SINGLE-STEP ALGORITHM FOR VARIATIONAL INEQUALITY PROBLEMS IN BANACH SPACES

A Thesis
Submitted to African University of Science and Technology
Abuja-Nigeria
In Partial fulfilment of the Requirements for

MASTER DEGREE IN PURE AND APPLIED MATHEMATICS

By

Halima Yusuf

Supervisor:

A. U. Bello, Ph.D

African University of Science and Technology Abuja
www.aust.edu.ng
P.M.B 681, Garki, Abuja F.C.T
Nigeria.

June, 2021.

Certification

This is to certify that the thesis titled "SINGLE-STEP ALGORITHM FOR VARIATIONAL INEQUALITY PROBLEMS IN BANACH SPACES" submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria, for the award of the Master's degree is a record of original research carried out by Halima Yusuf in the Department of Pure and Applied Mathematics under the supervision of Dr. A.U.Bello.

SINGLE-STEP ALGORITHM FOR VARIATIONAL INEQUALITIES IN BANACH SPACES.

By

Halima Yusuf

A THESIS APPROVED BY THE DEPARTMENT OF PURE AND APPLIED MATHEMATICS

RECOMMENDED: ______________________________

Dr. A. U. Bello (Supervisor)

____

Dr. A. U. Bello (Ag. Head of Department)

APPROVED: ______________________________

Chief Academic Officer

____

Date

Abstract

In this work, we propose a one-step algorithm for solving variational inequality problems in a 2-uniformly convex Banach space. Weak convergence of the scheme to a solution of variational inequality is established under reasonable assumptions. More precisely, we proved the following theorem:

Theorem *Let E be a real 2-uniformly convex and uniformly smooth Banach space. Let C be nonempty closed convex subset of E. $A : E \to E^*$ be monotone and Lipschitz with Lipschitz constant L. Let $x_0, x_{-1} \in E$ and defined the sequence $\{x_n\}$ by*

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n - \lambda_{n-1}(Ax_n - Ax_{n-1})), \quad n \geq 0;$$

where $\{\lambda_n\} \subseteq \left[\epsilon, \frac{1-2\epsilon}{2\mu L}\right]$ for some $\epsilon > 0$ and $\mu \geq 1$. Suppose Γ is nonempty and that the normalized duality mapping J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to an element of Γ.

Applications are also presented to show how our result can be applied to real life problems.

Acknowledgements

All praise be to Allah, the Almighty, on whom we ultimately depend for sustenance and guidance. If not for His special grace, the thought of writing this thesis would not have been conceived. For this reason, I express my gratitude to Him; may peace and blessings of Allah be upon our beloved prophet Muhammad (PBUH).

Special thanks go to my supervisor Dr. Usman Bello for his guidance, understanding, patience and tireless efforts that he put to supervise this thesis and make corrections which make this research project worthy of reading. It has been a great pleasure and honor to have him as my supervisor. May Allah, the Almighty, reward him and his entire family abundantly, Amin. My special regards also extends to the acting president of African University of Science and Technology, Abuja, Professor Charles E. Chidume and to Dr. Ma'aruf S. Minjibir (BUK), Prof. Bashir Ali (BUK) and all my lecturers for their assistance, mentorship and tutorship during my studies. Also, I would like to send my gratitude to the entire staff and AUST managements for their hospitality and kindness throughout my studies in AUST.

With pleasure, I will like to acknowledge my colleagues, Mrs. Maryam Alka, Mr. Akinola Tunde Paul and all my classmates for their collaboration and friendship. Furthermore, I am pleased to express my regards to all those who supported me in any respect during my work of this project. Moreover, this study wouldn't have been completed successfully without the immeasurable help rendered to me by my family, my father, Alhaji Yusuf Ahmad; my mum, Hajiya Hassana Atiku Kankia; my sisters; Sumayya, Maryam, Khadija and brother Ahmad, for their support in terms of prayers, advises and financial assistance given to me during the course of my training in AUST. May Allah reward them with jannatul Firdous, Amin.

Halima Yusuf
June, 2021.

Dedication

I dedicated this research work to my beloved parents, Alhaji Yusuf Ahmad and Hajiya Hassana Atiku Kankia for their support and encouragement towards compilation of this great achievement.

Contents

Chapter 1

General Introduction

1.1 Introduction

The contents of this thesis fall within the general area of functional analysis, an area which has attracted the attention of prominent mathematicians due to its diverse application in numerous field of sciences. The contribution of this thesis mainly concentrate on approximation of solutions of variational inequality problem in a 2-uniformly convex and uniformly smooth Banach spaces.

Let E be a real Banach space with its dual E^* and C be a nonempty closed convex subset of E. Let $A : E \to E^*$ be monotone and Lipschitz map with Lipsschitz constant L. We consider the following Variational inequality problem $VIP(A, C)$ which entails finding

$$x \in C \quad \text{such that} \quad \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1.1}$$

Let Γ denote the nonempty solution set of (1.1.1). We shall first offer a basic description of the nature of variational inequality.

Let E be a Banach space and $f : E \to \mathbb{R}$ be differentiable map. The problem of finding

$$u \in E \quad \text{such that} \quad f(u) \leq f(v) \quad \forall v \in E, \tag{1.1.2}$$

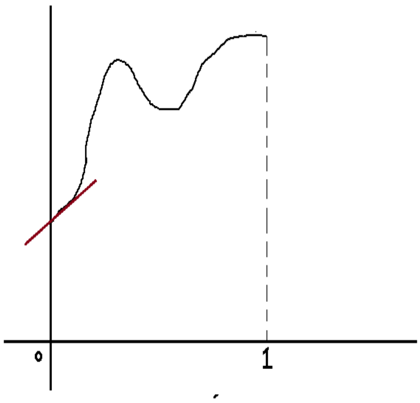
leads us to consider the equation

$$f'(u) = 0 \tag{1.1.3}$$

in E^*. If we consider a more general problem of finding a solution

$$u \in K \quad \text{such that} \quad f(u) \leq f(v) \quad \forall v \in K, \tag{1.1.4}$$

where K is a proper closed convex subset of E, then (1.1.3) may fail. Indeed, choose $K = [0, 1]$ and f as in the figure below



graph1

0 is a point where f achieves its minimum but $f'(0) \neq 0$.

Note: Its important to consider a closed set K in problem like (1.1.4) since on $(0,1)$, f does not achieve its minimum.

For problem (1.1.4), although a solution u may not satisfy (1.1.3), that is to say

$$\langle f'(u), v\rangle \;=\; 0 \quad \forall v \in K \tag{1.1.5}$$

(where $\langle ., .\rangle$ denotes the duality pairing between elements of E and E^*), but we can prove that u satisfies

$$\langle f'(u), v-u\rangle \geq 0, \quad \forall v \in K. \tag{1.1.6}$$

Indeed, for any $v \in K and$ t $\in [0,1]$, let $\gamma(t) = f(u + t(v-u))$,

Note: $u + t(v-u) \in K$ for any $t \in [0,1], u, v \in K$.

Now, since f is differentiable, then γ is differentiable and so, its Tailor expansion of the first order around $t_0 = 0$ is

$$\gamma(t) = \gamma(t_0) - \gamma'(t_0)(t-t_0) + o(t) = \gamma(0) + \gamma'(0)t + o(t), \quad \text{where} \quad o(t) \to 0 \quad \text{as} \quad t \to 0. \tag{1.1.7}$$

which implies

$$\gamma(t) - \gamma(0) = \gamma'(0)t + o(t) \quad \text{i.e.,} \quad \gamma(t) - \gamma(0) = t\langle f'(u), v-u\rangle + o(t). \tag{1.1.8}$$

But from (1.1.4), we have $\gamma(t) \geq \gamma(0), \quad \forall t \in [0,1]$. Hence,

$$0 \leq \lim_{t\to} \frac{\gamma(t) - \gamma(0)}{t} = \langle f'(u), v-u\rangle \;, \quad \forall v \in K$$

and we call (1.1.6) a variational inequality (VI) with respect to the function f and the set K. Moreover, if we assume that f is convex, then (1.1.6) characterizes exactly the points of K for which (1.1.4) holds. Indeed, in this case, we have (since γ is convex),

$$\gamma(1) \geq \gamma(0) - \gamma'(0) \Leftrightarrow f(v) \geq f(u) + \langle f'(u), v-u\rangle \geq f(u) \quad \forall v \in K. \tag{1.1.9}$$

We can summarize the above result in the following proposition

Proposition 1.1.1 *Let $f : E \to \mathbb{R}$ be convex and differentiable and K be a closed convex set of E, then the problem of finding*

$$u \in K : \quad f(u) \leq f(v) \quad \forall v \in K$$

is equivalent to the problem of finding

$$u \in K : \quad \langle f'(u), v-u\rangle \geq 0 \;, \quad \forall v \in K.$$

We shall now consider some problems connected to Variational inequalities. We first consider the connection between Variational inequality (1.1.1) and the following heat equation at steady state which requires us to find $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1.10}$$

where Ω is open, smooth domain in $\mathbb{R}^n$, $g \in H^1(\Omega)$. $f \in H^{-1}(\Omega)$ and $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$.
Suppose (1.1.10) has a solution, say, $u \in H^1(\Omega)$. Then, $-\Delta u = f$ and $u = g$ on $\partial\Omega$ (*i.e.*, $u - g \in H_0^1(\Omega)$)). So we have

$$\int_\Omega -\Delta u.v = \int_\Omega f.v, \quad \forall v \in H_0^1(\Omega).$$

By Regularity theorem; we have that $u \in H^2(\Omega)$. Thus, by Greens theorem, we have

$$\int_\Omega \nabla u.\nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial n}.v = \int_\Omega f.v, \text{ where } \frac{\partial u}{\partial n}(x) = \nabla u(x).\vec{n}(x) \quad \text{So, we have}$$

.

$$\int_\Omega \nabla u.\nabla v = \int_\Omega f.v, \quad \forall v \in H_0^1(\Omega). \tag{1.1.11}$$

