LaSalle Invariance Principle for Ordinary Differential Equations and Applications

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CERTIFICATION

This is to certify that the thesis titled "LaSalle Invariance Principle for Ordinary Differential Equations and Applications" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research work carried out by Okolie, Ogochukwu Patricia in the department of Pure and Applied Mathematics.

APPROVAL

LaSalle Invariance Principle for Ordinary Differential Equations and Applications

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A Thesis Presented to the Department of Pure and Applied Mathematics

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Date

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INTRODUCTION

The most popular method for studying **stability of nonlinear systems** is introduced by a Russian Mathematician named **Alexander Mikhailovich Lyapunov**. His work "The General Problem of Motion Stability " published in 1892 includes two methods: Linearization Method, and Direct Method. His work was then introduced by other scientists like Poincare and LaSalle .
In chapter one of this work, we focussed on the basic concepts of the ordinary differential equations. Also, we emphasized on relevant theroems in ordinary differential equations
In chapter two of this work, we study the existence and uniqueness of solutions of ordinary differential equations. Also, relevant theorems and concepts in ordinary differential equations was discussed in the chapter.
In chapter three, we study the stability of an equilibrium point and linearization principle. Also, relevant theorems and concepts in stability of an equilibrium point and linearization principle was discussed in the chapter
In chapter four, we study the various tools for determining stability of equilibrium points.
In chapter five, we discussed various applications of Lyapunov theorem, and LaSalle's invariance principle.

DEDICATION

To God almighty and to my family.

CONTENTS

CHAPTER 1

PRELIMINARIES

1.1 Definitions and basic Theorems

In this chapter, we focussed on the basic concepts of the ordinary differential equations. Also, we emphasized on relevant theroems in ordinary differential equations.

Definition 1.1.1 *An equation containing only ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation ODE. The order of an ODE is the order of the highest derivative in the equation. In symbol, we can express an n-th order ODE by the form*

$$x^{(n)} = f(t, x, ..., x^{(n-1)}) \tag{1.1.1}$$

Definition 1.1.2 *(**Autonomous ODE**) When f is time-independent, then (1.1.1) is said to be an autonomous ODE. For example,*

$$x'(t) = sin(x(t))$$

Definition 1.1.3 *(**Non-autonomous ODE**) When f is time-dependent, then (1.1.1) is said to be a non autonomous ODE. For example,*

$$x'(t) = (1 + t^2)y^2(t)$$

Definition 1.1.4 *$f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be locally Lipschitz, if for all $r > 0$ there exists $k(r) > 0$ such that*

$$\|f(x) - f(y)\| \leq k(r)\|x - y\|, \quad for \quad all \quad x, y \in B(0, r).$$

$f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be Lipschitz, if there exists $k > 0$ such that

$$\|f(x) - f(y)\| \leq k\|x - y\|, \quad for \quad all \quad x, y \in \mathbb{R}^n.$$

Definition 1.1.5 *(**Initial value problem (IVP)** Let I be an interval containing x_0, the following problem*

$$\begin{cases} x^{(n)}(t) = f(t, x(t), ..., x^{(n-1)}(t)) \\ x(t_0) = x_0, x'(t_0) = x_1, ..., x^{(n-1)}(t_0) = x_{n-1} \end{cases} \tag{1.1.2}$$

is called an initial value problem (IVP).

$$x(t_0) = x_0, x'(t_0) = x_1, ..., x^{(n-1)}(t_0) = x_{n-1}$$

are called initial condition.

Lemma 1.1.6 *[9](Gronwall's Lemma) Let* $u, v : [a,b] \to \mathbb{R}^+$ *be continuous such that there exists* $\alpha > 0$ *such that*

$$u(x) \le \alpha + \int_a^x u(s)v(s)ds, \quad for \quad all \quad x \in [a,b].$$

Then,

$$u(x) \le \alpha e^{\int_a^x v(s)ds}, \quad for \quad all \quad x \in [a,b].$$

Proof .

$$u(x) \le \alpha + \int_a^x u(s)v(s)ds$$

implies that

$$\frac{u(x)}{\alpha + \int_a^x u(s)v(s)ds} \le v(x).$$

So,

$$\frac{u(x)v(x)}{\alpha + \int_a^x u(s)v(s)ds} \le v(x),$$

which implies that

$$\int_a^x \frac{u(x)v(x)}{\alpha + \int_a^x u(s)v(s)ds} ds \le \int_a^x v(x)ds.$$

So, taking exponential of both side we get

$$u(x) \le \alpha + \int_a^x u(s)v(s)ds \le \alpha \int_a^x u(s)v(s)ds.$$

Thus,

$$u(x) \le \alpha e^{\int_a^x v(s)ds}, \quad x \in [a,b].$$

■

Corollary 1.1.7 *Let* $u, v : [a,b] \to \mathbb{R}^+$ *be continuous such that*

$$u(x) \le \int_a^x u(s)v(s)ds, \quad for \quad all \quad x \in [a,b].$$

Then, $u = 0$ *on* $[a,b]$.

Proof . Now,

$$u(x) \le \int_a^x u(s)v(s)ds$$

implies that

$$u(x) \le \int_a^x u(s)v(s)ds \le \frac{1}{n} + \int_a^x u(s)v(s)ds, \quad for \quad all \quad n \ge 1.$$

So, by Gronwall's lemma,

$$u(x) \le \frac{1}{n} e^{\int_a^x u(s)v(s)ds},$$

so as

$$n \to \infty, \quad u(x) \to 0.$$

Thus, $u(x) = 0$, since $u(x) \ge 0$. Hence, $u = 0$ on $[a,b]$. ■

1.2 Exponential of matrices

Definition 1.2.1 *Let $A \in M_{n\times n}(\mathbb{R})$, then e^A is an $n \times n$ matrix given by the power series*

$$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

The series above converges absolutely for all $A \in M_{n\times n}(\mathbb{R})$

Proof . The n-th partial sum is

$$S_n = \sum_{k=0}^{n} \frac{A^k}{k!}$$

So, let $n > m$ Then,

$$S_n - S_m = \sum_{k=m+1}^{n} \frac{A^k}{k!}.$$

So,

$$\|S_n - S_m\| \leq \sum_{k=m+1}^{n} \frac{\|A\|^k}{k!}.$$

So as

$$m \to \infty, \quad \|S_n - S_m\| \to 0$$

So, $(S_n)_n$ is Cauchy. Thus, converges. ■

Theorem 1.2.2 *[3]**(Cayley Hamilton Theorem)***
Let $A \in M_{n\times n}(\mathbb{R})$ and $\Delta(\lambda) = det(\lambda I - A)$ its characteristic polynomial then

$$\Delta(A) = 0.$$

Proof . Let $A \in M_{n\times n}(\mathbb{R})$,

$$\Delta(\lambda) = det(I - \lambda A) = c_0 + c_1\lambda + c_2\lambda^2 + ... + c_n\lambda^n.$$

$$adj(A - \lambda I) = B_0 + B_1\lambda + B_2\lambda^2 + ... + B_{n-2}\lambda^{n-2} + B_{n-1}\lambda^{n-1},$$

where $B_i \in M_{n\times n}(\mathbb{R})$ for $i = 0, 1, 2, ..., n,$ but, from linear algebra we have that

$$A^{-1} = \frac{adj(A)}{det(A)},$$

where $adj(A)$ denotes the adjugate or classical adjoint of A. So,

$$det(I - tA)I = (I - tA)adj(I - tA).$$

$$(A - \lambda I)(B_0 + B_1\lambda + B_2\lambda^2 + ... + B_{n-2}\lambda^{n-2} + B_{n-1}\lambda^{n-1}) = (c_0 + c_1\lambda + c_2\lambda^2 + ... + c_n\lambda^n)I.$$

Observe that the entries in $adj(I - tA)$ are polynomials in λ of degree at most $n - 1$. So, B_i is the zero matrix for $i = n$. Equating the coefficients of λ^n on both sides gives

$$c_0 I + c_1 A + c_2 A^2 + ... + c_n A^n = 0.$$

Thus,

$$\Delta(A) = 0.$$

■

Example 1.2.3 *(Application of Cayley Hamilton Theorem)*
Find e^{tA} *for* $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Solution:
The characteristic equation is $s^2 + 1 = 0$*, and the eigenvalues are* $\lambda_1 = i$*, and* $\lambda_2 = -i$*. So, by Theorem 1.2.2 we have that,*

$$e^{tA} = \alpha_0 I + \alpha_1 A,$$

where we are to find the value of α_0*, and* α_1*. So,*

$$e^{ti} = \cos t + i \sin t = \alpha_0 + \alpha_1 i$$
$$e^{-ti} = \cos t - i \sin t = \alpha_0 - \alpha_1 i$$

which implies that $\alpha_0 = \cos t$*, and* $\alpha_1 = \sin t$*. So,*

$$e^{tA} = \cos(t)I + \sin(t)A = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Theorem 1.2.4 *[11] Let* $A, B \in M_{n\times n}(\mathbb{R})$*. Then,*
(1) *If* 0 *denotes the zero matrix, then* $e^0 = I$*, the identity matrix.*
(2) *If* A *is invertible, then* $e^{ABA^{-1}} = Ae^B A^{-1}$*.*

Proof . Recall that, for all integers $s \geq 0$, we have $(ABA^{-1})^s = AB^sA^{-1}$. Now,

$$\begin{aligned} e^{ABA^{-1}} & \\ &= I + ABA^{-1} + \frac{(ABA^{-1})^2}{2!} + ... \\ &= I + ABA^{-1} + \frac{AB^2A^{-1}}{2!} + ... \\ &= A(I + B + \frac{B^2}{2!} + ...)A^{-1} \\ &= Ae^B A^{-1}. \end{aligned}$$

■

(3) If A is symmetric such that $A = A^T$, then

$$e^{(A^T)} = (e^A)^T.$$

Proof .

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Then

$$e^{A^T} = \sum_{k=0}^{\infty} \frac{(A^T)^k}{k!} = \sum_{k=0}^{\infty} \frac{(A^k)^T}{k!} = (\sum_{k=0}^{\infty} \frac{A^k}{k!})^T = (e^A)^T.$$

■

(4) If $AB = BA$, then

$$e^{A+B} = e^A e^B.$$

Proof .

$$\begin{aligned} e^A e^B &= (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ...)(I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + ...) \\ &= (\sum_{k=0}^{\infty} \frac{A^k}{k!})(\sum_{j=0}^{\infty} \frac{B^j}{j!}) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(A+B)^{k+j}}{j!k!} \end{aligned}$$

Put $m = j + k$, then $j = m - k$ then from the binomial theorem that

$$e^A e^B = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{A^m B^{m-k}}{(m-k)!k!} = \sum_{m=0}^{\infty} \frac{A^m}{m!} \sum_{k=0}^{\infty} \frac{m!}{(m-k)!} \frac{B^{m-k}}{k!} = \sum_{m=0}^{\infty} \frac{(A+B)^m}{m!} = e^{A+B}.$$

■

Theorem 1.2.5 *[9]*

$$\frac{de^{tA}}{dt} = Ae^{tA} = e^{tA}A, \quad for \quad t \in \mathbb{R}.$$

Proof . $x(t, x_0) = e^{tA}x_0$. Then,

$$\begin{aligned} &\frac{dx(t,x_0)}{dt} \\ &\quad = e^{tA}x_0 A = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} x_0 A = (\lim_{n\to\infty} \sum_{k=0}^{n} \frac{t^k A^k}{k!}) x_0 A \\ &\quad = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{t^k A^{k+1}}{k!} x_0 = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{A t^k A^k}{k!} x_0 = A \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} x_0 = Ae^{tA}x_0 \end{aligned}$$

So,

$$\frac{de^{tA}}{dt} = Ae^{tA} = e^{tA}A.$$

■

Proposition 1.2.6 *The solution $x(., x_0)$ of the following linear space*

$$\begin{cases} x'(t) = Ax(t), & t \in \mathbb{R} \\ x(0) = x_0 \in \mathbb{R}^n \end{cases}$$

where $A \in M_{n\times n}(\mathbb{R})$, is given by

$$x(t, x_0) = e^{tA}x_0.$$

CHAPTER 2

BASIC THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

In this chapter we give a broad discussion of the existence and uniqueness of solutions of ordinary differential equations. We discuss equilibrium points, stability, fundamental matrix and variation of constants formula, and other key concepts of dynamical systems. We start this chapter with the following definitions;

2.1 Definitions and basic properties

Definition 2.1.1 *Let I be an interval containing t_0, let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and Lipschitzian with respect to second variable, let $x : I \to \mathbb{R}^n$ be continuous, then x is a solution of the following ordinary differential equation*

$$\begin{cases} x'(t) = f(t, x(t)), & t \in I \\ x(t_0) = x_0, & t_0 \in I \end{cases} \tag{2.1.1}$$

on I, if
(i) x is a C^1 - function on I.
(ii) x satisfies the above ODE, for all $t \in I$.

Theorem 2.1.2 *[9](**Peano's Theorem**) Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous in a neighbourhood of (t_0, x_0) then there exists $a > 0$ such that the initial value problem*

$$\begin{cases} x' = f(t, x), & t \in \mathbb{R} \\ x(t_0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{2.1.2}$$

has at least one solution on the interval $I = [t_0 - a, t_0 + a] \subseteq \mathbb{R}$.

