ALGORITHMS FOR APPROXIMATING FIXED POINTS OF MULTIVALUED QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

A Thesis Presented to the Department of
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Master of Science

By

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Abuja, Nigeria

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Certification

This is to certify that the thesis titled "ALGORITHMS FOR APPROXIMATING FIXED POINTS OF MULTIVALUED QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research carried out by Chimezie Izuazu in the Department of Pure and Applied Mathematics.

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Abstract

Let E be a strictly convex real Banach space and let $D \subseteq E$ be a non-empty closed convex subset of E. Let $\{T\}_{i\in\mathcal{I}}$ be a finite family of quasi-nonexpansive multivalued map which are continuous with respect to the Hausdorff metric, where $T_i : D \longrightarrow \mathcal{PB}(D)$, for all $i \in \mathcal{I}$. Suppose that the family has a least one common fixed point, a Krasnolselskii-Mann-type sequence is shown to converge strongly to a common fixed point of T_i"s. Our result generalizes and complements some important results for single-valued and multivalued quasi-nonexpansive maps. Futhermore, we considered a countable family of quasi-nonexpansive multivalued map and proved a similar result.

Keywords Multivalued maps, quasi-nonexpansive maps, strictly convex space, Hausdorff metric, finite family, countable family, Krasnoselskii-Mann algorithm.

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Dedication

Dedicated to my mother Theresa Izuazu.

Contents

CHAPTER 1

General introduction

The content of this thesis fall within the general area of non-linear functional analysis. Our attention is focused on approximating fixed points of quasi-nonexpansive mappings. In this chapter, we give an introduction to the notion of fixed point of multivalued mappings and state some motivations for the study of fixed points of multivalued maps. Also, we collect basic definitions and some tools used in the thesis.

1.0.1 Introduction

Let D be a non-empty set, the power set of D is denoted by 2^D. Let B be another non-empty set, a map that sends a point to a set, $T : D \longrightarrow 2^B$, is called a multivalued mapping of D into B. The notation $D(T)$ means the subset of the domain such that $Tx \neq \emptyset$ for every $x \in D(T)$. An example of a multivalued map is the sub-differential of a convex functional: let f be a proper and convex map from a normed space E to $\mathbb{R} \cup \{\infty\}$. The sub-differential of f, $\quad \partial f : E \longrightarrow 2^{E^*}$, is defined by $\partial f(x) := \{x^* \in E^* : \langle x^*, y - x\rangle \leq f(y) - f(x), \forall y \in E\}$, where E^* is the dual of E. For instance, for $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = |x|$, we have

$$\partial(x) = \begin{cases} \{-1\}, & x < 0 \\ [-1, 1]\,, & x = 0 \\ \{1\}, & x > 0. \end{cases}$$

Remark 1.0.1 *Every singlevalued map can be seen as a multivalued map by viewing its images as singleton sets instead of points.*

The notion of a fixed point of a map, $T : D \longrightarrow 2^B$, makes sense if the intersection of its domain and codomain is non-empty. For such a map, a point $p \in D(T)$ is called a fixed point of T if $p \in Tp$. When T is singlevalued, p is a fixed point if $Tp = p$. Concerning fixed points (singlevalued or multivalued), two interesting questions are:

(1) Does T have fixed point(s)?

(2) If T has fixed point(s), how to get one?

The second question is addressed by iterative methods for fixed points of maps (singlevalued and multivalued) and this is the concern of this thesis.

The study of fixed points of multivalued maps has received the attention of many researchers for over thirty years and continues to do so. This is, perhaps, as a result of its applications in fields such as optimization, economics, game theory, non-smooth differential equation and so on.

- **Application to optimization:** Let f be a convex function from a normed space, E to $\mathbb{R}\bigcup\{\infty\}$. The sub-differential of f,

  $$\partial f : E \longrightarrow 2^{E^*}$$

  defined by

  $$\partial f(x) := \{x^* \in E^* : \langle x^*, y - x\rangle \le f(y) - f(x), \forall y \in E\}, \tag{1.0.1}$$

  is a multivalued map having the property that $0 \in \partial f(v)$ if and only if v is a minimizer of f over E. If $E = H$, a Hilbert space, then

  $$\partial f(x) := \{x^* \in H : \langle x^*, y - x\rangle \le f(y) - f(x), \forall y \in H\}. \tag{1.0.2}$$

  Setting $T = I - \partial f$, we have $0 \in \partial f(v)$ if and only if v is a fixed point of T. Hence solving for a fixed point of T is equivalent to solving for a minimizer of f.

- **Application to game theory:** The existence of equilibria for static non-cooperative games has been shown using the multivalued Brouwer or Kakutani fixed point theorem (see, e.g., [Nash, 1951]). In particular, under some conditions, given a game, there always exists a multivalued map whose fixed point(s) coincide with the equilibrium point(s) of the game (see, e.g., [Chidume et al., 2013a] and references therein). This made Nash a recipient of Nobel Prize in Economic Sciences. However, this theorem proves only existence and does not indicate a method of constructing a sequence starting from a nonequilibrium point which converges to an equilibrium solution. Hence, the need for iterative schemes to construct such sequences.

1.1 Basic definitions

Definition 1.1.1 (Contraction, Non-expansive and Quasi-nonexpansive mappings) *Let M_1 and M_2 be two metric spaces. A singlevalued map,*

$$f : D(f) \subseteq M_1 \longrightarrow M_2$$

is said to be

- *a contraction mapping if there exists $k \in [0, 1)$ such that*

$$\rho_2 (f(x), f(y)) \leq k\rho_1 (x, y), \forall x, y \in D(f), \quad (1.1.1)$$

- *non-expansive if*

$$\rho_2 (f(x), f(y)) \leq \rho_1 (x, y), \forall x, y \in D(f). \quad (1.1.2)$$

  Assume $M_1 = M_2$, then f is called

- *quasi-nonexpansive if for any $p \in F(f) := \{q \in D(f) : q = f(q)\}$,*

$$\rho_2 (f(x), p) \leq \rho_1 (x, p), \forall x \in D(f). \quad (1.1.3)$$

Definition 1.1.2 (Convex hull) *The convex hull of a set A is the smallest convex set containing A and it is denoted by $co(A)$. It consists exactly of all convex combinations of elements of A, i.e.,*

$$co(A) := \left\{ \sum_{i=1}^{m} \lambda_i x_i : \lambda_i \in [0, 1] \quad and \quad x_i \in A \quad for \quad each \quad i, \quad with \quad \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Definition 1.1.3 (Closed convex hull) *The closed convex hull of a set A, denoted by $\overline{co}(A)$, is the closure of the convex hull of A, $co(A)$.*

Definition 1.1.4 (Uniformly convex space) *A normed space E is said to be uniformly convex if for all $\varepsilon \in (0,2]$, there exists $\delta := \delta(\varepsilon) \in (0,1)$ such that $\|\frac{x+y}{2}\| \leq 1-\delta$, for all $x, y \in S_E$, with $\|x-y\| \geq \varepsilon$.*

Definition 1.1.5 (Proximinal Set) *Let D be a non-empty subset of $M, (M, \rho)$ a metric space. D is called proximinal if for each $x \in M$, there exists $u \in D$ such that*

$$\rho(x,u) = \inf_{y \in D} \rho(x,y) = dist(x,D). \tag{1.1.4}$$

In other words if for each $x \in M \setminus D$, the set

$$\mathcal{P}_D(x) := \{y \in D : \rho(x,y) = \rho(x,D)\},$$

is non-empty.

Concerning proximinality of sets, we have the following facts.

• A proximinal subset of a metric space is closed.

Proof Let Q be a proximinal subset of M. Let $\{q_n\}_{n\geq 1}$ be a sequence in Q such that $q_n \longrightarrow q^*$. Since $q^* \in M$, proximinality of Q implies that there exists $q^0 \in Q$ such that

$$\rho(q^*, q^0) = dist(q^*, Q) = \inf_{q \in Q} \rho(q^*, q) \leq \rho(q^*, q_n), \forall n \geq 1.$$

Letting n go to infinity, we have $\rho(q^*, q^0) = 0$. Thus $q^* = q^0 \in Q$ and hence Q is closed. ■

Remark 1.1.6 *The converse of the above fact is false in general. A counter example is given below.*

Example 1.1.7 *Consider $(\mathbb{C}_0, \|.\|_\infty)$ and $\Lambda := \left\{\{a_n\}_n \in \mathbb{C}_0 : \sum_{n=1}^{\infty} \frac{a_n}{2^n} = \frac{2}{3}\right\}$. Λ is a non-empty and closed subset of $\mathbb{C}_0$ but not proximinal.*

Proof

Non-empty: $\left\{\frac{1}{2^{n-1}}\right\}_n$ is in Λ.

Closure: Consider

$$f : \mathbb{C}_0 \longrightarrow \mathbb{R}$$

$$a = \{a_n\}_n \mapsto f(a) = \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

Clearly f is linear and bounded. In fact, $\|f(a)\| \leq \|a\|_\infty$ for all $a = \{a_n\}_n \in \mathbb{C}_0$. We note that $\Lambda = f^{-1}(\{\frac{2}{3}\})$. Hence, Λ is closed (as $\{\frac{2}{3}\}$ is closed in $\mathbb{R}$).

Let $a = \{a_n\}_n \in \Lambda$. We have $\frac{2}{3} = \left|\sum_{n=1}^{\infty} \frac{a_n}{2^n}\right| \leq \sum_{n=1}^{\infty} \frac{|a_n|}{2^n} \leq \|a\|_\infty$. Hence, $\inf_{a \in \Lambda} \|a\|_\infty \geq \frac{2}{3}$. Consider the sequence $\{w_n\}_n \subseteq \mathbb{C}_0$ where for each $n, w_n = \{a_{k,n}\}_{k \geq 1}$ is given by

$$a_{k,1} = \begin{cases} \frac{4}{3}, & k = 1 \\ 0, & k > 1 \end{cases}$$

and for $n \geq 2$,

$$a_{k,n} = \begin{cases} \frac{2}{3} + \frac{2}{3}\left(\frac{1}{2}\right)^{n-1}, & k = 1 \\ \frac{2}{3}, & 2 \leq k \leq n \\ 0, & k > n. \end{cases}$$

Clearly $\{a_{k,1}\}_{k \geq 1} \in \mathbb{C}_0$. For $n \geq 2$ we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{a_{k,n}}{2^k} &= \frac{\frac{2}{3}\left[1 + \left(\frac{1}{2}\right)^{n-1}\right]}{2} + \sum_{k=2}^{n} \frac{2}{3}\left(\frac{1}{2^k}\right) \\ &= \frac{1}{3}\left[1 + \left(\frac{1}{2}\right)^{n-1} + \sum_{k=1}^{n-1}\left(\frac{1}{2}\right)^k\right] \\ &= \frac{2}{3}, \forall n \geq 1. \end{aligned}$$

Also,

$$\|a_n\|_\infty = \frac{2}{3} + \frac{2}{3}\left(\frac{1}{2}\right)^{n-1}, \forall n \geq 2,$$

which implies

$$\inf_{a \in \Lambda} \|a\|_\infty \leq \frac{2}{3} + \frac{2}{3}\left(\frac{1}{2}\right)^{n-1}, \forall n \geq 2.$$

Letting n go to ∞, we have $\inf_{v \in \Lambda} \|v\|_\infty \leq \frac{2}{3}$. Thus, $dist(0, \Lambda) = \inf_{v \in \Lambda} \|v\|_\infty = \frac{2}{3}$. However, there is no $v \in \Lambda$ such that $\|v\|_\infty = \frac{2}{3}$. Suppose for contradiction that such an a exists. Then, $v = \{v_n\}_n \in \Lambda \Longrightarrow v_n \longrightarrow 0$ and hence there exists $N \in \mathbb{N}$ such that

$|a_n| < \frac{1}{2}, \forall n \geq N$. Thus,

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} \frac{a_n}{2^n}\right| &\leq \sum_{n=1}^{\infty} \frac{|a_n|}{2^n} \\
&= \sum_{n=1}^{N-1} \frac{|a_n|}{2^n} + \frac{|a_N|}{2^N} + \sum_{n=N+1}^{\infty} \frac{|a_n|}{2^n} \\
&\leq \sum_{n=1}^{N-1} \frac{2}{3}\left(\frac{1}{2^n}\right) + \frac{2}{3}\left(\frac{1}{2^N}\right) + \sum_{n=N+1}^{\infty} \frac{1}{2}\left(\frac{1}{2^n}\right) \\
&< \sum_{n=1}^{N-1} \frac{2}{3}\left(\frac{1}{2^n}\right) + \frac{2}{3}\left(\frac{1}{2^N}\right) + \frac{2}{3}\sum_{n=N+1}^{\infty} \left(\frac{1}{2^n}\right) \\
&= \frac{2}{3}.
\end{aligned}$$

Thus,

$$\frac{2}{3} = \left|\frac{2}{3}\right| = \left|\sum_{n=1}^{\infty} \frac{a_n}{2^n}\right| < \frac{2}{3},$$

a contradiction. Therefore Λ is not proximinal. ■

• Let K be a compact subset of M, then K is proximinal.

