MODIFIED FORWARD-BACKWARD SPLITTING METHOD WITHOUT COCOERCIVITY FOR THE SUM OF TWO MONOTONE OPERATORS IN BANACH SPACES

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By

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Certification

This is to certify that the thesis titled "MODIFIED FORWARD-BACKWARD SPLITTING METHOD WITHOUT COCOERCIVITY FOR THE SUM OF TWO MONOTONE OPERATORS IN BANACH SPACES" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research work carried out by Maryam Alka in the Department of Pure and Applied Mathematics.

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A THESIS APPROVED BY THE DEPARTMENT OF PURE AND APPLIED MATHEMATICS

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Abstract

In this thesis, we present an algorithm for solving a variation inclusion problem of sum of two monotone operators in real Banach spaces which uses variable stepsizes that are updated over each iteration by some cheap computations. These stepsizes are found without using Linesearch Procedure or prior knowledge of the Lipschitz constant. More precisely, we provide the following theorem.

Theorem. *Let E be a real 2-uniformly convex and uniformly smooth Banach space. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $B : E \to E^*$ be monotone and Lipschitz, and assume that $(A + B)^{-1}(0) \neq \emptyset$ and J is weakly sequentially continuous. For $x_0, x_{-1} \in E$ define the sequence x_n iteratively by*

$$x_{n+1} = J^A_{\lambda_n} \circ J^{-1}(Jx_n - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1})), \quad n \geq 0;$$
$$\lambda_{n+1} := \min\left\{\lambda_n, \frac{\theta\|x_{n+1} - x_n\|}{\|Bx_{n+1} - Bx_n\|}\right\}, \theta \in (0, \frac{1}{2\mu}), \mu \geq 1.$$

Then, the sequence $\{x_n\}$ converges weakly to s an element of $(A + B)^{-1}(0)$.
Finally, we give some real life physical applications to show how results can be applied.

Dedication

I dedicate this work to my late Father in-law Alhaji Disina Mohammed may Allah SWA continue to have mercy on your gentle soul.

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Contents

CHAPTER 1

General Introduction and Literature Review

The contents of this thesis fall within the general area of nonlinear functional analysis, an area which has attracted the attention of prominent mathematicians due to its diverse application in numerous fields of sciences. The contribution of this thesis focuses mainly on the approximation of solution of sum of two monotone operators in real Banach Spaces.

In this chapter, we give a general introduction on variational inclusion problem of the sum of two monotone operators. Let E be a real Banach space and E^* be its topological dual. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $B : E \to E^*$ a monotone and Lipschitz map. The Inclusion problem which is to

$$\text{find } x \in (A + B)^{-1}(0) \tag{1.0.1}$$

with $(A + B)^{-1}(0) \neq \emptyset$ plays a vital role in nonlinear analysis. The importance of problem (1.0.1) cannot be overemphasized as its application cuts across different areas of nonlinear analysis such as optimization, split feasibility problems, convex programming and saddle-point problems. For more on the inclusion problem of the form (1.0.1) see, Attouch *et al.* [5], Bruck [9], Censor and Elfvin [10], Chen and Rockafellar [12], Combettes and Wajs [15], Davis and Yim [16] Lions and Mercier [19], Moudafi and Thera [21], Passty [23], Peaceman and Rachford [24] to mention but a few.

1.1 Motivation of Research

There are many physical problems in application that can be cast as inclusion problem of the form (1.0.1). In this section, we mention some examples to show how problems in application can be transformed into an equation of the form (1.0.1):

Convex Optimization Problem

Consider, for instance, the optimization problem, introduced by Chen et al. in [11],

$$\min_{x \in H_1} \frac{1}{2}\|Tx - b\|^2 + \lambda\|x\|_1 \tag{1.1.1}$$

where H_1 and H_2 are real Hilbert spaces, $T : H_1 \to H_2$ is a bounded linear operator, $b \in H_2$ and $\lambda > 0$. Then, it is easy to see that (1.1.1) equivalent to (1.0.1) by setting $A(x) = \partial(\lambda\|x\|_1)$ and

.

$$\begin{aligned} B(x) \quad &= \nabla(\frac{1}{2}\|Tx - b\|^2) \\ &= \frac{1}{2}\nabla(\langle Tx - b, Tx - b\rangle) \\ &= \frac{1}{2}\nabla(\langle Tx, Tx\rangle - 2\langle Tx, b\rangle + \langle b, b\rangle) \\ &= \frac{1}{2}\nabla(x^*T^*Tx - 2x^*T^*b) \\ &= T^*(Tx - b). \end{aligned}$$

Split Feasibility Problem

Consider the Split Feasibility Problem introduced by Censor and Elfvin [10], which is to find

$$x \in C \text{ such that } Tx \in Q, \tag{1.1.2}$$

where $C \subset H_1$, $Q \subset H_2$ are nonempty closed and convex subsets of the Hilbert spaces H_1 and H_2, respectively, and $T : H_1 \to H_2$ is a bounded linear map. Similarly, by setting

$$A(x) = N_C(x) \text{ and } B(x) = \nabla\left(\frac{1}{2}\|Tx - P_QTx\|^2\right) = T^*(I - P_Q)Tx,$$

then, it is easily seen that (1.1.2) can be reformulated as (1.0.1).