Proof . Define the set

$$E = C([t_0 - a, t_0 + a], \mathbb{R}^n)$$

then E is a Banach space provided with the " sup " norm. Let

$$M = \max_{Q} \|f(t, x)\| \quad for \quad Q = \{(t, x) : -a \leq t - t_0 \leq a, \quad \|x - x_0\| \leq b\}$$

and define the set $A \subset E$ by

$$A := \{x \in E : \sup_{t \in I} \|x(t) - x_0\| \leq b\} = \overline{B}(x_0, b)_{C(I,\mathbb{R}^n} \subseteq E.$$

Then, A is a closed subset of E, as $x_n \in A$ implies that

$$\lim_{n \to \infty} x_n = x \in A$$

(this follows from the uniform convergence in E). Also, A is convex (every ball is convex). Thus, by the Ascoli-Arzela theorem, A is compact, and A is complete as a closed subset of a complete metric space with the sup norm.
Also, let $T : A \to E$ be defined by

$$(Tx)(t) := x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

Let, $(x_n)_{n\geq 1} \subseteq A \quad such \quad that \quad x_n \to x \in A$ with the "sup" norm, then,

$$x_n(t) \to x(t) \quad implies \quad \sup_{s\in I} \|x_n(s) - x(s)\| \to 0, \quad as \quad n \to \infty.$$

Therefore,

$$\begin{aligned} \|Tx_n(t) - Tx(t)\| &\leq \int_{t_0}^{t} \|f(s, x_n(s)) - f(s, x(s))\|ds \\ &\leq \int_{t_0-a}^{t_0+a} \|f(s, x_n(s)) - f(s, x(s))\|ds \\ &\leq 2a \sup_{s\in I} \|f(s, x_n(s)) - f(s, x(s))\|. \end{aligned}$$

f is continuous on $I \times \overline{B}(x_0, b)$ implies that f is uniformly continuous on $I \times \overline{B}(x_0, b)$. So, as

$$n \to \infty, \quad \|Tx_n(t) - Tx(t)\| \to 0.$$

Thus, T is continuous.
Let $Tz \in T(A)$,

$$\begin{aligned} \|Tz(t) - x_0\| &= \| \int_{t_0}^{t} f(s, z(s))ds\| \\ &\leq \int_{t_0}^{t} \|f(s, z(s))\|ds \\ &\leq \int_{t_0}^{t} Mds \\ &\leq \int_{t_0-a}^{t_0+a} Mds = 2aM \leq b \end{aligned}$$

for a small enough. Hence, $T(A) \subseteq A$.
We look for a fixed point of T, that is, we want to find

$$x \in E \quad such \quad that \quad Tx = x.$$

A fixed point of T solves the IVP(2.1.2), and T has a fixed point as a consequence of the following Schauder - Tychonoff's Theorem (If $T : X \to X$ is continuous and if $A \subset X$ is a convex compact subset of the normed linear space X and $T(A) \subset A$, then T has a fixed point in A). ∎

Example 2.1.3 *Consider*

$$\begin{cases} y' = \sqrt{|y(t)|}, \quad t \geq 0, \\ y(0) = 0. \end{cases}$$

Here, $f(y) = \sqrt{|y(t)|}$.
Solving the given IVP, we have that,

$$\begin{aligned} y(t) &= -\frac{t^2}{4}, \quad if \quad y(t) < 0, \\ y(t) &= 0, \quad if \quad y(t) = 0, \\ y(t) &= \frac{t^2}{4}, \quad if \quad y(t) > 0. \end{aligned}$$

So, the ODE does not have a unique solution. This is because f *is not Lipschitzian.*

.

Theorem 2.1.4 *[9] Let $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and Lipschitzian with respect to the second variable, x is a solution of the following equation.*

$$\begin{cases} x'(t) = f(t, x(t)), & t \geq 0 \\ x(0) = x_0 \end{cases} \tag{2.1.3}$$

on $[0, a]$ if and only if

$$x(t) = x_0 + \int_0^t f(s, x(s))ds, \quad t \in [0, a]. \tag{2.1.4}$$

Proof .
Assume x is a solution of the above IVP on $[0, a]$, then x is a C^1-function on $[0, a]$. Then, by mean value theorem, we get that

$$x(t) - x(0) = \int_0^t x'(t)ds \tag{2.1.5}$$

which implies that

$$x(t) = x_0 + \int_0^t f(s, x(s))ds, \quad t \in [0, a].$$

Assume that

$$x(t) = x_0 + \int_0^t f(s, x(s))ds$$

holds, then by fundamental law of Calculus

$$x'(t) = f(t, x(t)),$$

and

$$x(0) = x_0 + \int_0^0 f(s, x(s))ds = x_0.$$

Thus, x satisfies the IVP, and since $x'(t) = f(t, x(t))$ where f is a continuous function, we have that x is a C^1-function. Hence, x is a solution of the IVP. ■

Theorem 2.1.5 *[9](**Cauchy- Lipschitz Theorem**) Let $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, be continuous and Lipschitzian with respect to x. Then, the IVP has a unique solution on $\mathbb{R}^+$*

Proof.
Let $C([0, a], \mathbb{R}^n)$ be the Banach space provided with the " sup " norm
x is a solution of the above IVP on $[0, a]$ if and only if

$$x(t, x_0) = x_0 + \int_0^t f(s, x(s))ds, \quad t \in [0, a].$$

Let $a > 0$, and let $K : C([0, a], \mathbb{R}^n) \to C([0, a], \mathbb{R}^n)$ be defined as

$$(Kx)(t) = x_0 + \int_0^t f(s, x(s))ds, \quad t \in [0, a].$$

Then, x is a solution of the IVP on $[0, a]$ if and only if

$$x(t) = (Kx)(t), \quad on \quad C([0, a], \mathbb{R}^n).$$

f is Lipschitzian with respect to x implies that

$$\begin{aligned} \|f(t, x(t)) - f(t, y(t))\| &\leq k(t)\|x(t) - y(t)\| \\ &\leq \max_{t \in [0,a]} k(t)\|x(t) - y(t)\| = k_a\|x - y\|. \end{aligned}$$

Let $k_a = \max_{t\in[0,a]} k(t)$. Then,

$$\begin{aligned}\|(Ky)(t) - (Kx)(t)\| &= \| \int_0^t f(s, y(s)) - f(s, x(s))ds\| \\ &\leq \int_0^t \|f(s, y(s)) - f(s, x(s))\|ds.\end{aligned}$$

Thus,

$$\|(Ky)(t) - (Kx)(t)\| \leq k_a a\|x - y\|.$$

$$\begin{aligned}\|K^2y(t) - K^2x(t)\| &= \|K(Ky)(t) - K(Kx)(t)\| \\ &\leq k_a \int_0^t \|(Ky)(s) - (Kx)(s)\|ds \\ &\leq k_a \int_0^t k_a s\|y(s) - x(s)\|ds \\ &\leq k_a^2 \max_{s\in[0,a]} \|y(s) - x(s)\| \int_0^t sds \\ &\leq \frac{k_a^2 t^2}{2}\|y - x\|.\end{aligned}$$

So, we do this for any $p \geq 1$,

$$\|K^p y(t) - K^p x(t)\| \leq \frac{(k_a a)^p}{p!}\|y - x\|.$$

Then, we get that

$$p \geq 1, \quad \|K^p y(t) - K^p x(t)\| < \|y - x\|.$$

Thus, K^p is a contraction map. Hence, by Banach fixed point Theorem there exists uniquely $x \in C([0, a], \mathbb{R}^n)$ such that

$$(Kx)(t) = x(t) \iff x(t) = x_0 + \int_0^t f(s, x(s))ds, \quad t \in [0, a].$$

This holds for any $a > 0$. Thus, the solution of (2.1.3) exists uniquely on $\mathbb{R}^+$.

Theorem 2.1.6 *[9] Let $x(., x_0)$ and $x(., x_1)$ be solutions of the following ODE*

$$\begin{cases} x'(t) = f(t, x(t)), & t \geq 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2.1.6}$$

starting respectively from x_0 and x_1 if there exists $t_1 > 0$ such that $x(t_1, x_0) = x(t_1, x_1)$. Then,

$$x(t, x_0) = x(t, x_1), \quad for \quad all \quad t \geq 0.$$

Proof.
Let

$$E = \{t \in \mathbb{R}^+ : x(t, x_0) = x(t, x_1)\}.$$

Then, E is closed, and E is not empty, since $t_1 \in E$.
Claim:
E is open.
Infact,
Let $t_0 \in E$, then $x(t_0, x_0) = x(t_0, x_1) = y_0$.
Now, consider the following problem

$$\begin{cases} z'(t) = f(t, z(t)), & t \in [t_0 - \delta, t_0 + \delta], \delta > 0 \\ z(t_0) = y_0 \end{cases} \tag{2.1.7}$$

then, the IVP (2.1.7) has a unique solution on $[t_0-\delta, t_0+\delta]$.
$x(., x_1)$ and $x(., x_0)$ are solutions of the IVP (2.1.7). Thus,

$$x(t, x_0) = x(t, x_1), \quad t \in (t_0-\delta, t_0+\delta).$$

This implies that

$$(t_0-\delta, t_0+\delta) \subseteq E.$$

Thus, E is open, and thus E is an open and closed set in $\mathbb{R}^+$, which implies that

$$E = \mathbb{R}^+.$$

Thus,

$$x(t, x_0) = x(t, x_1), \quad for \quad all \quad t \geq 0.$$

2.2 Continuous dependence with respect to the initial conditions

$$\begin{cases} x'(t) = f(t, x(t)), & t \geq 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2.2.1}$$

$f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and Lipschitzian with respect to the second variable. Then, the IVP (2.2.1) has a unique solution $x(., x_0)$ on $\mathbb{R}^+$.

Theorem 2.2.1 *[9]*

$$\|x(t, x_0) - x(t, x_1)\| \leq \|x_0 - x_1\| e^{\int_0^t k(s)ds}, \quad for \quad all \quad t \geq 0.$$

Proof.

$$x(t, x_0) = x_0 + \int_0^t f(s, x(s, x_0))ds$$

and

$$x(t, x_1) = x_1 + \int_0^t f(s, x(s, x_1))ds,$$

which implies that,

$$\begin{aligned} \|x(t, x_0) - x(t, x_1)\| &\leq \|x_0 - x_1\| + \int_0^t \|f(s, x(s, x_0)) - f(s, x(s, x_1))\| ds \\ &\leq \|x_0 - x_1\| + \int_0^t k(s)\|x(s, x_0) - x(s, x_1)\| ds \end{aligned}$$

by Gronwall's Lemma we get,

$$\|x(t, x_0) - x(t, x_1)\| \leq \|x_0 - x_1\| e^{\int_0^t k(s)ds}, \tag{2.2.2}$$

for all $t \geq 0$

Semi group law for autonomous ODE

Consider the following ODE

$$\begin{cases} x'(t) = f(x(t)), & t \in \mathbb{R} \\ x(t_0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2.2.3}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz, and $T(t) : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$T(t)(x_0) = x(t, x_0),$$

where $x(., x_0)$ is the solution of (2.2.3). Then, we have what we call the semi group law.

Theorem 2.2.2 *[9] For all $t, s \in \mathbb{R}$.*

$$T(t + s) = T(t) \quad \text{o} \quad T(s).$$

Proof . Let $y(t) = x(t + s, x_0)$. Then,

$$y'(t) = x'(t + s, x_0) = f(x(t + s, x_0)) = f(y(t)),$$

and

$$y(0) = x(0 + s, x_0) = x(s, x_0).$$

Hence,

$$\begin{cases} y'(t) = f(y(t)), t \in \mathbb{R} \\ y(0) = y_0 = x(s, x_0) \end{cases} \tag{2.2.4}$$

$$y(t) = x(t, y_0) = x(t, x(s, x_0)),$$

which implies that

$$x(t + s, x_0) = x(t, x(s, x_0)).$$

Thus,

$$T(t+s)(x_0) = x(t+s, x_0) = x(t, x(s, x_0)) = T(t)(x(s, x_0)) = T(t)(T(s)(x_0)) = (T(t)\text{o}T(s))(x_0), \quad x_0 \in \mathbb{R}^n.$$

Hence,

$$T(t + s) = T(t) \quad \text{o} \quad T(s), \quad t, s \in \mathbb{R}.$$

■

2.3 Local existence and blowing up phenomena for ODEs

Definition 2.3.1 *Let $x : [0, a) \to \mathbb{R}^n$, we say that x blows up at a point a if*

$$\overline{\lim_{t\to a}}\|x(t)\| = +\infty \iff \|x(t)\| \quad is \quad not \quad bounded \quad on \quad [0, a)$$
$$\iff \exists (t_n)_n \quad such \quad that \quad \|x(t_n)\| \to +\infty, \quad as \quad t_n \to a.$$

Theorem 2.3.2 *[9] Consider the following ODE*

$$\begin{cases} x'(t) = f(x(t)), & t \geq 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2.3.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz.
Then, the ODE (2.3.1) has a unique maximal defined on $[0, t_{max})$, where $0 < t_{max} \leq +\infty$. If $t_{max} < +\infty$, then $\lim_{t\to t_{max}} \|x(t)\| = +\infty$.

Proof .
Existence of a maximal solution on $[0, t_{max})$:
By Peano's theorem, there exists at least one solution defined on some maximal interval $[0, tmax)$.
Uniqueness of maximal on $[0, t_{max})$:
Let x and y be two solutions of the given ODE on $[0, t_{max})$.
Let $0 < a < t_{max}$, and let

$$D = \{x(s), y(s), \quad s \in [0a]\}.$$

Then, D is compact since f is locally Lipschitz then it is globally Lipschitz on D.

$$x(t) = x_0 + \int_0^t f(x(s))ds, y(t) = x_0 + \int_0^t f(y(s))ds, \quad t \in [0, a],$$

then,

$$\|x(t) - y(t)\| = \| \int_0^t f(x(s)) - f(y(s))ds\| \leq \int_0^t \|f(x(s)) - f(y(s))\|ds.$$

But f is locally Lipschitz implies that there exists q such that

$$\|f(x(s)) - f(y(s))\| \leq q\|x(s) - y(s)\|,$$

for all $x, y \in D \subset \mathbb{R}^n$, and for all $s \in [0, a]$. So,

$$\|x(t) - y(t)\| \leq \int_0^t \|f(x(s)) - f(y(s))\|ds \leq q\|x(s) - y(s)\|,$$

for all $x, y \in D = \{x(s), y(s) : s \in [0, a]\}$. So, by Gronwall's Lemma we have that $\|x(s) - y(s)\| = 0$. Thus, $x = y$ on $[0, a]$, for all $0 < a < t_{max}$.
If $t_{max} < +\infty$, assume for contradiction, there exists $c > 0$ such that $\|x(t)\| \leq c$ for all $t \in [0, t_{max})$, then

$$\|x(s) - x(t)\| \leq \sup_{\omega \in [0, t_{max})} \|x'(\omega)\||s - t|$$

implies that,

$$\|x(s) - x(t)\| \leq \sup_{\omega \in [0, t_{max})} \|f(x(\omega))\||s - t| = M|s - t|, \quad M < \infty$$

Thus, x is uniformly continuous, and so $\lim_{t \to t_{max}} x(t) = x_{max} \in \mathbb{R}^n$.
Now we consider the following ODE

$$\begin{cases} z'(t) = f(z(t)), \quad t \geq t_{max} \\ z(t_{max}) = x_{max} \end{cases} \tag{2.3.2}$$

Using Banach fixed point theorem, there exists $\epsilon > 0$ such that (2.3.2) has a unique solution on $[t_{max}, t_{max} + \epsilon]$. Let

$$y = x\mathrm{V}z = \begin{cases} x(t), 0 \leq t \leq t_{max} \\ z(t), t_{max} \leq t < t_{max} + \epsilon \end{cases}$$

So,

$$\begin{aligned} y'_+(t_{max}) &= \lim_{h \to 0^+} \frac{y(t_{max} + h) - y(t_{max})}{h} \\ &= \lim_{h \to 0^+} \frac{z(t_{max} + h) - z(t_{max})}{h} \\ &= z'(t_{max}) = f(x_{max}). \end{aligned}$$

Also,

$$\begin{aligned} y'_-(t_{max}) &= \lim_{h\to 0^-} \frac{y(t_{max}+h) - y(t_{max})}{h} \\ &= \lim_{h\to 0^-} \frac{x(t_{max}+h) - x(t_{max})}{h} \\ &= x'(t_{max}) = f(x_{max}). \end{aligned}$$

