Bochner Integration in Banach spaces and integration in $\mathbb{R}^k$ and ℓ_1

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By

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Title of Thesis

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A THESIS PRESENTED TO THE DEPARTMENT OF PURE AND APPLIED MATHEMATICS

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Dedication

To My God, My Father, the giver of life, the provider of inspiration, knowledge and resources.

Acknowledgments and Thanks

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All notation

Contents

General Introduction

The history of integration began with an attempt to find the area under the graph of a function and the positive X- axis. In mathematics, an integral assigns numbers to functions describing area, volumes, displacement and other concepts. The process of finding the integrals is what we call Integration. We have different types of integrations.The first one we will talk about is the Riemann-Stieltjes integration.

The Riemann-Stieltjes integration is applicable to real valued functions but requires some properties such as, the function must be finite and bounded and it is always defined on a compact set. This is a major limitation of the Riemann-Stieltjes integration. We note that these conditions are not required for the Lebesgue integrals, so these type of integrals give us the privilege of computing the integral of a large range of real- valued functions. So we have that the Lebesgue integrals deal with the integrals of real-valued measurable functions.

Another type of integration is known as the Bochner integral, which was the focus of this research work. This type of integration deals with the integral of measurable functions in Banach Spaces. This research work focused on extending the concept of the Bochner integral to $\mathbb{R}^d$, $d \geq 1$ and also to ℓ_1. We proved a theorem that showed the relationship between the Bochner integrals on $\mathbb{R}^d$, $d \geq 1$ with the Lebesgue-Like integrals on $\mathbb{R}$.

Furthermore, to achieve the aim of this research work, we presented a brief reminder of the Lebesgue-Like Integration on $\mathbb{R}$ in chapter 1, in Chapter 2, we also presented the main steps of the construction of the space $\mathcal{L}^1(\Omega, \mathcal{A}, m, \mathbb{R})$ of Real- Valued Measurable and Integrable functions and retained them as milestones in the general case. The comparison of the Lebesgue-Like integral and the Bochner Integral in $\mathbb{R}$ were given with the measure being finite in chapter 3. Our main results were presented in chapter 4 and chapter 5. The Bochner integral on $\mathbb{R}^k$, $k \geq 1$ were defined in Chapter 4 and in Chapter 5, we defined the Bochner integral on ℓ_1.

CHAPTER 1

A brief reminder of Lebesgue-Like Integration on $\mathbb{R}$

This chapter is the summary of the integration of chapter 5 and elements of Chapter 6 in Lo (2017b) where all the properties and related limits theorems are proved as solutions of exercises.

Here, we recall the key results for an easy quotation.

I. Definition of the integral with respect to a measure.

Assume that we have a measure space $(\Omega, \mathcal{A}, m)$. We are going to construct the integral of a real-valued measurable function $f : (\Omega, \mathcal{A}) \rightarrow \overline{\mathbb{R}}$ with respect to the measure m (that may take infinite values) denoted by

$$\int f \ dm = \int_{\Omega} f(\omega) \ dm(\omega) = \int_{\Omega} f(\omega) \ m(d\omega).$$

into three steps. The first step concerns non-negative functions among the class of elementary functions which have the general representation :

$$f = \sum_{1 \leq i \leq k} \alpha_i \ 1_{A_i}, \ (\alpha_i \in \mathbb{R}_+, \ A_i \in \mathcal{A}, \ 1 \leq i \leq k), \ k \geq 1, \tag{0.1}$$

where the measurable sets A_i are pairwise disjoint (*pwd*). If it happens that the unions of the A_i, $1 \leq i \leq p$ is not Ω, we implicitly mean that $f = 0$ on the complement of $A_1 + \cdots + A_p$.

The class of real-valued elementary functions is denoted by $\mathcal{E}(\omega, \mathcal{A}, \mathbb{R})$ and $\mathcal{E}^+(\omega, \mathcal{A}, \mathbb{R})$ stands for the subclass of non-negative functions of $\mathcal{E}(\omega, \mathcal{A}, \mathbb{R})$.

The expression of an elementary function, as expressed in Formula (1.2) is not unique. But there exists one and only in which the coefficients α_i are disjoint, called the canonical representation. As a result, **that canonical representation is used with the summations of the A_i covering Ω**

unless the contrary is specified.

Let us begin to describe the construction.

Step 1M. Definition of the integral of a non-negative simple function : $f \in \mathcal{E}_+$.

The integral of a non-negative simple function

(0.2) $$f = \sum_{1 \leq i \leq k} \alpha_i \ 1_{A_i}, \ (\alpha_i \in \mathbb{R}_+, \ A_i \in \mathcal{A}, \ 1 \leq i \leq k), \ k \geq 1,$$

is defined by

(0.3) $$\int f \ dm = \sum_{1 \leq i \leq k} \alpha_i \ m(A_i).$$

Convention - Warning 1 In the definition (1.3), the product $\alpha_i \ m(A_i)$ is zero whenever $\alpha_i = 0$, even if $m(A_i) = +\infty$.

The definition (1.3) is coherent. This means that $\int f \ dm$ does not depend on one particular expression of f.

step 2M. Definition of the integral for a non-negative measurable function.

Let f be any non-negative measurable function. We have the following fact (see for example *Point (03-23) in Doc 03-02 in Chapter 4 in Lo (2017b))* : There exists a non-decreasing sequence $(f_n)_{n \geq 0} \subset \mathcal{E}_+$ such that

(0.4) $$f_n \uparrow f \text{as n } \uparrow +\infty.$$

We define

(0.5) $$\int f \ dm = \lim_{n \uparrow +\infty} \int f_n \ dm.$$

This definition (1.5) is also coherent since it does not depend on the sequence which is used in the definition. (See Chapter 4, Lo (2017b)).

Step 3M. Definition of the integral for a measurable function.

In the general case, the decomposition of f into its positive part and its negative part is used as follows :

$$f = f^+ - f^-, \quad |f| = f^+ + f^-, \quad and \quad f^+ f^- = 0, \tag{0.6}$$

where $f^+ = max(0, f)$ and $f^- = max(0, -f)$, which are measurable, form the unique couple of functions such that Formulas (1.6) holds.

By *Step 2M*, the numbers

$$\int f^+ \, dm \quad and \int f^- \, dm$$

exist in $\overline{\mathbb{R}}_+$. If one of them is finite, i.e.,

$$\int f^+ \, dm < +\infty \ \ \textbf{or} \int f^- \, dm < +\infty,$$

we say that f is quasi-integrable with respect to m and we define

$$\int f \, dm = \int f^+ \, dm \ - \ \int f^- \, dm. \tag{0.7}$$

Warning The integral of a real-valued and measurable function with respect to a measure m exists *if only if* : either it is of constant sign or the integral of its positive part or its negative part is finite.

By extension, the **Integration of a mapping over a measurable set**. If A is a measurable subset of Ω and $1_A f$ is quasi-integrable, we denote

$$\int_A f \, dm = \int 1_A f \, dm.$$

Convention - Warning 2 The function $1_A f$ is defined by $1_A f(\omega) = f(\omega)$ of $\omega \in A$, and *zero* otherwise.

II. Integrable functions.

The function f is said to be integrable if and only if the integral $\int f \, dm$ exist (in $\mathbb{R}$) and is finite, i.e.,

$$\int f^+ \, dm < +\infty \ \ \textbf{and} \int f^- \, dm < +\infty,$$

The set of all integrable functions with respect to m is denoted

$$\mathcal{L}^1(\Omega, \mathcal{A}, m).$$

III - Main properties.

Let f and g be measurable functions, such that $\int f\ dm$, $\int g\ dm$, $f+g$ is defined *a.e.*, $\int f+g\ dm$, exists and $\int f\ dm + \int g\ dm$ make senses, A and B be two disjoint measurable sets, and c be a finite non-zero real scalar. We have the following properties and facts.

III - 1. Linearity.

$$\int cf\ dm = c\int f\ dm, \quad (L1)$$

which is still valid for $c=0$ if f is integrable,

$$\int (f+g)\ dm = \int f\ dm + \int g\ dm, \quad (L2)$$

and

$$\int_{A+B} f\ dm = \int_A f\ dm + \int_B f\ dm, \ \ \int cf\ dm = c\int f\ dm, \quad (L3)$$

III - 2. Order preservation.

$$f \leq g \Rightarrow \int f\ dm \leq \int g\ dm, \quad (O1)$$

$$f = g\ a.e.\ \Rightarrow \int f\ dm = \int g\ dm, \quad (O2)$$

III - 3. - Integrability.

$$f\ integrable \Leftrightarrow |f|\ integrable \Rightarrow f\ \ finite\ a.e, \quad (I1)$$

$$|f| \leq g\ integrable \Rightarrow f\ integrable, \quad (I2)$$

and

$$\biggl((a,b)\in\mathbb{R}^2,\ f\ and\ g\ integrable\biggr) \Rightarrow \biggl(af+bg\ integrable\biggr). \quad (I3)$$

VI - Spaces of integrable functions.

VI - 1. Space $\mathcal{L}^p(\Omega, \mathcal{A}, m)$, $p \geq 1$.

$\mathcal{L}^p(\Omega, \mathcal{A}, m)$ is the space of all real-valued and measurable functions f defined on Ω such that $|f|^p$ is integrable with respect to m, that is

$$\mathcal{L}^p(\Omega, \mathcal{A}, m) = \{f \in \mathcal{L}_0(\Omega, \mathcal{A}) \int |f|^p \, dm < +\infty\}.$$

$\mathcal{L}_0(\Omega, \mathcal{A}, \mathbb{R})$ is the class of all real-valued and measurable functions defined on Ω.

VI - 2. Space of m-a.e. bounded functions $\mathcal{L}^\infty(\Omega, \mathcal{A}, m)$.

$\mathcal{L}^\infty(\Omega, \mathcal{A}, m)$ is the class of all real-valued and measurable functions which are bounded m-a.e., i.e,

$$\mathcal{L}^\infty(\Omega, \mathcal{A}, m) = \{f \in \mathcal{L}_0(\Omega, \mathcal{A})ss, \ \exists C \in \mathbb{R}_+, \ |f| \leq C \ m - a.e\}.$$

VI - 3. Equivalence classes.

When we only consider, in $\mathcal{L}^p$, the equivalence classes pertaining to the equivalence relation $\mathcal{R}$ defined $\mathcal{L}^p(\Omega, \mathcal{A}, m)$ by

$$f \ \mathcal{R} \ g \ \Leftrightarrow \ f = g \ m - a.e.,$$

which form :

$$L^p(\Omega, \mathcal{A}, m) = \mathcal{L}^p(\Omega, \mathcal{A}, m)/\mathcal{R}.$$

that is,

$$L^p(\Omega, \mathcal{A}, m) = \{\dot{f}, f \in \mathcal{L}_0(\Omega, \mathcal{A}), \ \int |f|^p \, dm \ < +\infty\}$$

and

$$L^\infty(\Omega, \mathcal{A}, m) = \mathcal{L}^{+\infty}(\Omega, \mathcal{A}, m)/\mathcal{R}$$

that is

$$L^{\infty}(\Omega, \mathcal{A}, m) = \{\dot{f}, f\mathcal{L}_0(\Omega, \mathcal{A}),\ |f|\ m - bounded\}.$$

VII - Norms on L^p.

