-(-2))y)\|^2 \\ &= \|x-y\|^2 + \frac{1}{3}\|(I-T)x-(I-T)y\|^2 \\ &\leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2, \quad \forall\, k \in \left[\frac{1}{3}, 1\right). \end{aligned}$$

Definition 1.4. The generalized mixed equilibrium problems (abbreviated GMEP) for operators $f,\ \Phi,\ B$ is a problem of finding $u^* \in K$ such that

$$f(u^*, y) + \Phi(y) - \Phi(u^*) + \langle Bu^*, y-u^*\rangle \geq\ 0, \forall\, y \in K, \tag{1.10}$$

where K is nonempty, closed and convex subset of a real Hilbert space H, f is a real valued bifunction with domain $K \times K$, Φ is a proper extended real valued function with domain K, that is, $\Phi : K \to \mathbb{R} \cup \{+\infty\}$ and B an operator defined from K to H. The solution set of (1.10) is denoted by

$$GMEP(f, \Phi, B)\ := \{u \in K : f(u, y) + \Phi(y) - \Phi(u) + \langle Bu, y-u\rangle \geq\ 0, \forall\, y \in K.$$

It is easy to see that $u^* \in GMEP(F, \Phi, B)$ implies that

$$u^* \in D(\Phi) := \{u \in H : \Phi(u) < +\infty\}.$$

If $\Phi \equiv 0 \equiv B$ in (1.10), then, inequality (1.10) reduces to the **Classical equilibrium problem** (*abbreviated* $EP(f)$), that is, the problem of finding $u^* \in K$ such that

$$f(u^*, y) \geq\ 0,\ \forall\, y \in K. \tag{1.11}$$

Solution set of (1.11) is denoted by

$$EP(f) := \{u \in K : f(u, y) \geq\ 0, \forall\, y \in K\}.$$

If $\Phi \equiv 0 \equiv f$ in (1.10), then (1.10) reduces to the **Classical variational inequality problem** $GMEP(0, 0, B)$, that is, the problem of finding $u^* \in K$ such that

$$\langle Bu^*, y-u^*\rangle \geq\ 0, \forall\, y \in K. \tag{1.12}$$

Solution set of (1.12) is denoted by

$$V.I(B, K) = \{u \in K : \langle Bu, y-u\rangle \geq\ 0\ , \forall\, y \in K\}.$$

If $B \equiv 0 \equiv f$ in (1.10), then (1.10) reduces to the following minimization problem: find $u^* \in K$ such that

$$\Phi(y) \geq \Phi(u^*), \forall\, y \in K. \tag{1.13}$$

Solution set of (1.13) is denoted by $Argmin(\Phi)$, where

$$Argmin(\Phi) := \{u \in K : \Phi(y) \geq \Phi(u), \forall\, y \in K\}.$$

If $B \equiv 0$ in (1.10), then (1.10) reduces to the **mixed equilibrium problem** (*abbreviated* $MEP(f, \Phi, 0)$, that is, the problem of finding $u^* \in K$ such that

$$f(u^*, y) + \Phi(y) - \Phi(u) + \geq\ 0, \forall\, y \in K. \tag{1.14}$$

Solution set of (1.14) is denoted by

$$MEP(f,\Phi) := \{u \in K : f(u,y) + \Phi(y) - \Phi(u) + \geq \ 0, \forall\, y \in K\}.$$

If $\Phi \equiv 0$ in (1.10), then (1.10) reduces to the **Generalized equilibrium problem**, that is, the problem of finding $u^* \in K$ such that

$$f(u^*,y) + \langle Bu^*, y - u^*\rangle \geq \ 0, \forall\, y \in K. \tag{1.15}$$

Solution set of (1.15) is denoted by

$$GEP(f,B) := \{u \in K : f(u,y) + \langle Bu, y - u\rangle \geq \ 0, \forall\, y \in K\}.$$

If $f \equiv 0$ in (1.10), then (1.10) reduces to the **Generalized variational inequality problems**, that is, the problem of finding $u^* \in K$ such that

$$\Phi(u^*) - \Phi(y) + \langle Bu^*, y - u^*\rangle \geq \ 0, \forall\, y \in K. \tag{1.16}$$

Solution set of (1.16) is denoted by

$$GVI(\Phi,B,K) := \{u \in K : \Phi(u) - \Phi(y) + \langle Bu, y - u\rangle \geq \ 0, \forall\, y \in K\}.$$

From the forgoing discussion so far, we observe that (1.10) solves three different types of problems simultaneously i.e., it solves problem of optimization, variational inequality and equilibrium problems.

Throughout this thesis, we assume that our bifunction f, satisfies the following conditions, namely:

A_1 $\quad f(x,x) = 0, \forall\, x \in K;$

A_2 $\quad$ f is monotone in the sense that

$$f(x,y) + f(y,x) \leq 0, \forall\, x, y \in K;$$

A_3 $\quad f$ is hemi-continuous, that is,

$$\limsup_{t\to 0^+} f(tz + (1-t)x, y) \leq f(x,y), \forall\, x, y, z \in K;$$

A_4 $\quad$ The function $f(x,.)$ is convex and lower semicontinous, $\forall\, x \in K$. Though the following definition is well known, we still present it here for clarity sake.

Definition 1.5. Let E be a real vector space.The map

1. $\|.\| : E \to [0,\infty)$ satisfying the following conditions:
   (i) $\|x\| \geq 0, \forall\, x \in E$ and $\|x\| = 0$ if and if $x = 0$,
   (ii) For any $\alpha \in \mathbb{R}$, $\|\alpha x\| = |\alpha|\|x\|, \forall\, x \in E$,
   (iii) $\|x + y\| \leq \|x\| + \|y\|, \forall\, x,\ y \in E$,
   is called a norm on E and the pair $(E, \|.\|)$ is called a normed vector space.
2. $\langle .,. \rangle : E \times E \to \mathbb{R}$ satisfying the following conditions:
   (i) $\langle x, x\rangle \geq 0, \forall\, x,\ y \in E$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,
   (ii) symmetricity, that is, $\langle x, y\rangle = \langle x, y\rangle, \forall\, x,\ y \in E$,

(iii) bilinear, that is, linear in both first and second argument.
is called real inner product on E and the pair $(E, \langle .,. \rangle)$ is called a real inner product space.

Remark 1.6. If $(E, \langle .,. \rangle)$ is an inner product space and we consider the map $\|.\| : E \to \mathbb{R}$ defined by $\|x\| = \sqrt{\langle x, x \rangle}$. One can easily verify that $\|.\|$ is a norm on E. It is called the norm induced by the inner product.

From now onward, we will always assume that:

(i) H is a real Hilbert space.

(ii) K is nonempty, closed and convex subset of H.

(iii) $\langle .,. \rangle$ is an inner product associated with H.

(iv) $\|.\|$ is the norm induced by the inner product.

(v) $F(T) = \{x \in D(T) : Tx = x\}$.

Definition 1.7. Let $\{x_n\}$ be a sequence in H. Then, $\{x_n\}$ is said to converge to $x^* \in H$

(i) strongly, if $\forall\, \epsilon > 0$, $\exists\, n_\epsilon \in \mathbb{N}$ such that $\forall\, n \geq n_\epsilon$, $\|x_n - x^*\| < \epsilon$;

(ii) weakly, if $\forall\, f \in H^*$, the sequence $\{f(x_n)\}_{n \geq 1}$ converges to $f(x^*)$ in $\mathbb{R}$ with the usual topology.

Definition 1.8. A net (or generalized sequence) in H indexed by $A := [0, 1]$ is an operator from A to H. It is denoted by $\{x_\alpha\}_{\alpha \in A}$.

Definition 1.9. (i) Let $\{x_\alpha\}_{\alpha \in A}$ be a net in H, $\{x_\alpha\}_{\alpha \in A}$ converges to a vector x^* as $\alpha \to 0$ if $\{x_\alpha\}_{\alpha \in A}$ lies eventually in every neighbourhood of x^*. i.e $\forall\, V \in Nbh(x^*), \exists\, b \in A$ such that $\alpha \leq b \Rightarrow x_\alpha \in V$.

