SPECTRAL DECOMPOSITION OF COMPACT OPERATORS ON HILBERT SPACES

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CERTIFICATION

This is to certify that the thesis titled "Spectral decomposition of compact operators on Hilbert spaces" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research carried out by Omogbhe, David in Department of Pure and Applied Mathematics.

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ABSTRACT

Compact operators are linear operators on Banach spaces that maps bounded set to relatively compact sets. In the case of Hilbert space H it is an extension of the concept of matrix acting on a finite dimensional vector space. In Hilbert space, compact operators are the closure of the finite rank operators in the topology induced by the operator norm. In general, operators on infinite dimensional spaces feature properties that do not appear in the finite dimension case; i.e matrices. The compact operators are notable in that they share as much similarity with matrices as one can expect from a general operator. Spectral decomposition of compact operators on Banach spaces takes the form that is very similar to the Jordan canonical form of matrices. In the context of Hilbert spaces, the spectral properties of compact operators resembles those of square matrices.

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DEDICATION

This work is research dedicated to God Almighty and to my late father Mr. Simple Omogbhe.

CONTENTS

INTRODUCTION

This research work focuses on the study of Compact linear operators and Spectral decompositions on Hilbert spaces. This is broken into three chapters where

In chapter one, the properties and characterization of bounded linear operators on finite and infinite dimensional vector space will be studied. The notion of complete normed vector space which is called a Banach space will be studied where we shall show that this notion of completeness is a metric space concept. Examples to illustrate this ideas will be given. All results results discussed in this chapter are preliminary and can be found in any material on the subject. The material for this chapter follows from Kreyszig (2007).

In chapter two, we shall study the concept of Compact linear operators on Banach spaces as a map $T \in \mathcal{L}(X, Y)$ such that for any bounded subset $A \subset X$, $\overline{T(A)}$ is compact in Y. Other characterization of this map T will also be discussed. Fredholm Alternative for Compact linear operators will be discussed and we shall also give examples of Compact Operators such as
- The injection of $H^1(\Omega)$ into $L^2(\Omega)$.
-The Hilbert-Schmidt Operator $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ with $K \in L^2(\Omega \times \Omega)$ defined by
$(Tf)x = \int_\Omega K(x, y)f(y)dy$ for a.e $x \in \Omega$, etc. Almost all the results from this chapter are from Brezis (2010), Kreyszig (2007), Rudin (1976) and C.E Chidume (2014).

In chapter three, spectral properties and decomposition of Compact linear operators, classification of $\lambda \in \sigma(T)$ ($\sigma(T)$ is spectrum of T) and examples,compact self-adjoint operators and their spectrum will also be discussed, where we obtain eigenvalues and eigenfunctions of compact linear operator from analytic and from their weak formulations. The materials for the chapter follows from Khalil (2010),Khalil (2017) and Joel (2011).

CHAPTER 1

LINEAR OPERATORS AND BOUNDEDNESS

In this chapter, some well-known results which will be needed in sequel are provided.

1.1 Definitions

Definition 1.1. (Norm): A non-negative function $||.||$ on a vector space X over $\mathbb{R}$ is called a norm on X if and only if the following are satisfied.

(N1) $||x|| \geqslant 0 \; \forall \; x \in X$ (positivity).

(N2) $||x|| = 0$ if and only if $x = 0$ (Nondegeneracy).

(N3) $||\lambda x|| = |\lambda| ||x|| \; \forall \; x \in X$, for all $\lambda \in \mathbb{R}$ (Homogeneity).

(N4) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$ (Sub-additivity).

A vector space X endowed with a norm $||.||$ denoted by (X, $||.||$) is called a normed linear space (or just a normed space).

Definition 1.2. A sequence $(x_n)_{n \geq 1}$ is said to be Cauchy if given $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $||x_n - x_m|| < \varepsilon$ for all $m, n \geq N_0$.

Definition 1.3. A space (X, d), where d is a metric is said to be complete if every Cauchy sequence in X converges to a point in it.

Remark

Completeness is a metric space concept. In a normed space, the metric is $d(x, y) = ||x - y||$ where it satisfies the following special properties:

(a) The underlying space is a vector space

(b) Homogenity: $d(\alpha x, \alpha y) = |\alpha| d(x, y)$

(c) Translation invariance $d(x + z, y + z) = d(x, y)$

Conversely, every metric satisfying those three conditions defines a norm: $||x|| = d(x, 0)$

Definition 1.4. A complete normed vector space is called a Banach space.

Definition 1.5. *Space* $C([a, b], \mathbb{R})$

The space $C([a, b], \mathbb{R})$ denotes the set of all real valued continous functions on $[a, b]$ into $\mathbb{R}$.

1.2 Examples of Banach spaces

1. The space $C([a, b], \mathbb{R})$ endowed with the sup-norm is Banach.

Proof. Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $C[a, b]$. This implies for every $x \in [a, b]$ and for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$||f_n - f_m||_C[a, b] = \sup |f_n(x) - f_m(x)| \leq \varepsilon$$

for all $x \in [a, b]$ and for all $m, n \geq N$.

This implies $|f_n(x) - f_m(x)| \leq \varepsilon$ for all $x \in [a, b]$ and $m, n \geq N$,

thus $(f_n(x))_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$ and since $\mathbb{R}$ is complete, it implies

$$f_n(x) \longrightarrow f(x) \in \mathbb{R} \quad as \quad n \longrightarrow \infty$$

this implies $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in [a, b]$ and for all $n \geq N$ we have $\sup |f_n(x) - f(x)| \leq \varepsilon$ for all $x \in [a, b]$ and for all $n \geq N$

thus

$$||f_n - f||_{C[a,b]} \leq \varepsilon \quad for \quad all \quad n \geq N,$$

this implies $f \in C[a, b]$

Hence $C[a, b]$ endowed with the sup-norm is Banach.

2.The space $\mathbb{R}^n$ with $||x||_{\mathbb{R}^n} = (\sum_{n=1}^{\infty} |x_i|^2)^{1/2}$ is Banach.

1.3 Linear operators

Definition 1.6. Let T be an operator from a vector space X to a vector space Y, then the domain $\mathcal{D}(T)$ is given by $\mathcal{D}(T) = \{ \text{ x} \in \text{X} : \text{Tx exists in Y } \}$
and the range $\mathcal{R}(T)$ is given by $\mathcal{R}(T) = \{\text{y} \in \text{Y} : \exists \text{ x} \in \text{X such that Tx} = \text{y}\}$.

Definition 1.7. (Null space)
Let T be an operator from a vector space X to a vector space Y, then the null space $\mathcal{N}(T)$ is given by $\mathcal{N}(T) = \{ \text{ x} \in \text{X} : \text{Tx} = 0 \ \}$.

Definition 1.8. (Injectivity)
An operator T from X to a vector space Y is said to be injective if $\forall \ x_1, x_2 \in \mathcal{D}(T)$, such that $Tx_1 = Tx_2$ implies $x_1 = x_2$.

Remark: If T is injective, then there exists an operator
$T^{-1} : \mathcal{R}(T) \subset \text{Y} \longrightarrow \mathcal{D}(T) \subset \text{X}$ such that $T^{-1}(y_0) = x_0 \Longrightarrow Tx_0 = y_0$.

Definition 1.9. (Continuity)
An operator T from a vector space X to a vector space Y said to be continous at a point $x_0 \in \text{X}$ if given any $\epsilon > 0 \ \exists \ \delta > 0$ such that

$$||x - x_0|| \leq \delta \Longrightarrow ||Tx - Tx_0|| \leq \epsilon$$

Definition 1.10. (Linear Operators)
Let X and Y be vector spaces. Let $T : X \longrightarrow Y$. Then T is said said to be linear if:

i. The domain $\mathcal{D}(T)$ is a vector space and the range $\mathcal{R}(T)$ lies in a vector space over the same field.

ii. $\forall \ x, y \in \mathcal{D}(T)$ and scalars α,

$$T(x + y) = Tx + Ty \tag{1.1}$$

$$T(\alpha x) = \alpha Tx \tag{1.2}$$

1.3.1 Examples of linear operators

1. Differential operator: Let X be the vector space of all polynomials on $[a, b]$. We define a linear operator T on X by setting $Tx(t) = x'(t) \ \forall \ x \in X$, where the prime denotes differentiation with respect to t. This operator maps X into itself.

2. Integral operator: A linear operator T from $C[a, b]$ into itself can be defined by $Tx(t) = \int_a^t x(s)ds, t \in [a, b]$.

3. Multiplication by t: This is linear operator from $C[a,b]$ into itself defined by: $Tx(t) = tx(t)$.

Theorem 1.11. *Let $T : X \longrightarrow Y$ be a linear operator space, then*

(a) *The range $\mathcal{R}(T)$ is a vector space.*

(b) *If $dim X = n < \infty$, then $dim \mathcal{R}(T) \leq n$.*

(c) *The null space $\mathcal{N}(T)$ is a vector space.*

Proof. (a). Let $y_1, y_2 \in \mathcal{R}(T)$, we show that $\alpha y_1 + \beta y_2 \in \mathcal{R}(T)$ for any scalars α , β. Since $y_1, y_2 \in \mathcal{R}(T)$, we have $y_1 = Tx_1, y_2 = Tx_2$ for some $x_1, x_2 \in \mathcal{D}(T)$, and $\alpha x_1 + \beta x_2 \in \mathcal{D}(T)$ because $\mathcal{D}(T)$ is a vector space. The linearity of T yields

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \alpha y_1 + \beta y_2.$$

Hence $\alpha y_1 + \beta y_2 \in \mathcal{R}(T)$. Since $y_1, y_2 \in \mathcal{R}(T)$ were arbitrary and so were the scalars, this prove that $\mathcal{R}(T)$ is a vector space.
(b). We choose $n+1$ elements $y_1, y_2, ..., y_{n+1}$ in $\mathcal{R}(T)$ arbitrary. Then we have $y_1 = Tx_1, ..., y_{n+1} = Tx_{n+1}$ for some $x_1, x_2, ..., x_{n+1}$ in X. Since $\dim X = n$, the set $\{x_1, ..., x_{n+1}\}$ must be linearly dependent. Hence

$$\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1} = 0 \tag{1.3}$$

for some scalars $\alpha_1, \cdots, \alpha_{n+1}$ not all zero. Since T is linear then $T(0) = 0$. Applying T to both sides of (1.3) gives $T(\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1}) = \alpha_1 y_1 + ... + \alpha_{n+1} y_{n+1} = 0$. This shows that $\{y_1, ..., y_{n+1}\}$ is linearly independent set because the α_i's are not all zero.
Remembering that this subset of $\mathcal{R}(T)$ was chosen arbitrary, we conclude that $\mathcal{R}(T)$ has no linearly independent subsets of n+1 or more elements, this implies $\dim \mathcal{R}(T) \leq$ n.

(c). Let $x_1, x_2 \in \mathcal{N}(T)$, then $Tx_1 = Tx_2 = 0$. Since T is linear then for any α, β we have

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = 0.$$

It implies $\alpha x_1 + \beta x_2 \in \mathcal{N}(T)$. Hence $\mathcal{N}(T)$ is a vector space.

Theorem 1.12. *(Inverse of a linear operator)*
Let X and Y be vector spaces over $\mathbb{R}$. Let $T : X \longrightarrow Y$ be linear operator then:

(a) *The inverse $T^{-1} : \mathcal{R}(T) \longrightarrow X$ exists if and only if $Tx = 0 \Longrightarrow x = 0$ (T is injective).*

(b) *If T^{-1} exists, then it is a linear operator.*

(c) *If $dim X = n < \infty$ and T^{-1} exists, then $dim \mathcal{R}(T) = dim X$*

Proof. (a).Suppose $Tx = 0 \Longrightarrow x = 0$. Let $Tx_1 = Tx_2$. Since T is linear,

$$T(x_1 - x_2) = Tx_1 - Tx_2 = 0,$$

so that $x_1 - x_2 = 0$ by hypothesis. Hence $Tx_1 = Tx_2 \Longrightarrow x_1 = x_2$ and T^{-1} exist by remark on Definition 1.4. Conversely T^{-1} exists then remark on definition 1.4 holds.
From Definition 1.4 with $x_2 = 0$, we obtain $Tx_1 = T0 = 0 \Longrightarrow x_1 = 0$.
(b). We assume T^{-1} exists and show that it is linear. The domain of T^{-1} is $\mathcal{R}(T)$ and it is a vector space, then by Theorem 1.7a, we consider any $x_1, x_2 \in \mathcal{D}(T)$ and their images
$y_1 = Tx_1$ and $y_2 = Tx_2$, then $x_1 = T^{-1}y_1$ and $x_2 = T^{-1}y_2$. T is linear so that for any scalar α and β, we have

$$\alpha y_1 + \beta y_2 = \alpha Tx_1 + \beta Tx_2 = T(\alpha x_1 + \beta x_2).$$

It implies $T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha T^{-1}y_1 + \beta T^{-1}y_2$. It implies T^{-1} is linear.
(c). We have $\dim\mathcal{R}(T) \leq dim X$ by Theorem 1.7b and $\dim X \leq dim\mathcal{R}(T)$ by the same theorem applied to T^{-1}. Hence, dim $X = dim\mathcal{R}(T)$.