Thus,

$$y'_+ = y'_- = f(x_{max}).$$

y is a C^1- function on $[0, t_{max}+\epsilon]$, and y satisfies

$$\begin{cases} y'(t) = f(y(t)), & t \in [0, t_{max}+\epsilon], \epsilon > 0 \\ y(0) = x_0 \end{cases} \tag{2.3.3}$$

this implies that

$$[0, t_{max}+\epsilon] \subseteq [0, t_{max}).$$

Contradiction
Thus,

$$\overline{\lim}_{t\to t_{max}} \|x(t)\| = +\infty, \quad if \quad t_{max} < +\infty.$$

■

Theorem 2.3.3 *[9] Consider the following ODE*

$$\begin{cases} x'(t) = f(t, x(t)), & t \geq 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2.3.4}$$

If there exists $k_1, k_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\|f(t,x)\| \leq k_1(t)\|x\| + k_2(t), \quad t \geq 0, \quad x \in \mathbb{R}^n.$$

Then,

$$t_{max} = +\infty, \quad for \quad all \quad x_0 \in \mathbb{R}^n.$$

Proof . Suppose for contradiction that $t_{max} < 0$, then

$$\overline{lim}_{t\to t_{max}} \|x(t)\| = +\infty.$$

But,

$$\begin{aligned} \|x(t)\| \leq \|x_0\| + \int_0^t \|f(s,x)\|ds &\leq \|x_0\| + \int_0^t (k_1(s)\|x(s)\| + k_2(s))ds \\ &\leq (\|x_0\| + \int_0^{t_{max}} k_2(s)ds) + \int_0^t k_1(s)\|x(s)\|ds. \end{aligned}$$

By Gronwall's Lemma,

$$\|x(t)\| \leq (\|x_0\| + \int_0^{t_{max}} k_2(s)ds) e^{\int_0^{t_{max}} k_1(s)ds},$$

for all $t \in [0, t_{max})$. This implies that x is bounded. Contradiction.
Hence, $t_{max} = +\infty$, for all $x_0 \in \mathbb{R}^n$. ■

2.4 Variation of constants formula

Consider the following ODE

$$\begin{cases} x'(t) = Ax(t) + f(t), & t \in \mathbb{R} \\ x(t_0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2.4.1}$$

where $f : \mathbb{R} \to \mathbb{R}^n$, continuous on $\mathbb{R}$, $A \in M_{n\times n}(\mathbb{R})$, $\quad n \geq 2$.

Theorem 2.4.1 *[9] The ODE (2.4.1) has a unique solution $x(., x_0)$ given by*

$$x(t, x_0) = \mathrm{e}^{(t-t_0)A}x_0 + \int_{t_0}^{t} \mathrm{e}^{(t-s)A} f(s)ds, \quad t \in \mathbb{R}.$$

Proof.
Let $y(t) = \mathrm{e}^{(t-t_0)A}x_0 + \int_{t_0}^{t} \mathrm{e}^{(t-s)A} f(s)ds$, then

$$\mathrm{e}^{(t-t_0)A}x_0 + \int_{t_0}^{t} \mathrm{e}^{(t-s)A} f(s)ds = \mathrm{e}^{(t-t_0)A}x_0 + \mathrm{e}^{tA}\int_{t_0}^{t} \mathrm{e}^{-sA} f(s)ds$$

implies that

$$y'(t) = A\mathrm{e}^{(t-t_0)A}x_0 + A\int_{t_0}^{t} \mathrm{e}^{(t-s)A} f(s)ds + f(t) = Ay(t) + g(t).$$

Also, $y(t_0) = x_0$. So, $y(t)$ satisfies (2.4.1). y is a C^1 function, and thus y is a solution of (2.4.1) on $\mathbb{R}$.
Claim:
y is unique.
Proof of claim:
Let $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by $g(t, x) = Ax(t) + f(t)$. Then, g is Lipschitzian with respect to the second variable. Thus, y must be the unique solution of (2.4.1) on $\mathbb{R}$.

Definition 2.4.2 *Let I be an interval in $\mathbb{R}$ $\quad t_0 \in I$ and $A : I \to M_{n\times n}(\mathbb{R})$ be continuous, $R(t, t_0)$ is called the fundamental matrix of*

$$\begin{cases} x'(t) = A(t)x(t), & t \in I \\ x(t_0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2.4.2}$$

If $R : I \times I \to M_{n\times n}(\mathbb{R})$ satisfies
(i) $R(s, s) = I$
(ii) $\frac{\partial R(t,s)}{\partial t} = A(t)\mathrm{o}R(t, s)$, for $t, s \in I$
It is also called the resolvent operator of (2.4.2).

Theorem 2.4.3 *[9] (i) For all $t, s, \sigma \in I$, $\quad R(t, s)\mathrm{o}R(s, \sigma) = R(t, \sigma)$.*
(ii) $R(t, s)$ is invertible and $(R(t, s))^{-1} = R(s, t)$, for all $t, s \in I$.
(iii) $\frac{\partial R(t,s)}{\partial s} = -R(t, s)A(s)$, for all $t, s \in I$.

Proof .
(i) Let $x_0 \in \mathbb{R}$ and let $y(t) = (R(t, s)\mathrm{o}R(s, t_0))(x_0)$, then,

$$y'(t) = \frac{\partial R(t, s)[R(s, t_0)](x_0)}{\partial t} = A(t)R(t, s)[R(s, t_0)](x_0) = A(t)y(t).$$

So,

$$y'(t) = A(t)y(t).$$

Also, let

$$z(t) = R(t, t_0)x_0,$$

then,

$$z'(t) = A(t)R(t, t_0)x_0 = A(t)z(t).$$

But

$$f(t, x) = A(t)x(t)$$

is continuous and Lipschitzian with respect to x. Thus, by uniqueness of solution $z = y$. Hence, for all $t, s \in I$, and $x_0 \in \mathbb{R}$,
$R(t, s)\text{o}R(s, t_0) = R(t, t_0)$

(ii) From (i), we have that

$$for \quad all \quad t, s \in I, \quad x_0 \in \mathbb{R}, \quad R(t, s)\text{o}R(s, t_0) = R(t, t_0).$$

So,

$$R(t, s)\text{o}R(s, t) = R(t, t) = I$$

Hence, $R(t, s)$ is invertible, and

$$(R(t, s))^{-1} = R(s, t), \quad for \quad all \quad t, s \in I.$$

(iii) Let $\theta : s \to R(s, t)^{-1}$, $\quad \varphi : s \to R(s, t)$, and $\quad \Phi : R(s, t) \to R(s, t)^{-1}$.
That is,

$$\varphi(s) = R(s, t), \quad and \quad \Phi(R(s, t)) = R(s, t)^{-1}.$$
$$\theta(s) = (\Phi \quad \text{o} \quad \varphi)(s) = R(s, t)^{-1}.$$

So,

$$\begin{aligned}
\theta'(s) &= \Phi'(\varphi(s))\varphi'(s) \\
&= \Phi'(R(s, t))A(s)R(s, t) \\
&= -R(s, t)^{-1}A(s)R(s, t)R(s, t)^{-1} \\
&= -R(s, t)^{-1}A(s)Id_{\mathbb{R}^n} \\
&= -R(s, t)^{-1}A(s).
\end{aligned}$$

Hence,

$$\frac{\partial R(t, s)}{\partial s} = -R(t, s)A(s), \quad for \quad all \quad t, s \in \mathbb{R}.$$

■

Proposition 2.4.4 *If* $A(t)A(s) = A(s)A(t), \quad for \quad all \quad t, s \in I$, *then*

$$R(t, t_0) = e^{\int_{t_0}^{t} A(s)ds}, \quad for \quad t, \quad t_0 \in I$$

Proof .
Let $S(t, t_0) = e^{\int_{t_0}^{t} A(s)ds}$. Then,

$$\frac{\partial S(t, t_0)}{\partial t} = e^{\int_{t_0}^{t} A(s)ds} \text{o}A(t).$$

From Riemann integration, we have that,

$$\int_{t_0}^{t} A(s)ds = \lim_{p\to\infty} \frac{t-t_0}{p} \sum_{k=0}^{p-1} A(t_0 + k\frac{t-t_0}{p})$$

so,

$$\begin{aligned} e^{\displaystyle\int_{t_0}^{t} A(s)ds} &= e^{\lim_{p\to\infty} \frac{t-t_0}{p} \sum_{k=0}^{p-1} A(t_0+k\frac{t-t_0}{p})} \\ &= \lim_{p\to\infty} e^{\frac{t-t_0}{p} \sum_{k=0}^{p-1} A(t_0+k\frac{t-t_0}{p})}, \end{aligned}$$

and then,

$$e^{\displaystyle\int_{t_0}^{t} A(s)ds} A(t) = \lim_{p\to\infty} \prod_{k=0}^{p-1} e^{\frac{t-t_0}{p} A(t_0+k\frac{t-t_0}{p})} A(t).$$

But, by Taylor's expansion of exponential funtions, we have that,

$$e^{\frac{t-t_0}{p} A(t_0+k\frac{t-t_0}{p})} A(t) = \sum_{j=0}^{\infty} (\frac{(t-t_0)^j}{p^j} \frac{A^j(t_0+k\frac{t-t_0}{p})}{j!} A(t)$$

Hence,

$$\begin{aligned} e^{\displaystyle\int_{t_0}^{t} A(s)ds} &= e^{\lim_{p\to\infty} \frac{t-t_0}{p} \sum_{k=0}^{p-1} A(t_0+k\frac{t-t_0}{p})} A(t) \\ &= \lim_{p\to\infty} e^{\frac{t-t_0}{p} \sum_{k=0}^{p-1} A(t_0+k\frac{t-t_0}{p})} \\ &= \lim_{p\to\infty} \sum_{j=0}^{\infty} (\frac{(t-t_0)^j}{p^j} \frac{A^j(t_0+k\frac{t-t_0}{p})}{j!} A(t) \\ &= A(t) \lim_{p\to\infty} \sum_{j=0}^{\infty} (\frac{(t-t_0)^j}{p^j} \frac{A^j(t_0+k\frac{t-t_0}{p})}{j!} \\ &= A(t) e^{\displaystyle\int_{t_0}^{t} A(s)ds}. \end{aligned}$$

Then,

$$\frac{\partial S(t,t_0)}{\partial t} = A(t)S(t,t_0),$$

and

$$S(t,t) = e^{\int_t^t A(s)ds} = I.$$

Thus,

$$S(t,t_0) = R(t,t_0) = e^{\displaystyle\int_{t_0}^{t} A(s)ds}.$$

■

Theorem 2.4.5 *[9] Let $x(.,t_0)$ be the solution of the following ODE*

$$\begin{cases} x'(t) = A(t)x(t) + f(t), & t \in I \\ x(t_0) = x_0 \in \mathbb{R}^n \end{cases} \qquad (2.4.3)$$

where $f : I \to \mathbb{R}^n$, is continuous, $A : I \to M_{n\times n}(\mathbb{R})$ is continuous. Then, the IVP (2.4.3) has a unique solution $x(.,x_0)$ given by

$$x(t,x_0) = R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)f(s)ds, \quad t \in I.$$

Proof . Let

$$y(t) = R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)f(s)ds,$$

then,

$$y(t_0) = R(t_0,t_0)x_0 = x_0,$$

and

$$\begin{aligned} y'(t) &= A(t)R(t,t_0)x_0 + (\frac{\partial R(t,0)}{\partial t})\int_{t_0}^{t} R(0,s)f(s)ds + R(t,0)\frac{\partial}{\partial t}\int_{t_0}^{t} R(0,s)f(s)ds \\ &= A(t)(R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)f(s)ds) + R(t,t)f(t) \\ &= A(t)y(t) + f(t). \end{aligned}$$

So, y satisfies (2.4.3), and y is a C^1 function. Thus, y is a solution of (2.4.3).

$$g(t,x) = A(t)x(t) + f(t)$$

is continuous and Lipschitz with respect to the second variable. Hence, by uniqueness of solution of (2.4.3),

$$x(t,x_0) = y(t) = R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)f(s)ds$$

is the unique solution of (2.4.3) on $I \subseteq \mathbb{R}$. ■

Formula of $R(t,t_0)$

Scalar ODE

$$\begin{cases} x'(t) = a(t)x(t), & t \in I \subseteq \mathbb{R} \\ x(t_0) = x_0 \in \mathbb{R} \end{cases} \tag{2.4.4}$$

where I is an interval of $\mathbb{R}$ and $a : I \to \mathbb{R}$ is continuous. Then,

$$R(t,t_0) = e^{\int_{t_0}^{t} a(s)ds} \quad for \quad all \quad t,t_0 \in \mathbb{R}.$$

Example 2.4.6 *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\alpha(t) = \sin t.$$

Let $B \in M_{n\times n}(\mathbb{R})$, and let $A : \mathbb{R} \to M_{n\times n}(\mathbb{R})$ be defined by

$$A(t) = \alpha(t)B.$$

Then,

$$A(t)A(s) = \sin t \sin s B^2 = \sin sB \sin tB = A(s)A(t).$$

Hence,

$$R(t,t_0) = e^{\int_{t_0}^{t} \sin s ds} = \sum_{n=0}^{\infty} \frac{B^n(\cos t_0 - \cos t)^n}{n!}, \quad for \quad all \quad t,t_0 \in \mathbb{R}.$$

Theorem 2.4.7 *[9] (**Estimation of** $\|R(t,t_0)\|$)*

$$\|R(t,t_0)\| \le e^{\int_{t_0}^{t} \|A(s)\|ds}, \quad t \ge t_0$$

Proof .

$$\begin{cases} \frac{\partial R(t,t_0)}{\partial t} = A(t) \text{o} R(t,t_0) \\ R(t_0,t_0) = I \end{cases}$$

$$\begin{aligned} R(t,t_0) &= R(t_0,t_0) + \int_{t_0}^{t} A(s)R(s,t_0)ds \\ &= I + \int_{t_0}^{t} A(s)R(s,t_0)ds. \end{aligned}$$

Then,

$$\begin{aligned} \|R(t,t_0)\| &\leq \|I\| + \|\int_{t_0}^{t} A(s)R(s,t_0)ds\| \\ &\leq 1 + \int_{t_0}^{t} \|A(s)\|\|R(s,t_0)\|ds. \end{aligned}$$

Hence, by Gronwall's Lemma we get that,

$$\|R(t,t_0)\| \leq \mathrm{e}^{\int_{t_0}^{t} \|A(s)\|ds}.$$

Hence,

$$\|R(t,t_0)\| \leq \mathrm{e}^{\int_{t_0}^{t} \|A(s)\|ds}, \quad t \geq t_0.$$

■

Theorem 2.4.8 *[9] Consider the following ODE*

$$\begin{cases} x'(t) = A(t)x(t) + f(t,x(t)), \quad t \geq 0 \\ x(t_0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2.4.5}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, continuous, and Lipschitzian with respect to the second variable, $A : I \to M_{n\times n}(\mathbb{R})$ is continuous.
Then, (2.4.5) has a unique solution $x(.,x_0)$ given by the following variation of constants formula

$$x(t,x_0) = R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)f(s,x(t,x_0))ds, \quad t \geq 0$$

Proof .
Let $y(t,x_0) = R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)f(s,x(t,x_0))ds, \quad t \in \mathbb{R}^+$, then,

$$\begin{aligned} y(t) &= R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)f(s,x(t,x_0))ds \\ &= R(t,t_0)x_0 + R(t,0)\int_{t_0}^{t} R(0,s)f(s,x(t,x_0))ds. \end{aligned}$$

So,

$$\begin{aligned} y'(t) &= A(t)(R(t,t_0))(x_0) + A(t)R(t,0)\int_{t_0}^{t} R(0,s)f(s,x(t,x_0))ds + f(t,x(t,x_0)) \\ &= A(t)(R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)f(s,x(t,x_0))ds) + f(t,x(t,x_0)) \\ &= A(t)y(t,x_0) + f(t,x(t,x_0)). \end{aligned}$$

Also,

$$y(t_0, x_0) = R(t_0, t_0)x_0 = x_0.$$

So, y is a solution of (2.4.5). By, uniqueness of solutions of (2.4.5), we have that

$$x(t, x_0) = R(t, t_0)x_0 + \int_{t_0}^{t} R(t, s)f(s, x(t, x_0))ds, \quad t \geq 0.$$

■

CHAPTER 3

STABILITY VIA LINEARIZATION PRINCIPLE

Introduction
An equilibrium point is said to be stable if all solutions starting at nearby points stay nearby. Moreover, if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity, then the equilibrium point is said to be asymptotically stable.
The most popular method for studying stability of nonlinear systems is introduced by a Russian Mathematician named Alexander Mikhailovich Lyapunov. His work the general problem of motion stability published in 1892 includes two methods namely; linearization method, and direct method. Linearization method studies nonlinear local stability around an equilibrium point from stability properties of its linear approximation.
This chapter is aimed at examining the stability of equilibrium points. Also, relevant theorems and concepts in stability of an equilibrium point and linearization principle will be discussed in this chapter.