The space $L^p(\Omega, \mathcal{A}, m)$, $1 \leq p < +\infty$, is equipped with the norm

$$||f||_p = (\int |f|^p\ dm)^{1/p}.$$

The space $L^{+\infty}(\Omega, \mathcal{A}, m)$ is equipped with the norm

$$||f||_{\infty} = \inf\{C \in \mathbb{R}_+,\ |f| \leq C\ m - .a.e\}.$$

IV - 5. Banach spaces.

For all $p \in [1, +\infty]$, $L^p(\Omega, \mathcal{A}, m)(+, ., ||f||_p)$ is a Banach space.

$L^2(\Omega, \mathcal{A}, m)(+, ., ||f||_2)$ is a Hilbert space.

(III) - Young-Fatou-Lebesgue Dominated Convergence Theorem in $\mathcal{L}^1(\Omega, \mathcal{A}, B, m)$.

We have :

THEOREM 1. *Let* $(f_n)_{n\geq 1}$ *be a sequence in*

$$L^1(\Omega, \mathcal{A}, m, E, \|\circ\|_{L^1(Bo,\ E)})$$

and f *a measurable mapping from* $(\Omega, \mathcal{A})$ *to* E m*-a.e. defined. Suppose that :*

(a) f_n *converges to* f m*-a.e or in measure.*

(b) There exists a a family $\{h, (h_n)_{n\geq 1}\} \subset L^1(\Omega, \mathcal{A}, m, \mathbb{R})$ *composed of non-negative functions such that :*

(b1) As $n \rightarrow +\infty$, $h_n \rightarrow h$ *and* $\int_{(\Omega,\mathbb{R})} h_n\ dm \rightarrow \int_{(\Omega,\mathbb{R})} h\ dm$

and—

(b2) $\|f_n\|_E \leq h_n$ *for all* $n \geq 1$.

Then $f \in L^1(\Omega, \mathcal{A}, m, \mathbb{R})$ *and , as* $n \rightarrow +\infty$,

$$\int_{(Bo,\ E)} f_n\ dm \rightarrow \int_{(Bo,\ E)} f\ dm$$

both in E *and in* $L^1(\Omega, \mathcal{A}, m, \mathbb{R})$.

Proof of Theorem of 1. Let us begin by supposing that f_n converges to f m-a.e. We have for any $p \geq 1$ and $q \geq 1$,

$$\|f_p - f_q\|_E \leq h_p + h_q$$

and as $(p, q) \rightarrow \{+\infty\}^2$,

$$0 \leq \|h_p + h_q\|_E \rightarrow 2h \text{ and } \int_{(\Omega,\mathbb{R})} (h_p + h_q)\ dm \rightarrow \int_{(\Omega,\mathbb{R})} (2h)\ dm.$$

We have $\|f_p - f_q\|_E \rightarrow 0$ as $(p, q) \rightarrow \{+\infty\}^2$. Hence, by applying the Young-Fatou-Lebesgue dominated convergence theorem,

$$\|f_p - f_q\|_{L^1(Bo,\ E)} = \int_{(\Omega,\mathbb{R})} \|f_p - f_q\|_E \ dm \to 0,$$

as $(p,q) \to \{+\infty\}^2$. So $(f_n)_{n\geq 1}$ is a Cauchy sequence in $L^1(\Omega, \mathcal{A}, m, E, \|\circ\|_{L^1(Bo,\ E)})$.

Now let f_n converges to f in measure. We have to prove that $f \in L^1(\Omega, \mathcal{A}, m, E)$ ans

$$\|f_n - f\|_{L^1(Bo,\ E)} \to 0. \tag{0.8}$$

But there exists a subsequence of $(f_n)_{n\geq 1}$ which converges m-a.e. to f. By applying the first part of this proof, we get that $f \in L^1(\Omega, \mathcal{A}, m, E)$. Now, to prove Formula 0.8, we use the Prohorov criterion for convergence in $\mathbb{R}$. Consider an arbitrary subsequence $\left(\|f_{n_k} - f\|_{L^1(Bo,\ E)}\right)_{n_k}$. The subsequence $(f_{n_k})_{k\geq 1}$ still converges to f in measure and thus, contains a subsequence $(\left(f_{n_{k_j}}\right))_{j\geq 1}$ that converges m-a.e. to f. By the first part, we have that the subsequence $\left(\left\|f_{n_{k_j}} - f\right\|_{L^1(Bo,\ E)}\right)_{n_{k_j}}$ of $\left(\|f_{n_k} - f\|_{L^1(Bo,\ E)}\right)_{n_k}$ converges to zero. By Prohorov criterion, Formula Formula 0.8 holds. ■

CHAPTER 2

Construction of the Bochner integral on a Banach space

1. Introduction

This chapter constitutes a generalization of the the integration theory for real-valued mappings we described it in details in Lo (2017b). In such an introductory book, we focused only on integration of real-valued measurable functions. In Lo (2018), for example, the integration of random vectors $X = (X_1, \cdots, X_d)^t, d \geq 1$, is considered as the vector of the integrals of the components, that is

$$\int_{\mathbb{R}^d} X \ dm = \left(\int_{\mathbb{R}} X_1 \ dm, \cdots, \int_{\mathbb{R}} X_d \ dm \right)^t.$$

This is rather a matrix extension of the integral than a new theory of integration.

The concepts of the modern integration theory of real-valued mappings with respect to a measure are naturally - but not easily - extended in more general situations, for example when the functions to be integrated takes values in a general normed space.

Here, we will follow recent works of Mikusiński (2015) to expose the Bochner integral of E-valued linear spaces, where E is Banach or locally convex. But we will **not** merely repeat them. Rather, we will take the results on real integrals as basis and then, shorten or better present the theory.

Therefore, again here, we present the main steps of the construction of the space $\mathcal{L}^1(\Omega, \mathcal{A}, m, \mathbb{R})$ of real-valued measurable integrable functions and retain them as milestones in the general case.

2. The Banach valued Bochner integral

In this section, we extend the method of Section 1 to construct an integral of functions with values in a real Banach space $(E, +, \| \circ \|_E)$ on $\mathbb{R}$. In our construction, we will repeat *Step 1M* (page 38) for real-valued integrals.But, we will not be able to have steps *Step 2M* and *Step 3M* (pages 39 and 39) since we are not supposed to have an order on E. Even we have one, we are not sure we get the rule (1.4).

An immediate remark is that using the elementary functions as a mean to construct the integral requires using bounded measures, unless we have corresponding notions of the *infinity*.

So, we are going to use again *Step 1M* and replace both *Step 2M* and *Step 3M* by one new step.

I - The construction of the integral.

A - The definition.

We are going to construct the Bochner integral of a measurable function

$$f : (\Omega, \mathcal{A}, m) \longrightarrow E.$$

in two steps.

For such a mapping $f : (\Omega, \mathcal{A}, m) \longrightarrow E$, we defined the real-valued mapping, still called the norm of f,

$$\|f\|_E : (\Omega, \mathcal{A}, m) \longrightarrow \mathbb{R},$$

defined by

$$\forall \omega \in \Omega, \ \|f\|_E(\omega) = \|f(\omega)\|_E.$$

Step 1B: Integral of an E-Valued elementary function.

The notion of elementary functions may be easily extended to any linear space E, by defining an elementary function E in the form

$$f = \sum_{j=1}^{p} x_j \ 1_{B_j}, \tag{2.1}$$

where $p \geq 1$, $x_j \in E$, $B_j \in \mathcal{A}$, $B_1 + B_2 + \cdots + B_p = \Omega$.

Here, only the linear structure of E is used since $f(\omega)$ is a finite linear combination. Let us denote by

$$\mathcal{E}(\Omega, \mathcal{A}, E),$$

the class of all elementary functions with values in E. The following fact will be used and exploited in the construction : for any $f \in \mathcal{E}(\Omega, \mathcal{A}, E)$, represented as in (1.8), we have

$$\|f\|_E = \sum_{j=1}^{p} \|x_j\| \ 1_{B_j} \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R}). \tag{2.2}$$

We summarize by saying

$$f \in \mathcal{E}(\Omega, \mathcal{A}, E) \ \Rightarrow \ \|f\|_E \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R}).$$

Now, let us define the the Bochner integral of $f \in \mathcal{E}(\Omega, \mathcal{A}, E)$, represented as in (1.8) by

$$\int_{(Bo, \ E)} f dm = \sum_{j=1}^{p} \ x_j m(A_j) \in E. \tag{2.3}$$

In the notation of the integral sign, the mention of $(Bo, \ E)$ helps us to remind that the mapping f is defined on Ω and the value of the integral is in E.

By using the techniques developed for the real-valued integral, we readily have the following facts.

(EF1) The definition (1.9) does not depends on the particular representation of f. As a consequence, that definition is coherent.

(EF2) The defined Bochner integral on $\mathcal{E}(\Omega, \mathcal{A}, E)$ is linear.

Step 2B: Since we do not have the simple property, as for $E = \mathbb{R}$, that any measurable function is a limit of a sequence of elementary functions, the existence of such a limit becomes an integrability condition.

DEFINITION 1. *A measurable function is said to be* $Bo-$ *integrable if and only if there exists* $(f_n)_{n\geq 1} \subset \mathcal{E}$ *such that*

$$\sum_{n\geq 1} \int_{(LL,\ \mathbb{R})} \|f_n\|_E \ dm < +\infty, \quad (I1)$$

and

$$f = \sum_{n\geq 1} f_n, \ m - a.e. \ \ (I2).$$

If Conditions (I1) and (I2) both hold, we write

$$f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E)).$$

NB. In the notation of $S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$, the mention of $\mathcal{E}(\Omega, \mathcal{A}, E)$ is important and tells us in which space the function f_n's are. Later, that space will change.

We have the following result.

LEMMA 1. *The implications below hold.*

(A) *If Condition (I1) and (I2) in Definition 1 both hold, then*

(a) $\sum_{n\geq 1} f_n dm$ *is defined* m*-a.e,*

and

(b) the series of Bochner integrals of $\sum_{n\geq 1} \int_{(Bo,\ E)} f_n \ dm$ *converges in* E*.*

(B) *If*

$$f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E)) \ \ and \ \ f \in S(g_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E)),$$

Then

$$\sum_{n\geq 1}\int_{(Bo,\ E)} f_n\ dm = \sum_{n\geq 1}\int_{(Bo,\ E)} g_n\ dm.$$

(C) *If*

$$f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E)),$$

then

$$\left\| \sum_{n\geq 1}\int_{(Bo,\ E)} f_n\ dm \right\|_E \leq \int_{(LL,\ \mathbb{R})} \|f\|_E\ dm. \tag{2.4}$$

(D) *If*

$$f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E)),$$

then, for any $\eta > 0$, there exists a sequence $(h_n)_{n\geq 1} \subset \mathcal{E}(\Omega, \mathcal{A}, E)$ such that

$$f \in S(h_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$$

and

$$\sum_{n\geq 1}\int_{(LL,\ \mathbb{R})} \|h_n\|_E\ \ dm \leq \int_{(LL,\ \mathbb{R})} \|f\|_E\ \ dm + \eta.$$

Based on Definition 1 and on Lemma 1, we may define :

The Bochner integral of $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$ **by** :

$$\int_{(Bo,\ E)} f\ dm = \sum_{n\geq 1}\int_{(Bo,\ E)} f_n\ dm. \tag{2.5}$$

Conclusion: We have entirely finished the construction of

$$\int_{(Bo,\ E)} f\ dm \in E,$$

in two steps. To finish the construction, we should prove Lemma 1. In particular, Point (A) of that lemma ensures that the integral of f does not depend on the sequence used to define it.

