INTEGRATION IN LATTICE SPACES

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In Partial Fulfillment of the Requirements for the Degree of

Master of Science

by

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Integration in Lattice Spaces

Certification

This is to certify that the thesis titled ”**Integration in Lattice Spaces**” submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master’s degree is a record of original research carried out by Fatima Doumbia in the department of Pure and Applied Mathematics.

Approval

INTEGRATION IN LATTICE SPACES

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Abstract

The goal of this thesis is to extend the notion of integration with respect to a measure to Lattice spaces. To do so the paper is first summarizing the notion of integration with respect to a measure on $\mathbb{R}$.
Then, a construction of an integral on Banach spaces called the Bochner integral is introduced and the main focus which is integration on lattice spaces is lastly addressed.

Key Words. *Banach spaces, Bochner Integral, Integration, Ordered vector space, Real-valued Mapping Modern Integral, Lattice space, Young-Fatou-Lebesgue Dominated Convergence Theorem,*

Dedication

Dedicated to my parents, Mr & Mrs Doumbia

Acknowledgements

I would like to thank the Almighty God for the grace of life and intellectual abilities which allowed my to work on this thesis.
My thanks also go to my supervisor, Professor Gane Samb Lo, for his patience and guidance throughout the whole time I have been working on this topic.

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Last but not least, I thank my parents and family for the unconditional love and support towards my choice of doing a postgraduate program in pure and applied Mathematics.

Contents

General Introduction

Integration is a mathematical technique used to find areas, volumes and so many other mathematical measures.
But to make the notion of integration easy to picture, we define it simply as a mean to find area under the curve of a function; and the result of the integration is called the integral of the function. Therefore, it is not a surprise that, there are different types of integration, since the functions to be integrated have variety of properties.

The first type of integration that comes to our mind, when we talk about area under a curve is the Riemann integration named as Riemann-Stieltjes integration under its general form. However, we will see that, this integration is applicable to real-valued functions and requires some specific properties that all real-valued functions need not have. Therefore, we will introduce another type of integration, called the Lebesgue-Stieltjes integration, that will address most of the limits of the Riemann-Stieltjes Integration. The Lebesgue -Stieltjes integration gives us a mean to compute the integral of a large range of real-valued function with an undeniable property called *measurability*. So, the Lebesgue-Stieljes Integration is addressed as the integration of real-valued measurable mappings with respect to a "'measure"'. While the Riemann-Stieljes integration focus on powerful computation tools, the Lebesgue-Stieljes theory adresses powerful results of existence, limits theory, integration and differentiation under the integral sign. As such, it is unavoidable in modern analysis.

Now, the goal of this thesis is not only to expand the integration of real-valued measurable mappings to Banach spaces (complete normed vector spaces) but also to introduce the notion of integration of measurable mappings with values in Lattice spaces.

In order to achieve that goal, we proceed as follows.

In Chapter 1, we introduce the Riemann-Stieljes integration as a first step.

In Chapter 2, the dissertation summarizes Integration with respect to measure on $\mathbb{R}$, following Lo (2018). This summary is very important because it lays the ground for our targeted generalization of the integration of measurable mappings.

Then, in chapter 3, we address the integration of measurable mappings with values in Banach spaces in general. This part mainly consists of constructing the integral of Banach-valued measurable mappings which satisfies two important conditions. The obtained integral is called the Bochner integral on Banach spaces. Our main source for that part is Mikusiński (2015) while the seminal work regarding Integration Vector spaces is Pettis (1938).
In Chapter 4, we open a window on integration with respect to a measure on lattice spaces. This chapter is just an introduction to integration in

lattice spaces and it uses results obtain from previous chapters to discuss some properties of the integral with respect to a measure of mappings with values in lattices spaces. We used the materials provide by van Rooij and van Zuijlen (2013).

We end this thesis by Chapter 4 by summarizing all what have been discussed in the different chapters and we give some perspectives as regards to integration on Lattice spaces. We also give perspective of future works.

CHAPTER 1

Introduction to Integration Theory

1.1. Riemann-Stieltjes Integration

Definition of the Riemann-Stieltjes integral on a compact set

Consider an arbitrary function $f : [a, b] \to \mathbb{R}$.
The Riemann-Stieltjes integral of f on $[a, b]$ associated with F, if it exists, is denoted by:

$$I = \int_a^b f(x)\, dF(x)$$

In establishing the existence of the Riemann-Stieltjes integral of a function, we need the function to be bounded.

Next, we define the Riemann-Stieltjes sums. To do so, for each $n \geq 1$, we divide $[a, b]$ into $l(n)$ sub-intervals ($l \geq 1$).

Let π_n be a subdivision of $[a, b]$ that divides$[a, b]$ into $l(n)$ sub-intervals.
So,

$$]a, b] = \sum_{i=0}^{l(n)-1}]x_{i,n}, x_{i+1,n}],$$

where $a = x_{0,n} < x_{1,n} < ... < x_{l(n),n} = b$.

The modulus of the subdivision π_n is defined by:

$$m(\pi_n) = \max_{0 \leq i \leq l(n)-1} (x_{i+1,n} - x_{i,n})$$

Then, in each sub-interval $]x_{i,n}, x_{i+1,n}]$, we pick an arbitrary point $c_{i,n}$, we therefore have the arbitrary sequence $(c_n)_{n \geq 1}$ where, $c_n = (c_{i,n})_{1 \leq i \leq l(n)-1}$. we now define a sequence of Riemann-Stieltjes sum associated to the sub-

division π_n and the vector c_n in the form:

$$S_n(f, F, a, b, \pi_n, c_n) = \sum_{i=0}^{l(n)-1} f(c_{i,n})(F(x_{i+1,n}) - F(x_{i,n})) \tag{1.1.1}$$

in short, $S_n(\pi_n, c_n)$

Definition 1.1. *A bounded function f is Riemann-Stieltjes integrable with respect to F if there exists a real number I such that any sequence of Riemann-Stieltjes sums $S_n(\pi_n, c_n)$ converges to I as $n \to \infty$ whenever $m(\pi_n) \to 0$ as $n \to \infty$.*
The number I is called the Riemann-Stieltjes integral of f on $[a, b]$

Now, in particular, if $F(x) = x,\ x \in \mathbb{R}$, I is called the Riemann Integral of f over $[a, b]$ and the sum in formula 1.1.1 is simply called the Riemann Sum.

For the sake of a later use, Let us introduce an important notion called "'Bounded Variation Functions"'.

1.2. Bounded Variation Functions

Consider a function $F : [a, b] \to \mathbb{R}$.
We define by $\mathcal{P}(a, b)$ the class of all partition π of the interval $[a, b]$ of the form:

(1.2.1) $$\pi = (a = x_0 < x_1 < ... < x_p = b), \quad p \geq 1$$

To each $\pi \in \mathcal{P}(a, b)$ represented as in formula 1.2.1, we associate the variation of F over π define by:

$$V_F(\pi, a, b) = \sum_{j=i}^{p} |F(x_{j+1}) - F(x_j)|$$

The total variation of F over $[a, b]$ is defined by:

$$V_F(a, b) = \sup_{\pi \in \mathcal{P}(a,b)} V_F(\pi, a, b)$$

Definition 1.2. *A function F is said to be of bounded variation if and only if its total bounded variation over $[a, b]$ is finite, that is:*

$$0 \leq V_F(a, b) = \sup_{\pi \in \mathcal{P}(a,b)} V_F(\pi, a, b)$$

Example 1.3. (1) *Any non-decreasing function $F : [a, b] \to \mathbb{R}$ is of bounded variation.*
We have, for all $\pi \in \mathcal{P}$, $V_F(\pi, a, b) = F(b) - F(a)$, So :

$$V_F(a, b) = F(b) - F(a) < +\infty$$

(2) *Any non-increasing function* $F : [a, b] \to \mathbb{R}$ *is of bounded variation.*
We have, for all $\pi \in \mathcal{P}$, $V_F(\pi, a, b) = F(a) - F(b)$, *So :*

$$V_F(a, b) = F(a) - F(b) < +\infty$$

(3) *Any continuously differentiable* (C^1) *function* $F : [a, b] \to \mathbb{R}$ *is of bounded variation.*
In fact, since $F' \in C[a, b]$, *then* $M := \sup_{x \in [a,b]} |F'(x)| < +\infty$ *Now, for all* $\pi \in \mathcal{P}(a, b)$, *by the Mean Value Theorem,* $\forall\ j = 1, ..., p;\ \ \exists\, \theta \in [0, 1]$ *such that:*

$$F(x_j) - F(x_{j-1}) = (x_j - x_{j-1})F'(x_{j-1} + \theta_j(x_j - x_{j-1})),$$

So,

$$\begin{aligned} V_F(\pi, a, b) &= \sum_{j=1}^{p} (x_j - x_{j-1})|F'(x_{j-1} + \theta_j(x_j - x_{j-1}))| \\ &\leq M(b - a) \end{aligned}$$

Therefore,

$$V_F(a, b) = \sup_{\pi \in \mathcal{P}}(a, b) V_F(\pi, a, b) \leq M(b - a) < +\infty$$

Lemma 1.4. *Any bounded variation function on* $[a, b]$ *is a difference of two non-decreasing function.*

Now, consider a continuous function $f : [a, b] \to \mathbb{R}$. Our interest here is to show the existence of the Riemann-Stieltjes integral of f. f being so

smooth, we should at least expect, for a strong theory of integration, f to be Riemann-Stieltjes integrable.
However, for what function F can we define the Riemann-Stieltjes integral of f.

THEOREM 1.5. If F is of bounded variation, every continuous function on $[a, b]$ is integrable, i.e, has a Riemann-Stieltjes integral I denoted by:

$$I = \int_a^b f(x)\, dF(x)$$

The Riemann-Stieltjes integration is limited. In fact, we started the construction by first assuming that our function f is bounded and is defined on the interval of the form $[a, b]$. Moreover, we also considered different parameters in establishing the Riemann Sum.
For example, Let $F(x) = x$. So to determine the Riemann integral of $f : [a, b] \to \mathbb{R}$, bounded, we need to compute the Riemann Sums. In fact, in the process of computing the Riemann sums, for a fixed n, we are technically computing sum of areas of small rectangles of width $w = x_i - x_{i-1},\ \ 1 \leq i \leq l(n)$.
However, to approximate the lengths of triangle, we arbitrarily choose a point c_i between x_{i-1} and x_i and we use the image $f(c_i)$ of the point c_i, in computing the areas of those triangle. That is, we can choose any c_i in $]x_{i-1}, x_i]$.
For our approximation to make sense, we need to have that for any two points arbitrarily chosen in the sub-interval $]x_{i-1}, x_i]$, the images of those points are not far from one another in terms of value. In order words, the

function f should be continuous.

However, in real-life situation, we hardly meet smooth functions. Therefore, we make use of the Lebesgue integration which mainly requires only measurality of functions.

The illustration is given below.

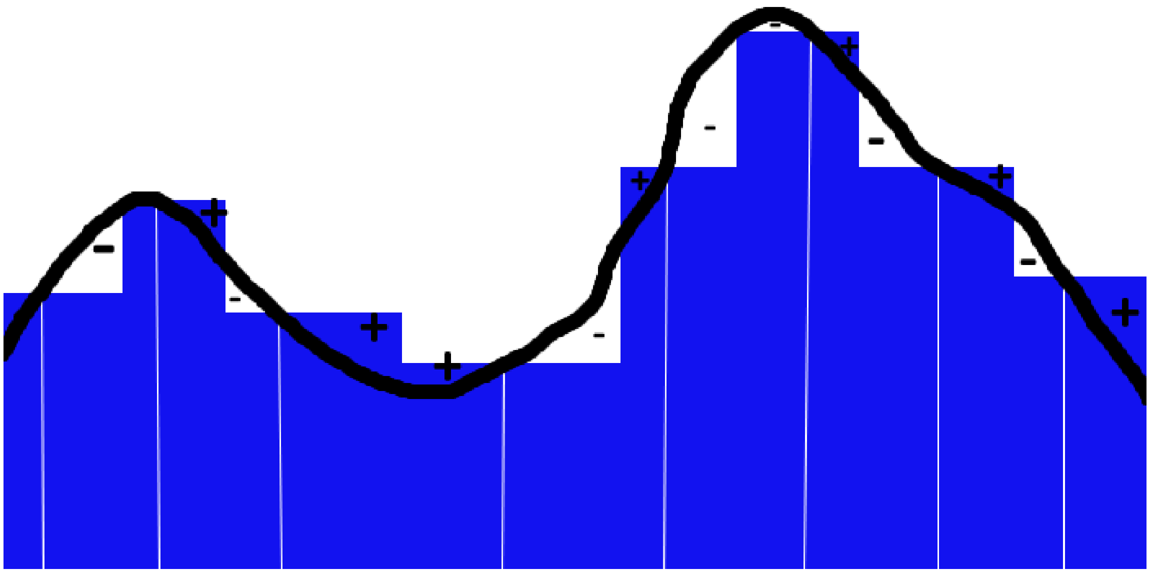

FIGURE 1. Geometric Interpretation of Riemann integration where we arbitrarily chose our c_i to be x_{i+1}.

1.3. Lebesgue Integration

1.3.1. Distribution function on $\mathbb{R}$.

Definition 1.6. *A function $F : \mathbb{R} \to \mathbb{R}$ is called a distribution function if and only if:*
(i) F is right continuous
(ii) F assigns to intervals non-negative lengths i.e $\forall\ a \leq b,\ F(b) - F(a) \geq 0$

1.3.2. Lebesgue-Stieltjes measure associated to F. We construct the Lebesgue-Stieltjes measure on $(\mathbb{R}, \mathcal{B}(R))$.
$\mathcal{B}(R) = \sigma(\mathcal{S})$
where $\mathcal{S} = \{]a,b], a < b\}$ is a semi algebra.