Define $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by $a(u, v) = \int_\Omega \nabla u.\nabla v$ and $f : H_0^1(\Omega) \to \mathbb{R}$ by $\langle f, v\rangle = \int_\Omega fv$. Then (1.1.11) becomes

$$a(u, v) = \langle f, v\rangle, \ \forall v \in H_0^1(\Omega). \tag{1.1.12}$$

Define

$$\mathbb{K} = \{v \in H^1(\Omega) : v - g \in H_0^1(\Omega)\}.$$

Then (1.1.12) is equivalent to the following variational inequality

$$a(u, v - u) \geq \langle f, v - u\rangle \quad \forall v \in \mathbb{K}. \tag{1.1.13}$$

Indeed, for any $v \in \mathbb{K}$, $v \in H^1(\Omega)$ and $v - g \in H_0^1(\Omega)$. Also, u being a solution to (1.1.10) implies $u \in H^1(\Omega)$ and $u - g \in H_0^1(\Omega)$. So, $v - u \in H^1$ and $(v - g) - (u - g) = v - u \in H_0^1(\Omega)$. Therefore, from (1.1.12), we have

$$a(u, v - u) \geq \langle f, v - u\rangle \ \ \forall v \in \mathbb{K}.$$

To show the other way round, suppose $\exists u \in \mathbb{K}$ such that

$$a(u, v - u) \geq \langle f, v - u\rangle, \ \ \forall v \in \mathbb{K}.$$

Obviously, the boundary condition of (1.1.10) is satisfied (since $u \in \mathbb{K}$). Now, take $v \in H_0^1(\Omega)$, then $u + v \in \mathbb{K}$ since $u \in \mathbb{K}$ (this follows from the fact that $v = 0$ on $\partial\Omega$ and $u - g = 0$ on $\partial\Omega$, which yields $u + v - g = 0$ on $\partial\Omega$). So, we have from (1.1.13) that

$$a(u, v) \geq \langle f, v\rangle \quad \forall v \in H_0^1. \tag{1.1.14}$$

Similarly, since $-v + u \in \mathbb{K}$, then

$$a(u, v) \leq \langle f, v\rangle, \ \ \forall v \in H_0^1(\Omega). \tag{1.1.15}$$

Combining (1.1.14) and (1.1.15) yields

$$a(u, v) = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega).$$

Finally, we consider the following optimization problem

$$\min\{\varphi(x) : x \in K\},$$

i.e., find $x \in K$ such that

$$\varphi(x) \leq \varphi(y), \quad \forall y \in K, \tag{1.1.16}$$

where $\varphi : K \subset E \to \bar{\mathbb{R}}$ is a proper sub-differentiable function such that $D_\varphi : E \to E^*$ is hemi-continuous. Hence, D_φ represents a single valued section of the multivalued sub-differential $\partial\varphi : E \to 2^{E^*}$. It is easy to see that (1.1) is equivalent to

$$x \in K \quad \text{such that} \quad \langle D_\varphi(x), y - x\rangle \geq 0 \quad \forall y \in K. \tag{1.1.17}$$

We note that

$$x \;\; solves \;\; (1.1.16) \quad \text{if and only if} \quad x \in \bigcap_{y \in K} A_y,$$

where

$$A_y = \{z \in K : \varphi(z) \leq \varphi(y)\}$$

is the level set with respect to y. Thus, the set of solution of (1.1.16) is closed and convex.

Consider the linearized version (weak version) of (1.1.17) which is to find

$$x \in E \quad \text{such that} \quad \langle D_\varphi(y), y - x\rangle \geq 0 \quad \forall y \in K. \tag{1.1.18}$$

Now, suppose (1.1.16) has a solution x, this implies that $0 \in \partial\varphi(x)$ and by monotonicity of $\partial\varphi$; we have

$$\langle D_{\varphi(y)} - D_{\varphi(x)}, y - x\rangle \geq 0 \;\; \forall x, y \in K. \tag{1.1.19}$$

So, (1.1.18) holds. Also, by setting $y_t = (1-t)x + ty$ in (1.1.18) in place of y, we see that

$$\langle D_\varphi(y_t), y_t - x\rangle = t\langle D_\varphi(y_t), y - x\rangle \geq 0 \;\; \forall y \in K, \tag{1.1.20}$$

which implies

$$\langle D_\varphi(ty + (1-t)x), y - x\rangle \geq 0 \tag{1.1.21}$$

and so, (1.1.17) holds as $t \to 0$ in (1.1.21), assuming that D_φ is hemi-continuous. Finally, (1.1.17) implies (1.1.16) as a consequence of the sub-differential inequality. So far, we summarized the what we have done above in the following theorem:

Theorem 1.1.2 *Let $\varphi : K \to \bar{\mathbb{R}}$ be proper, sub-differentiable function such that $D_\varphi : K \to X^*$ is hemi-continuous. Then the formulation* (1.1.16), (1.1.17) *and* (1.1.18) *are mutually equivalent.*

Consider also the fixed point problem

$$x = P_K(I - rD_\varphi)x, \quad r > 0. \tag{1.1.22}$$

If φ is proper and sub-differentiable function, then by the property of projection, P_K, it can be shown that (1.1.17) and (1.1.22) are equivalent. Next, we remark that the fixed point formulation suggests that the iterative algorithm

$$x_{n+1} = P_K(x_n - rD\varphi(x_n)) \tag{1.1.23}$$

may lead to approximation of solution of (1.1.17).

Variational inequality problems plays a crucial role in other areas of research including economics, mathematical modeling, transportation, to mention a but few (see [19], [8], [22], [9], [15]). Many algorithms have been proposed for solving (1.1.1)(see e.g, [11], [16]). One of the early methods for solving (1.1.1) is the extragradient method introduced in 1976 by Korpelevich [12] in a finite dimensional Euclidean space $\mathbb{R}^n$ which is given by

$$\begin{cases} x \in \mathbb{R}^n \\ y_n = P_C(x_n - \lambda A(x_n)) \\ x_{n+1} = P_C(x_n - \lambda A(y_n)), \quad \forall n \geq 1; \end{cases} \tag{1.1.24}$$

where the operator A was assumed to be monotone and L-Lipschitz and $\lambda \in (0, \frac{1}{L})$. The extragradient method (1.1.24) requires the computation of two orthogonal projection on to a closed convex subset C in each iteration process and so, affects the efficiency of the method and limits its application. To remedy this problem, [Censor et.all, 2011], proposed the "subgradient extragradient" method which replaces the second projection on to the closed convex subset C with a projection onto a half space. Moreover, the weak convergence of the method was obtained under mild assumptions and later on, the strong convergence was also obtained by Kraikaew and saejung, 2014 (see [5] and the references therein), both in a real Hilbert space H. The extragradient method have been modified in different ways by many Authors (see [2], [7], [17], [29], [30]). One of these modifications is that of Tseng (see [25]) proposed in Hilbert space H given below:

$$\begin{cases} x \in H \\ y_n = P_C(x_n - \lambda A(x_n)) \\ x_{n+1} = y_n - \lambda(Ay_n - Ax_n), \quad \forall n \geq 1; \end{cases} \tag{1.1.25}$$

which is more efficient than the subgradient extragradient method proposed by [Censor et.all, 2011] and many of its modifications because it requires neither the computation of two projections onto a closed convex subset C nor computation of projection on to a half space. The extended form of (1.1.25) in 2-uniformly convex uniformly smooth Banach space given by Yekini (see [28]) is:

$$\begin{cases} x_1 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)), \\ x_{n+1} = J^{-1}[Jy_n - \lambda_n(Ay_n - Ax_n)], \quad \forall n \geq 1, \end{cases} \tag{1.1.26}$$

where $A : E \to E^*$ is monotone and L-Lipschitz and $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of step-sizes satisfying

$$0 < a \leq \lambda_n \leq b < \frac{1}{\sqrt{2\mu\kappa L}},$$

(where μ is the 2-uniform convexity constant of E; κ is the 2-uniform smoothness constant of E^*); C is nonempty, closed and convex subset of E and J is the normalized duality mapping.

In this work, a one step recursion of (1.1.26) with less computational cost is proposed. We obtained its weak convergence under the assumption that J is weakly sequentially continuous in a 2-uniformly convex Banach spaces.

Chapter 2

Preliminaries

In this chapter, we give basic definitions of some terms and some fundamental results used in proving our main result.

2.1 Banach Spaces Geometry

Throughout this work, we shall denote by $\longrightarrow$, $\rightharpoonup$ and $\rightharpoonup^*$ the strong, weak and *weak** convergence respectively. E denotes a real normed linear space, S_E and B_E denote the unit sphere and the closed unit ball of E, respectively.

2.1.1 Uniform Convexity, Strict Convexity and Reflexivity

Definition 2.1.1 *A normed space X is called uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$ such that if x,$y \in X$ with $\|x\| = 1$, $\|y\| = 1$ and $\|x - y\| \geq \epsilon$, then*

$$\|\frac{1}{2}(x + y)\| \leq 1 - \delta.$$

Example 2.1.2 *Every Hilbert space is uniformly convex.*

Proof *Let $\epsilon \in (0, 2]$. Let $x, y \in H :$ $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \epsilon$. We find $\delta > 0$ such that $\|\frac{1}{2}(x+y)\| \leq 1-\delta$. Using paralellograms identity, we have:*

$$\|\frac{1}{2}(x + y)\|^2 = \frac{1}{2}(\|x\|^2 + \|y\|^2) - \frac{1}{4}\|x - y\|^2 \leq 1 - \frac{1}{4}\epsilon^2$$

so that

$$\|\frac{1}{2}(x + y)\| \leq \sqrt{1 - \frac{1}{4}\epsilon^2} = 1 - \Big(1 - \sqrt{1 - \frac{1}{4}\epsilon^2}\Big).$$

Choose $\delta = 1 - \sqrt{1 - \frac{1}{4}\epsilon^2}$, then the proof is complete.

Another example of uniformly convex space is l_p, for $(1 < p < \infty)$.