Proof Let $x \in M$ be fixed. Consider the function

$$g_x : K \longrightarrow \mathbb{R}$$

$$y \mapsto f_x(y) := \rho(x, y).$$

Since f_x is continuous (in fact, $|f_x(y_1) - f_x(y_2)| = |\rho(x, y_1) - \rho(x, y_2)| \leq \rho(y_1, y_2)$, for all $y_1, y_2 \in M$), there exists $y^0 \in K$ such that $f_x(y^0) = \inf_{y\in K} f_x(y)$, i.e., $\inf_{y\in K} \rho(x, y) = \rho(x, y^0)$. Hence, K is proximinal. ■

• Every non-empty and closed subset of a finite dimensional normed space is proximinal.

Proof Let E be a finite dimensional space and let D be a non-empty and closed subset of E. Let $x_0 \in E\backslash D$ be fixed. Let $r_0 := \inf_{d\in D} \|d - x_0\|$. Then, for all $\varepsilon > 0$ there exists $y_\varepsilon \in D$ such that $r_0 \leq \|x_0 - y_\varepsilon\| < r_0 + \varepsilon$. In particular, for all $n \geq 1$, there exists $y_n \in D$ such that $r_0 \leq \|x_0 - y_n\| < r_0 + \frac{1}{n}$. Hence $D_n := B(x_0; r_0 + \frac{1}{n}) \cap D$ is a non-empty and compact subset of E such that $D_{n+1} \subseteq D_n$ for all $n \geq 1$. Therefore, $\{D_n\}_n$ is a family of closed subsets of a compact set D_1 such that $\bigcap_{j=1}^{m} D_{i_j} = D_{i_0} \neq \emptyset, i_0 = \max_{1\leq j\leq m} i_j$. Hence $\bigcap_{n=1}^{\infty} D_n \neq \emptyset$. Let $d_0 \in \bigcap_{n=1}^{\infty} D_n$. Then,

$r_0 \leq \|x_0 - d_0\| < r_0 + \frac{1}{n}$ which implies $\|x_0 - d_0\| = r_0$. Therefore, D is proximinal. ∎

We give an example of proximinal set which is not compact.

Example 1.1.8 *Let $r \in \mathbb{R}$ be fixed and consider $(-\infty, r] \subseteq \mathbb{R}$ which is a non-empty closed subset of a finite dimensional space $(\mathbb{R})$, hence proximinal. However, $(-\infty, 0]$ is not compact.*

• Every non-empty closed and convex subset of a reflexive real Banach space E is proximinal.

Proof We first collect the following lemma:

Lemma 1.1.9 *(see, e.g., [Chidume, 2009]) Let E be a reflexive real Banach space and $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ be a convex, proper and lower semi-continuous function. Suppose $\lim\limits_{\|x\| \longrightarrow \infty} f(x) = \infty$. Then, there exists $x^* \in E$ such that $f(x^*) = \inf\limits_{x \in E} f(x)$.*

Let E be a reflexive real Banach space and let D be a non-empty closed convex subset of E. Let $x \in E \setminus D$ be arbitrary, we consider the set $D_x = D - \{x\} := \{d - x : d \in D\}$. Then D_x is closed and convex. Indeed, for convexity, given $u - x, v - x \in D_x$ and $\lambda \in [0, 1], \lambda(u - x) + (1 - \lambda)(v - x) = \lambda u + (1 - \lambda)v - x \in D_x$. For closedness, given $\{u_n - x\}_n \subseteq D_x$ such that $u_n - x \longrightarrow u \in E$. We have $u_n \longrightarrow u + x$. Since D is closed, $u + x \in D$ and $(u + x) - x \in D_x$. Hence, $u \in D_x$ and so D_x is closed. Let $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ be defined by

$$f(y) = \begin{cases} \|y\|, & y \in D_x \\ \infty, & y \in E \setminus D_x. \end{cases}$$

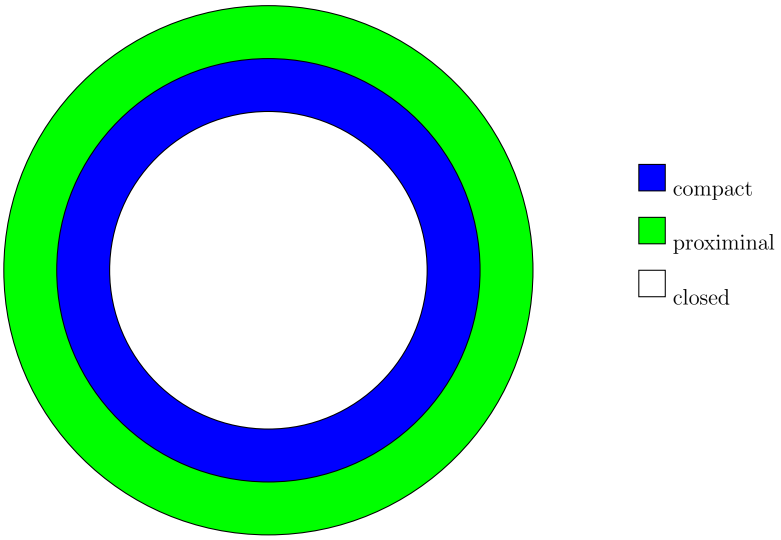
f is convex, proper and lower semi-continuous. Indeed, as $D \neq \emptyset, D_x \neq \emptyset$ and so f is proper. For convexity, let $y_1, y_2 \in E$. If $y_1 \in E \setminus D_x$, then $\lambda f(y_1) + (1 - \lambda) f(y_2) = +\infty \geq f(\lambda y_1 + (1 - \lambda) y_2)$. Also, if $y_1, y_2 \in D_x$, then the inequality follows from triangular inequality of the norm. For lower semi-continuity, let $y_0 \in E$. We show that for every sequence $\{y_n\} \subseteq E$, if $y_n \longrightarrow y_0$, then $f(y_0) \leq \liminf\limits_{n \longrightarrow \infty} f(y_n)$. So, let $\{y_n\} \subseteq E$ such that $y_n \longrightarrow y_0$. If $y_0 \in E \setminus D_x$, then $y_n \in E \setminus D_x$, for all $n \geq N$, for some $N \in \mathbb{N}$ (as D_x is closed). So, $f(y_n) = \|y_n\|$, for all $n \geq N$ and therefore, $f(y_n) \longrightarrow \|y_0\| = f(y_0)$. Thus, $f(y_0) = \liminf\limits_{n \longrightarrow \infty} f(y_n)$. Suppose $y_0 \in D_x$. Let $\{y_{n_j}\}_j$ be a subsequence of $\{y_n\}_n$ such that $\{f(y_{n_j})\}_j$ has a limit. If there are infinitely many terms of $\{y_{n_j}\}_j$ in D_x, then $f(y_{n_j}) \longrightarrow \|y_0\| = f(y_0)$. If this is not the case however, then there are infinitely many terms of $\{y_{n_j}\}_j$ in $E \setminus D_x$. In this case $f(y_{n_j}) \longrightarrow +\infty > f(y_0)$. In either case, $f(y_0) \leq f(y_{n_j})$ for any subsequence of $\{y_n\}_n$. Hence, $f(y_0) \leq \liminf_{n\longrightarrow\infty} f(y_n)$. Therefore, for any $y_0 \in E$, $f(y_0) \leq \liminf_{n\longrightarrow\infty} f(y_n)$ for every sequence $\{y_n\}_n \subseteq E$ such that $y_n \longrightarrow y_0$. It follows that f is lower semi-continuous on E. Moreover, $\lim_{\|y\|\longrightarrow\infty} f(y) = \infty$. Hence, we deduce from lemma 1.1.9 that there exists $y^* \in D$ such that $\|y^* - x\| = \inf_{y\in D} \|y - x\|$. Therefore, D is proximinal.

■

Corollary 1.1.10 *Every closed and convex subset of a uniformly convex real Banach space is proximinal.*

Proof We invoke the Milman-Pettis theorem (which says uniformly convex Banach spaces are reflexive, see, e.g., [Chidume, 2009]) and conclude. ■

The following figure depicts the relationship between the families of closed sets, proximinal sets and compact sets.



1.1.1 Hausdorff metric

Let D be a non-empty subset of a metric space M. We use $\mathcal{CB}(D), \mathcal{PB}(D)$ *and* $\mathcal{K}(D)$ to denote the families of non-empty closed bounded subsets of D, proximinal bounded subsets of D and non-empty compact subsets of D, respectively.

Definition 1.1.11 (Hausdorff Metric) *The metric* $d_H : \mathcal{CB}(D) \times \mathcal{CB}(D) \longrightarrow \mathbb{R}$ *defined by*

$$d_H(X,Y) := \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\right\}, \forall X, Y \in \mathcal{CB}(D)$$

is called Hausdorff metric.

Remark 1.1.12 *The Hausdorff metric is defined on a family of closed and bounded subsets of a given set not necessarily on the whole power set. We give examples below to emphasize the importance of closure and boundedness in the definition.*

Example 1.1.13 *Take* $M = D = \mathbb{R}$ *with the usual metric. We consider the following sets:* $X = [0,1]$ *and* $Y = (0,1)$. *Then* $X \neq Y$ *and* $d_H(X,Y) = 0$. *We note that* Y *is not closed in* D.

Example 1.1.14 *Take* $M = D = \mathbb{R}$ *with the usual metric. We consider the following sets:* $X = \{1\}$ *and* $Y = [0, +\infty)$. *Then* $d_H(X,Y) = +\infty$. *We note that* Y *is not bounded.*

With the aid of the Hausdorff metric, analogues of contraction, non-expansive and quasi-nonexpansive multivalued maps have been defined.

