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APPROXIMATION OF ZEROS OF M-ACCRETIVE OPERATORS; SOLUTIONS TO VARIATIONAL INEQUALITY AND GENERALIZED SPLIT FEASIBILITY PROBLEMS

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CERTIFICATE OF APPROVAL

Ph.D. Thesis

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for award of the degree of Doctor of Philosophy degree in Mathematics.

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Dedication

This thesis is dedicated to my father, Late Mr. Rufus I. Nnyaba.

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Abstract

In this thesis, the problem of solving the equation of the form

$$Au = 0, \tag{0.0.1}$$

where A is a nonlinear map (either mapping a Banach space, E to itself or mapping E to its dual, E^*), is considered. This problem is desirable due to its enormous applications in optimization theory,ecology, economics, signal and image processing, medical imaging, finance, agriculture, engineering, etc.

Solving equation (0.0.1) is connected to solving the following problems.

- In optimization theory, it is always desirable to find the minimizer of functions. Let $f : E \to \mathbb{R}$ be a convex and proper function. The subdifferential associated to f, $\partial f : E \to 2^{E^*}$ defined by

  $$\partial f(x) = \{u^* \in E^* : \langle u^*, y - x\rangle \leq f(y) - f(x) \quad \forall\, y \in E\}.$$

  It is easy to check that the subdifferential map ∂f is monotone on E and that $0 \in \partial f(x)$ *if and only if* x *is a minimizer of* f. Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$ is equivalent to solving for a minimizer of f. In the case where the operator A is single-valued, the inclusion $0 \in Au$ reduces to equation (0.0.1).

- The differential equation, $\dfrac{du}{dt} + Au = 0$, where A is an accretive-type map, describes the evolution of many physical phenomena that generate over time. At equilibrium state, $\dfrac{du}{dt} = 0$, thus the differential equation reduces to equation (0.0.1). Thus, solution of equation (0.0.1) correspond to equilibrium state of some dynamical system. Moreover, such equilibrium states are very desirable in many applications, e.g., economics, physics, agriculture and so on.

- In nonlinear integral equations, the Hammerstein integral equation which is of the form

$$u(x) + \int_{\Omega} k(x,y) f(y, u(y)) dy = w(x), \tag{0.0.2}$$

where $\Omega \subset \mathbb{R}^n$ is bounded, $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are measurable real-valued functions, and the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions, can be transformed into the form $u + KFu = 0$, without loss of generality. Thus, setting $A := I + KF$, where I is the identity map, will reduce to equation (0.0.1). Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear part posses Green's function can, as a rule, be transformed into the form (0.0.2).

Our objectives in this thesis are: studying and constructing new iterative algorithms; proving that the sequences generated by these algorithms approximate solutions of some nonlinear problems, such as, variational inequality problems, equilibrium problem, convex split feasibility problems, convex minimization problems and so on, and conducting numerical experiments to show the efficiency of our algorithms.

In particular, the following results are proved in this thesis.

- Let E be a uniformly smooth and uniformly convex real Banach space and let $A : E \to 2^E$ be a multi-valued m-accretive operator with $D(A) = E$ such that the inclusion $0 \in Au$ has a solution. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ by

$$x_{n+1} = x_n - \lambda_n u_n - \lambda_n \theta_n (x_n - x_1), \qquad u_n \in Ax_n, \quad n \geq 1.$$

Then the sequence $\{x_n\}$ converges strongly to a solution of the inclusion $0 \in Au$.

- Let E be a uniformly convex and uniformly smooth real Banach space and E^* be its dual. Let $A : E \to E^*$ be a generalized Φ-strongly monotone and bounded map and let $T_i : E \to E$, $i = 1, 2, 3, ..., N$ be a finite family of quasi-ϕ-nonexpansive maps such that $Q := \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence in E defined iteratively by $x_1 \in E$,

$$x_{n+1} = J^{-1}(J(T_{[n]}x_n) - \theta_n A(T_{[n]}x_n)), \quad \forall n \geq 1,$$

where $T_{[n]} := T_{n \ mod \ N}$. Assume $VI(A, Q) \neq \emptyset$, then $\{x_n\}$ converges strongly to some $x^* \in VI(A, Q)$.

- Let E be a uniformly smooth and 2-uniformly convex real Banach space. Let C be closed and convex subset of E. Suppose $A_i : C \to E^*,\ i = 1, 2, ..., N$ is a finite family of monotone and L-Lipschitz continuous maps and the solution set F is nonempty. Then, the sequences $\{x_n\},\ \{y_n^i\},\ \{z_n^i\}$ generated by

$$\begin{cases} x_0 \in E,\ 0 < \lambda < \dfrac{1}{L},\ C_0 = C, \\ y_n^i = \Pi_C J^{-1}(Jx_n - \lambda A_i(x_n)),\ i = 1, ..., N, \\ T_n^i = \{v \in E : \langle (Jx_n - \lambda A_i(x_n)) - Jy_n^i, v - y_n^i \rangle \leq 0\}, \\ z_n^i = \Pi_{T_n^i} J^{-1}(Jx_n - \lambda A_i(y_n^i)),\ i = 1, ..., N, \\ i_n = argmax\{||z_n^i - x_n|| : i = 1, ..., N\}, \quad \bar{z}_n := z_n^{i_n}, \\ C_{n+1} = \{v \in C_n : \phi(v, \bar{z}_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad n \geq 0. \end{cases}$$

converge strongly to $\Pi_F x_0$.

- Let K be a closed convex subset of E_1. Let E_1 and E_2 be uniformly smooth and 2-uniformly convex real Banach spaces, and E_1^*, E_2^* be their dual spaces respectively. Let $A : E_1 \to E_2$ be a bounded linear operator whose adjoint is denoted by A^* and $S : E_2 \to E_2$ be a nonexpansive map such that $F(S) \neq \emptyset$ and $T : K \to K$ be a relatively nonexpansive map such that $F(T) \neq \emptyset$. Let $B : E_1 \to 2^{E_1^*}$ be a maximal monotone mapping such that $B^{-1}0 \neq \emptyset$. Then the sequence generated by the following algorithm: for $x_1 \in K$ arbitrary and $\beta_n \in (0, 1)$,

$$\begin{cases} y_n = J_{E_1}^{-1}\big(J_{E_1}x_n - \gamma A^* J_{E_2}(I - S)Ax_n\big), \\ w_n = J_{E_1}^{-1}\big(\alpha_n J_{E_1}x_1 + (1 - \alpha_n)J_{E_1}J_\lambda^B y_n\big), \\ x_{n+1} = J_{E_1}^{-1}\big(\beta_n J_{E_1}x_n + (1 - \beta_n)J_{E_1}Tw_n\big), \quad \forall n \geq 1. \end{cases}$$

converges strongly to an element $z \in \Gamma$, where Γ is the solution set of some generalized split feasibility problem.

- Let K be a closed convex nonempty subset of a 2-uniformly convex and uniformly smooth real Banach space E with dual space E^*. Let $h_i : K \times K \to \mathbb{R}\ (i = 1, 2, 3, ...)$ be a sequence of bifunctions satisfying conditions $(A1) - (A4)$ and $G_i : K \to 2^E,\ i = 1, 2, 3, ...$ be a countable family of equally continuous and totally quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps with nonnegative real sequences $\{v_n^{(i)}\}, \{\mu_n^{(i)}\}$ and strictly increasing continuous functions $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $v_n^{(i)} \to 0, \mu_n^{(i)} \to 0$ as $n \to \infty$ and $\psi_i(0) = 0$. Let $A_i : K \to 2^{E^*},\ i = 1, 2, 3, ...$ be a countable family of γ_i-inverse strongly monotone multi-valued maps and let $\gamma = inf\{\gamma_i,\ i = 1, 2, 3, ...\} > 0$. Let $\Phi_i : K \to \mathbb{R}\ (i = 1, 2, 3, ...)$ be a sequence of lower semi-continuous convex functions and let $B_i : K \to E^*\quad (i = 1, 2, 3, ...)$ be a sequence of continuous monotone functions. Suppose $W := \Big(\cap_{i=1}^{\infty} F(G_i) \Big) \cap \Big(\cap_{i=1}^{\infty} A_i^{-1}(0) \Big) \cap \Big(\cap_{i=1}^{\infty} GMEP(h_i, \Phi_i, B_i) \Big) \neq \emptyset$ and the sequence $\{x_n\}$ in K is defined iteratively as follows:

$$\begin{cases} x_0 \in K_0 = K, \\ y_n = \Pi_K J^{-1}(Jx_n - \lambda \xi_{i_n}), \qquad (\xi_{i_n} \in A_{i_n} x_n), \\ z_n = J^{-1}\big(\alpha J x_n + (1-\alpha) J \eta_{m_n}^{(i_n)}\big), \qquad (\eta_{m_n}^{(i_n)} \in G_{i_n}(PG_{i_n})^{m_n - 1} y_n), \\ u_n = T_{r_n} z_n, \\ K_{n+1} = \{z \in K_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{K_{n+1}} x_0, \quad n \geq 0, \end{cases}$$

where $\theta_n := (1-\alpha)\big[v_{m_n}^{(i_n)} \sup_{p \in W} \psi_{i_n}\big(\phi(p, x_n)\big) + \mu_{m_n}^{(i_n)}\big]$; $\lambda \in (0, \frac{c_2}{2}\gamma)$, $c_2 > 0$ is a positive constant satisfying certain conditions and $\alpha \in (0, 1)$. Then, $\{x_n\}$ converges strongly to some element of W.

List of publications arising from the thesis and other peer-review publications

(A) **Papers Published/Accepted from the Thesis**

1. C.E. Chidume, **U.V. Nnyaba** and O.M. Romanus; Halpern-type iterative algorithms for generalized split feasibility problems in real Banach spaces, **Carpathian J. Math (Accepted)**.
2. C.E. Chidume, **U.V. Nnyaba** and O.M. Romanus; Some iterative algorithms for approximating solutions of nonlinear operator equations in Banach spaces. **Journal of Fixed Point Theory and Applications, (2019), doi:10.1007/s11784-019-0690-2**
3. C.E. Chidume, **U.V. Nnyaba** and O.M. Romanus; A new algorithm for variational inequality problems with a generalized phi-strongly monotone map over the set of common fixed points of a finite family of quasi-phi-nonexpansive maps, with applications. **Journal of Fixed Point Theory and Applications. DOI 10.1007/s11784-018-0502-0.**
4. C.E. Chidume, **U.V. Nnyaba**, O.M. Romanus and C.G. Ezea; Convergence theorems for strictly J-pseudo-contractions with application to zeros of gamma inverse-strongly monotone maps; **Pan Amer.Math.J., (2016) no.4, 57-76.**

(B) **Other Peer-reviewed Published/Accepted Papers**

5. C.E. Chidume, O.M. Romanus, and **U.V. Nnyaba**, an iterative algorithm for solving split equality fixed point problems for a class of nonexpansive-type mappings in Banach spaces, **Numerical Algorithms (2018), https://doi.org10.1007/s11075-018-0638-4**.
6. C.E. Chidume, O.M. Romanus and **U.V. Nnyaba**: An iterative algorithm for solving split equilibrium problems and split equality

variational inclusions for a class of nonexpansive-type maps, **Optimization, https://doi.org/10.1080/02331934.2018.1503270**.

7. C.E. Chidume, O.M. Romanus and **U.V. Nnyaba**, Relaxed iterative algorithms for a system of generalized mixed equilibrium problems and a countable family of totally quasi-Phi asymptotically nonexpansive multi-valued maps, with applications, **Fixed Point Theory and Appl (2017), 2017:21, doi.10.1186/s13663-017-0616-x**.
8. C.E. Chidume, O.M. Romanus and **U.V. Nnyaba**; Strong convergence theorems for a common zero of an infinite family of gamma-inverse strongly monotone maps with applications; **The Austr. Jour. of Math. Anal. and Appl. (2017)**.
9. C.E. Chidume, O.M. Romanus, and **U.V. Nnyaba**; A new iterative algorithm for zeros of generalized Phi-strongly monotone and bounded maps with application, **Brit. J. Math. Csc., vol. 17, iss. 1, DOI: 10.9734/BJMCS/2016/25884**.