Proposition 2.1.3 *Let X be a uniformly convex space and let $\alpha \in (0, 1)$ and $\epsilon > 0$. Then for any $d > 0$, if $x, y \in X$ are such that $\|x\| \leq d$, $\|y\| \leq d$, $\|x + y\| \geq \epsilon$, then there exists $\delta = \delta\Big(\frac{\epsilon}{d}\Big) > 0$ such that*

$$\|\alpha x + (1 - \alpha y)\| \leq \Big[1 - 2\delta\Big(\frac{\epsilon}{d}\Big)min\{\alpha, 1 - \alpha\}\Big]d.$$

Definition 2.1.4 *A normed space E is called strictly convex if for all $x, y \in E$, $x \neq y$, $\|x\| = \|y\| = 1$, we have*

$$\|\lambda x + (1-\lambda)y\| < 1 \quad \forall \lambda \in (0,1).$$

Proposition 2.1.5 *Every uniformly convex space is strictly convex.*

Proof Suppose E is uniformly convex. let $x, y \in E$ such that $x \neq y$ and take $\epsilon = \|x - y\| > 0$, and $d = 1$, using proposition (2.1.3), we have that for any $\lambda \in (0,1)$, $\|\lambda x + (1-\lambda)y\| < 1$.

Remark 2.1.6 *Examples of spaces that are not strictly convex are L_1, L_∞, C_0.Also, some examples of spaces that are not uniformly convex include $(\mathbb{R}^2, \ \|.\|_1)$ and $(R^2, \ \|.\|_\infty)$*

Definition 2.1.7 *A normed space E is called reflexive iff the cannonical embedding J is surjective, i.e., if $J(E) = E^{**}$, where J is defined from $J : E \to E^{**}$, $x \mapsto Jx$ $(Jx \in E^{**}, i.e., \ \ Jx : E^* \to \mathbb{R}, \ x$ fixed) defined by $\langle Jx, f\rangle = \langle f, x\rangle$.*

Remark 2.1.8 *One of the importance of uniformly convex spaces is that, it gives us a class of spaces that are reflexive. With regards to this, we shall consider an interesting result about uniform convexity called the $Milman - Petti's$ theorem.*

Theorem 2.1.9 (Milman-Petti's) *Every Uniformly convex Banach space is reflexive*

Definition 2.1.10 *Let E be a normed space with dim $E \geq 2$. The modulus of convexity of E , $\delta_E : [\,0,2] \to [\,0,2]$ is defined by*

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \ x, y \in B_E, \ \|x - y\| \geq \epsilon\right\}. \tag{2.1.1}$$

If E is an inner product space, then

$$\delta_E(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}}.$$

2.1.2 p-Uniformly Convex Spaces

Definition 2.1.11 *Let $p > 1$. A normed linear space E of dimension ≥ 2 is said to be p-uniformly convex if there exists $c > 0$ such that $\delta_E \geq c\epsilon^p$. In particular, if $p = 2$, then E is called 2-uniformly convex space.*

Example 2.1.12 *L_p spaces are p-uniformly convex Banach spaces, for $(1 < p < \infty)$. Indeed, for $1 < p < 2$, $\delta_E(\epsilon) \geq \frac{1}{2^{p+1}}\epsilon^2$ and for $2 \leq p < \infty$, $\delta_E(\epsilon) \geq \epsilon^p$.*

Remark 2.1.13 *It is obvious that every 2-uniformly convex Banach space is uniformly convex. It is known that all Hilbert spaces are 2-uniformly convex. It is also known that all the Lebesgue spaces L_p are 2-uniformly convex whenever $1 < p \leq 2$.*

Next, we consider some characterizations of $p-$uniformly convex Banach space.

Lemma 2.1.14 *let X be a real Banach space. Then $\delta_X \geq c\epsilon^p$ iff*

$$\exists c > 0 : \ \frac{1}{2}\Big(\|x+y\|^p + \|x-y\|^p\Big) \geq \|x\|^p + c\|y\|^p \quad \forall x, y \in X.$$

2.1.3 Smooth, Uniformly Smooth and $q-$ Uniformly Smooth Spaces

Definition 2.1.15 *The space E is said to be smooth if*

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \quad \text{exists for all } x,y \in S_E. \tag{2.1.2}$$

Definition 2.1.16 *The space E is also said to be uniformly smooth if*

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \text{ converges uniformly for } x,y\in S_E.$$

The modulus of smoothness of E, $\rho_E:[0,\infty)\to[0,\infty)$ is defined by

$$\rho_E(t):=\sup\left\{\frac{\|x+y\|+\|x-y\|}{2}-1 : x\in S_E, \|y\|=t\right\}. \tag{2.1.3}$$

Proposition 2.1.17 *Let E be a normed space with $dimE\geq 2$. Then*

$$\begin{aligned}\delta_E(\epsilon)&:=\inf\left\{1-\left\|\frac{x+y}{2}\right\| :\ x,y\in B_E,\ \|x-y\|\geq\epsilon\right\}\\ &=\inf\left\{1-\left\|\frac{x+y}{2}\right\| :\ x,y\in B_E,\ \|x-y\|=\epsilon\right\}.\end{aligned}$$

Remark 2.1.18 *We note that $\rho_E(0)=0$. Moreover, it can be proved that the modulus of convexity of a normed space E, δ_E is a convex and continuous function.*

Definition 2.1.19 *A space E is said to be q- uniformly smooth if there exists $c>0$ such that $\rho_E(t)\leq ct^q$. In particular, if $q=2$, then E is said to be 2-uniformly smooth space.*

Remark 2.1.20 *It is well known that every 2-uniformly smooth space is uniformly smooth. It is known that all Hilbert spaces are uniformly smooth. It is also known that all the Lebesgue spaces L_p are uniformly smooth for $1<p\leq\infty$.*

2.2 Duality Mappings

Definition 2.2.1 *Let ψ be a guage function, i.e., $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is continuous, strictly increasing with $\psi(0)=0$ and ${}_{t\to\infty}\psi(t)=\infty$, then the generalized duality mapping $J_\psi:E\to 2^{E^*}$ defined by*

$$J_\psi(x):=\left\{x^*\in E^*:\langle x^*,x\rangle=\|x\|\|x^*\| \text{ and } \psi(\|x\|)=\|x^*\|\right\}$$

where E is any normed linear space. In particular, if $\psi(t)=t$, the generalized duality map $J_\psi=J$ is called the normalized duality map. So, the normalized duality mapping of E into 2^{E^*} is defined by

$$Jx:=\{x^*\in E^*:\langle x^*,x\rangle=\|x^*\|^2=\|x\|^2\} \text{ for all } x\in E.$$

$J_\psi(x)$ is not empty for every $x\in E$. Indeed, if $x=0$, then choose $u^*=0\in E^*$. Now, for $x\neq 0$, then $x\psi(\|x\|)\neq 0$. So, by Hahn Banach theorem,

$$\exists x^*\in E^* \ :\ \|x^*\|=1 \quad \text{and} \quad \langle x^*, x\psi(\|x\|)\rangle=\|x\|\psi(\|x\|).$$

We note that $\|\psi(\|x\|)x^*\|=\psi(\|x\|)$ and $\langle x,\psi(\|x\|)x^*\rangle=\langle x\psi(\|x\|),x^*\rangle=\|x\|\psi(\|x\|)=\|x\|\|\psi(\|x\|)x^*\|$. So, take $u:=\psi(\|x\|)x^*$. Then, $u\in J_\psi(x)$ and therefore, $J_\psi(x)$ is not empty for all $x\in E$.

Lemma 2.2.2 *Let E be a real uniformly smooth Banach space with its dual E^*. Then the normalized duality map $J : E \to E^*$ is norm-to-norm uniformly continuous on bounded subsets of X.*

Lemma 2.2.3 *[27] Let E be a uniformly convex Banach space. Then, the normalized duality mapping, J, is uniformly monotone on every bounded set. That is, for every $R > 0$ and arbitrary $x, y \in E$ with $\|x\| \leq R$ and $\|y\| \leq R$ there exists a real non-negative and continuous function $\psi_R : [0, \infty) \to [0, \infty)$ such that $\psi_R(t) > 0$ for $t > 0$, $\psi_R(0) = 0$ and*

$$\langle Jx - Jy, x - y\rangle \geq \psi_R(\|x - y\|).$$

Remark 2.2.4 *Its known that if E is reflexive and strictly convex with the strictly convex dual space E^*, then J is single valued, one-to-one and onto mapping. In this case, we can define the single-valued mapping $J^{-1} : E^* \to E$ and we have $J^{-1} = J^*$, where J^* is the normalized duality mapping on E^**

2.3 Monotone Operators, Maximal Monotone Operators and Resolvent Operators

2.3.1 Monotone Operators

Definition 2.3.1 *Let E be a real normed space. A map $A : E \to 2^{E^*}$ is called monotone if for each $x, y \in E$,*

$$\big\langle u - v, x - y\big\rangle \geq 0, \quad \forall u \in Ax, \quad v \in Ay.$$

Example 2.3.2 *Let C be nonempty, closed and convex subset of E. A normal cone to C, $N_C : E \to 2^{E^*}$ defined as*

$$N_C(x) = \begin{cases} \{u \in E^* : \langle u, c - x\rangle \leq 0, \forall c \in C\}, \ if \ x \in C, \\ \\ \emptyset, \qquad\qquad\qquad\qquad\qquad\qquad , \ if \ x \notin C; \end{cases} \tag{2.3.1}$$

is monotone.

Proof .

let $(x, y) \in \ E \times E$, $u \in N_C(x)$, $\ v \in N_C(y)$. Then, we have, by definition of N_C, that

$$\langle u, c - x\rangle \leq 0 \ \ \forall c \in C \ \ \text{and} \ \ \langle v, c - y\rangle \leq 0 \ \ \forall c \in C \tag{2.3.2}$$

Which is equivalent to

$$\langle u, x - c\rangle \geq 0 \ \ \forall c \in C \quad \text{and} \ \ \langle -v, c - y\rangle \geq 0 \ \ \forall c \in C. \tag{2.3.3}$$

Adding up the above two inequalities yields $\langle u - v, x - y\rangle \geq 0 \ \ \forall c \in C$. Thus, N_C is monotone. .