Define:

$$\lambda_F : \qquad \mathcal{S} \to \bar{\mathbb{R}}_+$$
$$]a,b] \to \lambda_F(]a,b]) = F(b) - F(a)$$

λ_F is called the Lebesgue-Stieltjes measure.
If $F(x) = x, \lambda_F = \lambda$ is the Lebesgue measure on $\mathbb{R}$

1.3.3. The Lebesgue-Stieltjes Integral. Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function.
For f, measurable, the Lebesgue-Stieltjes integral of f with respect to the measure λ_F is denoted as:

$$I = \int f(x)\ d\lambda_F(x)$$

The construction of this type of integral, depending on some properties of f, is given in chapter 3.
In fact, this thesis is mainly about the integration of measurable mappings with respect to measure.

Also, for the coherence in the theory of integration, it is not a surprise that the Riemann-Stieltjes integration and the Lebesgue-Stieltjes integration sometimes coincide.

Example 1.7. (1) *Let* $f : [a, b] \to \mathbb{R}$*,* $a < b$*,*f *bounded.*
f *is Riemann integrable if and only if* f *is* $\lambda - a.e$ *continuous; and the Riemann and the Lebesgue integrals coincide.*

(2) *Any Riemann integral on the compact set* $[a, b]$ *is a Lebesgue integral on* $[a, b]$

Furthermore the notion of Lebesgue-Stieltjes integration is broader than the notion of Riemann-Stieltjes integration, because all Riemann-Stieltjes integrable functions are Lebesgue-Stieltjes integrable but not all Lebesgue-Stieltjes integrable functions are Riemann integrable.

Example 1.8. $f = \mathbf{1}_{[a,b]\bigcap\mathbb{Q}}$ *is Lebesgue integrable but not Riemann integrable.*

This chapter is a brief introduction to the theory of integration. All types of integration have not been discussed. Here, we only introduced the Riemann-Stieltjes integration and addressed a broader type of integration called the Lebesgue integration.

In fact, the Lebesgue-Stietjes integration is simply the integration of real-valued measurable mappings with respect to the Lebesgue-Stieltjes measure.
In coming chapters, we will discuss the integration of measurable functions with respect to any arbitrary measure on some specific cases. Depending on the space, we put a finiteness condition on the measure.

CHAPTER 2

Integration with respect to a measure on $\mathbb{R}$: a summary

In this part, Considering a measure space $(\Omega, \mathcal{A}, m)$ we are concerned with recalling the steps of the construction of the integral of a real-valued measurable function $f : (\Omega, \mathcal{A}) \rightarrow \bar{\mathbb{R}}$ with respect to a measure, denoted by:

$$\int f \, dm = \int_{\Omega} f(\omega) \, dm(\omega) = \int_{\Omega} f(\omega) \, m(d\omega)$$

Along the document, by "The Real-valued Mapping Modern Integrals (RVM-MI)", we mean the integrals of real-valued measurable functions.

2.1. The construction

STEP 1M: Definition of the integral for a non-negative elementary funtion f.

Let, $f = \sum_{i=1}^{p} \alpha_i \mathbf{1}_{A_i}$ $(p \geq 1, \ \ \alpha_i \in \mathbb{R}_+, A_i \in \mathcal{A}, \ \ A_1 + A_2 + ... + A_p = \Omega)$
be a non-negative elementary function.

The integral of f with respect to m is defined by:

$$\int f \, dm = \sum_{i=1}^{p} \alpha_i m(A_i) \tag{2.1.1}$$

Remark 2.1. *:*

(a) *(Convention) In definition (2.1.1), the product $\alpha_i m(A_i)$ is zero whenever $\alpha_i = 0$, even if $m(A_i) = +\infty$.*

(b) *The class of real-valued elementary functions is denoted by $\mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$ and $\mathcal{E}^+(\Omega, \mathcal{A}, \mathbb{R})$ stands for the subclass of non-negative functions of $\mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$.*

(c) *As an elementary function, f has various expressions, however, Definition (2.1.1) is coherent (i.e, $\int f \, dm$ does not depend on one particular expression of f).*

(d) *In Definition (2.1.1) we are using an expression of f in which the coefficient α_i are disjoint, called canonical representation of f.*

(e) *f is well defined,*
In fact, for $\omega \in \Omega$, $\exists ! i_0 : \omega \in A_{i_0}$. So $f(\omega) = \alpha_{i_0}$. Moreover, since the expression of f is the canonical one, α_{i_0} is unique.

(f)- *In these remarks, Point (c) - Points (e) will still hold in the rest of the paper even if the elementary functions mentioned are not real-valued.*

STEP 2M: Definition of the integral for a non-negative measurable function.

Let f be any non-negative measurable function. Then, there exists a non-decreasing sequence $(f_n)_{n\geq 1} \subseteq \mathcal{E}^+(\Omega, \mathcal{A}, \mathbb{R})$ such that :

$$f_n \uparrow f \ as \ n \rightarrow +\infty \tag{2.1.2}$$

(See point (03-23) in Doc 03-02 in Chapter 4 in "Measure Theory and Integration By and For the learner"(Gane Samb Lo))
We then define,

$$\int f \ dm = \lim_{n\rightarrow\infty} \int f_n \ dm \tag{2.1.3}$$

Definition (2.1.3) is coherent, since it does not depend on the sequence used in definition (2.1.2).
(See Chapter 4 in "Measure Theory and Integration By and For the learner"(Gane Samb Lo)).

STEP 3M: Definition of the integral for a measurable function f.

We recall that:

$$f = f^+ - f^- \ , \ \ |f| = f^+ + f^- \ , and \ \ f^+ f^- = 0 \tag{2.1.4}$$

Where $f^+ = max(0, f) \ \ and \ \ f^- = max(0, -f)$

Since f^+ and f^- are non-negative and measurable, By Step 2M, we have that:
$\int f^+ \, dm$ and $\int f^- \, dm$ exists in $\bar{\mathbb{R}}_+$.

Now if

$$\int f^+ \, dm \; < +\infty \quad or \quad \int f^- \, dm \; < +\infty,$$

We say that f is quasi-integrable with respect to m and the integral of f is defined by:

$$\int f \, dm = \int f^+ \, dm \; - \int f^- \, dm \tag{2.1.5}$$

Remark 2.2. **(a)** *The integral of a real-valued and measurable function with respect to a measure exists if and only if: either it is of constant sign or the integral of its positive part or its negative part is finite.*

(b) *(Integration of a measurable mapping over a measurable set.)*
If A is a measurable subset of Ω and $\mathbf{1}_A f$ is quasi-integrable, the integral of f over A is defined by:

$$\int_A f \, dm = \int \mathbf{1}_A f \, dm$$

Where, $\mathbf{1}_A f(\omega) = f(\omega) \;\; if \; \omega \in A$ and zero otherwise.

DEFINITION 2.3. *The real-valued measurable function f is said to be integrable if and only if $\int f\ dm$ exists and is finite, i.e,*

$$\int f^+\ dm\ < +\infty \quad and \quad \int f^-\ dm\ < +\infty,$$

The set of all integrable functions with respect to m is denoted $\mathcal{L}^1(\Omega, \mathcal{A}, m)$

2.2. Properties of Real-valued Integrable Functions

Let f and g be measurable functions with $f+g$ defined a.e such that $\int f\ dm$, $\int f\ dm$, and $\int f+g\ dm$ exist and make sense.
Let A and B be two disjoint measurable sets and c be a finite non-zero scalar. We have the following properties:

2.2.1. Linearity.

$$\int cf\ dm = c\int f\ dm \qquad (L1)$$

which is still valid for $c = 0$ if f is integrable;

$$\int (f+g)\ dm = \int f\ dm + \int g\ dm \qquad (L2)$$

and,

$$\int_{A+B} f\ dm = \int_a f\ dm + \int_B f\ dm \qquad (L3)$$

2.2.2. Order preservation.

$$f \leq g \implies \int f\ dm \leq \int g\ dm \quad (O1)$$

and,

$$f = g\ a.e \implies \int f\ dm = \int g\ dm \quad (O2)$$

2.2.3. Integrability.

$$f\ integrable \Leftrightarrow |f|\ integrable \implies f\ finite\ a.e, \quad (I1)$$

$$|f| \leq g\ \ integrable \implies f\ integrable \quad (I2)$$

and,

$$\left((a,b) \in \mathbb{R}^2,\ f\ and\ g\ integrable\right) \implies \left(af + bg\ integrable\right) \quad (I3)$$

2.3. Spaces of integrable functions

2.3.1. The space $\mathcal{L}^p(\Omega, \mathcal{A}, m, \mathbb{R}), p \geq 1$.

$$\mathcal{L}^p(\Omega, \mathcal{A}, m, \mathbb{R})$$

is the space of all real-valued and measurable functions f defined on Ω such that $|f|^p$ is integrable with respect to m, that is,

$$\mathcal{L}^p(\Omega, \mathcal{A}, m, \mathbb{R}) = \{f \in \mathcal{L}_0(\Omega, \mathcal{A}, m, \mathbb{R}), \int |f|^p\ dm < +\infty\}$$

, where $\mathcal{L}^0(\Omega, \mathcal{A}, m, \mathbb{R})$ is the class of all real-valued and measurable functions defined on Ω.

2.3.2. The space of m-a.e bounded functions $\mathcal{L}^{+\infty}(\Omega, \mathcal{A}, m, \mathbb{R})$**.**

$$\mathcal{L}^{+\infty}(\Omega, \mathcal{A}, m, \mathbb{R})$$

is the class of all real-valued and measurable functions which are bounded m-a.e, i.e,

$$\mathcal{L}^{+\infty}(\Omega, \mathcal{A}, m, \mathbb{R}) = \{f \in \mathcal{L}^0(\Omega, \mathcal{A}, m, \mathbb{R}), \exists C \in \mathbb{R}_+, \ |f| \leq C \ m - a.e\}$$

2.3.3. Equivalence classes. Let $L^p(\Omega, \mathcal{A}, m, \mathbb{R}$ be the set of equivalence classes of the binary equivalence relation $\mathcal{R}$ defined by:

$$\forall (f,g) \in (\mathcal{L}^p(\Omega, \mathcal{A}, m, \mathbb{R}))^2, f \ \mathcal{R} \ g \Leftrightarrow f = g \ m - a.e$$

So,

$$L^p(\Omega, \mathcal{A}, m, \mathbb{R}) = \mathcal{L}^p(\Omega, \mathcal{A}, m, \mathbb{R})/\mathcal{R}$$

that is

$$L^p(\Omega, \mathcal{A}, m, \mathbb{R}) = \{f \in \mathcal{L}_0(\Omega, \mathcal{A}, m, \mathbb{R}), \int |f|^p \ dm < +\infty\}$$

and,

$$L^{+\infty}(\Omega, \mathcal{A}, m, \mathbb{R}) = \mathcal{L}^{+\infty}(\Omega, \mathcal{A}, m, \mathbb{R})/\mathcal{R}$$

that is,

$$L^{+\infty}(\Omega, \mathcal{A}, m, \mathbb{R}) = \{f \in \mathcal{L}^0(\Omega, \mathcal{A}, m, \mathbb{R}), \exists C \in \mathbb{R}_+, \ |f| \leq C \ m - a.e\}$$

2.3.4. Norm on L^p. The space $L^p(\Omega, \mathcal{A}, m, \mathbb{R})$, $1 \leq < +\infty$, is equipped with the norm

$$\|f\|_p = \left(\int |f|^p \, dm \right)^{1/p}$$

The space $L^{+\infty}(\Omega, \mathcal{A}, m, \mathbb{R})$ is equipped with the norm

$$\|f\|_\infty = inf\{C \in \mathbb{R}_+, |f| \leq C \ m - a.e\}$$

2.3.5. Banach Spaces. For all $p \in [1, +\infty]$, $\left(L^p(\Omega, \mathcal{A}, m, \mathbb{R}), +, *, \|f\|_p \right)$ is a Banach space.

CHAPTER 3

Integration with respect to a measure on Banach spaces in general

In this part, we are going to construct an integral of functions with values in a Banach space $(E, +, *, \|.\|_E)$ over $\mathbb{R}$.
The construction will consist of repeating Step 1M (from the construction of the integral of Real-valued mappings) and defining a new step to replace both Step 2M and Step 3M.
In fact, we are replacing Step 2M ans Step 3M by one new step because they require an order that E need not to have. Moreover, for an E-valued function f, we are not certain of getting a sequence of elementary functions that converge to f.

Remark 3.1. *: Here we are considering bounded measure in the construction of the integral, unless we have corresponding notions of infinity.*

3.1. The construction of the integral

We are going to construct the Bochner integral of a measurable function $f : (\Omega, \mathcal{A}, m) \to E$ in two steps.

STEP 1B: Integral of an E-valued elementary function

The notion of elementary function may be easily extented to any linear space E, by defining an E-valued elementary function in the form:

$$f = \sum_{j=1}^{p} x_j \mathbf{1}_{B_j} \tag{3.1.1}$$

where $p \geq 1, x_j \in E, B_j \in \mathcal{A}, B_1 + B_2 + ... + B_p = \Omega \ , B_j \cap B_i = \emptyset \ (i \neq j)$

We are able to express f as a finite linear combination of vectors because it takes values in E, which is a linear space.
We then denote by $\mathcal{E}(\Omega, \mathcal{A}, E)$, the class of all elementary functions with values in E.