(ii) A point $x^* \in H$ is a cluster point of the net $\{x_\alpha\}_{\alpha \in A}$ if $\{x_\alpha\}_{\alpha \in A}$ frequently lies in every neighbourhood of x^*. i.e $\forall\, V \in Nbh(x^*)$, $\forall\, b \in A$, $\exists\, \alpha \in A$ such that $\alpha \leq b$ and $x_\alpha \in V$.

Definition 1.10. Let $f : H \to \mathbb{R} \cup \{\infty\}$ and and $x_0 \in H$, where H is a real Hilbert as we have pointed out before. Then, f is lower semicontinuous at x_0 if, for every net $(x_\alpha)_{\alpha \in A} \subseteq H$ such that $x_\alpha \to x_0$ as $\alpha \to 0^+$, Then, $f(x_0) \leq \liminf\limits_{\alpha \to 0^+} f(x_\alpha)$

1.2.1 Some Facts in Hilbert Spaces

(i) Given a nonempty, closed and convex subset K of H, let $P_K : H \to K$ be the projection operator. It is well known that for arbitrary vector $x \in H$, $z = P_K x$ if and only if

$$\langle x - z, y - z \rangle \leq 0, \forall\, y \in K. \tag{1.17}$$

The following identities are also well known in Hilbert spaces:

(ii) for any $t \in [0, 1]$ and for any $x, y \in H$,

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2. \tag{1.18}$$

(iii) for any $x, y \in H$

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle. \tag{1.19}$$

(iv) It is also well known that given any vector $y \in H$, there exists $f_y \in H^*$ such that

$$f_y(x) = \langle x, y \rangle, \quad \forall\, x \in H. \tag{1.20}$$

Where H^* denotes the dual space of H, i.e the set of all bounded linear operators from H to $\mathbb{R}$.

Remark 1.11. It is easy to see using equation (1.20) that $x_n \rightharpoonup x^*$ if and only if for any $y \in H$, $\langle x_n, y \rangle \to \langle x^*, y \rangle$.

1.3 Statement of Problem

Several Authors have published articles on how to approximate the solution of generalized equilibrium problems and common fixed points of finite family of strict pseudocontractions

For example, Marino and Xu (2007) proved that: Given a self mapping T from a nonempty, closed and convex subset K of a real Hilbert space H , the sequence $\{x_n\}$ defined recursively by the formula

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.21}$$

converges weakly to a fixed point of T. Where the initial guess $x_0 \in K$ is arbitrary, and $\{\alpha_n\}$ is a real control sequence in the interval $(0, 1)$. They proved the above result under the additional hypothesis that

(i) T is k-strictly pseudocontraction that admits at least a fixed point,

(ii) $k < \alpha_n < 1$, for all n ≥ 1 and

$$\sum_{n=0}^{\infty}(\alpha_n - k)(1 - \alpha_n) = \infty.$$

Hu and Cai (2011) proved the following theorem for class of asymptotically pseudocontractive mapping in the intermediate sense:

Theorem 1.12. *(Hu and Cai, 2011) Let C be a nonempty, closed and convex subset of a real Hilbert space H and $N \geq 1$ be an integer, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying A_1 - A_4 and A be an α-inverse strongly monotone mapping of C into H. Let, for each $1 \leq i \leq N$, $T_i : C \to C$ be a uniformly continuous k_i -strictly asymptotically pseudocontractive mapping in the intermediate sense for some $0 \leq k_i < 1$ with sequences $\{\gamma_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \gamma_{n,i} < \infty$ and $\{c_{n,i}\} \subset [0, \infty)$ such that $\lim_{n\to\infty} c_{n,i} = 0$. Let $k = max\{k_i : 1 \leq i \leq N\}$, $\gamma_n = max\{\gamma_{n,i} : 1 \leq i \leq N\}$ and $c_n = max\{c_{n,i} : 1 \leq i \leq N\}$. Assume that $F := \cap_{i=1}^{N} F(Ti) \cap EP$ is nonempty. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by arbitrary element $x_1 \in C$ and then by*

$$\begin{cases} f(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\, y \in K, \\ z_n = (1 - \beta_n)u_n + \beta_n T_{i(n)}^{k(n)} u_n, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)z_n, \end{cases} \tag{1.22}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $0 < a \leq \alpha_n \leq 1, \ \{\alpha_n\} \subseteq (0, 1)$;

(ii) $0 < \delta \leq \beta_n \leq 1 - k - \delta < 1, \quad \{\beta_n\} \subseteq (0, 1)$;

(iii) $\sum_{n=1}^{\infty} \beta_n c_n < \infty$;

(iv) $0 < b \leq r_n \leq c \leq 2\alpha$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of F.

Huang and Ma (2014) proved the following theorem by slightly adjusting scheme (1.12) and considering the class of strict pseudocontractions. They obtained the following theorem:

Theorem 1.13. *(Huang and Ma, 2014) Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to H$ be a λ-inverse-strongly monotone mapping. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions $(A_1) - (A_4)$. Let $S : C \to C$ be a k-strict pseudocontraction. Assume that $F := EP(F,T) \bigcap F(S)$ is not empty. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$. Let $\{r_n\}$ be a sequence in $(0, 2\lambda)$, and let $\{e_n\}$ be a bounded sequence in C. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x \in C, \\ F(u_n, u) + \langle Tx_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0 \quad \forall\, u \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n(\delta_n u_n + (1-\delta_n)Su_n) + \gamma_n e_n \quad n \geq 1, \end{cases} \tag{1.23}$$

Assume that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ nd $\{\delta_n\}, \{r_n\}$ satisfy the following restrictions: $0 < a \leq \alpha_n \leq a' < 1$, $0 \leq k \leq \delta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\lambda$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then, the sequence $\{x_n\}$ converges weakly to some point $x^* \in F$, where $x^* = \lim_{n\to\infty} P_F x_n$.

The problem is that all their results concluded weak convergence which seems to be less useful in applications compare to strong convergence. In this thesis, we studied the above problem and we constructed iterative algorithm by modifying the operators used in scheme (1.12) as Huang and Ma did, drop the error term introduced in scheme (1.13) and use a modified Halpern scheme which seems better than Mann's scheme in several ways to study the convergence analysis of the new problem.

1.4 Motivations

Our motivation arises from application point of view, the work of Huang and Ma (2014) and that of Hu and Cai (2011) precisely theorems (1.12) and (1.13), respectively.

1.5 Objectives

Our objectives are the following :

(i) to introduce a scheme that will have computational advantage over the existing ones.

(ii) to prove strong convergence theorem using our scheme which seems to be more useful in application.

1.6 Limitations

We proved our result in Hilbert space setting, so we are faced with the challenge of whether our result is valid in more general Banach spaces. It is difficult in practice to get operators that are inverse strongly monotone.This calls for further research on how to relax the inverse strongly monotone condition.

CHAPTER 2

LITERATURE REVIEW

In this chapter, we review people's works and establish link between theirs and our work.

The concept "strict pseudocontraction" was introduced in the literature by Browder and Petryshyn (1967) in a real Hilbert space.They proved weak and strong convergence theorems for strict pseudocontraction by using the following iterative algorithm

$$x_{n+1} = (1-\beta)x_n + \beta T x_n, \ n \in \mathbb{N}. \tag{2.1}$$

Before then, researchers were already working on how to approximate fixed point of some nonlinear maps whenever, it exists. In fact, Mann (1952) proposed the following iterative algorithm which generates sequence $\{x_n\}$ from an arbitrary initialization vector $x_0 \in K$ and the sequence $\{x_n\}$ is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T x_n, \ n \geq 0,$$

where T is a self map with domain K and $\{\alpha_n\} \subset [0,1]$ is a control sequence satisfying the following conditions: $\lim \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. The above construction of Mann attracted attention of several researchers (see Marino and Xu (2007) and the references therein) on how to construct iterative scheme for fixed point of nonexpansive maps whenever it exists.