Saddle Point Problem

Lastly, we consider the first order optimality condition for the saddle-points problems of the form

$$\min_{x\in H_1}\max_{x\in H_2} f(x) + \Phi(x, y) + g(x), \tag{1.1.3}$$

where $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ are proper convex and lower semicontinuous functions and $\Phi : H_1 \times H_2 \to \mathbb{R}$ is a smooth convex-concave function, is given by (1.0.1) with

$$A = \begin{pmatrix} \partial f(x) \\ \partial g(y) \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} \nabla_x\Phi(x, y) \\ -\nabla_y\Phi(x, y) \end{pmatrix}.$$

1.2 Approximation of zeros of sum of two monotone operators

Several authors have worked on the approximation of the solution of problem (1.0.1) see, e.g., [19, 23, 5, 9, 16]. A well-known method for solving Problem 1.0.1 in Hilbert spaces is the *forward-backward* splitting method introduced independently by Passty [23], and Lions and Mercier [19]. The method generates a sequence $\{x_n\}$ iteratively defined by

$$x_{n+1} = J^A_{\lambda_n}(x_n - \lambda_n B x_n), \; n \geq 0. \tag{1.2.1}$$

which converges weakly to some solution of (1.0.1) under the hypothesis that the operator B is μ-coercive, that is, there exists $\mu > 0$ such that

$$\langle x - y, B(x) - B(y) \rangle \geq \mu \|B(x) - B(y)\|^2, \quad \forall \; x, y \in E,$$

and that $\liminf \lambda_n > 0$ with $\limsup \lambda_n < 2\mu$. Under these assupmtions, the sequence $\{x_n\}$ generated by (1.2.1) converges weakly to a solution of (1.0.1). There are some important problems in application where the forward-backward splitting method fails to converge due to the lack of cocoercivity of one of the operators. For instance, problem (1.1.3) arises naturally in different areas of application such as statistics, machine learning and optimization to mention a few. Although the operator B is Lipschitz whenever $\nabla\Phi$ is, B is never coercive even when Φ is bilinear. Thus, the condition imposed on the operator B limits the class of operators for which the forward-backward splitting method is applicable.

In 2000, Tseng [31] introduced the *forward-backward-forward* splitting method for approximating solutions of (1.0.1) in which the cocoercivity of B is dispensed at the expense of its evaluation twice per iteration. The method generates a sequence $\{x_n\}$ iteratively defined by

$$\begin{cases} y_n = J^A_{\lambda_n}(x_n - \lambda_n B(x_n)), \\ x_{n+1} = y_n - \lambda_n B(y_n) + \lambda_n B(x_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.2.2}$$

with $\lambda_n \in (0, \frac{1}{L})$, where L is the Lipschitz constant of B. The weak convergence of the sequences $\{x_n\}$ and $\{y_n\}$ is established under the assumptions of monotonicity and Lipschitzness of B.

Recently, Malitsky and Tam [20], introduced the *forward-reflected-backward* splitting method generated by iteratively by

$$x_{n+1} = J^A_{\lambda_n}(x_n - \lambda_n B(x_n) - \lambda_{n-1}(B(x_n) - B(x_{n-1}))), \quad \forall n \in \mathbb{N},$$

with $\lambda_n \in (\epsilon, \frac{1-2\epsilon}{2L})$ and $\epsilon > 0$. The forward-reflected-backward splitting method requires only one evaluation of the operator B per iteration. Thus, improving on the computational cost when compared to Tseng's method which requires two evaluations of the operator B per iteration. It is worth mentioning that the forward-reflected backward method does not require B to be coersive

In 2020, Hieu *et al.* [17], also introduced same algorithm termed **M**odified Forward Reflected Backward Splitting Method (MFRBSM) generated iteratively by

$$x_{n+1} = J^A_{\lambda_n}(x_n - \lambda_n B(x_n) - \lambda_{n-1}(B(x_n) - B(x_{n-1})), \quad \forall n \in \mathbb{N},$$

with

$$\lambda_{n+1} := \min\left\{\lambda_n, \frac{\theta\|x_{n+1} - x_n\|}{\|Bx_{n+1} - Bx_n\|}\right\}, \theta \in (0, \tfrac{1}{2}).$$

One of the advantage of this scheme is that the variable stepsizes do not require prior knowlegde of Lipschitz constant or the use of Linesearch. Also, the operator B is not required to be coersive. It is worth mentioning that all the results mentioned above are done in the settings of Hilbert spaces. There are few results regarding the forward-backward method and its variants in Banach spaces, see, e.g., [28, 32]. One of the difficulties, perhaps, is the fact that the operators A and B go from the Banach space E to its dual E^*.The reason for this observation is that certain geometric properties which characterize Hilbert spaces make certain problems posed in Hilbert spaces easier to handle than in more general Banach Space. However, many problems fall naturally in Banach spaces more often than Hilbert spaces.

1.3 Our contribution

Motivated by the result of Hieu *et al.* [17], the contribution of this thesis is on approximation of zeros of variation inclusion problem of sum of two monotone operators in a real 2-uniformly convex and uniformly smooth Banach space. We also present application of our main theorem to some physical application problems.
Finally, our main theorem unifies, extends and complements many results in the literature.

CHAPTER 2

Preliminaries

The aim of this chapter is to recall some basic definitions and provide existing results of interest relating to geometric properties of Banach Space. However, we only give sketch of some of the proofs while the rest can be found in the mentioned references. Throughout this thesis, E is a real normed space with its dual space E^* and $\langle x, x^* \rangle$ denotes the duality pairing of the function $x^* \in E^*$ at $x \in E$. The norm in E is denoted by $\|.\|$, while the norm in E^* is denoted by $\|.\|_*$. If there is no danger of confusion we omit the asterisk from the notation $\|.\|_*$ and denote both norm in E and $E_*$by the symbol $\|.\|$.
If a sequence $\{x_n\}$ in E, converges strongly (weakly), we denote the convergence by $x_n \to x$ ($x_n \rightharpoonup x$), respectively.