3.1 Definitions and basic results

Definition 3.1.1 *Consider the following ODE:*

$$\begin{cases} x'(t) = f(x(t)), & t \geq 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{3.1.1}$$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz and affine or Lipschitzian. Then, equation (3.1.1) has a unique solution on $\mathbb{R}^+$.
x^* is an equilibrium point of (3.1.1) if $f(x^*) = 0$ and $x(t, x^*) = x^*$, for all $t \geq 0$.

Definition 3.1.2 *Let x^* be an equilibrium point of the IVP (3.1.1) , x^* is stable in Lyapunov sense if*

$$for \quad all \quad \epsilon > 0, \quad there \quad exists \quad \delta > 0 \quad such \quad that$$
$$\|x_0 - x^*\| < \delta \implies \|x(t, x_0) - x^*\| < \epsilon, \quad for \quad all \quad t \geq 0.$$

Definition 3.1.3 *x^* is said to be assymptotically stable if*
(i) x^* is stable
(ii) there exists $r > 0$ such that if $x_0 \in B(x^*, r)$ then $x(t, x_0) \to x^*, as \quad t \to \infty$.

Definition 3.1.4 *x^* is locally exponentially stable if x^* is assymptotically stable, and there exists $r > 0$ such that if*

$$x_0 \in B(x^*, r), \quad then \quad x(t, x_0) \to x^*, as \quad t \to \infty \quad exponentially.$$

Definition 3.1.5 *x^* is unstable if it is not stable. In other words, x^* is unstable if there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x_0 \in B(x^*, \delta)$ and there exists $t \geq 0$ such that*

$$\|x(t, x_0) - x^*\| \geq \epsilon.$$

Linear systems

$$\begin{cases} x'(t) = Ax(t), & t \geq 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{3.1.2}$$

where $A \in M_{n\times n}(\mathbb{R})$. Then,

$$x(t, x_0) = \mathrm{e}^{tA} x_0$$

Theorem 3.1.6 *[9] 0 is stable of equation (3.1.2) if and only if $\sup\limits_{t\geq 0} \|\mathrm{e}^{tA}\| < \infty$.*

Proof .
Assume 0 is stable. Then, for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x_0\| < \delta \quad implies \quad \|x(t, x_0)\| < \epsilon, \quad for \quad all \quad t \geq 0.$$

In particular,

$$for \quad \epsilon = 1 \quad there \quad exists \delta_1 > 0 \quad such \quad that \quad \|x_0\| < \delta_1 \quad implies \quad \|x(t, x_0)\| < 1,$$

and so

$$\sup_{t\geq 0} \|x(t, x_0)\| \leq 1, \quad implies \quad that \quad \sup_{t\geq 0} \|\mathrm{e}^{tA} x_0\| \leq 1 \quad for \quad all \quad t \geq 0, \quad and \quad for \quad all \quad x_0 \in B(0, \delta_1).$$

Let $y \in \mathbb{R}^n, \quad y \neq 0$ and let

$$\|\mathrm{e}^{tA}\| = \sup_{\|y\|\leq 1} \|\mathrm{e}^{tA} y\|.$$

Then,

$$\frac{y}{\|y\|} \frac{\delta_1}{2} \in B(0, \delta_1) \quad implies \quad that \quad \|\mathrm{e}^{tA} y\| < \frac{2}{\delta_1} \|y\|, \quad for \quad all \quad y \in \mathbb{R}^n.$$

Hence,

$$\|\mathrm{e}^{tA}\| \leq \frac{2}{\delta_1}, \quad for \quad all \quad t \geq 0 \quad implies \quad that \quad \sup_{t\geq 0} \|\mathrm{e}^{tA}\| < \infty.$$

Assume, $\sup\limits_{t\geq 0} \|\mathrm{e}^{tA}\| < \infty$, and let $M = \sup\limits_{t\geq 0} \|\mathrm{e}^{tA}\|$, let $\epsilon > 0$ be given, we seek $\delta > 0$ such that for all

$$\|x_0\| < \delta, \quad \|x(t, x_0)\| < \epsilon.$$

$$\|x(t, x_0)\| = \|\mathrm{e}^{tA} x_0\| \leq \sup_{t\geq 0} \|\mathrm{e}^{tA}\| \|x_0\| = M\|x_0\|.$$

Take $\delta = \frac{\epsilon}{M+1}$. Then,

$$\|x(t, x_0)\| < \epsilon, \quad for \quad all \quad \|x_0\| < \delta.$$

Hence, 0 is stable of equation (3.1.2)
Conclusion: 0 is stable of equation (3.1.2) if and only if $\sup\limits_{t\geq 0} \|\mathrm{e}^{tA}\| < \infty$. ∎

Definition 3.1.7 *Let, $A \in M_{n\times n}(\mathbb{R})$, then the set of all eigenvalues of A is denoted by $\sigma(A)$.*

Theorem 3.1.8 *[9] 0 is globally exponentially stable for equation (3.1.2) if and only if*

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} : Re(\lambda) < 0\}.$$

Theorem 3.1.9 *[9] Let $A \in M_{n\times n}(\mathbb{R})$. Consider the following ODE,*

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \geq 0 \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{3.1.3}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and locally Lipschitz with respect to the second variable. Moreover, we assume
(i) $f(t, 0) = 0, \quad for \quad all \quad t \geq 0$

(ii) $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : Re(\lambda) < 0\}$

(iii) There exists $k > 0, \quad a > 0$ such that

$$\|f(t, x)\| \leq k\|x\|^2, \quad t \geq 0, \quad \|x\| < a.$$

Then, there exists $c, b, \alpha > 0$ such that

$$\|x(t, x_0)\| \leq c\|x_0\|\mathrm{e}^{-\alpha t}, \quad for \quad \|x_0\| < b, \quad t \geq 0.$$

Theorem 3.1.10 *[9](**Linearization Principle**) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 function and x^* be the equilibrium point of f as in (3.1.4). Let $A = \frac{\partial f}{\partial x}|_{x=x^*}$ be the linearization of f, and $\sigma(A) \subseteq \{Re(\lambda) < 0\}$. Then, x^* is locally exponentially stable for (3.1.4). If there exists $\lambda_0 \in \sigma(A)$ such that $Re(\lambda_0) > 0$, then x^* is unstable for the following ODE,*

$$\begin{cases} x'(t) = f(x(t)), & t \geq 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{3.1.4}$$

Proof .
Without loss of generosity, we assume that, $x^* = 0$.
Let $\varphi(t) = f(ty), \quad t \in [0, 1]$. Then φ is differentiable and $y \in \mathbb{R}^n$.

$$\varphi'(t) = f'(ty)y$$

Let $g(y) = f(y) - Ay$. Then,

$$f(y) = Ay + f(y) - Ay = Ay + g(y) \tag{3.1.5}$$

But

$$\begin{aligned} f(y) &= f(y) - f(0) \\ &= \varphi(1) - \varphi(0) = \int_0^t \varphi'(s)ds. \end{aligned}$$

Then, $g(y) = \int_0^1 \varphi'(s)ds - Ay$

$$g(y) = \int_0^1 [f'(sy)y - f'(0)y]ds$$

$$\|g(y)\| \leq \int_0^1 \|f'(sy) - f'(0)\|ds\|y\|.$$

Let $r > 0, \quad z \in B(0, r)$,

$$\|f'(z) - f'(0)\| \leq \sup_{\sigma \in \overline{B}(0,r)} \|f'(\sigma)\|\|z\|.$$

Let $k = \sup_{\sigma \in \overline{B}(0,r)} \|f'(\sigma)\|$ and $y \in B(0,r)$, then, $sy \in B(0,r)$ because $0 \le s \le 1$. So for all

$$s \in [0,1], \quad \|f'(sy) - f'(0)\| \le k\|sy\| \le k\|y\|.$$

$$\begin{aligned}\|g(y)\| &\le \int_0^1 \|f'(sy) - f'(0)\| ds \|y\| \\ &\le \int_0^1 k\|y\|\|y\| ds \\ &\le \int_0^1 k\|y\|^2 ds = k\|y\|^2 \int_0^1 ds = k\|y\|^2.\end{aligned}$$

That is,

$$\|g(y)\| \le k\|y\|^2, \quad y \in B(0,r).$$

Hence, by Theorem 3.1.9, there exists

$$b, c, \alpha > 0 \quad \textit{such} \quad \textit{that} \quad \|x(t,x_0)\| \le c\|x_0\| \mathrm{e}^{-\alpha t}. \quad \textit{for} \quad \|x_0\| < b, \quad t \ge 0.$$

Hence, 0 is exponentially stable, and therefore, 0 is asymptotically stable.
For the instability, we refer to Theorem 3.1.8. ■

Theorem 3.1.11 *[9](**Scalar ODEs**) Consider the following ODE:*

$$\begin{cases} x'(t) = f(x(t)), t \ge 0 \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{3.1.6}$$

where $f : \mathbb{R} \to \mathbb{R}$ be C^1-function, let x^* such that $f(x^*) = 0$. Then, $f'(x^*) < 0$ implies that x^* is locally exponentially stable, $f'(x^*) > 0$ implies that x^* is unstable.

Example 3.1.12

$$\begin{cases} x'(t) = -x(t) + f(x(t)), \quad t \ge 0 \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{3.1.7}$$

$f : \mathbb{R} \to \mathbb{R}$ be a C^1-function, $f(0) = 0$.
Then, 0 is an equilibrium point of equation (3.1.7). if $-1 + f'(0) < 0 \implies 0$ is locally exponentially stable for (3.1.7). If $-1 + f'(0) > 0 \implies 0$ is unstable.

Example 3.1.13

$$\begin{cases} x'(t) = -x(t) + y(t) + x^2(t) + y^2(t), \quad t \ge 0 \\ y'(t) = -2y(t) + x^4(t) - y^4(t) \\ x(0) = x_0 \in \mathbb{R}, \quad y(0) = y_0 \in \mathbb{R} \end{cases} \tag{3.1.8}$$

Let $z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ Then, $z'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} z(t) + f(x(t), y(t))$, where $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$f(x,y) = \begin{pmatrix} x^2 + y^2 \\ x^4 - y^4 \end{pmatrix}.$$

Let, $g(z(t)) = Az(t) + f(z(t))$ where $A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$.

Then, $g'(z(t)) = A + f'(z(t))$. Where $f'(z(t)) = \begin{pmatrix} 2x(t) + 2y(t) \\ 4x^3(t) - 4y^3(t) \end{pmatrix}$.

So, at $(0,0)$, $\quad g(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *and* $g'(0,0) = A$.
But $\sigma(g'(0,0)) = \{-1,-2\} \subseteq \{Re(\lambda) < 0\}$. *Therefore,* $(0,0)$ *is locally exponentially stable for equation (3.1.8).*

Example 3.1.14

$$\begin{cases} x'(t) = x(t) - x^2(t), t \geq 0 \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{3.1.9}$$

Then, equilibrium points of (3.1.9) are 0, *and* 1.
Let $f(x(t)) = x(t) - x^2(t)$, *then* $f'(x(t)) = 1 - 2x(t) \implies f'(0) = 1 > 0$ *and* $f'(1) = -1 < 0$. *Therefore,* 0 *is unstable, and* 1 *is locally exponentially stable for equation (3.1.9)*

CHAPTER 4

LYAPUNOV FUNCTIONS AND LASALLE'S INVARIANCE PRINCIPLE

Introduction

A Lyapunov function $V : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is an energy-like function that can be used to determine stability of a system. It is a powerful tool for determining stability.
Therefore, it is the best way to study the asymptotic behaviour of solutions, but the construction of the Lyapunov functions depends on the nature of the ODE.
Moreover, the LaSalle invariance principle was then introduced by LaSalle, and thus it is an interesting working tool in dynamical systems and control theory.
In this chapter, our focus is to examine the various tools for determining stability of equilibrium points, like the Lyapunov functions and LaSalle's invariance principle.
Let us start this chapter by the following example

Example 4.0.1

$$x' = ax^3$$

Linearization about $x = 0$ yields:

$$A = \frac{\partial f}{\partial x}|_{x=0} = 3ax^2|_{x=0} = 0$$

Linearization fails to determine stability. If $a < 0$, then $x = 0$ is asymptotically stable. To see this, $V(x) = x^4$, implies that $V' = 4x^3x' = 4ax^6$.
If $a > 0$, $\quad x = 0$ is unstable. If $a \leq 0$, $\quad x = 0$ is stable, starting at any x, remains in x.

A powerful tool for determining stability is the use of Lyapunov functions.
A Lyapunov function $V : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is an energy-like function that can be used to determine stability of a system. Roughly speaking, if we can find a non-negative function that always decreases along trajectories of the system, we can conclude that the minimum of the function is a stable equilibrium point (locally).
To describe this more formally, we start with a few definitions.