Definition 1.1.15 *Let* T *be a multivalued map defined on a non-empty subset of* M *with closed and bounded images.* T *is said to be a contraction if there exists* $k \in [0,1)$ *such that*

$$d_H(Tx, Ty) \leq kd(x,y), \forall x, y \in D(T). \tag{1.1.5}$$

T *is said to be non-expansive if*

$$d_H(Tx, Ty) \leq d(x,y), \forall x, y \in D(T). \tag{1.1.6}$$

T *is said to be quasi-nonexpansive if for any* $p \in Tp$,

$$d_H(Tx, Tp) \leq d(x,p), \forall x \in D(T). \tag{1.1.7}$$

Remark 1.1.16 *Every contraction mapping is non-expansive and every non-expansive map is continuous and quasi-nonexpansive. However, the converses are false. We give examples below.*

Example 1.1.17 *Consider* $(\mathbb{R}, |\cdot|)$. *Let* $D = [0,1]$ *and let* $T : D \longrightarrow \mathcal{CB}(D)$ *be defined by* $Tx = [0, \frac{x}{2}]$. *For any* $x, y \in D$, *if* $x \leq y$, $\sup\limits_{a \in Tx} \inf\limits_{b \in Ty} |a - b| = 0$ *and* $\sup\limits_{b \in Ty} \inf\limits_{a \in Tx} |a-b| = \frac{y-x}{2}$. *Also, if* $y < x$, $\sup\limits_{a \in Tx} \inf\limits_{b \in Ty} |a-b| = \frac{x-y}{2}$ *and* $\sup\limits_{b \in Ty} \inf\limits_{a \in Tx} |a-b| = 0$. *Thus,*

$$d_H(Tx, Ty) = \max\left\{\frac{|x-y|}{2}, 0\right\} = \frac{|x-y|}{2}, \forall x, y \in D.$$

Hence, T *is a contraction multivalued map.*

Example 1.1.18 *Consider* $(\mathbb{R}, |\cdot|)$, *let* $D = [0,1]$ *and let* $S : D \longrightarrow \mathcal{CB}(D)$ *be defined by* $Sx = [0, x]$. *For any* $x, y \in D$, *if* $x < y$, $\sup\limits_{a \in Sx} \inf\limits_{b \in Sy} |a - b| = 0$, $\sup\limits_{b \in Sy} \inf\limits_{a \in Sx} |a - b| = y - x$ *and if* $y < x$, $\sup\limits_{a \in Sx} \inf\limits_{b \in Sy} |a - b| = x - y$, $\sup\limits_{b \in Sy} \inf\limits_{a \in Sx} |a - b| = 0$. *Thus,*

$$d_H(Sx, Sy) = \max\{|x-y|, 0\} = |x-y|, \forall x, y \in D.$$

Hence, S *is a non-expansive multivalued map but not a contraction.*

Example 1.1.19 *Consider* $(\mathbb{R}, |\cdot|)$, *let* $D = [0,5]$ *and let* $R : D \longrightarrow \mathcal{CB}(D)$ *be defined by*

$$Rx = \begin{cases} [0, \frac{x}{5}], & x \neq 5 \\ \{1\}, & x = 5. \end{cases}$$

Then, $F(R) = \{0\}$. *For any* $x \in D$, *if* $x = 5$, $\sup\limits_{a \in R5} \inf\limits_{b \in R0} |a-b| = 1$, $\sup\limits_{b \in R0} \inf\limits_{a \in R5} |a-b| = 1$ *and if* $x \neq 5$, $\sup\limits_{a \in Rx} \inf\limits_{b \in R0} |a - b| = \frac{x}{5}$, $\sup\limits_{b \in R0} \inf\limits_{a \in Rx} |a - b| = 0$. *Thus, in both cases we have*

$$d_H(Rx, R0) \leq |x| = |x - 0|.$$

Therefore R *is quasi-nonexpansive. However,* R *is not non-expansive since* $d_H(R5, R4.5) = 1 > 0.5 = |5 - 4.5|$.

Definition 1.1.20 (Strictly convex space) *A normed linear space* E *is said to be strictly convex if for all* $x, y \in S_E$ *with* $x \neq y$ *and for all* $\lambda \in (0,1)$, *we have* $\|(1-\lambda)x + \lambda y\| < 1$.

Proposition 1.1.21 *Let* E *be a normed linear space, then the following are equivalent.*

(1) $\|(1-\lambda)y + \lambda x\| < 1, \forall x, y \in S_E, x \neq y, \forall \lambda \in (0,1)$.

(2) $\|(1-\lambda)y+\lambda x\|<1, \forall x,y\in B_E, x\neq y, \forall\lambda\in(0,1)$.

(3) $\|\frac{1}{2}y+\frac{1}{2}x\|<1, \forall x,y\in S_E, x\neq y$.

Proof (1)$\Longrightarrow$(2). This follows since $B_E\setminus S_E$ is convex.
(2)$\Longrightarrow$(3). Since $S_E\subseteq B_E$ and $\frac{1}{2}\in(0,1)$, (2)$\Longrightarrow$(3).
(3)$\Longrightarrow$(1). Let $\lambda\in(0,1)$, without loss of generality, we assume $\lambda\in[\frac{1}{2},1)$ (if $\lambda<\frac{1}{2}$, then we replace λ with $1-\lambda\in[\frac{1}{2},1)$). Then, $\|\lambda x+(1-\lambda)y\|=\|\frac{1}{2}x+\frac{1}{2}\left((2\lambda-1)x+2(1-\lambda)y\right)\|$. Since $x,y\in S_E\Longrightarrow\|(2\lambda-1)x+2(1-\lambda)y\|\leq 1$, we have that $\|\lambda x+(1-\lambda)y\|<1$. ■

Hence with proposition 1.1.21, we obtain two characterizations of strict convexity of a normed linear space as follows:

(1) A normed linear space E is strictly convex if and only if for all $x,y\in S_E$ with $x\neq y, \lambda\in(0,1)$, we have $\|(1-\lambda)y+\lambda x\|<1$.

(2) A normed linear space E is strictly convex if and only if for all $x,y\in S_E$ with $x\neq y$, we have $\|\frac{1}{2}y+\frac{1}{2}x\|<1$.

We state another characterization of strict convexity of a normed linear space (see, e.g., [Dotson, 1970]).

Lemma 1.1.22 *A normed linear space E is strictly convex if and only if for all $x,y\in E, x\neq 0, y\neq 0$, we have that if $\|x+y\|=\|x\|+\|y\|$, then $y=cx$ for some $c>0$.*

Proof ($\Longrightarrow$) Let x,y be in E, such that $x\neq 0, y\neq 0$. Suppose $\|x+y\|=\|x\|+\|y\|$. We set $x^*=\frac{x}{\|x\|}, y^*=\frac{y}{\|y\|}$ which are both in S_E. Also we let $\lambda_0=\frac{\|x\|}{\|x+y\|}$ which is clearly in $(0,1)$. Observe that $\|\lambda_0x^*+(1-\lambda_0)y^*\|=1$. Hence, by strict convexity of E we have that $x^*=y^*$, i.e., $\frac{x}{\|x\|}=\frac{y}{\|y\|}$. Hence $y=cx$, where $c=\frac{\|y\|}{\|x\|}$.
($\Longleftarrow$) Let $x,y\in S_E, \lambda\in(0,1)$. Suppose $\|(1-\lambda)y+\lambda x\|=1$. We show that $x=y$. Indeed, we note that $\|(1-\lambda)y\|+\|\lambda x\|=1$. Hence, $\|(1-\lambda)y+\lambda x\|=\|(1-\lambda)y\|+\|\lambda x\|$. Therefore $(1-\lambda)y=c\lambda x$ for some $c>0$. Since x,y are in S_E, we have that $c=\frac{(1-\lambda)}{\lambda}$. Thus $y=x$ and so E is strictly convex. ■

• Let E be a finite dimensional normed space, then E is uniformly convex if and only if E is strictly convex.

Proof ($\Longrightarrow$) See remark 1.1.25.

($\Longleftarrow$) Suppose, for contradiction, that E is strictly convex but not uniformly convex. Then we obtain that there exist $\varepsilon_0 \in (0, 2]$, $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \subseteq S_E$ such that $\|x_n - y_n\| \geq \varepsilon_0$ and $\lim_{n\longrightarrow\infty} \|\frac{x_n+y_n}{2}\| = 1$. Since S_E is closed and bounded and E is finite dimensional, we have that S_E is compact. Hence, we can obtain $\{x_{n_j}\}_{j\geq 1} \subseteq \{x_n\}_{n\geq 1}$ and $\{y_{n_j}\}_{j\geq 1} \subseteq \{y_n\}_{n\geq 1}$, such that $x_{n_j} \longrightarrow x^* \in S_E$ and $y_{n_j} \longrightarrow y^* \in S_E$. Thus, $\lim_{j\longrightarrow\infty} \left\|\frac{x_{n_j}+y_{n_j}}{2}\right\| = \left\|\frac{x^*+y^*}{2}\right\|$. This implies $\|\frac{x^*+y^*}{2}\| = 1$, a contradiction to the fact that E is strictly convex. ■

Lemma 1.1.23 *Let E be a normed linear space and $\|\cdot\|_s$ be a norm which makes E strictly convex, then for any other norm, $\|\cdot\|$, on E the space $(E, \|\cdot\|_s + \|\cdot\|)$ is strictly convex.*

Proof We prove by contradiction. Suppose $(E, \|\cdot\|_s + \|\cdot\|)$ is not uniformly convex. Then, using lemma 1.1.22 there exist $x, y \in E, x \neq 0, y \neq 0$ and $y \neq cx$ for any $c > 0$ but $\|x + y\|_s + \|x + y\| = \|x\|_s + \|x\| + \|y\|_s + \|y\|$. Since $(E, \|\cdot\|_s)$ is strictly convex we have that $\|x + y\|_s < \|x\|_s + \|y\|_s$, as $y \neq cx$ for any $c > 0$. Hence, $\|x + y\| > \|x\| + \|y\|$ which is impossible. Therefore $(E, \|\cdot\|_s + \|\cdot\|)$ is strictly convex. ■

Remark 1.1.24 *Also, it is worthy of note that for any $\mu > 0$, the new space $\mathcal{E}_\mu := (E, \mu\|\cdot\|_s)$ is also strictly convex. Indeed for any $x, y \in S_{\mathcal{E}_\mu}, x \neq y, \lambda \in (0, 1)$, we have that $\mu\|\lambda x + (1 - \lambda)y\| = \|\lambda(\mu x) + (1 - \lambda)(\mu y)\| < 1$.*

Remark 1.1.25 *It is straightforward from definition 1.1.4 and proposition 1.1.21 above that every uniformly convex space is strictly convex. This gives a large class of strictly convex spaces, since Hilbert spaces (in fact inner product spaces), L_p and l_p are uniformly convex for each $p \in (1, \infty)$. For further studies on uniformly convex spaces, (see, e.g., [Chidume, 2009]). While it is true that for a finite dimensional space strict convexity and uniform convexity of a norm are the same, there are infinite dimensional strictly convex spaces which are not uniformly convex. We give below an example of a strictly convex space which is not uniformly convex.*

Example 1.1.26 *Let l_1 be endowed with the norm $\|\cdot\|_s$ defined for $x = \{x_n\}_n \in l_1$ by $\|x\|_s = \|x\|_1 + \|x\|_2$. Then $(l_1, \|\cdot\|_s)$ is strictly convex but not uniformly convex.*