2.1 Geometry of Banach Spaces

2.1.1 Smooth and uniformly Smooth Spaces

Definition 2.1.1. *A normed space E is called smooth if for every $x \in E$, $\|x\| = 1$, there exists a unique $x^* \in E^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.*

Definition 2.1.2. *Let $q > 1$ and $r > 0$, be two fixed real numbers. Then E is uniformly smooth if and only if there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+, g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^q \geq \lambda\|x\|^q + (1-\lambda)\|y\|^q - W_q(\lambda)g(\|x-y\|),$$

$\forall\ x, y \in B_r,\quad 0 \leq \lambda \leq 1,$ *where* $W_q(\lambda) = \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$.

2.1.2 Strictly and Uniformly Convex Spaces

Definition 2.1.3. *A normed linear space E is uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.*

Definition 2.1.4. *A normed space E is called strictly convex if for all $x, y \in E$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1-\lambda)y\| < 1$, $\forall \lambda \in (0, 1)$.*

Remark 2.1.5.

1. Every uniformly convex space is reflexive.
2. Every uniformly convex space is strictly convex. However, the converse may not hold.

It is important to observe that Uniform convexity of a Banach Space E is a property of the norm on E. For instance, $\mathbb{R}^2$ endowed with norm $\|.\|_2$ is uniformly convex whereas with neither the norm $\|.\|_1$ nor $\|.\|_\infty$ is uniformly convex.
We now give some examples to illustrate Uniformly and Strictly convex spaces.

Example 2.1.6. *let C[0,1] be endowed with the norm $\|.\|_\mu$ defined as follows:*

$$\|x\|_\mu := \|x\|_0 + \mu(\int_0^1 x^2(t)dt)^{\frac{1}{2}},$$

where $\|.\|_0$ is the usual sup norm. Then,

$$\|x\|_0 \leq \|x\|_\mu \leq (1+\mu)\|x\|_0, \; x \in C[0,1],$$

and the two norms are equivalent. However, (C[0, 1],$\|.\|_0$) is not strictly convex while for any $\mu > 0$, (C[0, 1], $\|.\|_\mu$) is strictly convex.
On the contrary, for any $\epsilon \in (0, 2]$ there exist functions $f, g \in$ C[0, 1] with $\|x\|_\mu = \|y\|_\mu = 1, \|x-y\| = \epsilon$, and $\|\frac{x+y}{2}\|$ arbitrarily near 1. Thus, (C[0, 1], $\|.\|_\mu$) is not uniformly convex.

Example 2.1.7. *Consider C[a, b], the space of real–valued continuous functions on the compact interval [a, b], with the "sup norm". Then C[a, b] is not strictly convex.*

Indeed, let $f, \; g$ be functions defined as follows:

$$f(t) := 1 \;\; for \;\; each \;\; t \in [a,b], \;\; g(t) := \frac{b-t}{b-a} \;\; for \;\; each \;\; t \in [a,b].$$

Take $\epsilon = \frac{1}{2}$. Clearly $f, g \in$ C[a, b], $\|f\| = \|g\| = 1$ and $\|f-g\| = 1 > \epsilon$. Also $\|\frac{1}{2}(f+g)\| = 1$. Thus, C[a, b] is not strictly convex.

2.1.3 P-Uniformly Convex Spaces

Definition 2.1.8. *Let $p > 1$ be a real number. Then, a normed space E is said to be p-uniformly convex if there is a constant $c > 0$ such that*

$$\delta_E(\epsilon) \geq c\epsilon^p,$$

Definition 2.1.9. *Let E be a normed space with $dim E \geq 2$. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by*

$$\delta_E := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\right\}.$$

Remark 2.1.10. *E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2)$.*

Example 2.1.11. *If $E = L_p($ or $l_p)$, $1 < p < \infty$, then E is $P-$uniformly Convex with*

(a) $\delta_E(\epsilon) \geq \frac{1}{2^{p+1}}\epsilon^2$ *if* $1 < p < 2;$ *and*

(b) $\delta_E(\epsilon) \geq \epsilon^p$, *if* $2 \leq p < \infty$.

2.2 Duality Mappings

The duality maps play a crucial role in solving nonlinear problems in the setting of Banach spaces due to the lack of an inner product on Banach spaces.

Definition 2.2.1. *Given a gauge function g, the map $J_g : E \to 2^{E^*}$ defined by*

$$J_g x := \{u^* \in E^* : \langle x, u^* \rangle = \|x\| \|u^*\|; \quad \|u^*\| = g(\|x\|)\},$$

is called the duality map with the gauge function g where E is any normed space.

It is easy to see that $J_g x$ is nonempty for each $x \in E$. Thus, indeed a consequence of the Hahn-Banach theorem.

Remark 2.2.2.

- In the particular case $g(x) = x$, the duality map $J = J_g$ is called the normalized duality map.
- If E is a reflexive, strictly convex and smooth real Banach space, then J is single-valued and bijective.
- In a Hilbert space H, the duality map J and it is inverse J^{-1} are the identity maps on H.
- If E is uniformly smooth and uniformly convex, then the dual space E^* is also uniformly smooth and uniformly convex and the normalized duality map J and its inverse, J^{-1}, are both uniformly continuous on bounded sets.
- In L_P Spaces, $1 < P < \infty$, the duality map is given by

$$J(f) = |f|^{p-1} .sgn \frac{f}{\|f\|^{p-2}}.$$

Proposition 2.2.3. *[3] Let E be a uniformly convex Banach space. Then, the normalized duality mapping, J, is uniformly monotone on every bounded set. That is, for every $R > 0$ and arbitrary $x, y \in E$ with $\|x\| \leq R$ and $\|y\| \leq R$ there exists a real non-negetive and continuous function $\psi_R : [0, \infty) \to [0, \infty)$ such that $\psi_R(t) > 0$ for $t > 0$, $\psi_R(0) = 0$ and*

$$\langle Jx - Jy, x - y \rangle \geq \psi_R(\|x - y\|).$$

2.3 Some nonlinear Functionals and Operators

2.3.1 Bregman Distance

Definition 2.3.1. *Let E be a smooth real Banach space with dual E^*. The functional, $\phi : E \times E \to \mathbb{R}$, defined by*

$$\phi(x, y) := \|x\|^2 - \langle x, Jy \rangle + \|y\|^2, \text{ for } x,y \in E, \tag{2.3.1}$$

where J is the normalized duality mapping on E will play a central role in the sequel. It was introduced by Alber and has been studied by Alber [1], Alber and Guerre-Delabriere [2], Kamimura and Takahashi [18], Reich [26], Chidume *et al.* [13, 14] and alot of other authors.