(C) **Other Papers Still in the Refereeing Process**

10. C.E. Chidume, **U.V. Nnyaba** and O.M. Romanus; New parallel and cyclic hybrid subgradient extragradient algorithms for solving variational inequality problems in Banach space, with applications, (Submitted), **Afrika Matematika**.
11. C.E. Chidume, **U.V. Nnyaba**, O.M. Romanus and A. Adamu; Approximation of zeros of m-accretive mappings, with applications to Hammerstein integral equations **(to appear)**.

Contents

CHAPTER 1

General Introduction

The contents of this thesis fall within the general area of functional analysis, in particular, nonlinear operator theory, a flourishing area of research for numerous mathematicians. In this thesis, we concentrate on the following three important topics, namely:

- Approximation of zeros of m-accretive maps with applications to Hammerstein integral equations.
- Approximation of solutions of variational inequality problems involving monotone-type maps.
- Approximation of solutions of generalized split feasibility problems.
- Approximation of solutions of some equilibrium problems.

1.1 Background

It is well known that many physically significant problems in several areas of research can be transformed as an equation of the form

$$Au = 0, \tag{1.1.1}$$

where A is a nonlinear monotone map. Consider for example, the differential equation

$$\frac{du}{dt} + Au = 0, \tag{1.1.2}$$

where A is monotone (accretive), describes the evolution of many physical phenomena that generate energy over time. At equilibrium state, $\frac{du}{dt} = 0$, thus equation (1.1.2) reduces to equation (1.1.1). Therefore, a solution of equation (1.1.1) (i.e., a zero of A) corresponds to the equilibrium state of the system described in equation (1.1.2). Such equilibrium points are very desirable in many applications, for example, in ecology, economics, physics and so on.

Also, in optimization theory, it is always desirable to find minimizers of a convex function, when they exist. It is known that if a function f is differentiable and has a minimizer $x^* \in D(f)$ (say), then $f'(x^*) = 0$. This is an explicit method of obtaining a minimizer of f. Unfortunately, most of the significant functions that arise in optimization problems are not always differentiable in the usual sense. For example, the absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$ has a minimizer, which in fact is 0, but it is not differentiable at 0. So, in a case where the operator under consideration is not differentiable, it becomes difficult to compute a minimizer with the above technique even when it exists.

Let E be a normed space and $f : E \to \mathbb{R}$ be a convex and proper function. The subdifferential map associated to f, $\partial f : E \to 2^{E^*}$, which always exists for any convex function f, is defined by

$$\partial f(x) = \{u^* \in E^* : \langle u^*, y - x\rangle \leq f(y) - f(x) \quad \forall\, y \in E\}.$$

It is easy to check that the subdifferential map ∂f is monotone on E, and that $0 \in \partial f(x)$ *if and only if x is a minimizer of f*. Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$ is equivalent to solving for a minimizer of f. In the case where the operator A is single-valued, the inclusion $0 \in Au$ reduces to equation (1.1.1).

Also, in nonlinear integral equations, consider the Hammerstein integral equation which is stated as follows:
Let $\Omega \subset \mathbb{R}^n$ be bounded and let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of Hammerstein-type has the form

$$u(x) + \int_\Omega k(x, y) f(y, u(y))dy = w(x), \tag{1.1.3}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions. If we define $F : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ and $K : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ by

$$Fu(y) = f(y, u(y)), \quad y \in \Omega$$

and

$$Kv(x) = \int_\Omega k(x, y)v(y)dy, \quad x \in \Omega$$

respectively, where $\mathcal{F}(\Omega, \mathbb{R})$ is a space of measurable real-valued functions defined from Ω to $\mathbb{R}$, then equation (1.1.3) can be put in an abstract form

$$u + KFu = 0, \tag{1.1.4}$$

where, without loss of generality, we have assumed that $w \equiv 0$. It can easily be observed that equation (1.1.4) is a special case of equation (1.1.1), where

$$A := I + KF.$$

Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear part posses Green's function can, as a rule, be transformed into the form (1.1.3) (see e.g., Pascali and Sburian [134], chapter p. 164). Among these is the problem of the forced oscillations of finite amplitude of a pendulum.

Example 1.1.1 *Consider the problem of pendulum*

$$\begin{cases} \dfrac{d^2v(t)}{dt^2} + a^2 \sin v(t) = w(t), \quad t \in [0,1] \\ v(0) = v(1) = 0, \end{cases} \tag{1.1.5}$$

where the driving force w is odd. The constant $a \neq 0$ depends on the length of the pendulum and gravity. Since the Green's function of the problem

$$v''(t) = 0, \quad v(0) = v(1) = 0$$

is the function

$$k(t,s) := \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Problem (1.1.5) is equivalent to the nonlinear integral equation

$$v(t) = -\int_0^1 k(t,s)[w(s) - a^2 \sin v(s)]ds, \quad t \in [0,1]. \tag{1.1.6}$$

If $g(t) := \int_0^1 k(t,s)w(s)ds$, $u(t) := v(t) + g(t)$, $t \in [0,1]$, then $v = u - g$. Equation (1.1.6) can be written as

$$u(t) + \int_0^1 k(t,s)a^2 \sin[g(s) - u(s)]ds = 0, \quad t \in [0,1],$$

which is in the form of (1.1.3) with

$$f(t,s) = a^2 \sin(g(t) - s), \quad t, \ s \in [0,1].$$

Equations of Hammerstein type play vital role in the theory of optimal control systems, in automation and in network theory (see, e.g., Dolezale [81])

1.2 Approximation of zeros of nonlinear m-accretive maps with applications to Hammerstein integral equations

Let E be a real Banach space with a dual space E^*. A mapping $A : E \to 2^E$ is called *accretive* if for each $x, y \in E, \eta \in Ax, \nu \in Ay$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle \eta - \nu, j(x-y) \rangle \geq 0,$$

where $J : E \to 2^{E^*}$ defined by $Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}$ is the normalized duality map on E. The mapping A is said to be *maximal accretive* if, in addition, its graph is not properly included in the graph of any other accretive mapping. Also, the mapping A is said to be *m-accretive* if, in addition to A being accretive, the following range condition holds: $R(I + \lambda A) = E$, for all $\lambda > 0$. It is noteworthy that in normed spaces, m-accretive implies maximal accretive. If E is a real Hilbert space, accretive mappings are called *monotone* and maximal accretive mappings are called *maximal monotone* maps.

Interest in the study of accretive mappings stems from their usefulness in several areas of applicable mathematics such as in economics, differential equations, calculus of variation, and so on (see e.g., Browder [17], Zeildler [182]). In nonlinear functional analysis, accretive operators appear mainly in *two problems*, in elliptic differential problems and in evolution equation problems. In the case of an elliptic problem, we are solving an inclusion of the form $y \in Tx$, where the operator T may be decomposed into a sum of operators among which are accretive operators. In the case of an evolution problem we study a time-dependent differential inclusion which contains, in one of its terms, an operator T which may be decomposed into a sum of operators containing an accretive operator.

In general, a fundamental problem in the study of accretive operators in Banach spaces is the following:

$$\text{find} \;\; u \in E \;\; \text{such that} \;\; 0 \in Au. \tag{1.2.1}$$

Several existence theorems for problem (1.2.1) have been proved (see e.g., Browder [17]). Also, methods of approximating solutions of the inclusion problem (1.2.1), when they are known to exist, have been studied extensively. One of the classical methods for approximating solutions of this inclusion problem where A is a maximal monotone operator on a real Hilbert space, is the celebrated *proximal point algorithm* introduced by Martinet [124] and studied extensively by Rockafellar [148] and numerous other authors. The algorithm is given by:

$$x_{n+1} = \Big(I + \frac{1}{\lambda_n} A\Big)^{-1} x_n + e_n, \tag{1.2.2}$$

where $\lambda_n > 0$ is a regularizing parameter. Rockafellar [148] proved that if the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is bounded from above, then the resulting sequence $\{x_n\}_{n=1}^{\infty}$ of proximal point iterates converges *weakly* to a solution of (1.1.1) when $E = H$, provided that a solution exists (see also Bruck and Reich [25]). Rockafellar [148] then posed the following question.

Question 1. *Does the proximal point algorithm always converge strongly?*

The question was resolved in the negative by Güler [87] who produced a proper closed convex function g in the infinite dimensional Hilbert space l_2 for which the proximal point algorithm converges *weakly but not strongly*, (see also Bauschke *et al.* [12]). This naturally raised the following questions.

Question 2. *Can the proximal point algorithm be modified to guarantee strong convergence?*

Question 3. *Can another iterative algorithm be developed to approximate a solution of* (1.1.1)*, assuming existence, such that the sequence of the algorithm converges strongly to a solution of* (1.1.1)*?*

Bruck [22] considered an iteration process of the Mann-type and proved that the sequence of the process converges strongly to a solution of (1.1.1) in a real Hilbert space where A is a maximal monotone map, *provided the initial vector is chosen in a neighbourhood of a solution of* (1.1.1). Chidume [45] extended this result to L_p spaces, $p \geq 2$ (see also Reich [141, 142, 143]). These results of Bruck [22] and Chidume [45] are not easy to use in any possible application because the neighborhood of a solution in which the initial vector must be chosen is not known precisely.

In connection with Question 2, Solodov and Svaiter [155] proposed a modification of the proximal point algorithm which guarantees strong convergence in a real Hilbert space. Their algorithm is as follows.

Choose any $x_0 \in H$ and $\sigma \in [0, 1)$. At iteration k, having x_k, choose $\mu_k > 0$, and find (y_k, v_k), an inexact solution of $0 \in Tx + \mu_k(x - x_k)$, with tolerance σ. Define

$$C_k = \{z \in H : \langle z - y_k, v_k \rangle \leq 0\}, \quad Q_k = \{z \in H : \langle z - x_k, x_0 - x_k \rangle \leq 0\}.$$

$$\text{Take} \qquad x_{k+1} = P_{C_k \cap Q_k} x_0, \quad k \geq 1.$$

The authors themselves noted ([155], p. 195) that " $\cdots$ at each iteration, there are two subproblems to be solved $\cdots$": (i) find an inexact solution of the proximal point algorithm, and (ii) find the projection of x_0 onto $C_k \cap Q_k$. They also acknowledged that these two subproblems constitute *a serious drawback in using their algorithm.*

Kamimura and Takahashi [101] extended this result of Solodov and Svaiter [155] to the framework of Banach spaces that are both uniformly convex and uniformly smooth, Reich and Sabach [145] extended this result to reflexive Banach spaces.

Xu [165] noted that "...Solodov and Svaiter's algorithm, though strongly convergent, does need more computing time due to the projection in the second subproblem...". He then proposed and studied the following algorithm:

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\big(I + c_n A\big)^{-1} x_n + e_n, \quad n \geq 0. \tag{1.2.3}$$

He proved that the sequence $\{x_n\}$ generated by algorithm (1.2.3) converges strongly to a solution of $0 \in Au$ provided that the sequences $\{\alpha_n\}$ and $\{c_n\}$ of real numbers and the sequence $\{e_n\}$ of errors are chosen appropriately. We note here, however, that the occurrence of errors is random and so the sequence $\{e_n\}$ cannot actually be chosen.

Lehdili and Moudafi [118] considered the *technique of the proximal map* and the *Tikhonov regularization* to introduce the so-called *Prox-Tikhonov method* which generates the sequence $\{x_n\}$ by the algorithm:

$$x_{n+1} = J_{\lambda_n}^{A_n} x_n, \quad n \geq 0, \tag{1.2.4}$$

where $A_n := \mu_n I + A, \quad \mu_n > 0$ and $J_{\lambda_n}^{A_n} := \big(I + \frac{1}{\lambda_n} A_n\big)^{-1}$. Using the notion of variational distance, Lehdili and Moudafi [118] proved strong convergence theorems for this algorithm and its perturbed version, under appropriate conditions on the sequences $\{\lambda_n\}$ and $\{\mu_n\}$.