Now, we consider $f \in \mathcal{E}(\Omega, \mathcal{A}, E)$ as expressed in (3.1.1).
Since the $B_j's$ are disjoints, $f(\omega) = x_j$ for $\omega \in B_j$.
Therefore, $\|f(\omega)\|_E = \|x_j\|_E$.
We now write: $\|f\|_E(\omega) := \|f(\omega)\|_E = \|x_j\|_E$

Consequently, the following fact will be used and exploited in the construction of the integral:
For any $f \in \mathcal{E}(\Omega, \mathcal{A}, E)$, we have the real valued mapping, still called the norm of f:

$$\|f\|_E : (\Omega, \mathcal{A}, m) \to \mathbb{R}$$

defined by

$$\forall \omega \in \Omega, \|f\|_E(\omega) = \|f(\omega)\|_E$$

,

where,

$$\|f\|_E = \sum_{j=1}^{p} \|x_j\|_E \mathbf{1}_{B_j} \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R}) \tag{3.1.2}$$

SUMMARY 1.

$$f \in \mathcal{E}(\Omega, \mathcal{A}, E) \implies \|f\|_E \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$$

As regards to integration in Measure Theory for a real-valued measurable mapping, we already know how to handle $\|f\|_E \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$. What we need to address now is the integral of f, where $f \in \mathcal{E}(\Omega, \mathcal{A}, E)$.

Let us define the Bochner integral of $f \in \mathcal{E}(\Omega, \mathcal{A}, E)$, represented as in (3.1.1) by:

$$\int_{(\Omega,E)} f\ dm = \sum_{j=1}^{p} x_j m(B_j)\ \in E \tag{3.1.3}$$

REMARK 3.2. *We mention (Ω, E) in the notation of the integral in order to put emphasis on the fact that f is defined on Ω and the value of the integral is in E.*

(EF1) Definition (3.1.3) does not depend on the particular representation of f. As a consequence, that definition is coherent. Proof

of (EF1):

We will use in this proof, the superposition principle seen in the Measure Theory and Integration Course.

(See point (03-22(a)) in Doc 03-02 in Chapter 4 in ”Measure Theory and Integration By and For the learner”(Gane Samb Lo)).

Let,

$$f = \sum_{j=1}^{p} x_j \mathbf{1}_{B_j} = \sum_{i=1}^{q} y_i \mathbf{1}_{A_i}$$

where, $x_j, y_i \in E$ and $\{B_j, j = 1, ..., p\}, \{A_i, i = 1, ..., q\}$ are partitions of Ω.

Take $I = \{(j,i) \in \{1, ..., p\} \times \{1, ..., q\} : B_j \bigcap A_i \neq \emptyset\}$. So we have that:

$f = \sum_{(j,i) \in I} z_{j,i} \mathbf{1}_{B_j A_i}$, where $z_{j,i} = x_j = y_i$ on $B_j A_i$.

In fact, using (3.1.3), we have:

$$\int_{(\Omega,E)} f \, dm = \sum_{j=1}^{p} x_j m(B_j) \ \ and \ \int_{(\Omega,E)} f \, dm = \sum_{i=1}^{q} y_i m(A_i)$$

.

Therefore, to show that Definition (3.1.3) is coherent, it suffices to show that

$$\sum_{j=1}^{p} x_j m(B_j) = \sum_{i=1}^{q} y_i m(A_i)$$

.

$$\begin{aligned}
\sum_{(j,i)\in I} z_{j,i} m(B_j A_i) &= \sum_{j=1}^{p} \sum_{i:(j,i)\in I} z_{j,i} m(B_j A_i) \\
&= \sum_{j=1}^{p} \sum_{i:(j,i)\in I} x_j m(B_j A_i) \\
&= \sum_{j=1}^{p} x_j \sum_{i:(j,i)\in I} m(B_j A_i) \\
&= \sum_{j=1}^{p} x_j \sum_{i=1}^{q} m(B_j A_i) \\
&= \sum_{j=1}^{p} x_j m(\sum_{i=1}^{q} B_j A_i) \\
&= \sum_{j=1}^{p} x_j m(B_j \bigcap \sum_{i=1}^{q} A_i) \\
&= \sum_{j=1}^{p} x_j m(B_j \bigcap \Omega) \\
&= \sum_{j=1}^{p} x_j m(B_j)
\end{aligned}$$

By the symmetry of the role of A_i and B_j, we will also get that: $\sum_{(j,i)\in I} z_{j,i} m(B_j A_i) = \sum_{i=1}^{q} y_i m(A_i)$. Therefore,

$$\sum_{j=1}^{p} x_j m(B_j) = \sum_{i=1}^{q} y_i m(A_i)$$

.

Hence, Definition (3.1.3) is coherent.

(EF2) The defined Bochner integral on $\mathcal{E}(\Omega, \mathcal{A}, E)$ is linear.

Proof of (EF2)

Let $f \in \mathcal{E}(\Omega, \mathcal{A}, E)$ and $g \in \mathcal{E}(\Omega, \mathcal{A}, E)$ such that,

$f = \sum_{j=1}^{p} x_j \mathbf{1}_{B_j}$ and $g = \sum_{i=1}^{q} y_i \mathbf{1}_{A_i}$.

So,

$$f + g = \sum_{(j,i)\in I} (x_j + y_i)\mathbf{1}_{B_j A_i} \in \mathcal{E}(\Omega, \mathcal{A}, E)$$

(Using superposition) and,

by Definition (3.1.3),

$$\int_{(\Omega,E)} f\ dm = \sum_{j=1}^{p} x_j m(B_j) \ \ and \ \ \int_{(\Omega,E)} g\ dm = \sum_{i=1}^{q} y_i m(A_i).$$

Therefore,

$$\begin{aligned}
\int_{(\Omega,E)} f + g\ dm &= \int_{(\Omega,E)} \Big(\sum_{(j,i)\in I} (x_j + y_i)\mathbf{1}_{B_j A_i}\Big)\ dm \\
&= \sum_{(j,i)\in I} (x_j + y_i) m(B_j A_i) \\
&= \sum_{(j,i)\in I} x_j m(B_j A_i) + \sum_{(j,i)\in I} + y_i m(B_j A_i) \\
&= \sum_{j=1}^{p} \sum_{i:(j,i)\in I} x_j m(B_j A_i) + \sum_{i=1}^{q} \sum_{j:(j,i)\in I} y_i m(B_j A_i) \\
&= \sum_{j=1}^{p} x_j m(B_{\mathrm{j}}) + \sum_{i=1}^{q} y_i m(A_{\mathrm{i}}) \\
&= \int_{(\Omega,E)} f\ dm + \int_{(\Omega,E)} g\ dm
\end{aligned}$$

Also, it is easy to see that for $\alpha \in \mathbb{R}$, $\int_{(\Omega,E)} \alpha f dm = \alpha \int_{(\Omega,E)} f dm$.

Hence, the defined Bochner integral on $\mathcal{E}(\Omega, \mathcal{A}, E)$ is linear.

Remark 3.3. *In the theory of integration of real valued measurable functions, we had, at Step 2M that for any non-negative measurable function f, we can find a sequence of non-negative elementary functions that converges to it. And we used that fact to construct the integral of f.*
For a Banach valued measurable function f, we cannot say that f is either positive or not (since f maps elements of Ω to vectors in E). Therefore, we cannot apply Step 2M of the theory of integration of real valued measurable function. We will therefore replace that step by a new one, namely Step 2B.

STEP 2B:

Definition 3.4. *A measurable function is said to be B-Integrable (Bochner-Integrable), if and only if there exists a sequence $(f_n)_{n \geq 1} \subseteq \mathcal{E}(\Omega, \mathcal{A}, E)$ such that,*

$$\sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm < \infty \qquad (I1)$$

and,

$$f = \sum_{n\geq 1} f_n \ , \quad m - a.e \qquad (I2)$$

If conditions (I1) and (I2) both hold, we write: $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$

Remark 3.5. *We mention $\mathcal{E}(\Omega, \mathcal{A}, E)$ in the notation of $S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$ to show that $f_n \in \mathcal{E}(\Omega, \mathcal{A}, E), \quad \forall n \geq 1$. Later, $(f_n)_{n\geq 1}$ might belong to another space.*

Lemma 3.6. *The implications below hold.*

(A) *If condition (I1) hold, then,*

(a) $\sum_{n\geq 1} f_n$ *is defined m-a.e in E, and*

(b) *The series of Bochner integrals $\sum_{n\geq 1} \int_{(\Omega,E)} f_n \, dm$, converges in E.*

(B) *If $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$ and $f \in S(g_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$, then,*

$$\sum_{n\geq 1} \int_{(\Omega,E)} f_n \, dm = \sum_{n\geq 1} \int_{(\Omega,E)} g_n \, dm$$

(C) *If $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$, then,*

$$\|\sum_{n\geq 1} \int_{(\Omega,E)} f_n \, dm\|_E \leq \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm \tag{3.1.4}$$

(D) *If $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$, then for any $\eta > 0$,*

there exists a sequence $(h_n)_{n\geq 1} \in \mathcal{E}(\Omega, \mathcal{A}, E)$, such that:

$$f \in S(h_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$$

and

$$\sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|h_n\|_E \, dm \leq \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm \quad + \eta$$

Based on Definition 3.4 and Lemma 3.6, we may define the Bochner integral of $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$ by:

$$\int_{(\Omega,E)} f \, dm = \sum_{n\geq 1} \int_{(\Omega,E)} f_n \, dm \tag{3.1.5}$$

SUMMARY 2. *We have entirely finished the construction of $\int_{(\Omega,E)} f \, dm \in E$ in two steps. Moreover, by part (B) of Lemma 1, we can say that Definition (3.1.5) is coherent.*

Before proving Lemma 3.6, we claim that for $h \in \mathcal{E}(\Omega, \mathcal{A}, E)$ we have:

$$\| \int_{(\Omega,E)} h \, dm\|_E \leq \int_{(\Omega,\mathbb{R})} \|h\|_E \, dm \tag{3.1.6}$$

Proof of claim: Consider f as defined in formula (3.1.1), then,

$$\begin{aligned} \| \int_{(\Omega,E)} f \; dm\|_E &= \| \sum_{j=1}^{p} x_j m(B_j)\|_E \\ &\leq \sum_{j=1}^{p} \|x_j m(B_j)\|_E \\ &= \sum_{j=1}^{p} \|x_j\|_E \; m(B_j) \\ &= \int_{(\Omega,\mathbb{R})} (\sum_{j=1}^{p} \|x_j\|_E \mathbf{1}_{B_j}) \; dm \\ &= \int_{(\Omega,\mathbb{R})} \|f\|_E \; dm \end{aligned}$$

This completes the proof.

Since a sum of E-valued elementary functions is an E-valued elementary function, then for any family $(h_j)_{1\leq j\leq p}$ of E-valued elementary functions, we have that: $\sum\limits_{j=1}^{p} h_j \in \mathcal{E}(\Omega, \mathcal{A}, E)$.

Then applying formula (3.1.6), we get:

$$\| \int_{(\Omega,E)} \sum_{j=1}^{p} h_j \; dm\|_E \leq \int_{(\Omega,\mathbb{R})} \| \sum_{j=1}^{p} h_j\|_E \; dm \tag{3.1.7}$$

Let us now prove Lemma 3.6

Proof of point (A)

(a) Suppose condition (I1) holds, i.e,

$$\sum_{n\geq 1}\int_{(\Omega,\mathbb{R})}\|f_n\|_E\, dm < \infty$$

We have that $\|f_n\|_E \in \mathcal{E}^+(\Omega,\mathcal{A},\mathbb{R}),\ \forall n \geq 1$. So By the Monotone Convergence Theorem and by Condition (I1),

$$\begin{aligned}
\sum_{n\geq 1}\int_{(\Omega,\mathbb{R})}\|f_n\|_E\, dm &= \int_{(\Omega,\mathbb{R})}\sum_{n\geq 1}\|f_n\|_E\, dm < \infty\\
&\Longrightarrow \int_{(\Omega,\mathbb{R})}\sum_{n\geq 1}\|f_n\|_E\, dm < \infty\\
&\Longrightarrow \sum_{n\geq 1}\|f_n\|_E \ \ is\ \ integrable\\
&\Longrightarrow \sum_{n\geq 1}\|f_n\|_E \ \ is\ \ finite\ \ m-a.e\\
&\Longrightarrow The\ \ series\ \ \sum_{n\geq 1} f_n\ \ is\ \ absolutely\ \ convergent\ \ m-a.e\\
&\Longrightarrow \sum_{n\geq 1} f_n\ \ is\ \ defined\ \ m-a.e
\end{aligned}$$

(b)

To show that $\sum\limits_{n\geq 1}\int_{(\Omega,E)} f_n\, dm$ converges in E, We just need to show that the sequence of partial sum $(S_n)_n \geq 1$ converges in E; where,

$$S_n = \sum_{i=1}^{n}\int_{(\Omega,E)} f_i\, dm.$$

Since E is Banach, to show that $(S_n)_n \geq 1$ converges in E, it suffices to show that $(S_n)_n \geq 1$ is Cauchy.

