APPROXIMATION OF SOLUTIONS OF SPLIT INVERSE PROBLEM FOR MULTI-VALUED DEMI-CONTRACTIVE MAPPINGS IN HILBERT SPACES

A Thesis Presented to the Department of Pure and Applied Mathematics

African University of Science and Technology

In Partial Fulfilment of the Requirements for the Degree of Master of Science

By

Isyaku Mustapha

Abuja, Nigeria

December, 2017.

Certification

This is to certify that the thesis titled "APPROXIMATION OF SOLUTIONS OF SPLIT INVERSE PROBLEM FOR MULTI-VALUED DEMI-CONTRACTIVE MAPPINGS IN HILBERT SPACES" submitted to the school of post-graduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research carried out by Isyaku Mustapha in the Department of Pure and Applied Mathematics.

APPROXIMATION OF SOLUTIONS OF SPLIT INVERSE PROBLEM FOR MULTI-VALUED DEMI-CONTRACTIVE MAPPINGS IN HILBERT SPACES

By

Isyaku Mustapha

A THESIS APPROVED BY THE DEPARTMENT OF PURE AND APPLIED MATHEMATICS

RECOMMENDED:

Dr. A. U. Bello (Supervisor)

Prof. C.E. Chidume (Co-supervisor)

Prof. C.E. Chidume (Head of Department)

APPROVED:

Chief Academic Officer

Date

Abstract

Let H_1 and H_2 be two Hilbert spaces and $A_j : H_1 \to H_2$ be bounded linear operators and $U_i : H_1 \to 2^{H_1}$, $T_j : H_2 \to 2^{H_2}$, $1 \leq i \leq n$, $1 \leq j \leq r$ be two multi-valued demi-contractive operators with demi-contractive constants β_i and μ_j, respectively, such that

$$\Gamma = \{x \in C = \cap_{i=1}^{n} F(U_i) \ : \ A_j x \in F(T_j)\} \neq \emptyset.$$

Moreover, suppose $U_i(x)$ and $U_j(y)$ are bounded $\forall x \in H_1, \ y \in H_2, \ 1 \leq i \leq n, \ 1 \leq j \leq r$ and such that $U_i(p) = \{p\}$ $\forall p \in F(U_i)$, $1 \leq i \leq n$ and $T_j(p) = \{p\}$ $\forall p \in F(T_j)$, $1 \leq j \leq r$. Then, for some $x_0 \in H_1$, the sequence $\{x_k\}$ defined by

$$\begin{cases} q_k = x_k + \gamma \sum_{j=1}^{r} A_j^*(b_{j,k} - A_j x_k), \ \textit{where} \ b_{j,k} \in T_j(A_j x_k) \ \forall 1 \leq j \leq r, \\ x_{k+1} = (1-\alpha_k) q_k + \frac{\alpha_k}{n} \sum_{i=1}^{n} u_{i,k}, \ \textit{where} \ u_{i,k} \in U_i(q_k) \ \forall \ 1 \leq i \leq n, \end{cases}$$

converges weakly to $x^* \in \Gamma$. Moreover, if there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle u_i - q, \sigma \rangle \geq 0 \quad \forall \ \ 1 \leq i \leq n, \ u_i \in U_i(q) \ \textit{and} \ q \in H_1, \\ \langle A_j^*(b_j - A_j y), \sigma \rangle \geq 0 \quad \forall \ \ 1 \leq j \leq r, \ b_j \in T_j(A_j y) \ \textit{and} \ y \in H_1, \end{cases}$$

then, the sequence $\{x_k\}$ converges strongly to $x^* \in \Gamma$.

Acknowledgements

All the praises and thanks be to God the Almighty, the Lord of mankind, Jinns and all that exists, whom out of his infinite mercy provided me with life, sound health and strength to pursue this program to the end. Peace and blessings be upon his favorite servant and Messenger, Muhammad, his household, companions and those who follow their footsteps till the day of Judgement.

I would like to express my sincere appreciation to my supervisor, Dr. Abdulmalik Usman Bello, for his supervision, encouragement, guidance and support. His joy, enthusiasm and lack of despair in resolving challenging research problems were motivational and contagious for me. It is indeed, a great honour to work with such a distinguished person.

My profound gratitude also goes to my co-supervisor, Professor Charles Ejike Chidume, FAS, for his supervision and guidance; and indeed, for sharing his vast research experience without hiding anything from me.

Special thanks to African University of Science and Technology (AUST), Nelson Mandela Institution (NMI), for awarding me scholarship during my M.Sc. program.

This journey would not have been completed without the immeasurable support of my beloved parents, Alhaji Isyaku Abubakar and Khadija Hamisu. They have been instrumental in my entire life. I sincerely appreciate their love, support and advices during my entire studies.

My wholeherted gratitude goes to Professor J. O. Fatokun, Dr. Yusuf Ahmed Omeiza, Mr. Lawal Haruna and Mr. Iliyasu Shehu for their unceasing help throughout my studies right from undergraduate up to M.Sc. level. Their support and kindness toward me will remain indelible in my memory.
Thanks to my entire M.Sc. colleques department of pure and applied mathematics, African University of Sciences and Technology. My appreciation also goes to my friends; Jabir Shehu Minjibir, Sulaiman Abdulfatah, Paul Iyoke, Mubarak Abubakar, and Aliyu Abubakar Runka for their support and prayers during the program.

Many thanks to my brothers and sisters; Muhammad, Zaharaddeed, Hafeez, Abdulhakim, Fatima, Ummulkhairi, Rukayya, Bilkisu, Zainab, Amina and Maryam.

Special thanks to my dear friend, Aisha Aminu Adam for her advice, encouragement and support during my M.Sc. program.

Isyaku Mustapha
December, 2017.

Dedication

To my parents, Alhaji Isyaku Abubakar and Khadija Hamisu.

Contents

Chapter 1

General Introduction

The contents of this thesis fall within the general area of nonlinear functional analysis, an area which has attracted the attention of prominent mathematicians due to its diverse applications in numerous fields of sciences. One of the most important problems in this area which has gained much attention in recent years is the split inverse problem (SIP).

The (SIP) concerns a model in which there are two spaces X and Y and a bounded linear operator $A : X \to Y$. In addition, two inverse problems are involved. The first one, denoted by IP_1, is formulated in the space X and the second one, denoted by IP_2, is formulated in the space Y.
Given these facts, the Split Inverse Problem (SIP) is formulated as follows:

$$\text{find a point } x^* \in X \text{ that solves } IP_1 \tag{1.0.1}$$

such that the point

$$y^* = Ax^* \in Y \text{ solves } IP_2 \tag{1.0.2}$$

Many models of inverse problems can be cast in this frame work by choosing different inverse problems for IP_1 and IP_2. The Split Convex Feasibility Problem (SCFP) which is to find a point

$$x^* \in C \text{ such that } y^* = Ax^* \in Q \tag{1.0.3}$$

where A is a bounded liner operator and C and Q are nonempty, closed and convex sets, is the first instance of a SIP in which the two problems IP_1 and IP_2 are Convex Feasibility Problems which is to

$$\text{find a point } x^* \in C := \cap_{i=1}^{p} C_i \tag{1.0.4}$$

where C_i is nonempty, closed and convex for each $1 \leq i \leq n$, each. This was used for solving an inverse problem in radiation therapy treatment planning in [20]. More work on the SCFP can be found in [6, 20, 25, 33, 42, 45, 48]. Two candidates for IP_1 and IP_2 that come to mind are the mathematical models of the Convex Feasibility Problem (CFP) and the problem of constrained optimization. In particular, the CFP formalism is in itself at the core of the modeling of many inverse problems in various areas of mathematics and the physical sciences; see, e.g., [16] and references therein for an early example.

Over the past four decades, the CFP has been used to model significant real world inverse problems in sensor networks, in radiation therapy treatment planning, in resolution enhancement, in antenna design, in computerized tomography, in materials science, in watermarking, in data compression, in color imaging, in optics and neural networks, in graph matching and adaptive filtering, see [15] for exact references to all the above. More work on the CFP can be found in [7, 5, 18].

In order to demonstrate the generality of this SIP modeling we wish to present several special cases which are studied intensively in the literature. We first start with the formulation of the Convex Feasibility Problem (CFP) which stands at the core of the modeling of many inverse problems in various areas of mathematics and the physical sciences; for example in sensor networks, in radiation therapy treatment planning, in color imaging and adaptive filtering, see e.g., [5, 7, 16, 18] and references therein.

Problem 1.1. The Convex Feasibility Problem (CFP)
Let H be real Hilbert space and $C_i \subseteq H$, $1 \leq i \leq p$ be closed and convex sets. The CFP is:

$$\text{find a point } x^* \in C := \cap_{i=1}^{p} C_i \tag{1.0.5}$$

The first instance of the split inverse problems is due to Censor and Elfving in [19] and called the *Split Convex Feasibility Problem* (SCFP) which is a SIP with IP_1 and IP_2 as CFPs. This reformulation was employed for solving an inverse problem in intensity-modulated radiation therapy (IMRT) treatment planning, see [17]. Other real-world application for the SCFP include the Multi-Domain Adaptive Filtering (MDAF) [52] and navigation on the Pareto frontier in Multi-Criteria Optimization, see [27]. The problem formulates as follows.

Problem 1.2. The Split Convex Feasibility Problem (SCFP)
Let H_1 and H_2 be real Hilbert spaces. Let $C \subseteq H_1$ and $Q \subseteq H_2$ be two non-empty, closed and convex sets, in addition given a bounded linear operator $A : H_1 \to H_2$, the SCFP is:

$$\text{find a point } x^* \in C \text{ such that } y^* = Ax^* \in Q \tag{1.0.6}$$

The next natural development is a MSSCFP which allows a finite number of non-empty, closed and convex sets in the spaces H_1 and H_2, respectively, [20].