Lemma 1.13. (Inverse of product)
Let $T : X \longrightarrow Y$ and $S : Y \longrightarrow Z$ be bijective linear operator, where X, Y, Z are vector spaces.Then the inverse $(ST)^{-1} : Z \longrightarrow X$ of the product (the composite) ST exists and $(ST)^{-1} = T^{-1}S^{-1}$.

Proof. The operator $ST : X \longrightarrow Z$ is bijective, so $(ST)^{-1}$ exists. We have

$$(ST)(ST)^{-1} = I_Z,$$

where I_Z is the identity operator on Z. Applying S^{-1} and using $S^{-1}S = I_Y$ (the identity operator on Y), we obtain

$$S^{-1}(ST)(ST)^{-1} = T(ST)^{-1} = S^{-1}I_Z = S^{-1}.$$

Applying T^{-1} and using $T^{-1}T = I_X$, we obtain the desired result

$$T^{-1}T(ST)^{-1} = (ST)^{-1} = T^{-1}S^{-1}.$$

Implies $(ST)^{-1} = T^{-1}S^{-1}$.

Theorem 1.14. *Every linear operator on a finite dimensional vector space can be represented by means of matrix.*

Proof. Let X and Y be finite dimensional vector spaces over thesame field. Let $T : X \longrightarrow Y$ be a linear operator, let dimX = n and dimY = r, then there exists a basis $\{e_1, e_2, ...e_n\}$ for X and a basis $\{b_1, b_2, ...b_r\}$ for Y.
Let $x \in X \Rightarrow x = \sum_{i=1}^{n} \xi_i e_i$ where $\xi_i' s$ are scalars . Since T is linear

$$y = T(x) = \sum_{i=1}^{n} \xi_i T(e_i).$$

So T is uniquely determined if the images $Te_i \quad 1 \leq i \leq n$ are prescribed. Since y and Te_i are in Y so $y = \sum_{j=1}^{r} \eta_j b_j$ and $Te_i = \sum_{j=1}^{r} \tau_{ji} b_j$ where η_j and τ_{ji} are scalars, thus

$$y = \sum_{j=1}^{r} \eta_j b_j = \sum_{i=1}^{n} \xi_i T(e_i) = \sum_{i=1}^{n} \xi_i \sum_{j=1}^{r} \tau_{ji} b_j = \sum_{j=1}^{r} (\sum_{i=1}^{n} \tau_{ji} \xi_i) b_j.$$

Hence

$$\eta_j = \sum_{i=1}^{n} \tau_{ji} \xi_i. \quad 1 \leq j \leq r.$$

1.4 Bounded linear operators

Definition 1.15. (Bounded linear operator): Let X and Y be normed spaces and T:$X \longrightarrow$ Y be linear operator. The operator T is said to be bounded if there exist a real number c >0 such that $||Tx|| \leq c||x||$ for all $x \in \mathcal{D}(T)$.

Theorem 1.16. *Let $T : X \longrightarrow Y$ be a bounded linear operator. Then*

$$||T|| := \sup_{x \in X; ||x||=1} ||Tx|| = \sup_{x \in X; x \neq 0} \frac{||Tx||}{||x||}.$$

Proof. Let $||x|| = a$, set $y = (\frac{1}{a})x$, where $x \neq 0$. Then $||y|| = \frac{||x||}{a} = 1$. Since T is linear, then

$$\sup_{x \in X; x \neq 0} \frac{||Tx||}{||x||} = \sup \frac{||Tx||}{a} = \sup ||T(\frac{1}{a})x)|| = \sup_{y \in X; ||y||=1} ||Ty|| := ||T||.$$

Remark: $||.||$ defines a norm on X.

Theorem 1.17. *(Finite dimension): If a normed space X is finite dimensional, then every linear operator on X is bounded.*

Proof. Let dim $X = n$ and $\{e_1, e_2, \ldots, e_n\}$ be a basis for X, then for all $x \in X$,

$$x = \sum_{i=1}^{n} \alpha_i e_i$$

α_i scalars. Since T is linear,

$$||Tx|| = ||\sum_{i=1}^{n} \alpha_i Te_i|| \leq \sum_{i=1}^{n} |\alpha_i| ||Te_i|| \leq \max_i ||Te_i|| \sum_{i=1}^{n} |\alpha_i| = \beta ||x||_1 \ where \ (\beta = \max ||Te_i||)$$

$= c||x||$ (where c= βk by equivalence of norms on finite dimensional vector space)

Theorem 1.18. *(Continuity and boundedness): Let $T : X \longrightarrow Y$ be a linear operator, where X and Y are normed spaces. Then:*

(a) T is continuous if and only if T is bounded.

(b) If T is continuous at the origin, then T is continuous.

Proof. (a) For $T = 0$, the statement is trivial. Let T $\neq 0$,then $||T|| \neq 0$. We assume T is bounded and consider $x_0 \in X$ such that $||x - x_0|| < \delta$ where $\delta = \epsilon/||T||$, we obtain

$$||Tx - Tx_0|| = ||T(x - x_0)|| \leq ||T||||x - x_0|| < ||T||\delta = \epsilon.$$

Since $x_0 \in X$ was arbitrary, this shows that T is continuous.
Conversely, assume that T is continuous at an arbitrary $x_0 \in X$, then given any $\epsilon > 0$, there exist $\delta > 0$ such that

$$||Tx - Tx_0|| \leq \epsilon$$

for all $x \in X$ satisfying

$$||x - x_0|| \leq \delta.$$

We now take $y \neq 0 \in X$ and set $x = x_0 + \frac{\delta}{||y||}y$. Then $x - x_0 = \frac{\delta}{||y||}y$. Hence $||x - x_0|| = \delta$. Since T is linear we have

$$||Tx - Tx_0|| = ||T(x - x_0)|| = ||T(\frac{\delta}{||y||}y)|| = \frac{\delta}{||y||}||Ty|| \leq \epsilon.$$

Thus $||Ty|| \leq \frac{\epsilon}{\delta}||y||$, $\quad ||Ty|| \leq c||y|| \Longrightarrow$ T is bounded, where $c = \frac{\epsilon}{\delta}$

(b) Suppose T is continuous at a point $x_0 = 0$, then it suffices to show that T is bounded (continuous). T is continuous at $x_0 = 0$, take $\epsilon = 1$, there exist $\delta > 0$. such that

$$||x|| \leq \delta \Longrightarrow ||Tx|| \leq 1.$$

Let $z \in \mathcal{D}(T)$ $z \neq 0$, then $||\frac{z}{||z||}\frac{\delta}{2}|| = \frac{\delta}{2} < \delta \Longrightarrow ||T(\frac{z}{||z||}\frac{\delta}{2})|| < 1$
(By linearity of T) $\Longrightarrow ||Tz|| \leq \frac{2}{\delta}||z||$ $\forall z \in \mathcal{D}(T) \Longrightarrow T$ is continuous.

Corollary 1.19. (Continuity and null space)
Let $T : X \longrightarrow Y$ be a bounded linear operator. Then
(a) $x_n \longrightarrow x$ implies $Tx_n \longrightarrow Tx$.
(b)The null space $\mathcal{N}(T)$ is closed.

Proof. (a) Suppose $x_n \longrightarrow x$ in X i.e $||x_n - x|| \longrightarrow 0$. Since T is linear and bounded, then

$$||Tx_n - Tx|| = ||T(x_n - x)|| \leq |||T||||x_n - x|| \longrightarrow 0.$$

It implies $Tx_n \longrightarrow Tx$ as $n \longrightarrow \infty$
(b) Let $x \in \overline{\mathcal{N}(T)}$ it implies there exist $(x_n)_{n\geq 1} \subset \mathcal{N}(T)$ such that $x_n \longrightarrow x$. Since T is bounded by corollary 1.19a $Tx_n \longrightarrow Tx$, but $x_n \in \mathcal{N}(T)$. It implies $Tx_n = 0$ $\forall n \geq 1$, thus $T(x) = 0$.
It implies $x \in \mathcal{N}(T)$. Hence $\mathcal{N}(T)$ is closed.

1.5 Examples of bounded operators on infinite dimensional spaces

1. Let $K : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be continuous. Let $T : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ be defined by

$$T(f)(x) = \int_0^1 K(x,y)f(y)dy.$$

Then $T \in \mathcal{L}(C[0,1])$ for $f \in C[0,1]$ and bounded.

Proof. Clearly T is linear. We next show boundedness.

$$|T(f)(x)| \leq \int_0^1 |K(x,y)||f(y)|dy \leq sup|f(y)| \int_0^1 |K(x,y)|dy \leq ||f|| \int_0^1 |K(x,y)|dy.$$

It implies

$$||T(f)(x)||_\infty \leq c||f||_\infty$$

where $\int_0^1 |K(x,y)| \leq c$ since K is continuous. It implies T is bounded.

2. Let $p \geq 1$, we define

$$l^p = \{(x_n)_{n\geq 1} \subset R : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$

Let $T : l^p \longrightarrow l^p$ be defined by

$T((x_n)_{n\geq 1}) = (x_{n+1})_{n\geq 1}$ (The left shift operator) is bounded

where $(x_n)_{n\geq 1} = (x_1, x_2, x_3, ...)$ and $T((x_n)_{n\geq 1}) = (x_2, x_3, ...)$

Proof.

$$||T((x_n)_{n\geq 1})|| = (\sum_{n=2}^{\infty} |x_n|^p)^{1/p} \leq (\sum_{n=1}^{\infty} |x_n|^p)^{1/p} = ||(x_n)_{n\geq 1}||.$$

Thus T is bounded with $||T|| \leq 1$

3. Let $T : L^2([0,1],\mathbb{R}) \longrightarrow L^2([0,1],\mathbb{R})$ be defined by

$$(Tf)(t) = tf(t) \quad for \quad a.e \quad t \in [0,1].$$

Then T is bounded.

Proof. $||Tf||^2_{L^2[0,1]} = \int_0^1 |(Tf)(t)|^2 dt = \int_0^1 |t|^2 |f(t)|^2 dt \leq \int_0^1 |f(t)|^2 dt = ||f||^2_{L^2[0,1]}$.

It implies $||Tf||_{L^2[0,1]} \leq ||f||_{L^2[0,1]}$. Hence T is bounded with $||T|| \leq 1$

1.6 Hilbert spaces

Definition 1.20. Let E be a real vector space. An inner product on E is a function,

$\langle .,. \rangle : E \times E \longrightarrow \mathbb{R}$ such that

(a) $||x||^2 \equiv \langle x, x \rangle \geq 0$ with equality $||x||^2 = 0$ iff $x = 0$

(b) $\langle x, y \rangle = \langle y, x \rangle$

(c)$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ i.e $x \longrightarrow \langle x, z \rangle$ is linear.

A real vector space E endowed with the inner product i.e $(E, \langle .,. \rangle)$ is called an inner product space.

Lemma 1.21. (Cauchy-Schwartz Inequality) Let E be an inner product space. Then for arbitrary $x, y \in E$,

$$|\langle x, y \rangle| \leq ||x|| ||y||$$

Lemma 1.22. (The Parallelogram Law) Let E be a real inner product space. Then for arbitary vector $x, y \in E$,

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2).$$

Proof. Expanding the LHS $||x+y||^2+||x-y||^2 = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle + \langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle$ $= 2(\langle x, x \rangle + \langle y, y \rangle) = 2(||x||^2 + ||y||^2) =$ RHS.

Definition 1.23. A complete inner product space is called a Hilbert space.

Definition 1.24. Let x, y be vectors in a Hilbert space H, then we say that x and y are orthogonal,written $x \perp y$, if $\langle x, y \rangle = 0$. We say that subsets A and B are orthogonal, written $A \perp B$, if $x \perp y$ for every $x \in A$ and $y \in B$. The orthogonal complement $A^\perp$ of a subset of A is the set of vectors orthogonal to A,

$$A^\perp = \{x \in H : x \perp y \ \ for \ \ all \ y \in A\}.$$

Definition 1.25. Let $\mathcal{M}$ and $\mathcal{N}$ be closed linear subspaces of a Hilbert space H, we define the orthogonal direct sum or simply the direct sum $\mathcal{M} \bigoplus \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ by

$$\mathcal{M} \bigoplus \mathcal{N} = \{y + z : y \in \mathcal{M} \ \ and \ \ z \in \mathcal{N}\}.$$

Definition 1.26. A subset U of nonzero vectors in a Hilbert space H is orthogonal if any two distinct elements in U are othorgonal. A set of vectors U is orthonormal if it is orthogonal and $||u|| = 1$ for all $u \in U$.

1.7 Some properties of Hilbert spaces

Theorem 1.27. *The orthogonal complement of a subset of a Hilbert space is a closed linear subspace.*

Proof. Let H be a Hilbert space and A a subset of H. if $y, z \in A^{\perp}$ and $\lambda, \mu \in \mathbb{R}$.Then the linearity of the inner product implies that

$$\langle x, \lambda y + \mu z\rangle = \lambda\langle x, y\rangle + \mu\langle x, z\rangle = 0$$

for all $x \in A$.

Therefore, $\lambda y + \mu z \in A^{\perp}$, so $A^{\perp}$ is a linear subspace.