Some years after Mann introduced the above iterative algorithm, Halpern (1967) studied the following iterative algorithm defined by arbitrary initialization vector x_1 and a fixed vector u$\in K$ called anchor and the sequence $\{x_n\}$ is generated as follows:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T x_n, \ n \geq 1. \tag{2.2}$$

He proved that the sequence $\{x_n\}$ generated by (2.2) converges weakly to a fixed point of T and converges strongly to a fixed point of T if $\{\alpha_n\} = \{n^{-\delta}\}$, $\delta \in (0,1)$ and $u = 0$. However, he noted that the necessary conditions: for (2.2) to converge strongly are

C_1 $\quad \lim \alpha_n = 0;$

C_2 $\quad \sum\limits_{n=1}^{\infty} \alpha_n = \infty.$

Lions (1977) improved Halpern's result, still in Hilbert space by putting a different restriction on $\{\alpha_n\}$. He investigated the strong convergence of the sequence $\{x_n\}$, where $\{\alpha_n\}$ satisfies the following conditions:

C_1 $\quad \lim \alpha_n = 0;$

$C_2 \quad \sum_{n=1}^{\infty} \alpha_n = \infty;$

$C_3 \quad \lim \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2}.$

In 1992, Wittmann (1992) proved, still in Hilbert spaces, the strong convergence of the sequence generated by (2.2) to a fixed point of T, where $\{\alpha_n\}$ satisfies the following conditions:

$C_1 \quad \lim \alpha_n = 0;$

$C_2 \quad \sum_{n=1}^{\infty} \alpha_n = \infty;$

$C_4 \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty.$

It was observed that both the Withmann and Lions' results excluded the canonical choice $\{\alpha_n\} = \{\frac{1}{n+1}\}$. Xu (2002b) improved the results of Lions (1977) and Withmann (1992). He improved Lions' result by weakening condition (C_3) of Lions' result. Precisely, he removed the square in the denominator of (C_3) so that it will accommodate the canonical choice $\{\alpha_n\} = \{\frac{1}{n+1}\}$.

The following question arises: Is a real sequence $\{\alpha_n\}$ satisfying the conditions C_1 and C_2 sufficient to guarantee the strong convergence of Halperns iteration (2.2) for nonexpansive mappings? It remains an open question, see Li *et al.* (2013). Some mathematicians considered the open question. Song and Chai (2009) proved that for a firmly nonexpansive mapping T, an important subclass of nonexpansive mappings, the answer to Halpern's open problem is affirmative. A partial answer to this question was given independently by Chidume and Chidume (2006) and Suzuki (2007). They defined the sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(1-\delta)x_n + \delta T x_n, \tag{2.3}$$

where $\delta \in [0, 1]$, and obtained the strong convergence of the iteration , where $\{\alpha_n\}$ satisfies the conditions C_1 and C_2 . Recently, Xu (2002a) gave another partial answer to this question. He obtained the strong convergence of the following iterative sequence

$$x_{n+1} = \alpha_n(1-\delta)u + \delta x_n + (1-\alpha_n)T x_n, \tag{2.4}$$

where $\delta \in [0, 1]$, and $\{\alpha_n\}$ satisfies the conditions C_1 and C_2 . In 2008, Hu (2008) introduced the following modified Halpern's iterative scheme: For arbitrary vectors $u, x_0 \in C$, the sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \ n \geq 0, \tag{2.5}$$

where $\{\alpha_n\}, \{\gamma_n\}, \{\beta_n\}$ are three real sequences in $[0, 1]$, satisfying $\alpha_n + \gamma_n + \beta_n = 1$. He showed that the sequence $\{\alpha_n\}$ satisfying the conditions C_1 and C_2 is sufficient to guarantee the strong convergence of the modified Halpern's iterative sequence (2.5) for nonexpansive mappings.

Convergence analysis of algorithm (2.1) has been investigated recently in literature see for example, Marino and Xu (2007) and the references therein. Related works can also be found in Marino and Xu (2007). There are several constructions by several authors on how to use algorithm (2.1) to approximate solution of equilibrium problems and fixed point of some nonlinear operators. Ceng (2009) constructed an iterative scheme for a k-strict pseudocontraction as follows:

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\, y \in K, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)T u_n, \ n \geq 1, \end{cases} \tag{2.6}$$

where $x_0 \in K$ is the initialization vector, $\{\alpha_n\}$ and $\{r_n\}$ are two nonnegative sequences of real numbers, satisfying the following conditions

(i) $\{\alpha_n\} \subset (\alpha, \beta)$ for some $\alpha, \beta \in (k, 1)$;

(ii) $\liminf r_n > 0$.

They proved that $\{x_n\}$ and $\{u_n\}$ converge weakly to $p \in \Omega := f(T) \cap EP(f)$, respectively. They further argued that the convergence of the sequences $\{x_n\}$ and $\{u_n\}$ in (2.6) is strong convergence if and if

$$\liminf d(x_n, F(T) \cap EP(f)) = 0,$$

where $d(x_n, f(T) \cap EP(f))$ denotes the metric distance from the point x_n to $F(T) \cap EP(f))$. It is necessary to observe that only weak convergence is obtained using algorithm (2.6). In order to get strong convergence, Jaiboon and Kumam (2010) constructed CQ algorithm that enable them to conclude strong convergence. Precisely, they constructed the following iterative scheme:

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\, y \in K, \\ y_{n+1} = \alpha_n u_n + (1-\alpha_n)Tu_n, \ n \geq 1, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_0. \end{cases} \tag{2.7}$$

where $x_0 \in H$, $C_1 = C$, $\{\alpha_n\}$ and $\{r_n\}$ are sequences of nonnegative real numbers, satisfying the following conditions:

(i) $\{\alpha_n\} \subset (\alpha, \beta)$ for some $\alpha, \beta \in (k, 1)$;

(ii) $\liminf r_n > 0$.

They proved that the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to $p = P_{f(T)\cap EP(f)}x_0$.

In 2011, Hu and Cai (2011) constructed an algorithm similar to that of Jaiboon and Kumam to approximate solution of generalized equilibrium problems and common fixed point of finite family of asymptotically k-strictly pseudocontractive mappings in the intermediate sense. They proved the following theorem:

Theorem 2.1. *(Hu and Cai, 2011) Let C be a nonempty closed convex subset of a real Hilbert space H and $N \geq 1$ be an integer, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and A be an α-inverse strongly monotone mapping of C into H. Let, for each $1 \leq i \leq N$, $T_i : C \to C$ be a uniformly continuous k_i -strictly asymptotically pseudocontractive mapping in the intermediate sense for some $0 \leq k_i < 1$ with sequences $\{\gamma_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \gamma_{n,i} < \infty$ and $\{c_{n,i}\} \subset [0, \infty)$ such that $\lim_{n\to\infty} c_{n,i} = 0$. Let $k = max\{k_i : 1 \leq i \leq N\}$, $\gamma_n = max\{\gamma_{n,i} \colon 1 \leq i \leq N\}$ and $c_n = max\{c_{n,i} \colon 1 \leq i \leq N\}$. Assume that $F := \cap_{i=1}^{N} F(Ti) \cap EP$ is nonempty and bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$ such that $0 < a \leq \alpha_n \leq 1$, $0 < \delta \leq \beta_n \leq 1 - k$, and $\{r_n\} \subset [0, \infty)$ satisfies $\liminf r_n > 0$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by arbitrary element $x_1 \in C$ and then by*

$$\begin{cases} f(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\, y \in K, \\ z_n = (1-\beta_n)u_n + \beta_n T_{i(n)}^{k(n)}u_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n z_n, \\ C_n = \{v \in H : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_0, \ \forall\, n \in \mathbb{N} \cup \{0\}. \end{cases} \tag{2.8}$$

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_F x_0$.

Though strong convergence is obtained using the above scheme but it will be difficult to implement the above scheme due to the projection operator involved. In fact, each of the iterates poses sub-problem of computing C_n and Q_n which may not be easy to handle. Furthermore, the choice of

$\{\alpha_n\}$ in both scheme excluded the natural choice $\{\frac{1}{n}\}$. The above two limitations are serious issues which may hinder the scope of its application.

He (2012) noted the above two limitations, He constructed an iterative scheme that included the canonical choice and at the same time, his scheme does not involve projection map. He proved the following theorem precisely.