Remark 2.3.2. *It is easy to see from the definition of ϕ that in a real Hilbert space H, equation (2.3.1) reduces to $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in H$.*

Lemma 2.3.3. *[1, 4] Let E be a real uniformly convex, smooth Banach space. Then, the following identities hold:*

(i) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E.$

(ii) $\phi(x, y) + \phi(y, x) = 2\langle x - y, Jx - Jy\rangle, \quad \forall x, y \in E.$

Furthermore, given $x, y, z \in E$, and $\tau \in (0, 1)$, we have the following properties (see, [22]):

P1 $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$,

P2 $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle z - x, Jy - Jz\rangle$,

P3 $\phi(\tau x + (1 - \tau)y, z) \leq \tau\phi(x, z) + (1 - \tau)\phi(y, z)$

Lemma 2.3.4. *[4] Let E be a real 2-uniformly convex Banach space. Then, there exists $\mu \geq 1$ such that*

$$\frac{1}{\mu}\|x - y\|^2 \leq \phi(x, y) \quad \forall x, y \in E.$$

Lemma 2.3.5. *[18] Let E be a uniformly convex and smooth Banach space, $\{x_n\}$ and $\{y_n\}$ be two sequence of E. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

2.3.2 Monotone Operators

Definition 2.3.6. *Let E be a real normed space. A map $A : E \to 2^{E^*}$ is called monotone if for each $x, y \in E$,*

$$\big\langle \sigma - \eta, x - y\big\rangle \geq 0, \quad \forall \quad \sigma \in Ax, \quad \eta \in Ay. \tag{2.3.2}$$

If A is single-valued, the map $A : E \to E^*$ is called monotone if

$$\big\langle Ax - Ay, x - y\big\rangle \geq 0, \quad \forall \;\; x, y \in E. \tag{2.3.3}$$

We now give some examples of monotone operators

Example 2.3.7. *Every non decreasing function on $\mathbb{R}$ is monotone.*

To see this, let $f : \mathbb{R} \to \mathbb{R}$ be a non decreasing function. Then for arbitrary $x, y \in R$ with $x \leq y$ we have $f(x) \leq f(y)$. Thus we see that $\langle y - x, f(y) - f(x)\rangle \geq 0$ for all $x, y \in \mathbb{R}$. Hence the monotonicity of f.

Example 2.3.8. *Let E be a real Banach space. Then the duality map defined in 2.2.1 is monotone.*

Indeed for any $(x_i, y_i) \in J$, i = 1, 2. we have

$$\begin{aligned} \langle x_1 - x_2, y_1 - y_2 \rangle = & \quad \|x_1\|^2 + \|x_2\|^2 - \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle \\ \geq & \quad \|x_1\|^2 + \|x_2\|^2 - \|x_1\|\|y_2\| - \|x_2\|\|y_1\| \\ = & \quad \|x_1\|^2 + \|x_2\|^2 - 2\|x_1\|\|x_2\| \\ = & \quad (\|x_1\| - \|x_2\|)^2 \end{aligned}$$

Hence J is monotone.

Example 2.3.9. *Let H be a real Hilbert space, I the identity map of H and $T : H \to H$ be a nonexpansive map that is,*

$$\|Tx - Ty\| \leq \|x - y\|.$$

Then the operator $I -\ T$ is monotone.

Let $x,\ y\ \in\ H$, then

$$\begin{aligned} \langle x - y, (I - T)x - (I - T)y \rangle &= \langle x - y, (x - y) - (Tx - Ty) \rangle \\ &= \|x - y\|^2 \langle x - y, (Tx - Ty) \rangle \\ &\geq \|x - y\|^2 - \|x - y\|\|Tx - Ty\| \\ &\geq \|x - y\|^2 - \|x - y\|^2 = 0. \end{aligned}$$

Here we have used Cauchy inequality and the fact that T is nonexpansive. Thus, we have $I - T$ is monotone on H.

Example 2.3.10. *Let A be $n \times n$ matrix with real entries. Consider the function $g : \mathbb{R}^n \to \mathbb{R}^n$ defined by $g(x) = Ax$. Then g is monotone if and only if A is positive semi definite.*

2.3.3 Maximally monotone operators

Definition 2.3.11. *A multivalued monotone operator $A : E \to 2^{E^*}$ is said to be maximal monotone if $A = B$ whenever $B : E \to 2^{E^*}$ is monotone and $G(A) \subset G(B)$, where $G(A) = \{(x, x^*) : x^* \in Ax\}$ is the graph of A.*

Lemma 2.3.12. *[6] Let $A : E \to 2^{E^*}$ be a maximal monotone mapping and $B : E \to E^*$ be a Lipschitz continuous and monotone mapping. Then the mapping $A+B$ is a maximal monotone.*

2.3.4 Resolvent Operator

Definition 2.3.13. *Let E be a real reflexive, strictly convex and smooth Banach space and let $A : E \to 2^{E^*}$ be a maximal monotone operator. Then for each $r > 0$ the resolvent of A, $J_r^A : E \to E$ is defined by*

$$J_r^A(x) = (J + rA)^{-1} Jx,$$

where J is the normalized duality mapping on E.