Definition 4.0.2 *We say that a continuous function $V(x)$ is positive definite if*

$$V(x) > 0, \quad for \quad all \quad x \neq 0 \quad and \quad V(0) = 0.$$

Definition 4.0.3 *We say that a continuous function $V(x)$ is negative definite if*

$$V(x) < 0, \quad for \quad all \quad x \neq 0 \quad and \quad V(0) = 0.$$

Definition 4.0.4 *We say that a continuous function $V(x)$ is positive semi-definite if*

$$V(x) \geq 0, \quad for \quad all \quad x.$$

Definition 4.0.5 *We say that a continuous function $V(x)$ is negative semi-definite if*

$$V(x) \leq 0, \quad for \quad all \quad x.$$

4.1 Definitions and basic results

Definition 4.1.1 *(* ***Lyapunov function****) The C^1 function $V : D \subset \mathbb{R}^n \to \mathbb{R}$ a C^1, positive definite function such that V' is negative semi-definite is called a Lyapunov function.*

Definition 4.1.2 *(****Strict Lyapunov function****) The C^1 function $V : D \subset \mathbb{R}^n \to \mathbb{R}$ a C^1, positive definite function such that V' is negative definite is called a strict Lyapunov function.*

Definition 4.1.3 *(* ***Lyapunov surface****) The surface $V(x) = c$, for some $c > 0$ is called a Lyapunov surface or level surface.*

If $V'(x) \leq 0$, when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$$

and traps inside Ω_c.
If $V' < 0$, trajectories move from one level surface to an inner level with smaller c till $V(x) = c$ shrinks to zero as time goes on.

Suppose V is C^1, we want to examine the derivative of V along trajectories of the system,

$$\begin{cases} x'(t) = f(x(t)), & t \geq 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{4.1.1}$$

where x and $f(x)$ are in $\mathbb{R}^n$, f is continuous and locally Lipschitzian. Suppose that x is a solution of (4.1.1), then we have that

$$\frac{dV(x(t))}{dt} = \sum_{j=1}^{n} \frac{\partial V(x(t))}{\partial x_j} x_j'(t) = \sum_{j=1}^{n} \frac{\partial V(x(t))}{\partial x_j} f_j(t). \tag{4.1.2}$$

Definition 4.1.4 *(****Quadratic functions****) A class of scalar functions for which sign definition can be easily checked is quadratic functions.*

Definition 4.1.5 *(****How to check the sign of quadratic functions****)*

$$V(x(t)) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j P_{ij} \tag{4.1.3}$$

where $P = P^T$ is a real matrix.
(i) $V(x(t))$ is positive definite/ positive semi-definite iff $\lambda_i(P) > 0/\lambda_i(P) \geq 0, i = 1, 2, 3, ..., n$ iff all leading principle minors of P are positive / non-negative, respectively.
If $V(x(t))$ is positive(positive semi-definite), we say the matrix P is positive(positive semi-definite) and write $(P > 0)(P \geq 0)$.

Theorem 4.1.6 *[4](****Lyapunov Theorem****) Let the origin be an equilibrium point for equation (4.1.1). Let D be a neighborhood of the origin. The origin is stable if there is $V : D \subset \mathbb{R}^n \to \mathbb{R}$ a C^1, positive definite function such that V' is negative semi definite. Then the origin is asymptotically stable if V' is negative stable.*

Proof.
Let $x_0 \in \mathbb{R}^n$, and $t \geq 0$, then

$$V(x(t, x_0)) \leq V(x_0),$$

and $V^{-1}([0, V(x_0)])$ is compact, so every solution curve stays bounded in forward time.
To show Lyapunov stability suppose $\epsilon > 0$ is given. Choose $\epsilon_1 \in (0, \epsilon)$. Since V is continuous and $[0, \epsilon_1]^n \subseteq (0, \epsilon)^n$ is compact, there exists $M \in \mathbb{R}$ such that V is bounded above by M on $[0, \epsilon_1]^n$. Since V is continuous, there exists $\delta > 0$ such that for all $x \in [0, \delta]^n, \quad V(x) < M$. Hence for every $x_0 \in [0, \delta]^n$, and every $t \geq 0$, since $V' < 0$, then

$$V(x(t, x_0)) = V(x_0) + \int_0^t V'(s, x_0) ds \leq V(x_0) \leq M$$

Hence for every $x_0 \in [0, \delta]^n$ and all $t \geq 0, \quad x(t, x_0) \in [0, \epsilon_1]^n \subseteq (0, \epsilon)^n$. So, for all $\epsilon > 0$ there is $\delta > 0$ such that $\|x(t, x_0)\| < \epsilon$, for all $t \geq 0$. Hence the origin is stable. To show that the origin is asymptotically stable, fix any $p \in D \subset \mathbb{R}^n$ and define

$$c = \liminf_{t \geq 0} V(x(t, x_0)) = \lim_{t \to \infty} V(x(t, x_0))$$

If $c = 0$ then we are done since V is positive definite.
Otherwise, choose $\delta > 0$ such that $0 < c - \delta < c + \delta < V(p)$, and consider the annular region $A = V^{-1}([c - \delta < c + \delta]) \subseteq D \subset \mathbb{R}^n . V'$ is continuous. V' is negative definite, then there exists $q > 0$ such that for all $x \in A, \quad V'(x) \leq -q$. By choice of c there exists $T \in \mathbb{R}$ such that $V(x(T, x_0)) < c + \delta$, that is, $x(T, x_0) \in A$. So,

$$V(x(T + \frac{2\delta}{q}, x_0)) = V(x(T, p)) + \int_0^{\frac{2\delta}{q}} V'(x(T + s, p)) ds \leq (c + \delta) + \int_0^{\frac{2\delta}{q}} -q ds \leq (c - \delta) \quad (4.1.4)$$

which contradicts the choice of c. Thus $c = 0$ and the origin is asymptotically stable.

Theorem 4.1.7 *[4]* ***(Babashin - Krasovskii Theorem)*** *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, $f(0) = 0$, and $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function, radially unbounded, positive definite such that V' is negative definite. Then the origin is a globally asymptotically stable equilibrium of $x' = f(x)$.*

Proof.
Let $x_0 \in \mathbb{R}^n$, and $t \geq 0$, then $V(x(t, x_0)) \leq V(x_0)$, and $V^{-1}([0, V(x_0)])$ is compact, so every solution curve stays bounded in forward time.
To show Lyapunov stability suppose $\epsilon > 0$ is given. Choose $\epsilon_1 \in (0, \epsilon)$. Since v is continuous and $[0, \epsilon_1]^n \subseteq (0, \epsilon)^n$ is compact, there exists $M \in \mathbb{R}$ such that V is bounded above by M on $[0, \epsilon_1]^n$. Since V is continuous, there exists $\delta > 0$ such that for all $x \in [0, \delta]^n, \quad V(x) < M$. Hence for every $x_0 \in [0, \delta]^n$, and every $t \geq 0$, since $V' < 0$, then

$$V(x(t, x_0)) = V(x_0) + \int_0^t V'(s, x_0) ds \leq V(x_0) \leq M.$$

Hence for every $x_0 \in [0, \delta]^n$ and all $t \geq 0, \quad x(t, x_0) \in [0, \epsilon_1]^n \subseteq (0, \epsilon)^n$. To show that the origin is attractive, fix any $x_0 \in \mathbb{R}^n$ and define

$$c = \underline{\lim}_{t \geq 0} V(x(t, x_0)) = \lim_{t \to \infty} V(x(t, x_0))$$

If $c = 0$ then we are done since V is positive definite.
Otherwise, choose $\delta > 0$ such that $0 < c - \delta < c + \delta < V(p)$, and consider the annular region

$$A = V^{-1}([c - \delta < c + \delta]) \subseteq \mathbb{R}^n.$$

Since V is proper, A is compact. V' is continuous. V' is negative definite, then there exists $q > 0$ such that for all

$$x \in A, \quad V'(x) \leq -q.$$

By choice of c there exists $T \in \mathbb{R}$ such that

$$V(x(T, x_0)) < c + \delta,$$

that is,

$$x(T, x_0) \in A.$$

Hence,

$$V(x(T + \frac{2\delta}{q}, x_0)) = V(x(T, p)) + \int_0^{\frac{2\delta}{q}} V'(x(T + s, p))ds \leq (c + \delta) + \int_0^{\frac{2\delta}{q}} -qds \leq (c - \delta)$$

which contradicts the choice of c. Thus $c = 0$ and the origin is globally asymptotically stable. The C^1 function $V(x)$ is called a Lyapunov function.
The surface $V(x) = c$, for some $c > 0$ is called a Lyapunov surface or level surface.
When $V'(x) \leq 0$, when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ and traps inside Ω_c.
When $V' < 0$, trajectories move from one level surface to an inner level with smaller c till $V(x) = c$ shrinks to zero as time goes on.

4.2 Instability Theorem

Theorem 4.2.1 *[2](**Chetaev's Theorem**) Let $x = 0$ be an equilibrium point of $x' = f(x)$. Let $V : D \to \mathbb{R}$ be a C^1 function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrary small $\|x_0\|$. Define a set $v = \{x \in B_r : V(x) > 0\}$ where $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$ and suppose that $V'(x)$ is positive definite in v. Then, $x = 0$ is unstable.*

Proof.
Let $\Omega = B_r$, and let $\epsilon > 0$ be so small that $\overline{B(0, \epsilon)}$ is contained in $\overline{B}(0, r)$ and let

$$M = \Omega \cap \{\|y\| < \epsilon\}.$$

We affirm that there are points arbitrarily close to the equilibrium point which move a distance at least ϵ from the equilibrium.
Note that M has points arbitrarily close to the origin, so for any $\delta > 0$ there is a point $x \in M$ with $\|x\| < \delta$ and $V(x) > 0$.
If $x(t, x_0)$ remains in M, for all $t \geq 0$ then $y(t)$ is increasing since $\Delta V(x(t, x_0)) > 0$ and so

$$y(t) \geq y(0) = V(x_0) > 0 \quad for \quad t \geq 0.$$

The closure of

$$\overline{\Omega} = \{x(t, x_0) : t \geq 0\}$$

is compact and then defining

$$\gamma = \min_{y \in \overline{\Omega}} \Delta V(y).$$

Since $\Delta V(y)$ is positive definite in $\overline{\Omega}$, it is clear that $\gamma > 0$. So, $x(t, x_0) \in \overline{\Omega}$, for every t implies that

$$V(x(t, x_0)) - V(x_0) = \sum_{s=0}^{t-1} \Delta V(x(s, x_0)) \geq \gamma t.$$

So, $V(x(t, x_0)) \to \infty$ as $t \to \infty$. Contradiction because $x(t, x_0)$ remains in $B(0, \epsilon)$ and V is continuous.
Therefore, there exists $t^* > 0$ such that $x(t, x_0)$ crosses the boundary of M for the first time at $t = t^*$, and

$$\Delta V(x(t, x_0)) > 0 \quad for \quad 0 \leq t < t^*$$

and so

$$y(t^*) \geq V(x_0) > 0.$$

Because the complment of M consist of points q where $V(q) \leq 0$ or where $\|q\| \geq \epsilon$, it follows that $\|x(t^*, x_0)\| \geq \epsilon$. Therefore, $x = 0$ is unstable.

4.3 How to search for a Lyapunov function (variable gradient method)

Idea is working backward:
Investigate an expression for $V'(x)$ and go back to choose the parameters of $V(x)$ so as to make $V'(x)$ negative definite.
Let $V = V(x)$ and $g(x) = (\frac{\partial V}{\partial x})^T$, then, $V' = \frac{\partial V}{\partial x} f = g^T f$.
Choose $g(x)$ such that it would be the gradient of a positive definite function V and make V' negative definite.
$g(x)$ is the gradient of a scalar function if and only if the Jacobian matrix $\frac{\partial g}{\partial x}$ is symmetric:

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad for \quad all \quad i, j = 1, 2, 3, ..., n.$$

Select $g(x)$ such that $g^T(x)f(x)$ is negative definite.
Then, $V(x)$ is computed from the integral;

$$V(x) = \int_0^x g(y)dy = \int_0^x \sum_{i=1}^{n} g_i(y)dy$$

The integration is taken over any path joining the origin to x. This can be done along the axes :

$$V(x) = \int_0^{x_1} g_1(y_1, 0, 0, ..., 0)dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, 0, ..., 0)dy_2 + ... + \int_0^{x_n} g_n(x_1, x_2, ..., y_n)dy_n$$

By leaving some parameters of g undetermined, one would try to choose them so that V is positive definite.

Definition 4.3.1 *(**Region of Attraction**) Let $\Phi(t, x)$ be the solution of the system (4.1.1) starting at x_0, then the Region of Attraction (R o A) is defined as the set of all points x such that* $\lim_{t\to\infty} \Phi(t, x) = 0$.

Lyapunov function can be used to estimate the R o A.
If there is a Lyapunov function satisfying asymptotic stability over domain D and the set

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$$

is bounded and contained in D, then all trajectories starting in Ω_c remains there and converges to 0 as $t \to \infty$.

4.4 LaSalle's invariance principle

Lasalle's theorem enables one to conclude asymptotic stability of an equilibrium point even when one can't find a function $V(x)$ such that $V'(x, t)$ is locally negative definite. However, it applies only to time-invariant or periodic systems. We will deal with the time-invariant case and begin by introducing a few more definitions.

Definition 4.4.1 *The set $S \subset \mathbb{R}^n$ is the limit set of a trajectory $x(t, x_0)$ if for every $p \in S$, there exists a strictly increasing sequence of times t_n such that $x(t_n, x_0) \to p$ as $t_n \to \infty$.*

Definition 4.4.2 *A set M is said to be a positive invariant set with respect to the system (4.1.1), if*

$$x_0 \in M \quad implies \quad x(t, x_0) \in M, \quad for \quad all \quad t \geq 0.$$

A set M is said to be an invariant set with respect to the system (4.1.1), if

$$x_0 \in M \quad implies \quad x(t, x_0) \in M, \quad for \quad all \quad t \in \mathbb{R}.$$

Example 4.4.3 *Set of equilibrium points, and set of limit cycles are invariant sets. Also, the set*

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \le c\} \quad withV' \le 0, \quad forall \quad x \in \Omega$$

is a positive invariant set.