B- Justification of the definition : Proof of Lemma 1.

Let us begin by remarking that $h \in \mathcal{E}(\Omega, \mathcal{A}, E)$, we have

$$\left\| \int_{(Bo,\ E)} h dm \right\| \leq \int_{(LL,\ \mathbb{R})} \|h\| dm. \tag{2.6}$$

To prove Formula (2.6), we consider an elementary function f, represented as in Formula (1.8) for example. We have

$$\begin{aligned}
\left\| \int_{(Bo,\ E)} f\ dm \right\|_E &= \| \sum_1^p x_j m(A_j) \|_E \\
&\leq \sum_1^p \|x_j\|_E m(A_j) \\
&= \int_{(LL,\ \mathbb{R})} \left(\sum_1^p \|x_j\|_E 1_{A_j} \right) dm, \ (L3) \\
&= \int_{(LL,\ \mathbb{R})} \| \sum_1^p x_j 1_{A_j} \|_E \ dm, \ (L4)
\end{aligned}$$

where we used the integration of real-valued functions (as developed in Section 1) in (L3) and Formula (2.2) in Line (L3). So, Formula (2.6) is proved.

Since a sum of E-valued elementary functions is an E-elementary function, we have that for any family $(h_j)_{1 \leq n \leq p}$ of p E-valued elementary functions, we have

$$\left\| \int_{(Bo,\ E)} \sum_{1 \leq j \leq p} h_j\ dm \right\| \leq \int_{(LL,\ \mathbb{R})} \left\| \sum_{1 \leq j \leq p} h_j \right\|_E \ dm.. \tag{2.7}$$

Let us begin the proof of lemma 1.

Proof of Part A. Point (a). Suppose $(I1)$ holds. By the Monotone Convergence Theorem for real-valued mapping (RVM), we have

$$\sum_{n=1}^{+\infty} \int_{(LL,\ \mathbb{R})} \|f_n\|_E \ dm = \int_{(LL,\ \mathbb{R})} \left(\sum_{n=1}^{+\infty} \|f_n\|_E \right) dm < +\infty.$$

So, $\sum_{n=1}^{+\infty} \|f_n\|_E$ is finite m-a.e. That is the series $\sum_{n=1}^{+\infty} f_n$ is m-a.e. absolutely convergent. So

$$\sum_{n=1}^{+\infty} f_n,$$

is defined m-a.e.

Point b. We have for all $1 \leq r \leq k$

$$\left\| \sum_{n=k}^{r} \int_{(Bo,\ E)} f_n \ dm \right\|_E \leq \int_{(LL,\ \mathbb{R})} \left\| \sum_{n=k}^{r} f_n \ dm \right\|_E \leq \sum_{n=k}^{r} \int_{(LL,\ \mathbb{R})} \|f_n\|_E \ dm.$$

By (I1), the extreme right member of that double inequality go to zero as $(r, k) \rightarrow (+\infty, \infty)$. So the sequence

$$\left(\sum_{n=1}^{r} \int_{(Bo,\ E)} f_n dm \right)_{k \geq 1}$$

is Cauchy and then

$$\sum_{n=1}^{r} \int_{(Bo,\ E)} f_n \ dm$$

converges in E as $r \rightarrow +\infty$. Point (A) is entirely proved.

Proof of Points (B) and (C).

Actually, Point (C) will be obtained in the proof of Point (B). Let us begin by supposing that $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$. Let us define

(2.8) $$g_0 = 0,\ g_n = f_1 + f_2 + \cdots + f_n,\ n \geq 1$$

and

$$\varphi_n = \|g_n\|_E - \|g_{n-1}\|_E, \quad n \geq 2.$$

It is clear that $\varphi_n \in \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$, for all $n \geq 1$ and that

$$\|g_n\|_E = \varphi_1 + \varphi_2 + \cdots + \varphi_n. \tag{2.9}$$

Next, for $n \geq 1$

$$|\varphi_n|_{\mathbb{R}} = |\|g_n\|_E - \|g_{n-1}\|_E| \leq \Big| \|g_n - g_{n-1}\|_E \Big| = \|f_n\|_E. \tag{2.10}$$

From Part A, from $f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$, from Formula (2.8) and by the continuity of the norm, we have $\|g_n\| \to \|f\|$, m-a.e. By combining this, Formulas (2.9) and (2.9), we have that $\varphi_1 + \varphi_2 + \cdots + \varphi_n$ converges (in $\mathbb{R}$) to $\|f\|$ m-a.e. while $\varphi_1 + \varphi_2 + \cdots + \varphi_n$ is bounded by the real-valued and integrable function $\sum_{n\geq 1} |\varphi_n|$. By the Fatou-Lebesgue theorem for (RVM), we get, on one side, that

$$\begin{aligned}
\int_{(LL,\ \mathbb{R})} \|f\|_E \, dm &= \sum_{k\geq 1} \int_{(LL,\ \mathbb{R})} \varphi_k \, dm \\
&= \int_{(LL,\ \mathbb{R})} \left(\sum_{k\geq 1} \varphi_k \right) dm \\
&= \lim_{n\to+\infty} \int_{(LL,\ \mathbb{R})} \left(\sum_{k=1}^{n} \varphi_k \right) dm \\
&= \lim_{n\to+\infty} \int_{(LL,\ \mathbb{R})} \|g_n\|_E \, dm \quad (L14)
\end{aligned}$$

On the other side, we have

$$\begin{aligned}\lim_{n\rightarrow+\infty}\left\|\sum_{k=1}^{n}\int_{(Bo,\ E)} f_k\ dm\right\|_E &= \lim_{n\rightarrow+\infty}\left\|\int_{(Bo,\ E)}\sum_{k=1}^{n} f_k\ dm\right\|_E \\ &\leq \lim_{n\rightarrow+\infty}\int_{(LL,\ \mathbb{R})}\left\|\sum_{k=1}^{n} f_k\right\|_E\ dm \\ &\leq \lim_{n\rightarrow+\infty}\int_{(LL,\ \mathbb{R})}\|g_n\|_E\ dm\ \ (L23).\end{aligned}$$

We get Point (C) by letting $n \rightarrow +\infty$ and by putting together the inequalities in lines (L14) and (L23) in the formulas above, i.e.,

$$\|\sum_{n\geq 1}\int_{(Bo,\ E)} f_n\ dm\|_E \leq \int_{(LL,\ \mathbb{R})}\|f\|_E\ dm. \tag{2.11}$$

Now suppose that $f \in S(f_n,\ n \geq 1)$ and $f \in S(g_n,\ n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$. We also have

$$\|\sum_{n\geq 1}\int_{(Bo,\ E)} h_n\ dm\|_E \leq \int_{(LL,\ \mathbb{R})}\|f\|_E\ dm.$$

But $0 \in S(f_n - g_n,\ n \geq 1,\ \mathcal{E}(\Omega, \mathcal{A}, E))$, which in turn implies

$$\begin{aligned}&\lim_{n\rightarrow+\infty}\left\|\left(\sum_{k=1}^{n}\int_{(Bo,\ E)} f_k\ dm\right)-\left(\sum_{k=1}^{n}\int_{(Bo,\ E)} g_k\ dm\right)\right\|_E \\ \leq\ &\lim_{n\rightarrow+\infty}\left\|\sum_{k=1}^{n}\int_{(Bo,\ E)}(f_k - g_k)\ dm\right\|_E \\ \leq\ &\int 0\ dm = 0.\end{aligned}$$

So the series $\sum_{n\geq 1}\int_{(Bo,\ E)} f_n\ dm$ and $\sum_{n\geq 1}\int_{(Bo,\ E)} g_n\ dm$, whose convergences are already proved, are equal. ■

Proof of Point (D). Suppose that

$$f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E)).$$

Given $\eta > 0$, the assumptions implies that there exists $n_0 > 0$ such that

$$\sum_{n>n_0} \int_{(LL,\ \mathbb{R})} \|f_n\|_E \ dm < \eta/2.$$

We set

$$h_1 = f_1 + \cdots + f_{n_0},\ h_n = f_{n_0+n-1}\ for\ n \geq 2.$$

We clearly have

$$f \in S(h_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$$

and

$$\begin{aligned}\int_{(LL,\ \mathbb{R})} \|f - h_1\|_E \ dm &= \int_{(LL,\ \mathbb{R})} \left\| \sum_{n>n_0} f_n \right\|_E \ dm \\ &\leq \sum_{n>n_0} \int_{(LL,\ \mathbb{R})} \|f_n\|_E \ dm \leq \eta/2.\end{aligned}$$

This and the second triangle inequality together lead to

$$\int_{(LL,\ \mathbb{R})} \|h_1\|_E \ dm - \int_{(LL,\ \mathbb{R})} \|f\|_E \ dm \leq \int_{(LL,\ \mathbb{R})} \|f - h_1\|_E \ dm \leq \eta/2.$$

Hence we get

$$\int_{(LL,\ \mathbb{R})} \|h_1\|_E \ dm \leq \int_{(LL,\ \mathbb{R})} \|f\|_E \ dm + \eta/2,$$

which in turn leads to

$$\begin{aligned}\sum_{n\geq 1}\int_{(LL,\ \mathbb{R})} \|h_n\|_E \ dm &= \sum_{n\geq 1}\int_{(LL,\ \mathbb{R})} \|h_1\|_E \ dm + \sum_{n\geq 2}\int_{(LL,\ \mathbb{R})} \|h_n\|_E \ dm \\ &\leq \int_{(LL,\ \mathbb{R})} \|h_1\|_E \ dm + \eta/2 \\ &\leq \int_{(LL,\ \mathbb{R})} \|f\|_E \ dm + \eta.\end{aligned}$$

3. The Bochner integral on $\mathbb{R}$

In the previous section, we used the Real-valued Mapping *(RVM)* integration scheme to get the Bochner integral in a complete normed space.

We already knew that the natural order on $\mathbb{R}$ was used in the general construction of the *(RVM)* integration in the step *2M* (see 39) and we saw how that approach allowed to integrate with respect to an infinite measure.

Now, we are going to see that by restricting ourselves to finite measure, the *Bo* and the *LL)* integrals are exactly the same.

We suppose that the *(LL)* integration with respect to a finite measure m is completely set. We have

THEOREM 2. *A real-valued measurable mapping and m-a.e. finite f : $(\Omega, \mathcal{A}, m) \to \overline{\mathbb{R}}$ is integrable in the sense of the LL scheme if and only if its is Bo-integrable and its (LL) and Bochner integrals coincide.*

Proof of Theorem 5. Let us consider a real-valued and measurable mapping and m-a.e. finite $f \ : \ (\Omega, \mathcal{A}, m) \to \overline{\mathbb{R}}$. Let use denote

$$\int_{(Bo,\ \mathbb{R})} f\ dm \ and \ \int_{(LL,\ \mathbb{R})} f\ dm$$

its Bochner integral and its in modern integral, whenever they make sense. By definition, we have

$$\int_{(Bo,\ \mathbb{R})} f\ dm = \int_{(LL,\mathbb{R})} f\ dm,$$

for any elementary function.