To show that $A^{\perp}$ is closed, we show that if $(y_n)_{n\geq 1}$ is a convergent sequence in $A^{\perp}$, then the limit y also belongs to $A^{\perp}$. Let $x \in A$ then by continuity of inner product we have

$$\langle x, y\rangle = \langle x, \lim_{n\longrightarrow\infty} y_n\rangle = \lim_{n\longrightarrow\infty}\langle x, y_n\rangle = 0.$$

Since $\langle x, y_n\rangle = 0$ for every $x \in A$ and $y_n \in A^{\perp}$. Hence $y \in A^{\perp}$

Theorem 1.28. *Let $\mathcal{M}$ be a closed linear subspace of a Hilbert space H*

(a) For every $x \in H$ there is a unique closest point $y \in \mathcal{M}$ such that

$$||x - y|| = min||x - z||, z \in \mathcal{M}$$

(b)The point $y \in \mathcal{M}$ closest to $x \in H$ is the unique element of $\mathcal{M}$ with the property that $(x-y) \perp \mathcal{M}$

Proof. (a). Let d be the distance of x from $\mathcal{M}$ i.e

$$d = \inf\{||x - z|| : z \in \mathcal{M}\}.$$

First, we prove that there is a closest point $y \in \mathcal{M}$ at which this infimum is attained, meaning that $||x - y|| = d$. From the definition of d, there is a sequence of elements $y_n \in \mathcal{M}$ such that

$$\lim_{n\longrightarrow\infty} ||x - y_n|| = d.$$

Thus, for any $\varepsilon > 0$, there is an N such that

$$||x - y_n|| \leq d + \varepsilon \quad when \quad n \geq N.$$

We show that the sequence $(y_n)_{n\geq 1}$ is Cauchy. From the parallelogram law, we have

$$||y_m - y_n||^2 + ||2x - y_m - y_n||^2 = 2||x - y_m||^2 + 2||x - y_n||^2.$$

Since $(y_m + y_n)/2 \in \mathcal{M}$, it implies that $||x - (y_m + y_n)/2|| \geq d$. Thus for all $m, n \geq N$

$$||y_m - y_n||^2 = 2||x - y_m||^2 + 2||x - y_n||^2 - ||2x - y_m - y_n||^2 \leq 4(d + \varepsilon)^2 - 4d^2 = 4\varepsilon(2d + \varepsilon).$$

Therefore,$(y_n)_{n\geq 1}$ is Cauchy.Since a Hilbert space is complete, there is a y such that $y_n \longrightarrow y$ and since $\mathcal{M}$ is closed, we have $y \in \mathcal{M}$. By continuity of norm we have

$$||x - y|| = \lim_{n\longrightarrow\infty} ||x - y_n|| = d$$

We prove the uniqueness of the vector $y \in \mathcal{M}$ that minimizes $||x-y||$. Suppose that y and y' both minimize the distance to x, meaning that $||x-y|| = d$, $||x-y'|| = d$.
Then the parallelogram law implies that

$$2||x-y||^2 + 2||x-y'||^2 = ||2x-y-y'||^2 + ||y-y'||^2.$$

Since $(y+y')/2 \in \mathcal{M}$,

$$||y-y'||^2 = 4d^2 - 4||x-(y+y')/2||^2 \leq 0.$$

Therefore, $||y-y'|| = 0$ so that $y = y'$. (b) We show that the unique $y \in \mathcal{M}$ found above satisfies the condition that the vector $x-y$ is orthogonal to $\mathcal{M}$. Since y minimizes the distance to x, we have for every $\lambda \in \mathbb{C}$ and $z \in \mathcal{M}$ that

$$||x-y||^2 \leq ||x-y+\lambda z||^2.$$

Expanding the right-hand side of this equation, we obtain that

$$2Re\lambda\langle x-y, z\rangle \leq |\lambda|^2||z||^2.$$

Suppose that $\langle x-y, z\rangle = |\langle x-y, z\rangle|e^{i\varphi}$. Choosing $\lambda = \varepsilon e^{-i\varphi}$, where $\varepsilon > 0$ and dividing by ε, we get

$$2|\langle x-y, z\rangle| \leq \varepsilon||z||^2.$$

Taking the limit as $\varepsilon \longrightarrow 0^+$, we get $\langle x-y, z\rangle = 0$ so $(x-y) \perp \mathcal{M}$.
Finally, we show that y is the only element in $\mathcal{M}$ such that $(x-y) \perp \mathcal{M}$. Suppose that y' is another such element in $\mathcal{M}$. Then $y-y' \in \mathcal{M}$, and for any $z \in \mathcal{M}$, we have

$$\langle z, y-y'\rangle = \langle z, x-y'\rangle - \langle z, x-y\rangle = 0.$$

In particular, we may take $z = y-y'$ and therefore we have $y = y'$

Definition 1.29. Let Ω be an open set in $\mathbb{R}^n$, then
$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}, measurable : \int_\Omega |f|^p dx < \infty\}$, $1 \leq p < \infty$.

Definition 1.30. Let Ω be an open set in R^n and $n \in N$, the Sobolev space $H^m(\Omega)$ is defined by $H^m(\Omega) = \{f \in L^2(\Omega), D^\alpha f \in L^2(\Omega), \alpha \in N^n, |\alpha| \leq m\}$ where $D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}} f$, $|\alpha| = \alpha_1 + ... + \alpha_n$ and $||u||_{H^m(\Omega)} = ||u||_{L^2} + \sum_{\alpha:|\alpha|\leq m} |D^\alpha u|_{L^2}, u \in H^m(\Omega)$

Definition 1.31. Let $\varphi : \Omega \longrightarrow R$ be continous, then the support of φ is defined by
$Supp(\varphi) = \overline{\{x \in \Omega : \varphi(x) \neq 0\}}$

Definition 1.32. $\mathcal{D}(\Omega)$ the space of test functions is defined by
$\mathcal{D}(\Omega) = \{f \in C^\infty : Supp(f) \text{ is compact in } \Omega\}$

Definition 1.33. A distribution is a continuous linear map $T : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$ such that

$$\lim_{n\longrightarrow\infty} T(\varphi_n) = T(\varphi)$$

for any sequence

$$\varphi_n \overset{\mathcal{D}(\Omega)}{\longrightarrow} \varphi.$$

The space of distribution on Ω is denoted by $\mathcal{D}'(\Omega)$.

Definition 1.34. A sequence of distribution $T_n \in \mathcal{D}'(\Omega)$ is said to converge in the sense of distribution to $T \in \mathcal{D}'(\Omega)$, if for every test function $\varphi \in \mathcal{D}(\Omega)$ one has

$$lim_{n\longrightarrow\infty}\langle T_n, \varphi\rangle = \langle T, \varphi\rangle$$

1.7.1 Examples of Hilbert spaces

(a). $L^2(\Omega)$ equipped with the norm $||f||_{L^2(\Omega)} = (\int_\Omega |f|^2 dx)^{1/2}$ is Hilbert.
(b). $H^1(\Omega)$ equipped with the norm $||u||^2_{H^1(\Omega)} = \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx$ is Hilbert, where

$$H^1(\Omega) = \{f \in L^2(\Omega) : \frac{\partial f}{\partial x_i} \in L^2(\Omega)\}$$

Proof. (a).Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $L^2(\Omega)$, then we can find a subsequence $(f_{n_k})_{k\geq 1}$ such that

$$||f_{n_k} - f_{n_{k+1}}|| < \frac{1}{2^k}, \quad k = 1, 2, 3, ...$$

Choose a function $g \in L^2(\Omega)$. By the Schwartz inequality,

$$\int_\Omega |g(f_{n_k} - f_{n_{k+1}}|d\mu \leq \frac{||g||}{2^k}.$$

Hence

$$\sum_{k=1}^{\infty}\int_\Omega |g(f_{n_k} - f_{n_{k+1}}|d\mu \leq ||g||.$$

Thus

$$|g(x)|\sum_{k=1}^{\infty}|f_{n_k}(x) - f_{n_{k+1}}(x)| < +\infty$$

almost everywhere on X.
It implies

$$\sum_{k=1}^{\infty}|f_{n_k}(x) - f_{n_{k+1}}(x)| < +\infty$$

almost everywhere on X.
Since the kth partial sum of the series $\sum_{k=1}^{\infty}(f_{n_k}(x) - f_{n_{k+1}}(x))$ which converges almost everywhere on X is $f_{n_k}(x) - f_{n_{k+1}}(x)$.
It implies

$$f(x) = \lim_{k\longrightarrow\infty} f_{n_k}(x).$$

Let $\varepsilon > 0$ be given, there exists $N_0 \in \mathbb{N}$ such that

$$||f - f_{n_k}|| \leq \lim_{j \longrightarrow \infty} \inf ||f_{n_j} - f_{n_k}|| \leq \varepsilon.$$

Thus $f - f_{n_k} \in L^2(\Omega)$, and since $f = (f - f_{n_k}) + f_{n_k}$, we see that $f \in L^2(\Omega)$.

Also, since ε is arbitrary,

$$\lim_{k \longrightarrow \infty} ||f - f_{n_k}|| = 0.$$

Finally, the inequality

$$||f - f_n|| \leq ||f - f_{n_k}|| + ||f_{n_k} - f_n||$$

shows that (f_n) converges to f in $L^2(\Omega)$.

(b). Let $(u_n)_{n \geq 1}$ be a Cauchy sequence in $H^1(\Omega)$ then given $\varepsilon > 0$ there exist $n_0 \in N$ such that $\forall m, n > n_0$

$$||u_n - u_m||_{H^1(\Omega)} < \varepsilon.$$

which implies that

$$\Big(\int_\Omega |u_n - u_m|^2 dx + \int_\Omega |\nabla u_n - \nabla u_m|^2 dx \Big)^{1/2} < \varepsilon \quad (**).$$

Thus $(u_n)_{n \geq 1}$ be a Cauchy sequence in $L^2(\Omega)$ and $(\nabla u_n)_{n \geq 1}$ is also a Cauchy sequence in $L^2(\Omega)$. Since $L^2(\Omega)$ is complete,

$$u_n \longrightarrow u \in L^2(\Omega) \ \ and \ \ \nabla u_n \longrightarrow w_i \in L^2(\Omega).$$

We need to show that $\nabla u = w_i$.

But

$$u_n \longrightarrow u \in L^2(\Omega) \Rightarrow u_n \longrightarrow u \in \mathcal{D}'(\Omega),$$

thus

$$\nabla u_n \longrightarrow \nabla u \in \mathcal{D}'(\Omega).$$

By uniqueness of limit in $\mathcal{D}'(\Omega)$, we have $\nabla u = w_i$

From (**), let n be fixed and let $m \longrightarrow \infty$ we have

$$\Big(\int_\Omega |u_n - u|^2 dx + \int_\Omega |\nabla u_n - \nabla u|^2 dx \Big)^{1/2} < \varepsilon,$$

thus $u_n \longrightarrow u$ in $H^1(\Omega)$. Hence $H^1(\Omega)$ is Hilbert.

Theorem 1.35. *(Riesz Theorem) Let H be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. If T is a bounded linear functional on H i.e T is a bounded operator from H to the field $\mathbb{R}$ or $\mathbb{C}$, then there exists some $g \in H$ such that for every $f \in H$ we have*
$T(f) = \langle f, g \rangle$. Moreover, $||T|| = ||g||$.

Proof. We can choose an orthonormal basis $\phi_j, j \geq 1$ for H. Let T be bounded linear functional and set $a_j = T(\phi_j)$. Choose $f \in H$, let $c_j = \langle f, \phi_j \rangle$ and define

$$f_n = \sum_{j=1}^{n} c_j \phi_j.$$

Since ϕ_j forms a basis we know that $||f_n - f|| \longrightarrow 0$ as $n \longrightarrow \infty$.
Since T is linear we have

$$T(f_n) = \sum_{j=1}^{n} a_j c_j \qquad (1)$$

Since T is bounded, say with norm $||T|| < \infty$ we have

$$||T(f_n) - T(f)|| \leq ||T|| ||f_n - f|| \qquad (2)$$

Because $||f_n - f|| \longrightarrow 0$ as $n \longrightarrow \infty$, we conclude from equations (1) and (2) that

$$T(f) = \lim_{n \longrightarrow \infty)} T(f_n) = \sum_{j=1}^{\infty} a_j c_j \qquad (3)$$

Infact, the sequence a_j must itself be square-summable. To see this, first note that since $|T(f)| \leq ||T|| ||f||$ we have

$$|\sum_{j=1}^{\infty} a_j c_j| \leq ||T|| (\sum_{j=1}^{\infty} c_j^2)^{1/2} \qquad (4)$$

Equation (4) must hold for any square-summable sequence c_j (since any c_j corresponds to some elements in H).Fix a positive integer N and define a sequence $c_j = a_j$ for $j \leq N$, $c_j = 0$ for $j \geq N$. Clearly such a sequence is square-summable and equation (4) then yields

$$|\sum_{j=1}^{N} a_j^2| \leq ||T|| (\sum_{j=1}^{N} a_j^2)^{1/2}$$

or

$$(\sum_{j=1}^{N} a_j^2)^{1/2} \leq ||T|| \qquad (5)$$

Thus a_j is square-summable the function $g = \sum_j a_j \phi_j$ is well defined as an element of H and

$$T(f) = \sum_j a_j c_j = \langle f, g \rangle.$$

Finally, equation (5) makes it clear that $||g|| \leq ||T||$. But from Cauchy-Schwartz we also have $|T(f)| = |\langle f, g \rangle| \leq ||f||||g||$ implying $||T|| \leq ||g||$, so $||T|| = ||g||$.