Theorem 2.2. *(He, 2012) Let K be a nonempty, closed and convex subset of a real Hilbert space H and $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $A_1 - A_4$.$T_i : K \to K$ be finite family of k_i-strictly pseudocontractive mappings $i = 1, 2, \cdots, N$ such that $\Omega = \bigcap_{i=1}^{N} F(T_i) \bigcap GEP(f) \neq \emptyset$. $0 \leq k_i < 1$.Suppose that v and x_1 are arbitrary in K, for some nonnegative real numbers λ_i, $i = 1, 2, \cdots, N$, $\sum_{i=1}^{N} \lambda_i = 1$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall \, y \in K, \\ x_{n+1} = \alpha_n v + (1 - \alpha_n) y_n, \\ y_n = (1 - \beta_n) x_n + \beta_n z_n, \\ z_n = (1 - \sigma) u_n + \sigma \sum_{i=1}^{N} \lambda_i T_i u_n, \end{cases} \tag{2.9}$$

where $\sigma \in (0, 1 - k)$, $k := \sup\{k_i : 1 \leq i \leq N\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\lim \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \ \{\alpha_n\} \subseteq (0, 1)$;

(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1, \quad \{\beta_n\} \subseteq (0, 1)$;

(iii) $\liminf r_n > 0, \ \lim |r_{n+1} - r_n| = 0$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* = P_{\Omega}(u)$.

We have to note that his scheme approximates the solution of the classical equilibrium problem and the common fixed point of finite family of strict pseudocontractions. Huang and Ma (2014) constructed a modified Mann's iterative algorithm that approximates the solution of the generalised equilibrium problem and the common fixed point of finite family of strict pseudocontractions. They proved the following theorem:

Theorem 2.3. *(Huang and Ma, 2014) Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a λ -inverse-strongly monotone mapping and Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies A_1 -A_4. Let $S : C \to C$ be a k-strict pseudocontraction. Assume that $F := EP(F, T) \bigcap F(S)$ is nonempty. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$. Let $\{r_n\}$ be a sequence in $(0, 2\lambda)$, and let $\{e_n\}$ be a bounded sequence in C. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in C, \\ F(u_n, u) + \langle T x_n, u - u_n \rangle + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0 \quad \forall \, u \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n (\delta_n u_n + (1 - \delta_n) S u_n) + \gamma_n e_n \quad n \geq 1, \end{cases} \tag{2.10}$$

Assume that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{r_n\}$ satisfy the following restrictions: $0 < a \leq \alpha_n \leq a^{'} < 1$, $0 \leq k \leq \delta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\lambda$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then, the sequence $\{x_n\}$ converges weakly to some point $x \in F$, where $x = \lim_{n \to \infty} P_F x_n$.

Though theorem 2.3 solves more general classes of problems compare to theorem 2.2 but there is still need to improve the scheme to enable one to conclude strong convergence which is more useful in terms of applications. This calls for further research on how to perturb the above scheme and get strong convergence. We have to note that, probably the anchor in scheme 2.2 is what that helped He (2012) to get strong convergence. Having seen the lapses in their works, we present to you in chapter four: a new iterative algorithm that approximates the solution of generalised equilibrium problem and the common fixed point of finite family of strict pseudocontractions of Halpern type and conclude strong convergence using our scheme.

CHAPTER 3

SOME AUXILIARY RESULTS

In this chapter, we are going to present some auxiliary results that will be useful in proving our main result in the next chapter. Here are the results.

Lemma 3.1. (Haung and Ma, 2014) Let K be a nonempty, closed and convex subset of H and $f : K \times K \to \mathbb{R}$ be a bifunction satisfying A_1-A_4. Let $r > 0$ *and* $x \in H$. Then, $\exists\, z \in K$ s.t

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in K.$$

Lemma 3.2. (Haung and Ma, 2014) Let K be a nonempty, closed and convex subset of a real Hilbert space H and Let $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $A_1 - A_4$. For $r > 0$ and $x \in H$, define a mapping T_r as follows:

$$T_r(x) = \{z \in K : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall\, y \in K\}.$$

Then, the following holds:

(i) T_r is single valued;

(ii) T_r is firmly nonexpansive i.e. $\forall x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle x - y, T_r x - T_r y\rangle$;

(iii) $F(T_r) = EP(f)$;

(iv) $EP(f)$ is closed and convex.

Proposition 3.3. Let f be a bifunction that satisfies A_1-A_4 and ϕ a lower semicontinuous and convex function from K to $\mathbb{R} \cup \{\infty\}$. Then, the bifunction Γ defined as follow :

$$\Gamma(x, y) = f(x, y) + \phi(y) - \phi(x). \tag{3.1}$$

satisfies conditions A_1 -A_4.

Proof. Claim: Γ satisfies conditions $A_1 - A_4$.

A_1

$$\begin{aligned} \Gamma(x, x) &= f(x, x) + \phi(x) - \phi(x) \\ &= 0. \end{aligned}$$

A_2

$$\begin{aligned} \Gamma(x, y) + \Gamma(y, x) &= (f(x, y) + \phi(y) - \phi(x)) + (f(y, x) + \phi(x) - \phi(y)) \\ &= f(x, y) + f(y, x) \\ &\leq 0. \end{aligned}$$

A_3. Using some properties of lim sup and definition 1.10, we obtain that

$$\begin{aligned}\limsup_{t\to 0^+}\Gamma(tw+(1-t)x,y) &= \limsup_{t\to 0^+}\big(f(tw+(1-t)x,y)+\phi(y)-\phi(tw+(1-t)x)\big)\\ &\le \limsup_{t\to 0^+}\big(f(tw+(1-t)x,y)\big)+\limsup_{t\to 0^+}\big(\phi(y)-\phi(tw+(1-t)x)\big)\\ &= \limsup_{t\to 0^+}\big(f(tw+(1-t)x,y)\big)+\phi(y)-\liminf_{t\to 0^+}\phi(tw+(1-t)x)\\ &\le f(x,y)+\phi(y)-\phi(x)\\ &= \Gamma(x,y).\end{aligned}$$

A_4. It is enough to show that $\Gamma(x,.)$ is convex and lower semicontinuous function. It follows from defintion of $\Gamma(.,.)$ that; $\Gamma(x,.)$ is a sum of two lower semicontinuous functions. So, it is enough to show that $\Gamma(x,.)$ is a convex function. Let $t\in[0,1]$ and $y,w\in K$ be arbitrary.

$$\begin{aligned}\Gamma(x,ty+(1-t)w)) &= f(x,ty+(1-t)w))+\phi(ty+(1-t)w))-\phi(x)\\ &\le tf(x,y)+(1-t)f(x,w)+t\phi(y)+(1-t)\phi(w)-\phi(x)\\ &= t\left(f(x,y)+\phi(y)-\phi(x)\right)+(1-t)\left(f(x,w)+\phi(w)-\phi(x)\right)\\ &= t\Gamma(x,y)+(1-t)\Gamma(x,w).\end{aligned}$$

□

Lemma 3.4. (Xu, 2002b) Suppose that $\{a_n\}_{n\ge 1}$ is a sequence of nonnegative numbers such that

$$a_{n+1}\le(1-\alpha_n)a_n+\sigma_n,\ \forall\, n\ge 1.$$

Where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ and $\{\sigma_n\}_{n\ge}$ is a sequence of nonnegative real numbers such that $\displaystyle\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\sigma_n=o(\alpha_n)$ (that is $\lim_{n\to\infty}\frac{\sigma_n}{\alpha_n}=0$) or $\displaystyle\sum_{n=1}^{\infty}\sigma_n<\infty$, then,

$$\lim_{n\to\infty}a_n=0.$$

Lemma 3.5. (Marino and Xu, 2007) Let C be a nonempty closed convex subset of a Hilbert space H, and let $S:C\to C$be a k-strict pseudocontraction. Then,
S is $\frac{1+k}{1-k}-$ *Lipschitz* and if $\{x_n\}$ is a sequence in K with $x_n\rightharpoonup x$ and $x_n-Sx_n\to 0$, then, $x\in F(S)$.