Lemma 4.4.4 *[10] Suppose $\Omega \subseteq \mathbb{R}^n$ is open, $f : \Omega \to \mathbb{R}^n$ is locally Lipschiptz function and $V : \Omega \to \mathbb{R}$ is a C^1 function and bounded from below. Suppose that $p \in \Omega$ and for all $t \ge 0$, $x(t,p)$ is defined and is contained in Ω. Then*

$$for \quad all \quad q \in L^+, \quad V'(q) = 0.$$

Proof .
If $L^+ = \emptyset$ then there is nothing to prove. Suppose $q_1 = q_3 \in L^+$, $q_2 \in L^+$ and fix any $\epsilon > 0$. Since V is continuous at q_1 and at q_2, there exists $\delta > 0$ such that for all $x \in \Omega$, for $i = 1, 2$, if

$$\|x - q_i\| < \delta \quad then \quad |V(x) - V(q_i)| < \frac{\epsilon}{4}.$$

Since $q_i \in L^+$, $i = 1, 2$, there exists $0 \le t_1 \le t_2 \le t_3$ such that for $i = 12, 3$, $\|x(t,p) - q_i\| < \delta$, and hence for

$$i = 1, 2, 3, \quad |V(x(t,p)) - V(q_i)| < \frac{\epsilon}{4}.$$

This implies that

$$|V(x(t_3,p)) - V(x(t_1,p))| \le |V(x(t_3,p)) - V(q_1)| + |V(q_1) - V(x(t_1,p))| < \frac{\epsilon}{2}.$$

Since V' does not change the sign this implies

$$\begin{aligned} |V(x(t_2,p)) - V(x(t_1,p))| &= |\int_{t_1}^{t_2} V'(x(s,p))ds| \\ &\le |\int_{t_1}^{t_3} V'(x(s,p))ds| \\ &\le |V(x(t_3,p)) - V(x(t_1,p))| < \frac{\epsilon}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} |V(q_2) - V(q_1)| &\le |V(q_2) - V(x(t_2,p))| + |V(x(t_2,p)) - V(x(t_1,p))| + |V(x(t_1,p)) - V(q_1)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this establishes that V is constant on L^+.
Now use that L^+ is invariant, $q \in L^+$ and all $t \in \mathbb{R}$, $V(x(t,q)) = V(q)$ and therefore

$$V'(q) = \frac{d}{dt}|_{t=0} \frac{1}{t}(V(x(t,q)) - V(q)) = 0,$$

proving that V' vanishes identically on L^+. ∎

Theorem 4.4.5 *[10](**LaSalle's Theorem**) Let f be a locally Lipschiptz function defined over a domain $D \subset \mathbb{R}^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to $x' = f(x)$. Let $V(x)$ be a C^1 function defined over D such that $V'(x) \le 0$ in Ω. Let E be the set of all points in Ω where $V'(x) = 0$, and M be the largest invariant set in E. Then, every solution starting in Ω approaches M as $t \to \infty$, means that $d(x(t,x_0), M) \underset{t\to\infty}{\to} 0$, for all $x_0 \in \Omega$.*

Proof .
Suppose $p \in \Omega$ and consider the curve $t \to x(t,p)$. Since V is continuous it is bounded from below from on the compact set Ω. Since Ω is invariant by hypothesis,

$$for \quad all \quad t \geq 0, \quad x(t,p) \in \Omega.$$

More over, using that Ω is compact , and hence bounded, the positive limit set L^+ is non-empty. Since Ω is closed,

$$L^+ \subseteq \Omega.$$

Using the Lemma above,

$$for \quad all \quad q \in L^+, \quad V'(q) = 0$$

and therefore,

$$L^+ \subseteq \{y : V'(y) = 0\} = E \subseteq M.$$

Since the positive limit sets are invariant,

$$L^+ \subseteq M.$$

Since

$$L^+ \subseteq M,$$

it follows that

$$q \in M.$$

Thus, the solution curve starting at an arbitrary $p \in \Omega$ approaches M as $t \to \infty$. ■

Theorem 4.4.6 *[4] Let f be a locally Lipschitz function defined over a domain $D \subset \mathbb{R}^n$; $\quad 0 \in D$. Let $V(x)$ be a C^1 positive definite function defined over D such that $V'(x) \leq 0$ in $D - \{0\}$. Let*

$$\Gamma = \{x \in D : V'(x) = 0\}$$

(i) If no solution can stay identically in Γ, other than the trivial solution $x(t) = 0$, then the origin is asymptotically stable.
(ii) Moreover, if $\Gamma \subset D$ is compact and positively invariant, then it is a subset of the region of attraction.
(iii) Furthermore, if $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable.

Lasalle's theorem can also extend the Lyapunov theorem in three different directions.
(1) It gives an estimate of the RoA not necessarily in the form of

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}, \tag{4.4.1}$$

the set can be a positively invariant set which leads to less conservative estimate.
(2) Can determine stability of equilibrium set, rather than isolated equilibrium points.
(3) The function $V(x)$ does not have to be positive definite.

4.5 Barbashin and Krasorskii Corollaries

Corollary 4.5.1 *[4] Let $x = 0$ be an equilibrium point of $x' = f(x)$. Let $V : D \to \mathbb{R}$ be a C^1, positive definite function on a domain D containing the origin $x = 0$, such that $V'(x) \leq 0$ in D. Let*

$$S = \{x \in D : V' = 0\}$$

and suppose that no solution can stay identically in S, other than the trivial solution $x(t) = 0$. Then, the origin is asymptotically stable.

Proof .
$$S = \{x \in D : V' = 0\} = \{0\}$$
since no solution can stay identically in S, other than the trivial solution $x(t) = 0$. So, we have that V is a C^1, positive definite with V' negative definite. Thus, by Lyapunov theorem the origin is asymptotically stable. ■

Corollary 4.5.2 *[4] Let $x = 0$ be an equilibrium point of $x' = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1, radially unbounded, positive definite function such that $V'(x) \leq 0$, for all $x \in \mathbb{R}^n$. Let*
$$S = \{x \in \mathbb{R}^n : V' = 0\}$$
and suppose that no solution can stay in S forever except at $x = 0$. Then, the origin is asymptotically stable.

Proof .
$$S = \{x \in D : V' = 0\} = \{0\}$$
since no solution can stay identically in S, other than the trivial solution $x(t) = 0$. So, we have that V is a C^1, positive definite, and a radially unbounded function with V' negative definite. Thus, by the Barbashin and Krasorskii theorem the origin is globally asymptotically stable. ■

Example 4.5.3 *Consider $x' = -g(x)$ where $g(x)$ is locally Lipschitz on $(-a, a), a > 0$ and*
$$g(0) = 0, \quad xg(x) > 0$$
for all $x \neq 0, x \in (-a, a)$. Origin is an equilibrium point. Then the origin is globally asymptotically stable.
Infact, consider the function,
$$V(x) = \int_0^x g(y)dy$$
over $D = (-a, a)$. Then, $V(x)$ is a C^1 function, and $V(0) = 0, V(x) > 0$, for all $x \neq 0$. Also,
$$\begin{aligned} V'(x) &= \frac{\partial V}{\partial x}.f(x) = g(x)f(x) = -g^2(x) \\ &\implies V'(x) = -g^2(x) < 0, \quad for \quad all \quad x \in D - \{0\}. \end{aligned}$$
Therefore, $V(x)$ is a valid Lyapunov function. Thus, the origin is asymptotically stable.

Example 4.5.4 ***(Pendulum with friction)***
$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{g}{I}\sin x_1 - \frac{k}{m}x_2 \end{cases} \tag{4.5.1}$$
the origin is an equilibrium point of the above system. The origin is asymptotically stable.

Infact, consider the function,
$$V(x) = \frac{1}{2}x^T Px + \frac{g}{I}(1 - \cos x_1)$$
where $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ is positive definite, that is $(P_{11} > 0, P_{22} > 0, P_{11}P_{22} - P_{12}^2 > 0)$
$$\begin{aligned} V' &= \frac{1}{2}((x^T)'Px + x^T Px') + (\frac{g}{I}\sin x_1).x_1' \\ &= \frac{g}{I}x_2 \sin x_1(1 - P_{22}) + x_2x_1(P_{11} - \frac{k}{m}P_{12}) + x_2^2(P_{12} - \frac{k}{m}P_{22}) - \frac{g}{I}P_{12}(x_1 \sin x_1) \end{aligned}$$

Let, $1 - P_{22} = 0, P_{11} - \frac{k}{m} P_{12} = 0$, and $0 < P_{12} < \frac{k}{m}$, then in particular,

$$P_{22} = 1, P_{11} = \frac{k}{m} P_{12} = \frac{k^2}{2m^2}, P_{12} = \frac{k}{2m}.$$

Then, we have that P is positive definite, and V' is reduced to

$$V' = -x_2^2 \frac{k}{2m} - \frac{gk}{2mI}(x_1 \sin x_1).$$

$x_1 \sin x_1 > 0, \forall x_1$ such that $0 < \|x_1\| < \pi$, defining a domain D by

$$D = \{x \in \mathbb{R}^2 : \|x_1\| < \pi\}.$$

Note: $k, m, g, I > 0$.
Therefore, $V(x)$ is positive definite and V' is negative definite over D. Thus, the origin is asymptotically stable by the Lyapunov Theorem.

Example 4.5.5

$$\begin{cases} x_1' = x_2 \\ x_2' = -h(x_1) - ax_2 \end{cases}$$

where $a > 0$, $h(.)$ *is locally Lipschitz,* $h(0) = 0, \quad yh(y) > 0, \quad for \quad all$
$y \neq 0,\ y \in (-b, c), \quad b, c > 0.$
Find the proper Lyapunov function.

Using variable gradient method

$$\begin{aligned} V'(x) = (\frac{\partial V}{\partial x}) f(x) &= (\frac{\partial V}{\partial x_1}) f_1(x) + (\frac{\partial V}{\partial x_2}) f_2(x) \\ &= g_1(x) f_1(x) + g_2(x) f_2(x) \\ &= g_1(x) x_2 + g_2(x)(-h(x_1) - ax_2) \end{aligned}$$

We select $g(x)$ such that $V' < 0$ and $V(x) = \int_0^x g^T(y) dy > 0$ for $x \neq 0$.
Choose $g(x) = \begin{pmatrix} \alpha(x) x_1 + \beta(x) x_2 \\ \gamma(x) x_1 + \delta(x) x_2 \end{pmatrix}$
where $\alpha, \beta, \gamma, \delta$ are to be determined.
To satisfy

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

we need

$$\beta(x) + \frac{\partial \beta(x)}{\partial x_2}.x_2 + \frac{\partial \alpha(x)}{\partial x_2}.x_1 = \gamma(x) + \frac{\partial \gamma(x)}{\partial x_1}.x_1 + \frac{\partial \delta(x)}{\partial x_1}.x_2$$

which implies that $\beta = \gamma$.

$$\begin{aligned} V'(x) = g_1(x)x_2 + g_2(x)(-h(x_1) - ax_2) &= (\alpha(x)x_1 + \beta(x)x_2)x_2 + (\gamma(x)x_1 + \delta(x)x_2)(-h(x_1) - ax_2) \\ &= x_2(\alpha(x)x_1 - a\gamma(x)x_1 - \delta(x)h(x_1)) - x_2^2(a\delta(x) - \beta(x)) - \gamma(x)h_1(x_1)x_1 \end{aligned}$$

Let, $\alpha(x)x_1 - a\gamma(x)x_1 - \delta(x)h(x_1) = 0$, then

$$V'(x) = -x_2^2(a\delta(x) - \beta(x)) - \gamma(x)h(x_1)x_1$$

and $g(x) = \begin{pmatrix} a\gamma(x)x_1 + \delta(x)h(x_1) + \gamma(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{pmatrix}$.
By integration, we have

$$\begin{aligned} V(x) &= \int_0^{x_1} (a\gamma(x)y_1 + \delta(x)h(y_1) + \gamma(x)x_2)dy_1 + \int_0^{x_2} (\gamma(x)x_1 + \delta(x)y_2)dy_2 \\ &= \frac{a\gamma(x)x_1^2}{2} + \frac{\gamma(x)x_1x_2}{2} + \frac{\delta(x)x_2^2}{2} + \delta(x)\int_0^{x_1} h(y)dy \\ &= \frac{x^T P x}{2} + \delta(x)\int_0^{x_1} h(y)dy \end{aligned}$$

where $P = \begin{pmatrix} a\gamma(x) & \gamma(x) \\ \gamma(x) & \delta(x) \end{pmatrix}$
choosing $\delta(x), a\gamma(x) > 0$, and $a\gamma(x)\delta(x) - \gamma(x)^2 > 0$. We choose $\gamma(x)$ such that $0 < \gamma(x) < a\delta(x)$. This implies that V is positive definite .
Also,

$$V'(x) = -x_2^2(a\delta(x) - \beta(x)) - \gamma(x)h(x_1)x_1 = -x_2^2(a\delta(x) - \gamma(x)) - \gamma(x)h(x_1)x_1 < 0, \quad x \neq 0$$

Thus, V' is negative definite, and V is positive definite.
Taking $P = \delta(x)\begin{pmatrix} ka^2 & ka \\ ka & 1 \end{pmatrix}$, $0 < k < 1, \quad \delta(x), \quad a > 0$

$$V(x) = \frac{x^T}{2}\delta(x)\begin{pmatrix} ka^2 & ka \\ ka & 1 \end{pmatrix} x + \delta(x)\int_0^{x_1} h(y)dy$$

over $D = \{x \in \mathbb{R}^n : -b < x_1 < c\}$. Conditions of the theorem are satisfied. Thus, the origin is asymptotically stabe.

Example 4.5.6 *Consider*

$$\begin{cases} x_1' = x_2 \\ x_2' = -g(x_1) - h(x_2) \end{cases}$$

where $g(.), h(.)$ are locally Lipschitz and satisfy $g(0) = 0, yg(y) > 0$, for all $y \neq 0, y \in (-a, a)$
$h(0) = 0, yh(y) > 0$, for all $y \neq 0, y \in (-a, a), \quad a > 0$
The system has an isolated equilibrium point at origin. The origin is globally asymptotically stable.

To see this, let

$$V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(y)dy$$

with $D = \{x \in \mathbb{R}^2 : -a < x_i < a, i = 1, 2\}$. Then, $V(x) \geq 0$ on D. So,

$$V'(x) = g(x_1)x_2 + x_2(-g(x_1) - h(x_2)) = -x_2h(x_2) < 0, \quad x_2 \neq 0.$$

Thus, $V'(x)$ is negative semi definite, and the origin is stable by Lyapunov theorem.
Using Lasalle's theorem, define

$$S = \{x \in D : V' = 0\}$$

$V' = 0$ implies that $x_2h(x_2) = 0$, and this implies that $x_2 = 0$.
Hence,

$$S = \{x \in D : x_2 = 0\}.$$

Suppose $x(t)$ is a trajectory in S, for all t, then $x_2 = 0 \implies x_1' = 0 \implies x_1 = c$, where c is a constant in $(-a, a)$. Also, $x_2 = 0$ implies that $x_2' = 0$ which implies that $g(c) = 0$, this then

implies that $c = 0$.
Therefore,, the only solution that can stay in S, for all $t \geq 0$ is the origin. Thus, the origin is asymptotically stable.
Now, let $a = \infty$ and assume g satisfy

$$\int_0^x g(y)dy \to \infty \quad as \quad \|x\| \to \infty,$$

then the Lyapunov function

$$V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(y)dy$$

is radially unbounded, and $V' \leq 0$ in $\mathbb{R}^2$. Thus, the origin is globally asymptotically stable.