Problem 1.3. The Multiple Set Split Convex Feasibility Problem (MSSCFP)
Let H_1 and H_2 be real Hilbert spaces and let r and p be two natural numbers. Given C_i, $1 \leq i \leq p$ and Q_j, $1 \leq j \leq r$, closed and convex subsets of H_1 and H_2 respectively, in addition given a bounded liner operator $A : H_1 \to H_2$, the MSSCFP is find a point x^* such that

$$x^* \in C := \cap_{i=1}^{p} C_i \textit{ and } Ax^* \in Q := \cap_{j=1}^{r} Q_j \tag{1.0.7}$$

Masad and Reich [39] generalized the MSSCFP in which several bounded and linear operators $A_j : H_1 \to H_2$ are involved.

Problem 1.4. The Constrained Multiple Set Split Convex Feasibility Problem (CMSSCFP)
Let H_1 and H_2 be real Hilbert spaces and let r and p be two natural numbers. Given C_i, $1 \leq i \leq p$ and Q_j, $1 \leq j \leq r$, closed and convex subsets of H_1 and H_2 respectively, further for $1 \leq j \leq r$ let $A_j : H_1 \to H_2$ be bounded liner operators. The CMSSCFP is to find a point x^*

$$x^* \in \cap_{i=1}^{p} C_i \textit{ and } A_j(x^*) \in Q_j \tag{1.0.8}$$

Censor and Segal [22] replaced the CFPs in the above split inverse problem with fixed points problems.

Problem 1.5. The Split Common Fixed Point Problem (SCFPP)
Let H_1 and H_2 be real Hilbert spaces and let r and p be two natural numbers. Given operators $U_i : H_1 \to H_1$, $1 \leq i \leq p$, and $T_j : H_2 \to H_2$, $1 \leq j \leq r$, with non-empty $Fix(U_i) = C_i$ and $Fix(T_j) = Q_j$ respectively, and a bounded liner operator $A : H_1 \to H_2$. The SCFPP is to find a point x^* such that

$$x^* \in \cap_{i=1}^{p} C_i \text{ and } Ax^* \in \cap_{j=1}^{r} Q_j \tag{1.0.9}$$

Censor, Gibali and Reich [21] introduced the following Split Variational Inequality Problem (SVIP).

Problem 1.6. The Split Variational Inequality Problem (SVIP)
Let H_1 and H_2 be real Hilbert spaces. Given operators $f : H_1 \to H_1$, $g : H_2 \to H_2$, a bounded liner operator $A : H_1 \to H_2$, non-empty closed and convex sets $C \subseteq H_1$ and $Q \subseteq H_2$. The SVIP is formulated as follows:

$$\text{find a point } x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C \tag{1.0.10}$$

and such that

$$\text{the point } \ y^* = Ax^* \in Q \textit{ solves } \langle g(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in Q \tag{1.0.11}$$

Moudafi [34] generalized the SVIP and introduced the Split Monotone Variational Inclusion (SMVI)

Problem 1.7. The Split Monotone Variational Inclusion (SMVI)
Let H_1 and H_2 be real Hilbert spaces. Given operators $f : H_1 \to H_1$, $g : H_2 \to H_2$, a bounded liner operator $A : H_1 \to H_2$, and two multi-valued mappings $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$, the SMVI is formulated as follows:

$$\text{find a point } x^* \in H_1 \text{ such that } 0 \in f(x^*) + B_1(x^*) \tag{1.0.12}$$

and such that the point

$$y^* = Ax^* \in H_2 \textit{ solves } 0 \in g(y^*) + B_2(y^*) \tag{1.0.13}$$

Byrne, Censor, Gibali and Reich [8] generalized and introduced the following Split Common Null Point Problem (SCNPP).

Problem 1.8. The Split Common Null Point Problem (SCNPP)
Let H_1 and H_2 be real Hilbert spaces. Given multi-valued mappings $B_i : H_1 \to 2^{H_1}$, $1 \leq i \leq p$ $F_j : H_2 \to 2^{H_2}$,$1 \leq j \leq r$ respectively, and a bounded linear operators $A_j : H_1 \to H_2$,$1 \leq j \leq r$ the SCNPP is formulated as follows:

$$\text{find a point } x^* \in H_1 \text{ such that } 0 \in \cap_{i=1}^{p} B_i(x^*) \tag{1.0.14}$$

and such that the points

$$y_j^* = A_j x^* \in H_2 \text{ solve } 0 \in \cap_{j=1}^{r} F_j(y_j^*) \tag{1.0.15}$$

Following the above SIPs we wish to study a new SIP which is a generalization of the split common fixed points problem.

Problem 1.9. The General Split Common Fixed Point Problem (GSCFPP)
Let H_1 and H_2 be real Hilbert spaces and let r and p be two natural numbers. Given operators $U_i : H_1 \to H_1$, $1 \leq i \leq p$, and $T_j : H_2 \to H_2$, $1 \leq j \leq r$, with non-empty $Fix(U_i) = C_i$ and $Fix(T_j) = Q_j$ respectively,in addition, for $1 \leq j \leq r$ let $A_j : H_1 \to H_2$ be bounded linear operators. The GSCFPP is

$$\text{find a point } x^* \in C := \cap_{i=1}^{p} C_i \text{ such that } A_j x^* \in Q_j. \tag{1.0.16}$$

It is easy to see that if $i = j = 1$ and T and U are projection operators onto C and Q, respectively, then problem (1.5) is reduced to the well-known split feasibility problem (SFP) [6, 19]. which is to find

$$x^* \in C \text{ such that } Ax^* \in Q, \tag{1.0.17}$$

where C and Q are nonempty closed convex subsets in H_1 and H_2, respectively. The SFP is called an inverse problem because it can transformed into finding

$$x^* \in C \text{ such that } x^* \in A^{-1}Q. \tag{1.0.18}$$

We use Γ to denote the solution set of the SFP (1.0.2), that is,

$$\Gamma = \{x^* \in C : Ax^* \in Q\} = C \cap A^{-1}Q \tag{1.0.19}$$

and assume the consistency of (1.0.19) so that Γ is nonempty, closed and convex. Censor and Segal [1] are the first to study the SCFP and approximation of its solution. They introduced the following iterative scheme:

$$x_{n+1} = U(x_n - \tau A^*(I - T)Ax_n), \quad n \geq 0. \tag{1.0.20}$$

where τ is a properly chosen stepsize and A^* is the corresponding adjoint operator of A. Algorithm (1.0.20) was originally designed to solve problem (1.0.9) for directed operators. It is shown that if the stepsize τ is chosen in the interval $(0, \frac{2}{\|A\|^2})$, then the iterative sequence generated by (1.0.20) converges weakly to a solution of the SCFP whenever such a solution exists. Subsequently, this iterative scheme was used to approximate solutions of SCFP for quasi-nonexpansive operators [24], demi-contractive operators [33] and finitely many directed operators [14, 47]. In [24], the constant stepsize in (1.0.20) was replaced by a variable stepsize that does not depend on the operator norm $\|A\|$ since the computation of the norm is in general not an easy work in practice. In a recent work [2, 30], a modification of (1.0.20) was presented so that it generates an iterative sequence with a norm convergent property. We note that all works mentioned above are conducted under the framework of algorithm (1.0.20).

In 2010, Moudafi [33] proved the following weak convergence result for demi-contractive operatorss;

Algorithm 1.1: [33]
Let $x_0 \in H_1$ be arbitrary and let the sequence $\{x_k\}$ be defined by

$$x_{k+1} = (1-\alpha_k)u_k + \alpha_k U(u_k), \ k \geq 0, \tag{1.0.21}$$

where $u_k = x_k + \gamma A^*(T-I)Ax_k$, $\gamma \in (0, \frac{1-\mu}{\lambda})$, λ being the spectral radius of the operator A^*A and $\alpha_k \in (0,1)$.

Theorem 1.0.1 *Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be demi-contractive (with constants β, μ, respectively) with nonempty $Fix(U) = C$ and $Fix(T) = Q$. Assume that $U - I$ and $T - I$ are demi-closed at 0. If $\Gamma \neq \emptyset$, then any sequence $\{x_k\}$ generated by algorithm (1.1) converges weakly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu}{L}\big)$ and $\alpha_k \in \big(\delta, 1-\beta-\delta\big)$ for small enough $\delta > 0$.*

Recently, in [26], Gibali prove the following strong convergence result for demi-contractive operators;

Theorem 1.0.2 *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be demi-contractive (with constants β, μ, respectively) with nonempty $Fix(U) = C$ and $Fix(T) = Q$. Assume that $U - I$ and $T - I$ are demi-closed at 0 and that there exists $\sigma \neq 0 \in H_1$, such that*

$$\begin{cases} \langle U(q) - q, \sigma \rangle \geq 0 \quad \forall \ \ q \in H_1, \\ \langle A^*(T-I)Ay, \sigma \rangle \geq 0 \quad \forall \ \ y \in H_1. \end{cases} \tag{1.0.22}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (1.1) converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu}{L}\big)$ and $\alpha_k \in \big(\delta, 1-\beta-\delta\big)$ for small enough $\delta > 0$.

Motivated by the previous works on generalizations of CFP, in this thesis, we propose an algorithm for solving The General Split Common Fixed Point Problem (GSCFPP) for multi-valued demi-contractive operators in Hilbert space. More precisely, we obtain the following;

Algorithm 1.2
Initialization: let $x^* \in H_1$ be arbitrary.
Iterative step: for $k \in \mathbb{N}$ set

$$\begin{cases} q_k = x_k + \gamma \sum_{j=1}^{r} A_j^*(b_{j,k} - A_j x_k), \ \textit{where} \ b_{j,k} \in T_j(A_j x_k) \ \forall 1 \leq j \leq r, \\ x_{k+1} = (1-\alpha_k)q_k + \frac{\alpha_k}{n} \sum_{i=1}^{n} u_{i,k}, \ \textit{where} \ u_{i,k} \in U_i(q_k) \ \forall \ 1 \leq i \leq n, \end{cases}$$

where $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ with L being the spectral radius of the operator A^*A and $\alpha_k \in (0,1)$.