Remark 2.3.3 *Other examples of multivalued operators which are monotone include: the normalized duality map, which was discussed in the previous section and subdifferential operator.*

In what follows, we shall give definition of single valued monotone operators and consider some example(s).

Definition 2.3.4 *If A is single valued, the map $A : E \to E^*$ is called*

(a) monotone if $\big\langle Ax - Ay, x - y\big\rangle \geq 0, \ \ \forall \ \ x, y \in E$.

(b) β-strongly monotone if $\exists\ \beta \in (0, 1)$: $\langle Ax - Ay, x - y\rangle \geq \beta\|x - y\|^2, \ \ \forall x, y \in E$.

(c) coercive if there exists $\delta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \delta\|Ax - Ay\|^2 \quad \forall x, y \in X;$$

(d) Lipschitz continuous if $\exists\ L > 0$: $\|Ax - Ay\| \leq L\|x - y\| \quad \forall \ x, y \in E$.

Remark 2.3.5 *We note that every coercive operator is Lipschitz.*

Next, we consider some examples of monotone operator in Hilbert space.

Example 2.3.6 *Let E be a Hilbert space, then the projection operator $P_C : H \to C$ defined by $P_C(x) = x_0$, is monotone.*

Proof *Let H be a Hilbert space and let $x, y \in H$. Let C be a non-empty closed convex subset of H. Define the projection map $P_C : H \to C$ by $P_C(x) = x_0$, then we have $\langle x_0 - x, c - x_0\rangle \geq 0 \ \ \forall c \in C$. Since $P_C(y) \in C$, we have*

$$\langle P_C(x) - x, P_C(y) - P_C(x)\rangle \geq 0. \tag{2.3.4}$$

Also, using the characterization of $P_C(y)$ we have

$$\langle y - P_C(y), P_C(x) - P_C(y)\rangle \geq 0. \tag{2.3.5}$$

Adding up (2.3.4) and (2.3.5) yields

$$\Big\langle P_C(y) - P_C(x), y - x\Big\rangle \geq \|P_C(x) - P_C(y)\|^2 \geq 0$$

and so, the result follows.

2.3.2 Maximal Monotone Operators

Definition 2.3.7 *A multivalued operator $A : E \to 2^{E*}$ is called maximally monotone if given any monotone operator, say, $B : E \to 2^{E^*}$ with $G(A) \subseteq G(B)$, then $G(A) = G(B)$, where $G(A) = \{(x, x^*) : x^* \in Ax\}$ is the graph of A.*

Lemma 2.3.8 *A subdifferential $\partial\varphi : E \to 2^E$ of a proper convex lower semicontinuous functional $\varphi : X \to \mathbb{R}$ is a maximal monotone operator.*

Example 2.3.9 *The normal cone to C, N_C, defined in example (2.3.2) is maximally monotone.*

Proof

Recall that the subdifferential of a function, say, $f : E \to 2^{E*}$, where E is a real Banach space is given by

$$\partial f := \{x^* \in E^* : f(y) - f(x) \geq \langle x^*, y - x\rangle, \quad \forall y \in E\}.$$

Also, we define as follows, the indicator function $l_C : E \to \bar{\mathbb{R}}$ by

$$l_C(x) \begin{cases} 0, & x \in C \\ +\infty, & x \notin C. \end{cases} \quad \text{where } C \text{ is nonempty closed convex subset of } E \tag{2.3.6}$$

So, $\partial l_C(x) := \{u \in E^* : l_C(y) - l_C(x) \geq \langle u, y - x\rangle, \quad \forall y \in E\}$. We note that if $x \in C$, then

$$\begin{aligned} \partial l_C \quad &:= \{u \in E^* : l_C(y) \geq \langle u, y - x\rangle, \quad \forall y \in E\} \\ &= \{u \in E^* : l_C(y) \geq \langle u, y - x\rangle, \quad y \in E\} \\ &= \{u \in E^* : 0 \geq \langle u, y - x\rangle, \quad \forall y \in E\} \\ &= N_C(x), \quad x \in C. \end{aligned}$$

However, for $x \notin C$, *then*

$$\partial l_C(x) := \{u \in E^* : l_C(y) - l_C(x) \geq \langle u, y - x\rangle, \quad \forall y \in E\} = \emptyset = N_C(x), \quad x \notin C.$$

Therefore, $\forall x \in E$, $\partial l_C(x) = N_C(x)$. *But it is known that* l_C *is proper, convex, lower semi-continuous (since* C *is closed). Therefore, by lemma*(2.3.8), ∂l_C *is maximally monotone. So,* N_C *is maximally monotone.*

2.3.3 Resolvent Operator

Definition 2.3.10 *Let* E *be a real reflexive, strictly convex and smooth.* $A : E \to 2^{E^*}$ *be a maximal monotone operator. Then for each* $\lambda > 0$, *the resolvent of* A, $J_\lambda^A : E \to E$ *is defined by*

$$J_\lambda^A(x) = (J + \lambda A)^{-1} Jx,$$

where J *is the normalized duality mapping on* E.

Proposition 2.3.11 *Let* E *be a real reflexive, strictly convex with strictly convex dual and let* $A : E \to 2^{E^*}$ *be a maximal monotone operator. Then* $A^{-1}(0) = F(J_\lambda^A)$ *for all* $\lambda > 0$, *where* $F(J_\lambda^A)$ *denotes the set of fixed points of* J_A.

Proof *Let* $\lambda > 0$. *Now,*

$$x \in A^{-1}(0) \Leftrightarrow 0 \in Ax \Leftrightarrow 0 \in \lambda Ax \Leftrightarrow Jx \in (\lambda A + J)x \Leftrightarrow (\lambda A + J)^{-1} \circ Jx = x \Leftrightarrow J_\lambda^A x = x. \tag{2.3.7}$$

Hence, $x \in A^{-1}(0) \;\Leftrightarrow\; Jx = x$. *The proof is complete.*

2.4 Lyapunov Function and Generalized Projection Operator

2.4.1 Lyapunov Function

Definition 2.4.1 *Let* E *be a smooth real Banach space with dual* E^*. *The functional,* $\phi : E \times E \to \mathbb{R}$, *defined by*

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \text{ for } x,y \in E, \tag{2.4.1}$$

is called Lyapunov function, where J *is the normalized duality mapping on* E. *Its clear that*

$$\phi(x, y) \geq (\|x\| - \|y\|)^2.$$

Lemma 2.4.2 *[3, 20] Let* E *be a real uniformly convex, smooth Banach space. Then, the following identities hold:*

(i) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E.$

(ii) $\phi(x, y) + \phi(y, x) = 2\langle x - y, Jx - Jy\rangle, \quad \forall x, y \in E.$

Lemma 2.4.3 *[20] Let* E *be a real* 2*-uniformly convex Banach space. Then, there exists* $\mu \geq 1$ *such that*

$$\frac{1}{\mu}\|x - y\|^2 \leq \phi(x, y) \quad \forall x, y \in X.$$

Lemma 2.4.4 *[24] Let* E *be a 2-uniformly convex and uniformly smooth Banach space, and* $C \neq \emptyset$, *closed, convex, subset of* E. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequence of* E. *If* $\lim_{n\to\infty} \phi(x_n, y_n) = 0$ *and either* $\{x_n\}$ *or* $\{y_n\}$ *is bounded, then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

2.4.2 Generalized Projection Operator

Definition 2.4.5 *Let C nonempty closed convex subset of a real uniformly convex, Banach Space. The Operator defined by $\Pi_C(z) = \bar{z}$ where $\bar{z}$ is the solution of the minimization problem $\inf_{y \in C} \phi(y, z)$ is called Generalized projection operator. The point $\bar{z} \in C$ is called a Generalized Projection of the point z onto C.*

Lemma 2.4.6 *(a) The point $\bar{z} \in C$ is a generalized projection of z on C iff*

$$\langle w - \bar{z}, J\bar{z} - Jz \rangle \leq 0, \quad \forall w \in C.$$

(b) The operator Π_C generates an optimal $z \in E$ relative to the functional $\phi(w, z)$, i.e.,

$$\phi(w, \bar{z}) \leq \phi(w, z) - \phi(\bar{z}, z).$$

Chapter 3

3.1 Main Result

In this section, we state and prove a weak convergence result for single projection algorithm for variational inequality problems in 2-uniformly convex, uniformly smooth real Banach spaces. We also provide some fundamental results which shall be used in the proof of our main result. We begin by establishing the boundedness of our scheme.

Theorem 3.1.1 *Let E be a 2-uniformly convex and uniformly smooth Banach space and C be nonempty closed convex subset of E. Let $A : E \to E^*$ be monotone and Lipschitz with Lipschtz constant L. Let $x_0, x_{-1} \in E$ and define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n - \lambda_{n-1}(Ax_n - Ax_{n-1})), \quad n \geq 0, \tag{3.1.1}$$

where $\{\lambda_n\} \subseteq \left[\epsilon, \frac{1-2\epsilon}{2\mu L}\right]$ for some $\epsilon > 0$, $n \geq 0$ and $\mu \geq 1$. Suppose Γ is nonempty, then the sequence $\{x_n\}$ is bounded.