4.6 Linear systems and linearization

Given $x' = Ax$, the equilibrium point is at origin. It is isolated iff $detA \neq 0$. System has an equilibrium subspace if $detA = 0$, the subspace is the null space of A.
The linear system cannot have multiple isolated equilibrium point since, linearity requires that if x_1 and x_2 are equilibrium points, then all points on the line connecting them should also be equilibrium points.
Recall that the equilibrium point $x = 0$ of $x' = Ax$ is stable iff all eigenvalues λ of A satisfy $Re(\lambda) < 0$, asymptotic stability can be verified by using Lyapunov method:
- Consider a quadratic Lyapunov function candidate

$$V(x) = x^T Px, \quad P = P^T > 0$$

then,

$$V' = (x^T)' Px + x^T Px' = (Ax)^T Px + x^T PAx = x^T(A^T P + PA)x = -x^T Qx,$$

where $-Q = A^T P + PA; \quad Q = Q^T$, Lyapunov equation.
- If Q is positive definite, then we conclude that $x = 0$ is globally asymptotically stable.
- We can proceed alternatively as follows:
- Start by choosing $Q = Q^T$, $Q > 0$, then solve the Lyapunov equation for P.
- If $P > 0$, then $x = 0$ is globally asymptotic stable.

Theorem 4.6.1 *[4] A matrix $A \in M_{n\times n}(\mathbb{R})$ is a stable one, that is , $Re(\lambda_i, i = 1, 2, ..., n) < 0$ if and only if for every given $Q = Q^T > 0$, there exists $P = P^T > 0$ that satisfies the Lyapunov equation. Moreover, if A is a stable matrix, then P is unique.*

Proof.
Assume that for every positive definite $Q = Q^T$ there exists a positive definite $P = P^T$ such that

$$A^T P + PA = -Q$$

is satisfied, consider

$$V(x) = x^T Px,$$

then V is positive definite, and V' is negative definite, and also V is radially unbounded. So $x = 0$ is asymptotically stable for $x' = Ax$. Thus,

$$Re(\lambda_i) < 0.$$

Conversely, suppose $Re(\lambda_i, i = 1, 2, ..., n) < 0$, and $Q = Q^T$ is positive definite. Define

$$P = \int_0^\infty e^{tA^T} Q e^{tA} dt$$

This integral converges absolutely because the norm of the integrand is bounded by a function of the form $Ct^N e^{-\alpha t}$ for some constants $C, \quad N, \quad \alpha$. To show that P is positive definite, suppose $x \in \mathbb{R}^n$ is such that $x^T P x = 0$. Then

$$\int_0^\infty (e^{tA}x)^T Q (e^{tA}x) dt = 0$$

since Q is positive definite this implies that for all $t \geq 0, \quad e^{tA}x = 0$. Since for every t, e^{tA} is invertible, this implies $x = 0$ proving that P is positive definite. To show that $V(x) = x^T P x$ has the desired derivative along solutions, first note that

$$V'(x) = x^T (PA + A^T P) x.$$

Now,

$$PA + A^T P = \int_0^\infty (e^{tA^T} Q e^{tA} A + A^T e^{tA^T} Q e^{tA}) dt = \int_0^\infty \frac{d(e^{tA^T} Q e^{tA})}{dt} dt = e^{tA^T} Q e^{tA}|_0^\infty = -Q$$

To show uniqueness, suppose $P, P^* \in \mathbb{R}^{n \times n}$ are both positive definite, and satisfy

$$PA + A^T P = P^* A + A^T P^* = -Q.$$

Then

$$(P - P^*)A + A^T (P - P^*) = 0,$$

and hence for every $t \geq 0$

$$e^{tA^T}((P - P^*)A + A^T (P - P^*))e^{tA} = 0$$

which implies that

$$\frac{d(e^{tA^T}(P - P^*)e^{tA})}{dt} = 0.$$

Thus,

$$for \quad all \quad t \geq 0, \quad e^{tA^T}(P - P^*)e^{tA} = e^{0.A^T}(P - P^*)e^{0.A} = P - P^*.$$

On the other hand, since we assumed that the origin is asymptotically stable,

$$P - P^* = \lim_{t \to \infty} e^{tA^T}(P - P^*)e^{tA} = 0.$$

Thus, $P = P^*$.

Example 4.6.2 *Let* $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Q^T > 0$, *and* $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} = P^T > 0$ *The Lyapunov equation* $A^T P + PA = -Q$ *becomes*

$$\begin{cases} 2P_{12} = -1 \\ -P_{11} - P_{12} + P_{22} = 0 \\ -2P_{12} - 2P_{22} = -1 \end{cases} \tag{4.6.1}$$

which implies $\begin{pmatrix} P_{11} \\ P_{12} \\ P_{22} \end{pmatrix} = \begin{pmatrix} 1.5 \\ -0.5 \\ 1 \end{pmatrix}$

so, $P = P^T = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{pmatrix} > 0$. *Hence,* $x = 0$ *is globally asymptotically stable.*

Remark 4.6.3 *Computationally, there is no advantage in computing the eigenvalues of A over solving Lyapunov equation.*

Consider $x' = f(x)$ where $f : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$, is continuously differentiable. Let $x = 0$ be in the interior of D and $f(0) = 0$. In a small neighborhood of $x = 0$, the nonlinear system

$$x' = f(x)$$

can be linearized by

$$x' = Ax.$$

Theorem 4.6.4 *[4] Let $x = 0$ be an equilibrium point for $x' = f(x)$ where $f : D \to \mathbb{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let $A = \frac{\partial f}{\partial x}|_{x=0}$, then*
(1) $x = 0$ is asymptotically stable if $Re(\lambda_i) < 0, i = 1, 2, 3, ..., n$
(2) $x = 0$ is unstable if $Re(\lambda_i) > 0$, for one or more eigenvalues.

Proof .
- Let D be a small neighborhood of $x = 0$, then

$$x' = f(x)$$

can be linearized by

$$x' = Ax.$$

So, the solution of the system

$$x' = Ax$$

is

$$x(t, x_0) = e^{tA} x_0.$$

Let λ be the eigenvalues of A, then for some $M \geq 1$, and $\alpha > 0$

$$\begin{aligned} \|x(t, x_0)\| &= \|e^{tA}\| \|x_0\| \\ &\leq M e^{-\alpha t} \| \|x_0\| \\ &< M \|x_0\|. \end{aligned}$$

Take $\delta = \frac{\epsilon}{M+1}$, then

$$\|x(t, x_0)\| \leq M \|x_0\| < M\delta < \epsilon.$$

So, $x = 0$ is stable. Also,

$$\|x(t, x_0)\| \leq M e^{-\alpha t} \|x_0\| \to 0, \quad as \quad t \to \infty,$$

since $Re(\lambda) < 0$. So,

$$x(t, x_0) \to 0, \quad as \quad t \to \infty.$$

Thus, $x = 0$ is asymptotically stable.
- If $Re(\lambda_0) > 0$, for one or more eigenvalues, say λ_0, then for λ_0,

$$\|x(t, x_0)\| = \|e^{tA} x_0\| = \|e^{\lambda_0 t}\| \|x_0\| = \|e^{Re\lambda_0 t}\| \|x_0\| \to \infty \quad as \quad t \to \infty.$$

Thus, $x = 0$ is unstable. ■

Example 4.6.5

$$x' = ax^3$$

Linearization about $x = 0$ yields:

$$A = \frac{\partial f}{\partial x}|_{x=0} = 3ax^2|_{x=0} = 0$$

Linearization fails to determine stability. If $a < 0$, then $x = 0$ is asymptotically stable. To see this, $V(x) = x^4$, implies that $V' = 4x^3 x' = 4ax^6$.
If $a > 0$, $x = 0$ is unstable. If $a \leq 0$, $x = 0$ is stable, starting at any x, remains in x.

Example 4.6.6

$$\begin{cases} x_1' = x_2 \\ x_2' = -(\frac{g}{I})\sin x_1 - (\frac{k}{m}x_2) \end{cases}$$

Linearization about 2 equilibrium points $(0,0)$, *and* $(\pi, 0)$*:*

$A = \frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{I}\cos x_1 & -\frac{k}{m} \end{pmatrix}.$

At $(0,0)$ $A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{I} & -\frac{k}{m} \end{pmatrix}$. *So, solving for eigenvalues we get*

$\lambda_{1,2} = -\frac{k}{2m} + \frac{\sqrt{(\frac{k}{m}^2 - \frac{4g}{I})}}{2}$. *Therefore, for all* $g, k, I, m > 0$ $\quad Re(\lambda_1, \lambda_2) < 0$. *Thus,* $x = 0$ *is asymptotically stable.*

If $k = 0$, *then* $Re(\lambda_1, \lambda_2) = 0$. *Therefore, stability cannot be determined.*

At $(\pi, 0)$, *change the variable to* $z_1 = x_1 - \pi, \quad z_2 = x_2$,

$A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{I} & -\frac{k}{m} \end{pmatrix}$, *implies that* $\lambda_{1,2} = -\frac{k}{2m} + \frac{\sqrt{(\frac{k}{m}^2 + \frac{4g}{I})}}{2}$. *Therefore, for all* $g, k, I, m > 0$, *there is one eigenvalue in the open right-half plane.Thus,* $x = 0$ *is unstable.*

CHAPTER 5

MORE APPLICATIONS

Introduction

In this chapter, we discussed various applications of Lyapunov Theorem, and LaSalle's invariance principle.

Example 5.0.1 *[6](**Robot Manipulator**) Dynamics:*

$$M(q)q'' + C(q,q')q' + Bq' + g(q) = u \tag{5.0.1}$$

where $M(q)$ is the $n \times n$ inertia matrix of the manipulator, $C(q,q')q'$ is the vector of coriolis and centrifugal forces, $g(q)$ is the term due to the gravity, Bq' is the viscous damping term, u is the input torque, usually provided by a DC motor.
Objective: To regulate the joint position q around desired position q_d.
A common control strategy PD + gravity:

$$u = K_P q^* - K_D q' + g(q) \tag{5.0.2}$$

where $q^* = q_d - q$ is the error between the desired and actual position. K_P and K_D are diagonal positive proportional and derivative gains.
Consider the following Lyapunov function candidate:

$$V = \frac{(q')^T M(q) q'}{2} + \frac{(q^*)^T K_P q^*}{2} \tag{5.0.3}$$

The first is the kinetic energy of the robot and the second term accounts for "artificial potential energy" associated with virtual spring in PD control law (proportional feedback $K_P q^*$)
Physical properties of a robot manipulator:
1. The inertia matrix $M(q)$ is positive definite
2. The matrix $M'(q) - 2C(q,q')$ is skew symmetric.
V is positive in $\mathbb{R}^n$ except at the goal position $q = q_d, \quad q' = 0$

$$V' = (q')^T M(q) q'' + \frac{(q')^T M'(q) q'}{2} + (q')^T K_P q^* \tag{5.0.4}$$

Substituting $M(q)q''$ from (5.0.1) into the above equation yields

$$\begin{aligned}
V' &= (q')^T(u - C(q,q')q' - Bq' - g(q)) + \frac{(q')^T M'(q) q'}{2} - (q')^T K_P q^* \\
&= (q')^T(u - C(q,q')q' - Bq' - g(q)) + \frac{(q')^T M'(q) q'}{2} - (q')^T (u + K_D q' - g(q)) \\
&= (q')^T(-C(q,q')q' - K_D q' - Bq') + \frac{(q')^T M'(q) q'}{2} \\
&= -\frac{(q')^T [2C(q,q')q'] q'}{2} + \frac{(q')^T M'(q) q'}{2} - (q')^T (Bq' + K_D q') \\
&= \frac{(q')^T [M'(q) - 2C(q,q')q'] q'}{2} - (q')^T (B + K_D) q' \\
&= -(q')^T (B + K_D) q' \leq 0.
\end{aligned}$$

So, V is non increasing, and thus the goal position is stable.
Use the invariant set Theorem:
Suppose $V' = 0$, then $V' = -(q')^T(B+K_D)q'$ implies that $q' = 0$ and hence $q'' = 0$. From equation (5.0.1) with (5.0.2), we have that

$$M(q)q'' + C(q,q')q' + Bq' = K_P q^* - K_D q' \tag{5.0.5}$$

we must then have $K_P q^* = 0$ which implies that $q^* = 0$. V is radially unbounded. Therefore, global asymptotic stability is ensured.
In case, the gravitational terms is not canceled, V' is modified to:

$$V' = -(q')^T((B + K_D)q' + g(q)) \tag{5.0.6}$$

The presence of gravitational term means PD control alone cannot guarantee asymptotic tracking. Assuming that the closed loop system is stable, the robot configuration q will satisfy

$$K_P(q_d - q) = g(q) \tag{5.0.7}$$

The physical interpretation of the above equation is that:
The configuration q must be such that the motor generates a steady state "holding torque" $K_P(q_d - q)$ sufficient to balance the gravitational torque $g(q)$. Therefore, the steady state error can be reduced by increasing K_P.

5.1 Control design based on lyapunov's direct method

Basically there are two approaches to design control using Lyapunov's direct method
- Choose a control law, then find a Lyapunov function to justify the choice
- Candidate a Lyapunov function, then find a control law to satisfy the Lyapunov stability conditions.

Both methods have a trial and error flavor. In robot manipulator example the first approach was applied:
First a PD controller was choosen based on physical intuition. Then a Lyapunov function is found to show globally asymptotic stability.

Example 5.1.1 *(**Regulator design**) Consider the problem of stabilizing the system:*

$$x'' - (x')^3 + x^2 = u \tag{5.1.1}$$

In other word, make the origin an asymptotically stable equilibrium point
Recall the example:

$$\begin{cases} x_1' = x_2 \\ x_2' = -g(x_1) - h(x_2) \end{cases} \tag{5.1.2}$$

where $g(.)$ and $h(.)$ are locally Lipschitz and satisfy $g(0), h(0) = 0, \quad yg(y), yh(y) > 0$ for all $y \in (-a, a), a > 0$.
Asymptotic stability of such system could be shown by selecting the following Lyapunov function:

$$V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(y)dy. \tag{5.1.3}$$

Let $x_1 = x, x_2 = x'$. The above example motivates us to select the control law u as

$$u = u_1(x') + u_2(x) \tag{5.1.4}$$

where

$$x'((x')^3 + u_1(x')) < 0, x' \neq 0, \quad x(u_2(x) - x^2) < 0, x \neq 0.$$

The globally stabilizing controller can be designed even in the presence of some uncertainties on the dynamics:

$$x'' + \alpha_1(x')^3 + \alpha_2 x^2 = u \tag{5.1.5}$$

where α_1 and α_2 are unknown, such that $\alpha_1 > -2$, and $\|\alpha_2\| < 5$. This system can be globally stabilized using the control law:

$$u = -2(x')^3 - 5(x + x^3) \tag{5.1.6}$$

Sometimes just knowing a system is asymptotically stable is not enough. At least an estimation of RoA is required.
Let $x = 0$ be an equilibrium point of $x' = f(x)$. Let $\Phi(t, x)$ be the solution starting at x at time $t = 0$. The region of attraction (RoA) of the origin denoted by R_A is defined by:

$$R_A = \{x \in \Re^n : \Phi(t, x) \to 0 \quad as \quad t \to \infty\}. \tag{5.1.7}$$

Example 5.1.2 ***(Van-der-Pol)*** *Dynamics of oscillator in reverse time*

$$\begin{cases} x_1' = -x_2 \\ x_2' = x_1 + (x_1^2 - 1)x_2 \end{cases} \tag{5.1.8}$$

Checking by linearization method
$A = \frac{\partial f}{\partial x}|_{x=0} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ *which implies that* $\lambda_1 = \frac{-1+j\sqrt{3}}{2}$ *and* $\lambda_2 = \frac{-1-j\sqrt{3}}{2}$. *Thus,*

$$Re(\lambda_i) < 0, i = 1, 2.$$

Hence, the origin is asymptotically stable.