Theorem 1.0.3 *Let H_1 and H_2 be two real Hilbert spaces and $A_j : H_1 \to H_2$, $1 \le j \le r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \le i \le n$ and $T_j : H_2 \to 2^{H_2}$, $1 \le j \le r$ be multi-valued demi-contractive (with constants β_i, μ_j, respectively) such that $U_i(p) = \{p\}$ for all $p \in F(U_i)$ and nonempty $Fix(U_i) = C_i$ and $Fix(T_j) = Q_j$.*
Assume that there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle u_i - q, \sigma \rangle \ge 0 \quad \forall \;\; 1 \le i \le n, \;\; u_i \in U_i(q) \text{ and } q \in H_1, \\ \langle A_j^*(b_j - A_j y), \sigma \rangle \ge 0 \quad \forall \;\; 1 \le j \le r, \;\; b_j \in T_j(A_j y) \text{ and } y \in H_1. \end{cases} \tag{1.0.23}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (1.2) converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ and $\alpha_k \in \left(\delta, 1 - \beta_{max} - \delta\right)$ for small enough $\delta > 0$.

Chapter 2

Preliminaries

In this chapter, we give some fundamental results in Hilbert spaces and some basic definitions as well as some well-known results on multi-valued demi-contractive mappings. Finally, we discuss some concepts of Split Feasibility Problem.

2.1 Some Basic Results in Hilbert Spaces

2.1.1 Inner Product Space (IPS)

Definition 2.1.1 *Let X be a linear space. An inner product on X is a mapping*

$$\langle , \rangle : X \times X \to F(F = \mathbb{R} \text{ or } \mathbb{C})$$

which satisfies the following conditions:

(i) $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(iii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

for all $x, y, z \in X$ $\alpha, \beta \in F$.

The pair $(X, \langle\cdot, \cdot\rangle)$ is called an IPS.

Remark 2.1.2 *(i) For* $x \in X$, *we define* $\|x\| = \sqrt{\langle x, x\rangle}$. *Hence,* IPS *is a normed linear space, thus a metric space.*

(ii) From (ii) *and* (iii) *of definition* 2.1.1, *we have*

$$\langle z, \alpha x + \beta y\rangle = \overline{\alpha}\langle z, x\rangle + \overline{\beta}\langle z, y\rangle$$

for each $x, y, z \in X$, $\alpha \in \mathbb{C}$.

Example 2.1.3 *The linear space* $\mathbb{R}^n$, *with the function* $\langle , \rangle$ *defined for arbitrary vectors* $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, *by*

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

is an inner product space.

Definition 2.1.4 *A complete inner product space is called Hilbert space*

Proposition 2.1.5 *Let $(X, \langle\cdot,\cdot\rangle)$ be an IPS. Then for any $x, y \in X,$ and $\alpha \in [0,1]$ the following inequality holds:*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 - \alpha(1-\alpha)\|x-y\|^2 + (1-\alpha)\|y\|^2 \tag{2.1.1}$$

indeed,

$$\begin{aligned}\|\alpha x + (1-\alpha)y\|^2 &= \langle \alpha x + (1-\alpha)y, \alpha x + (1-\alpha)y, \rangle \\ &= \alpha^2\|x\|^2 + 2\alpha(1-\alpha)\langle x, y\rangle + (1-\alpha)^2\|y\|^2\end{aligned}$$

Noting that $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2$,
we have

$$\begin{aligned}\|\alpha x + (1-\alpha)y\|^2 &= \alpha^2\|x\|^2 + \alpha(1-\alpha)\Big[\|x\|^2 + \|y\|^2 - \|x-y\|^2\Big] + (1-\alpha)^2\|y\|^2 \\ &= \alpha\|x\|^2 - \alpha(1-\alpha)\|x-y\|^2 + (1-\alpha)\|y\|^2\end{aligned}$$

which completes the proof.

2.1.2 Metric Projection

Let (H, d) be a metric space and K be a nonempty subset of H. For every $x \in H$, the distance between the point x and K is denoted by $d(x, K)$ and is defined by:

$$d(x, K) := \inf_{y\in K} \|x - y\|.$$

The metric projection operator (also called the nearest point mapping) P_K defined on H is a mapping from H to 2^K such that

$$P_K(x) := \{y \in K : d(x, y) = d(x, K)\} \ \forall\ x \in H.$$

Theorem 2.1.6 *(The Projection Theorem) Let H be a real Hilbert space and K a closed subspace of H. For arbitrary vector x in H, there exists a unique vector $x^* \in K$ such that $\|x - x^*\| \leq \|x - y\|$ for all $y \in K$. Furthermore, $x^* \in K$ is the unique vector if and only if $(x - x^*) \perp K$.*

Lemma 2.1.7 *Let $P_K : H \to K$ be the metric projection from H onto a nonempty closed and convex K of H. Then*

(i) $z = P_K x$ if and only if

$$\langle x - z, z - y\rangle \geq 0 \ \forall\ y \in K \tag{2.1.2}$$

(ii) For all $y \in H,\ x \in K$,

$$\|x - P_K y\|^2 + \|P_K y - y\|^2 \leq \|x - y\|^2 \tag{2.1.3}$$

(iii) P_K is 1 - inverse strongly monotone on H, i.e, for all $x, y \in H$,

$$\langle P_K x - P_K y, x - y\rangle \geq \|P_K x - P_K y\|^2 \tag{2.1.4}$$

2.1.3 Demi-contractive Operators

Definition 2.1.8 *Let $T : H \to H$ be an operator and $D \subseteq H$ and $F(T) = \{x \in K : x = Tx\}$.*

- *The operator T is called nonexpansive, if $\forall x, y \in D$*

$$\|Tx - Ty\| \le \|x - y\| \tag{2.1.5}$$

- *T is called quasi-nonexpansive, if $\forall (x, q) \in D \times F(T)$*

$$\|Tx - q\| \le \|x - q\| \tag{2.1.6}$$

- *T is called k-strictly pseudo-contractive (see e.g., [29]), if there exists $k \in [0, 1)$ such that $\forall (x, y) \in D$*

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2 \tag{2.1.7}$$

- *T is called demi-contractive (see e.g., [4, 21, 28, 40]), if there exists $\beta \in [0, 1)$ such that $\forall (x, q) \in D \times Fix(T)$*

$$\|Tx - q\|^2 \le \|x - q\|^2 + \beta\|x - Tx\|^2 \tag{2.1.8}$$

Definition 2.1.9 *Let H be a real Hilbert space, an operator T is called demi-closed at $q \in H$ (see e.g., [3]), if*
for any sequence $\{x_k\}_{k=1}^{\infty}$ such that $x_k \rightharpoonup x^*$ and $Tx_k \to q$, we have $Tx^* = q$.

It is easy to see that (2.1.8) is equivalent to

$$\langle x - Tx, x - a\rangle \ge \frac{1-\beta}{2}\|x - Tx\|^2 \quad \forall (x, a) \in D \times Fix(T), \tag{2.1.9}$$

indeed, Since T is a demi-contractive we have

$$\begin{aligned} \|Tx - p\|^2 &\le \|x - p\|^2 + \beta\|x - Tx\|^2 \ \forall p \in F(T) \\ \Leftrightarrow \ -\beta\|x - Tx\|^2 &\le \|x - p\|^2 - \|Tx - p\|^2 \ \forall p \in F(T) \ . \quad . \quad . \ (i) \end{aligned}$$

We observe that

$$\begin{aligned} 2\langle x - Tx, x - p\rangle &= \|x - Tx\|^2 + \|x - p\|^2 - \|Tx - p\|^2 \\ \Leftrightarrow \ \|x - p\|^2 - \|Tx - p\|^2 &= 2\langle x - Tx, x - p\rangle - \|x - Tx\|^2 \end{aligned}$$

Using this in (i) we have

$$\begin{aligned} &-\beta\|x - Tx\|^2 \le 2\langle x - Tx, x - p\rangle - \|x - Tx\|^2 \\ &\qquad \Leftrightarrow \ \langle x - Tx, x - p\rangle \ge \frac{1-\beta}{2}\|x - Tx\|^2. \end{aligned}$$

The class of nonexpansive operators is properly contained in the class of quasi-nonexpansive. More precisely, the following inclusion is obvious;
Nonexpansive $\subset$ Quasi-nonexpansive $\subset$ Strictly pseudo-contractive $\subset$ Demi-contractive. We now give example of demicontractive mapping which is not quasi-nonexpansive.

Example 2.1.10 *(see, e.g., [12])*

$$f : [-2,1] \to [-2,1], \quad f(x) = -x^2 - x.$$

Definition 2.1.11 *Let H be a real Hilbert space. The map $D : 2^H \times 2^H \to \mathbb{R}^+$ defined by*

$$\begin{aligned} D(A,B) \quad &= max\{\sup_{y\in A} d(y,B), \sup_{x\in B} d(x,A)\} \text{ for all } A,B \in 2^H, \\ where \quad & d(y,B) := \inf_{x\in A} d(x,y), \end{aligned}$$

is called Hausedorff distance.