Proof We first show that $(l_1, \|\cdot\|_s)$ is not uniformly convex. It suffices to show that there exists $\epsilon_0 \in (0, 2]$ and sequences $(x_n)_n, (y_n)_n$ in $S_{l_1}^{\|\cdot\|_s}$(i.e, the unit sphere in l_1) such that $\|x_n - y_n\|_s \geq \epsilon_0$ for all $n \in \mathbb{N}$ and $\left\|\frac{x_n+y_n}{2}\right\|_s \longrightarrow 1$. Take $x_n = \sum_{i=1}^{n} \frac{e_{2i}}{n+\sqrt{n}}$, $y_n = \sum_{i=1}^{n} \frac{e_{2i+1}}{n+\sqrt{n}}$, where $e_i = (0, 0, 0, ..., 0, 0, \underbrace{1}_{i^{th} term}, 0, 0, ...)$. Then,

$$\|x_n\|_s = \|x_n\|_1 + \|x_n\|_2 = \frac{n}{n+\sqrt{n}} + \frac{\sqrt{n}}{n+\sqrt{n}} = \frac{n+\sqrt{n}}{n+\sqrt{n}} = 1$$

and

$$\|y_n\|_s = \|y_n\|_1 + \|y_n\|_2 = \frac{n}{n+\sqrt{n}} + \frac{\sqrt{n}}{n+\sqrt{n}} = \frac{n+\sqrt{n}}{n+\sqrt{n}} = 1.$$

Also,

$$\begin{aligned}\|x_n - y_n\|_s &= \|x_n - y_n\|_1 + \|x_n - y_n\|_2 \\ &= \frac{2n}{n+\sqrt{n}} + \frac{\sqrt{2n}}{n+\sqrt{n}} \\ &\geq \frac{n+\sqrt{n}}{n+\sqrt{n}} + \frac{\sqrt{2n}}{n+\sqrt{n}} \\ &\geq 1 + \frac{\sqrt{2n}}{n+\sqrt{n}} > 1\end{aligned}$$

and

$$\begin{aligned}\left\|\frac{x_n+y_n}{2}\right\|_s &= \left\|\frac{x_n+y_n}{2}\right\|_1 + \left\|\frac{x_n+y_n}{2}\right\|_2 \\ &= \frac{n}{n+\sqrt{n}} + \frac{1}{2}\frac{\sqrt{2n}}{n+\sqrt{n}} = \frac{\sqrt{2}+\frac{1}{\sqrt{n}}}{\sqrt{2}\left(1+\frac{1}{\sqrt{n}}\right)}.\end{aligned}$$

Therefore, $\|x_n - y_n\| \geq 1$ for all $n \in \mathbb{N}$ and $\left\|\frac{x_n+y_n}{2}\right\| \longrightarrow 1$. Hence $(l_1, \|\cdot\|_s)$ is not uniformly convex. However, by lemma 1.1.23 $(l_1, \|\cdot\|_s)$ is strictly convex, as $\|\cdot\|_2$ is an inner product norm (on l_2 which contains l_1). ∎

CHAPTER 2

Literature review

The literature of fixed point theory of singlevalued and multivalued maps is substantial. Many existence theorems for singlevalued and multivalued maps have been proved (see, e.g.,[Deimling, 2010], [Xu, 2000]). However, there is always a question of approximating fixed point(s) when they exist. Consequently, iterative schemes for approximating fixed point(s) for certain kinds of maps have been introduced and studied by many mathematicians (see, e.g., [Dotson, 1970], [Banach, 1922], [Chidume and Minjibir, 2016], [Mann, 1953], [Edelstein, 1966], [Krasnoselskii, 1955], [Ishikawa, 1976], [Schaefer, 1957]). These schemes include the Picard scheme used in the proof of Banach contraction mapping principle, Halpern scheme, Krasnoselskii-Mann scheme, Ishikawa scheme and many others(see, e.g., [Picard, 1890], [Ishikawa, 1976], [Halpern, 1967], [Ishikawa, 1974]). These iterative schemes are used to construct sequences which are shown to converge strongly or weakly to a fixed point of a map in certain metric spaces.

2.0.2 Banach contraction mapping principle

Theorem 2.0.27 *Let (M, ρ) be a complete metric space and let $f : (M, \rho) \longrightarrow (M, \rho)$ be a contraction mapping, then f has a unique fixed point.*

This theorem was first proved by Stefan Banach in 1922 (see [Banach, 1922]), in the setting of a complete normed space (Banach space). The proof of the Banach contraction mapping principle is in many books on fixed point theory (see, e.g., [Bonsall, 1962], [Smart, 1980]). However, we still give the proof below to highlight the Picard iterative scheme (see [Picard, 1890]) used in the proof which gives a method of approximating fixed point(s) of some maps. In fact if the Picard sequence given by a continuous map converges, then the limit is a fixed point.

Proof For any $x_0 \in M$ fixed, we define a sequence $\{x_n\}_n$ iteratively using the Picard scheme by $x_{n+1} = f(x_n)$, for all $n \geq 0$. Using the definition of f, there exists $k \in [0, 1)$ such that

$$\rho(x_{n+1}, x_n) = \rho(f(x_n), f(x_{n-1})) \leq k\rho(x_n, x_{n-1}) \leq k^n \rho(x_1, x_0). \tag{2.0.1}$$

Let $m, n \in \mathbb{N}$ with $m > n$. This implies $m = n + p$ for some $p \geq 1$. Hence, applying triangle inequality we have

$$\begin{aligned} \rho(x_m, x_n) &= \rho(x_{n+p}, x_n) \\ &\leq \rho(x_{n+p}, x_{n+p-1}) + \rho(x_{n+p-1}, x_{n+p-2}) + \ldots + \rho(x_{n+1}, x_n). \end{aligned} \tag{2.0.2}$$

From (2.0.1) and (2.0.2) we obtain

$$\rho(x_m, x_n) \leq \rho(x_1, x_0) \sum_{i=n}^{n+p} k^i \leq \rho(x_1, x_0) \sum_{i=n}^{\infty} k^i.$$

Therefore, $\{x_n\}_n$ is a Cauchy sequence as $\sum_{i=n}^{\infty} k^i$ is a tail of a convergent series(a geometric series $\sum_{i=1}^{\infty} r^i$ with $-1 < r < 1$). Since (M, ρ) is complete, there exists $x^* \in X$ such that $x_n \longrightarrow x^*$. Hence $f(x_n) = x_{n+1} \longrightarrow x^*$. By continuity of f (contraction of f gives continuity), we have that $f(x_n) \longrightarrow f(x^*)$. Thus, by uniqueness of limit $f(x^*) = x^*$. We now prove uniqueness. Suppose that there exists a fixed point of f, say y^* different from x^*. By definition of f we have

$$\rho(x^*, y^*) = \rho(f(x^*), f(y^*)) \leq k\rho(f(x^*), f(y^*)).$$

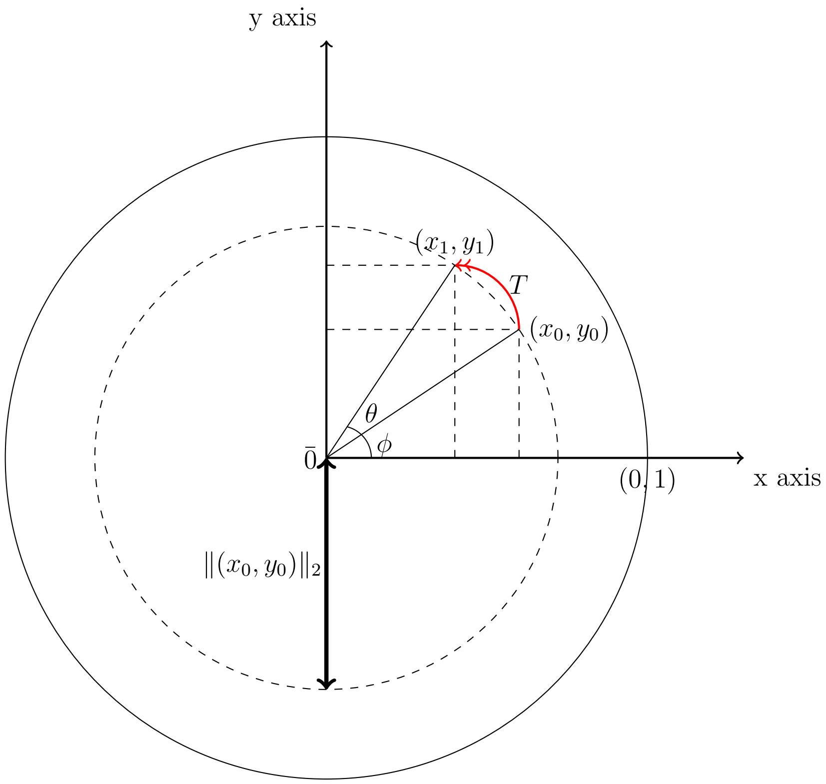
As $x^* \neq y^*$, we have $\rho(x^*, y^*) > 0$ and so dividing through yields $1 \leq k < 1$, a contradiction. ■

The Banach contraction mapping principle gives us four things: existence, uniqueness, method of approximation of the fixed point and how fast the sequence converges, we note that

$$\begin{aligned}\rho\left(x_{n+1}, x^*\right) &= \rho\left(f(x_n), f(x^*)\right) \\ &\leq k\rho\left(x_n, x^*\right) = k\rho\left(f\left(x_{n-1}\right), f\left(x^*\right)\right) \\ &\leq k^2\rho\left(x_{n-1}, x^*\right) \\ &\vdots \\ &\leq k^{n+1}\rho\left(x_0, x^*\right)\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. While the Picard sequence converges to the unique fixed point of contraction maps from a complete metric space to itself, it does not converge in general for nonexpansive maps. A counter-example has been given in $\mathbb{R}^2$.

Example 2.0.28 *We consider the map* $T : (B_{\mathbb{R}^2}, \|\cdot\|_2) \longrightarrow (B_{\mathbb{R}^2}, \|\cdot\|_2)$ *which rotates a point about an angle* θ, $0 < \theta < 2\pi$.



Given a point $(x_0, y_0), T$ rotates it through an angle of θ in the counter-clockwise direction and produces a point (x_1, y_1) with the same norm as (x_0, y_0), as shown in the diagram below. T is called a rotation map and it is given by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, 0 < \theta < 2\pi.$$

T is clearly linear. Also $F(T) = \{(0,0)\}$. If we pick an arbitrary point $(x_0, y_0) \neq (0,0)$,

$$\begin{aligned}
\left\| T^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\|_2 &= \left\| \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\|_2 \\
&= \left\| \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\|_2 \\
&= \left\| \begin{bmatrix} \cos(n\theta)x_0 - \sin(n\theta)y_0 \\ \sin(n\theta)x_0 + \cos(n\theta)y_0 \end{bmatrix} \right\|_2 \\
&= \sqrt{|\cos(n\theta)x_0 - \sin(n\theta)y_0|^2 + |\sin(n\theta)x_0 + \cos(n\theta)y_0|^2} \\
&= \sqrt{{x_0}^2 + {y_0}^2} \\
&= \left\| \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\|_2 > 0, \forall n \geq 0.
\end{aligned}$$

Thus the Picard sequence is a sequence with a non-zero constant norm. Therfore, it does not converge to $(0,0)$, the fixed point of T. We consider another example:

Example 2.0.29 *let $b > 0$ be fixed, we consider the function $f : [0,b] \longrightarrow [0,b]$*

$$x \mapsto f(x) = b - x.$$

$\|f(x) - f(y)\| = \|x - y\|, \forall x, y \in [0,b]$. *Hence f is non-expansive. Furthermore, f has a unique fixed point $\frac{b}{2}$. For any $x_0 \in [0,b] \setminus \{\frac{b}{2}\}$, the Picard sequence $\{x_n\}_n$ is $\{x_0, b - x_0, x_0, b - x_0, x_0, ...\}$, which obviously does not converge to $\frac{b}{2}$ because it is an oscillating sequence.*

Hence, one could infer that the Picard scheme is not appropriate for approximating fixed point(s) of non-expansive maps in general. Thus, other iterative schemes were introduced, among which are the so called Krasnoselskii-Mann schemes. It is the aim of this chapter to give a concise review of these schemes.