Let $p, q \in \mathbb{N}$ such that $p \geq q$. We define:

$$S_p = \sum_{n=1}^{p} \int_{(\Omega,E)} f_n \; dm \quad and \quad S_q = \sum_{n=1}^{q} \int_{(\Omega,E)} f_n \; dm.$$

So,

$$\begin{aligned} 0 \leq \|S_p - S_q\|_E &= \|\sum_{n=1}^{p} \int_{(\Omega,E)} f_n \; dm - \sum_{n=1}^{q} \int_{(\Omega,E)} f_n \; dm\|_E \\ &= \|\sum_{n=q+1}^{p} \int_{(\Omega,E)} f_n \; dm\|_E \quad (Since \; p \geq q) \\ &\leq \sum_{n=q+1}^{p} \|\int_{(\Omega,E)} f_n \; dm\|_E \\ &\leq \sum_{n=q+1}^{p} \int_{(\Omega,\mathbb{R})} \| f_n\|_E \; dm \end{aligned}$$

Now, as $p \to +\infty$, $\sum\limits_{n=q+1}^{p} \int_{(\Omega,E)} \| f_n\|_E \; dm \; \to \sum\limits_{n=q+1}^{+\infty} \int_{(\Omega,E)} \| f_n\|_E \; dm$,

and, as $q \to \infty$, $\sum\limits_{n=q+1}^{\infty} \int_{(\Omega,E)} \| f_n\|_E \; dm \; \to 0$ as the tail end of the series $\sum\limits_{n\geq 1} \int_{(\Omega,E)} \| f_n\|_E \; dm$, which converges in E by condition (I1) .

Therefore, as $p, q \to \infty$, $\sum\limits_{n=q+1}^{p} \int_{(\Omega,E)} \| f_n\|_E \; dm \to 0$.

So, $\|S_p - S_q\|_E \to 0$ and thus, $(S_n)_n \geq 1$ is Cauchy.

Hence $\sum\limits_{n\geq 1} \int_{(\Omega,E)} f_n \, dm$ converges in E.

Proof of points (B) and (C): In fact (C) will be obtained in the proof of (B).

We suppose that $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$ and $f \in S(g_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$ and we need to show that :

$$\sum_{n\geq 1} \int_{(\Omega,E)} f_n \, dm = \sum_{n\geq 1} \int_{(\Omega,E)} g_n \, dm$$

Now; Let $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$

We define:

(3.1.8) $g_0 = 0$, $g_n = f_1 + f_2 + ... + f_n, n \geq 1 \quad and \quad \varphi_n = \|g_n\|_E - \|g_{n-1}\|_E, \ \ n \geq 2.$

.

Since, $\|g_n\|_E \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$, and $\|g_{n-1}\|_E \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$,

then $\varphi_n \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$.

Also,

$\varphi_1 + \varphi_2 + ... + \varphi_n = \|g_1\| - \|g_0\| + \|g_2\| - \|g_1\| + ... + \|g_{n-1}\| - \|g_{n-2}\| + \|g_n\| - \|g_{n-1}\|.$

Since the terms are telescoping, we get:

$$\varphi_1 + \varphi_2 + ... + \varphi_n = \|g_n\|_E - \|g_0\|_E$$

$$\implies \varphi_1 + \varphi_2 + ... + \varphi_n = \|g_n\|_E \ \ (Since\ g_0 = 0)$$

Hence,

$$\|g_n\|_E = \varphi_1 + \varphi_2 + ... + \varphi_n. \tag{3.1.9}$$

Now, for $n \geq 1$,

$$|\varphi_n|_{\mathbb{R}} = |\ \|g_n\| - \|g_{n-1}\|\ |_{\mathbb{R}} \leq |\ \|g_n - g_{n-1}\|\ |_{\mathbb{R}} = \|f_n\|_E.$$

So,

$$|\varphi_n|_{\mathbb{R}} \leq \|f_n\|_E. \tag{3.1.10}$$

Next, Since $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$,

we have that $f = \sum_{n \geq 1} f_n$, $m - a.e.$

This implies that the sequence of partial sums $(S_n)_{n\geq 1}$ of the series $\sum_{n\geq 1} f_n$ converges to f $m - a.e.$ The sequence of partial sums is defined as :

$$S_n = \sum_{i=1}^{n} f_i = f_1 + f_2 + ... + f_n = g_n \quad (from(3.1.8)).$$

So $S_n = g_n, \forall n \geq 1$

Since $S_n \to f \;\; m-a.e$, then $g_n \to f \;\; m-a.e.$

By the continuity of norm, we have:

$\|g_n\|_E \to \|f\|_E, \;\; m-a.e$

This implies, $\varphi_1 + \varphi_2 + ... + \varphi_n \to \|f\|_E, \;\; m-a.e \qquad (using((3.1.9)))$

Futhermore,

$$\begin{aligned} |\varphi_1 + \varphi_2 + ... + \varphi_n| &\leq |\varphi_1| + |\varphi_2| + ... + |\varphi_n| \\ &\leq \sum_{n\geq 1} |\varphi_n| \\ &\leq \sum_{n\geq 1} \|f_n\|_E \quad (using(3.1.10)). \end{aligned}$$

Where, $\sum_{n\geq 1} \|f_n\|_E$ is integrable. (See proof of point (A) (a))

Hence by the Dominated Convergence Theorem for real-valued measurable function, we get:

$$\int_{(\Omega,\mathbb{R})} (\varphi_1 + \varphi_2 + ... + \varphi_n)\, dm \;\; \to \;\; \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm \qquad m-a.e$$

$$\Longrightarrow \lim_{n\to\infty} \int_{(\Omega,\mathbb{R})} \sum_{k=1}^{n} \varphi_k \, dm = \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm$$

$$\begin{aligned}
\Longrightarrow \int_{(\Omega,\mathbb{R})} \|f\|_E &= \lim_{n\to\infty} \int_{(\Omega,\mathbb{R})} \sum_{k=1}^{n} \varphi_k \; dm \\
&= \lim_{n\to\infty} \int_{(\Omega,\mathbb{R})} \|g_n\|_E \; dm \\
&= \lim_{n\to\infty} \int_{(\Omega,\mathbb{R})} \|\sum_{k=1}^{n} f_k\|_E \; dm.
\end{aligned}$$

Hence,

$$\int_{(\Omega,\mathbb{R})} \|f\|_E = \lim_{n\to\infty} \int_{(\Omega,\mathbb{R})} \|\sum_{k=1}^{n} f_k\|_E \; dm \qquad (*)$$

Now,

$$\|\sum_{k=1}^{n} \int_{(\Omega,E)} f_k \; dm\|_E = \|\int_{(\Omega,E)} \sum_{k=1}^{n} f_k \; dm\|_E \leq \int_{(\Omega,\mathbb{R})} \|\sum_{k=1}^{n} f_k\|_E \; dm$$

By letting $n \to \infty$, we get,

$$\lim_{n\to\infty} \|\sum_{k=1}^{n} \int_{(\Omega,E)} f_k \; dm\|_E \leq \lim_{n\to\infty} \int_{(\Omega,\mathbb{R})} \|\sum_{k=1}^{n} f_k\|_E \; dm.$$

This implies,

$$\begin{aligned}
\|\sum_{n\geq 1} \int_{(\Omega,E)} f_k \; dm\|_E \; &\leq \; \lim_{n\to\infty} \int_{(\Omega,\mathbb{R})} \|\sum_{k=1}^{n} f_k\|_E \; dm \quad (By\ continuity\ of\ norm) \\
\Longrightarrow \|\sum_{n\geq 1} \int_{(\Omega,E)} f_k \; dm\|_E \; &\leq \; \int_{(\Omega,\mathbb{R})} \|f\|_E \; dm \quad (using(*))
\end{aligned}$$

We then get point (C), that is,

for $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$, we have that

$$\| \sum_{n\geq 1} \int_{(\Omega,E)} f_n \; dm \|_E \; \leq \; \int_{(\Omega,\mathbb{R})} \|f\|_E \; dm \qquad (**)$$

Moreover, since $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$ and $f \in S(g_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$,

then,

$$0 \; \in \; S(f_n - g_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$$

Thus, by $(**)$,

$$0 \; \in \; S(f_n - g_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E)) \implies \| \sum_{n\geq 1} \int_{(\Omega,E)} (f_n - g_n) \; dm \|_E \leq \int_{(\Omega,\mathbb{R})} \|0\|_E$$

.

Now,

$$\begin{aligned} 0 \leq \lim_{n\to\infty} \| \, (\sum_{k=1}^{n} \int_{(\Omega,E)} f_k \; dm \; - \; \sum_{k=1}^{n} \int_{(\Omega,E)} g_k \; dm) \, \|_E \; &= \; \lim_{n\to\infty} \| \sum_{k=1}^{n} \int_{(\Omega,E)} (f_k - g_k) \; dm \|_E \\ &= \; \| \sum_{n\geq 1} \int_{(\Omega,E)} (f_n - g_n) \; dm \|_E \\ &\leq \; \int_{(\Omega,\mathbb{R})} \|0\|_E \; dm \\ &= \; 0 \end{aligned}$$

This implies,

$$\lim_{n\to\infty} \| \, (\sum_{k=1}^{n} \int_{(\Omega,E)} f_k \; dm \; - \; \sum_{k=1}^{n} \int_{(\Omega,E)} g_k \; dm) \, \|_E = 0.$$

Also, we already have that:

$\sum_{n\geq 1} \int_{(\Omega,E)} f_n \, dm$ and $\sum_{n\geq 1} \int_{(\Omega,E)} f_n \, dm$ converges in E.

Hence

$$\sum_{n\geq 1} \int_{(\Omega,E)} f_n \, dm \; = \; \sum_{n\geq 1} \int_{(\Omega,E)} g_n \, dm$$

Proof of point (D)

Suppose that $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$, we then recall that,

$$\sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm < \infty \qquad (I1)$$

and,

$$f = \sum_{n\geq 1} f_n \; , \quad m - a.e \qquad (I2)$$

$(I1)$ implies that the series of the integrals of $\|f_n\|_E$ converges.

Thus, $\forall \, 1 \leq q \leq n, \quad \sum_{n\geq q} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm \to 0 \;\; as \;\; q \to \infty$, i.e,

Given $\eta \geq 0, \exists \, n_0 \in \mathbb{N}$ such that,

$$\sum_{n\geq q} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm < \frac{\eta}{2} \; , \qquad \forall \, q \geq n_0$$

In particular for $q = n_0$ we have,

$$\sum_{n\geq n_0} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \; dm < \frac{\eta}{2}$$

So we have that,: Given $\eta \geq 0, \exists \; n_0 \in \mathbb{N}$ such that:

$$\sum_{n\geq n_0} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \; dm < \frac{\eta}{2}$$

We set:

$$h_1 = f_1 + ... + f_{n_0} \; ; \qquad h_n = f_{n_0+n-1} \quad , for \; n \geq 2$$

Now,

$$\begin{aligned}
\sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|h_n\|_E \; dm &= \int_{(\Omega,\mathbb{R})} \|h_1\|_E \; dm + \sum_{n\geq 2} \int_{(\Omega,\mathbb{R})} \|h_n\|_E \; dm \\
&= \int_{(\Omega,\mathbb{R})} \|f_1 + ... + f_{n_0}\|_E \; dm + \sum_{n\geq 2} \int_{(\Omega,\mathbb{R})} \|f_{n_0+n-1}\|_E \; dm \\
&\leq \sum_{k=1}^{n_0} \int_{(\Omega,\mathbb{R})} \|f_k\|_E \; dm + \sum_{k\geq n_0+1} \int_{(\Omega,\mathbb{R})} \|f_k\|_E \; dm \\
&= \sum_{k\geq 1} \int_{\Omega,\mathbb{R}} \|f_k\|_E \; dm < \infty
\end{aligned}$$

So,

$$\sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|h_n\|_E \; dm < \infty$$

Moreover,

$$\sum_{n\geq 1} h_n = h_1 + h_2 + h_3 + ... + h_n \quad = f_1 + ... + f_{n_0} + f_{n_0+1} + f_{n_0+2} + f_{n_0+n-1} + ...$$
$$= \sum_{n\geq 1} f_n$$
$$= f \qquad m - a.e$$

Hence, $f = \sum_{n\geq 1} h_n \qquad m - a.e.$

Therefore, $f \in S(h_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$

Now,

$$\begin{aligned}
\int_{(\Omega,\mathbb{R})} \|f - h_1\|_E \, dm &= \int_{(\Omega,\mathbb{R})} \|\sum_{n\geq 1} f_n - (f_1 + ... + f_{n_0})\|_E \, dm \\
&= \int_{(\Omega,\mathbb{R})} \|\sum_{n\geq n_0} f_n\|_E \, dm \\
&\leq \int_{(\Omega,\mathbb{R})} \sum_{n\geq n_0} \|f_n\|_E \, dm \\
&\leq \sum_{n\geq n_0} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm < \frac{\eta}{2}
\end{aligned}$$

$\implies \int_{(\Omega,\mathbb{R})} \|f - h_1\|_E \, dm < \frac{\eta}{2}$

But,

$$\int_{(\Omega,\mathbb{R})} \|f - h_1\|_E \, dm \geq \int_{(\Omega,\mathbb{R})} \|h_1\|_E \, dm - \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm$$

This implies,

$$\int_{(\Omega,\mathbb{R})} \|h_1\|_E \, dm - \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm \leq \int_{(\Omega,\mathbb{R})} \|f - h_1\|_E \, dm < \frac{\eta}{2}$$

Therefore,

$$\int_{(\Omega,\mathbb{R})} \|h_1\|_E \, dm < \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm + \frac{\eta}{2}$$

Now,

$$\begin{aligned}
\sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|h_n\|_E \, dm &= \int_{(\Omega,\mathbb{R})} \|h_1\|_E \, dm + \sum_{n\geq 2} \int_{(\Omega,\mathbb{R})} \|h_n\|_E \, dm \\
&\leq \int_{(\Omega,\mathbb{R})} \|h_1\|_E \, dm + \sum_{n\geq 2} \int_{(\Omega,\mathbb{R})} \|f_{n_0+n-1}\|_E \, dm \\
&\leq \int_{(\Omega,\mathbb{R})} \|h_1\|_E \, dm + \sum_{k\geq n_0+1} \int_{(\Omega,\mathbb{R})} \|f_k\|_E \, dm \\
&\leq \int_{(\Omega,\mathbb{R})} \|h_1\|_E \, dm + \frac{\eta}{2} \\
&\leq \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm + \frac{\eta}{2} + \frac{\eta}{2} \\
&= \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm + \eta
\end{aligned}$$

Hence,

$$\sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|h_n\|_E \, dm \leq \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm + \eta$$

We conclude that: If $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$, then for any $\eta > 0$,

there exists a sequence $(h_n)_{n\geq 1} \in \mathcal{E}(\Omega, \mathcal{A}, E)$, such that:

$$f \in S(h_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$$

and

$$\sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|h_n\|_E \, dm \leq \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm \quad + \eta$$

3.2. The Bochner integral on $\mathbb{R}$

The natural order on $\mathbb{R}$ was used in the general construction of the (RVM-MI) integration in the step 2M and we saw how that approach allowed to integrate with respect to a measure (finite or infinite).