CHAPTER 2

COMPACT LINEAR OPERATORS ON BANACH SPACES

2.1 INTRODUCTION

Compact linear operators are very important in applications. They play a central role in the theory of integral equations and various problems in mathematical physics.They have properties that closely resembles those operators on finite dimensional spaces. Many linear Operators in analysis are compact, a systematic theory of compact linear operators emerged from the theory of integral equation of the form:
$(T - \lambda I)x(s) = y(s)$ where $Tx(s) = \int_a^b K(s,t)x(t)dt$
where $\lambda \in \mathbb{C}$ is a parameter, y and the kernel K are given functions subject to certain conditions. The aim of this chapter is to study compact operators and their properties and also consider some examples.

2.2 Compact operators

Definition 2.1. Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$ then T is said to be compact if for any bounded subset $A \subset X$, then $T(A)$ is pre-compact in Y i.e $\overline{T(A)}$ is compact in Y.

Theorem 2.2. *(Riez). If $dim X < \infty$, A subset A in X is compact if it is closed and bounded.*

Theorem 2.3. *(Sequential Characterisation): If $T \in \mathcal{L}(X, Y)$, then T is compact if for any bounded sequence $(x_n)_{n\geq 1}$ in X the sequence $(T(x_n))_{n\geq 1}$ has convergent subsequence in Y.*

Definition 2.4. $\mathcal{K}(X,Y) = \{T : X \longrightarrow Y$ such that T is linear and compact$\}$.
Notation: $\mathcal{K}(X) = \mathcal{K}(X,X)$

Definition 2.5. $\mathcal{B}(X,Y) = \{T : X \longrightarrow$ Y such that T is linear and bounded$\}$.
Notation: $\mathcal{B}(X) = \mathcal{B}(X,X)$

Lemma 2.6. $\mathcal{K}(X,Y) \subset \mathcal{B}(X,Y)$

Proof. Suppose T $\in \mathcal{K}(X,Y)$, we want to show that T $\in \mathcal{B}(X,Y)$ i.e $\forall x \in X$ there exists M > 0 such that

$$||Tx|| \leq M||x||. \tag{2.1}$$

Let x $\in X$. If x $= 0$, then (2.1) holds trivially.
Assume $x \neq 0$, $\frac{x}{||x||} \in B_x(0,1)$, $\overline{T(B_x(0,1))}$ is compact since T $\in \mathcal{K}(X,Y)$.
So $T(B_x)$ is bounded, therefore there exist M > 0 such that $T(\frac{x}{||x||}) \leq M$

$$\begin{aligned}
&implies \quad \frac{1}{||x||}T(x) \leq M \quad \text{(By linearity of T)}\\
&implies \quad T(x) \leq M||x||\\
&implies \quad ||T(x)|| \leq M||x||
\end{aligned}$$

implies $T \in \mathcal{B}(X,Y)$, thus $\mathcal{K}(X,Y) \subset \mathcal{B}(X,Y)$.

Theorem 2.7. *Let X and Y be two normed vector spaces over $\mathbb{R}$. Assume Y is Banach, then $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{B}(X,Y)$*

Proof. Let $T \in \overline{\mathcal{K}(X,Y)}$ then there exist $T_n \in \mathcal{K}(X,Y)$ such that $\lim_{n\longrightarrow\infty} T_n = T$ in $\mathcal{B}(X,Y)$.
Let $\varepsilon > 0$ there exist $N \geq 1$ such that

$$||T_n - T|| < \varepsilon \quad for \quad all \quad n \geq N.$$

Fix N, $\overline{T_N(B_1)}$ is compact, then $T_N(B_1)$ is totally bounded so there exist $x_1, x_2, ..., x_k$ in B_1 such that

$$T_N(B_1) \subset \bigcup_{i=1}^{k} B(T_N(x_i), \varepsilon).$$

Let $x \in B_1$, then

$$||T_N(x) - T(x)|| \leq ||T_N - T||||x|| = \varepsilon||x|| = \varepsilon_1,$$

so $T(B_1) \subset \bigcup_{x\in B_1} B(T_N(x), \varepsilon_1) \Rightarrow T(B_1)$ is totally bounded. Since Y is Banach $\Rightarrow \overline{T(B_1)}$ is complete and hence compact in Y, thus $T \in \mathcal{K}(X,Y)$.
It implies $\mathcal{K}(X,Y)$ is a closed.

Theorem 2.8. *let X and Y be normed spaces, and $T : X \longrightarrow Y$ be a linear Operator, then*
(a)If T is bounded and $dimT(X) < \infty$, then the operator T is compact.
(b)If $dimX < \infty$, then the operator T is compact.

Proof. (a). Let (x_n) be any bounded sequence in X, then the inequality $||Tx_n|| \leq ||T||||x_n||$ shows that $T(x_n)$ is bounded. Hence $T(x_n)$ is relatively compact. Since $dimT(X) < \infty$, It follows that $T(x_n)$ has a convergent subsequence in X, thus T is compact.
(b). From (a) by noting that $dimX < \infty$, it implies boundedness of T and $dimT(X) \leq dimX$ thus T is compact.

Corollary 2.9. Let $T_n : X \to Y$ be compact linear operator for each n ≥ 1, where Y is Banach. Assume that $||T_n - T|| \longrightarrow 0$, then T is compact.

Proof. We show that for any bounded sequence $(x_m)_{m\geq 1}$ in X, the image $T(x_m)_{m\geq 1}$ has a convergent subsequence.We have that $(T_n)_{n\geq 1}$ is compact for each n.
Since T_1 is compact,$(x_m)_{m\geq 1}$ has a subsequence $(x_{1,m})$ such that $T_1(x_{1,m})$ is Cauchy .Similarly $(x_{1,m})$ has a convergent subsequence $(x_{2,m})$ such that $T_2(x_{2,m})$ is Cauchy. Continuing in this way, we observe that $(y_m) = (x_{m,m})$ is a subsequence of (x_m) such that for every fixed positive integer n, the sequence $(T_n(y_m))_{m\in N}$ is cauchy. (x_m) is bounded, say $||x_m|| \leq c$ for all m, hence $||y_m|| \leq c$ for all m. let $\varepsilon > 0$ since $T_n \longrightarrow T$, there is an $n = p$ such that $||T_p - T|| < \frac{\varepsilon}{3c}$
since $(T_p(y_m))_{m\in N}$ is Cauchy,there exist an N such that

$$||T_p y_j - T_p y_k|| < \frac{\varepsilon}{3}, \quad (j, k > N)$$

Hence we obtain for $(j, k > N)$

$$\begin{aligned} ||Ty_j - Ty_k|| &\leq ||Ty_j - T_p y_j|| + ||T_p y_j - T_p y_k|| + ||T_p y_k - Ty_k|| \\ &\leq ||T - T_p||||y_j|| + \frac{\varepsilon}{3} + ||T_p - T||||y_k|| \\ &< \frac{\varepsilon}{3c}.c + \frac{\varepsilon}{3} + \frac{\varepsilon}{3c}.c = \varepsilon \end{aligned}$$

This shows that (Ty_m) is Cauchy and converges since Y is complete. Remembering that (y_m) is a subsequence of the arbitrary bounded sequence (x_m), this implies compactness of T.

Examples

1. let $T : l^2 \longrightarrow l^2$ be defined by $Tx = \frac{\xi_j}{j}$ for $j = 1, 2, ...$ where $x = (\xi_j)_{j\geq 1} \in l^2$, we show that T is compact. Infact, clearly T is linear, we define $T_n : l^2 \longrightarrow l^2$ by

$$T_n x = (\xi_1, \frac{\xi_2}{2}, ..., \frac{\xi_n}{n}, 0, 0, ...)$$

T_n is linear, bounded and compact from theorem 2.9, thus

$$||(T - T_n)x||^2 = \sum_{j=n+1}^{\infty} \frac{|\xi_j|^2}{j^2} \leq \frac{1}{(n+1)^2} \sum_{j=n+1}^{\infty} |\xi_j|^2 \leq \frac{||x||^2}{(n+1)^2}$$

Taking the Sup norm over x with $||x|| = 1$, we get

$$||T - T_n|| \leq \frac{1}{n+1}.$$

Hence $T_n \longrightarrow T$ as $n \longrightarrow \infty$. $\Rightarrow T$ is compact.

2.Let $T : l^2 \longrightarrow l^2$ be defined by $Tx = (\lambda_n x_n)_{n\geq 1}$ such that $\lambda_n \longrightarrow 0$, we show that T is compact.

Solution.

To show this, we approximate T by compact operators T_N such that

$$T_N x = (\lambda_1 x_1, \lambda_2 x_2, ..., \lambda_N x_N, 0, 0, ...),$$

then

$$||(T_N - T)x||^2 = (\sum_{n>N} |\lambda_n x_n|^2) \leq (sup_{n>N}|\lambda_n|)^2 (\sum_{n>N} |x_n|^2) \leq (sup_{n>N}|\lambda_n|)^2 ||x||^2$$

Taking the Sup norm over x with $||x|| = 1$, we get

$$||T_N - T|| \leq sup_{n>N}|\lambda_n|.$$

It implies

$$||T_N - T|| \longrightarrow 0 \quad since \quad \lambda_n \longrightarrow 0.$$

Hence T is compact.

Definition 2.10. let X be a normed vector space, a sequence $\{x_n\} \in X$ is said to converge weakly to x (written as $x_n \rightharpoonup x$) if for every f in the dual space of X (i.e f a bounded and linear map on X), $f(x_n) \longrightarrow f(x)$.

Theorem 2.11. *Let T be a linear and continous operator from a reflexive Banach space X into a Banach space Y. Then T is compact if and only if it maps weakly convergent sequence in X to a strongly convergent sequence in Y i.e $(x_n \rightharpoonup x \ \ in \ X) \Rightarrow (Tx_n \longrightarrow Tx \ \ in \ Y)$*

Proof. ($\Rightarrow$) Assume that T $\in \mathcal{K}(X, Y)$ and let $(x_n)_{n\geq 1} \subset X$ such that $x_n \rightharpoonup x$, we show that $Tx_n \longrightarrow Tx$.

Suppose for contradiction that $(Tx_n)_{n\geq 1}$ does not converge strongly to Tx, then there exist $\varepsilon > 0$ and there exist $(Tx_{n_k})_{k\geq 1}$ subsequence of $(Tx_n)_{n\geq 1}$ such that

$$||Tx_{n_k} - Tx_n|| \geq \epsilon \tag{2.2}$$

$(x_{n_k})_{k\geq 1}$ is a subsequence of $(x_n)_{n\geq 1}$ so $x_{n_k} \rightharpoonup x$. Thus $(x_{n_k})_{k\geq 1}$ is bounded. Since T is compact, there exist $(x_{n_{k_m}})_{m\geq 1}$ a subsequence of$(x_{n_k})_{k\geq 1}$ such that $(Tx_{n_{k_m}}) \longrightarrow y$ in Y and so $(Tx_{n_{k_m}}) \rightharpoonup y$ (since strong convergence implies weak convergence). Moreover, the weak convergence $x_{n_k} \rightharpoonup x$ implies$(Tx_{n_{k_m}}) \rightharpoonup Tx$, we conclude that T(x)=y. so $(Tx_{n_{k_m}})_{m\geq 1}$ converges to Tx and is a subsequence of $(Tx_{n_k})_{k\geq 1}$ which satisfies (2.2), therefore,

$$\epsilon \leq ||Tx_{n_{k_m}} - Tx|| \longrightarrow 0 \text{ as } m \longrightarrow \infty \text{ (contradiction)}$$

Hence $Tx_n \longrightarrow Tx$.

($\Leftarrow$) Conversely, assume the image of weakly convergent sequence is strongly convergent, we show that T is compact.

Let D be a non-empty subset of X which is bounded. We want to prove that $\overline{T(D)}$ is compact, it is enough to prove that $(y_n)_{n\geq 1} \subset T(D)$ has convergent subsequence. Let $(y_n)_{n\geq 1} \subset T(D)$ there exist $(x_n)_{n\geq 1}$ in D such that $T(x_n)_{n\geq 1} = y_n$, $n \geq 1$, D is bounded so $(x_n)_{n\geq 1}$ is bounded. Since X is reflexive, then by Eberlyn Smulyan theorem there exist $(x_{n_k})_{k\geq 1} \subset (x_n)_{n\geq 1}$ which converges weakly. From our assumprion it implies that $(Tx_{n_k})_{k\geq 1}$ is strongly convergent sequence in Y, so $(y_{n_k})_{k\geq 1}$ is strongly convergent sequence in Y. This implies $\overline{T(D)}$ is compact. Hence T is compact.

Theorem 2.12. *(Riez): Let E be a normed vector space and let $M \subset E$ be a closed linear subspace of E, then for all $\epsilon \in (0, 1)$, there exists $x \in E$ with $||x|| = 1$ such that $d(x, M) \geq 1 - \epsilon$.*

Proof. Let $\varepsilon \in (0, 1)$, choose $v \in E$ such that v is not in M. Since M is closed then the set d:=$dist(x, M)$ >0. choose any $m_0 \in M$ such that

$$d \leq ||v - m_0|| \leq \frac{d}{1-\varepsilon} \tag{2.3}$$

Let u = $\dfrac{v - m_0}{||v - m_0||}$. Clearly $||u|| = 1$, indeed for any $m \in M$, we have

$$\begin{aligned} ||u - m|| &= ||\frac{v - m_0}{||v - m_0||} - m|| \\ &= ||\frac{v - (m_0 + m||v - m_0||)}{||v - m_0||}|| \\ &\geq \frac{d}{||v - m_0||} \\ &\geq 1 - \epsilon \qquad (from\ (2.3)) \end{aligned}$$

where $m_0 + m||v - m_0|| \in M$ since M is a subspace.