Remark 2.1.12 *In general, the map D is not a metric. However, it becomes a metric if it is defined on a set of closed and bounded subsets of H.*

Definition 2.1.13 *Let $T : H \to 2^H$ be a multi-valued mapping. An element $x^* \in H$ is said to be a fixed point of T if $x^* \in Tx^*$. We denote by $F(T)$ the fixed points set of T defined by*

$$F(T) := \{x \in H : x \in Tx\}. \tag{2.1.10}$$

Definition 2.1.14 *Let H be a real Hilbert space and $CB(H)$ be a set of closed and bounded subsets of H. $T : H \to 2^{CB(H)}$ be a multi-valued mapping. Then, T is said to be demi-closed at zero if for any sequence $\{x_k\} \subset H$ with $x_k \rightharpoonup x^*$, and $d(x_k, Tx_k) \longrightarrow 0$, we have $x^* \in Tx^*$.*

Definition 2.1.15 *Let H be a real Hilbert space.*

- *A multi-valued mapping $T : \mathcal{D}(T) \subseteq H \to 2^{CB(H)}$ is said to be nonexpansive (see e.g.,[23]), if*

$$D(Tx,Ty) \leq \|x-y\| \ \forall \ x,y \in \mathcal{D}(T) \tag{2.1.11}$$

- *The mapping $T : \mathcal{D}(T) \subseteq H \to 2^H$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and*

$$D(Tx,Tx^*) \leq \|x-x^*\| \ \forall \ x \in \mathcal{D}(T), \ x^* \in F(T). \tag{2.1.12}$$

- *The mapping $T : \mathcal{D}(T) \subseteq H \to 2^H$ is said to be k-strictly pseudo-contractive if there exists there exists a constant $k \in [0,1]$ such that for all $u \in Tx$, $v \in Ty$*

$$(D(Tx,Ty))^2 \leq \|x-y\|^2 + k\|x-y-(u-v)\|^2; \quad and \tag{2.1.13}$$

- *$T : \mathcal{D}(T) \subseteq H \to 2^H$ is said to be demi-contractive if $F(T) \neq \emptyset$ and there exists a constant $k \in [0,1]$ such that for all $x \in \mathcal{D}(T)$, $u \in Tx$*

$$(D(Tx,\{y\}))^2 \leq \|x-y\|^2 + k\|x-u\|^2. \tag{2.1.14}$$

The class of demi-contractive operators is a very important generalization of nonexpansive operators Also some operators that arise in optimization problems are of demi-contractive type. See for example, Chidume and Maruster [12].

Every multi-valued quasi-nonexpansive mapping is also multi-valued demi-contractive. However, the converse may not hold as shown in the following example.

Example 2.1.16 *(see e.g., [9]) Let $H = \mathbb{R}$ with the usual metric. Define $T : \mathbb{R} \to \mathbb{R}$ by*

$$Tx = \begin{cases} [-3x, -\frac{5x}{2}], & x \in [0, \infty), \\ [-\frac{5x}{2}, -3x], & x \in (-\infty, 0]. \end{cases} \tag{2.1.15}$$

We have that $F(T) = \{0\}$ and T is a multi-valued demi-contractive mapping which is not quasi-nonexpansive. In fact, for each $x \in (-\infty, 0) \cup (0, \infty)$, we have

$$\begin{aligned} \big(D(Tx, T0)\big)^2 &= |-3x - 0|^2 \\ &= 9|x - 0|^2, \end{aligned}$$

which implies that T is not quasi-nonexpansive.
Also, we have that

$$\begin{aligned} \big(d(x, Tx)\big)^2 &= |x - (-\frac{5x}{2})|^2 \\ &= \frac{49}{4}|x|^2. \end{aligned}$$

Thus,

$$\begin{aligned} \big(D(Tx, T0)\big)^2 &= |x - 0|^2 + 8|x - 0|^2 \\ &= |x - 0|^2 + \frac{32}{49}\big(d(x, Tx)\big)^2. \end{aligned}$$

Therefore, T is a demi-contractive mapping with constant $k = \frac{32}{49} \in (0, 1)$.

Lemma 2.1.17 *Let $A, B \in CB(X)$ and $a \in A$. For every $\gamma > 0$, there exixts $b \in B$ such that*

$$d(a, b) \leq D(A, B) + \gamma. \tag{2.1.16}$$

Lemma 2.1.18 *Let X be a reflexive real Banach space and $A, B \in CB(X)$. Assume that B is weakly closed. Then, for every $a \in A$, there exists $b \in B$ such that*

$$\|a - b\| \leq D(A, B). \tag{2.1.17}$$

Proof Let $a \in A$ and $\{\lambda_n\}$ be a sequence of positive real numbers such that $\lambda_n \to 0$ as $n \to \infty$.
from lemma 2.1.10, for each $n \geq 1$, there exists $b_n \in B$ such that

$$\|a - b_n\| \leq D(A, B) + \lambda_n. \tag{2.1.18}$$

It then follows that the sequence $\{b_n\}$ is bounded. Since X is reflexive and B is weakly closed, there exists a subsequence b_{n_k} of b_n that converges weakly to some $b \in B$. Now, using inequality (2.1.17), the fact that $\{a - b_{n_k}\}$ converges weakly to $a - b$ and $\lambda_{n_k} \to 0$, as $k \to \infty$, it follows that

$$\|a - b\| \leq \liminf \|a - b_{n_k}\| \leq D(A, B). \tag{2.1.19}$$

■

Proposition 2.1.19 *(see e.g., [11]) Let K be a nonempty subset of a real Hilbert space H and let $T : K \to CB(K)$ be a multi-valued β demi-contractive mapping. Assume that for every $p \in F(T)$, $Tp = \{p\}$. Then, there exists $L > 0$ such that*

$$D(Tx, Tp) \leq L\|x - p\| \ \forall\ x \in K,\ p \in F(T). \tag{2.1.20}$$

Proof Let $x, y \in \mathcal{D}(T)$ and $u \in Tx$. From Lemma 2.1.10, there exixts $v \in Tp$ such that

$$\|u - v\| \leq D(Tx, Ty). \tag{2.1.21}$$

Using the fact that T β demi-contractive, and inequality (2.1.17), we obtain the following estimates:

$$\begin{aligned}
(D(Tx, Tp))^2 &\leq \|x - p\|^2 + \beta\|x - u\|^2 \\
&\leq (\|x - p\| + \sqrt{\beta}\|x - u\|)^2 \\
\text{so that} & \\
D(Tx, Tp) &\leq \|x - p\| + \sqrt{\beta}\|x - u\| \\
&\leq \|x - p\| + \sqrt{\beta}[\|x - p\| + \|u - p\|] \\
&\leq \|x - p\| + \sqrt{\beta}\big[\|x - p\| + D(Tx, Tp)\big]. \\
\text{Hence,} & \\
D(Tx, Tp) &\leq \Big(\frac{1 + \sqrt{\beta}}{1 - \sqrt{\beta}}\Big)\|x - p\|.
\end{aligned}$$

Therefore, T is Lipchitzian with $L = \frac{(1+\sqrt{\beta})}{(1-\sqrt{\beta})}$. ■

Lemma 2.1.20 *(see e.g., [13]) Let E be a normed linear space, $B_1, B_2 \in CB(E)$ and $x_0, y_0 \in E$ arbitrary. Then the following hold;*

(i) $D(\{x_0\}, B_1) = \sup_{b_1 \in B_1} \|x_0 - b_1\|$

(ii) $D(\{x_0\}, B_1) = D(0, x_0 - B_1)$

(iii) $D(x_0 + B_1, y_0 + B_2) \leq \|x_0 - y_0\| + D(B_1, B_2)$

(iv) $D(B_1, B_2) = D(-B_1, -B_2)$

(v) $D(B_1, B_2) = D(x_0 + B_1, x_0 + B_2)$

Proof

(i) It is obvious that $d(x_0, B_1) = \sup_{x_0 \in x_0} d(x_0, B_1)$. On the other hand, for any $b_1 \in B_1$, we have $d(b_1, x_0) = \|b_1 - x_0\| \geq d(x_0, B_1)$.

Taking sup over B_1 we have

$\sup_{b_1 \in B_1} d(b_1, x_0) \geq d(x_0, B_1)$, and therefore,

$$\begin{aligned} D(\{x_0\}, B_1) &:= max\{\sup_{b_1 \in B_1} d(b_1, \{x_0\}), \sup_{x_0 \in \{x_0\}} d(x_0, B_1)\} \\ &= max\{\sup_{b_1 \in B_1} d(b_1, \{x_0\}), d(x_0, B_1)\} \\ &= \sup_{b_1 \in B_1} d(b_1, \{x_0\}). \end{aligned}$$

(ii)

$$\begin{aligned} D(\{x_0\}, B_1) &:= max\{\sup_{b_1 \in B_1} d(b_1, \{x_0\}), d(x_0, B_1)\} \\ &= max\{\sup_{b_1 \in B_1} \|x_0 - b_1\|, \inf_{b_1 \in B_1} \|x_0 - b_1\|\} \\ &= max\{\sup_{b_1 \in B_1} d(0, x_0 - B_1), d(0, x_0 - B_1)\} \\ &= D(\{x_0\}, x_0 - B_1). \end{aligned}$$

(iii) It is known that for any set $B \subseteq E, x, y \in E$ arbitrary, the inequality

$$d(x, B) \leq \|x - y\| + d(y, B) \text{ holds.}$$

Using this inequality we have

$$\begin{aligned} d(x_0 + b_1, y_0 + B_2) &\leq \|(x_0 + b_1) - (y_0 + b_1)\| + d(y_0 + b_1, y_0 + B_2) \\ &= \|x_0 - y_0\| + d(b_1, B_2), \end{aligned}$$

and similarly

$$d(y_0 + b_2, x_0 + B_1) \leq \|x_0 - y_0\| + d(b_1, B_2)$$

Therefore, taking supremum over B_1 and B_2 respectively, we have

$$\sup_{b_1 \in B_1} d(x_0 + b_1, y_0 + B_2) \quad \leq \|x_0 - y_0\| + \sup_{b_1 \in B_1} d(b_1, B_2)$$

and

$$\sup_{b_2 \in B_2} d(y_0 + b_2, x_0 + B_1) \quad \leq \|x_0 - y_0\| + \sup_{b_2 \in B_2} d(b_2, B_1).$$

Thus,

$$D(x_0 + B_1, y_0 + B_2) \leq \|x_0 - y_0\| + D(B_1, B_2).$$

(iv) We have

$$\begin{aligned} D(-B_1, -B_2) &= max\{\sup_{-b_1 \in -B_1} d(-b_1, -B_2), \sup_{-b_2 \in -B_2} d(-b_2, -B_1), \} \\ &= max\{\sup_{b_1 \in B_1} d(b_1, B_2), \sup_{b_2 \in B_2} d(b_2, B_1)\} \\ &= D(B_1, B_2). \end{aligned}$$

(v) By definition, we have

$$
\begin{aligned}
D(x_0+B_1, x_0+B_2) &= max\{\sup_{b_1\in B_1} d(x_0+b_1, x_0+B_2), \sup_{b_2\in B_2} d(x_0+b_2, x_0+B_1)\} \\
&= max\{\sup_{b_1\in B_1} d(b_1, B_2), \sup_{b_2\in B_2} d(b_2, B_1)\} \\
&= D(B_1, B_2).
\end{aligned}
$$

■

Lemma 2.1.21 *Let $T : \mathcal{D}(T) \subseteq H \to 2^H$ be a demi-contractive, then*

$$\langle x-u, x-p\rangle \geq \frac{1-\beta}{2}\|x-u\|^2 \ \forall u \in Tx. \tag{2.1.22}$$

Proof Definition of T gives

$$
\begin{aligned}
(D(Tx,p))^2 &\leq \|x-p\|^2 + \beta\|x-u\|^2 \ \forall u \in Tx \\
D(Tx,p) &\leq \sqrt{\|x-p\|^2 + \beta\|x-u\|^2} \ \forall u \in Tx
\end{aligned}
$$

We have by lemma (2.1.19(i)) that $D(Tx,p) = \sup_{u\in Tx}\|u-p\|$.