When it comes to real valued-mapping, by restricting ourselves to finte measure, we will see that the Bochner and the RVM-MI integrals are exactly the same.

THEOREM 3.7. A real valued measurable and finite m-a.e mapping, $f : (\Omega, \mathcal{A}, m) \to \bar{\mathbb{R}}$ is integrable in the sense of the Real Valued Mapping Modern Integration (RVM-MI) if and only if its Bochner integral and its RVM-MI integral coincide.

Proof of Theorem 1:
Let us consider a real-valued measurable and finite m-a.e mapping, $f : (\Omega, \mathcal{A}, m) \to \bar{\mathbb{R}}$.

We denote its Bochner integral by: $\int_{(\Omega,\mathbb{R},B)} f \ dm$ and its modern integral by: $\int_{(\Omega,\mathbb{R},MI)} f \ dm$, whenever they make sense.

Suppose $\exists! \alpha_i \in \bar{\mathbb{R}}+$ and a partition of $\Omega, \{A_i, i = 1, ..., q\}$ such that

$$f = \sum_{i=1}^{p} \alpha_i \mathbf{1}_{A\mathrm{j}} \ \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$$

Then,

$$\int_{(\Omega,\mathbb{R},B)} f\ dm\ = \sum_{i=1}^{p} \alpha_i m(A_{\mathrm{i}}) = \int_{(\Omega,\mathbb{R},MI)} f\ dm$$

So by definition, we have

$$\int_{(\Omega,\mathbb{R},B)} f\ dm\ = \int_{(\Omega,\mathbb{R},MI)} f\ dm$$

for elementary functions.

Thus in the rest of the proof, for an elementary function g, whether we are talking about, the Bochner integral or the modern integral, we will just write : $\int_{(\Omega,\mathbb{R})} g\ dm$

Let us now consider the case of measurable functions.

(a) Let f be integrable in the sense of RVM-MI. We want to show that f is Bochner integrable and that the two integrals coincide.

Since the modern integral of f exists (in $\mathbb{R}$), its positive part f^+ and negative part f^- are therefore integrable.

Now, Since f^+ is a real-valued non-negative function, there exists a non-decreasing sequence $(f_n^{(1)})_{n\geq 1}$ of non-negative real-valued elementary functions such that:

$$f_n^{(1)} \uparrow f^+ \ as\ n \to \infty$$

So, $f_n^{(1)} \to f^+ \ as\ n \to \infty \quad and \quad \forall\ n \geq 1,\ \ |f_n^{(1)}| \leq f^+$

So by the Dominated Convergence Theorem in the sense of RVM-MI, we have:

$$\int_{(\Omega,\mathbb{R})} f_n^{(1)}\, dm \;\to \int_{(\Omega,\mathbb{R},MI)} f^+\, dm$$

Now, set:

$$f_0^{(1)} = 0 \quad and \quad h_n^{(1)} = f_n^{(1)} - f_{n-1}^{(1)} \;\; for \;\; n \geq 1$$

So,

$$h_1^{(1)} + h_2^{(1)} + ... + h_{n-1}^{(1)} + h_n^{(1)} \;=\; f_1^{(1)} - f_0^{(1)} + f_2^{(1)} - f_1^{(1)} + ... + f_{n-1}^{(1)} - f_{n-2}^{(1)} + f_n^{(1)} - f_{n-1}^{(1)}$$

$$\implies \;\; h_1^{(1)} + h_2^{(1)} + ... + h_{n-1}^{(1)} + h_n^{(1)} \;=\; f_n^{(1)} - f_0^{(1)} = f_n^{(1)}$$

$$\implies f_n^{(1)} \;=\; h_1^{(1)} + ... + h_n^{(1)}$$

Since $f_n^{(1)} \quad and \quad f_{n-1}^{(1)} \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R}), \quad \forall\, n \geq 1$,
it is clear that $(h_n^{(1)})_{n\geq 1} \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$

Moreover,

$$f_n^{(1)} \to f^+ \quad \implies \quad h_1^{(1)} + ... + h_n^{(1)} \to f^+ \implies f^+ = \lim_{n\to\infty} \sum_{k=1}^{n} h_k^{(1)}$$

$$\implies f^+ = \sum_{n=1}^{\infty} h_n^{(1)}$$

Also, we have that, $(f_n^{(1)})_{n\geq 1}$ is a non-decreasing sequence.
This implies, $h_n^{(1)} = f_n^{(1)} - f_{n-1}^{(1)} \geq 0$.

So, for all $k \geq 1$,

$$\sum_{n=1}^{k} \int_{(\Omega,\mathbb{R})} |h_n^{(1)}| \, dm = \int_{(\Omega,\mathbb{R})} \sum_{n=1}^{k} h_n^{(1)} \, dm = \int_{(\Omega,\mathbb{R})} f_k^{(1)} \, dm$$

By letting $k \to \infty$, we get:

(3.2.1)
$$\sum_{n=1}^{+\infty} \int_{(\Omega,\mathbb{R})} |h_n^{(1)}| \, dm \leq \int_{(\Omega,\mathbb{R},MI)} f^+ \, dm$$

So we got a sequence $(h_n^{(1)})_{n\geq 1} \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$ such that $f^+ = \sum\limits_{n=1}^{+\infty} h_n^{(1)}$ and

$$\sum_{n=1}^{+\infty} \int_{(\Omega,\mathbb{R})} |h_n^{(1)}| \, dm \leq \int_{(\Omega,\mathbb{R},MI)} f^+ \, dm \; < +\infty$$

(Since f^+ is integrable)

Repeating the same for f^-, we also get a sequence $(h_n^{(2)})_{n\geq 1} \subseteq \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$ such that $f^- = \sum\limits_{n=1}^{+\infty} h_n^{(2)}$ and

$$\sum_{n=1}^{+\infty} \int_{(\Omega,\mathbb{R})} |h_n^{(2)}| \, dm \leq \int_{(\Omega,\mathbb{R},MI)} f^- \, dm \; < +\infty$$

Hence, by taking $h_n = h_n^{(1)} - h_n^{(2)}, \; n \geq 1$, we get, $(h_n)_{n\geq 1} \subseteq \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$ with

$$f = f^+ - f^- = \sum_{n=1}^{+\infty} h_n^{(1)} - \sum_{n=1}^{+\infty} h_n^{(2)} = \sum_{n=1}^{\infty} (h_n^{(1)} - h_n^{(2)}) = \sum_{n=1}^{+\infty} h_n$$

Moreover,

$$\begin{aligned}
\sum_{n=1}^{+\infty}\int_{(\Omega,\mathbb{R})}|h_n|\,dm &= \sum_{n=1}^{+\infty}\int_{(\Omega,\mathbb{R})}|h_n^{(1)}-h_n^{(2)}|\,dm \\
&\leq \sum_{n=1}^{+\infty}\int_{(\Omega,\mathbb{R})}|h_n^{(1)}|\,dm+\sum_{n=1}^{+\infty}\int_{(\Omega,\mathbb{R})}|h_n^{(2)}|\,dm \\
&\leq \int_{(\Omega,\mathbb{R},MI)}f^{+}\,dm+\int_{(\Omega,\mathbb{R},MI)}f^{-}\,dm \\
&\leq \int_{(\Omega,\mathbb{R},MI)}|f|\,dm<+\infty
\end{aligned}$$

So, we have that:

$$\sum_{n=1}^{\infty}\int_{(\Omega,\mathbb{R})}|h_n|\,dm<+\infty \quad and \quad f=\sum_{n=1}^{+\infty}h_n$$

We deduce that,

$$f\in\ S(h_n,n\geq 1,\mathcal{E}(\Omega,\mathcal{A},\mathbb{R}))$$

Hence f is Bochner integrable and its Bochner integral is :

$$\int_{(\Omega,\mathbb{R},B)}f\,dm=\sum_{n\geq 1}\int_{(\Omega,\mathbb{R})}h_n\,dm$$

Next,

$$\begin{aligned}
\sum_{n\geq 1}\int_{(\Omega,\mathbb{R})} h_n\, dm &= \lim_{k\to+\infty}\sum_{n=1}^{k}\int_{(\Omega,\mathbb{R})} h_n\, dm \\
&= \lim_{k\to+\infty}\int_{(\Omega,\mathbb{R})}\sum_{n=1}^{k} h_n\, dm \\
&= \lim_{k\to+\infty}\int_{(\Omega,\mathbb{R})}(\sum_{n=1}^{k} h_n^1 - \sum_{n=1}^{k} h_n^2)\, dm \\
&= \lim_{k\to+\infty}\int_{(\Omega,\mathbb{R})} f_k^{(1)} - f_k^{(2)}\, dm \\
&= (\lim_{k\to\infty}\int_{(\Omega,\mathbb{R})} f_k^{(1)}\, dm) - (\lim_{k\to+\infty}\int_{(\Omega,\mathbb{R})} f_k^{(2)}\, dm) \\
&= \int_{(\Omega,\mathbb{R},MI)} f^+\, dm - \int_{(\Omega,\mathbb{R},MI)} f^-\, dm \\
&= \int_{(\Omega,\mathbb{R},MI)} f\, dm
\end{aligned}$$

.

We then have that:

$$\int_{(\Omega,\mathbb{R},B)} f\, dm = \sum_{n\geq 1}\int_{(\Omega,\mathbb{R})} h_n\, dm = \int_{(\Omega,\mathbb{R},MI)} f\, dm$$

(b) Let f be Bochner integrable. We want to show that f is integrable in the sense of RVM-MI and that the two integrals coincide.

f is Bochner integrable implies that: $\exists\ (f_n)_{n\geq 1} \subseteq \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$ such that,

$$\sum_{n\geq 1}\int_{(\Omega,\mathbb{R})} |f_n|\, dm < +\infty \quad and, \quad f = \sum_{n\geq 1} f_n\ , \quad m - a.e$$

and,

$$\int_{(\Omega,\mathbb{R},B)} f\ dm = \lim_{k\to\infty}\sum_{n=1}^{k}\int_{(\Omega,\mathbb{R})} f_n\ dm \qquad (***)$$

Take $g_k = \sum_{n=1}^{k} f_n, \quad k \geq 1$

In the sense of RVM-MI, the $g_k^{'}\ s$ integrable (We recall that the measure is finite here) and

$$g_k = \sum_{n=1}^{k} f_n \rightarrow \sum_{n\geq 1}^{\infty} f_n = f \ \ as \ \ k \rightarrow \infty$$

Moreover; $\forall k \geq 1$,

$$|g_k| = |\sum_{n=1}^{k} f_n| \leq \sum_{n=1}^{\infty} |f_n|$$

($\sum_{n=1}^{\infty} |f_n|$ Integrable from (***) and the MCT in the RVM-MI sense)

Then by the Dominated Convergence Theorem of the RVM-MI scheme, we have that :

$$\int_{(\Omega,\mathbb{R},MI)} f\ dm = \lim_{k\to\infty}\int_{(\Omega,\mathbb{R})} g_k\ dm = \lim_{k\to\infty}\sum_{n=1}^{k}\int_{(\Omega,\mathbb{R})} f_n\ dm = \int_{(\Omega,\mathbb{R},B)} f\ dm \qquad by(***)$$

This complete the proof.

The main conclusions of this section are the following:

(1) The real-valued Bochner integral is exactly the modern real-valued integral with respect to a measure; provided the measure is bounded.

(2) The Banach valued Bochner integral is an extension of the modern integral with respect to finite measure to normed and complete space.

(3) The Bochner approach provides an alternative construction of the modern integral with respect to a bounded measure, independently of the natural order of $\mathbb{R}$.

3.3. Properties and limit theorems for Banach-Valued Bochner Integral

The main focus in this part is to discuss convergence theorems. We will also show that the class of Bochner integrable functions is a Banach space. We will end this part by establishing a dominated convergence theorem in the sense of Bochner Integration.