(a) Let f be integrable in the sense of the *(RVM-MI)* scheme. Its positive part f^+ and negative part f^- are integrable. In the context of $\mathbb{R}$, f is limit of a non-decreasing sequence $(f_n^{(1)})_{n\geq 1}$ of non-negative elementary functions such that

$$f_n^{(1)} \to f^+ \ as \ n \to +\infty \ \ and \ \ (\forall n \geq 1),\ |f_n^{(1)}| \leq f^+.$$

So, by the dominated Convergence theorem in the *(LL)* scheme, we have

$$\int_{(LL,\ \mathbb{R})} f_n^{(1)}\ dm \to \int_{(LL,\ \mathbb{R})} f^+\ dm.$$

Now set $f_0 = 0$ and $h_n^{(1)} = f_n^{(1)} - f_{n-1}^{(1)}$ for $n \geq 1$. We have, for all $n \geq 1$,

$$f_n^{(1)} = h_1^{(1)} + \cdots + h_n^{(1)}$$

It is clear that

$$\left(h_n^{(1)}\right)_{n\geq 1} \subset \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$$

and

$$f^+ = \sum_1^{+\infty} h_n^{(1)}$$

Further, we have for all $k \geq 1$,

$$\begin{aligned} \sum_1^k \int_{(LL,\mathbb{R})} |h_n^{(1)}|\ dm &= \sum_1^{+\infty} \int_{(\Omega,\mathbb{R})} h_n^{(1)}\ dm \\ &= \int_{(LL,\ \mathbb{R})} f_k^{(1)}\ dm. \end{aligned}$$

Hence, by taking the limit as $k \to +\infty$,

$$\sum_1^{+\infty} \int_{(LL,\ \mathbb{R})} |h_n^{(1)}|\ dm \leq \int_{(LL,\ \mathbb{R})} f^+\ dm < +\infty. \tag{3.1}$$

By doing the same for the negative part, we get a sequence

$$\left(h_n^{(2)}\right)_{n\geq 1} \subset \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$$

such that

$$f^- = \sum_1^{+\infty} h_n^{(2)}$$

and

$$\sum_{1}^{+\infty} \int_{(LL,\ \mathbb{R})} |h_n^{(2)}|\ dm \leq \int_{(LL,\ \mathbb{R})} f^-\ dm < +\infty.$$

Hence, by taking $h_n = h_n^{(1)} - h_n^{(2)}$, $n \geq 1$, we get

$$(h_n)_{n\geq 1} \subset \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R}),$$

$$f = f^+ - f^- = \sum_{1}^{+\infty} h_n$$

and

$$\sum_{1}^{+\infty} \int_{(LL,\ \mathbb{R})} |h_n|\ dm \leq \int_{(LL,\ \mathbb{R})} |f|\ dm < +\infty.$$

We conclude that

$$f \in S(f_n,\ n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R}))$$

and hence f is Bochner integrable and its Bochner integral is

$$\int_{(Bo,\mathbb{R})} f\ dm = \sum_{n\geq 1} \int_{(LL,\ \mathbb{R})} h_n\ dm.$$

Since

$$\begin{aligned}
\sum_{n\geq 1}\int_{(LL,\ \mathbb{R})} h_n\ dm &= \lim_{k\rightarrow +\infty}\sum_{1\leq n\leq k}\int_{(LL,\ \mathbb{R})} h_n\ dm \\
&= \lim_{k\rightarrow +\infty}\int_{(LL,\ \mathbb{R})} f_k^{(1)} - f_k^{(2)}\ dm \\
&= \left(\lim_{k\rightarrow +\infty}\int_{(LL,\ \mathbb{R})} f_k^{(1)}\right) - \left(\lim_{k\rightarrow +\infty}\int_{(LL,\ \mathbb{R})} f_k^{(1)}\ dm\right) \\
&= \int_{(LL,\ \mathbb{R})} f^+\ dm - \int_{(\Omega,\mathbb{R},MI)} f^-\ dm \\
&= \int_{(LL,\ \mathbb{R})} f\ dm,
\end{aligned}$$

We have

$$\int_{(Bo,\ \mathbb{R})} f\ dm = \int_{(LL,\mathbb{R})} f\ dm. \ \square$$

(a) Let f be integrable in the Bochner sense with :

$$f = \sum_{1}^{+\infty} f_n,$$

$$(f_n)_{n\geq 1} \subset \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R}),$$

$$\sum_{1}^{+\infty}\int_{(LL,\ \mathbb{R})} |f_n|\ dm < +\infty$$

and

$$\int_{(Bo,\mathbb{R})} f\ dm = \lim_{k\rightarrow +\infty}\sum_{1\leq n\leq k}\int_{(LL,\ \mathbb{R})} f_n\ dm.$$

We take $g_k = \sum_{1\leq n\leq k} f_n\ dm$, $k \geq 1$. In the *(LL)* scheme, the g_k's are integrable (we recall thet the measure if finite here!) and are bounded by

$$S = \sum_{n=1}^{+\infty} f_n,$$

which is integrable in the *(RVM-MI)* scheme. Then, by the dominated convergence theorem of the *(RVM-MI)* scheme, we have that f is integrable in that scheme and

$$\int_{(LL,\mathbb{R})} f \ dm = \lim_{k \rightarrow +\infty} \sum_{1 \leq n \leq k} \int_{(LL, \ \mathbb{R})} f_n \ dm = \int_{(\Omega,\mathbb{R},B)} f \ dm.$$

The proof is over. ■

The main conclusions of that section are the following.

(1) The real-valued Bochner integral is exactly the modern real-valued integral with respect to a measure, provided the **measure is finite**.

(2) The Banach valued Bochner integral is an extension of the modern integral with respect to a finite measure to normed and complete space.

(3) The Bochner approach provides an alternative construction of the modern integral with respect to a finite measure, independently of the natural order of $\mathbb{R}$.

4. Properties and limit theorems for Banach-valued Bochner integrals

In this section, we will expose the first convergence theorem, show that the class of Bochner integral functions is a Banach space and finally establish a dominated convergence theorem.

(I) - The σ-convergence theorem (SCT).

In the *(RVM-MI)* approach, the most important convergence theorem is the *Monotone Convergence Theorem (MCT)* which gives birth to the *Fatou-Lebesgue Dominated Convergence Theorem (DCT)*. In that approach, the *MCT* was just the generalization of the definition of the integral of a non-negative mapping, as exposed in Step 2M (see page 39). If the non-decreasing sequence $(f_n)_{n\geq 1}$ of non-negative elementary functions is only a sequence of non-negative measurable mappings, the formula defining the integral of the non-negative mapping f which is limit of the f_n's,

$$\int_{(LL,\ \mathbb{R})} f\ dm = \lim_{n\rightarrow+\infty} \int_{(LL,\ \mathbb{R})} f_n\ dm,$$

is still valid and is known as the monotone convergence theorem *(MCT)*.

Here again, in the Bochner integral, the first important convergence limit will consists in the generalization of the step 2B (page 41) where we do not require that the sequence $(f_n)_{n\geq 1}$ is not necessarily composed of elementary functions.

Let us denote by $\mathcal{L}^1(\Omega, \mathcal{A}, m, E)$, $\mathcal{L}^1(Bo,\ E)$ in short, the class of all Bochner integrable functions $f\ :\ (\Omega, \mathcal{A}) \rightarrow E$. We have :

THEOREM 3. *Let* $f\ :\ (\Omega, \mathcal{A}) \rightarrow E$ *be measurable mapping and suppose that there exists a sequence*

$$(f_n)_{n\geq 1} \subset \mathcal{L}^1(\Omega, \mathcal{A}, m, E),$$

such that

$$f \in S(f_n,\ n \geq 1, \mathcal{L}^1(\Omega, \mathcal{A}, m, E)).$$

Then $f \in \mathcal{L}^1(\Omega, \mathcal{A}, m, E)$ *and*

$$\int_{(Bo,\ E)} f\ dm = \sum_{n\geq 1} \int_{(Bo,\ E)} f_n\ dm. \tag{4.1}$$

and

$$\int_{(LL,\ \mathbb{R})} \left\| f - \sum_{1\leq n\leq k} f_n \right\|_E\ dm \rightarrow 0\ as\ k \rightarrow +\infty. \tag{4.2}$$

Proof of Theorem 3. We assume that the hypotheses of the theorem hold. We have

$$\sum_{n\geq p+1} \|f_n\| \rightarrow 0,\ m - a.e\ and\ \sum_{n\geq p+1} \int_{(LL,\ \mathbb{R})} \|f_n\|\ dm \rightarrow 0,\ as\ p \rightarrow +\infty.$$

Fix $\eta > 0$. Suppose that $(\varepsilon_n)_{n\geq 1}$ is a sequence of positive numbers such that

$$\sum_{n\geq 1} \varepsilon_n = 1.$$

(It suffices to take $\varepsilon_n = 2^{-n}$, $n \geq 1$). By using Point (D) in Lemma 1, we may consider, for each $n \geq 1$,

$$f_n \in S(f_{n,p},\ p \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E))$$

such that for each $n \geq 1$

$$\sum_{p\geq 1} \int_{(LL,\ \mathbb{R})} \|f_{n,p}\|_E\ dm \leq \int_{(LL,\ \mathbb{R})} \|f_n\|_E\ dm + \frac{\eta\varepsilon_n}{m(\Omega)}.$$

Let us consider two integers $r \geq 1$ and $s \geq 1$ and set

$$h_{r,s} = \sum_{1\leq n\leq r} \sum_{1\leq p\leq s} f_{n,p}.$$

We have

$$\|h_{r,s} - f\|_E \leq \left\| \sum_{1 \leq n \leq r} \left(\left(\sum_{1 \leq p \leq s} f_{n,p} \right) - f_n \right) \right\| + \sum_{p \geq r+1} \|f_p\|.$$

By assumption, we have for each $n \geq 1$,

$$\sum_{1 \leq p \leq s} f_{n,p} \rightarrow f_n \ in \ E, \ m - a.e, \ as \ s \rightarrow +\infty$$

So, there exists a measurable subset $\Omega_0 \subset \Omega$ such that $m(\Omega_0^c) = 0$ and such that for any $\omega \in \Omega_0$, for all $n \geq 1$, there exists $s(n, \omega)$ such that for all $s \geq s(n, \omega)$, we have for

$$\left\| f_n - \left(\sum_{1 \leq p \leq s} f_{n,p} \right) \right\| \leq \eta \varepsilon_n.$$

Applying this to the previous formula, for $\omega \in \Omega_0$ and $s \geq s(n, \omega)$, we have

$$\|h_{r,s}(\omega) - f(\omega)\|_E \leq \eta \sum_{1 \leq n \leq r} \varepsilon_n + \sum_{p \geq r+1} \|f_p(\omega)\|.$$