Proof *Let $x^* \in \Gamma$, then*

$$\langle Ax^*, z - x^* \rangle \geq 0, \quad \forall z \in C. \tag{3.1.2}$$

From (3.1.1), *using lemma 2.4.6 (a), we have*

$$\langle Jx_{n+1} - Jx_n + \lambda_n Ax_n + \lambda_{n-1}(Ax_n - Ax_{n-1}), z - x_{n+1} \rangle \geq 0 \quad \forall \ z \in C. \tag{3.1.3}$$

Adding inequalities (3.1.2) *and* (3.1.3), *we obtain*

$$\langle Jx_{n+1} - Jx_n + \lambda_n(Ax_n - Ax^*) + \lambda_{n-1}(Ax_n - Ax_{n-1}), x^* - x_{n+1} \rangle \geq 0. \tag{3.1.4}$$

Now, by Lemma 2.4.2 (i), we have

$$2\langle Jx_{n+1} - Jx_n, x^* - x_{n+1} \rangle = \phi(x^*, x_n) - \phi(x^*, x_{n+1}) - \phi(x_{n+1}, x_n). \tag{3.1.5}$$

Also

$$\langle Ax_n - Ax^*, x^* - x_{n+1} \rangle = \langle Ax_{n+1} - Ax^*, x^* - x_{n+1} \rangle + \langle Ax_n - Ax_{n+1}, x^* - x_{n+1} \rangle \tag{3.1.6}$$

and

$$\langle Ax_n - Ax_{n-1}, x^* - x_{n+1} \rangle = \langle Ax_n - Ax_{n-1}, x^* - x_n \rangle + \langle Ax_n - Ax_{n-1}, x_n - x_{n+1} \rangle. \tag{3.1.7}$$

Substituting (3.1.5), (3.1.6) *and* (3.1.7) *in* (3.1.4) *we have*

$$\begin{aligned}
0 &\leq \langle Jx_{n+1} - Jx_n +_n (Ax_n - Ax^*) + \lambda_{n-1}(Ax_n - Ax_{n-1}), x^* - x_{n+1} \rangle \\
&\leq 2\langle Jx_{n+1} - Jx_n, x^* - x_{n+1} \rangle + 2\lambda_n \langle Ax_n - Ax^*, x^* - x_{n+1} \rangle + 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x^* - x_{n+1} \rangle \\
&= \phi(x^*, x_n) - \phi(x^*, x_{n+1}) - \phi(x_{n+1}, x_n) + 2\lambda_n \langle Ax_{n+1} - Ax^*, x^* - x_{n+1} \rangle + 2\lambda_n \langle Ax_n - Ax_{n+1}, x^* - x_{n+1} \rangle \\
&+ 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x^* - x_n \rangle + 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x_n - x_{n+1} \rangle.
\end{aligned} \tag{3.1.8}$$

Rearranging (3.1.8), *we obtain*

$$\begin{aligned}
&\phi(x^*, x_{n+1}) + \phi(x_{n+1}, x_n) + 2\lambda_n\langle Ax_{n+1} - Ax_n, x^* - x_{n+1}\rangle \\
&\leq \phi(x^*, x_n) + 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x^* - x_n\rangle + 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x_n - x_{n+1}\rangle \\
&+2\lambda_n\langle Ax_{n+1} - Ax^*, x^* - x_{n+1}\rangle \\
&\leq \phi(x^*, x_n) + 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x^* - x_n\rangle + 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x_n - x_{n+1}\rangle.
\end{aligned} \tag{3.1.9}$$

We have, in the last inequality, used the fact that A is Monotone. Using the Lipschitz property of A, we have

$$\begin{aligned}
2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x_n - x_{n+1}\rangle &\leq 2\lambda_{n-1}\|Ax_n - Ax_{n-1}\|\|x_n - x_{n+1}\| \\
&\leq 2\lambda_{n-1}L\|x_n - x_{n-1}\|\|x_n - x_{n+1}\| \\
&\leq \lambda_{n-1}L(\|x_n - x_{n-1}\|^2 + \|x_n - x_{n+1}\|^2).
\end{aligned} \tag{3.1.10}$$

Using Lemma (2.4.3), *we have*

$$\begin{aligned}
2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x_n - x_{n+1}\rangle &\leq \lambda_{n-1}L(\|x_n - x_{n-1}\|^2 + \|x_n - x_{n+1}\|^2) \\
&\leq \lambda_{n-1}L\mu(\phi(x_n, x_{n-1}) + \phi(x_n, x_{n+1})).
\end{aligned} \tag{3.1.11}$$

Substituting (3.1.11) *in* (3.1.9) *we have*

$$\begin{aligned}
&\phi(x^*, x_{n+1}) + 2\lambda_n\langle Ax_{n+1} - Ax_n, x^* - x_{n+1}\rangle + \phi(x_{n+1}, x_n) \\
&\leq \phi(x^*, x_n) + 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x^* - x_n\rangle + \lambda_{n-1}\mu L(\phi(x_n, x_{n-1}) + \phi(x_{n+1}, x_n)).
\end{aligned} \tag{3.1.12}$$

Rearranging, we have:

$$\begin{aligned}
&\phi(x^*, x_{n+1}) + 2\lambda_n\langle Ax_{n+1} - Ax_n, x^* - x_{n+1}\rangle + (1 - \lambda_{n-1}\mu L)\phi(x_{n+1}, x_n) \\
&\leq \phi(x^*, x_n) + 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x^* - x_n\rangle + \lambda_{n-1}\mu L\phi(x_n, x_{n-1}).
\end{aligned} \tag{3.1.13}$$

Now, $\lambda_n \in \left[\epsilon, \frac{1-2\epsilon}{2\mu L}\right]$ *implies* $\epsilon + \frac{1}{2} \leq 1 - \lambda_n\mu L$. *So,* (3.1.13) *becomes*

$$\phi(x^*, x_{n+1}) + 2\lambda_n\langle Ax_{n+1} - Ax_n, x^* - x_{n+1}\rangle + \left(\epsilon + \frac{1}{2}\right)\phi(x_{n+1}, x_n) \tag{3.1.14}$$

$$\leq \phi(x^*, x_n) + 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x^* - x_n\rangle + \frac{1}{2}\phi(x_n, x_{n-1}). \tag{3.1.15}$$

Thus, we have

$$\begin{aligned}
\epsilon\phi(x_{n+1}, x_n) &\leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x^* - x_n\rangle \\
&- 2\lambda_n\langle Ax_{n+1} - Ax_n, x^* - x_{n+1}\rangle + \frac{1}{2}\phi(x_n, x_{n-1}) - \frac{1}{2}\phi(x_{n+1}, x_n).
\end{aligned} \tag{3.1.16}$$

Hence, for each n, *we have*

$$\begin{aligned}
\epsilon\sum_{i=0}^{n}\phi(x_{i+1}, x_i) &\leq \phi(x^*, x_0) - \phi(x^*, x_{n+1}) + 2\lambda_{-1}\langle Ax_0 - Ax_{-1}, x^* - x_0\rangle \\
&- 2\lambda_n\langle Ax_{n+1} - Ax_n, x^* - x_{n+1}\rangle + \frac{1}{2}\phi(x_0, x_{-1}) - \frac{1}{2}\phi(x_{n+1}, x_n).
\end{aligned} \tag{3.1.17}$$

Rearranging (3.1.17), *we obtain*

$$\begin{aligned}
&\phi(x^*, x_{n+1}) + 2\lambda_n\langle Ax_{n+1} - Ax_n, x^* - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + \epsilon\sum_{i=0}^{n}\phi(x_{i+1}, x_i) \\
&\leq \phi(x^*, x_0) + 2\lambda_{-1}\langle Ax_0 - Ax_{-1}, x^* - x_0\rangle + \frac{1}{2}\phi(x_0, x_{-1}).
\end{aligned} \tag{3.1.18}$$

Again, using the Lipschitz property of A and Lemma 2.4.3 we obtain

$$\begin{aligned}2\lambda_n\langle Ax_{n+1}-Ax_n,x_{n+1}-x^*\rangle &\leq 2\lambda_n\|Ax_{n+1}-Ax_n\|\|x^*-x_{n+1}\|\\ &\leq 2\lambda_nL\|x_{n+1}-x_n\|\|x^*-x_{n+1}\|\\ &\leq \lambda_nL(\|x_{n+1}-x_n\|^2+\|x^*-x_{n+1}\|^2)\\ &\leq \lambda_nL\mu(\phi(x_{n+1},x_n)+\phi(x^*,x_{n+1})). \qquad (3.1.19)\end{aligned}$$

Set $K=\phi(x^*,x_0)+2\lambda_{-1}\langle Ax_0-Ax_{-1},x^*-x_0\rangle+\phi(x_0,x_{-1})$ and substitute (3.1.19) in (3.1.18), we obtain

$$\phi(x^*,x_{n+1})-\lambda_nL\mu(\phi(x_{n+1},x_n)+\phi(x^*,x_{n+1}))+\frac{1}{2}\phi(x_{n+1},x_n)+\epsilon\sum_{i=0}^{n}\phi(x_{i+1},x_i)\leq K, \quad i.e., \qquad (3.1.20)$$

$$(1-\lambda_n\mu L)\phi(x^*,x_{n+1})+(\frac{1}{2}-\lambda_n\mu L)\phi(x_{n+1},x_n)+\epsilon\sum_{i=0}^{n}\phi(x_{i+1},x_i)\leq K. \qquad (3.1.21)$$

But

$$\lambda_n\in\left[\epsilon,\frac{1-2\epsilon}{2\mu L}\right] \quad \textit{implies} \quad \frac{1}{2}<1-\lambda_n\mu L \quad \textit{and} \quad \frac{1}{2}-\lambda_n\mu L>0. \textit{ So, we have}$$

$$\frac{1}{2}\phi(x^*,x_{n+1})+\epsilon\sum_{i=0}^{n}\phi(x_{i+1},x_i)\leq K, \quad \forall\ n\geq 0. \qquad (3.1.22)$$

Therefore, the sequence $\{\phi(x^*,x_{n+1})\}$ is bounded. Consequently, $\{x_n\}$ is bounded.

Remark 3.1.2 *: We note that from (3.1.22), we have $\phi(x_{n+1},x_n)\longrightarrow 0$ as $n\longrightarrow\infty$ and by Lemma 2.4.4, we have*

$$\|x_{n+1}-x_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

Lipschitzness of A implies $\|Ax_n-Ax_{n-1}\|\longrightarrow 0$ as $n\longrightarrow\infty$.