3.3.1. The σ-convergence theorem (SCT). In the construction of the Modern real valued integral of measurable function, if the non-decreasing sequence $(f_n)_{n\geq 1}$ of non-negative elementary function in Step 2M is only a sequence of non-negative mappings, then we still have that:

$$\int_{(\Omega,\mathbb{R})} f\ dm = \lim_{n\to\infty} \int_{(\Omega,\mathbb{R})} f_n\ dm \ \ ,$$

and this formula defining the integral of f is known as the Monotone Convergence Theorem (MCT).
In the sense of RVM-MI, the MCT (Which gives birth to the Dominated Convergence Theorem (DCT)) is the most important convergence theorem.

Our goal here, as regards to Bochner integration, is to give a generalization of Step 2B, where we do not require that the sequence $(f_n)_{n\geq 1}$ is necessarily composed of elementary functions.
We can even notice that, this generalization of Step 2B, is an "analogue" of the MCT (in the RVM-MI sense), since here we will just require f_n to be B-integrable for each n.

Let us denote by $\mathcal{L}^1(\Omega, \mathcal{A}, m, E)$,in short $\mathcal{L}^1(\Omega, E)$, the class of all Bochner integrable functions $f : (\Omega, \mathcal{A}, m) \to E$.
We then have:

Theorem 3.8. Let $f : (\Omega, \mathcal{A}, m) \to E$ be measurable mapping and suppose that there exists a sequence $(f_n)_{n\geq 1} \subseteq \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$ such that

$$f \in S(f_n, n \geq 1, \mathcal{L}^1(\Omega, \mathcal{A}, m, E)).$$

Then, $f \in \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$ and

$$\int_{(\Omega,E)} f\ dm = \sum_{n\geq 1} \int_{(\Omega,E)} f_n\ dm \tag{3.3.1}$$

$$\int_{(\Omega,\mathbb{R})} \|f - \sum_{n=1}^{k} f_n\|_E\ dm \to 0\ as\ k \to +\infty \tag{3.3.2}$$

Proof of Theorem 2:

Let $f : (\Omega, \mathcal{A}, m) \to E$ be a measurable mapping such that:

$\exists (f_n)_{n\geq 1} \subseteq \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$ such that $f \in S(f_n, n \geq 1, \mathcal{L}^1(\Omega, \mathcal{A}, m, E))$.

We then have that:

$$(I1) : \sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm \; < \infty \quad and \quad (I2) : f = \sum_{n\geq 1} f_n$$

So,

$$(I1) \implies \sum_{n\geq p+1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm \to 0 \; as \; p \to \infty$$

Moreover, applying the MCT, we get that:

(I1) implies that $\sum\limits_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm = \int_{(\Omega,\mathbb{R})} \sum\limits_{n\geq 1} \|f_n\|_E \, dm < \infty$

$\implies \sum\limits_{n\geq 1} \|f_n\|_E \;\; is \; finite \; m - a.e$

$\implies \sum\limits_{n\geq p+1} \|f_n\|_E \; \to 0 \; m - a.e \; as \; p \to \infty$

Therefore, we have that:

$$\sum_{n\geq p+1} \|f_n\|_E \; \to 0 \; m - a.e \;\; and \quad \sum_{n\geq p+1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm \to 0 \; as \; p \to \infty$$

Now, fix $\eta > 0$. Suppose that $(\varepsilon_n)_{n\geq 1}$ is a sequence of positive number such that:

$\sum_{n\geq 1} \varepsilon_n = 1$. (For example, we take $\varepsilon_n = 2^{-n},\ n \geq 1$).
So we have the geometric progression $\sum_{n\geq 1} (\frac{1}{2})^n = 1$

Now, since f_n is Bochner integrable for each $n \geq 1$, by using point (D) in Lemma 3.6, for each n, we deduce that: $\exists (f_{n,p})_{p\geq 1} \subseteq \mathcal{E}(\Omega, \mathcal{A}, E)$ such that:

$f_n \in S(f_{n,p}; p \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))\ \ \forall\, n \geq 1$ and

$$\sum_{p\geq 1} \int_{(\Omega,\mathbb{R})} \|f_{n,p}\|_E \, dm \ \leq \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm + \eta\varepsilon_n$$

Let us consider two integers $r \geq 1$ and $s \geq 1$ and set:

$$h_{r,s} = \sum_{n=1}^{r} \sum_{p=1}^{s} f_{n,p},$$

we have:

$$\begin{aligned}\|h_{r,s}-f\|_E &\leq \|\sum_{n=1}^{r}\sum_{p=1}^{s}(f_{n,p})-(\sum_{n=1}^{r}f_n+\sum_{p\geq r+1}f_p)\|_E\\ &\leq \|\sum_{n=1}^{r}(\sum_{p=1}^{s}(f_{n,p})-f_n)\|_E+\sum_{p\geq r+1}\|f_p\|_E\\ &\leq \sum_{n=1}^{r}\|(\sum_{p=1}^{s}(f_{n,p})-f_n)\|_E+\sum_{p\geq r+1}\|f_p\|_E\end{aligned}$$

This implies that:

$$\|h_{r,s}-f\|_E \leq \sum_{n=1}^{r}\|(\sum_{p=1}^{s}(f_{n,p})-f_n)\|_E+\sum_{p\geq r+1}\|f_p\|_E \qquad (a)$$

Now, Since $\forall n \geq 1$ we have that $f_n \in S(f_{n,p}; p \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$, we have that: for each $n \geq 1$,

$$\sum_{p=1}^{s} f_{n,p} \to f_n \;\; in\ E, \;\; m-a.e \;\; as\ s \to \infty$$

i.e, there exists a measurable subset $\Omega_0 \subset \Omega$ such that $m(\Omega_0^c) = 0$ and that for any $\omega \in \Omega_0$, for all $n \geq 1$, $\exists\ s(n,\omega)$ such that,
$\forall s \geq s(n,\omega)$ we have that:

$$\|f_n - (\sum_{p=1}^{s} f_{n,p})\|_E \leq \ \eta\varepsilon_n$$

Applying this to the previous formula (a), for $\omega \in \Omega_0$ and for $s \geq s(n, \omega)$, we have,

$$\|h_{r,s}(\omega) - f(\omega)\|_E \leq \sum_{n=1}^{r} \eta\varepsilon_n + \sum_{p \geq r+1} \|f_p(\omega)\|_E$$

$$\Longrightarrow \|h_{r,s}(\omega) - f(\omega)\|_E \leq \eta \sum_{n=1}^{r} \varepsilon_n + \sum_{p \geq r+1} \|f_p(\omega)\|_E$$

So by letting $(r, s) \to \{\infty\}^2$, we get:

$$\lim_{(r,s) \to \{\infty\}^2} \|h_{r,s}(\omega) - f(\omega)\|_E \leq \eta$$

(Since, $\sum_{n \geq 1} \varepsilon_n = 1$ and $\sum_{n \geq p+1} \|f_n\|_E \ \to 0 \ m - a.e \ as \ p \to \infty$)

Next, By letting $\eta \to 0$, we get,

$$\lim_{(r,s)\to\{\infty\}^2} \|h_{r,s}(\omega) - f(\omega)\|_E \leq 0$$

$$\Longrightarrow \quad \lim_{(r,s)\to\{\infty\}^2} \|h_{r,s}(\omega) - f(\omega)\|_E = 0$$

$$\Longrightarrow \quad \|h_{r,s}(\omega) - f(\omega)\|_E \to 0 \;\; as(r,s) \to \{\infty\}^2$$

$$\Longrightarrow \quad f = \lim_{(r,s)\to\{\infty\}^2} h_{r,s} \quad m - a.e$$

$$\Longrightarrow \quad f = \lim_{(r,s)\to\{\infty\}^2} \sum_{n=1}^{r}\sum_{p=1}^{s} f_{n,p} \quad m - a.e$$

So,

$$f = \sum_{n\geq 1,\; p\geq 1} f_{n,p} \quad m - a.e \tag{3.3.3}$$

We also have:

$$\begin{aligned}
\sum_{n\geq 1,\; p\geq 1} \int_{(\Omega,\mathbb{R})} \|f_{n,p}\|_E \, dm &= \sum_{n\geq 1} (\sum_{p\geq 1} \int_{(\Omega,\mathbb{R})} \|f_{n,p}\|_E \, dm) \\
&\leq \sum_{n\geq 1} (\int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm + \eta\varepsilon_n) \\
&= \sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm + \sum_{n\geq 1} \eta\varepsilon_n \\
&= \sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm + \eta < +\infty
\end{aligned}$$

(Since $f \in S(f_n, n \geq 1, \mathcal{L}^1(\Omega, E))$)

We deduce that,

$$\sum_{n\geq 1,\ p\geq 1} \int_{(\Omega,\mathbb{R})} \|f_{n,p}\|_E \ dm < +\infty$$

So we have that:

(3.3.4) $f = \sum_{n\geq 1,\ p\geq 1} f_{n,p} \quad m - a.e \quad and \quad \sum_{n\geq 1,\ p\geq 1} \int_{(\Omega,\mathbb{R})} \|f_{n,p}\|_E \ dm < +\infty$

We conclude that, $f \in S(f_{n,p}, n \geq 1, p \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E)) \qquad (4.4)$

Hence,

$$f \in \mathcal{L}^1(\Omega, \mathcal{A}, E) \ \ and \ \ \int_{(\Omega,E)} f \ dm = \sum_{n\geq 1,\ p\geq 1} \int_{(\Omega,E)} f_{n,p} \ dm$$

Moreover, Since $\forall n \geq 1,\ f_n$ is Bochner integrable and

$$f_n \in S(f_{n,p}, p \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$$

then,

$$\int_{(\Omega,E)} f_n \ dm = \sum_{p\geq 1} \int_{(\Omega,E)} f_{n,p} \ dm$$

Therefore,

$$\int_{(\Omega,E)} f \ dm = \sum_{n\geq 1} \sum_{p\geq 1} \int_{(\Omega,E)} f_{n,p} \ dm = \sum_{n\geq 1} \int_{(\Omega,E)} f_n \ dm$$

We deduce that,

$$\int_{(\Omega,E)} f\ dm = \sum_{n\geq 1} \int_{(\Omega,E)} f_n\ dm \quad (Which\ is\ (3.3.1))$$

Furthermore,

$$\begin{aligned}
\int_{(\Omega,\mathbb{R})} \|f - \sum_{n=1}^{k} f_n\|_E\ dm &\leq \int_{(\Omega,\mathbb{R})} \|\sum_{n\geq 1} f_n - \sum_{n=1}^{k} f_n\|_E\ dm \\
&= \int_{(\Omega,\mathbb{R})} \|\sum_{n\geq k+1} f_n\|_E\ dm \\
&\leq \sum_{n\geq k+1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \to 0 \quad as\ k \to \infty
\end{aligned}$$

(as the tail end of the converging series $\sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E$)

We thus obtained (3.3.2)

3.3.2. The space $\mathcal{L}^1(\Omega, \mathcal{A}, m, E)$. We begin this part by reformulating the notion of limits of measurable mappings in the real case into the Banach case. We will just replace the absolute value by the Banch norm $\|.\|_E$ and use the properties of Banach spaces.
Here again, we consider bounded measure, since the notion of infiniteness is not defined on the Banach space E.

(A)- Different Limits

(1)-Limit in measure: A sequence $(f_n)_{n\geq 1}$ of m-a.e defined E-valued measurable mappings, defined on $(\Omega, \mathcal{A}, m)$, converges in measure to $f : (\Omega, \mathcal{A}, m) \to E$, m-a.e defined, as $n \to +\infty$ denoted by:

$$f_n \to_m f \ \ as \ \ n \to +\infty$$

if and only if, for any $\eta > 0$,

$$m(\|f_n - f\|_E > \eta) \to 0 \ \ as \ \ n \to +\infty \tag{3.3.5}$$

The sequence $(f_n)_{n\geq 1}$ is Cauchy in measure

if and only if, for any $\eta > 0$,

$$m(\|f_p - f_q\|_E > \eta) \to 0 \ \ as \ \ (p, q) \to \{+\infty\}^2 \tag{3.3.6}$$

(2)-Limit almost everywhere with respect to a measure (m-a.e):

A sequence $(f_n)_{n\geq 1}$ of m-a.e defined E-valued measurable mappings, defined on $(\Omega, \mathcal{A}, m)$, converges almost everywhere with respect to m (m-a.e) to $f : (\Omega, \mathcal{A}, m) \to E$, m-a.e defined, as $n \to +\infty$ denoted by:

$$f_n \to f \ , m - a.e \ \ as \ \ n \to +\infty$$

if and only if,

$$m(\|f_n - f\|_E \nrightarrow 0 \ as \ n \to +\infty) = 0 \tag{3.3.7}$$

(where, ($\|f_n - f\|_E \nrightarrow 0$) is the measurable subset of Ω where f_n does not converge to f).

if and only if, for any $\eta > 0$,

(3.3.8) $$m\ (\bigcap_{n\geq 1}\ \bigcup_{p\geq n} \|f_n - f\|_E > \eta) = 0$$

The sequence $(f_n)_{n\geq 1}$ is Cauchy m-a.e

if and only if,

(3.3.9) $$m(\|f_p - f_q\|_E \nrightarrow 0\ \ as\ \ (p,q) \rightarrow \{+\infty\}^2) = 0$$

if and only if, for any $\eta > 0$,

(3.3.10) $$m\ (\bigcap_{n\geq 1}\ \bigcup_{p\geq 0}\ \bigcup_{q\geq 0} \|f_{n+p} - f_{n+q}\|_E > \eta) = 0$$

(3)- Relationship between convergence in measure and convergence almost everywhere:

This part is also the reformulation of the relationship of these two modes of convergence in the real case into the Banach case. We then have that: for a sequence $(f_n)_{n\geq 1}$ of m-a.e defined E-valued measurable mappings,

(3-a) $(f_n)_{n\geq 1}$ converges in measure if and only if $(f_n)_{n\geq 1}$ is Cauchy in measure.