Xu also studied the recurrence relation (1.2.4). He used the technique of nonexpansive mappings to get convergence theorems for the perturbed version of the algorithm (1.2.4), under much relaxed conditions on the sequences $\{\lambda_n\}$ and $\{\mu_n\}$.

In response to Question 3, Chidume [44] recently proved the following theorem.

Theorem 1.2.1 (Chidume [44]) *Let E be a uniformly smooth real Banach space with modulus of smoothness ρ_E, and let $A : E \to 2^E$ be a multi-valued* ***bounded*** *m-accretive operator with $D(E) = E$ such that the inclusion $0 \in Au$ has a solution. For arbitrary $u_1 \in E$, define a sequence $\{u_n\}$ iteratively by,*

$$u_{n+1} = u_n - \lambda_n \eta_n - \lambda_n \theta_n (u_n - u_1)\big), \quad \eta_n \in Au_n, \quad n \geq 1, \tag{1.2.5}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying certain conditions. There exists a constant $\gamma_0 > 0$ such that $\frac{\rho_E(\lambda_n)}{\lambda_n} \leq \gamma_0 \theta_n$. Then, the sequence $\{u_n\}$ converges strongly to a zero of A.

We remark that Theorem 1.2.1 of Chidume resolves Questions 1, 2 and 3 but in the special case in which the operator A is m-accretive and **bounded**. This restriction eliminates several important operators, for example, the differential operator. Hence, the following question is of interest.

Question 4. *Can the requirement that A be bounded imposed in Theorem 1.2.1 be dispensed with?*

In Chapter 3 of this thesis, an affirmative answer to Question 4 is given. This is achieved by means of a new important result concerning accretive operators, which was recently proved by Chidume *et al.* [60]. Furthermore, the convergence theorem proved is applied to approximate a solution of *a Hammerstein integral equation.*

1.3 Approximation of solutions of variational inequality problems of monotone-type maps

A map $A : D(A) \subset E \to E^*$ is called

- *monotone* if for each $x, y \in D(A)$, the following inequality holds: $\langle Ax - Ay, x - y\rangle \geq 0$.
- *β-strongly monotone* if for each $x, y \in D(A)$, there exists $\beta > 0$ such that the following inequality holds: $\langle Ax - Ay, x - y\rangle \geq \beta||x - y||^2$.
- *ϕ-strongly monotone* if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that $\langle Ax - Ay, x - y\rangle \geq \phi(\|x - y\|)\|x - y\|, \ \forall\, x, y \in D(A)$.
- *generalized Φ-strongly monotone* if there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that $\langle Ax - Ay, x - y\rangle \geq \Phi(\|x - y\|), \ \forall\, x, y \in D(A)$.

It is easy to see that the class of *β-strongly monotone maps* is a proper subclass of the class of *ϕ-strongly monotone maps* (with $\phi(t) = \beta t^2$), which in turn is a proper subclass of the class of *generalized Φ-strongly monotone maps* (with $\Phi(t) = t\phi(t)$). It is well known that the class of generalized Φ-strongly monotone maps is the largest class of monotone maps for which if the equation $Au = 0$ has a solution, then the solution is necessarily unique. This class of maps has been studied by various authors.

Let C be a nonempty subset of E. A map $T : C \to E$ is called K- *Lipschitzian* if there exists $K > 0$ such that

$$||Tx - Ty|| \leq K||x - y||, \ \forall x, y \in C. \tag{1.3.1}$$

If in inequality (1.3.1) $K = 1$, then the map T is called *nonexpansive*. It is known that the set $F(T) := \{x \in E : Tx = x\}$ is closed and convex whenever T is nonexpansive.

Let $A : C \subset E \to E^*$ be a nonlinear operator. The classical variational inequality problem (VIP) is the following:

$$\text{find } x^* \in C \text{ such that } \langle Ax^*, y - x^*\rangle \geq 0, \ \forall\, y \in C. \tag{1.3.2}$$

The set of solutions of problem 1.3.2 is denoted by $VI(A, C)$. Variational inequality problem was first introduced and studied by Stampacchia [156] in 1964 and is central in the study of nonlinear analysis. Precisely, such a problem is connected with convex minimization problems, complementarity problems, fixed point problems, zeros of nonlinear operators and so on (see e.g., Shi [153], Noor [129], Yao [173], Stampacchia [156]).

Several existence results for problem (1.3.2) have been proved when A is a monotone mapping defined on certain real Banach spaces (see e.g., Hartman and Stampacchia [92], Browder [17], Barbu and Precupanu [9]). Iterative methods of approximating solutions, assuming existence, have been studied extensively.

A typical iterative procedure for approximating solutions of 1.3.2 in a real Hilbert space for a Lipschitz β-strongly monotone operator is the *projected gradient method* (see for example, Goldstein [98], Zeidler [182]) expressed as follows.

Theorem 1.3.1 (Projected Gradient Method, ([98], [182])) *Let C be a closed convex subset of a Hilbert space H and $A : C \to H$ be a K- Lipschitz and β-strongly monotone operator. Let $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and*

$$x_{n+1} = P_C(I - \lambda A)x_n, \tag{1.3.3}$$

for $n = 1, 2, ...,$ where P_C is the metric projection onto C, I is the identity mapping on H and $\lambda \in (0, \frac{2\beta}{L^2})$. Then, $\{x_n\}$ converges strongly to the unique $x^* \in VI(C, A)$.

A more general form of (1.3.3) is given by

$$x_{n+1} = P_C(I - \lambda_{n+1} A)x_n. \tag{1.3.4}$$

The projected gradient method relies on the fact that $P_C : H \to C$ is a nonexpansive map with a nonempty set of fixed points and that for any fixed $x \in H$, $\langle P_C x - x, y - P_C x\rangle \geq 0, \forall \;\; y \in C$. Other properties of the metric projection can be found, for example, in section 3 of the book by Goebel and Reich [97].
We remark that the projected gradient method is especially appealing in applications when the explicit form of P_C is known (e.g., when C is a closed ball or a closed cone). In order to reduce the possible difficulty with the use of P_C especially when P_C is not very easy to compute, Yamada [171] introduced a *hybrid steepest descent method* for solving problems of this type.

Theorem 1.3.2 (Hybrid steepest descent method, [171]) *Let $T : H \to H$ be a nonexpansive mapping with $F(T) = \{x \in H : Tx = x\} \neq \emptyset$. Suppose that a mapping $A : H \to H$ is k-Lipschitzian and β-strongly monotone over $T(H)$. Then with any $x_0 \in H$, any $\mu \in (0, \frac{2\eta}{L^2})$ and any sequence*

$\{\lambda_n\}$ *in* $(0,1]$ *satisfying:* *(C1)* $\lim \lambda_n = 0$; *(C2)* $\sum \lambda_n = \infty$; *(C3)* $\lim \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0$, *the sequence* $\{x_n\}$ *generated by*

$$x_{n+1} = Tx_n - \lambda_{n+1} A(Tx_n), \; n \geq 0, \tag{1.3.5}$$

converges strongly to the uniquely existing solution of $VIP(A, F(T))$.

For $C = \cap_{k=1}^r F(T_k) \neq \emptyset$, where $\{T_k\}_{k=1}^r$ is a finite family of nonexpansive mappings, Yamada [171] also studied the following algorithm:

$$x_{n+1} = T_{[k]}x_n - \lambda_n \mu A(T_{[k]}x_n), \; n \geq 1, \tag{1.3.6}$$

where $T_{[k]} = T_{k \; mod \; r}$, for $k \geq 1$ and the sequence $\{\lambda_n\}$ satisfies conditions (C1), (C2) and (C4): $\sum |\lambda_n - \lambda_{n+N}| < \infty$, and proved the strong convergence of $\{x_n\}$ to the unique solution of $VIP(A, C)$.

In the case where $f : H \to \mathbb{R}$ is a convex functional and $A := \nabla f$, he obtained that $x_n \to x^* \in \arg\inf_{x \in F(T)} f(x)$, where $\nabla f : H \to H$ is the gradient of the convex functional f.

Xu and Kim [168] studied the convergence of the algorithms (1.3.6) and (1.3.5), still in the framework of Hilbert spaces, with the condition (C3) replaced by $\lim \dfrac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}} = 0$ and the condition (C4) replaced by $\lim \dfrac{\lambda_n - \lambda_{n+r}}{\lambda_{n+r}} = 0$. The theorems of Xu and Kim [168] are improvements on the results of Yamada [171] because the canonical choice $\lambda_n = \frac{1}{n+1}$ is applicable in their theorem but is not applicable in the result of Yamada [171]. Other extensions of the theorem of Yamada, still in Hilbert spaces, can be found in Wang [170], Zeng and Yao [183] and Yamada *et al.* [172].

Chidume *et al.* [56] extended the result of Yamada [171] to q-uniformly smooth spaces, $q \geq 2$, in particular, to L_p spaces, $2 \leq p < \infty$. They proved the following theorem.

Theorem 1.3.3 (Chidume *et al.*, [56]) *Let* E *be a* q*-uniformly smooth real Banach space with constant* d_q, $q \geq 2$. *Let* $T_i : E \to E$, $i = 1, 2, 3, ..., r$ *be a finite family of nonexpansive mappings with* $K := \cap_{i=1}^r F(T_i) \neq \emptyset$. *Let* $G : E \to E$ *be an* η*-strongly accretive map which is also* k*-Lipschitzian. Let* $\{\lambda_n\}$ *be a sequence in* $[0,1]$ *satisfying:* *(C1)* $\lim \lambda_n = 0$; *(C2)* $\sum \lambda_n = \infty$; *(C6)* $\lim \frac{\lambda_n - \lambda_{n+r}}{\lambda_{n+r}} = 0$. *For* $\delta \in \left(0, \min\{\frac{q}{4\eta}, (\frac{q\eta}{d_q k^q})^{\frac{1}{(q-1)}}\}\right)$, *define a sequence* $\{x_n\}$ *iteratively in* E *by* $x_0 \in E$,

$$x_{n+1} = T_{[n+1]}x_n - \delta \lambda_{n+1} G(T_{[n+1]}x_n), \; n \geq 0, \tag{1.3.7}$$

where $T_n = T_{n \; mod \; r}$. *Assume also that* $K = F(T_r T_{r-1} ... T_1) = F(T_1 T_r ... T_2) = ... = F(T_{r-1} T_{r-2} ... T_r)$. *Then,* $\{x_n\}$ *converges strongly to the unique solution* x^* *of the variational inequality* $VI(G, K)$.

We remark that in Theorem 1.3.3, the operator G maps the space E into itself, however, it is desirable to consider when the operator G maps the space into its dual as it is known to have applications in convex optimization problems.

In Chapter 4 of this thesis, we obtained an analogue of Theorem 1.3.3 in a uniformly convex and uniformly smooth real Banach space for a more general class of operators, $A : E \to E^*$, *a generalized* Φ*-strongly monotone and bounded map*. Furthermore, we applied the result obtained to a convex optimization problem.

1.3.1 Parallel and Cyclic Hybrid Subgradient Extragradient Algorithms for Solutions of VIP for Lipschitz Monotone Maps

In this section, we study iterative methods for approximating a solution of a VIP for a Lipschitz monotone map. In this case, the solution of VIP is not necessarily unique.