Theorem 2.13. *(Riez) Let E be a normed vector space with $B_E(0, 1)$ compact, then E is finite dimensional.*

Proof. Suppose for contradiction that E is infinite dimensional then there exist $(E_n)_n$ of finite dimensional subspace of E such that $E_{n-1} \subset E_n$ and $E_{n-1} \neq E_n$, from Riez theorem 2.13

there exists $(u_n) \subset E_n$ such that $||u_n|| = 1$, $n \geq 1$ and $dist(u_n, E_{n-1}) \geq 1 - \varepsilon$, take $\varepsilon = 1/2 \quad \Rightarrow dist(u_n, E_{n-1}) \geq 1/2$. In particular, let $m < n$ for $(u_m)_{m\geq 1} \subset E_{n-1}$, we have

$$||u_m - u_n|| \geq 1/2.$$

Thus $(u_n)_{n\geq 1}$ has no convergent subsequence (contradiction since $B_E(0,1)$ is compact). Hence E is finite dimensional.

Lemma 2.14. let X be normed space, if $dimX = \infty$, then the identity Operator $I : X \longrightarrow X$ is not compact.

Proof. Let $M = \{x \in X : ||x|| \leq 1\}$, then M is bounded. Since $dimX = \infty$, $\Rightarrow M$ cannot be compact, thus $I(M) = M$ is not compact.

Theorem 2.15. *(a).* $\mathcal{N}(T^n) \subset \mathcal{N}(T^{n+1})$ *i.e* $\mathcal{N}(T^n)$ *is increasing.*
(b). $\mathcal{R}(T^{n+1}) \subset \mathcal{R}(T^n)$ *i.e* $\mathcal{R}(T^n)$ *is decreasing.*
where $T^0 = I$, $T^1 = T$ *and* $T^n = T \circ T \circ T ... \circ T$ *(n- times)*

Proof (a). Let $x \in \mathcal{N}(T^n) \Rightarrow T^n(x) = 0$. Then
$T^{n+1}(x) = T(T^n(x)) = T(0) = 0$,
implies $x \in \mathcal{N}(T^{n+1}) \Rightarrow \mathcal{N}(T^n) \subset \mathcal{N}(T^{n+1})$.
(b). Let $y \in \mathcal{R}(T^{n+1})$. Then there exists $x \in X$ such that
y $= T^{n+1}(x) = T^n(T(x)) = T^n(z)$ where $z = T(x)$
implies $y = T^n(z) \Rightarrow y \in \mathcal{R}(T^n)$.
Hence $\mathcal{R}(T^{n+1}) \subset \mathcal{R}(T^n)$.

Theorem 2.16. *(Fredholm's Alternative)*
Let X be a Banach space and $T \in \mathcal{K}(X)$. Then for any $\lambda \neq 0$, the following hold:
(a) $\mathcal{N}(\lambda I - T) = \mathcal{K}er(\lambda I - T)$ *is finite dimensional.*
(b)space $\mathcal{R}(\lambda I - T)$ *is closed.*
(c) $\mathcal{N}(\lambda I - T) = \{0\} \Longleftrightarrow \mathcal{R}(\lambda I - T) = X$.

Proof. (a) We show that $\mathcal{N}(\lambda I - T)$ is finite dimensional.
Let $B^* = \{x \in \mathcal{N}(\lambda I - T) : ||x|| \leq 1\}$, it suffices to show that B^* is compact. Let $\{x_n\}$ *be in* B^*. Then

$$(\lambda I - T)(x_n) = 0.$$

It implies $T(x_n) = \lambda x_n$, but T is compact $\Rightarrow$ there exist $(x_{n_k})_{k\geq 1}$ such that $T(x_{n_k})$ is convergent, $(x_{n_k})_{k\geq 1} = 1/(\lambda)T(x_{n_k})_{k\geq 1}$ is also convergent. So B^* is compact, Hence $\mathcal{N}(\lambda I - T)$ is finite dimensional.

(b) we show that $\mathcal{R}(\lambda I - T)$ is closed. Let $y \in \overline{\mathcal{R}(\lambda I - T)}$, then there exist $(x_n)_{n\geq 1} \in X$ such that $y = \lim_{n\to\infty}(\lambda I - T)(x_n)$. Since T is compact then there exists $(x_{n_k})_{k\geq 1} \subset (x_n)_{n\geq 1}$ such that $T(x_{n_k})_{n\geq 1}$ is convergent.
Let $z = \lim_{k\to\infty} T(x_{n_k})$. Then

$$(\lambda I - T)(x_{n_k}) = \lambda(x_{n_k}) - T(x_{n_k})$$

$$x_{n_k} = \frac{1}{\lambda}((\lambda I - T)(x_{n_k}) + T(x_{n_k}))$$

It implies $\lim_{k\to\infty}(x_{n_k}) = \frac{1}{\lambda}(y + z)$, thus $(x_{n_k})_{k\geq 1}$ is convergent in X.
It implies $y = (\lambda I - T)(1/\lambda)(y + z) \in \mathcal{R}(\lambda I - T)$. Hence$\mathcal{R}(\lambda I - T)$ is closed.
(c). Suppose $\mathcal{N}(\lambda I - T) = \{0\}$, we show that $\mathcal{R}(\lambda I - T) = X$. Suppose for contradiction that $\mathcal{R}(\lambda I - T) \neq X$.Now $\mathcal{N}(\lambda I - T) = \{0\} \Rightarrow \lambda I - T$ is one to one has a bounded inverse

$$(\lambda I - T)^{-1} : \mathcal{R}(\lambda I - T) \longrightarrow X.$$

Let X_1 =$(\lambda I - T)X$, $X_2 = (\lambda I - T)X_1 = (\lambda I - T)^2X$,..., $X_n = (\lambda I - T)^nX$. Clearly, X_{n+1} is a closed proper subset of X_n $\quad (n = 0, 1, 2, ...)$, since $(\lambda I - T)$ is one - to -one, therefore by Riez there exist $x_n \in X_n$ with $||x_n|| = 1$, $dist(x_n, X_{n+1}) \geq 1/2$. For $n > m$, $X_n \subsetneq X_m$,

$$||Tx_n - Tx_m|| = ||\lambda x_m - \lambda x_m + \lambda x_n - \lambda x_n + Tx_n - Tx_m|| = ||(\lambda I - T)x_m - (\lambda I - T)x_n + \lambda x_n - \lambda x_m||.$$

We observe that $(\lambda I - T)x_m - (\lambda I - T)x_n + \lambda x_n - \lambda x_m \in X_{m+1}$, so

$$||Tx_n - Tx_m|| \geq |\lambda| dist(x_m, x_{m+1}) \geq \frac{|\lambda|}{2}$$

implies $(Tx_n)_{n\geq 1}$ has no convergent subsequence and this contradicts the fact that T is compact, thus
$\mathcal{N}(\lambda I - T) = \{0\} \Rightarrow \mathcal{R}(\lambda I - T) = X$.
Conversely: Suppose $\mathcal{R}(\lambda I - T) = X$, we show that $\mathcal{N}(\lambda I - T) = \{0\}$.
Claim:$\mathcal{N}(\lambda I - T^*) = \{0\}$.
Proof of the claim: suppose $\mathcal{N}(\lambda I - T^*) \neq \{0\}$ then there exist $f \in X^* - \{0\}$ such that $(\lambda I - T^*)f = 0 \Rightarrow \lambda f(x) - (f \circ T)(x) = 0\ \forall x \in X$.
It implies

$$f(\lambda x - T(x)) = f((\lambda I - T)(x)) = 0\ \forall x \in X$$

$f((\lambda I - T)(X)) = \{0\} \Rightarrow f(X) = \{0\}$ since $(\mathcal{R}(\lambda I - T) = X$

implies $f = 0$ (contradicts the fact that $f \in X^* - 0$)

implies

$$\mathcal{N}(\lambda I - T^*) = \{0\}.$$

implies $\mathcal{R}(\lambda I - T^*) = X^*$. Hence $\mathcal{N}(\lambda I - T) = \{0\}$.

Theorem 2.17. *Let $\lambda \neq 0$ and $T \in K(X)$. Then there exist $r, s \in N$ such that*
(a) $\mathcal{N}((\lambda I - T)^k) \neq \mathcal{N}((\lambda I - T)^{k+1})$ for k = 0,1,2,...,r-1 and
$\mathcal{N}((\lambda I - T)^k) = \mathcal{N}((\lambda I - T)^r)$ for $k \geq r$.
(b) $\mathcal{R}((\lambda I - T)^k) \neq \mathcal{R}((\lambda I - T)^{k+1})$ for k = 0,1,2,...,s-1 and
$\mathcal{R}((\lambda I - T)^k) = \mathcal{N}((\lambda I - T)^s)$ for $k \geq s$.

Theorem 2.18. *Let X be a Banach space and let $T \in K(X)$, then*

$$X = \mathcal{R}((\lambda I - T)^s) \bigoplus \mathcal{N}((\lambda I - T)^s)$$

for some $s \in \mathbb{N}$.

Proof. Let $x \in X$, $(\lambda I - T)^s(x) \in \mathcal{R}((\lambda I - T)^s) = \mathcal{R}((\lambda I - T)^{2s})$.
Then there exist $z \in X$ such that $(\lambda I - T)^s(x) = (\lambda I - T)^{2s}(z)$.
It implies

$$(\lambda I - T)^s(x) - (\lambda I - T)^{2s}(z) = 0$$

It implies

$$(\lambda I - T)^s(x - (\lambda I - T)^s(z)) = 0$$

It implies

$$x - (\lambda I - T)^s(z) \in \mathcal{N}((\lambda I - T)^s)),$$

but

$$X = (\lambda I - T)^s(z) + X - (\lambda I - T)^s(z).$$

Thus,

$$X = \mathcal{R}((\lambda I - T)^s) + \mathcal{N}((\lambda I - T)^s).$$

Now, let $x \in \mathcal{R}((\lambda I - T)^s) \cap \mathcal{N}((\lambda I - T)^s)$. Let $x \in \mathcal{R}((\lambda I - T)^s)$ it implies there exist $x_1 \in X$ such that x = $(\lambda I - T)^s(x_1)$, for any $n \geq 1$, $x \in \mathcal{R}((\lambda I - T)^{ns})$, there exist $x_n \in X$ such that $x = (\lambda I - T)^s(x_n)$, since $x \in \mathcal{N}((\lambda I - T)^s)$ there exist $n_0 \geq 1$ such that $x \in \mathcal{N}((\lambda I - T)^{n_0 s})$ and

$n_0 s \geq r$.

$$(\lambda I - T)^s(x) = (\lambda I - T)^{ns+s}(x_n) = (\lambda I - T)^{(n+1)s}(x_n) = 0,$$

so $x_n \in \mathcal{N}((\lambda I - T)^{(n+1)s})$

Assume $X \neq 0$, then x_n is not in $\mathcal{N}((\lambda I - T)^{ns})$

thus

$$\mathcal{N}((\lambda I - T)^{ns}) \neq \mathcal{N}((\lambda I - T)^{(n+1)s})$$

for n large enough

$$\mathcal{N}((\lambda I - T)^{n_0 s}) = \mathcal{N}((\lambda I - T)^{(n_0+1)s}) = \mathcal{N}((\lambda I - T)^r),$$

so $x = 0$ i.e

$$x \in \mathcal{R}((\lambda I - T)^s) \cap \mathcal{N}((\lambda I - T)^s) = \{0\}$$

Hence

$$X = \mathcal{R}((\lambda I - T)^s) \bigoplus \mathcal{N}((\lambda I - T)^s).$$

Definition 2.19. Let $\mathcal{F}$ be a collection of functions on $D \subseteq \mathbb{R}^n$, then $\mathcal{F}$ is said to be uniformly bounded on D if there exists $M \in \mathbb{R}$, $M > 0$ such that $|f(x)| \leq M$, for all $x \in D$ and for all $f \in \mathcal{F}$. Furthermore, $\mathcal{F}$ is said to be Equicontinuous on D if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$, for all $x, y \in D$ and $f \in \mathcal{F}$.

Theorem 2.20. *(Arzela-Ascoli) Let $D \subseteq \mathbb{R}^n$ be compact and let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of continuous functions defined on D with values in $\mathbb{R}^n$ If $\mathcal{F} = \{f_n : n \in N\}$ is uniformly bounded and equicontinuous on D, then there exists a subsequence $\{f_{n,k}(x)\}_{k=1}^{\infty}$ that converges uniformly to a function $f \in C(D, \mathbb{R})$.*

Theorem 2.21. *The injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, where*

$$H^1(\Omega) = \{u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega)\}$$

Rudin (1976)

Theorem 2.22. *$K(X)$ is an Ideal.*

Proof. Let g be compact and f continuous. Let B be bounded.

$$(gof)(B) = g(f(B)).$$

f is continous, then $f(B)$ is bounded. Since g is compact then $(gof)(B)$ is pre-compact.

$$(fog)(B) = f(g(B)).$$

g is compact, then $g(B)$ is pre-compact. But f is continuous, then the image $f(g(B))$ is pre-compact.