(3-b) $(f_n)_{n\geq 1}$ converges m-a.e if and onl if $(f_n)_{n\geq 1}$ is Cauchy in m-a.e.

(3-c) If $(f_n)_{n\geq 1}$ converges m-a.e, then it converges in measure to f.

(3-d) If $(f_n)_{n\geq 1}$ is Cauchy in measure, then it converges in measure to a measurable and m-a.e defined function f, and there exists a subsequence $(f_{n_k})_{k\geq 1}$ that converges to f m-a.e.

(3-d) If $(f_n)_{n\geq 1}$ converges to f in measure to f, then there exists a subsequence $(f_{n_k})_{k\geq 1}$ that converges to f m-a.e.

3.4. The space $L^1(\Omega, \mathcal{A}, m, E)$, in short $L^1(\Omega, E)$

We recall that the space $\mathcal{L}^1(\Omega, \mathcal{A}, m, E)$ is a linear space.
In fact, for $f \in \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$, the mapping

$$N(f) = \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm \in \mathbb{R}$$

is well defined.

Let $L^1(\Omega, \mathcal{A}, m, E)$ be the space of equivalence classes of the binary equivalence relation $\mathcal{R}$ defined by:

$$\forall (f,g) \in (\mathcal{L}^1(\Omega, \mathcal{A}, m, E))^2, \;\; f \, \mathcal{R} \, g \Leftrightarrow f = g, \; m - a.e$$

We define by $\overset{\circ}{f}$ and $\overset{\circ}{g}$, the classes of $f \in \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$ and $g \in \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$.

For $\lambda \in \mathbb{R}$, we define the operations:

$$\overset{\circ}{f} + \overset{\circ}{g} = \overset{\circ}{\overbrace{f+g}}, \quad \lambda \overset{\circ}{f} = \overset{\circ}{\overbrace{\lambda f}}$$

Also, $g \in \overset{\circ}{f} \implies f = g, \; m - a.e$. We then deduce that, for $g \in \overset{\circ}{f}$,

$$\int_{(\Omega,\mathbb{R})} \|f\|_E \, dm = \int_{(\Omega,\mathbb{R})} \|g\|_E \, dm$$

In fact the mapping

$$N(\overset{\circ}{f}) = \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm$$

is a norm on $L^1(\Omega, \mathcal{A}, m, E)$

Proof:

(i)

$$\begin{aligned} N(\overset{\circ}{f}) = 0 \quad &\Leftrightarrow \quad \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm = 0 \\ &\Leftrightarrow \quad \|f\|_E = 0 \; a.e \\ &\Leftrightarrow \quad f = 0 \; a.e \\ &\Leftrightarrow \quad \overset{\circ}{f} = 0 \end{aligned}$$

(ii)

$$N(\lambda \overset{\circ}{f}) = N(\overset{\circ}{\overbrace{\lambda f}}) = \int_{(\Omega,\mathbb{R})} \|\lambda f\|_E \, dm = |\lambda| \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm = |\lambda| N(\overset{\circ}{f})$$

(iii)

$$\begin{aligned} N(\overset{\circ}{f} + \overset{\circ}{g}) = N(\overbrace{f+g}^{\circ}) &= \int_{(\Omega,\mathbb{R})} \|f+g\|_E \, dm \\ &\leq \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm + \int_{(\Omega,\mathbb{R})} \|g\|_E \, dm \\ &\leq N(\overset{\circ}{f}) + N(\overset{\circ}{g}) \end{aligned}$$

By (i), (ii), and (iii), we conclude that:$N(\overset{\circ}{f}) = \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm$ is a norm on $L^1(\Omega, \mathcal{A}, m, E)$,
and we can write: $N(\overset{\circ}{f}) = \|f\|_{L^1(\Omega,\mathcal{A},m,E)}$.
This completes the proof.

Now for the sake of simplicity, we use f instead of $\overset{\circ}{f}$ and we write $N(\overset{\circ}{f}) = \|f\|$
We then have:

$$\|f\|_{L^1(\Omega,\mathcal{A},m,E)} = \|f\|_{L^1(\Omega,E)} = \int_{(\Omega,\mathbb{R})} \|f\|_E \, dm$$

We endow the space $L^1(\Omega, \mathcal{A}, m, E)$ with the norm $\|.\|_{L^1(\Omega,E)}$.
Now, the question that come to our mind is:

Is the normed linear space $(L^1(\Omega, \mathcal{A}, m, E), +, *, \|.\|_{L^1(\Omega,E)})$ **complete?**

To answer this question, we first need to know what convergence in $L^1(\Omega, E)$ means.

Definition 3.9. *The sequence* $(f_n)_{n\geq 1}$ *converges to* f *in* $L^1(\Omega, E)$ *if and only if*

$$\|f_n - f\|_{L^1(\Omega,E)} \to \quad 0 \;\; n \to +\infty$$

Consequences of Definition 3.9:

(1)- We first recall the Markov Inequality that states:
for a real-valued measurable function f, and $\forall \lambda > 0$,

$$m(|f| > \lambda) \leq \frac{1}{\lambda} \int |f| \, dm$$

We will here applied the Markov inequality to E-valued mapping.

Now, if $(f_n)_{n\geq 1}$ converges to f in $L^1(\Omega, E)$, then by Markov Inequality, for any $\eta > 0$, we have:

$$m(\|f_n - f\|_E > \eta) \leq \frac{1}{\eta} \int_{(\Omega,\mathbb{R})} \|f_n - f\|_E \, dm = \frac{1}{\eta} \|f_n - f\|_{L^1(\Omega,E)} \to 0 \; as \; n \to +\infty$$

which implies that f_n converges in measure to f, and then, there exists a subsequence $(f_{n_k})_{k\geq 1}$ that converges to f m-a.e.

(2)- If $(f_n)_{n\geq 1}$ converges to f in $L^1(\Omega, E)$, then, as $n \to +\infty$,

$$| \; \|f_n\|_{L^1(\Omega,E)} - \|f_n\|_{L^1(\Omega,E)} \; |_{\mathbb{R}} \leq \|f_n - f\|_{L^1(\Omega,E)} \to 0$$

3.4.1. The space $L^1(\Omega, \mathcal{A}, m, E)$ is Banach. We first consider a direct consequence of theorem 2:

PROPOSITION 3.10. *Consider a sequence $(f_n)_{n\geq 1} \subseteq \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$ such that:*

$$\sum_{n\geq 1} \|f_n\|_{L^1(\Omega,E)} < +\infty$$

Then,

$$f = \sum_{n\geq 1} f_n \;\; is\, defined\, m - a.e$$

and as $k \to +\infty$,

$$\sum_{n=1}^{k} f_n \to f \;\; in\; L^1(\Omega, E)$$

Proof of Proposition 3.10:

Let $(f_n)_{n\geq 1} \subseteq \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$,
By assumption,

$$\sum_{n\geq 1} \|f_n\|_{L^1(\Omega,E)} = \sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \, dm < +\infty$$

Then

$$\sum_{n\geq 1} f_n \;\; is\, defined\, m - a.e$$

(See proof of point(A) in Lemma 3.6)

Let $f = \sum_{n\geq 1} f_n$ m-a.e.
Then,

$$f \in S(f_n, n \geq 1, \mathcal{L}^1(\Omega, \mathcal{A}, m, E))$$

So applying theorem 2, we get that: $f \in \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$ with:

$$\int_{(\Omega,E)} f\ dm = \sum_{n\geq 1} f_n \int_{(\Omega,E)} f_n\ dm$$

and,

$$\int_{(\Omega,\mathbb{R})} \|f - \sum_{n=1}^{k} f_n\|_E\ dm \to 0\ \ as\ k \to +\infty$$

This implies, $\|f - \sum_{n=1}^{k} f_n\|_{L^1(\Omega,E)} \to 0\ \ as\ k \to +\infty$
Hence,

$$\sum_{n=1}^{k} f_n \to f\ \ in\ L^1(\Omega, E)\ \ as\ k \to +\infty$$

This complete the proof.

THEOREM 3.11. The space $(L^1(\Omega, \mathcal{A}, m, E), +, *, \|.\|_{L^1(\Omega,E)})$ is a Banach space,

More precisely, we have:

If $(f_n)_{n\geq 1}$ is a Cauchy sequence in $(L^1(\Omega, \mathcal{A}, m, E), +, *, \|.\|_{L^1(\Omega,E)})$,
then, there exists $f \in L^1(\Omega, \mathcal{A}, m, E)$ such that, as $n \to +\infty$,

(3.4.1) $$\|f_n - f\|_{L^1(\Omega,E)} \to 0$$

and,

(3.4.2) $$\|\int_{(\Omega,E)} f_n\ dm - \int_{(\Omega,E)} f\ dm\|_E \to 0$$

Proof of theorem 3:

Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $(L^1(\Omega, \mathcal{A}, m, E), +, *, \|.\|_{L^1(\Omega,E)})$.
That is

$$\|f_p - f_q\|_{L^1(\Omega,E)} \to 0 \ \ as \ \ (p,q) \to \{+\infty\}^2$$

Now, applying Markov's inequality, for $\eta > 0$ fix, we get:

$$m(\|f_p - f_q\|_E > \eta) \leq \frac{1}{\eta} \int_{(\Omega,\mathbb{R})} \|f_p - f_q\|_E \; dm \leq \frac{1}{\eta} \|f_p - f_q\|_{L^1(\Omega,E)} \ \to 0$$

as $(p,q) \to \{+\infty\}^2$

This implies that $(f_n)_{n\geq 1}$ is Cauchy in measure.
Therefore, $(f_n)_{n\geq 1}$ converges in measure to a m-a.e defined function f, and there exists a subsequence $(f_{n_k})_{k\geq 1}$ which converges both in measure and m-a.e.

The proof of the construction of $(f_{n_k})_{k\geq 1}$ is given in "Solution of exercise 7, Doc 06-08, chapter 7" in the book "Measure Theory and Integration By and For the learner" by Prof Samb Lo.

At step 2 of the construction, by replacing the absolute value by the norm $\|.\|_{L^1(\Omega,E)}$, We get that:

$$\forall k \geq 1, \quad \|f_{n_k} - f_{n_k - 1}\|_{L^1(\Omega,E)} \leq \frac{1}{2^k}, \qquad with \ \ f_{n_0} = 0$$

We set: $g_k = f_{n_k} - f_{n_k - 1}, \quad k \geq 1$

So $(g_k)_{k\geq 1} \subseteq L^1(\Omega, E)$ and $f_{n_k} = \sum_{j=1}^{k} g_j$

Also,

$$\|g_k\|_{L^1(\Omega,E)} = \|f_{n_k} - f_{n_k - 1}\|_{L^1(\Omega,E)} \leq \frac{1}{2^k}$$

We deduce that:

$$\sum_{k\geq 1} \|g_k\|_{L^1(\Omega,E)} \leq \sum_{k\geq 1} \frac{1}{2^k} < +\infty$$

Applying proposition 3.10, we get that:

$f = \sum_{k\geq 1} g_k$ is defined m-a.e, with,

$$\int_{(\Omega,E)} f_{n_k}\, dm = \int_{(\Omega,E)} \sum_{j=1}^{k} g_j\, dm \to \int_{(\Omega,E)} f\, dm \quad as\ k \to +\infty$$

and,

$$f_{n_k} = \sum_{j=1}^{k} g_j \to f \quad in\ L^1(\Omega, E)$$

Therefore, $\forall k \geq 1, \forall n \geq 1$,

$$\|f_n - f\|_{L^1(\Omega,E)} \leq \|f_n - f_{n_k}\|_{L^1(\Omega,E)} + \|f_{n_k} - f\|_{L^1(\Omega,E)}$$

and

$$\begin{aligned}
\| \int f_n \, dm - \int f \, dm \|_E &= \| \int_{(\Omega,E)} f_n \, dm - \int_{(\Omega,E)} f_{n_k} \, dm + \int_{(\Omega,E)} f_{n_k} \, dm + \int_{(\Omega,E)} f \, dm \|_E \\
&\leq \int_{(\Omega,\mathbb{R})} \|f_n - f_{n_k}\|_E \, dm + \int_{(\Omega,\mathbb{R})} \|f_{n_k} - f\|_E \, dm \\
&= \|f_n - f_{n_k}\|_{L^1(\Omega,E)} + \|f_{n_k} - f\|_{L^1(\Omega,E)}
\end{aligned}$$

as $k \to +\infty$ and $n \to +\infty$, we obtain:

$$\|f_n - f\|_{L^1(\Omega,E)} \to 0$$

and

$$\| \int_{(\Omega,E)} f_n \, dm - \int_{(\Omega,E)} f \, dm \|_E \to 0$$

This ends the proof.

3.5. Young-Fatou-Lebesgue Convergence Theorem in $L^1(\Omega, \mathcal{A}, m, E)$

We have:

THEOREM 3.12. Let $(f_n)_{n\geq 1}$ be a sequence in $L^1(\Omega, \mathcal{A}, m, E, \|.\|_{L^1(\Omega,E)})$, and f a measurable mapping from $(\Omega, \mathcal{A})$ to E m-a.e defined. Suppose that:

(a) f_n converges to f m-a.e or in measure.