Korpelevich [113] in 1976, proposed a projection method in a real Hilbert space H, called the *extragradient method*, for solving saddle point problems. The extragradient method is designed as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_C(x_n - \lambda A(y_n)), \quad n \geq 0, \end{cases}$$

where P_C is the metric projection onto C, λ is a suitable parameter and C is a nonempty, closed and convex subset of H. In the case when C has a simple structure and the projections onto it can be evaluated readily, the extragradient method is very useful. However, if C is *any* closed and convex set, one has to solve two distance optimization problems in the extragradient method to obtain the next approximation x_{n+1}. This can affect the efficiency of the method. In 2011, Censor *et al.* [39] proposed the following algorithm, called the *subgradient extragradient method*, for variational inequality problem in a real Hilbert space H,

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_{T_n}(x_n - \lambda A(y_n)), \quad n \geq 0, \end{cases} \tag{1.3.8}$$

where T_n is a half-space defined as $T_n := \{v \in H : \langle (x_n - \lambda A(x_n)) - y_n, v - y_n \rangle \leq 0\}$. They proved that the sequences $\{x_n\}$, $\{y_n\}$ generated by (1.3.8) converge weakly to a a point in $VI(C, A)$.

Moreover, to obtain strong convergence of these iterative sequences, Censor *et al.* [40] proposed a hybrid algorithm which combines the subgradient extragradient method and the outer approximation method.

They also considered the problem of finding a common solution to variational inequality problems $CSVIP$. The $CSVIP$ is stated as follows: Let $K_i, \quad i = 1, ..., N$ be a finite family of nonempty closed and convex subsets of H such that $K := \cap_{i=1}^N K_i \neq \emptyset$. Let $A_i : H \to H, \quad i = 1, ..., N$ be nonlinear maps. The $CSVIP$ is to find $x^* \in K$ such that

$$\langle x - x^*, A_i(x^*) \rangle \geq 0, \quad \forall x \in K_i, \; i = 1, ..., N. \tag{1.3.9}$$

If $N = 1$, the $CSVIP$ (1.3.9) becomes VIP. The $CSVIP$ is motivated by the fact that it includes as special cases, convex feasibility problems, common fixed point problems, common minimization problems, common saddle-point problems, variational inequality problems over the intersection of convex sets. These problems have practical applications in signal processing , networking, resource allocation, image recovery and many other problems, (see e.g.,[56], [40], [114], [95] and the references contained in them).

In 2015, Anh and Hieu [4, 55] proposed a *parallel monotone hybrid algorithm* for finding a common fixed point of a finite family of *quasi ϕ-nonexpansive mappings* $\{S_i\}_{i=1}^N$ in a real Hilbert space, H. They considered the following algorithm:

$$\begin{cases} x_0 \in C \subset H, \\ y_n^i = \alpha_n x_n + (1 - \alpha_n) S_i x_n, \; i = 1, ..., N, \\ i_n = argmax\{||y_n^i - x_n|| : i = 1, ..., N\}, \quad \bar{y_n} := y_n^{i_n}, \\ C_{n+1} = \{v \in C_n : ||v - \bar{y_n}|| \leq ||v - x_n||\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0, \end{cases} \tag{1.3.10}$$

where $0 < \alpha_n < 1$, $\limsup_{n\to\infty} \alpha_n < 1$. In this algorithm, the intermediate approximations y_n^i can be found simultaneously. Then, among all y_n^i, the furthest element from x_n, denoted by $\bar{y}_n$, is chosen. After that, using this element, the closed convex set C_{n+1} is constructed. Finally, the next approximation x_{n+1} is defined as the projection of x_0 onto C_{n+1}. However, it seems difficult to find the explicit form of the sets C_n and perform numerical experiments.

In 2016, Hieu [96] proposed two parallel and cyclic hybrid subgradient extragradient algorithms for $CSVIP$ in Hilbert spaces for a class of Lipschitz continuous and monotone maps. In this algorithm, the additional computation of index i_n, allowed to compute explicitly the next iterate. He proved the following theorem for parallel hybrid subgradient extragradient method:

Theorem 1.3.4 (Hieu, [96]) *Let $K_i, \; i = 1, 2, ..., N$ be closed and convex subsets of a real Hilbert space H such that $K = \cap_{i=1}^N K_i \neq \emptyset$. Suppose that $\{A_i\}_{i=1}^N : H \to H$ is a finite family of monotone and L-Lipschitz continuous mappings. In addition, the solution set $F := \cap_{i=1}^N VI(A_i, K_i)$ is nonempty.*

Then, the sequences $\{x_n\}$, $\{y_n^i\}$, $\{z_n^i\}$ *generated by the algorithm:*

$$\begin{cases} x_0 \in H, \; 0 < \lambda < \dfrac{1}{L}, \\ y_n^i = P_{K_i}(x_n - \lambda A_i(x_n)), \; i = 1, ..., N, \\ T_n^i = \{v \in H : \langle (x_n - \lambda A_i(x_n)) - y_n^i, v - y_n^i \rangle \leq 0\}, \\ z_n^i = P_{T_n^i}(x_n - \lambda A_i(y_n^i)), \; i = 1, ..., N, \\ i_n = argmax\{||z_n^i - x_n|| : i = 1, ..., N\}, \quad \bar{z_n} := z_n^{i_n}, \\ C_n = \{v \in H : ||\bar{z}_n - v|| \leq ||x_n - v||\}, \\ Q_n = \{v \in H : \langle v - x_n, x_n - x_0 \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.3.11}$$

converge strongly to $P_F x_0$.

We remark that the results of Hieu, [96] and other authors (see e.g., Anh and Hieu [4, 55]) are in Hilbert space. However, as has been rightly observed by Hazewinkle [93], "... many and probably most mathematical objects and models do not naturally live in Hilbert space".

In Chapter 5 of this thesis, we study the parallel and cyclic hybrid subgradient extragradient algorithms and prove that the sequences generated by these algorithms converge strongly to a common element of the set of solutions of variational inequality problems in a *uniformly smooth and* 2*-uniformly convex real Banach space.*

1.4 Approximation of solution of generalized split feasibility problem

Let E_1 and E_2 be real Banach spaces. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. The *split feasibility problem* (SFP) is formulated as:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q, \tag{1.4.1}$$

where $A : E_1 \to E_2$ is a bounded linear operator. In finite dimensional Hilbert space, the SFP was first introduced by Censor and Elfving [38] for modeling inverse problems which arise from phase retrievals, medical image reconstruction and recently in modeling of intensity modulated radiation therapy. There has been growing interest in recent years in the theory of split feasibility problem due to its application in signal processing (see e.g., [6, 41, 40, 35, 127, 128, 114, 160, 161, 162] and the references therein for further details). In the recent past, several split type problems have been introduced and studied. Byrne et al. [106] considered and studied the split common null point problem (SCNPP) in the setting of Hilbert spaces. They developed some algorithms for finding the approximate solutions of SCNPP. Very recently, Takahashi and

Yao [163] introduced SCNPP in the setting of Banach spaces. By using hybrid method and Halpern-type method, they proposed some iterative algorithms for computing the approximate solutions of SCNPP. They also established some strong and weak convergence theorems for such algorithms under some suitable conditions.

Another split type problem is the generalized split feasibility problem (GSFP), which in the setting of Banach spaces, is as follows:

$$\text{Find } x^* \in F(T) \cap B^{-1}0 \text{ such that } Ax^* \in F(S), \tag{1.4.2}$$

where E_1 and E_2 are real Banch spaces, $B : E_1 \to 2^{E_1^*}$ is a nonlinear map such that $B^{-1}0 \neq \emptyset$, $S : E_2 \to E_2$ is a map such that $F(S) \neq \emptyset$, $A : E_1 \to E_2$ be a bounded linear map, $F(S)$ denotes the set of fixed points of S and $T : K \to K$ is a map such that $F(T) \neq \emptyset$ and K is nonempty closed convex subset of E_1. If we consider $T \equiv I$, the identity mapping, then problem (1.4.2) reduces to the following generalized split feasibility problem:

$$\text{Find } x^* \in B^{-1}0 \text{ such that } Ax^* \in F(S). \tag{1.4.3}$$

We denote by Γ and Ω the solution set of problem (1.4.2) and (1.4.3), respectively, and assume that $\Omega \neq \emptyset$ and $\Gamma \neq \emptyset$.

In 2014, Takahashi *et al.* [162] studied problems (1.4.2) and (1.4.2) in the setting of Hilbert space. They constructed the following iterative algorithms; for $x_1 \in H_1$,

$$x_{n+1} = J_{\lambda_n}(I - \gamma_n A^*(I - T)A)x_n, \quad \forall\ n \in \mathbb{N},$$

and,

$$x_{n+1} = x_n + (1 - \beta_n)VJ_{\lambda_n}(I - \gamma_n A^*(I - T)A)x_n, \quad \forall\ n \in \mathbb{N},$$

where $A : H_1 \to H_2$ is a bounded linear map, $T : H_1 \to H_1$ is a nonexpansive map, $B : H_1 \to 2^{E_1}$ is maximal map and $J_{\lambda_n} = (I + \lambda_n B)^{-1}$, $V : H_1 \to H_1$ and obtained weak convergence theorems for finding a solution of the generalized split feasibility problems.

In 2017, Ansari and Rehan [7] studied and extended the result of Takahashi *et al.* [162] from Hilbert space to uniformly convex and 2-uniformly smooth real Banach spaces. They proposed the following iterative algorithms: For $x_1 \in K$ and $\beta_n \in (0, 1)$,

$$x_{n+1} = J_{E_1}^{-1}\big(\beta_n J_{E_1}(x_n) + (1 - \beta_n)J_{E_1}(y_n)\big) \tag{1.4.4}$$

and,

$$x_{n+1} = J_\lambda^B\big(J_{E_1}^{-1}(x_n) - \gamma A^* J_{E_2}(I - S)Ax_n\big) \tag{1.4.5}$$

where $y_n = VJ_\lambda^B(J_{E_1}^{-1}(x_n) - \gamma A^* J_{E_2}(I - SAx_n)\big)$ and proved that if E_1 and E_2 are uniformly convex and 2-uniformly smooth real Banach spaces, J_{E_1} is weakly sequentially continuous and, B, V, A, S and T are certain maps. Then the sequences generated by Algorithms (1.4.4) and (1.4.5) converge weakly to a point in Ω and Γ, respectively.

We note that the restriction that J_{E_1} is weakly sequentially continuous eliminates some very important real Banach spaces. It is known that for l_p spaces, $1 < p < \infty$, J_{l_p} is weakly sequentially continuous but J_{L_p}, $1 < p < \infty$, $p \neq 2$ is **not** weakly sequentially continuous.

In Chapter 6 of this thesis, by dispensing with the condition that the normalized duality map on E_1, J_{E_1} is weakly sequentially continuous, we construct Halpern-type iterative algorithms and prove that the sequences generated by these algorithms converge **strongly** to a point in Ω and Γ in a *uniformly smooth and* 2-*uniformly convex* real Banach space.

1.5 Approximation of solutions of some equilibrium problems

Let E be a real Banach space with dual space E^* and K be a nonempty closed convex subset of E. Let $h : K \times K \to \mathbb{R}$ be a bifunction. The *classical equilibrium problem (EP)* for a bifunction h is to find $x^* \in K$ such that

$$h(x^*, y) \geq 0, \ \forall y \in K. \tag{1.5.1}$$

We denote the set of solutions for problem (1.5.1) by $EP(h) = \{x^* \in K : h(x^*, y) \geq 0, \ \forall y \in K\}$.

The classical equilibrium problem (EP) includes (see [85] and [29]) as special cases the monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, vector equilibrium problems, Nash equilibria in noncooperative games. Furthermore, there are several other problems, for example, the complementarity problems and fixed point problems, which can also be written in the form of the classical equilibrium problem. In other words, the classical equilibrium problem is a unifying model for several problems arising from engineering, physics, statistics, computer science, optimization theory, operations research, economics, and many other fields.