Using this result we get

$$-\beta\|x-u\|^2 \leq \|x-p\|^2 - \|u-p\|^2 \ \forall u \in Tx \ . \quad . \quad . \ (i)$$

We observe that $2\langle x-u, x-p\rangle = \|x-u\|^2 + \|x-p\|^2 - \|u-p\|^2$,

this implies $\|x-p\|^2 - \|u-p\|^2 = 2\langle x-u, x-p\rangle - \|x-u\|^2$.

Using this in (i) we have

$$-\beta\|x-u\|^2 \leq 2\langle x-u, x-p\rangle - \|x-u\|^2,$$

hence,

$$\frac{1-\beta}{2}\|x-u\|^2 \leq \langle x-u, x-p\rangle \ \forall u \in Tx,$$

which completes the proof. ■

2.1.4 Split Feasibility Problem (SFP)

Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ a bounded linear operator. Suppose that C and Q are nonempty, closed and convex subsets of H_1 and H_2, respectively. The SFP is formulated as follows:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q \tag{2.1.23}$$

It is easy to see that x^* solves (2.1.23) if and only if it is a fixed point of $P_C(I - rT^*(I-P_Q)T)$, where T^* is the adjoint operator of T , P_C and P_D are the metric projections from H_1 onto C and from H_2 onto Q, respectively, and $r > 0$ is a positive constant.

Indeed,
suppose $x^* \in C \cap T^{-1}Q$

$$\Rightarrow\ x^* \in C \text{ and } Tx^* \in Q.$$

Thus,

$$\begin{aligned} P_C(I - rT^*(I - P_Q)T)x^* &= P_C(x^* - rT^*(Tx^* - P_QTx^*) \\ &= P_C(x^* - rT^*(Tx^* - Tx^*) \\ &= P_Cx^* \\ &= x^*. \end{aligned}$$

Conversely, assume $x^* = P_C(I - rT^*(I - P_Q)T)x^*$, we show that $x^* \in C \cap T^{-1}Q$.
Clearly, $x^* \in C$. It is remain to show that $Tx^* \in Q$.
To show this, it suffices to show that $P_QTx^* = Tx^*$.
Now,

$$\begin{aligned} x^* &= P_C(I - rT^*(I - P_Q)T)x^* \\ &\Leftrightarrow \langle (I - rT^*(I - P_Q)T)x^* - x^*, x^* - y\rangle \geq 0\ \forall\ y \in C \\ &\Rightarrow \langle -rT^*(I - P_Q)Tx^*, x^* - y\rangle \geq 0\ \forall\ r > 0, y \in C \\ &\Rightarrow \langle T^*(I - P_Q)Tx^*, x^* - y\rangle \leq 0\ \forall\ y \in C \\ &\Rightarrow \langle Tx^* - P_QTx^*, Tx^* - Ty\rangle \leq 0\ \forall\ y \in C\ .\ \ .\ \ .\ (1) \end{aligned}$$

Also $P_QTx^* = z$

$$\begin{aligned} &\Leftrightarrow \langle Tx^* - P_QTx^*, P_QTx^* - h\rangle \geq 0\ \forall\ h \in Q\ .\ \ .\ \ .\ (2) \\ (1) &= \langle Tx^* - P_QTx^*, Ty - Tx^*\rangle \geq 0\ \forall\ y \in C\ .\ \ .\ \ .\ (3) \end{aligned}$$

Since $C \cap T^{-1}Q \neq \emptyset$

$$\begin{aligned} &\Rightarrow\ \exists\ q \in C \cap T^{-1}Q \\ &\Rightarrow\ q \in C \text{ and } Tq \in Q. \end{aligned}$$

In particular, for $y = q$ and $h = Tq$ equation (2) and (3) become

$$\begin{aligned} \langle Tx^* - P_QTx^*, P_QTx^* - Tq\rangle \geq 0\ .\ \ .\ \ .\ (4) \\ \langle Tx^* - P_QTx^*, Tq - Tx^*\rangle \geq 0\ .\ \ .\ \ .\ (5) \end{aligned}$$

Adding (4) and (5) gives

$$\langle Tx^* - P_QTx^*, P_QTx^* - Tx^*\rangle \geq 0$$

$$\begin{aligned} &\Rightarrow \langle Tx^* - P_QTx^*, Tx^* - P_QTx^*\rangle \leq 0 \\ &\Rightarrow 0 \leq \|Tx^* - P_QTx^*\|^2 \leq 0 \\ &\Rightarrow P_QTx^* = Tx^*. \end{aligned}$$

So $Tx^* \in Q$,
therefore, $x^* \in T^{-1}Q$ and $x^* \in C$. Thus, $x^* \in C \cap T^{-1}Q$.

Chapter 3

In this chapter, we prove weak convergence of our proposed iterative scheme for approximation of solutions of Generalized Split Common Fixed Point problem (GSCFPP). Moreover, under additional assumption we prove strong convergence for the scheme. For the strong convergence, we follow the method of proof in [26].

3.1 Main Result

3.1.1 Weak Convergence Result

Algorithm 3.1
Let $x^* \in H_1$ be arbitrary and for $k \in \mathbb{N}$ set

$$\begin{cases} q_k = x_k + \gamma \sum_{j=1}^{r} A_j^*(b_{j,k} - A_j x_k), \ \textit{where}\ b_{j,k} \in T_j(A_j x_k)\ \forall 1 \leq j \leq r \\ x_{k+1} = (1-\alpha_k) q_k + \frac{\alpha_k}{n} \sum_{i=1}^{n} u_{i,k}, \ \textit{where}\ u_{i,k} \in U_i(q_k)\ \forall\ 1 \leq i \leq n, \end{cases}$$

where U_i and T_j are multi-valued demi-contractive for each $1 \leq i \leq n$, $1 \leq j \leq r$, respectively, $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ with μ_{max} the maximum of demi-contractive constants of U_i and L being the spectral radius of the operator A^*A and $\alpha_k \in (0,1)$.
We start with the following lemma.

Lemma 3.1.1 *Let $A_j : H_1 \to H_2$, $1 \leq j \leq r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \leq i \leq n$ and $T_j : H_2 \to 2^{H_2}$, $1 \leq j \leq r$ be multi-valued demi-contractive (with constants β_i, μ_j, respectively) such that $U_i(p) = \{p\}$ for all $p \in F(U_i)$ and nonempty $Fix(U_i) = C_i$ and $Fix(T_j) = Q_j$ with $U_i(x)$ and $T_j(y)$ closed and bounded $\forall i$ and j and $\forall x \in H_1$, $y \in H_2$. Then any sequence $\{x_k\}$ generated by algorithm* (3.1) *is Féjer monotone with respect to Γ, that is for every $x \in \Gamma$,*

$$\|x_{k+1} - x\| \leq \|x_k - x\| \quad \forall k \in \mathbb{N},$$

provided that $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ and $\alpha_k \in (0,1)$.

Proof Set $L := sup_{\substack{1\leq i\leq n \\ 1\leq j\leq r}} A_i^* A_j$, $\mu_{max} := sup_{1\leq i\leq n} U_i$, $\beta_{max} := sup_{1\leq j\leq r} T_j$; where U_i and T_j are demi-contractive constants of U_i and T_j, respectively.

Let $p \in \Gamma$ then from (3.1), we have

$$\begin{aligned}
\|x_{k+1}-p\|^2 &= \|(1-\alpha_k)q_k + \frac{\alpha_k}{n}\sum_{i=1}^{n} u_{i,k} - p\|^2 \\
&= \|q_k - p + \frac{\alpha_k}{n}\sum_{i=1}^{n}(u_{i,k}-q_k)\|^2 \\
&= \|q_k-p\|^2 + 2\frac{\alpha_k}{n}\langle q_k - p, \sum_{i=1}^{n}(u_{i,k}-q_k)\rangle \\
&+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k}-q_k)\|^2 \\
&= \|q_k-p\|^2 + 2\frac{\alpha_k}{n}\sum_{i=1}^{n}\langle u_{i,k}-q_k, q_k-p\rangle \\
&+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k}-q_k)\|^2 \\
&= \|q_k-p\|^2 - 2\frac{\alpha_k}{n}\sum_{i=1}^{n}\langle q_k-u_{i,k}, q_k-p\rangle \\
&+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k}-q_k)\|^2
\end{aligned}$$

Using (2.1.22), we have

$$\begin{aligned}
&\leq \|q_k-p\|^2 - \frac{\alpha_k}{n}\sum_{i=1}^{n}(1-\beta_i)\|q_k-u_{i,k}\|^2 \\
&+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k}-q_k)\|^2 \\
&\leq \|q_k-p\|^2 - \frac{\alpha_k}{n}(1-\beta_{max_i})\sum_{i=1}^{n}\|q_k-u_{i,k}\|^2 \\
&+ \frac{\alpha_k^2}{n}\sum_{i=1}^{n}\|(u_{i,k}-q_k)\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{k+1}-p\|^2 &\leq \|q_k-p\|^2 - \frac{\alpha_k}{n}(1-\beta_{max})\sum_{i=1}^{n}\|q_k-u_{i,k}\|^2 \\
&+ \frac{\alpha_k^2}{n}\sum_{i=1}^{n}\|(u_{i,k}-q_k)\|^2 \\
&= \|q_k-p\|^2 - \frac{\alpha_k}{n}((1-\beta_{max})-\alpha_k)\sum_{i=1}^{n}\|u_{i,k}-q_k\|^2 \quad . \quad . \quad . \ (3.1)
\end{aligned}$$