So, by letting $(r, s) \rightarrow \{+\infty\}^2$, we get

$$\lim_{(r,s) \rightarrow \{+\infty\}^2} \|h_{r,s}(\omega) - f(\omega)\|_E \leq \eta.$$

Next, by letting $\eta \rightarrow 0$, we get

$$f = \sum_{n \geq 1, \ p \geq 1} f_{n,p}, \ m - a.e. \tag{4.3}$$

We also have

$$\begin{aligned}\sum_{r\geq 1,\ s\geq 1}\int_{(LL,\ \mathbb{R})}\|f_{n,p}\|_E\ dm &= \sum_{r\geq 1}\left(\sum_{s\geq 1}\int_{(LL,\ \mathbb{R})}\|f_{n,p}\|_E\ dm\right)\\ &\leq \sum_{r\geq 1}\left(\int_{(LL,\ \mathbb{R})}\|f_n\|_E+\frac{\eta\varepsilon_n}{m(\Omega)}\right)\ dm\\ &\leq \sum_{r\geq 1,\ s\geq 1}\int_{(LL,\ \mathbb{R})}\|f_n\|_E+\eta<+\infty.\end{aligned}$$

We conclude that

$$f\in S(f_{n,p},\ n\geq 1,\ p\geq 1,\mathcal{E}(\Omega,\mathcal{A},m)). \tag{4.4}$$

Hence $f\in\mathcal{L}^1(\Omega,\mathcal{A},m)$ and

$$\int_{(LL,\ \mathbb{R})} f\ dm=\sum_{n\geq 1,\ p\geq 1}\int_{(LL,\ \mathbb{R})} f_{n,p}\ dm,$$

which, by regrouping the terms, is

$$\int_{(LL,\ \mathbb{R})} f\ dm=\sum_{n\geq 1}\int_{(LL,\ \mathbb{R})} f_n\ dm.$$

So Formula (4.1) holds. Formula (4.2) holds too, since

$$\int_{(LL,\ \mathbb{R})}\left\|f-\sum_{1\leq n\leq k} f_n\right\|_E\ dm\leq\sum_{n\geq k+1}\|f_n\|_E\ dm\rightarrow 0, \tag{4.5}$$

as $k\rightarrow+\infty$, as the sequence of tails of a converging series. ■

II - Linear transform of the Bochner Integral.

For $Y\in L^1(\Omega,\mathcal{A},m,E)$, we may be interested in the intagral of $\ell(Y)$, ℓ being a bounded linear transform from E to another Banach space. The proposition below settles the case.

PROPOSITION 1. *Let* $(\Omega, \mathcal{A}, m)$ *be a finite measure space,* E_1 *and* E_2 *be two Banach spaces. Let* $Y \in L^1(\Omega, \mathcal{A}, m, E_1)$ *and* ℓ *be bounded linear space from* E_1 *and* E_2*. Then*

$$\ell(Y) \in \mathcal{L}^1(\Omega, \mathcal{A}, m, E_2)$$

and

$$\int_{Bo} \ell(Y)\, dm = \ell\left(\int_{Bo} Y dm\right).$$

Proof. We assume that the hypotheses hold. We have in $Y \in L^1(\Omega, \mathcal{A}, m, E_1)$, that is there exists $(Y_n)_{n\geq 1} \subseteq \varepsilon(\Omega, E_1)$:
(i) $\sum_{n\geq 1} \int \|Y_n\|_{E_1} dm \leq \infty$,
(ii) $Y = \sum_{n\geq 1} Y_n$, *m.a.e*

and

$$\int_{Bo} Y dm = \sum_{n\geq 1} \int_{Bo} Y_n dm.$$

Let us break the rest of the proof into steps.

Step 1. Since $Y_n \in \mathcal{E}(\Omega, \mathcal{A}, E_1)$, we also have $\ell(Y_n) \in \mathcal{E}_{\in}(\Omega, \mathcal{A}, E_2)$. Let us prove this for $X \in \mathcal{E}(\Omega, \mathcal{A}, E_1)$, is of the form

$$X = \sum_{j=1}^{r} x_j 1_{B_j}$$

with

$$(B_1, \ldots, B_r) \in \mathcal{A}^r, (x_1, \ldots, x_r) \in E_1^r, B_1 + \ldots + B_r = \Omega.$$

By linearity, we get

$$\ell(X) = \sum_{j=1}^{r} \ell(x_j)\, 1_{B_j}.$$

Thus $\ell(X) \in \mathcal{E}(\Omega, \mathcal{A}, E_2)$. Besides, we have

$$\begin{aligned}\int_{Bo,E_2} \ell(X)\,dm &= \sum_{j=1}^{r} \ell(x_j)\, m(B_j) \\ &= \ell\left(\sum_{j=1}^{r} x_j m(B_j)\right) \\ &= \ell\left(\int X\ dm\right) \qquad (4.6)\end{aligned}$$

Step 2. Next (ii) means that, as $r \to +\infty$

$$\|Y - \sum_{n=1}^{r} Y_n\|_\infty \to 0$$

Since ℓ us bounded, we have

$$\eta = \|\ell\| = \sup_{\|x\|=1} \frac{\|\ell(x)\|_{E_L}}{\|x\|_{E_1}} < +\infty.$$

We also have

$$\begin{aligned}\|\ell(Y) - \sum_{n=1}^{r} \ell(Y_n)\|_{E_2} &= \|\ell\left(Y_n - \sum_{n=1}^{r} Y_n\right)\|_{E_2} \qquad (4.7) \\ &\leq \eta\|Y - \sum_{n=1}^{r} Y_n\|_{E_1} \qquad (4.8) \\ &\to 0. \qquad (4.9)\end{aligned}$$

So, we get the half-way:

$$\ell(Y) = \sum_{n=1}^{\infty} \ell(Y_n)\, m - a.e.$$

To conclude, we must prove that

$$\sum_{n=1}^{\infty}\int \|\ell\left(Y_n\right)\|_{E_1} dm < +\infty.$$

But

$$\sum_{n=1}^{\infty}\int \|\ell\left(Y_n\right)\|_{E_1} dm \leq \eta \sum_{n=1}^{\infty}\int \|Y_n\| \ dm < +\infty$$

Hence, $\ell\left(X\right) \in \mathcal{L}^1\left(\Omega, \mathcal{A}, E_1\right)$ and

$$\int \ell\left(X\right) = \sum \int \ell\left(Y_n\right) dm$$

But, by Equation 4.6, we get

$$\begin{aligned}\int \ell\left(X\right) &= \sum_{n\geq 1}\int \ell\left(\int Y_n\right) dm \\ &= \sum_{n=1}^{r}\int \ell\left(Y_n\right) dm + \sum_{n=r+1}^{\infty}\int \ell\left(Y_n\right) dm \\ &= \ell\left(\sum_{n=1}^{r}\int Y_n dm\right) + R_{n,2},\end{aligned}$$

with

$$\|R_{n,2}\|_{E_2} \prec \eta \sum_{n=r+1}^{\infty}\int \|Y_n\| dm \to 0$$

as $r \to +\infty$. So, $r \to +\infty$,

$$\| \int \ell\left(Y\right) - \ell\left(\sum_{n=1}^{r}\int Y_n\right) dm\|_{E_1} \to 0$$

and

$$\ell\left(\sum_{n=1}^{r}\int Y_n dm\right) \to \ell\left(\sum_{n=1}^{\infty} Y_n\right) = \ell\left(\int Y dm\right),$$

since, $r \to +\infty$,

$$\left\| \ell \left(\sum_{n=1}^{r} \int Y_n dm \right) - \ell \left(\sum_{n=1}^{\infty} \int Y_n \right) \right\|_{E_2}$$

$$= \left\| \ell \left(\sum_{n=1}^{r} Y_n \right) - \ell \left[\sum_{n=1}^{r} \int Y_n dm \right] - \ell \left(\sum_{n=r+1}^{\infty} \int Y_n \right) \right\|_{E_2}$$

$$= \left\| \ell \left(\sum_{n=r+1}^{r} \int Y_n dm \right) \right\| \leq n \left\| \sum_{n=r+1}^{\infty} \int Y_n \right\| dm$$

$$\eta \sum_{n=r+1}^{\infty} \int \|Y_n\| dm \to 0.$$

We get the second half:

$$\int \ell\left(Y\right)\ dm = \sum_{n=1}^{+\infty} \int \ell(Y_n)\ dm.$$

The proof is over. ■

CHAPTER 3

Bochner integration on $\mathbb{R}$

1. Introduction

This simple note focuses on the comparison between the LL-integral and the Bochner integral of a random variable f defined on a measure space $(\Omega, \mathcal{A}, m)$ and taking values in $E = \mathbb{R}$, whenever they make sense. The LL-integral uses the order of $\mathbb{R}$ to completely describe the construction of the LL-integral, that we denote as

$$\int_{(LL),\mathbb{R}} f \ dm \quad or \quad \int_{(LL)} f \ dm$$

and call the (LL)-integral of f, by using a the three step method (f elementary function, f measurable and non-negative, f simply measurable. The Bochner integral, that we denote as

$$\int_{(Bo),\mathbb{R}} f \ dm \quad or \quad \int_{(Bo)} f \ dm$$

and call the (Bo)-integral of f, which is constructed in a general Banach spaces ignores the order structure and use a two step methods.

Since both approaches are available on $\mathbb{R}$, we want to compare the two integral. So, we begin by summarizing the key elements of the construction of the two types of integrals, restricting ourselves on the construction stages, to be in a position to make comparisons.

1.1. Construction of the LL-integral.

Assume that we have a measure space $(\Omega, \mathcal{A}, m)$. We are going to construct the integral of a real-value measurable function $f : (\Omega, \mathcal{A}) \to \overline{\mathbb{R}}$ with respect to the measure m (that may take infinite values) denoted by

$$\int_{(LL)} f\ dm = \int_{(LL),\Omega} f(\omega)\ dm(\omega) = \int_{(LL),\Omega} f(\omega)\ m(d\omega).$$

into three steps. The first step concerns non-negative functions among the class of elementary functions which have the general representation :

$$f = \sum_{1\leq i\leq k} \alpha_i\ 1_{A_i},\ (\alpha_i \in \mathbb{R}_+,\ A_i \in \mathcal{A},\ 1 \leq i \leq k),\ k \geq 1, \tag{1.1}$$

where the measurable sets A_i are pairwise disjoint (*pwd*). If it happens that the unions of the A_i, $1 \leq i \leq p$ is not Ω, we implicitly mean that $f = 0$ on the complement of $A_1 + \cdots + A_p$.

The class of real-valued elementary functions is denoted by $\mathcal{E}(\omega, \mathcal{A}, \mathbb{R})$ and $\mathcal{E}^+(\omega, \mathcal{A}, \mathbb{R})$ stands for the subclass of non-negative functions of $\mathcal{E}(\omega, \mathcal{A}, \mathbb{R})$.

The expression of an elementary function, as expressed in Formula (1.2) is not unique. But there exists one and only in which the coefficients α_i are disjoint, called the canonical representation. As a result, **that canonical representation is used with the summations of the A_i covering Ω** unless the contrary is specified.

Let us begin to describe the construction.

Step 1M. Definition of the integral of a non-negative simple function : $f \in \mathcal{E}_+$.