A more robust and unifying equilibrium-type problem is the following: find $x^* \in K$ such that

$$h(x^*, y) + \psi(y) - \psi(x^*) + \langle y - x^*, Bx^* \rangle \geq 0 \ \ \forall \ y \in K, \tag{1.5.2}$$

where $h : K \times K \to \mathbb{R}$ is a bifunction, $\psi : K \to \mathbb{R} \cup \{\infty\}$ is a proper extended real-valued function and $B : K \to E^*$ is a nonlinear map. Problem (1.5.2) is called the *generalized mixed equilibrium problem*. We denote the set of solutions of problem (1.5.2) by $GMEP(h, \psi, B) = \{x^* \in K : h(x^*, y) + \psi(y) - \psi(x^*) + \langle y - x^*, Bx^* \rangle \geq 0 \ \ \forall \ y \in K\}$.

The generalized mixed equilibrium problem contains as special cases: the *classical equilibrium problem* $EP(h)$ (when $\psi \equiv 0 \equiv B$) studied by Fan [85], Blum and Oettli [29] and a host of other authors, the *classical variational inequality problem* (when $h \equiv 0 \equiv \psi$) studied by Stampacchia [156] and a host other authors, the *mixed equilibrium problem (MEP)* (when $B \equiv 0$) studied by Ceng and Yao [33], the *generalized equilibrium problem (GEP)* (when $\psi \equiv 0$) studied by Takahashi and Takahashi [158] and a host of other authors, the *generalized variational inequality problem (GVIP)* (when $h \equiv 0$), and the convex minimization problem (when $h \equiv 0 \equiv B$).

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$ for all $x \in E$. It is well known that every nonempty, closed, convex subset of a uniformly convex Banach space E is a retract of E (see e.g., Kopecká and Reich [112], for more information on nonexpansive retracts and retractions). In what follows, we assume that K is a retract of E and $P : E \to K$ is a nonexpansive retraction.

A map $A : D(A) \subset E \to E^*$ is called

- *monotone*, if for all $x, y \in D(A)$, we have that $\langle x - y, Ax - Ay \rangle \geq 0$.
- γ *-inverse strongly monotone*, if there exists a positive real number γ such that for all $x, y \in D(A)$, $\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2$.

It is easy to see that if A is γ- inverse strongly monotone, then A is Lipschitz continuous with Lipschitz constant $\frac{1}{\gamma}$.

Let E be a smooth, strictly convex and reflexive real Banach space and let K be a nonempty closed convex subset of E. Let $G : K \to 2^E$ be any map. Define the Lyapunov function $\phi : E \times E \to \mathbb{R}$ by $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$, for $x, y \in E$. (See e.g., Bo and Yi [30], Yi [174], Chang *et al.* [34], Butnariu *et al.* [31]) A countable family of multi-valued nonself maps, $G_i : K \to 2^E, i = 1, 2, ...,$ is said to be

- *uniformly quasi-ϕ-nonexpansive* if $\cap_{i=1}^{\infty} F(G_i) \neq \emptyset$,
$$\phi(p, \eta_x) \leq \phi(p, x) \ \ \forall \ p \in \cap_{i=1}^{\infty} F(G_i), x \in K, \eta_x \in G_i(PG_i)^{n-1}x;$$
- *uniformly quasi-ϕ-asymptotically nonexpansive* if $\cap_{i=1}^{\infty} F(G_i) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty), k_n \downarrow 1$ such that
$$\phi(p, \eta_n^i) \leq k_n \phi(p, x) \ \ \forall \ p \in \cap_{i=1}^{\infty} F(G_i), x \in K, \eta_n^i \in G_i(PG_i)^{n-1}x, \ n \geq 1;$$
- *uniformly totally quasi-ϕ-asymptotically nonexpansive* if $\cap_{i=1}^{\infty} F(G_i) \neq \emptyset$ and there exist nonnegative real sequences $\{\upsilon_n\}, \{\mu_n\}$ with $\upsilon_n \to 0, \mu_n \to$

0 $(n \to \infty)$ and a strictly increasing and continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(0) = 0$ such that

$$\begin{aligned}\phi(p, \eta_n^i) \quad &\leq \quad \phi(p, x) + \upsilon_n \rho\big[\phi(p, x)\big] + \mu_n \quad \forall\, p \in \cap_{i=1}^{\infty} F(G_i), x \in K,\\ &\qquad \eta_n^i \in G_i(PG_i)^{n-1}x, \;\; n \geq 1.\end{aligned}$$

It is easy to see from the above definitions that a countable family of uniformly quasi-ϕ-nonexpansive multi-valued nonself maps is a countable family of uniformly quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps, and a countable family of uniformly quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps is a countable family of uniformly totally quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps, but the converse need not be true. A motivation for the study of totally quasi-ϕ-asymptotically nonexpansive self or nonself maps is to unify various definitions of classes of maps associated with the class of relatively nonexpansive self or nonself maps which are extensions, to arbitrary real Banach spaces, of nonexpansive maps with nonempty fixed point sets in Hilbert spaces; and also, to prove general convergence Theorems applicable to all of these classes.

Recently, some iterative algorithms for approximating fixed points of *self-maps* satisfying certain contractive conditions and *zeros* of monotone and monotone-type operators have been studied extensively by various authors. These problems have also been applied to solve several nonlinear problems such as integral equations of Hammerstein-type, variational inequality problems and equilibrium problems involving nonlinear maps (see e.g., Ofoedu and Malonza [132], Zegeye *et al.* [179], Zegeye and Shahzad ([180], [178]), Wang *et al.* [169], Deng [76] and the references contained in them).

For approximating *a common element* of set of solutions for a system of generalized mixed equilibrium problems, the set of common fixed points of a countable family of **uniformly Lipschitzian** and **uniformly** totally quasi-ϕ-asymptotically nonexpansive **self-maps** and set of common zeros of a finite family of inverse strongly monotone maps, Wang *et al.* [169] proposed the following projection algorithm:

$$\begin{cases} x_0 \in K_0 = K,\\ y_n = \Pi_K J^{-1}[Jx_n - \lambda A_{n+1} x_n],\\ z_n = J^{-1}\big[\alpha_{n,0} J x_n + \sum\limits_{i=1}^{\infty} \alpha_{n,i} J G_i^n y_n\big],\\ u_n = K_{r_{M,n}}^{\Lambda_M} K_{r_{M-1,n}}^{\Lambda_{M-1}} \ldots K_{r_{2,n}}^{\Lambda_2} K_{r_{1,n}}^{\Lambda_1} z_n,\\ K_{n+1} = \{v \in K_n : \phi(v, u_n) \leq \phi(v, x_n) + \eta_n\},\\ x_{n+1} = \Pi_{K_{n+1}} x_0, \quad n \geq 0.\end{cases}$$

The authors established in a 2-uniformly convex and uniformly smooth real Banach space, strong convergence of the sequence generated by the above algorithm to a solution of the problem they considered.

We remark that this result of Wang *et al.* [169] is an improvement of several recent results, for example, the results of Ofoedu and Malonza [132], Zhang [176], Su *et al.* [157] to mention a few.

In order to dispense with the infinite sum $\sum_{i=1}^{\infty} \alpha_{n,i} JG_i y_n$ in the above algorithm of Wang *et al.* [169], hence simplifying the algorithm; Deng [76] recently studied the following *relaxed hybrid shrinking iterative algorithm*:

$$\begin{cases} x_0 \in K_0 = K, \\ y_n = \Pi_K J^{-1}[Jx_n - \lambda A_{i_n} x_n], \\ z_n = J^{-1}[\alpha_{i_n} Jx_n + (1 - \alpha_{i_n}) JG_{(i_n)}^{m_n} y_n], \\ u_n \in K \quad \text{such that} \quad H(u_n, z_n, y) \geq 0 \quad \forall y \in K, \\ K_{n+1} = \{v \in K_n : \phi(v, u_n) \leq \phi(v, x_n) + \eta_n\}, \\ x_{n+1} = \Pi_{K_{n+1}} x_0, \quad n \geq 0, \end{cases}$$

and proved strong convergence of the sequence of the above algorithm to *a common element* of set of solutions for a system of generalized mixed equilibrium problems, the set of common fixed points of a countable family of **uniformly Lipschitzian** and **closed** totally quasi-ϕ-asymptotically nonexpansive **self-maps** and set of common zeros of an infinite family of inverse strongly monotone maps.

It is worth noting that if the domain $D(G)$ of the operator, G studied in the papers discussed above is a proper subset of E and G maps $D(G)$ into E (which is the case in several applications), then the iterative algorithms proposed in the paper of Wang *et al.* [169], Deng [76], Ofoedu and Malonza [132], Zegeye *et al.* [179], Zegeye and Shahzad ([180], [178]), may fail to be well defined (see e.g., [110, 111, 42] for more details).

In [42], Chidume employed the concept of *retraction* to study nonexpansive **non-self maps** as the generalization of nonexpansive **self-maps**. A number of algorithms have also been proposed for approximating fixed points of certain nonlinear non-self maps (see e.g., [107], [108], [109], and [146]). However, little or none has been done for approximating common elements in the set of common fixed points of **non-self maps** and solutions set of other nonlinear problems.

It is our purpose in Chapter 7 of this thesis to construct iterative schemes of *Krasnoselkii-type* and *Halpern-type* that approximate common elements in the set of common fixed points of a *countable* family of **equally continuous** and totally quasi-ϕ-asymptotically nonexpansive **nonself** maps, common zeros of a *countable* family of inverse strongly monotone maps and a solution of a system of generalized mixed equilibrium problems. The theorems proved in this Chapter are significant improvements of the results of Deng [76], Wang *et al.* [169], Bo and Yi [30], Ofoedu and Malonza [132] and results of a host of other authors (see Remark 2 in Chapter 7 for more details).

CHAPTER 2

PRELIMINARIES

In this chapter, we give some definitions, lemmas and examples of some non-linear mappings used in this thesis; most of which could be found in standard monographs and papers of researchers working in this area of research for example, Chidume [59], Cioranescu [74], Alber and Ryazantseva [3] and Berinde [14].

2.1 Some of normed linear spaces

Definition 2.1.1 *A normed space E is said to be uniformly convex if and only if for all $\varepsilon \in (0, 2]$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for $x, y \in E$ with $||x|| \leq 1, ||y|| \leq 1$ and $||x - y|| \geq \varepsilon$, we have*

$$\left\|\frac{x+y}{2}\right\| < 1 - \delta.$$

Definition 2.1.2 *A normed space E is said to be strictly convex if and only if for all $x, y \in E, x \neq y, ||x|| = ||y|| = 1$, we have that $||\lambda x + (1-\lambda)y|| < 1, \quad \forall \lambda \in (0, 1)$.*

Remark 2.1.1 *Every uniformly convex space is strictly convex. However the converse may not hold.*

Remark 2.1.2 *Geometrically, a normed space E is uniformly convex if and only if the unit ball centred at the origin is "uniformly round". We list some examples of uniformly convex spaces.*

1. *Let E be the cartesian plane, $\mathbb{R}^2$ with the norm defined for each $x = (x_1, x_2) \in \mathbb{R}^2$ by $||x||_2 = \left[|x_1|^2 + |x_2|^2\right]^{\frac{1}{2}}$. Then $\mathbb{R}^2$ endowed with this norm is uniformly convex. But the space $\mathbb{R}^2$ defined for each $x = (x_1, x_2) \in \mathbb{R}^2$ by $||x||_1 = |x_1| + |x_2|$ and $||x||_\infty = \max |x_1|, |x_2|$ are not uniformly convex.*

2. *Every real inner product space H is uniformly convex (see e.g., Chidume, [59] p.163).*

3. *L_p (or l_p) spaces, $1 < p < \infty$, are uniformly convex.*

Definition 2.1.3 *The modulus of convexity of a normed space E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by*

$$\delta_E(\varepsilon) := \inf \Big\{ 1 - \|\frac{x+y}{2}\| : ||x|| = ||y|| = 1, \varepsilon = ||x-y|| \Big\},$$

Some of the properties of modulus of convexity are:

- The modulus of convexity δ_E is a non-decreasing function.
- The modulus of convexity is continuous (see e.g., V.I. Gurarri, [89]).