Also,

$$\begin{aligned}
\|q_k - p\|^2 &= \|x_k - p + \gamma \sum_{j=1}^{r} A_j^*(b_{j,k} - A_j x_k)\|^2 \\
&= \|x_k - p\|^2 + 2\gamma \sum_{j=1}^{r} \langle x_k - p, A_j^*(b_{j,k} - A_j x_k)\rangle \\
&+ \gamma^2 \| \sum_{j=1}^{r} (b_{j,k} - A_j x_k)\|^2 \\
&= \|x_k - p\|^2 - 2\gamma \sum_{j=1}^{r} \langle A_j x_k - A_j p, A_j x_k - b_{j,k})\rangle \\
&+ \gamma^2 \| \sum_{j=1}^{r} (b_{j,k} - A_j x_k)\|^2
\end{aligned}$$

Using (2.1.22), we have

$$\begin{aligned}
&\leq \|x_k - p\|^2 - \gamma \sum_{j=1}^{r} (1 - \mu_j)\|b_{j,k} - A_j x_k\|^2 \\
&+ \gamma^2 r L \|b_{j,k} - A_j x_k\|^2
\end{aligned}$$

Hence,

$$\begin{aligned}
\|q_k - p\|^2 &\leq \|x_k - p\|^2 - \gamma \sum_{j=1}^{r} (1 - \mu_{max})\|b_{j,k} - A_j x_k\|^2 \\
&+ \gamma^2 r L \|b_{j,k} - A_j x_k\|^2 \\
&\leq \|x_k - p\|^2 - \gamma(1 - \mu_{max}) \sum_{j=1}^{r} \|b_{j,k} - A_j x_k\|^2 \\
&+ \gamma^2 r L \|b_{j,k} - A_j x_k\|^2 \\
&\leq \|x_k - p\|^2 - \gamma((1 - \mu_{max}) - \gamma r L) \sum_{j=1}^{r} \|b_{j,k} - A_j x_k\|^2.
\end{aligned}$$

Substituting this in (3.1) we have,

$$\begin{aligned}
\|x_{k+1} - p\|^2 &\leq \|x_k - p\|^2 - \gamma((1 - \mu_{max}) - \gamma r L) \sum_{j=1}^{r} \|b_{j,k} - A_j x_k\|^2 \\
&- \frac{\alpha_k}{n}((1 - \beta_{max}) - \alpha_k) \sum_{i=1}^{n} \|u_{i,k} - q_k\|^2 \quad . \quad . \quad . \ (3.2) \\
&\leq \|x_k - p\|^2
\end{aligned}$$

provided $\gamma \in \big(0, \frac{1-\mu_{max}}{rL}\big)$ and $\alpha_k \in \big(0, 1 - \beta_{max}\big)$.

Hence, $\{x_k\}$ is Féjer monotone. ■

Lemma 3.1.2 ***(Opial's lemma)*** *Let H be a real Hilbert space and $\{x_k\}$ a sequence in H such that there exists a nonempty set $\Gamma \subset H$ satisfying the following;*

i) For every $y \in \Gamma$, $\lim \|x_k - y\|$ *exists.*

ii) Any weak-cluster point of the sequence x_k *belong to* Γ.
Then, there exists $\bar{x} \in \Gamma$ *such that* $\{x_k\}$ *converges weakly to* $\bar{x}$.

Theorem 3.1.3 *Let* $A_j : H_1 \to H_2$, $1 \le j \le r$ *be bounded linear operators,* $U_i : H_1 \to 2^{H_1}$, $1 \le i \le n$ *and* $T_j : H_2 \to 2^{H_2}$, $1 \le j \le r$ *be multi-valued demi-contractive (with constants* β_i, μ_j, *respectively) such that* $U_i(p) = \{p\}$ *for all* $p \in F(U_i)$ *and nonempty* $Fix(U_i) = C_i$ *and* $Fix(T_j) = Q_j$ *with* $U_i(x)$ *and* $T_j(y)$ *closed and bounded* $\forall i$ *and* j *and* $\forall x \in H_1,\ y \in H_2$.
If $\Gamma \neq \emptyset$, *then any sequence* $\{x_k\}$ *generated by algorithm* (3.1) *converges weakly to a split common fixed point* $x^* \in \Gamma$, *provided that* $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ *and* $\alpha_k \in \left(\delta, 1 - \beta_{max} - \delta\right)$ *for small enough* $\delta > 0$.

Proof
From (3.2), we obtained that $\{\|x_k - p\|\}$ is monotone decreasing thus, $\{x_k\}$ is bounded and $\lim \|x_k - p\|$ exists say, y^*.
Since $\{x_k\}$ is bounded, we have that there exists $\{x_{k_v}\}$ such that

$$x_{k_v} \rightharpoonup x^* \quad as\ v \to \infty, \text{ which implies that}$$
$$A_j x_{k_v} \longrightarrow A_j x^*\ as\ v \to \infty, \text{ and thus}$$
$$A_j x_{k_v} \rightharpoonup A_j x^* \quad . \quad . \quad . \ (3.3)$$

From (3.2) also, we have

$$\lim \|b_{j,k} - A_j x_k\| = 0\ as\ k \to \infty,$$
which implies that $d(T_j(A_j x_k), A_j x_k) \le \|b_{j,k} - A_j x_k\| \longrightarrow 0 \ \ \forall 1 \le j \le r$,
then, $d(T_j(A_j x_k), A_j x_k) \longrightarrow 0$,
thus, $d(T_j(A_j x_{k_v}), A_j x_{k_v}) \longrightarrow 0 \ \ \forall 1 \le j \le r \quad . \quad . \quad . \ (3.4)$

Since $(T_j - I)$ is demi-closed at 0, we have from (3.3) and (3.4) that

$$A_j x^* \in T_j(A_j x^*)$$
$$\Rightarrow A_j x^* \in F(T_j) \ \ \forall 1 \le j \le r$$

We also have that

$$q_{k_v} = x_{k_v} + \gamma \sum_{j=1}^{r} A_j^*(b_{j,k} - A_j x_{k_v})$$

Therefore,

$$q_{k_v} \longrightarrow x^* \quad . \quad . \quad . \ (3.5)$$

From (3.2), we have $\|u_{i,k} - q_k\| \longrightarrow 0$ as $k \longrightarrow 0$

this implies that $d(U_i(q_k), q_k) \le \|u_{i,k} - q_k\| \ \ \forall 1 \le i \le n$,
then, $d(U_i(q_k), q_k) \longrightarrow 0 \ \ \forall 1 \le i \le n$,
hence, $d(U_i(q_{k_v}), q_{k_v}) \longrightarrow 0 \ \ \forall 1 \le i \le n$.

This together with (3.5) imply that

$$x^* \in U_i(x^*) \quad \Rightarrow x^* \in F(U_i) \ \ \forall 1 \leq i \leq n$$

hence, $x^* \in \cap_{i=1}^n F(U_i)$ and $A_j x^* \in F(T_j) \ \ \forall 1 \leq j \leq r \quad \Rightarrow x^* \in \Gamma$.
We have shown for any subsequence $\{x_{k_v}\}$ of $\{x_k\}$ such that $x_{k_v} \rightharpoonup x^*$ that $x^* \in \Gamma$.
Thus, by Opial's lemma there exists $x^{**} \in \Gamma$ such that the sequence $x_k \rightharpoonup x^{**}$.
Hence, weak convergence for $\{x_k\}$ is established. ■

We now prove strong convergence for our iterative scheme.

3.1.2 Strong Convergence Result

Theorem 3.1.4 *Let H_1 and H_2 be two real Hilbert spaces and $A_j : H_1 \to H_2$, $1 \leq j \leq r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \leq i \leq n$ and $T_j : H_2 \to 2^{H_2}$, $1 \leq j \leq r$ be multi-valued demi-contractive (with constants β_i, μ_j, respectively) such that $U_i(p) = \{p\}$ for all $p \in F(U_i) = C_i$ and $T_j(p) = \{p\}$ for all $p \in F(T_j) = Q_j$ with $U_i(x)$ and $T_j(y)$ closed and bounded $\forall i = 1, 2, ..., n$ and $j = 1, 2, ..., r$ and $\forall x \in H_1, \ y \in H_2$.*
Suppose that there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle u_i - q, \sigma \rangle \geq 0 \quad \forall \ \ 1 \leq i \leq n, \ u_i \in U_i(q) \text{ and } q \in H_1, \\ \langle A_j^*(b_j - A_j y), \sigma \rangle \geq 0 \quad \forall \ \ 1 \leq j \leq r, \ b_j \in T_j(A_j y) \text{ and } y \in H_1. \end{cases} \tag{3.1.1}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (3.1) converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ and $\alpha_k \in \left(\delta, 1 - \beta_{max} - \delta\right)$ for small enough $\delta > 0$.