It is easy to show that $A^{-1}0 = F(J_r^B)$ for all $r > 0$, where $F(J_r^A)$ denotes the set of fixed points of J_r^A.

CHAPTER 3

Main Results

In this section,we present an explicit algorithm with a new stepsize rule for approximating a solution of 1.0.1. We assume that A is maximally monotone and B is monotone and Lipschitz. However, the prior knowledge or an estimate of the constant is not needed. Moreover, the solution set $(A+B)^{-1}(0)$ of problem 1.0.1 is assumed to be nonempty. The algorithm is described as follows:

Algorithm 3.1	:	(Modified FRBSM with New Stepsizes)
Initialisation	:	choose $x_{-1}, x_0 \in E, \lambda_{-1}\lambda_0 > 0$.
Iterative Step	:	Assume x_{n-1}, x_n are known, calculate x_n as follows : $x_{n+1} = J^A_{\lambda_n} \circ J^{-1}(Jx_n - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1})), n \geq 0$ $\lambda_{n+1} := \min\left\{\lambda_n, \frac{\theta\|x_{n+1}-x_n\|}{\|Bx_{n+1}-Bx_n\|}\right\}, \theta \in (0, \frac{1}{2\mu}), \mu \geq 1.$
	:	If $x_{n+1} = x_n = x_{n-1}$ stop.

Theorem 3.0.1. *Let E be a real 2-uniformly convex and uniformly smooth Banach space. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $B : E \to E^*$ be monotone and Lipschitz. For $x_0, x_{-1} \in E$ define the sequence x_n iteratively by*

$$x_{n+1} = J^A_{\lambda_n} \circ J^{-1}(Jx_n - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1})), \quad n \geq 0; \tag{3.0.1}$$

where $\lambda_{n+1} := \min\left\{\lambda_n, \frac{\theta\|x_{n+1}-x_n\|}{\|Bx_{n+1}-Bx_n\|}\right\}, \theta \in (0, \frac{1}{2\mu}), \mu \geq 1$. Suppose $(A+B)^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ is bounded.

Proof. Let $x^* \in (A+B)^{-1}(0)$, So that

$$-Bx^* \in Ax^*. \tag{3.0.2}$$

From (3.0.1) we have that

$$\frac{1}{\lambda_n}(Jx_n - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1}) - Jx_{n+1}) \in Ax_{n+1}. \tag{3.0.3}$$

Using (3.0.2) and (3.0.3) and the monotonicity of A, we obtain

$$\langle Jx_{n+1} - Jx_n + \lambda_n(Bx_n - Bx^*) + \lambda_{n-1}(Bx_n - Bx_{n-1}), x^* - x_{n+1}\rangle \geq 0. \tag{3.0.4}$$

By Lemma 2.3.3 (i), we have

$$2\langle Jx_{n+1} - Jx_n, x^* - x_{n+1}\rangle = \phi(x^*, x_n) - \phi(x^*, x_{n+1}) - \phi(x_{n+1}, x_n). \tag{3.0.5}$$

Also

$$\begin{aligned} \langle Bx_n - Bx^*, x^* - x_{n+1}\rangle &= \langle Bx_{n+1} - Bx^*, x^* - x_{n+1}\rangle \\ &+ \langle Bx_n - Bx_{n+1}, x^* - x_{n+1}\rangle, \end{aligned} \tag{3.0.6}$$

and

$$\begin{aligned} \langle Bx_n - Bx_{n-1}, x^* - x_{n+1}\rangle &= \langle Bx_n - Bx_{n-1}, x^* - x_n\rangle \\ &+ \langle Bx_n - Bx_{n-1}, x_n - x_{n+1}\rangle. \end{aligned} \tag{3.0.7}$$

Substituting (3.0.5), (3.0.6) and (3.0.7) in (3.0.4) we have:

$$\begin{aligned} \phi(x^*, x_{n+1}) &+ 2\lambda_n\langle Bx_{n+1} - Bx_n, x^* - x_{n+1}\rangle \\ &\leq \phi(x^*, x_n) + 2\lambda_{n-1}\langle Bx_n - Bx_{n-1}, x^* - x_n\rangle \\ &+ 2\lambda_{n-1}\langle Bx_n - Bx_{n-1}, x_n - x_{n+1}\rangle \\ &- \phi(x_{n+1}, x_n) + 2\lambda_n\langle Bx_{n+1} - Bx^*, x^* - x_{n+1}\rangle. \end{aligned} \tag{3.0.8}$$

we have, in the last inequality, used the fact that B is monotone. So we have

$$\begin{aligned} \phi(x^*, x_{n+1}) &+ 2\lambda_n\langle Bx_{n+1} - Bx_n, x^* - x_{n+1}\rangle + \phi(x_{n+1}, x_n) \\ &\leq \phi(x^*, x_n) + 2\lambda_{n-1}\langle Bx_n - Bx_{n-1}, x^* - x_n\rangle \\ &+ 2\lambda_{n-1}\langle Bx_n - Bx_{n-1}, x_n - x_{n+1}\rangle. \end{aligned} \tag{3.0.9}$$

Using the definition of λ_n and Lemma 2.3.4 we have

$$\begin{aligned} 2\lambda_{n-1}\langle Bx_n - Bx_{n-1}, x_n - x_{n+1}\rangle &\leq 2\lambda_{n-1}\|Bx_n - Bx_{n-1}\|\|x_n - x_{n+1}\|. \\ &\leq 2\theta\frac{\lambda_{n-1}}{\lambda_n}\|x_n - x_{n-1}\|\|x_n - x_{n+1}.\| \\ &\leq \theta\frac{\lambda_{n-1}}{\lambda_n}(\|x_n - x_{n-1}\|^2 + \|x_n - x_{n+1}\|^2). \\ &\leq \mu\theta\frac{\lambda_{n-1}}{\lambda_n}(\phi(x_n, x_{n-1}) \\ &+ \phi(x_{n+1}, x_n)). \end{aligned} \tag{3.0.10}$$