The integral of a non-negative simple function

$$f = \sum_{1\leq i\leq k} \alpha_i\ 1_{A_i},\ (\alpha_i \in \mathbb{R}_+,\ A_i \in \mathcal{A},\ 1 \leq i \leq k),\ k \geq 1, \tag{1.2}$$

is defined by

$$\int_{(LL)} f\ dm = \sum_{1\leq i\leq k} \alpha_i\ m(A_i). \tag{1.3}$$

Convention - Warning 1 In the definition (1.3), the product $\alpha_i \ m(A_i)$ is zero whenever $\alpha_i = 0$, event if $m(A_i) = +\infty$.

The definition (1.3) is coherent. This means that $\int_{(LL)} f \ dm$ does not depend on one particular expression of f.

step 2M. Definition of the integral for a non-negative measurable function.

Let f be any non-negative measurable function. By we have the following fact (see for example *Point (03-23) in Doc 03-02 in Chapter 4 in Lo (2017b))* : There exists a non-decreasing sequence $(f_n)_{n\geq 0} \subset \mathcal{E}_+$ such that

$$f_n \uparrow f \text{ as n } \uparrow +\infty. \tag{1.4}$$

We define

$$\int_{(LL)} f \ dm = \lim_{n\uparrow+\infty} \int_{(LL)} f_n \ dm. \tag{1.5}$$

This definition (1.5) is also coherent since it does not depends of the sequence which is used in the definition. (See Chapter 4, Lo (2017b)).

Step 3M. Definition of the integral for a measurable function.

In the general case, the decomposition of f into its positive part and its negative part is used as follows :

$$f = f^+ - f^-, \quad |f| = f^+ + f^-, \quad and \quad f^+ f^- = 0, \tag{1.6}$$

where $f^+ = max(0, f)$ and $f^- = max(0, -f)$, which are measurable, form the unique couple of functions such that Formulas (1.6) holds.

By *Step 2M*, the numbers

$$\int_{(LL)} f^+ \ dm \quad and \int_{(LL)} f^- \ dm$$

exist in $\overline{\mathbb{R}}_+$. If one of them is finite, i.e.,

$$\int_{(LL)} f^+ \, dm < +\infty \ \ \textbf{or} \int_{(LL)} f^- \, dm < +\infty,$$

we say that f is quasi-integrable with respect to m and we define

$$\int_{(LL)} f \, dm = \int f^+ \, dm \ - \ \int_{(LL)} f^- \, dm. \tag{1.7}$$

Warning The integral of a real-valued and measurable function with respect to a measure m exists *if only if* : either it is of constant sign or the integral of its positive part or its negative part is finite.

By extension, the **Integration of a mapping over a measurable set**. If A is a measurable subset of Ω and $1_A f$ is quasi-integrable, we denote

$$\int_{(LL),a} f \, dm = \int_{(LL)} 1_A f \, dm.$$

Convention - Warning 2 The function $1_A f$ is defined by $1_A f(\omega) = f(\omega)$ of $\omega \in A$, and $zero$ otherwise.

The function f is said to be (LL)-integrable if and only if the integral $\int f \, dm$ exist (in $\mathbb{R}$) and is finite, i.e.,

$$\int_{(LL)} f^+ \, dm < +\infty \ \ \textbf{and} \int_{(LL)} f^- \, dm < +\infty,$$

The set of all integrable functions with respect to m is denoted

$$\mathcal{L}^1(\Omega, \mathcal{A}, m).$$

Let us move to the Bochner integral as given in Mikusiński (2015).

1.2. Bochner Integration on Banach Sapces.

Given a finite measure space $(\Omega, \mathcal{A}, m)$, the Banach valued Bochner integral of measurable function

$$f : (\Omega, \mathcal{A}, m) \longrightarrow (E, \mathcal{B}),$$

where $(E, +, ., \|\circ\|_E)$ is a real Banach endowed with its Borel σ-algebra, is defined through two steps. Below, the *norm* of f, still denoted by $\|f\|_E$, is the measurable real-valued function defined by

$$\forall \omega \in \Omega, \ \|f\|_E(\omega) = \|f(\omega)\|_E.$$

In the sequel, we denote by $\mathcal{E}(\Omega, \mathcal{A}, E)$ the class of all Banach-elementary functions.

Step 1B: The integral of an Banach-valued elementary function of the form

$$f = \sum_{j=1}^{p} x_j \ 1_{B_j}, \tag{1.8}$$

where $p \geq 1$, $x_j \in E$, $B_j \in \mathcal{A}$, $B_1 + B_2 + \cdots + B_p = \Omega$. At this stage, only the linear structure of E is used since $f(\omega)$ is a finite linear combination. Let us denote by

$$\int_{(Bo)} f \ dm = \int_{(Bo} f \ dm = \sum_{j=1}^{p} \ x_j m(A_j) \in E. \tag{1.9}$$

Step 2B: A function (1.2) is Banach-Bochner integrable, denoted (Bo)-integrable, (denoted by $f \in \mathcal{L}^1(\Omega, \mathcal{A}, E, m)$) if and only if the two following conditions hold.

(a) There exists $(f_n)_{n\geq 1} \subset \mathcal{E}(\Omega, \mathcal{A}, E)$ such that

$$\sum_{n\geq 1} \int_{((LL)} |f_n|_E \ dm < +\infty. \tag{1.10}$$

(b) We have

$$f = \sum_{n \geq 1} f_n, \ m - a.e. \tag{1.11}$$

If so, we write

$$f \in S(f_n, n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, E)). \tag{1.12}$$

and the Banach-valued Bochner integral is defined by

$$\int_{(Bo)} f \ dm = \sum_{n \geq 1} \int_{(\Omega, E)} f_n \ dm. \tag{1.13}$$

A complete round on the subject is available in Mikusiński (2015), where the consistency of the definitions of both steps has been been proved and the justification of both Formula (1.10) and (1.11).

In the same paper, the space $\mathcal{L}^1(\Omega, \mathcal{A}, E, m)$ modulo the class of m-null-sets and denoted by $L^1(\Omega, \mathcal{A}, E, m)$ is proved to be a Banach space when endowed by the norm

$$|f|_{L^1(\Omega, \mathcal{A}, E, m)} = \int_{(LL)} |f|_E \ dm.$$

Also, a dominated convergence theorem (DCT) is given therein.

As wll, the limits theory of sequences Bochner integrals is also very interesting. That theory is based on two key-ideas. First, we may replace $\mathcal{E}(\Omega, \mathcal{A}, E)$ by $\mathcal{L}^1(\Omega, \mathcal{A}, E)$ in Formula (1.2) and still get the same decomposition of the integral as follows

THEOREM 4. *Suppose that we have*

$$f \in S(f_n, n \geq 1, \mathcal{L}^1(\Omega, \mathcal{A}, E)),$$

then f is integrable and we have

$$\int_{(Bo)} f \ dm = \sum_{n \geq 1} int_{(\Omega, E)} f_n \ dm. \tag{1.14}$$

The second idea is that, in *Step 1B*, for any $\eta > 0$, we can choose a sequence $(f_n)_{n\geq 1}$ such that Formula holds and

$$|f|_E \leq \sum_{n\geq 1} |f_n|_E \leq |f|_E + \eta. \tag{1.15}$$

We have finished describing two approaches. Let us proceed to their comparisons.

2. Comparison of the two Integrals on $\mathbb{R}$

In the previous section, we used the Real-valued Mapping *(RVM)* integration scheme to get the Bochner integral in a complete normed space.

We already knew that the natural order on $\mathbb{R}$ was used in the general construction of the *(RVM)* integration in the step *2M* (see 39) and we saw how that approach allowed to integrate with respect to an infinite measure.

Now, we are going to see that by restricting ourselves to finite measure, the Bochner and the *(RVM)* integrals are exactly the same.

We suppose that the *(RVM)* integration with respect to a finite measure m is completely set. We have

THEOREM 5. *A real-valued measurable mapping and m-a.e. finite f : $(\Omega, \mathcal{A}, m) \rightarrow \overline{\mathbb{R}}$ is (LL)-integrable if and only if its is (Bo)-integrable and its its (LL)-integral and its (Bo)-integral coincide.*

Proof of Theorem 5. Let us consider a real-valued and measurable mapping and m-a.e. finite $f : (\Omega, \mathcal{A}, m) \rightarrow \overline{\mathbb{R}}$. We have : for any elementary function,

$$\int_{(Bo)} f \, dm = \int_{(LL)} f \, dm.$$

Let us proceed by step.

(a) Let f be (LL)-integrable. Its positive part f^+ and negative part f^- are integrable. In the context of $\mathbb{R}$, f is limit of a non-decreasing sequence $(f_n^{(1)})_{n\geq 1}$ of non-negative elementary functions such that

$$f_n^{(1)} \to f^+ \ as \ n \to +\infty \quad and \quad (\forall n \geq 1), \ |f_n^{(1)}| \leq f^+.$$

So, by the Dominated Convergence Theorem in the *(RVM-MI)* scheme, we have

$$\int_{(LL)} f_n^{(1)} \ dm \to \int_{(LL)} f^+ \ dm.$$

Now set $f_0 = 0$ and $h_n^{(1)} = f_n^{(1)} - f_{n-1}^{(1)}$ for $n \geq 1$. We have, for all $n \geq 1$,

$$f_n^{(1)} = h_1^{(1)} + \cdots + h_n^{(1)}$$

It is clear that

$$\left(h_n^{(1)}\right)_{n\geq 1} \subset \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$$

and

$$f^+ = \sum_1^{+\infty} h_n^{(1)}$$

Further, we have for all $k \geq 1$,

$$\begin{aligned} \sum_1^k \int_{(LL)} |h_n^{(1)}| \ dm &= \sum_1^{+\infty} \int_{(LL)} h_n^{(1)} \ dm \\ &= \int_{(LL)} f_k^{(1)} \ dm. \end{aligned}$$

Hence, by taking the limit as $k \to +\infty$,

$$\sum_1^{+\infty} \int_{(LL)} |h_n^{(1)}| \ dm \leq \int_{(LL)} f^+ \ dm < +\infty. \tag{2.1}$$

By doing the same for the negative part, we get a sequence

$$\left(h_n^{(2)}\right)_{n\geq 1} \subset \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R})$$

such that

$$f^- = \sum_1^{+\infty} h_n^{(2)}$$

and

$$\sum_1^{+\infty} \int_{(LL)} |h_n^{(2)}| \ dm \leq \int_{(LL)} f^- \ dm < +\infty.$$