The following deep result proved by Mainge is crucial in the prove of our result.

Lemma 3.6. (Mainge, 2008) Let $\{\gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{\gamma_{n_j}\}$ such that $\gamma_{n_j}<\gamma_{n_j+1},\ \forall j\in\mathbb{N}$. Consider the sequence of integers $\{\tau(n)\}$ defined by

$$\tau(n)=\max_k\{k\le n:\gamma_k<\gamma_{k+1}\}.$$

Then,

(i) $\{\tau(n)\}$ is a non decreasing sequence for all $n\ge n_0$

(ii) $\lim_{n\to\infty}\tau(n)=\infty$;

(iii) $\gamma_{\tau(n)}<\gamma_{\tau(n)+1},\quad\forall\, n\ge n_0$;

(iv) $\gamma_n<\gamma_{\tau(n)+1},\quad\forall\, n\ge n_0$.

CHAPTER 4

MAIN RESULTS

In this chapter, we introduce our iterative scheme, present our main result and its detailed proof. We shall also demonstrate how our main result can be used to approximate the solution of generalised mixed equilibrium problems and common fixed point of finite family of strict pseudocontractions.

We immediately turn to the iterative scheme introduced in this thesis. Let K be a nonempty, closed and convex subset of a real Hilbert space H, let $f : K \times K \to \mathbb{R}$ and $A : K \to H$ be given functions. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$. The sequence $\{x_n\}_{n\geq 1}$ is defined from arbitrary elements $x_1, u \in K$ by

$$\begin{cases} f(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\, y \in K, \\ z_n = (1 - \beta_n)u_n + \beta_n T_{i(n)}u_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \end{cases} \tag{4.1}$$

where $\{r_n\}_{n\geq 1}$ is a sequence of positive real numbers, and $\{u_n\}_{n\geq 1}$ is a sequence guaranteed by Lemma 3.2, provided the mappings f and A satisfy appropriate conditions.

To appreciate how $\{z_n\}_{n\geq 1}$ in (4.1) runs, we compute some terms of the sequence . Observe that

$$\begin{aligned}
z_1 &= (1 - \beta_1)u_1 + \beta_1 T_1 u_1 \\
z_2 &= (1 - \beta_2)u_2 + \beta_2 T_2 u_2 \\
&\vdots \\
z_N &= (1 - \beta_N)u_N + \beta_N T_N u_N \\
z_{N+1} &= (1 - \beta_{N+1})u_{N+1} + \beta_{N+1} T_1 u_{N+1} \\
z_{N+2} &= (1 - \beta_{N+2})u_{N+2} + \beta_{N+2} T_2 u_{N+2} \\
&\vdots \\
z_{2N} &= (1 - \beta_{2N})u_{2N} + \beta_2 N T_N u_{2N} \\
z_{2N+1} &= (1 - \beta_{2N+1})u_{2N+1} + \beta_{2N+1} T_1 u_{2N+1} \\
&\vdots \\
z_{kN+i} &= (1 - \beta_{kN+i})u_{kN+i} + \beta_{kN+i} T_i u_{kN+i} \qquad k = 0, 1, 2, \cdots \ and \ i = 1, 2, \cdots, N, \\
&\vdots
\end{aligned}$$

where $u_n = T_{r_n}(I - r_n A)x_n$.

Remark 4.1. Thus, the sequence $\{z_n\}_{n\geq 1}$ can be written in compact form as

$$z_n = (1-\beta_n)u_n + \beta_n T_{i(n)}u_n, \qquad i(n) \in \{1, 2, \cdots, N\}.$$

We are now ready to present our main result.

Theorem 4.2. *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $A_1 - A_4$ and A be an α-inverse strongly monotone mapping of K into H. Let $T_i : K \to K$ be finite family of k_i-strictly pseudocontractive mappings $i = 1, 2, \cdots, N$ and let $k := \max\{k_i : 1 \leq i \leq N\}$. Assume that $\Omega = \bigcap_{i=1}^{N} F(T_i) \bigcap GEP(f, A) \neq \emptyset$. Let $\{x_n\}, \{u_n\}$ be defined by (4.1), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following conditions:*

(i) $\lim \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \ \{\alpha_n\} \subseteq (0, 1);$

(ii) $0 < \delta \leq \beta_n \leq 1 - k - \delta < 1, \quad \{\beta_n\} \subseteq (0, 1);$

(iii) $0 < b \leq r_n \leq c \leq 2\alpha.$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* = P_\Omega(u)$

Remark 4.3. It is clear from lemmas 3.1, 3.2 and remark 4.1 that our scheme is well defined.

Proof. Let $x^* = P_\Omega(u)$. We show that

$$\|z_n - x^*\| \leq \|x_n - x^*\|, \ \forall \ n \geq 1.$$

Using equation (1.18), definition of T_i and k, we have that

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|(1-\beta_n)u_n + \beta_n T_{i(n)}u_n - x^*\|^2 \\
&= \|(1-\beta_n)(u_n - x^*) + \beta_n(T_{i(n)}u_n - x^*)\|^2 \\
&= (1-\beta_n)\|u_n - x^*\|^2 + \beta_n\|T_{i(n)}u_n - x^*\|^2 - \beta_n(1-\beta_n)\|T_{i(n)}u_n - u_n\|^2 \\
&\leq (1-\beta_n)\|u_n - x^*\|^2 + \beta_n[\|u_n - x^*\|^2 + k\|T_{i(n)}u_n - u_n\|^2] \\
&- \beta_n(1-\beta_n)\|T_{i(n)}u_n - u_n\|^2 \\
&= \|u_n - x^*\|^2 - \beta_n(1-k-\beta_n)\|T_{i(n)}u_n - u_n\|^2.
\end{aligned}$$

From condition (ii), we have that

$$0 < \delta \leq \beta_n \leq 1 - k - \delta < 1. \tag{4.2}$$

Multiplying inequality (4.2) by -1 and adding $1 - k$ to the resulting inequality we obtain that

$$\delta \leq 1 - k - \beta_n,$$

which implies that

$$-\beta_n(1-k-\beta_n) \leq -\delta^2.$$

Thus,

$$\|z_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \delta^2\|T_{i(n)}u_n - u_n\|^2. \tag{4.3}$$

Using the fact that T_r is firmly nonexpansive from Lemma 3.2 and definition of u_n, we have that $u_n = T_{r_n}(I - r_nA)x_n$, furthermore, using the fact that $x^* \in GEP(f, A)$, we also have that $x^* = T_{r_n}(I - r_nA)x^*$. Using the above, nonexpansiveness of T_{r_n} and condition (iii) we have that

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)x^*\|^2 \\
&\leq \|(I - r_nA)x_n - (I - r_nA)x^*\|^2 \\
&= \|x_n - x^* - r_n(Ax_n - Ax^*)\|^2 \\
&= \|x_n - x^*\|^2 - 2r_n \langle x_n - x^*, Ax_n - Ax^*\rangle + {r_n}^2\|Ax_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\alpha r_n\|Ax_n - Ax^*\|^2 + {r_n}^2\|Ax_n - Ax^*\|^2 \\
&= \|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ax^*\|^2 \qquad (4.4) \\
&\leq \|x_n - x^*\|^2. \qquad (4.5)
\end{aligned}$$

Using inequalities (4.3) and (4.4), we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(u - x^*) + (1 - \alpha_n)(z_n - x^*)\| \\
&\leq \alpha_n\|u - x^*\| + (1 - \alpha_n)\|z_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\|. \qquad (4.6)
\end{aligned}$$

Claim : $\|x_n - x^*\| \leq M, \quad \forall\, n \geq 1$, where $M, := max\{\|u - x^*\|, \|x_1 - x^*\|\}$.

We proceed by induction. Clearly, the claim holds for n=1. Assume it is true for some $n \geq 1$ i.e $\|x_n - x^*\| \leq M$.