2.1 Singlevalued maps

Let E be a normed space, D be a non-empty convex subset of E and let $T : D \longrightarrow D$ be continuous. In [Mann, 1953], Robert Mann considered an infinite triangular matrix A,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{bmatrix}$$

whose entries satisfy the following:

1. $a_{ij} \geq 0, \forall i, j.$

2. $a_{ij} = 0, \forall i, j$ with $j > i$.

3. $\sum_{j=1}^{i} a_{ij} = 1, \forall i.$

Using this matrix, he introduced the following iterative scheme

$$\begin{cases} x_0 \in D; \\ v_n = \sum_{k=1}^{n} a_{nk} x_k; \\ x_{n+1} = T v_n, n \geq 0. \end{cases} \tag{2.1.1}$$

Using (2.1.1), he proved the following theorem.

Theorem 2.1.1 *If either of the sequences $\{x_n\}_n$ or $\{v_n\}_n$ converges, then the other also converges to the same point, and their common limit is a fixed point of T.*

Furthermore, he considered a special case where A is the Cesaro matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & . & . & . \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ \frac{1}{n} & \frac{1}{n} & . & . & . & \frac{1}{n} & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{bmatrix}$$

which satisfies all the assumptions above. Hence, (2.1.1) becomes

$$\begin{cases} x_0 \in D; \\ v_n = \frac{1}{n} \sum_{k=1}^{n} x_k; \\ x_{n+1} = T v_n, n \geq 0. \end{cases}$$

We have that

$$\begin{aligned} v_{n+1} &= \frac{1}{n+1} \sum_{k=1}^{n+1} x_k \\ &= \frac{1}{n+1} \sum_{k=1}^{n} x_k + \frac{1}{n+1} x_{n+1} \\ &= \frac{1}{n+1} \sum_{k=1}^{n} x_k + \frac{1}{n+1} T v_n \\ &= \frac{n}{n+1} \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right) + \frac{1}{n+1} T v_n \\ &= \left(1 - \frac{1}{n+1} \right) v_n + \frac{1}{n+1} T v_n, \forall n \geq 0. \end{aligned}$$

Setting $\lambda_n = \frac{1}{n+1}$, for all $n \geq 0$, we have

$$v_{n+1} = (1 - \lambda_n) v_n + \lambda_n T v_n, \forall n \geq 0. \tag{2.1.2}$$

Using (2.1.2), he proved the theorem

Theorem 2.1.2 *If T is a continuous function carrying the interval $a \leq x \leq b$ into itself and having a unique fixed point, $p \in [a, b]$, then the sequence $\{v_n\}_n$ converges to p for all choices of $x_0 \in [a, b]$.*

2.1.1 Non-expansive maps

In [Krasnoselskii, 1955], Krasnoselskii introduced the iterative scheme

$$\begin{cases} x_0 \in D; \\ x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}Tx_n, n \geq 0 \end{cases} \tag{2.1.3}$$

and proved the theorem

Theorem 2.1.3 *Suppose E is uniformly convex, D closed and T is non-expansive with $T(D)$ contained in a compact subset of D. Then $\{x_n\}_n$ given by (2.1.3) converges strongly to a fixed point of T.*

This theorem resolves the problem of the two examples mentioned at the beginning of this chapter.

In [Schaefer, 1957], Schaefer observed that the same result holds for the sequence given by the iterative scheme

$$\begin{cases} x_0 \in D; \\ x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, n \geq 0, \lambda \in (0,1). \end{cases} \tag{2.1.4}$$

In [Edelstein, 1966], Edelstein proved that if E is strictly convex, the same result holds. One can associate (2.1.4) with the infinite matrix $A_\lambda = [a_{ij}]$ where $a_{n1} = \lambda^{n-1}, a_{nj} = \lambda^{n-j}(1-\lambda)$ for $j = 2, 3, \ldots, n$ and $a_{nj} = 0$ for $j > n, n \geq 2$.

$$A_\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ \lambda & 1-\lambda & 0 & 0 & 0 & 0 & 0 & . & . & . \\ \lambda^2 & \lambda(1-\lambda) & 1-\lambda & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ \lambda^{n-1} & \lambda^{n-2}(1-\lambda) & . & . & . & 1-\lambda & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{bmatrix}.$$

Hence, $\lambda_n = a_{n+1,n+1} = 1 - \lambda$, for all $n \geq 1$. If $\lambda = \frac{1}{2}$, the infinite matrix $A_{\frac{1}{2}}$ is associated to the iterative scheme of (2.1.3). Also, if we consider the case of $\lambda = 0$, the infinite identity matrix is obtained, which is associated to the Picard iterative scheme.

2.1.2 Quasi-nonexpansive maps

In [Dotson, 1970], Dotson introduced a class of maps which is a proper superclass of non-expansive maps. He called them quasi-nonexpansive maps (see definition 1.1.1) and proved some convergence theorem for a Krasnoselskii-Mann sequence to a fixed point of such maps. One of the theorems he proved was

Theorem 2.1.4 *Suppose E is a strictly convex Banach space, D is a closed convex subset of $E, T : D \longrightarrow D$ is continuous and quasi-nonexpansive on D, and $T(D) \subseteq K \subseteq D$ where K is compact. For any $x_0 \in D$ define a sequence $\{x_n\}_n$ by*

$$x_{n+1} = (1 - t_n)x_n + t_n T x_n, n \geq 0, \{t_n\}_n \subseteq (0, 1). \tag{2.1.5}$$

Assume $\{t_n\}_n$ clusters at some $t \in (0, 1)$, then $\{x_n\}_n$ converges (strongly) to a fixed point of T.

The theorem above improves the theorem of Krasnoselskii (Theorem 2.1.3) both with regard to the map and the space as the classes of quasi-nonexpansive maps and strictly convex spaces are proper superclasses of the classes of non-expansive maps and uniformly convex spaces, respectively.

In recognition to the works of Krasnoselskii and Mann, any iterative scheme of the form

$$\begin{cases} x_0 \in D; \\ x_{n+1} = (1 - a_n)x_n + a_n T x_n, n \geq 0, \{a_n\}_n \subseteq (0, 1), \end{cases} \tag{2.1.6}$$

is usually referred to as a Krasnoselskii-Mann scheme in the literature.

The question of whether Krasnoselskii-Mann sequence converges in spaces more general than strictly convex spaces was open for so many years. In 1976, Ishikawa

[Ishikawa, 1976] gave an affirmative answer for non-expansive maps. He proved the following theorem

Theorem 2.1.5 *If the sequence $(x_n)_n$ given by (2.1.6) is bounded, then*

$$\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0,$$

i.e., $\{x_n\}_n$ is an approximate fixed point sequence for any non-expansive map in any Banach space.

Under some compactness conditions, a sequence $\{x_n\}_n$ being an approximate fixed point sequence yields convergence to a fixed point. However, for quasi-nonexpansive maps, a counter example was given by Chidume (see [Chidume, 1986]) to show that in general Banach spaces, the same result of Ishikawa fails to hold. He considered l_∞ and defined a continuous quasi-nonexpansive map $T : B_{l_\infty} \longrightarrow B_{l_\infty}$ with $F(T) = \{0, 0, 0, \dots\}$ by $Tx = \{0, {x_1}^2, {x_2}^2, {x_3}^2, \dots\}$, where $x = \{x_1, x_2, x_3, \dots\} \in l_\infty$.

2.2 Multivalued maps

In [Nadler, 1969], an analogue of Banach contraction mapping principle was proved. He considered an iterative process similar to the Picard scheme and proved the following theorem:

Theorem 2.2.1 ([Nadler, 1969]) *Let (M, ρ) be a complete metric space. If $T : M \longrightarrow \mathcal{CB}(M)$ is a multivalued contraction mapping, then T has a fixed point.*

This theorem gives only existence of a fixed point and a scheme to approximate it as follows;

$$\begin{cases} x_0 \in M; \\ x_{n+1} \in Tx_n \, with \, \rho(x_n, x_{n+1}) \leq d_H(Tx_n, Tx_{n+1}) + k^n, n \geq 1. \end{cases} \tag{2.2.1}$$

A lot of works have been done for fixed points of non-expansive multivalued mappings using the Hausdorff metric (see, e.g., [Markin, 1973], [Abbas et al., 2011], [Chidume et al., 2013b], [Djitte and Sene, 2014], [Khan et al., 2010], [Panyanak, 2007], [Sastry and Babu, 2005]). Recently, Chidume and Minjibir studied convergence of a Krasnolskii-Mann type algorithm for multivalued quasi-nonexpansive mappings in uniformly convex real Banach spaces [Chidume and Minjibir, 2016]. They proved the following theorem:

Theorem 2.2.2 ([**Chidume and Minjibir, 2016**]) *Let D be a non-empty closed convex subset of a uniformly convex real Banach space E. Suppose that $T : D \longrightarrow \mathcal{CB}(D)$ is a multivalued quasi-nonexpansive mapping such that $Tp = \{p\}$ for some $p \in F(T)$. Then for any $x_0 \in D$ and arbitrary $\lambda \in (0,1)$, define a sequence $\{x_n\}_n$ iteratively, by*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, y_n \in Tx_n, n \geq 0. \tag{2.2.2}$$

Then, $\lim_{n\longrightarrow\infty} dist(x_n, Tx_n) = 0.$

[Diop et al., 2014] proved the following theorem which is an extension of the result of Chidume and Minjibir [Chidume and Minjibir, 2016] to a finite family of quasi-nonexpansive. They proved the following theorem:

Theorem 2.2.3 ([**Diop et al., 2014**]) *Let D be a nonempty closed convex subset of a uniformly convex real Banach space E. Let $\mathcal{I}$ be a finite collection and $T_i :\longrightarrow \mathcal{CB}(D)$ is a multivalued quasi-nonexpansive mapping for each $i \in \mathcal{I}$ such that $T_i p = \{p\}$ for all i and for some $p \in F$. For any $x_0 \in D$ define a sequence by*

$$\begin{cases} x_0 \in D; \\ x_{n+1} = \lambda_0 x_n + \sum_{i\in\mathcal{I}} \lambda_i y_n, y_n \in T_i x_n, n \geq 0; \\ \lambda_0, \lambda_i \in (0,1), \sum_{i\in\mathcal{I}} \lambda_i + \lambda_0 = 1. \end{cases} \tag{2.2.3}$$

Then, $\lim_{n\longrightarrow\infty} dist(x_n, T_i x_n) = 0.$ *for each* $i \in \mathcal{I}$.