Thus $(gof)(B)$ and $(fog)(B)$ are pre-compact. Hence $K(X)$ is an ideal.

CHAPTER 3

SPECTRAL DECOMPOSITION OF COMPACT OPERATORS ON HILBERT SPACES

3.1 INTRODUCTION

In this chapter, we shall study the spectral theory and decomposition of bounded and compact operators on Hilbert spaces developed by a Hungarian mathematician Frigyes Riez(22 January 1880 - 28 February 1956) which is concern with problems that arises naturally in solving differential and Integral equations.This helps us in understanding the properties of the operators themselves.Here we shall obtain eigen values and eigen functions of compact linear and self-adjoint operators both from their analytic and weak formulations.We shall also consider some applications.

3.2 Spectral theory

Definition 3.1. (The spectrum of a linear operator)
Let T be linear map from X into X, the set denoted by $\rho(T)$ is defined by

$$\rho(T) = \{\lambda \in R : (\lambda I - T) \text{ is bijective from X into X}\}$$

The spectrum denoted by $\sigma(T)$ is the complement of the resolvent set i.e
$\sigma(T) = \mathbb{R} - \{\rho(T)\}$.

Theorem 3.2. *Let X be a Banach space over $\mathbb{R}$ and $T \in \mathcal{B}(X)$, then the following hold:*
(a) The spectrum $\sigma(T)$ is closed subset of $\mathbb{R}$.

(b) The spectrum $\sigma(T) \subset B(0, ||T||)$.

(c) The spectrum $\sigma(T)$ is a compact subset of $\mathbb{R}$.

Proof. (a). It suffices to show that $\rho(T)$ is open. Let $\lambda \in \rho(T)$, we need to show that there exist r>0 such that $B(\lambda, r) \subset \rho(T)$

Claim: $B(\lambda, \frac{1}{||R(\lambda,T)||}) \subseteq \rho(T)$ where $R(\lambda, T) = (\lambda I - T)^{-1}$

Proof of claim: Let $\mu \in B(\lambda, \frac{1}{||R(\lambda,T)||}) \Rightarrow ||\lambda - \mu|| < \frac{1}{||R(\lambda,T)||}$, we show that $\mu \in \rho(T)$ i.e $(\mu I - T)^{-1}$ exist. one has that

$$\mu I - T = \mu I - \lambda I + \lambda I - T = \lambda I - T + \mu I - \lambda I$$

$$= (\lambda I - T)[(\lambda I - T)^{-1}(\mu - \lambda) + I] = (\lambda I - T)[R(\lambda, T)(\mu - \lambda) + I],$$

since $||R(\lambda, T)||||(\mu - \lambda)|| < 1$.

Then, $I + R(\lambda, T)(\mu - \lambda)$ is invertible.

It implies $I + R(\lambda, T)(\mu - \lambda) \in ISO(X)$ i.e a bijective linear map from X into itself, and since $\lambda \in \rho(T)$ then $\lambda I - T \in ISO(X)$.

It implies $\mu I - T \in ISO(X)$. Thus $B(\lambda, \frac{1}{||R(\lambda,T)||}) \Rightarrow ||\lambda - \mu|| < \frac{1}{||R(\lambda,T)||}$, which implies $\rho(T)$ is open, hence $\sigma(T)$ is closed.

(b) **Claim:** If $||T|| < |\lambda|$ then $\lambda I - T$ is invertible.

Proof of the claim: If

$$||T|| < |\lambda| \Rightarrow \left(I - \frac{T}{\lambda}\right)^{-1} \quad exists$$

and

$$\left(I - \frac{T}{\lambda}\right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$$

$$(\lambda I - T)^{-1} = \frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}},$$

we have $(\lambda I - T)$ is invertible $\Rightarrow \lambda \in \rho(T)$.

Now if $T \in \mathcal{B}(X)$ and $\lambda \in \mathbb{R}$ such that $||T|| < |\lambda|$

then $\lambda I - T$ is invertible i.e $\{\lambda \in \mathbb{R} : ||T|| < |\lambda|\} \subset \rho(T)$. Consequently

$$\rho(T)^c \subset \{\lambda \in \mathbb{R} : ||T|| < |\lambda|\}^c \Rightarrow \sigma(T) \subset \{\lambda \in \mathbb{R} : |\lambda| < ||T||\}.$$

It implies$\sigma(T) \subset B(0, ||T||)$

(c). From part (b) $\sigma(T)$ is bounded and from part (a) $\sigma(T)$ is closed, Hence $\sigma(T)$ is compact.

Theorem 3.3. *Let $R(\lambda, T) = (\lambda I - T)^{-1}$ and $||T|| < |\lambda|$. Then*
(a) $||R(\lambda, T)|| \longrightarrow 0$ as $\lambda \longrightarrow \infty$
(b)$R(\lambda, T) - R(\mu, T) = (\lambda - \mu)R(\lambda, T)R(\mu, T)$

Proof. (a). $R(\lambda, T) = (\lambda I - T)^{-1} = \frac{1}{\lambda}(I - \frac{T}{\lambda})^{-1}$, we get

$$\begin{aligned} ||R(\lambda, T)|| = \frac{1}{|\lambda|}||(I - \frac{T}{\lambda})^{-1}|| &\leq \frac{1}{\lambda}\left(\frac{1}{1 - ||\frac{T}{\lambda}||}\right) \\ &= \frac{1}{|\lambda| - ||T||} \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty \end{aligned}$$

(b)

$$\begin{aligned} (\lambda - \mu)R(\lambda, T)R(\mu, T) &= (\mu I - T + T - \lambda I)(\mu I - T)^{-1}(\lambda I - T)^{-1} \\ &= [(\mu I - T) - (\lambda I - T)](\mu I - T)^{-1}(\lambda I - T)^{-1} \\ &= (\mu I - T)(\mu I - T)^{-1})(\lambda I - T)^{-1} - (\lambda I - T)(\mu I - T)^{-1}(\lambda I - T)^{-1} \\ &= (\lambda I - T)^{-1} - (\mu I - T)^{-1} = R(\lambda, T) - R(\mu, T) \end{aligned}$$

Note: $R(\lambda, T)R(\mu, T) = R(\mu, T)R(\lambda, T)$

Theorem 3.4. *Let X be an infinite dimensional space, $T \in \mathcal{K}(X)$. Then the following hold:*
(a) $0 \in \sigma(T)$
(b) $\sigma(T) - \{0\}$ consists of eigen values of finite multiplicity i.e the dimension of λ- eigen space ($\mathcal{K}er(\lambda I - T)$) has finite dimension for all $\lambda \in \rho(T) - \{0\}$
i.e $\sigma(T) - \{0\} = \sigma_p(T)$

Remark: In an infinite dimensional space, a compact operator is never invertible.
Proof. (a) If $0 \notin \sigma(T)$, then $0 \in \rho(T) \Rightarrow T$ is invertible, since X is Banach then $T^{-1} \in B(Y, X)$. Therefore $TT^{-1} = I$ is compact, this is a contradiction because X is infinite dimensional, thus $0 \in \sigma(T)$
(b) Let $\lambda \in \rho(T) - \{0\}$ then $\lambda \neq 0$, we show that $\lambda \in \sigma_p(T)$. Assume $\lambda \notin \sigma_p(T)$ we have

$$\mathcal{K}er(\lambda I - T) = 0$$

It implies $R(\lambda I - T) = X$ (from theorem 2.9 c),thus $(\lambda I - T)$ is invertible, this contradicts the choice of λ.

3.3 Classification of $\lambda \in \sigma(T)$

(a) The point spectrum: The set of all eigenvalues of T is defined by

$$\sigma_p(T) = EV(T) = \{\lambda \in \mathbb{R} : \mathcal{K}er(\lambda I - T) \neq 0\}.$$

(b) The residual spectrum of T is the set defined by

$$\sigma_r(T) = \{\lambda \in \mathbb{R} : \mathcal{K}er(\lambda I - T) = 0 \text{ and } \operatorname{Im}\overline{(\lambda I - T)} \subset X\}.$$

(c) The continous spectrum of T is the set defined by

$$\sigma_c(T) = \{\lambda \in \mathbb{R} : \mathcal{K}er(\lambda I - T) = 0 \text{ and } \operatorname{Im}\overline{(\lambda I - T)} = X\}.$$

Note:

$$\sigma_p(T) \cap \sigma_c(T) \cap \sigma_r(T) = \phi \ , \ \sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) \text{ and } \mathcal{C} = \rho(T) \cup \sigma(T)$$

3.3.1 Examples

1. (Operators with spectral value which are not eigen value).

On the Hilbert space $X = l^2$ where $l^2 = \{(x_n)_{n\geq 1} : \sum_n = 1^\infty (x_n)^2 < \infty\}$

Define: $T : l^2 \longrightarrow l^2$ by $T(x_1, x_2, x_3, ...) = (0, x_1, x_2, \dots)$.

We show that $\sigma_p(T) = \phi$.

Now suppose $\sigma_p(T) \neq \phi \Rightarrow$ there exist $\lambda \in \sigma_p(T)$. It follows there exist $x = (x_1, x_2, x_3, ...) \in l^2 - \{0\}$ such that $Tx = \lambda x$

$\Rightarrow (0, x_1, x_2, ...) = (\lambda x_1, \lambda x_2, ...)$, if $\lambda = 0$ then $x = 0$ (impossible since $x \neq 0$).

Suppose $\lambda \neq 0$ then

$$\lambda x_1 = 0 \Rightarrow x_1 = 0,$$

$$\lambda x_2 = 0 \Rightarrow x_2 = 0,$$

$$\dots$$

$$\lambda x_n = 0 \Rightarrow x_n = 0$$

$\forall n \in N, x_n = 0 \Rightarrow x = 0$ (impossible)

$\sigma_p(T) = \phi$.

2. Given $T : l^2 \longrightarrow l^2$ such that for $x = (\xi_j)_{j\geq 1} \in l^2$ and

$Tx = (\xi_1 + \xi_2, \xi_2, \xi_3 + \xi_4, \xi_4, \xi_5 + \xi_6, \xi_6, \xi_7 + \xi_8, ...)$. we determine $\rho(T), \sigma_p(T), \sigma_c(T), \sigma_r(T)$.

Solution:

Clearly T is linear, we need to show that T is bounded, using the Cauchy-Schartz inequality in $\mathcal{C}^2$ i.e

$$|a_1 b_1 + a_2 b_2|^2 \leq (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2) \ \forall a_1, a_2, b_1, b_2 \in \mathcal{C} \tag{3.1}$$

(we use this with $a_1 = a_2 = 1$ and $b_1 = \xi_1, b_2 = \xi_2$) we have

$$\begin{aligned} ||Tx||^2 &= |\xi_1+\xi_2|^2 + |\xi_2|^2 + |\xi_3+\xi_4|^2 + |\xi_4|^2 + \cdots \\ &\leq (1+1)(|\xi_1|^2+|\xi_2|^2) + |\xi_2|^2 + (1+1)(|\xi_3|^2+|\xi_4|^2) + |\xi_4|^2 + \cdots \\ &= 2|\xi_1|^2 + 3|\xi_2|^2 + 2|\xi_3|^2 + 3|\xi_4|^2 + \cdots \\ &\leq 3\sum_{n=1}^{\infty} |\xi_n|^2 \\ &= 3||x||^2 \end{aligned}$$

$\Rightarrow ||T|| \leq \sqrt{3}$.

We investigate if $T - \lambda I$ is injective, for $x = (\xi_j)_{j\geq 1} \in l^2$, $y = (T-\lambda I)x = (\eta_j) \in l^2$

$$\Rightarrow y = (\eta_j) = Tx - \lambda x = \{(\xi_1+\xi_2, \xi_2, \xi_3+\xi_4, \xi_4, \xi_5+\xi_6, \xi_6, \xi_7+\xi_8, ...) - \lambda(\xi_1, \xi_2, \xi_3, ...)\}$$

$$\Rightarrow (\eta_1, \eta_2, \eta_3, ...) = ((1-\lambda)\xi_1 + \xi_2, (1-\lambda)\xi_2, (1-\lambda)\xi_3 + \xi_4, (1-\lambda)\xi_4, ...)) \tag{3.2}$$

Given $Tx - \lambda x = 0$ and $\lambda = 1$ we have $\xi_2 = \xi_4 = \xi_6 = ... = 0$.

Thus there exist many such vectors which are not zero. This shows that if $\lambda = 1$, then $T - \lambda I$ is not injective; hence $1 \in \sigma_p(T)$. Clearly we observe that if $\lambda = 0$

then $x = 0$ (contradiction)

$\Rightarrow 0 \notin \sigma_p(T)$.

From now on we assume $\lambda \neq 1$, then we obtain from (3.2) we obtain that for any $x = (\xi_j)_{j\geq 1} \in l^2$ we have

$$\begin{cases} \xi_1 = \dfrac{1}{1-\lambda}\eta_1 - \dfrac{1}{(1-\lambda)^2}\eta_2 \\ \xi_2 = \dfrac{1}{1-\lambda}\eta_2 \\ \xi_3 = \dfrac{1}{1-\lambda}\eta_3 - \dfrac{1}{(1-\lambda)^2}\eta_4 \\ \xi_4 = \dfrac{1}{1-\lambda}\eta_4 \\ \vdots \end{cases} \tag{3.3}$$

Now if some $x' = (\xi'_j)_{j\geq 1} \in l^2$, let $Tx' = Tx = (\eta_j)$ then each ξ'_j is given by the same formula as ξ_j in (3.3) i.e $\xi'_j = \xi_j \ \forall\ j$ and hence $x = x'$.