Finally, since J is norm-to-norm uniformly continuous on each bounded subset of uniformly smooth spaces, then we have

$$\lim_{n\longrightarrow\infty}\|Jx_n-Jx_{n+1}\|=0.$$

Lemma 3.1.3 *Let C nonempty closed convex subset of E. $A:C\to E^*$ be continuous and monotone. Then,*

$$z\in\Gamma\Longleftrightarrow\langle Ax,x-z\rangle\geq 0 \quad \forall x\in C.$$

Proof *Suppose $z\in\Gamma$, then $\langle Az,x-z\rangle\geq 0 \ \forall x\in C$. So, A being monotone implies $\langle Ax-Az,x-z\rangle\geq 0$ for all $x,y\in C$ so that $\langle Ax,x-z\rangle\geq\langle Az,x-z\rangle\geq 0$ for all $x,y\in C$. Thus, the result follows.*

To prove the other direction, Suppose $\langle Ax,x-z\rangle\geq 0$, for all $x\in C$. Since $z\in C$, then for any $x\in C$, $\alpha x+(1-\alpha)z\in C$ for any $\alpha\in(0,1)$ and so, we have:

$$0\leq\langle A\alpha x+(1-\alpha)z,\alpha x+(1-\alpha)z-z\rangle=\langle A\alpha x+(1-\alpha)z,\alpha(x-z)\rangle \quad \forall x\in C. \qquad (3.1.23)$$

So, we have $0\leq\langle A\alpha x+(1-\alpha)z,x-z\rangle, \quad \forall x\in C$. Take limit as $\alpha\to 0$, and thus, the result follows.

Theorem 3.1.4 *Let E be a 2-uniformly convex and uniformly smooth real Banach space. Let C be nonempty closed convex subset of E. $A:E\to E^*$ be monotone and Lipschitz with Lipschitz constant L. Let $x_0,x_{-1}\in E$ and defined the sequence iteratively $\{x_n\}$ by*

$$x_{n+1}=\Pi_CJ^{-1}(Jx_n-\lambda_nAx_n-\lambda_{n-1}(Ax_n-Ax_{n-1})), \quad n\geq 0; \qquad (3.1.24)$$

where $\{\lambda_n\}\subseteq\left[\epsilon,\frac{1-2\epsilon}{2\mu L}\right]$ for some $\epsilon>0$ and $\mu\geq 1$. Suppose Γ is nonempty and that the normalized duality mapping J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to an element of Γ.

Proof *Since $\{x_n\}$ is bounded from Theorem 3.1.1, let x^* be a weak sequential cluster point of $\{x_n\}$, then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. We show that $x^* \in \Gamma$.*

From (3.1.27), using Lemma 2.4.6, we have $\forall\ y \in C$, that

$$\begin{aligned}
&0 \le \langle Jx_{n+1} - Jx_n - \lambda_n Ax_n + \lambda_{n-1}(Ax_n - Ax_{n-1}), y - x_{n+1}\rangle \\
&= \langle Jx_{n+1} - Jx_n, y - x_{n+1}\rangle + \lambda_{n-1}\langle Ax_n - Ax_{n-1}, y - x_{n+1}\rangle - \lambda_n\langle Ax_n, y - x_{n+1}\rangle \\
&= \langle Jx_{n+1} - Jx_n, y - x_{n+1}\rangle + \lambda_{n-1}\langle Ax_n - Ax_{n-1}, y - x_{n+1}\rangle - \lambda_n\langle Ax_n, x_n - x_{n+1}\rangle - \lambda_n\langle Ax_n, y - x_n\rangle \\
&\le \langle Jx_{n+1} - Jx_n, y - x_{n+1}\rangle + \lambda_{n-1}\langle Ax_n - Ax_{n-1}, y - x_{n+1}\rangle - \lambda_n\langle Ax_n, x_n - x_{n+1}\rangle - \lambda_n\langle Ay, y - x_n\rangle.
\end{aligned} \tag{3.1.25}$$

The last inequality holds because A is monotone. Now, passing $\liminf$ and using the fact that $\{\lambda_n\}$ is bounded, we have

$$\liminf_{n\to\infty}\langle Ay, y - x_n\rangle \ge 0, \quad \forall y \in C.$$

So, we have

$$0 \le \liminf_{n\to\infty}\langle Ay, y - x_n\rangle \le \lim_{n\to\infty}\langle Ay, y - x_{n_k}\rangle = \langle Ay, y - x^*\rangle, \quad \forall y \in C.$$

By Lemma 3.1.3, we have that $x^* \in \Gamma$. Now, set $a_n = \phi(x^*, x_n) + 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x^* - x_n\rangle + \frac{1}{2}\phi(x_n, x_{n-1})$. So, from (3.1.14) we have

$$a_{n+1} + \epsilon\phi(x_{n+1}, x_n) \le a_n,\ \forall n \ge 0.$$

Claim: $0 \le a_{n+1}$, for all n so that $a_{n+1} \le a_n$, $\forall n$. But

$$\begin{aligned}
2\lambda_{n-1}\langle Ax_{n-1} - Ax_n, x^* - x_n\rangle &\le 2\lambda_{n-1}\|Ax_{n-1} - Ax_n\|\|x^* - x_n\| \\
&\le 2\lambda_{n-1}L\|x_{n-1} - x_n\|\|x^* - x_n\| \\
&\le \lambda_{n-1}L(\|x_{n-1} - x_n\|^2 + \|x^* - x_n\|^2) \\
&\le \lambda_{n-1}L\mu(\phi(x_n, x_{n-1}) + \phi(x^*, x_n))
\end{aligned} \tag{3.1.26}$$

$$\textit{implies}\ -\lambda_{n-1}L\mu(\phi(x_n, x_{n-1}) + \phi(x^*, x_n)) \le 2\lambda_{n-1}\langle Ax_n - Ax_{n-1}, x^* - x_n\rangle \quad \forall n.$$

Therefore, we have

$$\begin{aligned}
a_{n+1} &= \phi(x^*, x_{n+1}) + 2\lambda_n\langle Ax_{n+1} - Ax_n, x^* - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) \\
&\ge (1 - \lambda_n\mu L)\phi(x^*, x_{n+1}) + (\frac{1}{2} - \lambda_n\mu L)\phi(x_{n+1}, x_n) \\
&\ge \frac{1}{2}\phi(x^*, x_{n+1}) \ge 0.
\end{aligned}$$

Hence, $a_{n+1} \le a_n \ \forall\ n$. So, $\lim_{n\to\infty} a_n$ exists.

So, $\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0$ and $\lim_{n\to\infty}\langle Ax_{n+1} - Ax_n, x^* - x_{n+1}\rangle = 0$ implies $\lim_{n\to\infty}\phi(x^*, x_n)$ exists. Now, let x^* and z^* be two distinct weak cluster points of $\{x_n\}$, then

$$\phi(x^*, x_n) = \|x^*\|^2 - 2\langle x^*, Jx_n\rangle + \|x_n\|^2$$

and

$$\phi(z^*, x_n) = \|z^*\|^2 - 2\langle z^*, Jx_n\rangle + \|x_n\|^2,$$

so that

$$2\langle x^* - z^*, Jx_n\rangle = \phi(z^*, x_n) - \phi(x^*, x_n) + \|x^*\|^2 - \|z^*\|^2.$$

So, $\lim_{n\to\infty}\langle x^* - z^*, Jx_n\rangle$ exists. Since J is weakly sequentially continuous, we have

$$\langle x^* - z^*, Jx^*\rangle = \lim_{j\to\infty}\langle x^* - z^*, Jx_{n_j}\rangle = \lim_{k\to\infty}\langle x^* - z^*, Jx_{n_k}\rangle = \langle x^* - z^*, Jz^*\rangle$$

for some $\{x_{n_k}\}$ and $\{x_{n_j}\}$ such that $x_{n_k} \rightharpoonup z^*$ and $x_{n_j} \rightharpoonup x^*$.

$$\text{So we have } \langle x^* - z^*, Jx^* - Jz^* \rangle = 0.$$

Using Lemma 2.2.3 we conclude that $x^* = z^*$. The proof is complete.

Now, if we fix $\lambda_n = \lambda$ for all $n \geq 1$ in Theorem 3.1.1 we obtain the following corollary.

Corollary 3.1.5 *Let E be a real 2-uniformly convex uniformly smooth Banach space. Let $A : E \to E^*$ be monotone Lipschitz. For $x_0, x_{-1} \in E$ define the sequence $\{x_n\}$ by*

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - 2\lambda A x_n + \lambda A x_{n-1}), \quad n \geq 0; \tag{3.1.27}$$

where $\lambda \in \left[\epsilon, \frac{1-2\epsilon}{2\mu L}\right]$ for some $\epsilon > 0$ and $\mu \geq 1$. Suppose $\Gamma \neq \emptyset$ and that the normalized duality mapping J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to an element of Γ.

Corollary 3.1.6 *Let H be a real Hilbert space. Let C be nonempty closed convex subset of H. $A : H \to H$ be monotone and Lipschitz with Lipschitz constant L. Let $x_0, x_{-1} \in H$. Then the iterative sequence $\{x_n\}$ defined by*

$$x_{n+1} = P_C(x_n - \lambda_n A x_n - \lambda_{n-1}(Ax_n - Ax_{n-1})), \quad n \geq 0; \tag{3.1.28}$$

where $\{\lambda_n\} \subseteq \left[\epsilon, \frac{1-2\epsilon}{2\mu L}\right]$ for some $\epsilon > 0$ and $\mu \geq 1$ converges weakly to an element of Γ whenever Γ is nonempty and the normalized duality mapping J is weakly sequentially continuous.

3.2 Applications

In this chapter, we give some real life applications of our main result. We begin with " Nash-Cournot Equilibrium Model of Oligopoly Markets".

3.2.1 Nash-Cournot Equilibrium Model of Oligopoly

We consider the "Nash-Cournot equilibrium model in oligopolistic electricity markets". This model has been studied in see (e.g, [18]). In this model, it is assumed that there are $i = 1, 2, 3...n$ number of companies competing within themselves. Each company generates x_i amount of power/electricity. All companies, when acting/ working together, can generate a collective power/electricity, which is a vector $x = (x_1, x_2, x_3, ...x_n)$. Denote by K_i an action set/ strategy set of company i. Then, the collective action set of the model is

$$C := K_1 \times K_2 \times K_3 \times ... \times K_n.$$

Suppose the price $P_i(s)$ is decreasing, affine function of s, where $s := \sum_{i=1}^n x_i$. Thus, $P_i(s) := \alpha - \beta_i s$, then the profit for company i is given by $f_i(x) = x_i P_i(s) - h_i(x_i)$, where $h_i(x_i)$ denote the cost function of company i for generating x_i power.