Also, Bunyawat and Suantai proved the following theorem:

Theorem 2.2.4 ([**Bunyawat and Suantai, 2013**]) *Let E be a real Banach space and D a non-empty, closed and convex subset of E. Let $\{T_i : i = 1, 2, ..., m\}$ be a finite family of multivalued quasi-nonexpansive mappings from D into $\mathcal{CB}(D)$ with $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}_n$ defined iteratively by*

$$\begin{cases} x_0 \in D; \\ x_{n+1} = \lambda_{0,n} x_n + \sum_{i=1}^m \lambda_{i,n} x_{i,n}, n \geq 0; \\ \{\lambda_{i,n}\}_n \subseteq [0,1) \, for\, each\, i \in \mathbb{I} \bigcup \{0\}, \sum_{i=0}^m \lambda_{i,n} = 1; \\ x_{i,n} \in T_i x_n \, with\, d(p, x_{i,n}) = d(p, T_i x_n) \, for\, each\, i \in \mathbb{I}, n \geq 1 \end{cases} \tag{2.2.4}$$

converges strongly to a common fixed point of T_i*'s if and only if* $\liminf_{n \longrightarrow \infty} dist(x_n, F) = 0$.

The study of fixed points of multivalued maps is relatively more difficult when compared to that of singlevalued maps. Nevertheless, many results for multivalued maps analogous to those for singlevalued maps have been gotten.

2.3 Statement of the problem

Let E be a strictly convex real Banach space and let D be a non-empty, closed and convex subset of E. Given T, a continuous, quasi-nonexpansive and singlevalued self-map on D with non-empty fixed point set, Dotson proved strong convergence of a Krasnoselskii-Mann sequence to a fixed point of T in [Dotson, 1970]. We consider a similar problem when T is a multivalued map. In fact we consider a problem involving a countable family of quasi-nonexpansive multivalued mappings in a strictly convex real Banach space, E, i.e., we investigate whether the result of Dotson [Dotson, 1970] holds for a countable family of quasi-nonexpansive multivalued mappings, assuming they have a common fixed point.

CHAPTER 3

Theory of Methods

Lemma 3.0.1 (see, e.g., [Megginson, 1998]) *A normed space E is Banach if and only if every absolutely convergent series is conditionally convergent.*

Proof $(\Longrightarrow)$. Suppose E is a Banach space. Let $\{x_n\}_n \subseteq E$ be a sequence such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Set $S_k = \sum_{n=1}^{k} x_n$. Let $k, j \in \mathbb{N}$ with $k > j$. This implies $k = j + p$ for some $p \in \mathbb{N}$. Hence, we have

$$\begin{aligned}
\|S_k - S_j\| &= \|S_{j+p} - S_j\| \\
&= \left\| \sum_{n=1}^{j+p} x_n - \sum_{n=1}^{j} x_n \right\| \\
&= \left\| \sum_{n=j+1}^{j+p} x_n \right\| \\
&\leq \sum_{n=j+1}^{j+p} \|x_n\| .
\end{aligned}$$

Thus,

$$0 \leq \|S_k - S_j\| \leq \sum_{n=j+1}^{j+p} \|x_n\| .$$

Since $\sum_{n=1}^{\infty} \|x_n\| < \infty$, we have that $\sum_{n=j+1}^{j+p} \|x_n\| \longrightarrow 0$ as j goes to infinity. Which yields the fact that $\{S_k\}_k$ is a Cauchy sequence and hence $\sum_{n=1}^{\infty} x_n = \lim_{k \longrightarrow \infty} S_k$ exists in E.

$(\Longleftarrow)$. Suppose E is a normed space with the property that every absolutely convergent series is conditionally convergent. Let $\{x_n\}_n$ be a Cauchy sequence. Suppose for contradiction that $\{x_n\}_n$ is not convergent. $\{x_n\}_n$ being Cauchy implies that for each $j \in \mathbb{N}$ there exists $n_j \in \mathbb{N}$ with $n_j > j$ such that

$$\|x_n - x_m\| < \frac{1}{2^j}, \forall n, m \geq n_j$$

and $\{x_{n_j}\}_j \subseteq \{x_n\}_n$ has no limit. Let $n_{j+1} > n_j$ for each j. We have that

$$\sum_{j=1}^{k} \left(x_{n_{j+1}} - x_{n_j}\right) = x_{n_{k+1}} - x_{n_1}.$$

Thus, $\sum_{j=1}^{\infty} \left(x_{n_{j+1}} - x_{n_j}\right)$ does not exists, but $\sum_{j=1}^{\infty} \left\|x_{n_{j+1}} - x_{n_j}\right\| \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1$, which is a contradiction. ■

Lemma 3.0.2 (**[Dotson, 1970]**) *Let E be a strictly convex Banach space and let $x, y \in E$ with $\|x\| \leq \|y\|$, and $\|(1-\lambda)y + \lambda x\| = \|y\|$, for some $\lambda \in (0,1)$. Then $y = x$.*

Proof Let $x, y \in E$ such that $\|x\| \leq \|y\|$. If $y = 0$, then $x = 0 = y$. Assume $y \neq 0$. We prove the contrapositive. Suppose $x \neq y$. Set $\hat{x} = \frac{x}{\|y\|}$ and $\hat{y} = \frac{y}{\|y\|}$. Then $\hat{x}, \hat{y} \in B_E$ and $\hat{x} \neq \hat{y}$. Since E is a strictly convex Banach space, it follows that $\|(1-\lambda)\hat{y} + \lambda\hat{x}\| < 1,\ \forall\ \lambda \in (0,1)$. This implies $\|(1-\lambda)y + \lambda x\| < \|y\|, \forall\ \lambda \in (0,1)$ and so there does not exist $\lambda \in (0,1)$ with $\|(1-\lambda)y + \lambda x\| = \|y\|$. ■

Lemma 3.0.3 (**[Chidume and Minjibir, 2016]**) *If $x, y, z \in D$ such that $Ty = \{z\}$, then*

$$\|u - z\| \leq d_H(Tx, Ty), \forall u \in Tx. \tag{3.0.1}$$

With these two lemmas, we prove the two more lemmas. Henceforth, we let $m \in \mathbb{N}$ such that $m \geq 2$ be fixed and let $\mathbb{I} := \{1, 2, \ldots, m\}$.

Lemma 3.0.4 *Let E be strictly convex space and let $\{x_n\}_n \subseteq E$.*

(1) *If $\{\lambda_i\}_{i=1}^m \subseteq (0,1)$ such that $\sum_{i=1}^m \lambda_i = 1, \|x_i\| \leq \|x_1\|$, for all i and $\|\sum_{i=1}^m \lambda_i x_i\| = \|x_1\|$, then $x_i = x_1$, for all i.*

(2) *If $\{\lambda_i\}_i \subseteq (0,1)$ such that $\sum_{i=1}^\infty \lambda_i = 1, \|x_i\| \leq \|x_1\|$, for all i and $\|\sum_{i=1}^\infty \lambda_i x_i\| = \|x_1\|$, then $x_i = x_1$, for all i.*

Proof

(1) We prove using induction argument. For $m = 2$, we have exactly lemma 3.0.2. Assume true for some $N \geq 2$, we now show it is true for $m = N + 1$. Let $W_k := \sum_{i=1}^{k} \lambda_j x_j$, we have that

$$\|W_{N+1}\| = \left\| W_{N-1} + (\lambda_N + \lambda_{N+1})[\frac{\lambda_N}{(\lambda_N + \lambda_{N+1})} x_N + \frac{\lambda_{N+1}}{(\lambda_N + \lambda_{N+1})} x_{N+1}] \right\|$$
$$= \|x_1\|.$$

Since

$$\left\| \frac{\lambda_N}{(\lambda_N + \lambda_{N+1})} x_N + \frac{\lambda_{N+1}}{(\lambda_N + \lambda_{N+1})} x_{N+1} \right\| \leq \|x_1\|$$

(consequent upon the fact that $\|x_N\|, \|x_{N+1}\| \leq \|x_1\|$), by inductive hypothesis we have

$$x_1 = x_\cdot = \ldots = x_{N-1} = \frac{\lambda_N}{(\lambda_N + \lambda_{N+1})} x_N + \frac{\lambda_{N+1}}{(\lambda_N + \lambda_{N+1})} x_{N+1}. \qquad (3.0.2)$$

We now show $x_{N+1} = x_N$. Indeed, if this is not the case, then $\frac{x_N}{\|x_1\|}, \frac{x_{N+1}}{\|x_1\|} \in B_E$ and $\frac{x_N}{\|x_1\|} \neq \frac{x_{N+1}}{\|x_1\|}$. By strict convexity of E, we must have

$$\left\| \frac{\lambda_N}{(\lambda_N + \lambda_{N+1})} \left(\frac{x_N}{\|x_1\|} \right) + \frac{\lambda_{N+1}}{(\lambda_N + \lambda_{N+1})} \left(\frac{x_{N+1}}{\|x_1\|} \right) \right\| < 1.$$

Which yields

$$\left\| \frac{\lambda_N}{(\lambda_N + \lambda_{N+1})} x_N + \frac{\lambda_{N+1}}{(\lambda_N + \lambda_{N+1})} x_{N+1} \right\| < \|x_1\|,$$

a contradiction, since from (3.0.2),

$$x_1 = \frac{\lambda_N}{(\lambda_N + \lambda_{N+1})} x_N + \frac{\lambda_{N+1}}{(\lambda_N + \lambda_{N+1})} x_{N+1}.$$

Therefore, $x_N = x_{N+1}$. It then follows from (3.0.2) that $x_i = x_j$, for all $i, j \in \mathbb{I}$.

(2) We first prove the following fact

- Let E be a normed space and $\{u_i\}_i \subseteq E$. If $\sum_{i=1}^{\infty} u_i = L$, then for any $N > 1$,

$$\sum_{i=N}^{\infty} u_i = L - (u_1 + u_2 + \cdots + u_{N-1}).$$

Indeed,

$$\sum_{i=1}^{\infty} u_i = \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} u_i = \lim_{n \longrightarrow \infty} \left(u_1 + u_2 + \cdot + u_{N-1} + \sum_{i=N}^{n} u_i \right) = L.$$

Hence, we have

$$(u_1 + u_2 + \cdot + u_{N-1}) + \lim_{n \longrightarrow \infty} \sum_{i=N}^{n} u_i = L,$$

which yields

$$\sum_{i=N}^{\infty} u_i = L - (u_1 + u_2 + \cdots + u_{N-1}).$$

■

Let $i_0 \in \mathbb{N}$. We have

$$\left\| \sum_{i=1}^{i_0} \lambda_i x_i + (1 - \lambda^*) \sum_{i=i_0+1}^{\infty} \frac{\lambda_i}{(1 - \lambda^*)} x_i \right\| = \|x_1\|,$$

where $\lambda^* = \sum_{i=1}^{i_0} \lambda_i$. Noting that

$$\left\| \sum_{i=i_0+1}^{\infty} \frac{\lambda_i}{(1 - \lambda^*)} x_i \right\| \leq \left(\frac{1}{(1 - \lambda^*)} \sum_{i=i_0+1}^{\infty} \lambda_i \right) \|x_1\| = \|x_1\|.$$

Applying (1) yields $x_{i_0} = x_1$. Since $i_0 \in \mathbb{N}$ was arbitrarily chosen, it follows that $x_i = x_1$, for all i.

■

Lemma 3.0.5 *Let D be a nonempty, closed and convex subset of a normed space E. Let $T_i \colon D \longrightarrow \mathcal{CB}(D)$ be quasi-nonexpansive for all $i \in \mathbb{I}$. Let $\{\lambda_{i,n}\}_n \subseteq (0,1)$, for all $i \in \mathbb{I} \cup \{0\}$ with $\sum_{i=0}^{m} \lambda_{i,n} = 1$, for all $n \in \mathbb{N}$. For any $x_0 \in D$, define the sequence $\{x_n\}_n$ iteratively by $x_{n+1} = \lambda_{0,n} x_n + \sum_{i=1}^{m} \lambda_{i,n} u_{i,n}$, where $u_{i,n} \in T_i x_n$, for all $n \geq 0$. Suppose $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and there exists $p \in D$ such that $T_i p = \{p\}$, for all $i \in \mathbb{I}$. Then,*

(1) $\|x_{n+1} - p\| \leq \|x_n - p\|$, *for all* $n \geq 1$.