Proposition 2.1.1 *A normed space is uniformly convex space if and only if $\delta_E(\varepsilon) > 0 \quad \forall \varepsilon \in (0, 2]$.*

Proof.
Let E be a uniformly convex space. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\delta \leq 1 - \|\frac{x+y}{2}\|$ for every x and y such that $||x|| = ||y|| = 1$ and $\varepsilon \leq ||x - y||$. Therefore, $\delta_E(\varepsilon) > 0$.
For the converse, assume $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$. Fix $\varepsilon \in (0, 2]$ and take $||x|| = ||y|| = 1$ and $\varepsilon \leq ||x - y||$, then

$$0 < \delta_E(\varepsilon) \leq 1 - \|\frac{x+y}{2}\|.$$

Thus, $\|\frac{x+y}{2}\| \leq 1 - \delta$ with $\delta = \delta_E(\varepsilon)$.

Definition 2.1.4 *A normed space is called smooth if and only if for all $x \in E$ with $||x|| = 1$, there exists a unique $x^* \in E^*$ such that $||x^*|| = 1$ and $\langle x, x^* \rangle = ||x||$.*

Recall that in any smooth space E, $\rho_E(\tau) \leq \tau$ for all $\tau \geq 0$, where $\rho_E : [0, \infty) \to [0, \infty)$ is the modulus of smoothness of E.

Definition 2.1.5 *Let E be a normed linear space with dim$(E) \geq 2$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by*

$$\begin{aligned} \rho_E(\tau) &:= \sup \Big\{ \frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1; ||y|| = \tau \Big\} \\ &= \sup \Big\{ \frac{||x+\tau y|| + ||x-\tau y||}{2} - 1 : ||x|| = 1; ||y|| = 1 \Big\}. \end{aligned}$$

Definition 2.1.6 *A normed space E is said to be uniformly smooth if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $||x|| = 1$ and $||y|| \leq \delta$, then*

$$||x+y|| + ||x-y|| < 2 + \varepsilon||y||. \tag{2.1.1}$$

Definition 2.1.7 *Let E be a Banach space and let $J : E \to E^{**}$ be the canonical injection from E into E^{**}, that is $\langle J(x), f\rangle = \langle f, x\rangle$, $\forall\, x \in E, f \in E^*$. Then E is said to be reflexive if J is surjective, i.e., $J(E) = E^{**}$.*

Definition 2.1.8 *Let E be a real normed linear space and $p > 1$, Then, the generalized duality map $J_p : E \to 2^{E^*}$ is defined by*

$$J_p(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|,\ \|x\| = \|x^*\|^{p-1}\}, \tag{2.1.2}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between elements of E and E^*.

In particular, for $p = 2$, we have from (2.1.2) that,

$$J_2(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|,\ \|x\| = \|x^*\|\}. \tag{2.1.3}$$

J_2 is called the normalized duality map and it is simply denoted as J. Hence, we make the following remark about J.

Remark 2.1.3 *The following basic properties for Banach space E and for the normalized duality mapping J can be found in Cioranescu, [74]:*

(i) If E is an arbitrary Banach space, then J is monotone and bounded.

(ii) If E is strictly convex Banach space, then J is strictly monotone.

(iii) If E is a smooth Banach space, then J is single-valued and hemi-continuous, i.e., J is continuous from the strong topology of E to the weak star topology of E.

(iv) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.

(v) If E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$.

(vi) A Banach space E is uniformly smooth if and only if E^* is uniformly convex. If E is uniformly smooth, then it is smooth and reflexive.

(vii) If $E = L_p$ space $(2 \le p < \infty)$, then $J : L_p \to L_p^*$ is Lipschitz.

(viii) If $E = L_p$ space $(1 < p < 2)$, then $J : L_p \to L_p^*$ is Hölder continuous. i.e., $\forall\, x, y \in E$, $||Jx - Jy|| \le M||x - y||^\alpha$, for some constants $M > 0$ and $\alpha \in (0, 1]$.

Remark 2.1.4 *If $E = L_p$ spaces $1 < p < \infty$, the formulas for the normalized duality map $J : E \to E^*$ is known precisely (see e.g., Alber [3], p. 36, Cioranescu [74]), and is given by*

$$Jx = ||x||_{l_p}^{2-p}|x|^{p-2}x.$$

Remark 2.1.5 *If $E = l_p$ spaces, $1 < p < \infty$, the formula for the normalized duality map $J : E \to E^*$ is known precisely (see Alber [3], p. 36) and is given by*

$$Jx = ||x||_{l_p}^{2-p}(|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...),$$

for any $x = (x_1, x_2, ...) \in l_p$. For example, let $\bar{x} = (1, \frac{1}{2}, \frac{1}{3}, ...) \in l_3$. Then,

$$J(\bar{x}) = \frac{1}{\Big(\sum\limits_{n=1}^{\infty} \frac{1}{n^3}\Big)^{\frac{1}{3}}} \Big(1, \frac{1}{2}, \frac{1}{3}, ...\Big).$$

Lemma 2.1.1 (Xu, [167]) *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, such that for every $x, y \in B_r(0)$, the following inequality is satisfied*

$$||\lambda x + (1 - \lambda)y||^2 \leq \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)g(||x - y||), \quad \lambda \in (0, 1)$$

.

Lemma 2.1.2 (Xu, [167]) *Let $p > 1$ be a given real number. Then the following are equivalent in a Banach space:*

1. *E is p-uniformly convex;*
2. *There is a constant $c > 0$ such that for every $x, y \in E$ and $jx \in J_p(x)$, The following inequality holds: $\|x + y\|^p \geq \|x\|^p + p\langle y, jx\rangle + c\|y\|^p$;*
3. *There is a constant $c_2 > 0$ such that for every $x, y \in E$ and $jx \in J_p(x), jy \in J_p(y)$, the following inequality holds: $\langle x - y, jx - jy\rangle \geq c_2\|x - y\|^p$.*

Lemma 2.1.3 (Xu, [167]) *Let E be a q-uniformly smooth real Banach space. Then for any $x, y \in E$, there exists $C_q > 0$ such that*

$$||x - y||^q \leq ||x||^q - q\langle y, J_q(x)\rangle + C_q||y||^q.$$

Lemma 2.1.4 (Xu, [167]) *Let E be a 2-uniformly convex real Banach space. Then for all x, $y \in E$, the inequality $||x - y|| \leq \frac{2}{c_2}||Jx - Jy||$ holds, where J is the normalized duality map on E and $0 < c_2 \leq 1$ is the 2-uniformly convex constant of E.*

2.2 Some classes of nonlinear maps

In this section, unless otherwise specified, we let E be a real Banach space with dual E^*. Let $T : D(T) \subset E \to R(T) \subset E$ be a mapping, where $D(T)$ denotes the domain of T and $R(T)$ denotes the range of T.

Definition 2.2.1 *A mapping T with domain $D(T)$ and range $R(T)$ in E is called a contraction if and only if there exists a constant $k \in [0, 1)$ such that for all $x, y \in D(T)$,*

$$||Tx - Ty|| \leq k||x - y||.$$

For $k = 1$, T is called a *nonexpansive mapping.*
Observe that every contraction is nonexpansive, but the converse is false.

Let $D(T)$ and $R(T)$ denote the domain and range of a mapping T, respectively.

Definition 2.2.2 *A mapping $T : D(T) \subset E \to R(T) \subset E$ is called pseudo-contractive if and only if for every $x, y \in D(T)$ and $r > 0$, the following inequality holds:*

$$||x - y|| \leq ||(1 + r)(x - y) - r(Tx - Ty)||.$$

Proposition 2.2.1 *Let E be a real normed space. Then, the duality map $J : E \to 2^{E^*}$ is well defined. That is, for every $x \in E$, $Jx \neq \emptyset$.*

Proof: Let $x \in E$. If $x = 0$, take $x^* = 0$ and the argument follows. Suppose $x \neq 0$, then $x||x|| \neq 0$. By consequences of Hahn Banach theorem, there exists $u^* \in E^*$ such that $||u^*|| = 1$ and $\langle u^*, x||x||\rangle = ||x||^2$.
Now,

$$\langle ||x||u^*, x\rangle = \langle u^*, x||x||\rangle = ||x||^2. \tag{2.2.1}$$

Take $x^* = ||x||u^*$. Then, $x^* \in Jx$. Hence, $Jx \neq \emptyset \quad \forall\, x \in E$.

Definition 2.2.3 (Monotone mapping) *A map $A : D(A) \subset E \to 2^{E^*}$ is said to be monotone if $\forall\ x, y \in D(A),\ x^* \in Ax,\ y^* \in Ay$, we have*

$$\langle x^* - y^*,\ x - y\rangle \geq 0.$$

From the definition above, a single-valued map $A : D(A) \subset E \to E^*$ is monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0,\ \forall\ x, y \in D(A).$$

Definition 2.2.4 (Accretive mapping) *A map $A : D(A) \subset E \to 2^E$ is said to be accretive if $\forall\ x, y \in D(A),\ x^* \in Ax,\ y^* \in Ay$, there exists $j(x - y) \in J(x - y)$ such that*

$$\langle x^* - y^*,\ j(x - y)\rangle \geq 0.$$

From the definition above, a single-valued map $A : D(A) \subset E \to E$ is accretive if

$$\langle Ax - Ay, j(x - y)\rangle \geq 0,\ \forall\ x, y \in D(A).$$

Definition 2.2.5 *A map $A : D(A) \subset E \to E$ is called strongly accretive if there exists $k \in (0, 1)$ such that $\forall\ x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that*

$$\langle Ax - Ay, j(x - y)\rangle \geq k||x - y||^2.$$

Definition 2.2.6 *A map $A : D(A) \subset E \to E$ is γ-inverse strongly accretive if $\forall\ x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that*

$$\langle Ax - Ay, j(x-y)\rangle \geq \gamma ||Ax - Ay||^2,$$

for some $\gamma > 0$.

Theorem 2.2.1 (Kato,1967) *Let E be a real Banach space with dual E^*. Then, the following are equivalent: for all $x, y \in E$,*
1. $||x|| \leq ||x + \lambda y||, \quad \lambda > 0$,
2. there exists $j(x) \in J(x)$ such that $\langle y, j(x)\rangle \geq 0$.

As a consequence of this, the pseudo-contractive mappings can be defined in terms of the normalized duality mappings as follows:

Definition 2.2.7 *A mapping $T : D(T) \subset E \to R(T) \subset E$ is called pseudo-contractive if and only if for every $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that*

$$\langle Tx - Ty, j(x-y)\rangle \leq ||x-y||^2. \tag{2.2.2}$$

The following proposition shows that every nonexpansive mapping is pseudo-contractive.

Proposition 2.2.2 *Let $T : D(T) \subset E \to R(T) \subset E$ be a nonexpansive mapping, then the mapping T is pseudo-contractive.*

Proof: Let T with domain $D(T)$ and range $R(T)$ in E be a nonexpansive mapping, then for all $r > 0$ and $x, y \in D(T)$,

$$\begin{aligned} ||(1+r)(x-y) - r(Tx-Ty)|| &\geq (1+r)||x-y|| - r||Tx-Ty|| \\ &\geq (1+r)||x-y|| - r||x-y|| \\ &= ||x-y||. \end{aligned}$$

Hence, T is pseudo-contractive.
The converse of this proposition is however, not true. To see this, we consider the following example.