Actually, each company seeks to maximize its profit by choosing the corresponding production level under the assumption that the production of the other companies are parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept.

Definition 3.2.1 *A point $x^* = (x_1^*, x_2^*, ..., x_n^*) \in C$ is called a Nash equilibrium point if for each i,*

$$f_i(x^*) = \max_{(x_i|x^*) \in C} f_i(x_i|x^*), \quad \text{where } (x_i|x^*) = (x_1^*, x_2^*, x_3^*..., x_{i-1}^*, x_i, x_{i+1}^*, ..., x_n^*) \tag{3.2.1}$$

equivalently, if

$$f_i(x^*) \geq f_i(x_i|x^*), \quad \forall x_i \in K_i, \ i = 1, 2, 3..., n. \tag{3.2.2}$$

In order to compute the Nash equilibrium, we define

$$f(x,y) = \Psi(x,y) - \Psi(x,x), \quad \textit{where} \;\; \Psi(x,y) := -\sum_{i=1}^{n} f_i(y_i|x). \tag{3.2.3}$$

The problem of finding Nah equilibrium can be formulated as

$$\textit{find} \;\; x^* \in C: \; f(x^*,x) \geq 0, \quad \forall x \in C. \tag{3.2.4}$$

The task is to find a production x^* *such that it is a Nash equilibrium point. Now, suppose the cost* h_j *for production is increasingly convex, i.e., the cost for producing a unit production increases as the quantity of the production gets larger(i.e., the demand increases). Under this convexity assumption, its not hard to see that* (3.2.4) *is equivalent to*

$$\textit{find} \quad x^* \in C: \langle \bar{B_1}x^* - \bar{a}, x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0, \;\forall x \in C, \tag{3.2.5}$$

where

$$\bar{B_1} = \begin{pmatrix} 0 & \beta_1 & \beta_1 & \beta_1 & \cdots & \beta_1 \\ \beta_2 & 0 & \beta_2 & \beta_2 & \cdots & \beta_2 \\ \beta_3 & \beta_3 & 0 & \beta_3 & \cdots & \beta_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_n & \beta_n & \beta_n & \beta_n & \cdots & 0 \end{pmatrix}, B_1 = \begin{pmatrix} \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \beta_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta_3 & & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & & 0 & 0 & \cdots & \beta_n \end{pmatrix}, \bar{a} = (\alpha,\alpha,\alpha,...\alpha)^T,$$

$\varphi(x) := x^T B_1 x + h(x)$ *where* $h(x) = \sum_{i=1}^{n} h_i(x_i)$,

and when h_i *is differentiable convex function* $\forall i$, *then* (3.2.5) *is equivalent to the following V.I*

$$\textit{find} \;\; x^* \in C: \langle \bar{B_1}x^* - \bar{a} + \nabla\varphi(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{3.2.6}$$

Indeed,

$$f(x^*,x) \geq 0 \;\forall x \in C \iff -\sum_{i=1}^{n} f_i(x_i|x^*) + \sum_{i=1}^{n} f_i(x^*) \geq 0 \iff \sum_{i=1}^{n} f_i(x^*) - \sum_{i=1}^{n} f_i(x_i|x^*) \geq 0 \tag{3.2.7}$$

But

$$f_i(x^*) = x_i^* p_i(s) - h_i(x_i{}^*) = x_i^*\Big(\alpha - \beta_i s\Big) - h_i(x_i^*) = x_i^*\Big(\alpha - \beta_i \sum_{j=i}^{n} x_j^*\Big) - h_i(x_i^*)$$

Therefore,

$$\sum_{i=1}^{n} f_i(x^*) = \sum_{i=1}^{n} \Big[x_i^*\Big(\alpha - \beta_i \sum_{j=1}^{n} x_j^*\Big] - h(x^*) \tag{3.2.8}$$

Also,

$$f_i(x_i|x^*) = x_i p_i(s) - h_i(x_i) = x_i\Big[\alpha - \beta_i\Big(\sum_{j\neq i}^{n} x_j^* + x_i\Big)\Big] - h(x_i)$$

So,

$$\sum_{i=1}^{n} f_i(x_i|x^*) = \sum_{i=1}^{n} x_i\Big[\alpha - \beta_i\Big(\sum_{j\neq i}^{n} x_j^* + x_i\Big)\Big] - h(x). \tag{3.2.9}$$

Substituting (3.2.8) *and* (3.2.9) *in* (3.2.7), *we obtain*

$$0 \leq \sum_{i=1}^{n} f_i(x^*) - \sum_{i=1}^{n} f_i(x_i|x^*) = \sum_{i=1}^{n} \Big[x_i^*\Big(\alpha - \beta_i \sum_{j=1}^{n} x_j^*\Big)\Big] - h(x^*) - \sum_{i=1}^{n} x_i\Big[\alpha - \beta_i\Big(\sum_{j\neq i}^{n} x_j^* + x_i\Big)\Big] + h(x)$$

$= \sum_{i=1}^{n} \alpha(x_i^* - x_i) + \sum_{i=1}^{n} x_i\beta_i\Big(\sum_{j\neq i}^{n} x_j^* + x_i\Big) - \sum_{i=1}^{n} x_i^*\beta_i\Big(\sum_{j=1}^{n} x_j^*\Big) + h(x) - h(x^*).$(3.2.10)

But

$$\sum_{i=1}^{n} x_i^*\beta_i\Big(\sum_{j=1}^{n} x_j^*\Big) = \sum_{i=1}^{n} x_i^*\beta_i\Big(\sum_{j\neq i}^{n} x_j^* + x_i^*\Big) = \sum_{i=1}^{n} x_i^*\beta_i\Big(\sum_{j\neq i}^{n} x_j\Big) + \sum_{i=1}^{n} \beta_i {x_i^*}^2$$

So,

$$\sum_{i=1}^{n} x_i\beta_i\Big(\sum_{j\neq 1}^{n} x_j^* + x_i\Big) - \sum_{i=1}^{n} x_i^*\beta_i\Big(\sum_{j=1}^{n} x_j^*\Big) = \sum_{i=1}^{n} x_i\beta_i\Big(\sum_{j\neq i}^{n} x_j^*\Big) + \sum_{i=1}^{n} \beta_i x_i^2 - \sum_{i=1}^{n} x_i^*\beta_i\Big(\sum_{j\neq i}^{n} x_j^*\Big) - \sum_{i=1}^{n} \beta_i {x_i^*}^2$$
$$= \sum_{i=1}^{n} \beta_i\Big(\sum_{j\neq i}^{n} x_j^*\Big)(x_i - x_i^*) + \sum_{i=1}^{n} \beta_i(x_i^2 - {x_i^*}^2).$$

Thus,

$$\sum_{i=1}^{n} x_i\beta_i\Big(\sum_{j\neq i}^{n} x_j^* + x_i\Big) - \sum_{i=1}^{n} x_i^*\beta_i\Big(\sum_{j=1}^{n} x_j^*\Big) = \sum_{i=1}^{n} \beta_i\Big(\sum_{j\neq i}^{n} x_j^*\Big)(x_i - x_i^*) + \sum_{i=1}^{n} \beta_i(x_i^2 - {x_i^*}^2). \tag{3.2.11}$$

Using definition of inner product, its trivial to see that

$$\langle \bar{B}_1 x^*, x - x^*\rangle = \sum_{i=1}^{n} \beta_i\Big(\sum_{j\neq i}^{n} x_j^*\Big)(x_i - x_i^*). \tag{3.2.12}$$

But, we also have that $x^T B_1 x = \langle x, B_1 x\rangle = \sum_{i=1}^{n} \beta_i x_i^2$. *So,*

$$\varphi(x) - \varphi(x^*) = x^T B_1 x - {x^*}^T B_1 x^* + h(x) - h(x^*) = \sum_{i=1}^{n} \beta_i(x_i^2 - {x_i^*}^2) + h(x) - h(x^*) \tag{3.2.13}$$

Substituting (3.2.11), (3.2.12) *and* (3.2.13) *in* (3.2.10), *we obtain*

$$\langle -a, x - x^*\rangle + \langle \bar{B}_1 x^*, x - x^*\rangle + \varphi(x) - \varphi(x^*) \geq 0 \ \forall x \in C, i.e., \ \ \langle \bar{B}_1 x^* - \bar{a}, x - x^*\rangle + \varphi(x) - \varphi(x^*) \geq 0 \ \forall x \in C.$$

In particular, since C *is convex, let* $x \in C$. *Take* $x^* + t(x - x^*) \in C, \quad \forall t \in [0,1]$. *Then*

$$\langle \bar{B}_1 x^* - \bar{a}, t(x - x^*)\rangle + \varphi(x^* + t(x - x^*)) - \varphi(x^*) \geq 0 \tag{3.2.14}$$

Dividing through by $t \neq 0$ *and taking limit as* $t \longrightarrow 0$, *we obtain*

$$\langle \bar{B}_1 x^* - \bar{a}, x - x^*\rangle + \langle \nabla\varphi(x^*), x - x^*\rangle \geq 0, \quad i.e., \ \langle \bar{B}_1 x^* - \bar{a} + \nabla\varphi(x^*), x - x^*\rangle \geq 0 \ \ \forall x \in C. \tag{3.2.15}$$

So, to prove the other way round, since φ *is subdifferentiable, then the following subdifferential inequality hold, i.e.,*

$$\varphi(x) - \varphi(x^*) \geq \langle \bar{a} - \bar{B}_1 x^*, x - x^*\rangle \quad \forall x \in C, \tag{3.2.16}$$

$$i.e., \ \langle \bar{B}_1 x^* - \bar{a}, x - x^*\rangle + \varphi(x) - \varphi(x^*) \geq 0 \ \forall x \in C, \tag{3.2.17}$$

$$i.e., \ \langle -a, x - x^*\rangle + \langle \bar{B}_1 x^*, x - x^*\rangle + \varphi(x) - \varphi(x^*) \geq 0 \ \forall x \in C. \tag{3.2.18}$$

Substituting the equivalent form of $\varphi(x)$, $\varphi(x^*)$ and $\langle \bar{B_1}x^*, x-x^*\rangle$, $\langle -a, x-x^*\rangle + \langle \bar{B_1}x^*, x-x^*\rangle$ in (3.2.18), we have

$$\sum_{i=1}^{n}\alpha(x_i^*-x_i)+\sum_{i=1}^{n}\beta_i\Big(\sum_{j\neq i}^{n}x_j^*\Big)(x_i-x_i^*)+\sum_{i=1}^{n}\beta_i(x_i^2-x_i^{*2})+h(x)-h(x^*)\geq 0 \quad \forall x\in C,$$

$$i.e.,\quad f(x^*,x)\geq 0 \quad \forall x\in C.$$

Hence, the proof is complete.