Example 5.1.3

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 + \frac{x_1^3}{3} - x_2 \end{cases} \tag{5.1.9}$$

There are 3 isolated equilibrium points $(0, 0), (\sqrt{3}, 0), (-\sqrt{3}, 0)$.
Checking by linearization method
At $(0, 0)$ $\quad A = \frac{\partial f}{\partial x}|_{x=(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ *which implies that* $\lambda_1 = \frac{-1+j\sqrt{3}}{2}$ *and* $\lambda_2 = \frac{-1-j\sqrt{3}}{2}$. *Thus, $Re(\lambda_i) < 0, i = 1, 2$. Hence, the origin is asymptotically stable.*
At $(\sqrt{3}, 0)$ $\quad A = \frac{\partial f}{\partial x}|_{x=(\sqrt{3},0)} = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}$ *which implies that $\lambda_1 = 1 > 0$ and $\lambda_2 = -2$. Thus, $(\sqrt{3}, 0)$ is not stable.*
At $(-\sqrt{3}, 0)$ $\quad A = \frac{\partial f}{\partial x}|_{x=(-\sqrt{3},0)} = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}$ *which implies that $\lambda_1 = 1 > 0$ and $\lambda_2 = -2$. Thus, $(-\sqrt{3}, 0)$ is not stable.*

Example 5.1.4 *Recall the example:*

$$\begin{cases} x_1' = x_2 \\ x_2' = -h(x_1) - ax_2 \end{cases} \tag{5.1.10}$$

$$V = \frac{\delta}{2} x^T \begin{pmatrix} ka^2 & ka \\ ka & 1 \end{pmatrix} x + \delta \int_0^{x_1} h(y)dy \tag{5.1.11}$$

Let,

$$V = \frac{\delta}{2} x^T \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} x + \int_0^{x_1} (y - \frac{1}{3}y^3)dy = \frac{3x_1^2}{4} - \frac{x_1^4}{12} + \frac{1}{2}x_1x_2 + \frac{x_2^2}{2}.$$

We get,

$$V' = -\frac{x_1^2}{2}(1 - \frac{x_1^2}{3}) - \frac{1}{2}x_2^2.$$

Define

$$D = \{x \in \mathbb{R}^2 : -\sqrt{3} < x_1 < \sqrt{3}\}.$$

Therefore, $V(x) > 0$ and $V'(x) < 0$ in $D - \{0\}$. D is not a subset of R_A. Trajectory starting in D move from one Lyapunov surface to $V(x) = c_1$ to an inner surface $V(x) = c_2$ with $c_2 < c_1$. However, there is no guarantee that the trajectory will remain in D forever. Once, the trajectory leaves D, no guarantee that V' remains negative. This problem does not occur in R_A since R_A is an invariant set. The simplest stimate is given by the set

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\} \tag{5.1.12}$$

where Ω_c is bounded and connected and $\Omega_c \in D$. To find RoA, first we need to find a domain D in which V' is negative definite, then a bounded set $\Omega_c \subset D$ shall be sought. We are interested in largest set Ω_c, that is the largest value of c since Ω_c is an estimate of R_A. V is positive definite everywhere in $\mathbb{R}^2$. If $V(x) = x^T Px$, let

$$D = \{x \in \mathbb{R}^2 : \|x\| \leq r\}.$$

Once D is obtained, then select

$$\Omega_c \subset D$$

by

$$c < \min_{\|x\|=r} V(x).$$

In other words, the smallest $V(x) = c$ which fits into D. Since

$$x^T Px \geq \lambda_{\min}(P)\|x\|^2 \tag{5.1.13}$$

We can choose

$$c < \lambda_{\min}(P)r^2 \tag{5.1.14}$$

To enlarge the estimate of R_A implies find largest ball on which V' is negative definite.

Example 5.1.5

$$\begin{cases} x_1' = -x_2 \\ x_2' = x_1 + (x_1^2 - 1)x_2 \end{cases} \tag{5.1.15}$$

From the linearization principle
$A = \frac{\partial f}{\partial x}|_{x=0} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, *the origin is stable.*
Taking $Q = I$ and then solving the Lyapunov equation

$$PA + A^T P = -I$$

we give

$$P = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{pmatrix}.$$

$\lambda_{\min}(P) = 0.69,$

$$\begin{aligned} V' &= -(x_1^2 + x_2^2) - (x_1^3 x_2 - 2x_1^2 x_2^2) \\ &\le -\|x\|_2^2 + \|x_1\| \|x_1 x_2\| \|x_1 - 2x_2\| \\ &\le -\|x\|_2^2 + \frac{\sqrt{5}}{2} \|x\|_2^4 \end{aligned}$$

where $|x_1| \le \|x\|_2, \quad |x_1 x_2| \le \frac{\|x\|_2^2}{2}, \quad |x_1 - 2x_2| \le \sqrt{5}\|x\|_2$*.* V' *is negative definite on a ball* D *of radius* $r^2 = \frac{2}{\sqrt{5}} = 0.894$*, so* $c < 0.894 \times 0.69 = 0.617$*. To find less conservative estimate of* Ω_c
Let $x_1 = \alpha \cos\theta, \quad x_2 = \alpha \sin\theta$

$$\begin{aligned} V' &= -\alpha^2 + \alpha^4 \cos^2\theta \sin\theta (2\sin\theta - \cos\theta) \\ &\le -\alpha^2 + \alpha^4 |\cos^2\theta \sin\theta| |2\sin\theta - \cos\theta| \\ &\le -\alpha^2 + \alpha^4 (0.3849)(2.2361) \\ &\le -\alpha^2 + 0.861\alpha^4 < 0 \end{aligned}$$

for $\alpha^2 < \frac{1}{0.861}, \quad c < 0.8 < \frac{0.69}{0.861} = 0.801$*. Thus the set:*

$$\Omega_c = \{x \in \mathbb{R}^2 : V(x) \le 0.8\} \tag{5.1.16}$$

is an estimate of R_A*.*

Example 5.1.6

$$\begin{cases} x_1' = -2x_1 + x_1 x_2 \\ x_2' = -x_2 + x_1 x_2 \end{cases} \tag{5.1.17}$$

There are two equilibrium points, $(0,0), \quad (1,2)$*.*

$$At \quad (1,2), \quad A = \begin{pmatrix} -2 + x_2 & x_1 \\ x_2 & -1 + x_1 \end{pmatrix}_{x=(1,2)} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},$$

which implies that $\lambda_1 = \sqrt{2}$*, and*
$\lambda_2 = -\sqrt{2}$*. Thus* $(1,2)$ *is unstable.*

$$At \quad (0,0), \quad A = \begin{pmatrix} -2 + x_2 & x_1 \\ x_2 & -1 + x_1 \end{pmatrix}_{x=(0,0)} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},$$

which implies that $\lambda_1 = -1$*, and* $\lambda_2 = -2$*. Thus* $(0,0)$ *is asymptotically stable.*
Taking $Q = I$ *and solving Lyapunov equation* $A^T P + PA = -I$
$\implies P = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$*. Therefore the Lyapunov function is*

$$V(x) = x^T P x,$$

which implies that

$$V' = -(x_2^2 + x_1^2) + \frac{1}{2}(x_1^2 x_2 + 2x_2^2 x_1).$$

Now we find largest D such that V' is negative definite in D.
Let $x_1 = \alpha\cos\theta, \quad x_2 = \alpha\sin\theta$

$$\begin{aligned} V' &= -\alpha^2 + \alpha^3\cos\theta\sin\theta(\sin\theta + \frac{1}{2}\cos\theta) \\ &\leq -\alpha^2 + \frac{1}{2}\alpha^3|\sin 2\theta||\sin\theta + \frac{1}{2}\cos\theta| \\ &\leq -\alpha^2 + \frac{\sqrt{5}}{4}\alpha^3 < 0 \end{aligned}$$

for $\alpha < \frac{4}{\sqrt{5}}$.
Since $\lambda_{\min}(P) = \frac{1}{4}$, *then we choose* $c = 0.79 < \frac{1}{4}\times(\frac{4}{\sqrt{5}})^2 = 0.8$. *Thus the set:*

$$\Omega_c = \{x \in \mathbb{R}^2 : V(x) \leq 0.79\} \subset R_A \tag{5.1.18}$$

Estimating RoA by the set Ω_c is simple but conservative, alternatively LaSalle's theorem can be used. It provides an estimate of R_A.

Example 5.1.7

$$\begin{cases} x_1' = x_2 \\ x_2' = -4(x_1+x_2) - h(x_1+x_2) \end{cases} \tag{5.1.19}$$

where $h : \mathbb{R} \to \mathbb{R}$ *such that* $h(0) = 0$, *and* $xh(x) \geq 0$, *for all* $\|x\| \leq 1$
Consider the Lyapunov function candidate:

$$V(x) = x^T \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x = 2x_1^2 + 2x_1x_2 + x_2^2.$$

Then

$$\begin{aligned} V' &= -2x_1^2 - 6(x_1+x_2)^2 - 2(x_1+x_2)h(x_1+x_2) \\ &\leq -2x_1^2 - 6(x_1+x_2)^2 \\ &= -x^T \begin{pmatrix} 8 & 6 \\ 6 & 6 \end{pmatrix} x, \quad for \quad all \quad |x_1+x_2| \leq 1. \end{aligned}$$

Therefore V' is negative definite in the set

$$G = \{x \in \mathbb{R}^2 : |x_1+x_2| \leq 1\},$$

and thus $(0,0)$ is asymptotically stable, to estimate R_A, first we do it from Ω_c. Find the largest c such that $\Omega_c \subset G$. Now, c is given by $c = \min_{|x_1+x_2|=1} V(x)$ or $c = \min\{\min_{x_1+x_2=1} V(x), \min_{x_1+x_2=-1} V(x)\}$
The first minimization yields

$$\min_{x_1+x_2=1} V(x) = \min_{x_1}\{2x_1^2 + 2x_1(1-x_1) + (1-x_1)^2\} = 1$$

and

$$\min_{x_1+x_2=-1} V(x) = 1.$$

Hence, Ω_c with $c = 1$ is an estimate of R_A.
A better estimate of R_A is possible. The key point is to observe that trajectory inside G cannot leave it through certain segment of the boundary $|x_1+x_2| = 1$.
Let

$$\omega = x_1 + x_2$$

then ∂G is given by $\omega = 1$ and $\omega = -1$.
We have

$$\frac{d}{dt}\omega^2 = 2\omega(x_1' + x_2') = 2\omega x_2 - 8\omega^2 - 2\omega h(\omega) \leq 2\omega x_2 - 8\omega^2, \quad for \quad all \quad |\omega| \leq 1.$$

On the boundary

$$\omega = 1 \quad implies \quad that \quad \frac{d}{dt}\omega^2 \leq 2x_2 - 8 \leq 0, \quad for \quad all \quad x_2 \leq 4.$$

Hence, the trajectory on $\omega = 1$ for which $x_2 \leq 4$ cannot move outside the set G since ω^2 is non-increasing. Similarly, on the boundary $\omega = -1$ we have

$$\frac{d}{dt}\omega^2 \leq -2x_2 - 8 \leq 0, \quad for \quad all \quad x_2 \geq -4$$

Hence, the trajectory on $\omega = -1$ for which $x_2 \geq -4$ cannot move outside the set G. To define the boundary of G, we need to find two other segments to close the set. We can take them as the segments of Lyapunov function surface. Let c_1 be such that

$$V(x) = c_1$$

intersects the boundary of

$$x_1 + x_2 = 1 \quad at \quad x_2 = 4$$

let c_2 be such that

$$V(x) = c_2$$

intersects the boundary of

$$x_1 + x_2 = -1 \quad at \quad x_2 = -4.$$

Then, we define

$$V(x) = \min(c_1, c_2),$$

we have

$$c_1 = V(x)|_{x_1=-3, \quad x_2=4} = 10,$$

and

$$c_2 = V(x)|_{x_1=3, x_2=-4} = 10$$

The set ω is defined by

$$\omega = \{x \in \mathbb{R}^2 : V(x) \leq 10, |x_1 + x_2| \leq 1\}$$

This set is closed and bounded and positively invariant. Also, V' is negative definite in Ω since $\Omega \subset G$, which implies that

$$\Omega \subset R_A.$$

Let $x(t)$ be the integral curve of f starting at x_0. Suppose $x(t)$ remains in D for $0 \leq t < T$. The equation

$$x' = Ax(t) + R(x(t)) \tag{5.1.20}$$

with initial conditions $x(0) = x_0$ has a solution that satisfies the variation of constants formula, namely

$$x(t) = \mathrm{e}^{tA}x_0 + \int_0^t \mathrm{e}^{(t-s)A}R(x(s))ds \tag{5.1.21}$$

and so

$$\|x(t)\| \leq M\mathrm{e}^{-t\epsilon}\|x_0\| + \alpha\int_0^t \mathrm{e}^{-(t-s)\epsilon}\|x(s)\|ds \tag{5.1.22}$$

letting $f(t) = \mathrm{e}^{t\epsilon}\|x(t)\|$, the previous inequality becomes

$$f(t) \leq M\|x_0\| + \alpha\int_0^1 f(s)ds \tag{5.1.23}$$

and so, by Gronwall's inequality,

$$f(t) \leq M\|x_0\|e^{\alpha t}$$

and thus,

$$\|x(t)\| \leq Me^{(\alpha-\epsilon)t}\|x_0\| = M\|x_0\|e^{-\frac{1}{2}t\epsilon}. \tag{5.1.24}$$

Therefore, $x(t) \in D, \quad 0 \leq t < T$, *so as* $x(t)$ *may be indefinitely extended in* t *and the foregoing estimates holds.*

CONCLUSION

The aim of this thesis is to study a long time behaviour solution using 3 approaches .
The first one is the linearization principle: If it works that fine, but most of cases it does not work well, like we have seen in many examples.
The second one is the Lyapunov functions, it is the best way to study the asymptotic behaviour of solutions, but the construction of the Lyapunov functions depends on the nature of the ODE.
The third one is based on LaSalle invariance principle, it is an interesting working tool in dynamical systems and control theory.

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