This shows that $T - \lambda I$ is injective (under our present assumption $\lambda \neq 1$.)

From (3.2) we have that

$$Im((T-\lambda I)^{-1}) = \{(\eta_j) \in l^2 : \sum_{j=1}^{\infty} |\xi_j|^2 < \infty\}$$

and

$$(T-\lambda I)^{-1}((\eta_j)) = (\xi_j), \forall (\eta_j) \in \mathrm{Im}((T-\lambda I)^{-1})$$

To decide whether λ belongs to $\sigma_c(T), \sigma_p(T)$ or $\rho(T)$ we must determine whether $\mathrm{Im}((T-\lambda I)^{-1})$ is dense in l^2 and whether $(T-\lambda I)^{-1}$ is bounded.

$$\begin{aligned}\sum_{j=1}^{\infty}|\xi_j|^2 &= |\frac{1}{1-\lambda}\eta_1 - \frac{1}{(1-\lambda)^2}\eta_2|^2 + |\frac{1}{1-\lambda}\eta_2|^2 + |\frac{1}{1-\lambda}\eta_3 - \frac{1}{(1-\lambda)^2}\eta_4|^2 + |\frac{1}{1-\lambda}\eta_4|^2 + \cdots \\ &\leq (|\frac{1}{1-\lambda}|^2 + |\frac{1}{(1-\lambda)^2}|^2)(|\eta_1|^2 + |\eta_2|^2) + |\frac{1}{1-\lambda}|^2||\eta_2|^2) \\ &+ (|\frac{1}{1-\lambda}|^2 + |\frac{1}{(1-\lambda)^2}|^2)(|\eta_3|^2 + |\eta_4|^2) + |\frac{1}{1-\lambda}|^2||\eta_4|^2) + \ldots \quad \text{(using (3.3.1))} \\ &= (|\frac{1}{1-\lambda}|^2 + |\frac{1}{(1-\lambda)^2}|^2 + |\frac{1}{1-\lambda}|^2)(|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2, \cdots)\end{aligned}$$

then,

$$\sum_{j=1}^{\infty}|\xi_j|^2 \leq C_k.\sum_{j=1}^{\infty}|\eta_j|^2$$

where $C_k = |\frac{1}{1-\lambda}|^2 + |\frac{1}{(1-\lambda)^2}|^2 + |\frac{1}{1-\lambda}|^2$

In particular, for every $(\eta_j) \in l^2$ we have $\sum_{j=1}^{\infty}|\xi_j|^2 < \infty$, i.e $(\xi_j) \in l^2$. In view of our earlier formular for $\mathrm{Im}((T-\lambda I)^{-1})$ it shows that $\mathrm{Im}((T-\lambda I)^{-1}) = l^2$ i.e.,

$\mathrm{Im}((T-\lambda I)^{-1})$ is dense in l^2.

$$||(T-\lambda I)^{-1}((\eta_j))|| = \sqrt{\sum_{j=1}^{\infty}|\xi_j|^2} \leq \sqrt{C_k}||(\eta_j)||$$

for all $(\eta_j) \in l^2$ i.e the operator $(T-\lambda I)^{-1}$ is bounded with $||(T-\lambda I)^{-1}|| \leq \sqrt{C_k}$

$\Rightarrow \lambda \in \rho(T)\ \forall \lambda \neq 1$.

Hence $\sigma_p(T) = \{1\}; \sigma_c(T) = \{\phi\}; \sigma_r(T) = \{\phi\}; \rho(T) = \mathbb{R} - \{1\}$.

3. (Spectrum of Multiplication Operators)

Let $(X, \mathcal{M}, \mu)$ be a $\sigma - finite$ measurable space,$1 \leq p \leq \infty$ and $a : X \longrightarrow \mathbb{R}$ be a bounded measurable function on X.

Define the bounded operator A:$L^p(X, \mathcal{M}, \mu) \longrightarrow L^p(X, \mathcal{M}, \mu)$ by

$$(A\varphi)(x) = a(x)\varphi(x)$$

we determine $\sigma_p(A), \rho(A)$ and $\sigma(A)$.

Let $\lambda \in \mathbb{R}$. Then $\lambda I - A$ is injective $\Longleftrightarrow \{\varphi \in L^p(X, \mathcal{M}, \mu), (\lambda - a(x))\varphi(x) = 0\ a.e \Rightarrow \varphi(x) = 0\ a.e\}$.

$$\Longleftrightarrow \lambda - a(x) \neq 0\ a.e$$

$$\Longleftrightarrow \mu(\{x \in X : a(x) = \lambda\}) = 0$$

Hence

$$\sigma_p(A) = \{\lambda \in \mathbb{R} : \mu(\{x \in X : a(x) = \lambda\}) > 0\}.$$

Let $\lambda \in \mathbb{R} - \sigma_p(A)$ then $\lambda I - A$ has an inverse operator, which we now determine.

$$(\lambda I - A)\varphi = \psi \Longleftrightarrow (\lambda - a(x))\varphi(x) = \psi(x) a.e \Longleftrightarrow \varphi(x) = \frac{1}{\lambda - a(x)}\psi(x) \; a.e$$

Hence $(\lambda I - A)^{-1}$ is the operator of multiplication by $\frac{1}{\lambda - a(x)}$ with the domain consisting of $\{\varphi \in L^p(X, \mathcal{M}, \mu) : \frac{1}{\lambda - a(x)}\psi(x) \in L^p(X, \mathcal{M}, \mu)\}$. This is a bounded operator if and only if there is a $K > 0$ such that $\frac{1}{\lambda - a(x)} \leq K$ almost everywhere. Thus

$$\rho(A) = \{\lambda \in \mathbb{R} : \exists \; \varepsilon > 0 \; such \; that \; |\lambda - a(x)| \geq \varepsilon \; a.e\}.$$

Since $\sigma(A)$ is the complement of $\rho(A)$ then,

$$\sigma(A) = \{\lambda \in \mathbb{R} : \nexists \; \varepsilon > 0 \; such \; that \; |\lambda - a(x)| \geq \varepsilon \; a.e\}.$$

3.4 Spectral decomposition

Definition 3.5. Let $T : H \longrightarrow H$ be bounded linear operator on a real Hilbert space H. Then the Hilbert adjoint operator $T^* : H \longrightarrow H$ is defined by $\langle Tx, y\rangle = \langle x, T^*y\rangle$.
Furthermore, T is said to be self- adjoint or Hermitian if $T^* = T$, hence $\langle Tx, y\rangle = \langle x, Ty\rangle$.

Definition 3.6. let H_1 and H_2 be Hilbert spaces and let $h : H_1 \times H_2 \longrightarrow K$ be a bounded sesquilinear form, then h has a representation $h(x, y) = \langle Sx, y\rangle$ where $S : H_1 \longrightarrow H_2$ is a bounded linear Operator and S is uniquely determined by h and has norm $||S|| = ||h||$

Theorem 3.7. *(Existance of adjoint)*
The Hilbert adjoint Operator $T^* : H_2 \longrightarrow H_1$ of $T : H_1 \longrightarrow H_2$ exists and its bounded with $||T^*|| = ||T||$ such that $\forall x \in H_1$ and $y \in H_2$, $\langle Tx, y\rangle = \langle x, T^*y\rangle$

Proof. The formula $h(y, x) = \langle y, Tx\rangle$ defines a sesquilinear form on $H_2 \times H_1$, because the inner product is sesquilinear and T is linear. Infact, conjugate linearity of the form is seen from $h(y, \alpha x_1 + \beta x_2) = \langle y, T(\alpha x_1 + \beta x_2)\rangle = \langle y, \alpha Tx_1 + \beta Tx_2\rangle = \overline{\alpha}\langle y, Tx_1\rangle + \overline{\beta}\langle y, Tx_2\rangle = \overline{\alpha}h(y, x_1) + \overline{\beta}h(y, x_2)$
h is bounded, by the Schwartz inequality $|h(y, x)| = |\langle y, Tx\rangle| \leq ||y||||Tx|| \leq ||T||||x||||y||$.

It implies$||h|| \leq ||T||$.

Moreover, we have $||h|| = sup_{x\neq 0, y\neq 0}\frac{|\langle y,Tx\rangle|}{||y||||x||} \geq sup\frac{|\langle Tx,Tx\rangle|}{||Tx||||x||} = ||T||$

It implies $||h|| \geq ||T||$, thus we have $||h|| = ||T||$

By the representation for h, writing T^* for S, we have $h(y, x) = \langle T^*y, x\rangle$

From Definition 3.7, we have that $T^* : H_2 \longrightarrow H_1$ is uniquely determined and defines a linear operator such that

$$||T^*|| = ||h|| = ||T||$$

This proves that $\langle Tx, y\rangle = \langle x, T^*y\rangle$.

Theorem 3.8. *Let $T \in \mathcal{B}(H)$ be a self - adjoint operator, then the following hold.*
(a) All the eigenvalues of T (if they exist) are real.
(b) Eigenvectors corresponding to different eigen values of T are orthogonal.
(c)If λ is an eigenvalue of T, then $|\lambda| \leq ||T||$.

Proof. (a) Let λ be an eigen value of T and x a corresponding eigen vector, then $x \neq 0$ and $Tx = \lambda x$.

$$\lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle = \langle x, \lambda x\rangle = \overline{\lambda} \langle x, x\rangle .$$

It implies $\lambda = \overline{\lambda}$, hence λ is real.

(b) Let λ and μ be eigen vectors of T and let x and y be corresponding eigen vectors, then $Tx = \lambda x$ and $Ty = \mu y$.

Since T is self adjoint then λ and μ are real

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle = \langle x, \mu y\rangle = \mu\langle x, y\rangle$$

It implies

$$(\lambda - \mu)\langle x, y\rangle = 0 \Rightarrow \langle x, y\rangle = 0$$

(since $\lambda \neq \mu$).

(c)

$$|\lambda|||x|| = ||\lambda x|| = ||Tx|| \leq ||T||||x||.$$

It implies

$$|\lambda| \leq ||T||$$

Theorem 3.9. *Let $T \in \mathcal{B}(H)$ be a self adjoint operator and*

$$m = \inf_{u \in H ||u||=1} < Tu, u >, M = \sup_{u \in H ||u||=1} < Tu, u > .$$

Then $\sigma(T) \subset [m, M]$, moreover $||T|| = max\{|m|, |M|\}$.

Proof.

Let $\lambda > M = \sup_{u \in H ||u||=1} \langle Tu, u \rangle$, we prove that $\lambda \in \rho(T)$.
We have that $\langle Tu, u \rangle \ \leq M||u||^2, \forall u \in H$
therefore

$$\langle \lambda u - Tu, u \rangle \ = \ \langle \lambda u, u \rangle - \langle Tu, u \rangle \ \geq (\lambda - M)||u||^2 = \alpha ||u||^2, \forall u \in H, \alpha > 0$$

By Lax-Milgram's theorem, since $\lambda I - T$ is bounded and coercive, we deduce that $\lambda I - T$ is bijective and thus $\lambda \in \rho(T)$. Similarly $\lambda < m$ belongs to $\rho(T)$, thus $\sigma(T) \subset [m, M]$.
Next, we show that $||T|| = \mu$ where $\mu = max\{|m|, |M|\}, \forall u, v \in H$. Now,

$$\langle T(u+v), u+v \rangle = \langle Tu, u \rangle + \langle Tv, v \rangle + 2\langle Tu, v \rangle \tag{3.4}$$

$$\langle T(u-v), u-v \rangle = \langle Tu, u \rangle + \langle Tv, v \rangle - 2\langle Tu, v \rangle \tag{3.5}$$

from (3.4) - (3.5), we have

$$\begin{aligned} 4\langle Tu, v \rangle \ &= \ \langle T(u+v), u+v \rangle - \langle T(u-v), u-v \rangle \\ &\leq \ M||u+v||^2 - m||u-v||^2 \\ &\leq \ M||u+v||^2 + m||u-v||^2 \\ &\Rightarrow \ 4\langle Tu, v \rangle \ \leq \mu(||u+v||^2 + ||u-v||^2), \mu = \max\{|m|, |M|\} \\ &= \ 2\mu(||u||^2 + ||v||^2) \end{aligned}$$

replacing v by $\alpha v, \alpha > 0$ yields
$4\langle Tu, v \rangle \ \leq 2\mu \left(\frac{||u||^2}{\alpha} + \alpha ||v||^2 \right)$
choose $\alpha = \frac{||u||}{||v||}$
$\Rightarrow 4|\langle Tu, v \rangle| \leq 2\mu \left(||u||^2 \frac{||v||}{||u||} + \frac{||u||}{||v||} ||v||^2 \right)$

$\Rightarrow |\langle Tu, v \rangle| \leq \mu ||u|| ||v|| \forall u, v \in H$

$$\Rightarrow ||T|| \leq \mu \tag{3.6}$$

It is clear that $|\langle Tu, v \rangle| \leq ||T|| ||u||^2$ so that $|m| \leq ||T||$ and $|M| \leq ||T||$

$$\Rightarrow \mu \leq ||T|| \tag{3.7}$$

from (3.6) and (3.7) we have that $||T|| = \mu = \max\{|m|, |M|\}$

Definition 3.10. (Seperable space)

A topological space, say $(X, ||.||)$ is called seperable if it contains a countable dense subset i.e there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every non-empty open subset of the space contains atleast one element of the sequence. Example R^n, $L^2(a, b)$, $C[a, b]$ are seperable.