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + (1 - \alpha_n)max\{\|u - x^*\|, \|x_1 - x^*\|\} \\
&\leq \alpha_n max\{\|u - x^*\|, \|x_1 - x^*\|\} + (1 - \alpha_n)max\{\|u - x^*\|, \|x_1 - x^*\|\} \\
&= max\{\|u - x^*\|, \|x_1 - x^*\|\}.
\end{aligned}$$

Hence, $\|x_{n+1} - x^*\| \leq M, \quad \forall\, n \geq 1$. Hence, $\{x_n\}_{n\geq 1}$ is bounded. Consequently, $\{z_n\}, \{u_n\}$ are bounded. We now consider the following two cases:

<u>Case 1</u>: $\|x_{n+1} - x^*\| \leq \|x_n - x^*\|$, $\forall\, n \geq n_0$, for some $n_0 \geq 1$.
Therefore, $\{||x_n - x^*||\}$ is a bounded monotone nonincreasing sequence, hence its limit exists.

Next we show that

$$\lim_{n\to\infty} ||z_n - u_n|| = \lim_{n\to\infty} ||x_n - z_n|| = 0.$$

We proceed as follows. Using inequalities (4.3) and (4.4), we have that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||\alpha_n(u - x^*) + (1 - \alpha_n)(z_n - x^*)||^2 \\
&\leq \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\|z_n - x^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\left(\|u_n - x^*\|^2 - \delta^2\|T_{i(n)}u_n - u_n\|^2\right) \qquad (4.7) \\
&\leq \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\left(\|x_n - x^*\|^2 - \delta^2\|T_{i(n)}u_n - u_n\|^2\right) \\
&\leq \alpha_n||u - x^*||^2 + \|x_n - x^*\| - (1 - \alpha_n)\delta^2\|T_{i(n)}u_n - u_n\|^2,
\end{aligned}$$

which implies that

$$(1 - \alpha_n)\delta^2||T_{i(n)}u_n - u_n||^2 \leq \alpha_n||u - x^*||^2 + ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2. \qquad (4.8)$$

Thus,

$$\lim_{n\to\infty} \|T_{i(n)}u_n - u_n\| = 0. \qquad (4.9)$$

Since $\|z_n - u_n\| = \beta_n \|T_{i(n)}u_n - u_n\|$ and $\{\beta_n\}$ is bounded, we have that

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{4.10}$$

Also, for some $M_0 > 0$,

$$\begin{aligned} ||x_{n+1} - z_n|| &= \alpha_n ||u - z_n|| \\ &\leq \alpha_n M_0 \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{4.11}$$

Next, we show that $\lim ||Ax_n - Ax^*|| = 0$.
Using convexity of $\|.\|^2$, inequalities (4.3) and (4.4), we obtain that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq \alpha_n \|u - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2 \\ &\leq \alpha_n ||u - x^*||^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\ &\leq \alpha_n ||u - x^*||^2 + (1-\alpha_n)\|x_n - x^*\|^2 + r_n(r_n - 2\alpha)||Ax_n - Ax^*||^2, \end{aligned}$$

which implies that

$$-r_n(r_n - 2\alpha)||Ax_n - Ax^*||^2 \leq \alpha_n ||u - x^*||^2 + (1-\alpha_n)||x_n - x^*||^2 - ||x_{n+1} - x^*||^2.$$

Using condition (iii), we obtain that

$$b(2\alpha - c)||Ax_n - Ax^*||^2 \leq \alpha_n ||u - x^*||^2 + (1-\alpha_n)||x_n - x^*||^2 - ||x_{n+1} - x^*||^2,$$

hence,

$$\lim_{n\to\infty} ||Ax_n - Ax^*||^2 = 0. \tag{4.12}$$

Using the fact that $u_n = T_{r_n}(I - r_n A)x_n$, $x^* = T_{r_n}(I - r_n A)x^*$, $x^* \in GEP(f, A)$, firmly nonexpansiveness of T_{r_n}, equation (1.19) and the fact that $I - r_n A$ is nonexpansive, we obtain that

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n}(I - r_n A)x_n - T_{r_n}(I - r_n A)x^*\|^2 \\ &\leq \langle (x_n - r_n Ax_n) - (x^* - r_n Ax^*), u_n - x^* \rangle \\ &= \frac{1}{2}\left(\|(x_n - r_n Ax_n) - (x^* - r_n Ax^*)\|^2 + \|u_n - x^*\|^2\right) \\ &\quad -\frac{1}{2}\left(\|\left((x_n - r_n Ax_n) - (x^* - r_n Ax^*)\right) - (u_n - x^*)\|^2\right) \\ &\leq \frac{1}{2}\left(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|(x_n - u_n) - r_n(Ax_n - Ax^*)\|^2\right), \end{aligned}$$

which implies that

$$\begin{aligned} \|u_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|(x_n - u_n) - r_n(Ax_n - Ax^*)\|^2 \\ &= \|x_n - x^*\|^2 - \|x_n - u_n\|^2 - {r_n}^2\|Ax_n - Ax^*\|^2 \\ &\quad +2r_n \langle x_n - u_n, Ax_n - Ax^* \rangle \\ &\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|.\|Ax_n - Ax^*\|. \end{aligned} \tag{4.13}$$

Substituting inequality (4.13) in inequality (4.7),using the facts that $\{x_n\}, \{u_n\}$ are bounded and equation (4.12), we obtain that

$$\|x_{n+1} - x^*\|^2 \leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ax^*\|.$$

Thus,

$$\|x_n - u_n\|^2 \leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2r_n\|x_n - u_n\|.\|Ax_n - Ax^*\|,$$

consequently,

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{4.14}$$

Since we have from (4.10) and (4.14) that $\lim_{n\to\infty} \|u_n - z_n\| = \lim_{n\to\infty} \|u_n - x_n\| = 0$, it follows that

$$\begin{aligned}\|x_n - z_n\| &= \|x_n - u_n + u_n - z_n\| \\ &\leq \|x_n - u_n\| + \|u_n - z_n\| \to 0 \; as \; n \to \infty.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \|x_n - z_n\| = 0 \tag{4.15}$$

Next, we show that $\lim_{n\to\infty} \|u_{n+j} - u_n\| = 0 \; and \; \lim_{n\to\infty} \|x_{n+j} - x_n\| = 0, \; \forall \; j \in \{1, 2, \cdots, N\}$. Using the following equations: (4.11), (4.14) and (4.10), we obtain that

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - u_n\| \\ &\leq \|u_{n+1} - x_{n+1}\| + \|x_{n+1} - z_n\| + \|z_n - u_n\| \to 0 \; as \; n \to \infty.\end{aligned}$$

Consequently,

$$\begin{aligned}\|u_{n+j} - u_n\| \leq \; & \|u_{n+j} - u_{n+j-1}\| + \|u_{n+j-1} - u_{n+j-2}\| + \|u_{n+j-2} - u_{n+j-3}\| \\ & + \cdots + \|u_{n+1} - u_n\| \to 0 \; as \; n \to \infty.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|u_{n+j} - u_n\| = 0, \; \forall \; j \in \{1, 2, \cdots, N\}. \tag{4.16}$$

Similarly, using (4.14) and (4.16), we obtain that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - u_{n+1}\| + \|u_{n+1} - u_n\| + \|u_n - x_n\| \to 0 \; as \; n \to \infty. \tag{4.17}$$

Consequently,

$$\begin{aligned}\|x_{n+j} - x_n\| &\leq \|x_{n+j} - x_{n+j-1}\| + \|x_{n+j-1} - x_{n+j-2}\| + \|x_{n+j-2} - x_{n+j-3}\| \\ &+ \cdots + \|x_{n+1} - x_n\| \to 0 \; as \; n \to \infty.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|x_{n+j} - x_n\| = 0, \; \forall \; j \in \{1, 2, \cdots, N\}.$$

Next, we show that $\lim_{n\to\infty} \|u_n - T_j u_n\| = 0 = \lim_{n\to\infty} \|x_n - T_j x_n\|, \; \forall \; j \in \{1, 2, \cdots, N\}$.