Example 2.2.1 *Consider the map $T : [0,1] \to \mathbb{R}$ defined by, $Tx = 1 - x^{\frac{2}{3}}$. Then, T is a pseudo-contractive but not nonexpansive.*
To see that T is pseudo-contractive, let $x, y \in [0,1]$, $r > 0$, then

$$\begin{aligned} ||(1+r)(x-y) - r(Tx-Ty)|| &= ||(1+r)(x-y) - r(1 - x^{\frac{2}{3}} - 1 + y^{\frac{2}{3}})|| \\ &= ||x - y + r(x + x^{\frac{2}{3}} - (y + y^{\frac{2}{3}}))|| \end{aligned}$$

Observe that $A : [0,1] \subset \mathbb{R} \to \mathbb{R}$ defined by $Ax = x + x^{\frac{2}{3}}$ is monotone i.e., for all $x, y \in [0,1]$,

$$\begin{aligned} \langle Ax - Ay, x - y\rangle &= \langle x - y + (x^{\frac{2}{3}} - y^{\frac{2}{3}}), x - y\rangle \\ &= ||x-y||^2 + \langle x^{\frac{2}{3}} - y^{\frac{2}{3}}, x - y\rangle \\ &\geq 0. \end{aligned}$$

Hence, by Kato's theorem, T is pseudo-contractive.
We next show that T is not nonexpansive. Suppose T is nonexpansive, i.e., for all $x, y \in [0,1]$, $||Tx - Ty|| \leq ||x - y||$.
Take $x = 0$ and $y = \frac{1}{8}$, then

$$||Tx - Ty|| = ||1 - (1 - \frac{1}{4})|| = \frac{1}{4} \leq \frac{1}{8}$$

This is a contradiction. Hence, T is not nonexpansive.

2.3 Some useful tools

Definition 2.3.1 *A continuous, strictly increasing function ω: (0,∞)→(0,∞) is called modulus of continuity if $\omega(t) \to 0$ as $t \to 0$. It follows that a function is uniformly continuous if and only if it has a modulus of continuity.*

In the sequel, we shall need the following definitions and results. Let E be a smooth real Banach space with dual E^*. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall\, x, y \in E, \tag{2.3.1}$$

where J is the normalized duality mapping from E into E^* will play a central role in the sequel. It was introduced by Alber and has been studied by Alber [1], Alber and Guerre-Delabriere [2], Kamimura and Takahashi [101], Reich [144] and a host of other authors. If $E = H$, a real Hilbert space, equation (2.3.1) reduces to $\phi(x, y) = \|x - y\|^2 \quad \forall\, x, y \in H$. From the definition of the function ϕ, we have that

$$\begin{aligned} \phi(x, y) &= ||x||^2 - 2\langle x, Jy\rangle + ||y||^2 \\ &\leq ||x||^2 + 2||x||||y|| + ||y||^2 \\ &= (||x|| + ||y||)^2, \end{aligned}$$

also, we have

$$\begin{aligned} \phi(x, y) &= ||x||^2 - 2\langle x, Jy\rangle + ||y||^2 \\ &\geq ||x||^2 - 2||x||||y|| + ||y||^2 \\ &= (||x|| - ||y||)^2. \end{aligned}$$

Then, combining the two inequalities, we obtain

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2 \quad \forall\, x,y \in E. \tag{2.3.2}$$

Define a map $V : X \times X^* \to \mathbb{R}$ by

$$V(x,x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2. \tag{2.3.3}$$

From this definition, we obtain

$$\begin{array}{rcl} V(x,x^*) & = & \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \\ & = & ||x||^2 - \langle x, J(J^{-1}x^*)\rangle + \|J^{-1}x^*\|^2 \\ & = & \phi(x, J^{-1}(x^*)). \end{array}$$

Thus,

$$V(x,x^*) = \phi(x, J^{-1}(x^*)) \quad \forall \;\; x \in X, \;\; x^* \in X^*. \tag{2.3.4}$$

Lemma 2.3.1 *(see e.g., [3], p.36) Let E be a reflexive strictly convex Banach space with strictly convex dual space E^*. If $J_p : E \to E^*$ and $J_q^* : E^* \to E$ are the duality mappings on E and E^*, respectively, such that $\frac{1}{p} + \frac{1}{q} = 1$, then $J_p^{-1} = J_q^*$, for $p \in (0,\infty)$.*

Lemma 2.3.2 *Let $f : E \to R \cup \{+\infty\}$ be a function defined by*

$$f(x) = \frac{1}{2}||x||^2 \quad \forall\, x \in E.$$

Then, for each $x \in E$, $\partial f(x) = J(x)$, where J is the duality map on E.

Proof:
Let $x^* \in J(x)$. Then, for any $y \in E$, we have

$$\begin{array}{rcl} \langle y - x, x^*\rangle & = & \langle y, x\rangle - ||x||^2 \\ & \le & ||y||||x|| - ||x||^2 \\ & \le & \frac{1}{2}||y||^2 - \frac{1}{2}||x||^2 \\ & = & f(y) - f(x). \end{array}$$

Thus, we have $x^* \in \partial f(x)$.
Conversely, for $x^* \in \partial f(x)$, we have

$$\langle y - x, x^*\rangle \le f(y) - f(x) \quad \forall\, y \in E.$$

For $t \in (0,1)$, set $y = x + ty$, then we have

$$\langle y, x^*\rangle \le \frac{1}{2t}(||x+ty||^2 - ||x||^2) \le ||x||||y|| + \frac{t}{2}||y||^2.$$

As $t \to 0^+$, we have $\langle y, x^* \rangle \leq ||x|| ||y||$, which implies $||x^*|| \leq ||x||$. Also, using the fact that $x^* \in \partial f(x)$ and setting $y = x - tx$, $t \in (0,1)$, we have

$$2t\langle -x, x^* \rangle \leq ||x - tx||^2 - ||x||^2 = (t^2 - 2t)||x||^2.$$

So, we have $(2-t)||x||^2 \leq 2\langle x, x^* \rangle$. Now, as $t \to 0^+$ we obtain

$$||x||^2 \leq \langle x, x^* \rangle \leq ||x|| ||x^*||,$$

which implies $||x|| \leq ||x^*||$.
Therefore, we have $||x|| = ||x^*||$ and $\langle x, x^* \rangle = ||x||^2$. Thus, $x^* \in J(x)$.
Hence, $\partial f(x) = J(x)$.

Lemma 2.3.3 (Alber, [1]) *Let X be a reflexive striclty convex and smooth Banach space with X^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*) \tag{2.3.5}$$

for all $x \in X$ and $x^*, y^* \in X^*$.

Proof:
For arbitrary $x \in E$, $x^*, y^* \in E^*$, we have

$$\begin{aligned} V(x, x^*) &= ||x||^2 - 2\langle x, x^* \rangle + ||x^*||^2 \\ &= ||x||^2 - 2\langle x, x^* + y^* \rangle + ||x^* + y^*||^2 + ||x^*||^2 - ||x^* + y^*||^2 + 2\langle x, y^* \rangle \\ &= V(x, x^* + y^*) + ||x^*||^2 - ||x^* + y^*||^2 + 2\langle x, y^* \rangle. \end{aligned}$$

Using the subdifferential inequality and the fact that $\partial(\frac{1}{2}||\cdot||^2) = J_* = J^{-1}$ (see lemmas 2.3.1 and 2.3.2), where $\|\cdot\|_*$ and J_* are the norm and the normalized duality map on E^* respectively. Thus, we have

$$\begin{aligned} V(x, x^*) &\leq V(x, x^* + y^*) - 2\langle J^{-1}x^*, y^* \rangle - 2\langle -x, y^* \rangle \\ &= V(x, x^* + y^*) - 2\langle J^{-1}x^* - x, y^* \rangle. \end{aligned}$$

Hence, this completes the proof.

Lemma 2.3.4 (Kamimura and Takahashi, [101]) *Let X be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 2.3.5 (Alber and Ryazantseva, [3]) *Let X be a uniformly convex Banach space. Then, for any $R > 0$ and any $x, y \in X$ such that $||x|| \leq R$, $||y|| \leq R$, the following inequality holds:*

$$\langle Jx - Jy, x - y \rangle \geq (2L)^{-1}\delta_X(c_2^{-1}||x - y||),$$

where $c_2 = 2\max\{1, R\}$, $1 < L < 1.7$.

Observe that from lemma 2.3.5, we obtain

$$||x-y|| \leq c_2\delta_X^{-1}(2L||Jx-Jy||||x-y||). \tag{2.3.6}$$

Lemma 2.3.6 (Alber and Ryazantseva, [3]) *Let E be a uniformly convex and smooth Banach space. Then for any $x, y \in E$ such that $||x|| \leq R$, $||y|| \leq R$, the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \leq 8||Jx - Jy||^2 + c_1\rho_{E^*}(||Jx - Jy||),$$

where $c_1 = 8\max\{L, R\}$ and $\rho_{E^*} : [0, \infty) \to [0, \infty)$ is the modulus of smoothness of E^*.

Lemma 2.3.7 (Alber, [1]) *Let C be nonempty closed convex subset of a smooth Banach space E, $x_0 \in C$ and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C$.*

Lemma 2.3.8 (Alber, [1]) *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then, $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C$, where Π_C is the generalized projection of E onto C.*

Lemma 2.3.9 *Let E be a 2-uniformly convex and smooth Banach space. Then, for every $x,\ y \in E$, $\phi(x, y) \geq c_1||x - y||^2$, where $c_1 > 0$ is the 2-uniformly convexity constant of E.*

Lemma 2.3.10 (Tan and Xu, [164]) *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq a_n + \sigma_n, \quad n \geq 0, \tag{2.3.7}$$

such that $\sum_{n=1}^{\infty}\sigma_n < \infty$. Then, $\lim\limits_{n\to\infty} a_n$ exists. If, in addition, the sequence $\{a_n\}$ has a subsequence that converges to 0, then the sequence $\{a_n\}$ converges to 0.

Lemma 2.3.11 (Chidume, [59]) *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, there exists a continuous strictly increasing convex function*

$$g : [0, \infty) \to [0, \infty), g(0) = 0,$$

such that for every $x, y \in B_r(0)$, the following inequalities is satisfied

$$\langle Jx - Jy, x - y\rangle \geq g(||x - y||),$$

where J is the single-valued normalized duality map.

Lemma 2.3.12 (Chidume, [67]) *Let E be uniformly convex real Banach space. For arbitrary $d > 0$, let $B_d := \{x \in E : ||x|| \leq d\}$. Then, for arbitrary $x, y \in B_d(0)$, the following inequality holds:*

$$\phi(x,y) \leq ||x-y||^2 + ||x||^2. \tag{2.3.8}$$

Lemma 2.3.13 *(Xu and Roach, [167]). Let E be a uniformly smooth real Banach space. Then, there exist constants D and C such that for all $x,\ y \in E,\ j(x) \in J(x)$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + D\max\left\{\|x\| + \|y\|, \frac{1}{2}C\right\}\rho_E(\|y\|),$$

where ρ_E denotes the modulus of smoothness of E.

Lemma 2.3.14 *Let E be a normed real linear space. Then, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \ \forall\ j(x+y) \in J(x+y),\ \forall\ x,y \in E. \tag{2.3.9}$$

Lemma 2.3.15 (Xu, [166]) *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, there exists a continuous strictly increasing convex function*

$$g : [0,\infty) \to [0,\infty), g(0) = 0,$$

such that for every $x, y \in B_r(0)$, the following inequalities is satisfied

$$\langle Jx - Jy, x - y\rangle \geq g(||x-y||),$$

where J is the single-valued normalized duality map.

CHAPTER 3

Approximation of zeros of m-accretive maps with application to Hammerstein integral equations

3.1 Introduction

In this chapter, we first used a new important result concerning accretive operators which was recently proved by Chidume *et al.* [60] to prove a strong convergence to a zero of an m-accretive map in a uniformly smooth real Banach space. Furthermore, the convergence result obtained is applied to approximate a solution of a Hammerstein integral equation. Finally, some numerical examples are presented to illustrate the convergence of the sequence of our algorithm.

We shall use the following lemma in the sequel.