Now, we test our proposed algorithm with the cost function given by $h_i(x_i)=\sum_{i=1}^{n}\frac{1}{2}x_i^2$. Take

- $n=5$,
- *Take* $\beta_1=0.01,\ \beta_2=0.2,\ \beta_3=0.5,\ \beta_4=0.02,\ \beta_5=0.3.$
- $C:=[0,100]\times[0,100]\times ...\times[0,100]$ *which is closed(by Tychonov) and convex*
- $A:=\bar{B_1}-\bar{a}+\nabla\varphi$ *which is monotone and Lipschitz.*
- $\epsilon=\frac{1}{20}$, $\mu=1$ *and suppose the normalize duality map J is weakly sequentially continuous.*

$$\text{We note that}\quad \varphi(x)=x^TB_1x+h(x)=\sum_{i=1}^{n}\beta_ix_i^2+\sum_{i=1}^{n}\frac{1}{2}x_i^2=\sum_{i=1}^{n}(\beta_i+\frac{1}{2})x_i^2.\quad \text{So,}$$

$$\nabla\varphi(x)=((2\beta_1+1)x_1,(2\beta_2+1)x_2,(2\beta_3+1)x_3...,(2\beta_n+1)x_n)=((2\beta_i+1)x_i)_{i=1,2,..n}$$

which is monotone (this follows directly from definition of monotonicity)

Since all the hypothesis of Corollary (3.1.6) holds, (i.e., the operator A is monotone and Lipschitz; the mother space, $\mathbb{R}^4$, is of finite dimension (therefore Hilbert); the set C is nonempty closed and convex subset of $\mathbb{R}^4$), then its conclusion follows, i.e., the following iterative sequence

$$x_{n+1}=P_C(x_n-\lambda_nAx_n-\lambda_{n-1}(Ax_n-Ax_{n-1})),\quad n\geq 0;$$

where $\{\lambda_n\}\subseteq\Big[\epsilon,\frac{1-2\epsilon}{2\mu L}\Big]$ converges weakly to the equilibrium point in Γ.

Example 3.2.2 (2) *Consider the space $H=L_2([0,1])$, with norm and inner product function defined as*

$$\|x\|_2=\Big(\int_0^1|x(t)|^2\ dt\Big)^{\frac{1}{2}}\qquad \text{and}\qquad \langle x,y\rangle=\int_0^1x(t)y(t)\ dt\quad \text{respectively.}$$

Define $C:=\{u\in H: c\leq\langle a,u\rangle\leq b\}$, where $0\neq a\in H$, $b\in\mathbb{R}$. Also, consider the projection map $P_C:H\to C$ defined by

$$P_C(u)=\begin{cases}u+\Big(\frac{c-\langle a,u\rangle}{\|a\|^2}\Big)a, & \text{if}\quad \langle a,u\rangle<c,\\ u\qquad\qquad\qquad\ , & \text{if}\quad c\leq\langle a,u\rangle\leq b,\\ u+\Big(\frac{b-\langle a,u\rangle}{\|a\|^2}\Big)a, & \text{if}\quad \langle a,u\rangle>b.\end{cases}\tag{3.2.19}$$

Also, consider the operator $A:H\to H$ by

$$Ax(t)=\int_0^1\Big[x(s)-\Big(\frac{2tse^{t+s}}{e\sqrt{e^2-1}}\Big)cosx(s)\Big]\ ds+\frac{2te^t}{e\sqrt{e^2-1}},\quad x\in L_2([0,1]),$$

then, A is monotone and lipschitz with lipsghitz constant $L=2$. Setting $c=-3$, $b=5$, let $a:[0,1]\to\mathbb{R}$, $t\longmapsto t$. Now, let $\varphi\in H^*$, then by Reiz representation theorem, since $0\in\Gamma$, we show that $x_n\rightharpoonup 0$ as $n\to\infty$, i.e.,

$$\langle f_\varphi,x_n\rangle=\langle\varphi,x_n\rangle=\int_0^1\varphi(t)x_n(t)\ dt\rightharpoonup 0\quad \text{as}\quad n\to\infty.$$

Bibliography

[1] *A.U.Bello et.al* Foward-Baackward splitting method without coercivity for the sum of maximal monotone operators in Banach spaces. *200 Mathematics Subject Classification: 47H09; 47H10; 49J20; 49J40.*

[2] *Al-Mazrooei A E, Bin Dehaish B A, Latif A, Yao J C,* On general system of Variational inequalities in Banach Spaces. *J Nonlinear convex Anal, 2015* **16***(4): 639-658.*

[3] *Alber Y I.* Metric and generalized projection operator in Banach spaces: properties and applications. Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. *178 of Lecture Notes in Pure and Applied Mathematics. New York, NY: Dekker, 1996: 15–50.*

[4] *B. Beauzamy,* Introduction to Banach Spaces and Their Geometry*, 2nd edn. North-Holland Mathematics Studies, 68. Notas de Matemtica [Mathematical Notes], 86. North-Holland Publishing Co., Amsterdam (1985). xv+338 pp. ISBN:0-444-87878-5.*

[5] C.E. Chidume, A. Adamu and L.O.Chinwendu,, A Krasnoselskii-type algorithm for approximating solutions of variational inequality problems and convex feasibility problems*, Jornal of Nonlinear Variational Analysis, Volume 2, Issue 2, Page 203-218.*

[6] *Censor, Y., Gibali, A., and Reich S. (2011).* The Subgradient Extragradient Method for Solving Variational Inequalities in Hilbert Space. *Journal of Optimization Theory and Applications, 148(2): 318- 335.*

[7] *Cai G, Bu S.* Modified extragradient methods for variational inequality problems and fixed point problems for an infinite family of nonexpansive mappings in Banach spaces. *J Global Optim, 2013, 55(2): 437–457.*

[8] *C. Baiocchi and A. Capelo,* Variational and Quasivariational Inequalities; Applications to Free Boundary Problems, *Wiley, New York, 1984.*

[9] *D. Kinderlehrer and G. Stampacchia,*An Introduction to Variational inequalities and their Applications, *Academic press, New York, 1980.*

[10] *D. Davis, W.T. Yin* A three-operator splitting scheme and its optimization applications. *Set-Valued Var. Anal.(2017) 25:829-858.*

[11] *F. Facchinei and J.-S. Pang,* Finite-Dimensional Variational inequalities and Complementarity problems, *Volume I. Springer series in operations research, Springer, New York, 2003.*

[12] *G.M Korpelevich,* The extragradient method for finding saddle points and other problems and other problems, *Ekonomika i Mat. Metody,* **12** *(1976), 747-756.*

[13] *G. B. Passty, Ergodic convergence to a zero of the sum of monotone operators in Hilbert spaces, J. Math. Anal. Appl., 72,(1979) 383–390.*

[14] *H. Attouch, J. Peypouquet, and P. Redont,* Backward–forward algorithms for structured monotone inclusions in Hilbert spaces. *J Math Anal Appl(2018) 457:1095-1117.*

[15] *I.V. Konnov,* Combined Relaxation Methods for Variational inequalities, *Springer-Verlag, Berlin, 2001.*

[16] *I.V. Konnov,* Combined Relaxation Methods for Variational Inequalities, *Springer-Verlag, Berlin,2001.*

[17] *Iusem A N, Nasri M.* Korpelevich's method for variational inequality problems in Banach spaces. *J Global Optim, 2011,* **50***: 59–76.*

[18] *J, Klusch M, Krawczyk JB (2004)* Numerical solution to Nash -Cournot equilibria in coupled constraint electricity markets. *EEE Trans Power Syst 19:195-206.*

[19] *J.-P.Aubin and I. Ekeland,* Applied Nonlinear Analysis, *Wiley, New York, 1984.*

[20] K. Aoyama, F. Kohsaka, Strongly relatively nonexpansive sequences generated by firmly nonexpansive-like mappings, *Fixed Point Theory Appl.,* **95**, *(2014). 13pp*

[21] *P. L. Lions, B. Mercier,* Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal., 16, (1979) 964-979 .*

[22] *R. Glowinski, J.-L.Lions and R. Trémolières,* Numerical Analysis of Variational Inequalities, *North-Holland, Amsterdam, 1981.*

[23] *R. Bruck, On* the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space. *J. Math. Anal. Appl(1977) 61:159-164.*

[24] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim.,* **13** *(3), (2003) 938-945.*

[25] *Tseng P. A modified forward-backward splitting method for maximal monotone mappings. SIAM J Control Optim, 2000, 38: 431–446*

[26] *Y. Malitsky, M. K. TAM,* A forward-Backward spliting method for monotone inclusion without cocoercivity, *SIAM J. Optim.,***30***(2) 1451-1472.*

[27] *Y. Alber , I. Ryazantseva,* Nonlinear Ill Posed Problems of Monotone Type*, Springer, London, UK, 2006.*

[28] *Yekini SHEHU,* Single projection algorithm for variational inequalities in Banach space with application to contact problem, *Acta Math Sci* ***40****, 1045–1063 (2020).*

[29] *Yao Y, Aslam Noor M, Inayat Noor K, Liou Y -C, Yaqoob H.* Modified extragradient methods for a system of variational inequalities in Banach spaces. *Acta Appl Math, 2010,* 1*10(3): 1211–1224.*

[30] *Zegeye H, Shahzad N.* Extragradient method for solutions of variational inequality problems in Banach spaces. *Abstr Appl Anal, 2013,* **2013***: Art ID 832548, 8 pp.*