Observe that for any $n \in \mathbb{N}$, there exist $k(n) \in \mathbb{N} \cup \{0\}, i(n) \in \{1, 2, \cdots, N\}$ such that $n = k(n)N + i(n)$. Now for any $n \geq 1$, set $T_n = T_{i(n)}$. Then, it follows that

$$T_{n+j} = \begin{cases} T_{(i(n)+j)}, & i(n) + j \leq N, \\ T_{(i(n)+j-N)}, & \text{otherwise}. \end{cases} \tag{4.18}$$

Hence, using equation (4.9) we have that

$$\|u_n - T_n u_n\| = \|u_n - T_{i(n)} u_n\| \to 0 \; as \; n \to \infty. \tag{4.19}$$

Consequently, for any $j \in \{1, 2, \cdots, N\}$, we have using equation (4.16), (4.19) and Lemma 3.5 that

$$\|u_n - T_{n+j}u_n\| \leq \|u_n - u_{n+j}\| + \|u_{n+j} - T_{n+j}u_{n+j}\| + \|T_{n+j}u_{n+j} - T_{n+j}u_n\| \to 0 \; as \; n \to \infty.$$

Thus,

$$\lim \|u_n - T_j u_n\| = 0, \quad \forall\, j \in \{1, 2, \cdots, N\}. \tag{4.20}$$

Moreover, for each $j \in \{1, 2, \cdots, N\}$, we observe from equation (4.14) and (4.20) and Lemma 3.5 that

$$\|x_n - T_j x_n\| \leq \|x_n - u_n\| + \|u_n - T_j u_n\| + \|T_j u_n - T_j x_n\| \to 0 \; as \; n \to \infty. \tag{4.21}$$

Next, we show that $\limsup_{n\to\infty} \langle u - x^*, x_{n+1} - x^* \rangle \leq 0$. From the definition of limit superior of a sequence of real numbers, we have that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle = \lim_{k\to\infty} \langle u - x^*, x_{n_k} - x^* \rangle . \tag{4.22}$$

Since $\{x_{n_k}\}_{k\geq 1}$ is bounded, there exists a subsequence $\{x_{n_{k_m}}\}_{m\geq 1}$ of $\{x_{n_k}\}_{k\geq 1}$ s.t $x_{n_{k_m}} \rightharpoonup p^* \in K, \; as \; m \to \infty$.
claim : $p^* \in \Omega$.

We first show that $p^* \in F(T_j), \quad \forall\, j \in \{1, 2, \cdots, N\}$. It follows from (4.21) that

$$\lim_{i\to\infty} \|x_{n_{k_m}} - T_j x_{n_{k_m}}\| = 0, \quad \forall\, j \in \{1, 2, \cdots, N\}.$$

Hence by Lemma 3.5, we have that $p^* \in F(T_j), \quad \forall\, j \in \{1, 2, \cdots, N\}$.

Since $u_n = T_{r_n}(I - r_n A)x_n$, for any $y \in K$, we have that

$$f(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \; \forall\, y \in K.$$

From A_2, we have

$$f(u_n, y) + f(y, u_n) \leq 0 \leq f(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle , \; \forall\, y \in K.$$

This implies that

$$\langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq f(y, u_n), \; \forall\, y \in K.$$

Replacing n with n_{k_m} we obtain

$$\left\langle Ax_{n_{k_m}}, y - u_{n_{k_m}} \right\rangle + \frac{1}{r_{n_{k_m}}} \left\langle y - u_{n_{k_m}}, u_{n_{k_m}} - x_{n_{k_m}} \right\rangle \geq f(y, u_{n_{k_m}}), \; \forall\, y \in K.$$

put $z_t = ty + (1 - t)p^*, \; t \in (0, 1) \; and \; y \in K$. Clearly, $z_t \in K$. Therefore, using monotonicity of A and the conditions on r_n, we have that

$$\begin{aligned}
\left\langle z_t - u_{n_{km}}, Az_t \right\rangle &\geq \left\langle z_t - u_{n_{km}}, Az_t \right\rangle - \left\langle Ax_{n_{km}}, z_t - u_{n_{km}} \right\rangle \\
&\quad - \frac{1}{r_{n_{ki}}} \left\langle z_t - u_{n_{ki}}, u_{n_{ki}} - x_{n_{ki}} \right\rangle + f(z_t, u_{n_{ki}}) \\
&= \left\langle z_t - u_{n_{ki}}, Az_t - Au_{n_{ki}} \right\rangle + \left\langle Au_{n_{ki}} - Ax_{n_{ki}}, z_t - u_{n_{ki}} \right\rangle \\
&\quad - \frac{1}{r_{n_{ki}}} \left\langle z_t - u_{n_{ki}}, u_{n_{ki}} - x_{n_{ki}} \right\rangle + f(z_t, u_{n_{ki}}) \\
&\geq \left\langle Au_{n_{ki}} - Ax_{n_{ki}}, z_t - u_{n_{ki}} \right\rangle \\
&\quad - \frac{1}{r_{n_{ki}}} \left\langle z_t - u_{n_{ki}}, u_{n_{ki}} - x_{n_{ki}} \right\rangle + f(z_t, u_{n_{ki}}) \\
&\geq \left\langle Au_{n_{ki}} - Ax_{n_{ki}}, z_t - u_{n_{ki}} \right\rangle \\
&\quad - \frac{1}{b} \|z_t - u_{n_{ki}}\|.\|u_{n_{ki}} - x_{n_{ki}}\| + f(z_t, u_{n_{ki}}).
\end{aligned}$$

Using the fact that $\lim \|x_n - u_n\| = 0$, A is uniformly continuous and from A_4, we obtain that

$$\langle z_t - p^*, Az_t \rangle \quad \geq \quad f(z_t, p^*) \quad as\ i \to \infty. \tag{4.23}$$

From A_1 *and* A_4, we have that

$$\begin{aligned}
0 &= f(z_t, z_t) \\
&= f(z_t, ty + (1-t)p^*) \\
&\leq tf(z_t, y) + (1-t)f(z_t, p^*) \\
&\leq tf(z_t, y) + (1-t)\langle z_t - p^*, Az_t \rangle \\
&= tf(z_t, y) + t(1-t)\langle y - p^*, Az_t \rangle,
\end{aligned}$$

this implies that

$$0 \leq f(z_t, y) + (1-t)\langle y - p^*, Az_t \rangle.$$

As $t \to 0$ we obtain using A_3 that

$$f(p^*, y) + \langle y - p^*, Ap^* \rangle \geq 0, \quad \forall\, y \in K.$$

This implies that $p^* \in EP(f, A)$. Hence, $p^* \in \Omega$.
It follows from equation (4.22), remark (1.11) and inequality (1.17) that

$$\begin{aligned}
\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle &= \lim_{k\to\infty} \langle (u - x^*), x_{n_k} - x^* \rangle \\
&= \lim_{i\to\infty} \left\langle u - x^*, x_{n_{ki}} - x^* \right\rangle \\
&= \langle u - x^*, p^* - x^* \rangle \\
&\leq 0.
\end{aligned}$$

But,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle x_{n+1} - x^*, x_{n+1} - x^* \rangle. \\
&= \langle \alpha_n(u - x^*) + (1-\alpha_n)(z_n - x^*), x_{n+1} - x^* \rangle \\
&= \alpha_n \langle u - x^*, x_{n+1} - x^* \rangle + (1-\alpha_n)\langle z_n - x^*, x_{n+1} - x^* \rangle \\
&\leq \alpha_n \langle u - x^*, x_{n+1} - x^* \rangle + (1-\alpha_n)\|z_n - x^*\|.\|x_{n+1} - x^*\| \\
&\leq \alpha_n \langle u - x^*, x_{n+1} - x^* \rangle + \frac{1-\alpha_n}{2}(\|z_n - x^*\|^2 + \|x_{n+1} - x^*\|^2).
\end{aligned}$$

Thus,

$$2\|x_{n+1} - x^*\|^2 \leq 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle + (1 - \alpha_n)(\|z_n - x^*\|^2 + \|x_{n+1} - x^*\|^2),$$

which implies that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq 2\alpha_n \langle (u - x^*), x_{n+1} - x^* \rangle + (1 - \alpha_n)\|x_n - x^*\|^2 \\ &\leq \sigma_n + (1 - \alpha_n)\|x_n - x^*\|^2, \end{aligned}$$

where $\sigma_n = max\{0, 2\alpha_n \langle (u - x^*), x_{n+1} - x^* \rangle\}$. Clearly, $\sigma_n \to 0 \; as \; n \to \infty$. Hence by Lemma 3.4 we have that $x_n \to x^* = P_\Omega u$

Case 2: there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - x^*\| < \|x_{n_k+1} - x^*\|, \; \forall \; k \geq 1$. By Mainge's result, there exists a sequence of integers $\{\tau(n)\}$ that satisfies

(i) $\{\tau(n)\}$ is nondecreasing, $\quad \forall \; n \geq n_0$;

(ii) $\lim_{n\to\infty} \tau(n) = \infty$;

(iii) $\|x_{\tau(n)} - x^*\| < \|x_{\tau(n)+1} - x^*\|, \quad \forall \; n \geq n_0$;

(iv) $\|x_n - x^*\| < \|x_{\tau(n)+1} - x^*\|, \quad \forall \; n \geq n_0$.