(2) *If $\{x_n\}_n$ clusters at some p with $T_i p = \{p\}$, for all $i \in \mathbb{I}$, then $\{x_n\}_n$ converges strongly to p.*

(3) *If $\{x_n\}_n$ clusters at y and z, then $\|y - p\| = \|z - p\|$.*

Proof

(1) By lemma 3.0.3 and quasi-non expansiveness of T_i's, we have

$$\begin{aligned}\|x_{n+1}-p\| &= \left\|\lambda_{0,n}(x_n-p)+\sum_{i=1}^{m}\lambda_{i,n}(u_{i,n}-p)\right\| \\ &\leq \lambda_{0,n}\|x_n-p\|+\sum_{i=1}^{m}\lambda_{i,n}\|u_{i,n}-p\| \\ &\leq \lambda_{0,n}\|x_n-p\|+\sum_{i=1}^{m}\lambda_{i,n}d_H(T_ix_n,Tp) \\ &\leq \lambda_{0,n}\|x_n-p\|+\sum_{i=1}^{m}\lambda_{i,n}\|x_n-p\| \\ &= \|x_n-p\|, \forall n\geq 1.\end{aligned}$$

Hence, $\|x_{n+1}-p\|\leq\|x_n-p\|, \forall n\geq 1$.

(2) It follows trivially from (1).

(3) Let $\{w_n\}_n$ and $\{v_n\}_n$ be two subsequences of $\{x_n\}_n$ such that $w_n\longrightarrow y$ and $v_n\longrightarrow z$ From (1), we have that $\lim_{n\longrightarrow\infty}\|x_n-p\|$ exists. By triangle inequality, we have

$$\|y-p\|\leq\|y-w_n\|+\|w_n-p\|.$$

Letting n go to infinity, we have

$$\|y-p\|\leq\lim_{n\longrightarrow\infty}\|x_n-p\|.$$

Also,

$$\|w_n-p\|\leq\|w_n-y\|+\|y-p\|.$$

Letting n go to infinity, we have

$$\lim_{n\longrightarrow\infty}\|x_n-p\|\leq\|y-p\|.$$

It then follows that $\|y-p\|=\lim_{n\longrightarrow\infty}\|x_n-p\|$. Similarly, we obtain $\|z-p\|=\lim_{n\longrightarrow\infty}\|x_n-p\|$. Hence, $\|y-p\|=\|z-p\|$.

■

Lemma 3.0.6 *Let D be a non-empty, closed and convex subset of a normed space E. Let $T_i\colon D\longrightarrow\mathcal{CB}(D)$ be quasi-nonexpansive for all $i\in\mathbb{N}$. Let $\lambda_i\in(0,1)$, for all $i\in\mathbb{N}\cup\{0\}$ with $\sum_{i=0}^{\infty}\lambda_i=1$. Suppose D is bounded or $\{T_i\}_i$ is uniformly bounded.*

For any $x_0 \in D$, define the sequence $\{x_n\}_n$ iteratively by $x_{n+1} = \lambda_0 x_n + \sum_{i=1}^{\infty} \lambda_i u_{i,n}$, where $u_{i,n} \in T_i x_n$, for all $n \geq 0$. Suppose $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exists $p \in D$ such that $T_i p = \{p\}$, for all $i \in \mathbb{N}$. Then,

(1) $\|x_{n+1} - p\| \leq \|x_n - p\|$, *for all* $n \geq 1$.

(2) *If $\{x_n\}_n$ clusters at some p with $T_i p = \{p\}$, for all $i \in \mathbb{N}$ then $\{x_n\}_n$ converges strongly to p.*

(3) *If $\{x_n\}_n$ clusters at y and z, then $\|y - p\| = \|z - p\|$.*

Proof We first note that using lemma 3.0.1 the sequence is well-defined if D is bounded or $\{T_i\}_i$ is uniformly bounded.

(1) By lemma 3.0.3 and quasi-non expansiveness of T_i's, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\| \lambda_{0,n}(x_n - p) + \sum_{i=1}^{\infty} \lambda_i (u_{i,n} - p) \right\| \\
&\leq \lambda_0 \|x_n - p\| + \sum_{i=1}^{\infty} \lambda_i \|(u_{i,n} - p)\| \\
&\leq \lambda_0 \|x_n - p\| + \sum_{i=1}^{\infty} \lambda_i d_H(T_i x_n, Tp) \\
&\leq \lambda_0 \|x_n - p\| + \sum_{i=1}^{\infty} \lambda_i \|x_n - p\| \\
&= \|x_n - p\|, \forall n \geq 1.
\end{aligned}$$

Hence, $\|x_{n+1} - p\| \leq \|x_n - p\|, \forall n \geq 1$.

(2) It follows trivially from (1).

(3) Let $\{w_n\}_n$ and $\{v_n\}_n$ be two subsequences of $\{x_n\}_n$ such that $w_n \longrightarrow y$ and $v_n \longrightarrow z$ From (1), we have that $\lim\limits_{n\longrightarrow\infty} \|x_n - p\|$ exists. By triangle inequality, we have

$$\|y - p\| \leq \|y - w_n\| + \|w_n - p\|.$$

Letting n go to infinity, we have

$$\|y - p\| \leq \lim_{n\longrightarrow\infty} \|x_n - p\|.$$

Also,

$$\|w_n - p\| \leq \|w_n - y\| + \|y - p\|.$$

Letting n go to infinity, we have

$$\lim_{n\longrightarrow\infty}\|x_n-p\|\leq\|y-p\|.$$

It then follows that $\|y-p\|=\lim_{n\longrightarrow\infty}\|x_n-p\|$. Similarly, we obtain $\|z-p\|=\lim_{n\longrightarrow\infty}\|x_n-p\|$. Hence, $\|y-p\|=\|z-p\|$.

$\blacksquare$

Lemma 3.0.7 ([Mazur, 1930]) *If K is a compact subset of a Banach space E, then the closed convex hull of $K, \overline{co}(K)$ is compact.*

Proof The proof is done in two steps:

Step 1: If K is a finite set, say $\{x_1,x_2,\dots,x_m\}$, then

$$co(K)=\left\{\sum_{i=1}^m\lambda_ix_i:\{\lambda_i\}_{i=1}^m\subseteq[0,1],\sum_{i=1}^m\lambda_i=1\right\}.$$

Setting $D:=\{\{\lambda_i\}_{i=1}^m\subseteq[0,1]:\sum_{i=1}^m\lambda_i=1\}$ which is compact (it is a closed and bounded subset of a compact set $\underbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}_{finite}$) and $D_K:=D\times\{x_1\}\times\{x_2\}\times\cdots\times\{x_m\}$. We consider the function

$$f:D_K\longrightarrow E$$

$$x\mapsto f(x)=\sum_{i=1}^m\lambda_ix_1,$$

where $x=\{\lambda_i\}_{i=1}^m\times\{x_1\}\times\{x_2\}\times\cdots\times\{x_m\}$ with $\{\lambda_i\}_{i=1}^m\in D$. Indeed, the function f is clearly continuous and D_K is compact as a finite Cartesian product of compact sets. Thus, $co(K)=f(D_k)$ must be compact.

Step 2: Suppose K is any compact set. Let $\epsilon_0>0$ be fixed. Clearly, $K\subseteq\bigcup_{x\in K}B(x;\frac{\epsilon_0}{2})$ and hence, by compactness of K, there exists $F\subseteq K$, finite such that $K\subseteq\bigcup_{x\in F}B(x;\frac{\epsilon_0}{2})$. By **step 1**, we have that $co(F)$ is compact, hence there exists $G\subseteq co(F)$, finite such that $co(F)\subseteq\bigcup_{y\in G}B(y;\frac{\epsilon_0}{2})$ (consequence of $\bigcup_{y\in co(F)}B(y;\frac{\epsilon_0}{2})$ being an open cover for $co(F)$). We now show that $co(K)\subseteq\bigcup_{x\in G}B(x;\epsilon_0)$. Let $x\in co(K)$, we have that $x=\sum_{i=1}^p t_ix_i$ where $\{t_i\}_{i=1}^p\subseteq[0,1]$ with $\sum_{i=1}^p t_i=1$ and $\{x_i\}_{i=1}^p\subseteq K$. For each i, there exists $y_i\in F$ such that $x_i\in B(y_i;\frac{\epsilon_0}{2})$, i.e.,

$\|x_i - y_i\| < \frac{\epsilon_0}{2}$). Setting $y = \sum_{i=1}^{p} t_i y_i \in co(F)$, we have that $y \in B(y_0, \frac{\epsilon_0}{2})$ for some $y_0 \in G$.

$$\begin{aligned}
\|x - y_0\| &= \|x - y + y - y_0\| \\
&\leq \|x - y\| + \|y - y_0\| \\
&= \left\| \sum_{i=1}^{p} t_i x_i - \sum_{i=1}^{p} t_i y_i \right\| + \|y - y_0\| \\
&\leq \sum_{i=1}^{p} t_i \|x_i - y_i\| + \|y - y_0\| \\
&< \frac{\epsilon_0}{2} \sum_{i=1}^{p} t_i + \frac{\epsilon_0}{2} \\
&= \epsilon_0.
\end{aligned}$$

Thus, $co(K) \subseteq \bigcup_{x \in G} B(x; \epsilon_0)$ and so $co(K)$ is totally bounded, hence $\overline{co}(K)$ is compact since E is Banach.

■

Lemma 3.0.8 (**[Schauder, 1950]**) *Let K be a non-empty, closed, bounded and convex subset of a Banach space E. Let $f : K \longrightarrow K$ be completely continuous (i.e., f is continuous and $f(K)$ is compact), then there exists $x^* \in K$ such that $f(x^*) = x^*$.*

CHAPTER 4

Main Results

Theorem 4.0.9 *Let E be a strictly convex real Banach space and D be a nonempty, closed and convex subset of E. Let $T_i\colon D \longrightarrow \mathcal{PB}(D)$ be quasi-nonexpansive and continuous with respect to the Hausdorff metric, for all $i \in \mathbb{I}$ with $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$, for all $p \in \bigcap_{i=1}^{m} F(T_i)$. Suppose $T_i(D)$ is contained in a compact set K for all $i \in \mathbb{I}$. For any $x_0 \in D$ define a sequence $\{x_n\}_n$ iteratively, by*

$$x_{n+1} = \lambda_{0,n} x_n + \sum_{i=1}^{m} \lambda_{i,n} u_{i,n}, u_{i,n} \in T_i x_n, n \geq 0, \tag{4.0.1}$$

where $\{\lambda_{i,n}\}_n \subseteq (0,1), i \in \mathbb{I} \cup \{0\}, \sum_{i=0}^{m} \lambda_{i,n} = 1, n \geq 0$. If for each $i \in \mathbb{I} \cup \{0\}$, $\{\lambda_{i,n}\}_n$ clusters at some point of $(0,1)$, then $\{x_n\}_n$ converges strongly to a common fixed point of T_i's.