Hence, by taking $h_n = h_n^{(1)} - h_n^{(2)}$, $n \geq 1$, we get

$$(h_n)_{n\geq 1} \subset \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R}),$$

$$f = f^+ - f^- = \sum_1^{+\infty} h_n$$

and

$$\sum_1^{+\infty} \int_{(LL)} |h_n| \ dm \leq \int_{(LL)} |f| \ dm < +\infty.$$

We conclude that

$$f \in S(f_n, \ n \geq 1, \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R}))$$

and hence f is Bochner integrable and its Bochner integral is

$$\int_{(Bo)} f \ dm = \sum_{n\geq 1} \int_{(LL)} h_n \ dm.$$

Since

$$\begin{aligned}
\sum_{n\geq 1}\int_{(LL)} h_n\ dm &= \lim_{k\rightarrow+\infty}\sum_{1\leq n\leq k}\int_{(LL)} h_n\ dm\\
&= \lim_{k\rightarrow+\infty}\int_{(LL)} f_k^{(1)} - f_k^{(2)}\ dm\\
&= \left(\lim_{k\rightarrow+\infty}\int_{(LL)} f_k^{(1)}\right) - \left(\lim_{k\rightarrow+\infty}\int_{(LL)} f_k^{(1)}\ dm\right)\\
&= \int_{(LL)} f^+\ dm - \int_{(LL)} f^-\ dm\\
&= \int_{(LL)} f\ dm,
\end{aligned}$$

We have

$$\int_{(Bo)} f\ dm = \int_{(LL)} f\ dm. \square$$

(a) Let f be (Bo)-integrable with :

$$f = \sum_{1}^{+\infty} f_n,$$

$$(f_n)_{n\geq 1} \subset \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R}),$$

$$\sum_{1}^{+\infty}\int_{(LL)} |f_n|\ dm < +\infty$$

and

$$\int_{(Bo)} f\ dm = \lim_{k\rightarrow+\infty}\sum_{1\leq n\leq k}\int_{(LL)} f_n\ dm.$$

We take $g_k = \sum_{1\leq n\leq k} f_n\ dm$, $k \geq 1$. In the *(LL)* scheme, the g_k's are integrable (we recall that the measure is finite here!) and are bounded by

$$S = \sum_{n=1}^{+\infty} f_n,$$

which is (LL)-integrable. Then, by the dominated convergence theorem of the *(LL)* scheme, we have that f is integrable in that scheme and

$$\int_{(LL)} f \, dm = \lim_{k \to +\infty} \sum_{1 \leq n \leq k} \int_{(LL)} f_n \, dm = \int_{(Bo)} f \, dm.$$

The proof is over. ■

The main conclusions of that section are the following.

(1) The real-valued Bochner integral is exactly the modern real-valued integral with respect to a measure, provided the **measure is finite**.

(2) The Banach valued Bochner integral is an extension of the modern integral with respect to a finite measure to normed and complete space.

(3) The Bochner approach provides an alternative construction of the modern integral with respect to a finite measure, independently of the natural order of $\mathbb{R}$.

CHAPTER 4

Bochner integration on $\mathbb{R}^k$ ($k \geq 1$)

In this chapter we focus on the Bochner integration in $\mathbb{R}^k$, $k \geq 1$, endowed with its usual norms. Unless the contrary is specified we use the max norm

$$\|(x_1, \ldots, x_k)\| = \max_{1 \leq i \leq k} |x_i|$$

We have the following simple characterization

THEOREM 6. *Let* $(\Omega, \mathcal{A}, m)$ *be a finite measure space and*

$$Y = (Y_1, \ldots, Y_k)^T : (\Omega, \mathcal{A}) \to \mathbb{R}^k,$$

a measurable random vector. Here A^T *stands for the transpose of any matrix.*

Then Y *is Bo-integrable with respect to* m *if and only if each component* Y_i*,* $1 \leq i \leq k$*, is LL-integrable and*

$$\int Y \, dm = \left(\int_{LL} Y_i \, dm, 1 \leq i \leq k \right)^T.$$

Proof of Theorem 6. Let us split the proof into a direct and an indirect part.

1. **Direct implication.** Let $Y = (Y_1, \ldots, Y_k)^T : (\Omega, \mathcal{A}) \to \mathbb{R}^k$ be Bo-integrable. By lemma 1, for any $\eta > 0$ fixed, we can find a sequence of elementary functions

$$\left(Y_n = (Y_{1,n}, \ldots, Y_{k,n})^T\right)_{n \geq 1} \subset \mathcal{E}(\Omega, \mathcal{A}, \mathbb{R}^k)$$

such that:

(ia) $\sum_{n\geq 1} \int \|Y_n\|_\infty \, dm < +\infty$

(iia) $Y = \sum_{n\geq 1} Y_n$, m-a.e

and

(iiia) $\|Y_n\|_\infty \leq \sum_{n\geq 1} \|Y_n\|_\infty \leq \|Y_n\|_\infty + \eta$.

The Projections

$$\pi_i(Y) = Y_i, 1 \leq i \leq k$$

are bounded linear mappings since, $\forall X, Y \in \mathbb{R}^k$, $\lambda \in \mathbb{R}$.

$$\begin{aligned} \pi_i(\lambda X + Y) &= (\lambda X + Y)_i \\ &= \lambda X_i + Y_i \\ &= \lambda \pi_i(X) + \pi_i(Y) \end{aligned}$$

$$|\pi_i(Y)| = |Y_i| \leq \|Y\|_\infty$$

Then by Proposition 1

$$\pi_i(Y) = Y_i \in \mathcal{L}^1(\Omega, \mathcal{A}, m, LL) \quad 1 \leq i \leq k.$$

We have proved the direct implication, we go ahead to prove the indirect implication.

2. **Indirect implication.** Let $Y \in \mathcal{L}^1$ for any $1 \leq i \leq k$. Theorem 2 in Lo *et al.* (2019), for any $i \in \{1, \cdots, k\}$ fixed, there exists a sequence of elementary functions $(Y_{i,n})_{n\geq 1} \subset \varepsilon(\Omega, \mathcal{A}, \mathbb{R})$ such that:

(1b) $\forall i \in \{1, \cdots, k\}$, $\sum_{n\geq 1} \int |Y_{i,n}|_\infty \, dm < +\infty$

(2b) $\forall i \in \{1, \cdots, k\}$,$Y_i = \sum_{n\geq 1} Y_{i,n}$, m-a.

Let us set

$$Y_n = (Y_{1,n}, \ldots, Y_{k,n})^T.$$

We want to show that

(i) $\sum_{n\geq 1} \int \|Y_n\|_\infty dm < +\infty$ (L1)

(ii) $Y = \sum_{n\geq 1} Y_n$, m-a.e (L2)

Since we are dealing with a finite dimensional space, $\mathbb{R}^{\mathbb{K}}, K \geq 1$, we now that all norms are equivalent and so there exist $\eta > 0$ such that

$$\begin{aligned} \sum_{n\geq 1} \int \|Y_n\|_\infty dm &\leq \eta \sum_{n\geq 1} \int \|Y_n\|_1 dm \\ &= \eta \sum_{n\geq 1} \int \sum_{i=1}^{k} |Y_{i,n}| dm \\ &= \eta \sum_{i=1}^{k} \sum_{n=1}^{\infty} \int |Y_{i,n}| dm < \infty, \end{aligned}$$

since $\sum_{n=1}^{\infty} \int |Y_{i,n}|$dm $< +\infty$.

Hence we have that $\sum_{n\geq 1} \int \|Y_n\|_\infty dm < \infty$

We now show Condition (L2)

$$\left\| Y - \sum_{n=1}^{p} Y_n \right\|_1 = \sum_{i=1}^{k} |Y_i - \sum_{n=1}^{p} Y_{i,n}| \rightarrow_{p\rightarrow+\infty} 0$$

Since all norms are equivalent here, we have that there exists $\gamma > 0$ such that

$$\left\|Y - \sum_{n=1}^{p} Y_n\right\|_{\infty} \leq \gamma \left\|Y - \sum_{n=1}^{p} Y_n\right\|_{1},$$

which implies that

$$\left\|Y - \sum_{n=1}^{p} Y_n\right\|_{\infty} \rightarrow_{p \rightarrow +\infty} 0$$

proving condition (L2)

Finally, by (L1) and (L2) we get $Y \in \mathcal{L}^1$ with

$$\begin{aligned}
\int_{Bo} Y dm &= \sum_{n=1}^{\infty} \int_{Bo} Y_n dm \\
&= \int_{Bo} \sum_{n \geq 1} Y_n dm \quad (By\ DCT) \\
&= \int_{Bo} \sum_{n \geq 1} (Y_{1,n}, \ldots, Y_{k,n})^T \, dm \\
&= \int_{Bo} \sum_{n \geq 1} (Y_{i,n}, \quad 1 \leq i \leq k)^T \, dm \\
&= \int_{LL} \sum_{n \geq 1} (Y_{i,n}, \quad 1 \leq i \leq k)^T \, dm \\
&= \int_{LL} \left(\sum_{n \geq 1} Y_{i,n}, \quad 1 \leq i \leq k \right)^T dm \\
&= \int_{LL} (Y_i, \quad 1 \leq i \leq k)^T \, dm,
\end{aligned}$$

where *DCT* stands for *dominated convergence theorem*. Hence we have that

$$\int Y dm = \left(\int Y_1 dm, \cdots, \int Y_k dm \right)^T.$$

The proof is over. ■

LEMMA 2. *Let* $Y_n = (Y_{1,n}, \ldots, Y_{k,n})^T$ *such that* $\forall 1 \leq i \leq k$, $Y_{i,n} \subset \varepsilon(\Omega, \mathcal{A}, \mathbb{R})$, *then*

$$(Y_n)_{n\geq 1} \subset \varepsilon\left(\Omega, \mathcal{A}, \mathbb{R}^k\right)$$

Proof of lemma. Let $Y_{i,n} \subset \varepsilon(\Omega, \mathcal{A}, \mathbb{R}) \, \forall 1 \leq i \leq k$. Each $Y_{i,n}, \forall 1 \leq i \leq k$, is of the form

$$Y_{i,n} = \sum_{j=1}^{\ell(i)} y_{j,n}^i 1_{B_{j,n}}^i.$$

Using the superposition of Ω, for

$$I_i = \{1, \ldots, \ell(i)\},$$

$$\sum_{(i_1,\cdots,i_k)\in\prod_{i=1}^k I_i} \bigcap_{j=1}^k B_{i_{j,n}}^{i_j} = \sum_{(i_1,\cdots,i_k)\in K} \bigcap_{j=1}^k B_{i_{j,n}}^{i_j},$$

where

$$K = \left\{ (i_1, \cdots, i_k), \bigcap_{j=1}^k B_{i_{j,n}}^{i_j} \neq \emptyset \right\}.$$

It is not difficult to see that

$$Y_{i,n} = y_{i_j,n}^{i_j} on \bigcap_{j=1}^k B_{i_{j,n}}^{i_j}$$

By denoting

$$\bigcap_{j=1}^k B_{i_{j,n}}^{i_j}, \quad (i_1, \cdots, i_k) \in K$$

as

$$\{C_{j,n}, 1 \leq j \leq p(n)\}$$

and by denoting $z_{j,n}^{(i)}$, the values of $Y_{i,n}$ on $C_{j,n}$, all the $Y_{i,n}$ are based on the same partition of Ω using the $C_{j,n}$'s and we have

$$\forall 1 \leq i \leq n, \quad Y_{i,n} = \sum_{j=1}^{p(n)} z_{j,n}^{(i)} 1_{C_{j,n}}$$

By an abuse of language, we still denote $z_{j,n}^{(i)} = y_{j,n}^{(i)}$. Hence, we have $z_{j,n}^{(i)} \equiv y_{j,n}^{(i)}$.