Substituting (3.0.10) in (3.0.9) we have

$$\begin{aligned} \phi(x^*, x_{n+1}) &+ 2\lambda_n\langle Bx_{n+1} - Bx_n, x^* - x_{n+1}\rangle + \phi(x_{n+1}, x_n) \\ &\leq \mu\theta\frac{\lambda_{n-1}}{\lambda_n}(\phi(x_n, x_{n-1}) \\ &+ \phi(x_{n+1}, x_n)) + \phi(x^*, x_n) + 2\lambda_{n-1}\langle Bx_n - Bx_{n-1}, x^* - x_n\rangle. \end{aligned}$$

Rearranging the above inequality, we obtain

$$\begin{aligned} \phi(x^*, x_{n+1}) &+ 2\lambda_n\langle Bx_{n+1} - Bx_n, x^* - x_{n+1}\rangle + (1 - \mu\theta\frac{\lambda_{n-1}}{\lambda_n})\phi(x_{n+1}, x_n) \\ &\leq \mu\theta\frac{\lambda_{n-1}}{\lambda_n}\phi(x_n, x_{n-1}) + \phi(x^*, x_n) + 2\lambda_{n-1}\langle Bx_n - Bx_{n-1}, x^* - x_n\rangle. \end{aligned}$$

Now, define

$$\Gamma_n(x^*) = \phi(x^*,x_n)+\mu\theta\frac{\lambda_{n-1}}{\lambda_n}\phi(x_n,x_{n-1}) + 2\lambda_{n-1}\langle Bx_n - Bx_{n-1}, x^*-x_n\rangle. \quad (3.0.11)$$

Therefore,

$$\Gamma_{n+1}(x^*) \leq \Gamma_n(x^*) - (1-\mu\theta\frac{\lambda_{n-1}}{\lambda_n} - \mu\theta\frac{\lambda_n}{\lambda_{n+1}})\phi(x_{n+1},x_n). \quad (3.0.12)$$

Now let $\epsilon \in (0, 1-2\mu\theta)$ be fixed, since $\lambda_n \to \lambda > 0$, we derive

$$\lim_{n\to\infty}(1-\mu\theta\frac{\lambda_{n-1}}{\lambda_n} - \mu\theta\frac{\lambda_n}{\lambda_{n+1}}) = 1-2\mu\theta > \epsilon.$$

Thus, there exists $n_1 \geq 1$:

$$1-\mu\theta\frac{\lambda_{n-1}}{\lambda_n} - \mu\theta\frac{\lambda_n}{\lambda_{n+1}} \geq \epsilon \quad \forall n \geq n_1. \quad (3.0.13)$$

It follows from (3.0.12) and (3.0.13) that

$$\Gamma_{n+1}(x^*) \leq \Gamma_n(x^*) - \epsilon\phi(x_{n+1},x_n) \quad \forall n \geq n_1. \quad (3.0.14)$$

Thus,

$$\Gamma_{n+1}(x^*) \leq \Gamma_n(x^*) - \epsilon\phi(x_{n+1},x_n) \leq \Gamma_n(x^*), \quad \forall n \geq n_1. \quad (3.0.15)$$

Therefore,the sequence $\{\Gamma_n\}_{n\geq n_1}$ is non-increasing. Now, from the definition of Γ_n and λ_n for each $n \geq n_1$ we see that

$$\begin{aligned}
\Gamma_n(x^*) &= \phi(x^*,x_n) + 2\lambda_{n-1}\langle Bx_n - Bx_{n-1}, x^*-x_n\rangle + \mu\theta\frac{\lambda_{n-1}}{\lambda_n}\phi(x_n,x_{n-1})\\
&\geq \phi(x^*,x_n) - 2\lambda_{n-1}\|Bx_n - Bx_{n-1}\|\|x^*-x_n\| + \mu\theta\frac{\lambda_{n-1}}{\lambda_n}\phi(x_n,x_{n-1})\\
&\geq \phi(x^*,x_n) - 2\theta\frac{\lambda_{n-1}}{\lambda_n}\|x_n - x_{n-1}\|\|x^*-x_n\| + \mu\theta\frac{\lambda_{n-1}}{\lambda_n}\phi(x_n,x_{n-1})\\
&\geq \phi(x^*,x_n) - \mu\theta\frac{\lambda_{n-1}}{\lambda_n}(\|x_n - x_{n-1}\|^2 + \|x^*-x_n\|^2) + \mu\theta\frac{\lambda_{n-1}}{\lambda_n}\phi(x_n,x_{n-1})\\
&\geq \phi(x^*,x_n) - \mu\theta\frac{\lambda_{n-1}}{\lambda_n}(\phi(x_n,x_{n-1}) + \phi(x^*,x_n)) + \mu\theta\frac{\lambda_{n-1}}{\lambda_n}\phi(x_n,x_{n-1})\\
&= (1-\mu\theta)\frac{\lambda_{n-1}}{\lambda_n}\phi(x^*,x_n)\\
&\geq (1-\mu\theta\frac{\lambda_{n-1}}{\lambda_n} - \mu\theta)\frac{\lambda_n}{\lambda_{n+1}})\phi(x^*,x_n)\\
&\geq \epsilon\phi(x^*,x_n) \geq 0.
\end{aligned}$$

Thus, the limit of $\{\Gamma_n\}$ exists. Also, $\{\phi(x^*,x_n)\}$ is bounded which inturn implies $\{x_n\}$ is bounded. ∎