Proof Since $\{\lambda_{i,n}\}_n$ clusters at some point, say $\lambda_i \in (0,1)$ for each $i \in \mathbb{I} \cup \{0\}$ and $\{x_n\}_{n\geq 1} \subseteq \overline{co}(K \cup \{x_0\})$ which is compact by lemma 3.0.7 and the fact that finite union of compact sets is compact, we obtain $\{\lambda_{i,n_k}\}_k \subseteq \{\lambda_{i,n}\}_n$ and $\{x_{n_k}\}_k \subseteq \{x_n\}_{n\geq 1}$ such that $\{\lambda_{i,n_k}\}$ converge to points in $(0,1)$, for each i and $x_{n_k} \longrightarrow x^* \in \overline{co}(K \cup \{x_0\}) \subseteq D$. This implies $d_H(T_i x_{n_k}, T_i x^*) \longrightarrow 0$. Indeed, since the corresponding sequences $\{u_{i,n_k}\}_k \subseteq K$, it follows that there exists $\{u_{i,n_{k_j}}\}_j \subseteq \{u_{i,n_k}\}_k$ such that $u_{i,n_{k_j}} \longrightarrow u_i{}^* \in D$. Let $w_i{}^* \in T_i x^*$ such that $\|w_i{}^* - u_i{}^*\| = \inf\limits_{u_i \in T_i x^*} \|u_i - u_i{}^*\|$ (such a

${w_i}^*$ exists, since T_ix^* is proximinal for each i, by assumption). Hence,

$$\begin{aligned}\|{w_i}^* - {u_i}^*\| &\leq \inf_{u_i \in T_ix^*} \|u_i - u_{i,n_{k_j}}\| + \|u_{i,n_{k_j}} - {u_i}^*\| \\ &\leq \sup_{v_i \in T_ix_{n_{k_j}}} \inf_{u_i \in T_ix^*} \|u_i - v_i\| + \|u_{i,n_{k_j}} - {u_i}^*\| \\ &\leq d_H(T_ix_{n_{k_j}}, T_ix^*) + \|u_{i,n_{k_j}} - {u_i}^*\| \forall j \geq 1.\end{aligned}$$

Letting j go to infinity, we have $\|{w_i}^* - {u_i}^*\| = 0$. Hence ${u_i}^* = {w_i}^* \in T_ix^*$. Therefore,

$$x_{n_{k_j}+1} = \lambda_{0,n_{k_j}} x_{n_{k_j}} + \sum_{i=1}^{m} \lambda_{i,n_{k_j}} u_{i,n_{k_j}} \longrightarrow \lambda_0 x^* + \sum_{i=1}^{m} \lambda_i {u_i}^*.$$

Thus, $\{x_n\}_n$ clusters at x^* and $\lambda_0 x^* + \sum_{i=1}^{m} \lambda_i {u_i}^*$. By lemma 3.0.5(3), we have

$$\left\| \lambda_0(x^* - p) + \sum_{i=1}^{m} \lambda_i({u_i}^* - p) \right\| = \|x^* - p\|.$$

Also, by lemma 3.0.3 and definition of quasi-nonespansive multivalued map, we have that $\|{u_i}^* - p\| \leq \|x^* - p\|$, for each $i \in \mathbb{I}$. Since E is strictly convex we have by lemma 3.0.4(1) that $x^* - p = {u_i}^* - p$, for all i. This implies $x^* = {u_i}^* \in T_ix^*$, for all i. Thus, $x^* \in \bigcap_{i=1}^{m} F(T_i)$ and so $T_ix^* = \{x^*\}$, for all i. Using lemma 3.0.5(2) we conclude that $x_n \longrightarrow x^*$. Hence the sequence defined above converges strongly to a common fixed point of T_i's. ∎

Corollary 4.0.10 *Let E be a strictly convex real Banach space and D be a nonempty, closed and convex subset of E. Let $T_i \colon D \longrightarrow \mathcal{PB}(D)$ be quasi-nonexpansive and continuous with respect to the Hausdorff metric, for all $i \in \mathbb{I}$ with $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and $T_ip = \{p\}$, for all $p \in \bigcap_{i=1}^{m} F(T_i)$. Suppose $T_i(D)$ is contained in a compact set K for all $i \in \mathbb{I}$. For any $x_0 \in D$ define a sequence $\{x_n\}_n$ iteratively, by*

$$x_{n+1} = \lambda_0 x_n + \sum_{i=1}^{m} \lambda_i u_{i,n}, u_{i,n} \in T_ix_n, n \geq 0, \tag{4.0.2}$$

where $\lambda_i \in (0,1), i \in \mathbb{I} \cup \{0\}, \sum_{i=0}^{m} \lambda_i = 1$. Then $\{x_n\}_n$ converges strongly to a common fixed point of T_i's.

Proof We take $\{\lambda_{i,n}\}_n$ to be the constant sequence $\{\lambda_i\}_n$ for each $i \in \mathbb{I}$. Then $\{\lambda_{i,n}\}_n$ clusters at λ_i for each i and the proof follows from theorem 4.0.9. ∎

Corollary 4.0.11 *Let E be a strictly convex real Banach space and D be a nonempty, closed and convex subset of E. Let $T\colon D \longrightarrow \mathcal{PB}(D)$ be quasi-nonexpansive and continuous with respect to the Hausdorff metric, with $F(T) \neq \emptyset$ and $Tp = \{p\}$, for all $p \in F(T)$. Suppose $T(D)$ is contained in a compact set K. For any $x_0 \in D$ define a sequence $\{x_n\}_n$ iteratively by*

$$x_{n+1} = (1 - t_n)x_n + t_n y_n, y_n \in Tx_n, n \geq 0,$$

where $\{t_n\} \subseteq (0, 1)$ clusters at some $t \in (0, 1)$. Then $\{x_n\}_n$ converges strongly to a fixed point of T.

Corollary 4.0.12 **([Krasnoselskii, 1955])** *Let E be a uniformly convex normed space and D be a nonempty, closed and convex subset of E. Let $f : D \longrightarrow D$ be nonexpansive and $f(D) \subseteq K \subseteq D, K$ compact. For any $x_0 \in D$, let a sequence $\{x_n\}_n$ be defined iteratively, by*

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}f(x_n), n \geq 0. \tag{4.0.3}$$

Then $\{x_n\}_n$ converges strongly to a fixed point of f.

Proof By lemma 3.0.8, we have that $F(f) \neq \emptyset$. Also define $T : D \longrightarrow \mathcal{PB}(D)$ by $Tx = \{f(x)\}$. Then, the proof follows from corollary 4.0.10. ■

Corollary 4.0.13 **([Dotson, 1970])** *Let E be a strictly convex normed space and D be a nonempty, closed and convex subset of E. Let $f : D \longrightarrow D$ be continuous and quasi-nonexpansive and $f(D) \subseteq K \subseteq D, K$ compact. For any $x_0 \in D, \{t_n\}_n \subseteq (0, 1)$ such that $\{t_n\}_n$ clusters at some $t \in (0, 1)$ let a sequence $\{x_n\}_n$ be defined iteratively, by*

$$x_{n+1} = (1 - t_n)x_n + t_n f(x_n), n \geq 0. \tag{4.0.4}$$

Then $\{x_n\}_n$ converges strongly to a fixed point of f.

Proof By lemma 3.0.8, we have that $F(f) \neq \emptyset$. Also define $T : D \longrightarrow \mathcal{PB}(D)$ by $Tx = \{f(x)\}$. Then, the proof follows from corollary 4.0.10. ■

Theorem 4.0.14 *Let E be a strictly convex real Banach space and D be a nonempty, closed and convex subset of E. Let $T_i\colon D \longrightarrow \mathcal{PB}(D)$ be quasi-nonexpansive and* *continuous with respect to the Hausdorff metric, for all $i \in \mathbb{N}$ with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$, for all $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Suppose $T_i(D)$ is contained in a compact set K for all $i \in \mathbb{N}$. For any $x_0 \in D$ define a sequence $\{x_n\}_n$ iteratively, by*

$$x_{n+1} = \lambda_0 x_n + \sum_{i=1}^{\infty} \lambda_i u_{i,n}, u_{i,n} \in T_i x_n, n \geq 0, \tag{4.0.5}$$

where $\lambda_i \in (0,1), i \in \mathbb{N} \cup \{0\}, \sum_{i=0}^{\infty} \lambda_i = 1$. Then $\{x_n\}_n$ converges strongly to a common fixed point of T_i's.

Proof Since $\{x_n\}_{n\geq 1} \subseteq \overline{co}(K \cup \{x_0\})$ which is compact by lemma 3.0.7 and the fact that finite union of compact sets is compact, we obtain $\{x_{n_k}\}_k \subseteq \{x_n\}_{n\geq 1}$ such that $x_{n_k} \longrightarrow x^* \in \overline{co}(K \cup \{x_0\}) \subseteq D$. This implies $d_H(T_i x_{n_k}, Tx^*) \longrightarrow 0$. Indeed, since the corresponding sequences $\{u_{i,n_k}\}_k \subseteq K$, it follows that there exists $\{u_{i,n_{k_j}}\}_j \subseteq \{u_{i,n_k}\}_k$ such that $u_{i,n_{k_j}} \longrightarrow {u_i}^* \in D$. Let ${w_i}^* \in T_i x^*$ such that $\|{w_i}^* - {u_i}^*\| = \inf_{u_i \in T_i x^*} \|u_i - {u_i}^*\|$ (such a ${w_i}^*$ exists, since $T_i x^*$ is proximinal for each i, by assumption). Hence,

$$\begin{aligned}
\|{w_i}^* - {u_i}^*\| &\leq \inf_{u_i \in T_i x^*} \|u_i - u_{i,n_{k_j}}\| + \|u_{i,n_{k_j}} - {u_i}^*\| \\
&\leq \sup_{v_i \in T_i x_{n_{k_j}}} \inf_{u_i \in T_i x^*} \|u_i - v_i\| + \|u_{i,n_{k_j}} - {u_i}^*\| \\
&\leq d_H(T_i x_{n_{k_j}}, T_i x^*) + \|u_{i,n_{k_j}} - {u_i}^*\|, \forall j \geq 1.
\end{aligned}$$

Letting j go to infinity, we have $\|{w_i}^* - {u_i}^*\| = 0$. Hence ${u_i}^* = {w_i}^* \in T_i x^*$. Therefore,

$$x_{n_{k_j}+1} = \lambda_{0,n_{k_j}} x_{n_{k_j}} + \sum_{i=1}^{\infty} \lambda_{i,n_{k_j}} u_{i,n_{k_j}} \longrightarrow \lambda_0 x^* + \sum_{i=1}^{\infty} \lambda_i {u_i}^*.$$

Thus, $\{x_n\}_n$ clusters at x^* and $\lambda_0 x^* + \sum_{i=1}^{\infty} \lambda_i {u_i}^*$. By lemma 3.0.6(3), we have that

$$\left\| \lambda_0(x^* - p) + \sum_{i=1}^{\infty} \lambda_i({u_i}^* - p) \right\| = \|x^* - p\|.$$

Also, by lemma 3.0.3 and definition of quasi-nonespansive multivalued map, we have that $\|{u_i}^* - p\| \leq \|x^* - p\|$, for each $i \in \mathbb{N}$. Again by lemma 3.0.6(3). Since E is strictly convex we have by lemma 3.0.4(2) that $x^* - p = {u_i}^* - p$ for all i. This implies $x^* = {u_i}^* \in T_i x^*$. Thus, $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$ and so $T_i x^* = \{x^*\}$. Using lemma 3.0.6(2) we conclude that $x_n \longrightarrow x^*$. Hence the sequence defined above converges strongly to a common fixed point of T_i's. ∎

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