(b) There exists a family $\{h, (h_n)_{n\geq 1}\} \subset L^1(\Omega, \mathcal{A}, m, E)$ composed of non-negative functions such that:

$$(b1)\ As\ n \to \infty,\ h_n \to h\ \ and\ \ \int_{(\Omega,\mathbb{R})} h_n\ dm \to \int_{(\Omega,\mathbb{R})} h\ dm$$

and

$$(b2)\ \|f_n\|_E \leq h_n\ for\ all\ n \geq 1$$

Then $f \in L^1(\Omega, \mathcal{A}, m, E)$ and $as\ n \to \infty$,

$$\int_{\Omega,E} f_n\ dm \to \int_{\Omega,E} f\ dm$$

Proof of Theorem:

1- Suppose f_n converges to $f\ m-a.e$.

We have that $as\ \{p;q\} \to \{+\infty\}^2$,

(a)- $|\ \|f_p - f_q\|_E\ | \leq h_p + h_q$ **(b)-** $h_p + h_q \to 2h\ \ and\ \ \int_{(\Omega,\mathbb{R})} h_p + h_q\ dm \to \int_{(\Omega,\mathbb{R})} 2h\ dm$.

Also, since $(f_n)_{n\geq 1}$ converges to f m-a.e, then $(f_n)_{n\geq 1}$ is Cauchy m-a.e.

So $\|f_p - f_q\|_E \to 0 \;\; m - a.e \;\; as \; \{p; q\} \to \{+\infty\}^2$

Applying the Dominated Convergence Theorem, we have:

$$\|f_p - f_q\|_{L^1(\Omega,E)} = \int_{(\Omega,\mathbb{R})} \|f_p - f_q\|_E \to 0 \; as \; \{p; q\} \to \{+\infty\}^2$$

So, $(f_n)_{n\geq 1}$ is Cauchy in $(L^1(\Omega, E), \|.\|_{L^1(\Omega,E)})$, which is complete. Therefore, $(f_n)_{n\geq 1}$ converges in $L^1(\Omega, E)$ to f and $f \in L^1(\Omega, E)$. Hence,

$$\int_{(\Omega,E)} f_n \; dm \to \int_{(\Omega,E)} f_n \; dm \;\; in \; E$$

;,

2- Suppose $(f_n)_{n\geq 1}$ converges to f in measure. We prove that $f \in L^1(\Omega, E)$, $\|f_n - f\|_{L^1(\Omega,E)} \to 0 \; as \; n \to \infty$ and $\int_{\Omega,E} f_n \; dm \to \int_{(\Omega,E)} f \; dm$ in E.

But $(f_n)_{n\geq 1}$ converges in measure to f implies that $\exists$ a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ which converges to f m-a.e and in measure.

So, $(f_{n_k})_{k\geq 1}$ converges m-a.e to f implies that $f \in L^1(\Omega, E)$ and $\|f_{n_k} - f\|_{L^1(\Omega,E)} \to 0 \; as \; n \to \infty$ using part 1

Now consider an arbitrary subsequence $(\|f_{n_k} - f\|_{L^1(\Omega,E)})_{k\geq 1}$ of $(\|f_n - f\|_{L^1(\Omega,E)})_{n\geq 1}$.

We recall that the subsequence $(f_{n_k})_{k\geq 1}$ still converges to f in measure.

Thus $\exists$ a subsequence $(f_{n_{k_j}})_{j\geq 1}$ of $(f_{n_k})_{k\geq 1}$ that converges to f m-a.e.

Therefore, by still using part 1, we have that, $f \in L^1(\Omega, E)$ and $\|f_{n_{k_j}} - f\|_{L^1(\Omega,E)} \to 0 \; as \; n \to \infty$.

So by the Prohorov criterion for convergence,

$$\|f_n - f\|_{L^1(\Omega,E)} \to 0 \; as \; n \to \infty$$

Hence,

$$\int_{\Omega,E} f_n \, dm \to \int_{\Omega,E} f \;\; in \; E.$$

CHAPTER 4

Integration of mappings with respect to a measure on lattice spaces

In this part we are going to discuss the construction of the Bochner integral in Ordered vector spaces. This part is just an introductory part to Integration in Lattice spaces.

4.1. Another view on the construction of the Bochner integral

Ordered Banach spaces are Banach spaces by definition, so in this setion we are just recalling the construction done in in chapter 3, we will also recall some of the properties of the Bochner integral.

Let us recall the construction of the Bochner integral of functions with values in Banach spaces.

Consider the measure space$(\Omega, \mathcal{A}, m)$ with m a bounded measure.
Consider the Banach space $(E, +, *, \|.\|_E)$ over $\mathbb{R}$
We constructed the Bochner integral of a measurable function $f : (\Omega, \mathcal{A}, m) \to E$ in two steps.

Step 1: Integral of an E-valued elementary function.

Let

$$f=\sum_{j=1}^{p} x_j \mathbf{1}_{B_j}, \ \ \textit{be an elementary function}$$

where $p \geq 1, x_j \in E, B_j \in \mathcal{A}, B_1 + B_2 + ... + B_p = \Omega \ , B_j \cap B_i = \emptyset \ (i \neq j)$.

The class of all elementary functions with values in E is denoted by $\mathcal{E}(\Omega, \mathcal{A}, E)$.

We also recall that for any $f \in \mathcal{E}(\Omega, \mathcal{A}, E)$ we have the real valued mapping, still called the norm of f:

$$\|f\|_E : (\Omega, \mathcal{A}, m) \to \mathbb{R}$$

defined by $\forall \omega \in \Omega, \ \|f\|_E(\omega) = \|f(\omega)\|_E$.

So, $\|f\|_E = \sum_{j=1}^{p} \|x_j\|_E \mathbf{1}_{B_j} \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$.

We define the Bochner integral of $f \in \mathcal{E}(\Omega, \mathcal{A}, E)$ by:

$$\int_{(\Omega,E)} f dm = \sum_{j=1}^{p} x_j m(B_j) \in E.$$

Step 2:

Definition 4.1. *A measurable function is said to be B-Integrable (Bochner-Integrable), if and only if there exists a sequence* $(f_n)_{n\,\geq\,1} \subseteq \mathcal{E}(\Omega, \mathcal{A}, E)$ *such that,*

$$\sum_{n\geq 1} \int_{(\Omega,\mathbb{R})} \|f_n\|_E \; dm < \infty \qquad (I1)$$

and,

$$f = \sum_{n\geq 1} f_n \;, \quad m - a.e \qquad (I2)$$

If conditions (I1) and (I2) both hold, we write: $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$.

We define the Bochner integral of $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$ by:

$$\int_{(\Omega,E)} f \; dm = \sum_{n\geq 1} \int_{(\Omega,E)} f_n \; dm$$

The space of the classes of Bochner integrable function defined on Ω with values in E is denoted by:

$$L^1(\Omega, \mathcal{A}, m, E) \;\; in\ short\ L^1(\Omega, E)$$

We endow the space $L^1(\Omega, E)$ with the norm $\|.\|_{L^1(\Omega,E)}$ defined by:

$$\|f\|_{L^1(\Omega,E)} = \int_{\Omega,\mathbb{R}} \|f\|_E \; dm$$

The space $(L^1(\Omega, E), +, *, \|.\|_{L^1(\Omega,E)})$ is a Banach space.

The Bochner integral construted has a lot of properties. let us recall some of the properties of the Bochner integral that still hold on Ordered Banach spaces.

(1) Let $f : (\Omega, \mathcal{A}) \rightarrow E$ be measurable mapping and suppose that there exists a sequence $(f_n)_{n\geq 1} \subseteq \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$ such that

$$f \in S(f_n, n \geq 1, \mathcal{L}^1(\Omega, \mathcal{A}, m, E)).$$

Then, $f \in \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$ and

$$\int_{(\Omega,E)} f \, dm = \sum_{n\geq 1} \int_{(\Omega,E)} f_n \, dm$$

(2) Let $(f_n)_{n\geq 1}$ be a sequence in $L^1(\Omega, \mathcal{A}, m, E, \|.\|_{L^1(\Omega,E)})$, and f a measurable mapping from $(\Omega, \mathcal{A}, m)$ to E m-a.e defined. Suppose that:

(a) f_n converges to f m-a.e or in measure.

(b) There exists a family $\{h, (h_n)_{n\geq 1}\} \subset L^1(\Omega, \mathcal{A}, m, E)$ composed of non-negative functions such that:

$$(b1)\ As\ n \rightarrow \infty,\ h_n \rightarrow h\ \ and\ \ \int_{(\Omega,\mathbb{R})} h_n \, dm \rightarrow \int_{(\Omega,\mathbb{R})} h \, dm$$

and

$$(b2)\ \|f_n\|_E \leq h_n\ for\ all n \geq 1$$

Then $f \in L^1(\Omega, \mathcal{A}, m, E)$ and $as\ n \rightarrow \infty$,

$$\int_{(\Omega,E)} f_n \, dm \rightarrow \int_{(\Omega,E)} f \, dm$$

There is one property of the Bochner integral that has not been discussed in chapter 3. That property is concerned with bounded linear maps between two Banach Spaces.

THEOREM 4.2. Let E be a Banach space and $T : D \to E$ be a linear and continuous map, where D is a Banach space with norm $\|.\|_D$.
If $f \in L^1(\Omega, \mathcal{A}, D)$, then $Tof \in L^1(\Omega, \mathcal{A}, E)$ and

$$T\bigg(\int_{(\Omega,D)} f\ dm\bigg) = \int_{(\Omega,E)} Tof\ dm$$

4.2. Properties of Ordered Vector Spaces

Since this chapter is just an introduction to integration on Lattice spaces especially Ordered Banach Spaces, it is important to define some important notions.

(1) A preorder is a binary relation that is reflexive and transitive.

(2) A partial order is an antisymmetric preorder.

(3) Given a vector space X over $\mathbb{R}$, $(X, \leq)$ is called an ordered vector space, if $\leq$ is a partial order.

(4) A subset K of a vector space X over $\mathbb{R}$ is called a cone if it satisfies:

$$K \bigcap \{-K\} = \{0\}, \;\; K + K \subseteq K, \;\; rK \subseteq K, \; \forall\, r > 0$$

(5) A cone K is said to be generating if $K - K = X$

(6) The set $X^+ = \{x \in X : 0 \leq x\}$ is the positive cone of X and is convex.

(7) Given a cone X^+ in X, we say that (X, X^+) is an ordered vector space. The partial order leq on X is defined by:

$$x \leq y \ \ if \ \ y - x \in X^+$$

(8) The space (X, X^+) is also denoted by $(X, \leq)$ or simply by X if $\leq$ is well understood.

(9) Let X be an odered vector space:
   (i) For every $x, y \in X$, the set:

$$[x, y] = \{z \in X : x \leq z \leq y\}$$

   is called order interval.
   (ii) A subset $A \subseteq X$ is said to be order convex if

$$\forall a, b \in A, [a, b] \in A$$

   (iii) An ordered vector space X is said to be Archimedean (We also say that X^+ has the Archimedian property) if:

$$\forall n \geq 1, \ ny \leq x \implies y \leq 0$$

   for $x, y \in X$.
   (iv) A partially ordered set $(L, \leq)$ is called a lattice if each two-elements subset $\{a, b\} \in L$ has a supremum and an infimum.

(v) An odered vector space X is said to be a vector lattice (or Riesz space) if every non-empty finite subset of X has a least upper bound.

The following theorem gives us an important property of the Bochner integral on ordered vector spaces.

THEOREM 4.3. If E is an ordered Banach space, then $L^1(\Omega, E)$ is an ordered Banach space under the partial ordering given by:

$$f \leq g \ (inL^1(\Omega, E) \ \Leftrightarrow f \leq g \ m - a.e$$

4.3. Two main Results of the integration on Ordered Banach Spaces

THEOREM 4.4. Let E be a Banach lattice. Then $L^1(\Omega, \mathcal{A}, E)$ is a Banach lattice and the Bochner integral on E is linear and order preserving.

THEOREM 4.5. Let E be an ordered Banach space for which E^+ is closed. Then the Bochner integral on E is order preserving.

CHAPTER 5

Conclusion and Perspectives

Using the knowledge of Measure Theory, the integration of real-valued measurable mappings can be extended to an integration of measurable mappings with values in Banach spaces, called Bochner Integration.

On $\mathbb{R}$ the Bochner integral and the Modern Integral coincide when using bounded measure.

We have been able to establish limit theorems for Banach-valued measurable mappings and we also establish an important result which is the Dominated Convergence theorem on Banach spaces in general.

In fact, the integration in linear spaces goes back to Bochner (1933), Dunford (1936a), Dunford (1936b), Birkoff (1935), Birkoff (1937).

But these works are summarized in Pettis (1938) whose paper is considered as the seminal introduction to vector valued integration. Currently, the topic of integration in Banach spaces, locally convex spaces and in other abstract spaces is very popular, and this thesis is part of this trend.

Moreover, it is important to emphasize on the fact that this thesis fully discussed the notion of Bochner integration in general Banach Spaces; and also extended some well-know theorems about real-valued functions to functions with values in Banach spaces. Also, since the last chapter was just an introductory part to integration in Lattice spaces, We discussed the notion of ordered Banach spaces and gave the properties on the Bochner integral on Banach Lattice spaces.

As perspectives, this thesis can be more complete by discussing in depth Bochner integration in Lattice spaces and Locally convex spaces. Random set integration on Banach spaces using Bochner integrals and applications is an interesting research topic after reading this thesis that gives us a good idea of Bochner integration.

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