Consequently,

$$\begin{aligned} 0 &\leq \liminf(\|x_{\tau(n)+1} - x^*\| - \|x_{\tau(n)} - x^*\|) \\ &\leq \limsup(\|x_{\tau(n)+1} - x^*\| - \|x_{\tau(n)} - x^*\|) \\ &\leq \limsup(\|x_{n+1} - x^*\| - \|x_n - x^*\|) \\ &= \limsup(\|\alpha_n u + (1 - \alpha_n)z_n - x^*\| - \|x_n - x^*\|) \\ &\leq \limsup(\alpha_n\|u - x^*\| + (1 - \alpha_n)\|z_n - x^*\| - \|x_n - x^*\|) \\ &\leq \limsup(\alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| - \|x_n - x^*\|) \\ &= 0. \end{aligned}$$

Therefore,

$$\lim(\|x_{\tau(n)+1} - x^*\| - \|x_{\tau(n)} - x^*\|) = 0. \tag{4.24}$$

$$\tag{4.25}$$

It follows from inequality (4.8) that

$$\left(1 - \alpha_{\tau(n)}\right) \delta^2 \|T_{i\tau(n)} u_{\tau(n)} - u_{\tau(n)}\|^2 \leq \alpha_{\tau(n)}\|u - x^*\|^2 + \|x_{\tau(n)} - x^*\|^2 - \|x_{\tau(n)+1} - x^*\|^2. \tag{4.26}$$

Using equation (4.24) and inequality (4.26), we obtain that

$$\lim_{n\to\infty} \|T_{i(\tau(n))} u_{\tau(n)} - u_{\tau(n)}\| = 0$$

Using the same pattern of computation as in Case 1, we obtain that $x_{\tau(n)} \to x^*$. Using part (iv) of Lemma 3.6, we have that $x_n \to x^* \; as \; n \to \infty$. This completes the proof. □

Corollary 4.4. Let K be a nonempty, closed and convex subset of a real Hilbert space H and $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $A_1 - A_4$ and A be an α-inverse strongly monotone mapping of K into H. Let $T_i : K \to K$ be a finite family of nonexpansive mappings $i = 1, 2, \cdots, N$ Assume that $\Omega := \bigcap_{i=1}^N F(T_i) \bigcap GEP(f, A)$ is nonempty. Let $\{x_n\}, \{u_n\}$ be sequences defined from arbitrary elements $x_1, u \in K$ by

$$\begin{cases} f(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \; \forall \; y \in K, \\ z_n = (1 - \beta_n)u_n + \beta_n T_{i(n)} u_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \end{cases} \tag{4.27}$$

where $\{\alpha_n\}, \; \{\beta_n\} \; and \; \{r_n\}$ satisfy the following conditions:

(i) $\lim \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \ \{\alpha_n\} \subseteq (0,1);$

(ii) $0 < \delta \le \beta_n \le 1 - k - \delta < 1, \quad \{\beta_n\} \subseteq (0,1);$

(iii) $0 < b \le r_n \le c \le 2\alpha.$

Then, $\{x_n\}$ *and* $\{u_n\}$ converge strongly to $x^* = P_\Omega(u)$

Proof. Taking $k_i = 0, \ \forall i = 1, 2, \cdots, N$ in theorem 4.2, Then, the result follows. □

4.1 Application

In this section, we will apply our main result to approximate the solution of the generalized mixed equilibrium problems and the common fixed point of a finite family of strict pseudocontractions. Precisely, we prove the following theorem:

Theorem 4.5 (Generalised Mixed Equilibrium Problem)**.** *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $A_1 - A_4$, A be an $\alpha -$ inverse strongly monotone mapping of K into H and ϕ be a lower semicontinuous and convex functional. Let $T_i : K \to K$ be a finite family of k_i-strictly pseudocontractive mapping $i = 1, 2, \cdots, N$, $0 \le k_i < 1$ and $k := \sup\{k_i : 1 \le i \le N\}$ Assume that $\Omega := \bigcap_{i=1}^N F(T_i) \bigcap GEP(f, A)$ is nonempty. Let $\{x_n\}, \{u_n\}$ be a sequences generated by arbitrary elements $x_1, u \in K$ and then by*

$$\begin{cases} f(u_n, y) + \phi(y) - \phi(u_n)) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0 \ \forall \ y \in K, \\ z_n = (1 - \beta_n)z_n + \beta_n T_{i(n)} z_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n. \end{cases} \tag{4.28}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}\{\alpha_n\}$ satisfy the following conditions:

(i) $\lim \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \{\alpha_n\}, \ \{\alpha_n\} \subseteq (0,1);$

(ii) $0 < \delta \le \beta_n \le 1 - k - \delta < 1;$

(iii) $0 < b \le r_n \le c \le 2\alpha$

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* = P_F(u)$.

Proof. The proof follows from proposition 3.3 and theorem 4.2. □

CHAPTER 5

CONCLUSION

In this thesis, we have made significant improvements on some results that were recently announced by several other authors in the following ways. We have to note that theorem 4.2 improves theorem of He (2012) in several ways, namely: First, theorem 4.2 approximates solution of generalized equilibrium problems and common fixed point of finite family of strict pseudocontractions while theorem 2.2 approximates solution of classical equilibrium problems and common fixed point of finite family of strict pseudocontractions. Secondly, In each iterate of theorem 4.2, we are faced with only subproblem of computing z_n while theorem 2.2 poses two subproblems of computing z_n and y_n, hence each iterate in theorem 2.2 requires more processing time. Furthermore, the scheme will be difficult to implement because of the summation involved, if N is large enough and $T_i, i = 1, 2, \cdots, N$ not simple enough.

We have to observe that theorem 4.2 is also significant improvement of theorem 2.3 in the following sense,namely: First, the choice of $\{\alpha_n\}$ in theorem 2.3 excludes the canonical choice $\{\frac{1}{n}\}$, considering application point of view, theorem 4.2 concluded strong convergence which is more applicable than weak convergence. Furthermore, the rate of convergence of our approximating sequence in theorem 4.2 is faster than that of theorem 2.3 and it holds for any finite family of strict pseudocontractions.

We also applied theorem 4.2 to solve generalized mixed equilibrium problems. It is worthy to note that our main result solves mixed equilibrium problem as a particular case of theorem 4.5 by assuming that $A \equiv 0$ theorem 4.5.

Remark 5.1. Our theorems improves several other results that were recently announced by several authors but to reduce volume, we decide not to enumerate them.

Remark 5.2. We observed from our judicious application of *theorem 4.2 in solving generalized mixed equilibrium problems that the generalized equilibrium problems coincides with generalized mixed equilibrium problems.* Thus, the generalized equilibrium problems includes the mixed equilibrium problem as special case when $A \equiv 0$ in theorem 4.5. We emphasis that *generalized mixed equilibrium problems is not in any way a generalization of the generalized equilibrium problems.*

We are recommending the content of this work to those in pure and applied sciences especially those who are into research of how to: maximize or minimize some certain resources, approximate fixed point of some nonlinear maps which turns out to be the solution of some important problems in sciences, engineering etc.

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