So the Y_n can be written as

$$Y_n = \sum_{j=1}^{p(n)} y_{j,n} \quad on \quad C_{j,n} \quad with \quad y_{j,n} = (y_{j,n}^i, \ 1 \leq i \leq k)^T.$$

The proof is over ■

CHAPTER 5

Bochner integration in ℓ_1

In this chapter, we study the Bochner integration in ℓ_1. The characterization is as follows.

Theorem 7. *Let*
$Y : (\Omega, \mathcal{A}, m) \to \left(\ell_1 \, \|.\|_{\ell_1}\right)$ *be strongly measurable with* m *finite. Then*

$$Y = (Y_j)_{j \geq 0} \in \mathcal{L}^1(\Omega, \mathcal{A}, m, \ell_1, Bo)$$

if and only if
(ic) $\forall j \geq 0, Y_j \in \mathcal{L}^1(\Omega, \mathcal{A}, m, \ell_1, LL)$*,*
(iic) $\sum_{j=0}^{\infty} \int \|Y_j\| dm \leq \infty$*.*

Moreover, the Bo-integral of Y*, if it exist, is*

$$\left(\int_{LL,\mathbb{R}} Y_j \; dm, j \geq 0\right)^T = \int_{Bo,\ell_1} Y \; dm \in \ell_1$$

Proof. We split the proof into a direct and an indirect part.

Proof of the direct implication. Let $Y = (Y_j)_{j \geq 0} \in \ell_1$. The projections

$$\pi_j(Y) = Y_j, j \geq 1,$$

are bounded linear mappings. By Proposition 1,

$$\forall j \geq 1, \quad \pi_j(Y) = Y_j \in \mathcal{L}^1(\Omega, \mathcal{A}, m, LL)$$

and

$$\pi\left(\int Y dm\right) = \int Y_j \ dm.$$

So,

$$\int Y dm = \left(\int Y_j dm, j \geq 0\right)^T \in \ell_1$$

and this holds whenever Y is Bo-integrable. Now by definition of the Bochner integral,we have

$$\int \|Y\|_{\ell_1} dm < +\infty.$$

Since

$$\int \|Y\|_{\ell_1} dm = \int \left(\sum_{j=0}^{\infty} \|Y_j\|\right) dm < +\infty,$$

We get by the monotone convergence theorem

$$\sum_{j=0}^{\infty} \int \|Y_j\| \ dm < +\infty,$$

which is (iic). The proof of the direct implication is over.

Proof of the indirect part. Let $Y = (Y_j)_{j \geq 0} \in \ell_1$. Suppose that condition (ic) and (iic) of the theorem hold. Let $\eta > 0$. Then for any $j \geq 1$ fixed,by the definition of the integral and by Formula 1.15, we can find $(Y_{j,n})_{n \geq 1} \subset \varepsilon(\Omega, \mathcal{A}, \mathbb{R})$ such that, for $j \geq 1$,

$$Y_j = \sum_{n \geq 1} Y_{j,n}, \quad \sum_{n \geq 1} \int \|Y_{j,n}\| dm < \infty$$

and

$$\sum_{n \geq 1} \int \|Y_{j,n}\| dm \leq \|Y_j\| + \frac{\eta}{2^j m(\Omega)}.$$

Hence,

$$\sum_{j=0}^{\infty}\sum_{n\geq 1}\|Y_{j,n}\| \leq \sum_{j=0}^{\infty}\|Y_j\| \leq \frac{\eta}{m(\Omega)}. \tag{0.1}$$

We want to show that $Y = (Y_j, j \geq 1)^T$ is Bo-integrable. We set

$$Y_n = (Y_{j,n}, j \geq 1)^T \quad n \geq 0.$$

We are going to derive our conclusion from the following three facts to be proved:

(C1) The Y_n's are integrable;

(C2) $\sum_{n\geq 1}\int Y_n$ dm exists in ℓ_1;

(C3) $Y = \sum_{n\geq 0} Y_n$.

Indeed applying Theorem 3 to (C1), (C2) and (C3) achieves the proof. Let us prove them one by one,

Proof of (C1). Here, n is fixed. We split Y at the rank $p \geq 1$, i.e,

$$Y_n = (Y_{1,n}, \ldots, Y_{p,n}, 0, \ldots) + (0, \ldots, 0, Y_{p+1,n}, Y_{p+2,n}, \ldots) =: Y_{n,1,p} + Y_{n,2,p}.$$

Remark. To form $Y_{n,1,p}$ from Y_n, we replace all elements of Y_n of rank at least $p+1$ by zero's, and to form $Y_{n,2,p}$ from Y_n, we replace all elements of Y_n of rank up to p by zeros.

By the finiteness of the double-indexed series (of non-negative terms) in formula 0.1, we have that as $p \to +\infty$,

$$\|Y_{n,2,p}\| \leq \sum_{j=0}^{\infty}\sum_{n\succ p}\|Y_{j,n}\| \to 0. \tag{0.2}$$

Now, for r being an infinite vectors of zeros, we have

$$Y_{n,1,p} = \Big((Y_{1,n}, \dots, Y_{p,n}), r\Big)^T =: \Big(Y^*_{n,p}, r\Big)^T.$$

$(Y^*_{n,p})^T$ is a function taking its values in $\mathbb{R}^p$, and by Lemma 2, $(Y^*_{n,p})^T$ is an elementary function of the form

$$(Y^*_{n,p})^T = \sum_{s=1}^{q(p,n)} Y^*_s 1_{B_{n,p,n}}.$$

By expanding $Y^*_s \in \mathbb{R}^p$ to an element Y_s of ℓ_1 by completing with zero's, we get that

$$Y_{n,1,p} = \sum_{s=1}^{q(p,n)} Y_s 1_{B_{n,p,n}}$$

is an elementary function on ℓ_1, and by combining 0.1 and 0.2, we have that $(Y_{n,1,p})_{p\geq 1}$ is a sequence of elementary functions converging to Y_n and

$$\|Y_{n,1,p}\| \leq g = \sum_{n\geq 1 j\geq 0} \|Y_{i,j}\| \in \mathcal{L}^1$$

By Theorem 3, we conclude that Y_n is integrable for each $n \geq 1$.

Proof of (C2). We have already proved that the Y_n's are integrable in ℓ_1, hence

$$\left(\int Y_n dm\right) = \left(\int Y_{j,n}, dm,\ j \geq 1\right)^T.$$

Hence,

$$\begin{aligned}
\left\|\sum_{n\geq 1}\int Y_n\ dm\right\|_{\ell_1} &= \sum_{n=1}^{\infty}\int\sum_{j=0}^{\infty}\|Y_{j,n}\|\ dm \\
&= \int\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\|Y_{j,n}\|\ dm \quad (L2) \\
&\leq \int\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\|Y_{j,n}\|\ dm \\
&\leq \int\left(\sum_{j=1}^{\infty}\|Y_j\|\right)+\frac{\eta}{2^j m(\Omega)}dm \quad (L4) \\
&\leq \sum_{j=1}^{\infty}\int\|Y_j\|\,dm+\eta<\infty, \quad (L5)
\end{aligned}$$

Here we used the monotone convergence theorem in line (L2), Formula (0.1) in (L4) and assumption (iic) in line (L5).

Proof of (C3). We have

$$\begin{aligned}
\left\|Y-\sum_{n\geq 1}Y_n\right\|_{\ell_1} &= \left\|Y-\left(\sum_{n\geq 1}Y_{j,n}, j\geq 0\right)\right\|_{\ell_1} \\
&= \sum_{j=0}^{\infty}\|Y_j-\sum_{n\geq 1}Y_{j,n}\| \\
&= \sum_{j=0}^{\infty}\|Y_j-\sum_{n=1}^{k}Y_{j,n}\|+\sum_{j=0}^{\infty}\sum_{n\geq k}\|Y_{j,n}\| \\
&= R_k(1)+R_k(2).
\end{aligned}$$

By the Fubini's law for non-negative series, we get

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{n=k}^{\infty}\|Y_{j,n}\| &= \sum_{n=k}^{\infty}\sum_{j=1}^{\infty}\|Y_{j,n}\| \\ &= \sum_{n=k}^{\infty} b_n,\end{aligned}$$

with,

$$b_n = \sum_{j=1}^{\infty}\|Y_{j,n}\|, n \geq 1.$$

So, by formula (0.1) (as the tail of a converging series), we have, $k \to +\infty$,

$$0 \leq \sum_{n=k}^{\infty} b_n = \sum_{n=k}^{\infty}\sum_{j=1}^{\infty}\|Y_{j,n}\| \to 0,$$

that is as $k \to +\infty$,

$$R_k(2) = \sum_{j=0}^{\infty}\sum_{n=k}^{\infty}\|Y_{j,n}\| \to 0.$$

Now,

$$R_k(1) = \sum_{j=0}^{\infty}\|Y_j - S_{j,k}\|$$

with

$$S_{j,k} = \sum_{n=1}^{k} Y_{j,n}.$$

Let us write $R_k(1)$ as an integral w.r.t. the counting measure

$$v = \sum_{j=0}^{\infty} \delta_j,$$

and denote

$$Y = (Y_j)_{j \geq 0}, \quad S_k = (S_{j,k})_{k \geq 1}.$$

We have,

$$R_k(1) = \int_{\mathbb{R}} \|Y - S_k\| dv.$$

But,

(1) $\forall j \geq 0, S_k(j) = S_{j,k} \to Y(j) = y_j$

and

(2) $\forall j \geq 0, \|S_k(j)\| \leq \sum_{n=1}^{k} \|Y_{j,n}\| \leq \|Y_j\| + \frac{\eta}{2^j m(\Omega)}$

By defining the function $a(o)$ as follows

$$a(j) = \frac{\eta}{2^j m(\Omega)}, j \geq 1,$$

It is clear that $Y \in \mathcal{L}^1(v)$ and hence

$$\|S_k\| \leq Y + a.$$

In Summary,

(3) $S_k \to Y$ everywhere

and

(4) $\|S_k\| \leq Y + a \in \mathcal{L}^1(v)$.

By the dominated convergence theorem, we obtain

$$\int_{\mathbb{R}} \|Y - S_k\| dv \to 0.$$

Hence

$$R_k(1) \to 0 \ as \ k \to +\infty.$$

We have proved (C3).

hence the proof is over ■

Conclusion and perspectives

Although the Bochner integration was initiated around 1930 with the seminal work of Pettis (1038), the research on it is still active due to the extraordinary advances of Functional Analysis. Currently, the interest in it grows in this area of Big Data which treats the most complex data. As a consequence, the usual methods on $\mathbb{R}^d$ are nolonger sufficient.
Another reason of our interest in Bochner integration is the development of Aumann integration of set-valued functions which uses Bochner integral a lots. Aumann integration of set-valued functions, by their own, has significant applications in many disciplines such as Image processing, Economics, etc.

On a concluding note we have been able to extend the concept of the Bochner integals to Banach Spaces such as $\mathbb{R}^K$, $k \geq 1$ and the sequence space ℓ_1, we were able to do this with the help of our foundational knowledge in mathematics, knowledge of measure theory and functional analysis. As a Perspective to this, the concept of the Bochner integrals may also be extended to the sequence spaces ℓ_p, $p \geq 2$.

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