Definition 3.11. (Hilbert-Schmidt Operator)

Let H be a seperable Hilbert space. A bounded linear operator T is called a Hilbert-Schmidt if there exists a Hilbert basis $(e_n)_{n\geq 1}$ in H such that $||T||^2_{\mathcal{HS}} = \sum_{n\geq 1} |Te_n|^2 < \infty$.

Remark: Obviously $||.||_{\mathcal{HS}}$ defines a norm.

Theorem 3.12. *Let $H = L^2(\Omega)$ and $K(x, y) \in L^2(\Omega \times \Omega)$, then the operator $u \mapsto (Ku)(x) = \int_\Omega K(x, y)u(y)dy$ is a Hilbert-Schmidt operator.*
Conversely, every Hilbert-Schmidt operator on $L^2(\Omega)$ is of the preceeding form for some unique function $K(x, y) \in L^2(\Omega \times \Omega)$.

Corollary 3.13. Let $T \in \mathcal{L}(H)$ be a self-adjoint operator such that $\sigma(T) = \{0\}$ then $T = 0$

Theorem 3.14. *Let H be a seperable Hilbert space and let T be a compact self-adjoint operator. Then there exists a Hilbert basis composed of eigenvectors of T.*

Proof.

Let $(\lambda_n)_{n\geq 1}$ be the sequence of all (distinct) nonzero eigenvalues of T.
Set $\lambda_0 = 0$, $E_0 = N(T)$ and $E_n = N(T - \lambda_n I)$. Recall that $0 \leq dimE_0 \leq \infty$ and $0 < dimE_n < \infty$.
We claim that H is the Hilbert sum of the $E_n's, n = 0, 1, 2, ...$
(i) The spaces $(E_n)_{n\geq 0}$ are mutually orthogonal.
Indeed, if $u \in E_m$ and $v \in E_n$ with $m \neq n$, then

$$Tu = \lambda_m u \ \ and \ \ Tv = \lambda_n v$$

so that $\langle Tu, v\rangle = \lambda_m\langle u, v\rangle = \langle u, Tv\rangle = \lambda_n\langle u, v\rangle$.
Therefore $\langle u, v\rangle = 0$
(ii) Let F be the vector space spanned by the spaces $(E_n)_{n\geq 0}$. We shall prove that F is dense in H.
Clearly,$T(F) \subset F$. It foolows that $T(F^\perp) \subset F^\perp$; indeed, given $u \in F^\perp$ we have

$$\langle Tu, v\rangle = \langle u, Tv\rangle = 0 \ \forall v \in F,$$

so that $Tu \in F^{\perp}$. The operator T restricted to $F^{\perp}$ is denoted by T_0. This is a self-adjoint compact operator on $F^{\perp}$.

We claim that $\sigma(T_0) = \{0\}$.

Suppose not; suppose that some $\lambda \neq 0$ belongs to $\sigma(T_0)$. Since $\lambda \in EV(T_0)$, there exists some $u \in F^{\perp}$, $u \neq 0$ such that $T_0 u = \lambda u$.

Therefore λ is one of the eigen values of T, say $\lambda = \lambda_n$ with $n \geq 1$.

Thus $u \in E_n \subset F$, since $u \in F^{\perp} \bigcap F$, we deduce that $u = 0$; a contradiction.

Applying Corollary 3.13, we duce that $T_0 = 0$ i.e T vanishes on $F^{\perp}$. It follows that $F^{\perp} \subset N(T)$.

On the other hand $N(T) \subset F$ and consequently $F^{\perp} \subset F$.

This implies that $F^{\perp} = \{0\}$ and so F is dense in H.

Finally, we choose in each subspace $(E_n)_{n \geq 0}$ a Hilbert basis, obviously they are finite dimensional.

The union of those bases is clearly a Hilbert basis for H, composed of eigenvectors of T.

Remark.

Let T be a compact self-adjoint operator. From the preceeding analysis we may write any element $u \in H$ as

$u = \sum_{n=0}^{\infty} u_n$ with $u_n \in E_n$.

Then $Tu = \sum_{n=0}^{\infty} \lambda_n u_n$. Given an integer $k \geq 1$, set

$$T_k u = \sum_{n=1}^{k} \lambda_n u_n.$$

Clearly, T_k is a finite-rank operator and

$||T_k - T|| \leq sup_{n \geq k+1} |\lambda_n| \longrightarrow 0$ as $k \longrightarrow \infty$.

Recall that in fact, in a Hilbert space, every compact operator not necessarily self-adjoint is the limit of a sequence of finite-rank operators.

3.5 Applications

Definition 3.15. We define the gradient of the scalar function $f \in R^n$ as the vector field of partial derivative of f denoted by ∇f i.e

$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \frac{\partial f}{\partial x_n} \right)$

Definition 3.16. The Laplacian of the vector field F with the co-ordinate $(x_1, x_2, ..., x_n)$ is a map $F : \Omega \longrightarrow \mathbb{R}$ defined by $\Delta F = \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial x_2^2} + ... + \frac{\partial^2 F}{\partial x_n^2}$

Definition 3.17. Let Ω be open and bounded set. Then

$H_0^1(\Omega) = \overline{D(\Omega)}|_{H^1(\Omega)}$.

Theorem 3.18. *Let Ω be bounded and smooth. Then*
$H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$.

Lemma 3.19. (Green's formula)
Let Ω be a bounded smooth open set in R^n, let $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ then
$-\int_\Omega \Delta u.v = \int_\Omega \nabla u \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial \eta}.v d\sigma$ where $\Delta = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$

Lemma 3.20. The Laplacian operator $\Delta : H_0^1(\Omega) \cap H^2(\Omega) \longrightarrow L^2$ is self adjoint operator

Proof.
Let $\phi \in H_0^1(\Omega), \psi \in H^2(\Omega)$. Then
$-<\phi, \Delta\psi> \;=\; -\int_\Omega \Delta\psi.\phi = \int_\Omega \nabla\psi\nabla\phi - \int_{\partial\Omega} \frac{\partial \phi}{\partial \eta}.\psi d\sigma$
$\Rightarrow <\phi, \Delta\psi> \;=\; -\int_\Omega \nabla\psi\nabla\phi$ since $\phi \;=\; 0$ on the boundary
similarly $<\psi, \Delta\phi> \;=\; -\int_\Omega \nabla\psi\nabla\phi$
hence Δ is self adjoint.

Theorem 3.21. *Let $p \in C^1(\overline{\Omega})$ with $\Omega = (0,1)$ and $p \geq \alpha > 0$; let $q \in C(\overline{\Omega})$ then there exists a sequence (λ_n) of real numbers and a Hilbert basis (e_n) of $L^2(\Omega)$ such that $e_n \in C^2(\overline{\Omega}) \forall n$ and*

$$\begin{cases} -(pe_n')' + qe_n = \lambda_n e_n \text{ on } \Omega \\ e_n(0) = e_n(1) = 0 \end{cases} \tag{3.8}$$

Furthermore, $\lambda_n \longrightarrow +\infty$ as $n \longrightarrow \infty$
One says that the (λ_n) are the eigen values of the differential operator
$Au = -(pu')' + qu$ with the Dirichlet boundary condition and the (e_n) are the associated eigen functions.

Proof.
We can always assume $q \geq 0$, if not, pick any constant c such that $q + c \geq 0$, which amounts to replacing λ_n by $\lambda_n + c$ in (1). For every $f \in L^2(\Omega)$ there exist a unique $u \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfying

$$\begin{cases} -(pu')' + qu = f \text{ on } \Omega \\ u(0) = u(1) = 0 \end{cases} \tag{3.9}$$

Denote by T the operator $f \mapsto u$ considered as an operator from $L^2(\Omega)$ into $L^2(\Omega)$.
We claim that T is self-adjoint and compact.
First, the compactness. Because of (2) we have
$\int_\Omega pu'^2 + \int_\Omega qu^2 = \int_\Omega fu$
and thus $\alpha||u'||_{L^2}^2 \leq ||f||_{L^2}||u||_{L^2}$ (since $0 < \alpha \leq p$ and applying cauchy schwartz)
It follows that $||u||_{H^1} \leq c||f||_{L^2}$ where c is a constant depending on α.
Thus, this can be written as

$||Tf||_{H^1} \leq c||f||_{L^2} \forall f \in L^2(\Omega)$.

Since the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact (because Ω is bounded), we deduce that T is a compact operator from $L^2(\Omega)$ into $L^2(\Omega)$.

Next, we show that T is self -adjoint i.e

$\int_\Omega (Tf)g = \int_\Omega f(Tg)$ for all $f, g \in L^2(\Omega)$.

Setting $u = Tf$ and $v = Tg$ we have

$$pu')' + qu = f \tag{3.10}$$

$$-(pv')' + qv = g \tag{3.11}$$

multiply (3.10) by v and (3.11) by u and then integrating, we obtain

$$\int_\Omega pu'v' + \int_\Omega quv = \int_\Omega fv = \int_\Omega gu$$

$\Rightarrow \int_\Omega (Tf)g = \int_\Omega f(Tg) \forall f, g \in L^2(\Omega)$

Finally, we note that

$$\int_\Omega (Tf)f = \int_\Omega pu'^2 + qu^2 \geq 0 \ \forall f \in L^2(\Omega)$$

and also that $\mathcal{N}(T) = \{0\}$, since $Tf = 0$ implies $u = 0$ and so $f = 0$

We know that $L^2(\Omega)$ admits a Hilbert basis $(e_n)_{n\geq 1}$ consisting of eigen vectors of T with corresponding eigen values $(\mu_n)_{n\geq 1}$ we have $\mu_n > 0 \ \forall n$ ($\mu_n \neq 0$, since $\mathcal{N}(T) = \{0\}$).

we also know that $\mu_n \longrightarrow 0$, writing $Te_n = \mu_n e_n$ we obtain

$$\begin{cases} -(pe_n')' + qe_n = \lambda_n e_n \ with \ \lambda_n = \frac{1}{\mu_n} \\ e_n(0) = e_n(1) = 0 \end{cases}$$

In addition, we have $e_n \in C^2(\overline{\Omega})$ since $f = \lambda_n e_n \in C(\overline{\Omega})$

(infact $e_n \in C^\infty(\overline{\Omega}) \ if \ p, q \in C^\infty(\overline{\Omega})$)

1. Consider when $p \equiv 1$ and $q \equiv 0$ we have the problem

$$\begin{cases} -(e_n')' = \lambda_n e_n \\ e_n(0) = e_n(1) = 0 \end{cases}$$

Claim :λ_n is positive for each n.

Proof of claim

Multiply the equation by e_n and integrate over (0,1), we have

$$-\int_0^1 e_n'' e_n dx = \lambda_n \int_0^1 e_n^2 dx$$

$$\Rightarrow \int_0^1 (e_n')^2 dx = \lambda_n \int_0^1 e_n^2 dx \qquad (after\ integrating\ by\ part)$$

$$\Rightarrow \lambda_n = \frac{\int_0^1 (e_n')^2 dx}{\int_0^1 e_n^2 dx}$$

$$\Rightarrow \lambda_n = \frac{||u'||^2_{L^2((0,1))}}{||u||^2_{L^2((0,1))}} \geq 0$$

Now let $\lambda_n = \sigma_n^2$

from the characteristic equation of the Differential equation we have $r = \pm i\sigma_n$

$\Rightarrow e_n(x) = C_1 cos\sigma_n x + C_2 sin\sigma_n x$

Applying the conditions $e_n(0) = e_n(1) = 0$ we obtain that $\sigma_n = n\pi$

$\Rightarrow \lambda_n = n^2\pi^2$ and the eigen function $e_n(x) = sin(n\pi x)$ where $n = 1, 2, 3, \ldots$

2.Let $\Omega = [0, 1]$. Use the function $v(x) = x(1 - x)$ to approximate the first eigenvalue for this region Ω

Solution:

Clearly $v(x)$ satisfies the Dirichlet boundary condition and we know that the eigen values for $\Omega = [0, 1]$ are given by $\lambda_n = (n\pi)^2$. from previous formula we have:

$$\lambda = \frac{\int_0^1 v'^2 dx}{\int_0^1 v^2 dx} = \frac{\int_0^1 (1-2x)^2 dx}{\int_0^1 (x - x^2)^2 dx} = \frac{1/3}{1/30} = 10$$

The first eigen value is actually $\pi^2 \approx 9.8696$, thus we have a fairly good approximation. This implies that this formula is actually a weak formulation.

3.5.1 CONCLUSION

We have studied bounded linear operators on infinite dimensional vector spaces. We also investigate the spectral properties of bounded and compact operators where we observe that on finite dimensional vector space, the spectrum of an operator consists of all its eigenvalues while on infinite dimensional vector space, the spectrum consists of the continuous, residual and the point spectrum. In Hilbert spaces, the spectral properties of compact operators resembles those of square matrices.In general, compact operators on infinite dimensional spaces feature properties that do not appear in the finite dimension case; i.e matrices.

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