Proof Let $x^* \in \Gamma$ and choose $x_0 \in H_1$ such that

$$\langle x_0 - x^*, \sigma \rangle > 0,$$

then there exists $\epsilon > 0$ such that

$$\langle x_0 - x^*, \sigma \rangle \geq \epsilon \|x_0 - x^*\|^2.$$

We now proof by induction that

$$\langle x_{k+1} - x^*, \sigma \rangle \geq \epsilon \|x_{k+1} - x^*\|^2 \ \ \forall \ k \geq 0. \tag{3.1.2}$$

Indeed, assume it holds up to some $k \geq 0$, then

$$\begin{aligned} \langle x_{k+1} - x^*, \sigma \rangle &= \langle x_{k+1} - x_k + x_k - x^*, \sigma \rangle \\ &= \langle x_{k+1} - x_k, \sigma \rangle + \langle x_k - x^*, \sigma \rangle \\ &= \langle \gamma \sum_{j=1}^r A_j^*(b_{j,k} - A_j x_k) + \frac{\alpha_k}{n} \sum_{i=1}^n (u_{i,k} - q_k), \sigma \rangle \\ &+ \langle x_k - x^*, \sigma \rangle \\ &= \gamma \sum_{j=1}^r \langle A_j^*(b_{j,k} - A_j x_k), \sigma \rangle + \frac{\alpha_k}{n} \sum_{i=1}^n \langle (u_{i,k} - q_k), \sigma \rangle \\ &+ \langle x_k - x^*, \sigma \rangle. \end{aligned}$$

Since $\gamma > 0$, $\alpha_k > 0$ and by (3.1.1) we get

$$\langle x_{k+1} - x^*, \sigma \rangle \geq \langle x_k - x^*, \sigma \rangle$$

by the induction assumption we have that

$$\langle x_{k+1} - x^*, \sigma \rangle \geq \epsilon \|x_k - x^*\|^2,$$

by lemma (3.1.1) the sequence $\{x_k\}$ generated by algorithm (3.1) is Féjer monotone with respect to Γ, so that

$$\langle x_{k+1} - x^*, \sigma \rangle \geq \epsilon \|x_{k+1} - x^*\|^2.$$

Therefore, (3.1.2) holds for all $k \geq 0$.
By theorem (3.1.3) we have

$$\begin{aligned} &x_k \rightharpoonup x^*, \text{ so that} \\ &\langle g, x_k \rangle \longrightarrow \langle g, x^* \rangle \ \ \forall\, g \in H_1. \end{aligned}$$

In particular, for $g = \sigma \in H_1$ we get

$$\langle \sigma, x_k \rangle \longrightarrow \langle \sigma, x^* \rangle which implies \langle \sigma, x_k - x^* \rangle \longrightarrow 0 \ \ as \ \ k \longrightarrow +\infty.$$

From (3.1.2) we have

$$\|x_k - x^*\|^2 \leq \frac{1}{\epsilon} \langle x_k - x^*, \sigma \rangle \longrightarrow 0 \ \ as \ \ k \longrightarrow +\infty.$$

Thus $\|x_k - x^*\|^2 \longrightarrow 0$
$as\ k \longrightarrow +\infty$.

Consequently, $\|x_k - x^*\| \longrightarrow 0 \ \ as \ \ k \longrightarrow +\infty$; and hence $x_k \longrightarrow x^* \in \Gamma$
This completes the proof. ■

The following corollary is a special case of theorem (3.1.4) when $i = j = 1$

Corollary 3.1.5 *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be multi-valued demi-contractive (with constants β, μ, respectively) such that $U(p) = \{p\}$ for all $p \in F(U)$ and nonempty $Fix(U) = C$ and $Fix(T) = Q$ with $U(x)$ and $T(y)$ closed and bounded $\forall x \in H_1,\ y \in H_2$.*
Assume that there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle u - q, \sigma \rangle \geq 0 \quad \forall \ \ u \in U(q) \ and \ q \in H_1, \\ \langle A^*(b - Ay), \sigma \rangle \geq 0 \quad \forall \ \ b \in T(Ay) \ and \ y \in H_1. \end{cases} \tag{3.1.3}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (3.1) converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \left(0, \frac{1-\mu}{L}\right)$ and $\alpha_k \in \left(\delta, 1 - \beta - \delta\right)$ for small enough $\delta > 0$.

The following result generalizes theorem of Moudafi [33] which is a special case of theorem (3.1.4) where $n = r = 1$, and U and T are single-valued demi-contractive.

Corollary 3.1.6 *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be demi-contractive (with constants β, μ, respectively) with nonempty $Fix(U) = C$ and $Fix(T) = Q$. Assume that $U - I$ and $T - I$ are demi-closed at 0 and that there exists $\sigma \neq 0 \in H_1$, such that*

$$\begin{cases} \langle U(q) - q, \sigma \rangle \geq 0 \quad \forall \;\; q \in H_1, \\ \langle A^*(T - I)Ay, \sigma \rangle \geq 0 \quad \forall \;\; y \in H_1. \end{cases} \tag{3.1.4}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (3.1) converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu}{L}\big)$ and $\alpha_k \in \big(\delta, 1 - \beta - \delta\big)$ for small enough $\delta > 0$.

Corollary 3.1.7 *Let H_1 and H_2 be two real Hilbert spaces and $A_j : H_1 \to H_2$, $1 \leq j \leq r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \leq i \leq n$ and $T_j : H_2 \to 2^{H_2}$, $1 \leq j \leq r$ be multi-valued quasi-nonexpansive such that $U_i(p) = \{p\}$ for all $p \in F(U_i) = C_i$ and $T_j(p) = \{p\}$ for all $p \in F(T_j = Q_j$ with $U_i(x)$ and $T_j(y)$ closed and bounded $\forall i = 1, 2, ..., n$ and $j = 1, 2, ..., r$ and $\forall x \in H_1, \; y \in H_2$. Suppose that there exists $\sigma \neq 0 \in H_1$, such that*

$$\begin{cases} \langle u_i - q, \sigma \rangle \geq 0 \quad \forall \;\; 1 \leq i \leq n, \;\; u_i \in U_i(q) \text{ and } q \in H_1, \\ \langle A_j^*(b_j - A_j y), \sigma \rangle \geq 0 \quad \forall \;\; 1 \leq j \leq r, \;\; b_j \in T_j(A_j y) \text{ and } y \in H_1. \end{cases} \tag{3.1.5}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (3.1) converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu_{max}}{rL}\big)$ and $\alpha_k \in \big(\delta, 1 - \beta_{max} - \delta\big)$ for small enough $\delta > 0$.

3.1.3 Numerical Examples

In order to illustrate numerical application, we consider a special case of our scheme for $i = j = 1$ and $H_1 = H_2 = \mathbb{R}^3$.

All computations in this section were performed using python 3.5.2 terminal based on linux running 64-bit. The first 100 iterations of our scheme are presented in Table 1, and relationship between $\|x - x^*\|$ - values and number of iterations are given in Figure 1, where $x^* = 0 \in \Gamma$.

Now, for $x_0 = (1, 1, 1) \in \mathbb{R}^3$, $\gamma = 0.2$, $\alpha_k = 1 - \alpha_k = 0.5$, $\forall k \geq 1$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} \sqrt{\frac{3}{20}} & \sqrt{\frac{1}{20}} & 0 \\ \sqrt{\frac{1}{20}} & \sqrt{\frac{3}{20}} & \sqrt{\frac{3}{10}} \\ 0 & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{10}} \end{bmatrix}, \quad \text{and } U = \begin{bmatrix} \sqrt{\frac{1}{10}} & 0 & \sqrt{\frac{3}{10}} \\ 0 & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{10}} \\ \sqrt{\frac{3}{20}} & 0 & \sqrt{\frac{3}{20}} \end{bmatrix}$$

we have the following iterations for $k = 100$.

Iterations	$\|x - x^*\|$
10	$1.09e^{-01}$
20	$7.00e^{-03}$
30	$4.00e^{-04}$
40	$3.37e^{-05}$
50	$2.30e^{-06}$
60	$1.54e^{-07}$
70	$1.04e^{-08}$
80	$6.10e^{-10}$
90	$4.72e^{-11}$
100	$3.20e^{-12}$

Table 1. The first 100 iterations generated by (3.1.6).

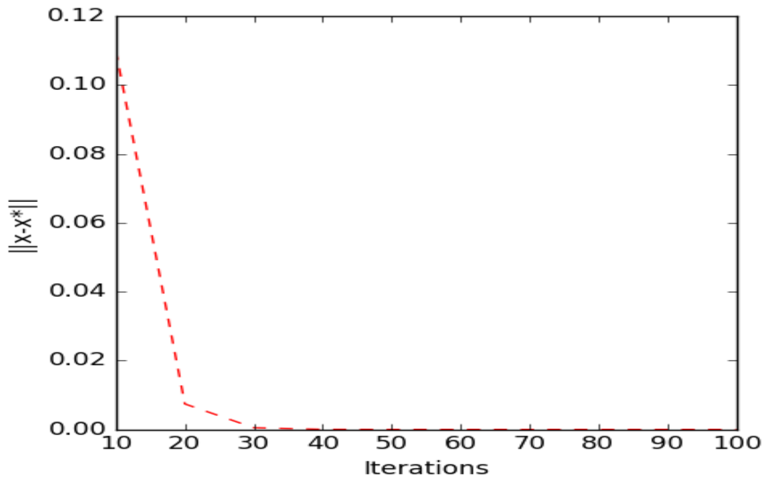

Figure 1. Relationship between $\|x - x^*\|$ - values and number of iterations.

Chapter 4

Summary, Conclusion and Recommendation

4.1 Summary

In this thesis, we have succesfully introduced a new iterative scheme for the approximation of solutions of generalized split common fixed point problem (GSCFPP) for multi-valued demi-contractive mappings in Hilbert spaces. We first proved weak convergence for our scheme and further proved strong convergence under additional mild condition. Finally, a numerical examples were presented to illustrate our scheme.

4.2 Conclusion

Our theorems and corollaries are important generalizations of several important recent results in the following sense:

- The class of operators considered in this thesis is larger than the class considered in [26] and [33].
- The algorithm for the split common fixed point problem considered in this thesis is new and generalizes that of [26] and [33].
- In [26] and [33], authors considered one bounded linear operator and single-valued operators whereas in this thesis, we consisdered finite family of bounded linear operators and multi-valued operators.

We conclude, by saying that the condition $T(p) = \{p\}$ for all $p \in F(P)$, which is imposed in our theorems and corollaries is not crucial. However, some works in the litrature show that this condition can actually be replaced by another condition (see, e.g., Shahzad and Zegeye [57]).

4.3 Recomendation

A more delicate problem is to prove theorem (3.1.4) in some real Banach spaces. The use of Alber function (see e.g., [1]) could be helpful in this direction.