Remark 3.0.2. *From (3.0.14), we obtain*

$$\begin{aligned}\Gamma_{n+1}(x^*) &\leq \Gamma_{n_1}(x^*) - \epsilon\sum_{n=n_1}^{\infty}\phi(x_{n+1},x_n)\\ \epsilon\sum_{n=n_1}^{\infty}\phi(x_{n+1},x_n) &\leq \Gamma_{n_1}(x^*) - \Gamma_{n+1}(x^*)\\ \epsilon\sum_{n=n_1}^{\infty}\phi(x_{n+1},x_n) &\leq \Gamma_{n_1}(x^*) - \lim_{n\to\infty}\Gamma_{n+1}(x^*) < +\infty.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\phi(x_{n+1},x_n) = 0. \tag{3.0.16}$$

Using the fact that B is Lipschitz and $\lambda_n \to \lambda > 0$ then from (3.0.16) and Lemma(2.3.5),we obtain

$$\lim_{n\to\infty}[2\lambda\langle Bx_n - Bx_{n-1}, x^* - x_n\rangle + \mu\theta\frac{\lambda_{n-1}}{\lambda_n}\phi(x_n,x_{n-1})] = 0$$

using definition of Γ_n, we find that

$$\lim_{n\to\infty}\Gamma_n(x^*) = \lim_{n\to\infty}\phi(x^*,x_n)$$

That is, the limit of $\phi(x^*,x_n)$ exists for each x^* in the solution set of the variational inclusion problem.

Theorem 3.0.3. *Let E be a real 2-uniformly convex and uniformly smooth Banach space. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $B : E \to E^*$ be monotone and Lipschitz, and assume that $(A+B)^{-1}(0) \neq \emptyset$ and J is weakly sequentially continuous. Then, the sequence $\{x_n\}$ generated by algorithm 3.1 converges weakly to some solution of (1.0.1).*

Proof. From Theorem 3.0.1, we have that the sequence $\{x_n\}$ is bounded. let ρ be a weakly cluster point of $\{x_n\}$, then there exists $\{x_{n_j}\} \subseteq \{x_n\} such that x_{n_j} \rightharpoonup \rho$.
We show that $\rho \in (A+B)^{-1}(0)$. Now,from the definition of x_n, we have

$$\frac{1}{\lambda_n}(Jx_n - Jx_{n+1} - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1})) \in Ax_{n+1}. \tag{3.0.17}$$

Let $(u, v) \in \text{Graph}(A+B)$. That is $v - Bu \in Au$. Monotonicity of A implies

$$
\begin{aligned}
\langle u - x_{n+1}, v\rangle &\geq \langle u - x_{n+1}, Bu + \frac{1}{\lambda_n}(Jx_n - Jx_{n+1} - \lambda_n Bx_n \\
&- \lambda_{n-1}(Bx_n - Bx_{n-1}))\rangle \\
&= \frac{1}{\lambda_n}\langle u - x_{n+1}, Jx_n - Jx_{n+1}\rangle - \frac{\lambda_{n-1}}{\lambda_n}\langle u - x_{n+1}, Bx_n - Bx_{n-1}\rangle \\
&+ \langle u - x_{n+1}, Bu - Bx_n\rangle \\
&\geq \frac{1}{\lambda_n}\langle u - x_{n+1}, Jx_n - Jx_{n+1}\rangle - \frac{\lambda_{n-1}}{\lambda_n}\langle u - x_{n+1}, Bx_n - Bx_{n-1}\rangle \\
&+ \langle u - x_{n+1}, Bx_{n+1} - Bx_n\rangle. \qquad (3.0.18)
\end{aligned}
$$

where in the last inequality we have used the fact that $\langle u - x_{n+1}, Bu - Bx_{n+1}\rangle \geq 0$. Now from (3.0.16), we have that $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Thus by Lemma 2.3.5, we have $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$, and consequently, Lipschitz of B implies that $\|Bx_{n+1} - Bx_n\| \to 0$ as $n \to \infty$ and using the fact that J is uniformly continuous on bounded sets on uniformly smooth spaces, hence we have

$$\underline{\lim}\langle u - x_{n+1}, v\rangle \geq 0.$$

So that $\langle u - \rho, v\rangle = \lim_{k\to+\infty}\langle u - x_{n_k}, v\rangle \geq \underline{\lim}\langle u - x_n, v\rangle \geq 0$. Since $A+B$, by Lemma (2.3.12), is maximal monotone, we have $\rho \in (A+B)^{-1}(0)$. Hence,

$$0 \in (A+B)(\rho).$$

Next we show that the whole sequence $\{x_n\}$ converges weakly to ρ. Indeed, assume that $\rho^{'} \neq \rho$ is another weak cluster point of $\{x_n\}$ then there exists$\{x_{n_k}\} \subseteq \{x_n\}$ such that $x_{n_k} \rightharpoonup \rho^{'}$ in the solution set of the variational inclusion problem. Since $\rho^{'}, \rho$ are two weak cluster points of $\{x_n\}$, then, we have

$$\phi(\rho, x_n) = \|\rho\|^2 - 2\langle \rho, Jx_n\rangle + \|x_n\|^2 \quad and \quad \phi(\rho^{'}, x_n) = \|\rho^{'}\|^2 - 2\langle \rho^{'}, Jx_n\rangle + \|x_n\|^2.$$

Now,

$$\phi(\rho^{'}, x_n) - \phi(\rho, x_n) = \|\rho^{'}\|^2 - \|\rho\|^2 - 2\langle \rho^{'} - \rho, Jx_n\rangle,$$

so that

$$2\langle \rho^{'} - \rho, Jx_n\rangle = \phi(\rho, x_n) - \phi(\rho^{'}, x_n) + \|\rho^{'}\|^2 - \|\rho\|^2.$$