Bibliography

[1] Y. I. Alber; *Metric and generalized projecton operators in Banach spaces* properties and applications in kartsators AG editor. The theory and applications of nonlinear operators of accretive and monotone type. New York (NY): Marcel Dekker; 1996. p. 15-50.

[2] O.A. Boikanyo; *A strongly convergent algorithm for the split common fixed point problem*, Appl. Math. Comput. **265** (2015), 844-853.

[3] F. E. Browder; *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201-255.

[4] F. E. Browder and W. V. Petryshyn; *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 297-228.

[5] C. Byrne; *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl. **20**, (2004) 103-120.

[6] C. Byrne; *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Prob. **18** (2002), 441-453. http://dx.doi.org/10.1088/0266-5611/18/2/310.

[7] C. Byrne; *Iterative projection onto convex sets using multiple Bregman distances*, Inverse Probl. **15**, (1999) 1295-1313.

[8] C. Byrne, Y. Censor, A. Gibali and S. Reich; *The split common null point problem*, J. Nonlinear Convex Anal. **13** (2012), 759-775.

[9] C.E. Chidume *et al.*; *The multiple-sets split equality fixed point problem for countable families of multi-valued demi-contractive mappings*, Int. J. Math. Anal. Vol. **9** (2015), 453-469. http://dx.doi.org/10.12988/ijma.2015.412209.

[10] C.E. Chidume *et al.*; *The split equality fixed point problem for demi-contractive mappings*, (Accepted to appear 2014): J. Nonlinear Anal. and Opt., Theory and Applications.

[11] C. E. Chidume, C. O. Chidume, N. Djitt, and M. S. Minjibir; *Convergence Theorems for Fixed Points of Multi-valued Strictly-Pseudocontractive Mappings in Hilbert Spaces*, (2012, to appear). http://dx.doi.org/10.1155/2013/629468.

[12] C.E. Chidume and S. Maruster; *Iterative methods for the computation of fixed points of demi-contractive mappings*, J. Comput. Appl. Math. Vol. **234** (2010), 861-882. http://dx.doi.org/10.1016/j.cam.2010.01.050.

[13] C.E. Chidume, M.E. Okpala; *On a general class of Multi-valued Strictly-Pseudocontractive Mappings*, J. of Nonlinear Anal. and Optim. Vol. **5** (2014), 07-20.

[14] A. Cegielski; *General method for solving the split common fixed point problem*, J. Optim. Theory Appl. **165** (2015), 385-404.

[15] Y. Censor *et al.*; *On the effectiveness of projection methods for convex feasibility problems with linear inequality constraints*, Comput. Optim. Appl. (2011, accepted for publication). doi:10.1007/s10589-011-9401-7. http://arxiv.org/abs/0912.4367.

[16] Y. Censor, M.D. Altschule, W.D. Powlis; *On the use of Cimminos simultaneous projections method for computing a solution of the inverse problem in radiation therapy treatment planning*, Inverse Probl. **4**, (1988) 607-623.

[17] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov; *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol. **51**, (2006) 2353-2365. http://dx.doi.org/10.1088/0031-9155/51/10/001.

[18] Y. Censor, R. Davidi,G.T. Herman; *Perturbation resilience and superiorization of iterative algorithms*, Inverse Probl. **26**, (2010) 065008 17 pp.

[19] Y. Censor, T. Elfving; *A multiprojection algorithms using Bregman projection in a product space*, Numer. Algorithms. **8**, (1994) 221-239. http://dx.doi.org/10.1007/bf02142692.

[20] Y. Censor, T. Elfving, N. Kopf, T. Bortfeld; *The multiple-sets split feasibility problem and its applications for inverse problems*, Inverse Probl. **21**, (2005) 2071-2084.

[21] Y. Censor, A. Gibali and S. Reich; *Algorithm for the split variational inequality problem*, Numer. Algorithms **59** (2012), 301-323.

[22] Y. Censor, A. Segal; *The split common fixed point problem for directed operators*, J. Convex Anal. **16**, (2009) 587-600.

[23] S. S. Chang *et al.*; *The split feasibility problem for quasi-nonexpansive multi-valued mappings and total asymptotically strict pseudo-contractive mappings*, Appl. Math. Comput. Vol. **219** (2013), 10416-10424. http://dx.doi.org/10.1016/j.amc.2013.04.020.

[24] H. Cui, F. Wang; *Iterative methods for the split common fixed point problem in Hilbert spaces*, Fixed Point Theory Appl. (2014), 1-8.

[25] Y. Dang, Y. Gao; *The strong convergence of a KM-CQ-like algorithm for a split feasibility problem*, Inverse Probl. **27**, (2011) 015007.

[26] A. Gibali; *A new Split Inverse Problem and an application to least intensity feasible solutions*, J. funct. Anal. and Appl. **2** (2017), 243-258.

[27] A. Gibali, K.H. Kúfer, P. Súass; *Reformulating the pascoletti-Serafini problem as a bi-level optimization problem*, Contemp. Math. **636**, (2015) 121-129.

[28] T. L. Hicks and J. R. Kubicek; *On the Mann iteration process in Hilbert space*, J. Math. Anal. Appl. **59** (1977), 498-504.

[29] T. Jifang; *Split common fixed point problem for a family of quasi-nonexpansive multi-valued mappings and total asymptotically strict pseudo-contractive mapping*, Acta Mathematicae Applicatae Sinica **37** (2014), 119-126.

[30] P. Kraikaew, S. Saejung; *On split common fixed point problems*, J. Math Anal. Appl. **415** (2014), 513-524.

[31] A. Moudafi; *A note on the split common fixed-point problem for quasi-nonexpansive operators*, Nonlinear Anal. **74** (2011), 4083-4087.

[32] A. Moudafi; *A relax CQ algorithm for convex feasibility problems*, Nonlinear Anal., **79** (2013), 117-121. http://dx.doi.org/10.1016/j.na.2012.11.013.

[33] A. Moudafi; *The split common fixed point problem for demi-contractive mappings*, Inverse Probl. **26**, (2010) 1-6.

[34] A. Moudafi; *Split monotone variational inclusions*, J. Optim. Theory Appl. **150** (2011), 275-283.

[35] A. Moudafi, Eman Al-Shemas; *Simultaneous iterative methods for split equality problem*, Transactions on Mathematical Programming and Applications, **1** (2013), 1-11.

[36] St. Maruster; *Quasi-nonexpansivity and the convex feasibility problem*, An. Univ. Al. **15** (2005), 47-56.

[37] St. Maruster, and C. Popirlan; *On the Mann-type iteration and convex feasibility problem*, J. Comput. Appl. Math. **212** (2008), 390-396.

[38] S. Maruster and C. Popirlan; *On the Mann-type iteration and convex feasibility problem*, J. Comput. Appl. Math. **212** (2008), 390-396. http://dx.doi.org/10.1016/j.cam.2006.12.012

[39] E. Masad and S. Reich; *A note on the multiple-set split convex feasibility problem in Hilbert space*, J. Non. Con. Anal. **8**, (2007) 367-371.

[40] M. O. Osilike and A. Udomene; *Demiclosedness principle and convergence theorems for stictly pseudocontractive mappings of the Browder-Petryshyn type*, J. Math. Anal. Appl. **256** (2001), 431-445.

[41] R. J. Palta and T. R. Mackie, Eds; *Intensity modulated radiation therapy: the state of art*, in Medical Physics Monograph, **29**, Medical Physics Publishing, Madison, Wis, USA, (2003).

[42] B. Qu, N. Xiu; *A note on the CQ algorithm for the split feasibility problem*, Inverse Probl. **21**, (2005) 1655-1666.

[43] Y. C. Tang, J.G. Peng, L.W. Liu; *A cyclic algorithm for the split common fixed point problem of demi-contractive mappings in Hilbert spaces*, Math. Model. Anal. **17**, (2012) 457-466.

[44] Y. Shehu, P. Cholamjiak; *Another look at the split common fixed point problem for demi-contractive operators*, (Accepted to appear 2015): Inverse Probl. (2015) 1-18.

[45] F. Schopfer, T. Schuster, A.K. Louis; *An iterative regularization method for the solution of the split feasibility problem in Banach spaces*, Inverse Probl. **21**, (2005) 1655-1666.

[46] F. Wang, H. Cui; *Convergence of a cyclic algorithm for the split common fixed point problem without continuity assumption*, Math. Model. Anal. **18**, (2013) 537-542.

[47] F. Wang, H.K. Xu; *Cyclic algorithms for split feasibility problems in Hilbert spaces*, Nonlinear Anal. **74** (2011), 4105-4111.

[48] H.K. Xu; *A variable Krasnoselskii-Mann algorithm and the multiple-set split feasibility problem*, Inverse Probl. **22**, (2006) 2021-2034.

[49] H.K. Xu; *Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces*, Inverse Probl. **26**, (2010) 105018.

[50] I. Yamada; *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, Inherently Parallel Algorithms in Feasibility and Optimization and their Applications, Editors: D. Butnariu, Y. Censor, S. Reich, Elsevier, Amsterdam, (2001), 473-504.

[51] I. Yamada, N. Ogura; *Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings*, Numer. Funct. Anal. Optim. **25** (2004), 619-655.

[52] M. Yukawa, K. Slavakis, and I. Yamada; *Multi-domain adaptive filtering by feasibility spliting*, ICASSP (2010), 3814-3817.

[53] Q. Yang; *The relaxed CQ algorithm solving the split feasibility problem*, Inverse Probl. **20**, (2004) 1261-1266.

[54] Yuang-Fang *et al.*; *Strong and weak convergence theorems for a new feasibility problems*, Int. Math. Forum, **8** (2013), 1621-1627. http://dx.doi.org/10.12988/imf.2013.38160.

[55] J. Zhao, Q. Yang; *Several solution methods for the split feasibility problem*, Inverse Probl. **21**, (2005) 1791-1800.

[56] W. Zhang, D. Han, Z. Li; *A self-adaptive projection method for solving the multiple-sets split feasibility problem*, Inverse Probl. **25**, (2009) 115001.

[57] H. Zegeye and N. Shahzad; *On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces*, Nonlinear Anal. **71**, 838-844 (2009).