Therefore, the limit of $\langle \rho^{'} - \rho, Jx_n\rangle$ exists.
Since J is weakly sequentially continuous, we have

$$
\begin{aligned}
\langle \rho^{'} - \rho, J\rho\rangle &= \lim_{j\to\infty}\langle \rho^{'} - \rho, Jx_{n_j}\rangle \\
&= \lim_{k\to\infty}\langle \rho^{'} - \rho, Jx_{n_k}\rangle \\
&= \langle \rho^{'} - \rho, J\rho^{'}\rangle
\end{aligned}
$$

for some $\{x_{n_j}\}, \{x_{n_k}\}$ subsequences of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \rho$ and $x_{n_k} \rightharpoonup \rho^{'}$. We have that

$$\langle \rho^{'} - \rho, J\rho^{'} - J\rho\rangle = 0$$

using Lemma(2.2.3),

$$\rho^{'} - \rho = 0$$
$$\rho^{'} = \rho$$

Hence, $\{x_n\}$ converges weakly to ρ. ■

Corollary 3.0.4. *Let H be a real Hilbert space. Let $A : H \to 2^H$ be a maximal monotone operator and $B : H \to H$ be monotone and Lipschitz. Choose $x_{-1}, x_0 \in H, \lambda_{-1}, \lambda_0 > 0$. Let $\{x_n\}$ be the sequence define by*

$$\begin{cases} x_{n+1} = J^A_{\lambda_n}(x_n - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1})), \quad n \geq 0, \\ \lambda_{n+1} := \min\left\{\lambda_n, \frac{\mu\|x_{n+1}-x_n\|}{\|Bx_{n+1}-Bx_n\|}\right\}, \mu \in (0, \frac{1}{2}). \end{cases}$$

Suppose $(A+B)^{-1}(0) \neq \emptyset$. Then the sequence $\{x_n\}$ converges weakly to an element of $(A+B)^{-1}(0)$.

Corollary 3.0.5. *Let C be a closed convex subset of a real Hilbert space, H and $B : H \to H$ be monotone and Lipschitz. For $x_{-1}, x_0 \in H, \lambda_{-1}, \lambda_0 > 0$ and $\mu \in (0, \frac{1}{2})$, define the sequence x_n iteratively by*

$$\begin{cases} x_{n+1} = P_C(x_n - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1})), \quad n \geq 0, \\ \lambda_{n+1} := \min\left\{\lambda_n, \frac{\mu\|x_{n+1}-x_n\|}{\|Bx_{n+1}-Bx_n\|}\right\}. \end{cases}$$

Suppose $(A+B)^{-1}(0) \neq \emptyset$. Then, the sequence $\{x_n\}$ converges weakly to an element of $(A+B)^{-1}(0)$.

CHAPTER 4

Application

In this section, we give some physical applications to show how our results can be applied.

Application 4.0.1. *Consider the l_1-regularized least squares problem, suppose we want to solve the following minimization problem (popularly known as the LASSO problem in statistics and basis pursuit in signal processing):*

$$\min \frac{1}{2}\|Tx - b\|_2^2 + \gamma\|x\|_1 \tag{4.0.1}$$

where is $T \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $\gamma > 0$ is the regularization parameter and $x \in \mathbb{R}^n$.

In this case,

$$J_A^\lambda(x) = (A^T A + \lambda^{-1} I)^{-1}(A^T b + \lambda^{-1} x),$$

while

$$J_B^\lambda(x) = (sgn(x_i).\max\{0, |x_i| - \lambda\gamma\})_i, \; i = 1, 2, ..., n.$$

Then, this problem is equivalent to (1.0.1) by setting

$$A = \nabla(\frac{1}{2}\|Tx - b\|_2^2) \text{ and } B = \partial(\gamma\|x\|_1).$$

We can perform a sparse signal recovery $x \in R^n$ which contains N randomly replaced $\pm$ 1 spikes. The matrix $T \in R^{m\times n}$ is randomly generated with mean zero and one variance.
The starting points can also be randomly generated in the interval $(-1, 1)$.

Application 4.0.2. *Let $H = L^2([0,1])$ and let C be a closed convex subset of H. $C = \{x \in H : \|x - y\| \leq r\}$ Then the following optimization probelm: find $x \in L^2[0,1]$ such that*

$$0 \in \partial(\|x\|) + N_C(x) \tag{4.0.2}$$

(4.0.2) is equivalent to problem (1.0.1) by setting $A = \partial(\|.\|)$ and $B = N_C$.

The resolvents of A and B are respectively given by

$$J_A^\lambda(x) = (I + \lambda\partial(\|.\|))^{-1}(x),$$

and

$$J_B^\lambda(x) = (I + \lambda N_C)(x) = P_C(x).$$

Furthermore,

$$J_A^\lambda(x) = (I + \lambda\partial\|.\|)^{-1}(x) = Prox_{\lambda\|.\|}(x) = x - P_{B_{\|\cdot\|}}(\frac{x}{\lambda}),$$

where,

$$Prox_{\lambda\|.\|}(x) = argmin_y \ \{\|y\| + \frac{1}{2}\|x-y\|^2\},$$

$P_{B_{\|\cdot\|}}$ is the projection operator and $B_{\|\cdot\|}$ is the dual norm unit ball. Moreover,

$$P_{B_{\|\cdot\|}}(x) = \begin{cases} x, & \|x\| \leq 1, \\ \frac{x}{\|x\|}, & \|x\| > 1. \end{cases}$$

Then we have,

$$P_C(x) = \begin{cases} x, & \|x-y\| \leq r, \\ y + \frac{r(x-y)}{\|x-y\|}, & \|x-y\| > r. \end{cases}$$

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