Lemma 3.1.1 (Fitzpatrick, Hess and Kato, [86]) *Let E be a real reflexive Banach space, $A : D(A) \subset E \to E$ be an accretive mapping. Then A is locally bounded at any interior point of $D(A)$.*

Lemma 3.1.2 (Chidume *et al.* [60]) *Let E be a reflexive Banach space and $A : E \to 2^E$ be an accretive map with $0 \in IntD(A)$. Then, for any $M > 0$, there is exists $C > 0$ such that:*

(i) $(y, v) \in G(A)$;

(ii) $\langle v, j(x) - j(x - y)\rangle \leq M(2\|x\| + \|y\|)$;

(iii) $\|y\| \leq M$,

imply $\|v\| \leq C$.

3.2 Main results

In Theorem 3.2.1 below, $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\theta_n = 0, \quad \{\theta_n\|$ is decreasing;

(ii) $\lim_{n\to\infty}\left[\frac{\frac{\theta_{n-1}}{\theta_n}-1}{\lambda_n\theta}\right] = 0;$

(iii) $\dfrac{\rho_E(M_0\lambda_n)}{M_0\lambda} \le \gamma_0\theta_n,$

for some constants $\gamma_0 > 0$ and $M_0 > 0$; where ρ_E is the modulus of smoothness.

Prototypes for $\{\lambda_n\}$ and $\{\theta_n\}$ are:

$$\lambda_n = \frac{1}{(n+1)^a} \qquad \text{and} \qquad \theta_n = \frac{1}{(n+1)^b},$$

where $a+b<1$ and $0<b<a$ (see e.g., Chidume and Idu, [70]).

We now prove the following theorem.

Theorem 3.2.1 *Let E be a uniformly smooth real Banach space and let $A : E \to 2^E$ be a multi-valued m-accretive operator with $D(A) = E$ such that the inclusion $0 \in Au$ has a solution. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = x_n - \lambda_n u_n - \lambda_n\theta_n(x_n - x_1), \qquad u_n \in Ax_n, \quad n \ge 1. \tag{3.2.1}$$

Then the sequence $\{x_n\}$ converges strongly to a solution of the inclusion $0 \in Au$.

Proof:
First, we show that $\{x_n\}$ is bounded. Let x^* be a solution of the inclusion $0 \in Au$. Then there exists $r > 0$ such that $x_1 \in B(x^*, \frac{r}{2}) := \{x \in E : \|x - x^*\| \le \frac{r}{2}\}$. Define $B = B(x^*, r)$. Then for any $x \in B$, we have that $\|x\| \le r + \|x^*\|$.

Let $x,\ y \in E$ and $u_y \in Ay$ be arbitrary. Since A is locally bounded at $0 \in E = int(D(A))$, there exist $\delta > 0$, $K > 0$ such that $\|u_e\| \le K$, for all $w \in B(0,\delta)$, $u_w \in Aw$. Therefore, we have,

$$\begin{aligned} \langle u_y, j(x) - j(x-y)\rangle &\le \|u_y\|\|j(x) - j(x-y)\| \\ &\le K\|j(x) - j(x-y)\|, \quad \text{for } y \in B(0,\delta) \\ &\le K(2\|(x)\| + \|y\|), \quad \text{for } \|y\| \le \delta. \end{aligned}$$

Define

$M := \max\{r + \|x^*\|, \delta, K\}$. So, $\|y\| \le M$ and $\langle u_y, j(x) - j(x-y)\rangle \le M(2\|(x)\| + \|y\|),$

which implies, by Lemma 3.1.2, that there exists $L > 0$ such that $\|u_y\| \leq L$.

Now, define the following:

$$M_0 := \sup\{\|u_x + \theta(x - x_1)\| : x \in B, u \in Ax; 0 < \theta < 1\} + 1.$$

$$M_1 := \sup\Big\{D \max\Big\{\|x\| + \lambda M_0, \frac{C}{2}\Big\} : x \in B, \lambda \in (0,1)\Big\}.$$

$$\gamma_0 := \frac{1}{2}\min\Big\{1, \frac{r^2}{4M_1M_0}\Big\},$$

where D and C are the constants in Lemma 2.3.13.

Claim: $x_n \in B, \quad \forall\, n \geq 1.$

We prove this by induction. By construction, $x_1 \in B$. Assume $x_n \in B$ for some $n \geq 1$. We prove $x_{n+1} \in B$. Using the recursion formula (3.2.1), Lemma 2.3.13, the fact that $h(\tau) := \frac{\rho_E(\tau)}{\tau}$ is non-decreasing, and denoting $0 \in Ax^*$ by 0^*, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|x_n - x^* - \lambda_n(u_n + \theta_n(x_n - x_1))\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\lambda_n\langle u_n + \theta_n(x_n - x_1), j(x_n - x^*)\rangle \\
&\quad + D\max\Big\{\|x_n - x^*\| + \lambda_n\|u_n + \theta_n(x_n - x_1)\|, \frac{C}{2}\Big\} \times \rho_E(\lambda_n\|u_n + \theta_n(x_n - x_1)\|) \\
&\leq \|x_n - x^*\|^2 - 2\lambda_n\langle u_n - 0^*, j(x_n - x^*)\rangle - 2\lambda_n\theta_n\langle x_n - x_1 . j(x_n - x^*)\rangle \\
&\quad + M_1\rho_E(\lambda_n\|u_n + \theta_n(x_n - x_1)\|) \\
&\leq \|x_n - x^*\|^2 - 2\lambda_n\theta\|x_n - x^*\|^2 + \lambda_n\theta_n(\|x^* - x_1\|^2 + \|x_n - x^*\|^2) \\
&\quad + + M_1\rho_E(\lambda_n\|u_n + \theta_n(x_n - x_1)\|) \\
&\leq (1 - \lambda_n\theta_n)\|x_n - x^*\|^2 + \lambda_n\theta_n\|x^* - x_1\|^2 + M_1\frac{\rho_E(\lambda_n M_0)}{\lambda_n M_0}\lambda_n M_0 \\
&\leq (1 - \lambda_n\theta_n)\|x_n - x^*\|^2 + \lambda_n\theta_n\|x^* - x_1\|^2 + M_1\gamma_0\lambda_n\theta_n M_0 \\
&\leq \Big(1 - \frac{1}{2}\lambda_n\theta_n\Big)r^2 \leq r^2.
\end{aligned}$$

Hence, $x_n \in B \quad \forall \quad n \geq 1$, and so $\{x_n\}$ is bounded. The rest of the proof of the convergence of $\{x_n\}$ to a zero of A follows the same method as in the proof of Theorem 3.2 in [44].

3.3 Applications to Hammerstein integral equations

Definition 3.3.1 *Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of Hammerstein-type has the form*

$$u(x) + \int_\Omega k(x,y)f(y,u(y))dy = w(x), \tag{3.3.1}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions.

By simple transformation (3.3.1) can put in the form

$$u + KFu = w. \tag{3.3.2}$$

which, without loss of generality can be written as

$$u + KFu = 0. \tag{3.3.3}$$

Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear part posses Green's function can, as a rule, be transformed into the form (3.3.1) (see e.g., Pascali and Sburian [134], chapter p. 164).

Many iterative methods for approximating solutions of problem (3.3.3) have been studied extensively (see e.g., Chidume and Zegeye [46, 47], Chidume and Djitte [48], Ofoedu and Onyi [132], Shehu [154], Chidume and Idu [70], Djitte and Sene [80], Chidume and Shehu, [73], Chidume and Bello [72]) and the references therein.

We shall apply Theorem 3.2.1 to approximate a solution of problem (3.3.3). To do this, the following lemma would be needed in what follows.

Lemma 3.3.1 (Barbu, [11]) *Let E be a real Banach space, A be m-accretive set of $E \times E$ and let $B : E \to E$ be a continuous, m-accretive operator with $D(B) = E$. Then $A + B$ is m-accretive.*

Lemma 3.3.2 *Let E be a uniformly convex and uniformly smooth real Banach space and $X = E \times E$. Let $F, K : E \to E$ be m-accretive mappings. Let $A : X \to X$ be defined by $A([u, v]) = [Fu - v, Kv + u]$. Then, A is m-accretive.*

Proof:
Define $S, T : E \times E \to E \times E$ as

$$S[u, v] = [Fu, Kv] \quad T[u, v] = [-v, u].$$

Then $A = S + T$. It is easy to verify that S is m-accretive and that T is m-accretive, continuous and $D(T) = E$. Hence, by Lemma 3.3.1, A is m-accretive.

Remark 3.3.1 *We remark that for A defined in Lemma 3.3.2, $[u^*, v^*]$ is a zero of A if and only if u^* solves (3.3.3), where $v^* = Fu^*$.*

We now prove the following theorem.

Theorem 3.3.1 *Let X be a uniformly smooth and uniformly convex real Banach space. Let F, $K : X \to X$ be m-accretive mappings. Let $E := X \times X$ and $A : E \to E$ be defined by $A([u, v]) := [Fu - v, Kv + u]$. For arbitrary x_1, $u_1 \in E$, define the sequences $\{u_n\}$ in E by*

$$u_{n+1} = u_n - \lambda_n A u_n - \lambda_n \theta_n (u_n - x_1), \quad n \geq 1, \tag{3.3.4}$$

Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}_{n=1}^{\infty}$ converge strongly to a solution of $u + KFu = 0$.

Proof:
By Lemma 3.3.2, E is uniformly convex and uniformly smooth, and by Lemma 3.3.1, A is m-accretive. Hence, the conclusion follows from Theorem 3.2.1 and Remark 3.3.1.

Theorem 3.3.1 can also be stated as follows.

Theorem 3.3.2 *Let X be a uniformly smooth and uniformly convex real Banach space and let F, $K : X \to X$, be m-accretive mappings. For (x_1, y_1), $(u_1, v_1) \in X \times X$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E, by*

$$u_{n+1} = u_n - \lambda_n (F u_n - v_n) - \lambda_n \theta_n (u_n - x_1), \quad n \geq 1,$$

$$v_{n+1} = v_n - \lambda_n (K v_n + u_n) - \lambda_n \theta_n (v_n - y_1), \quad n \geq 1.$$

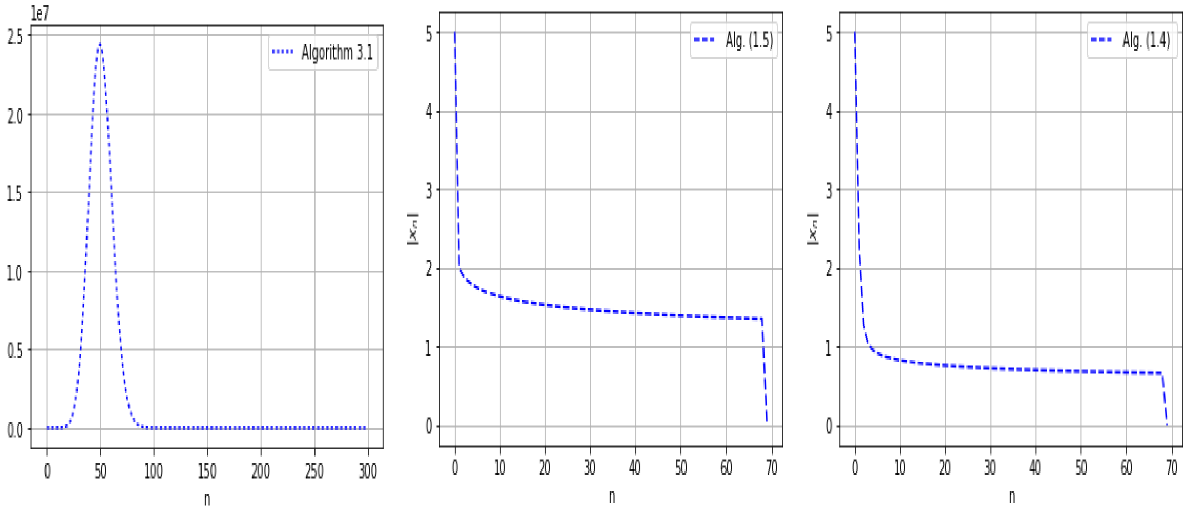
Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

3.4 Numerical Experiment

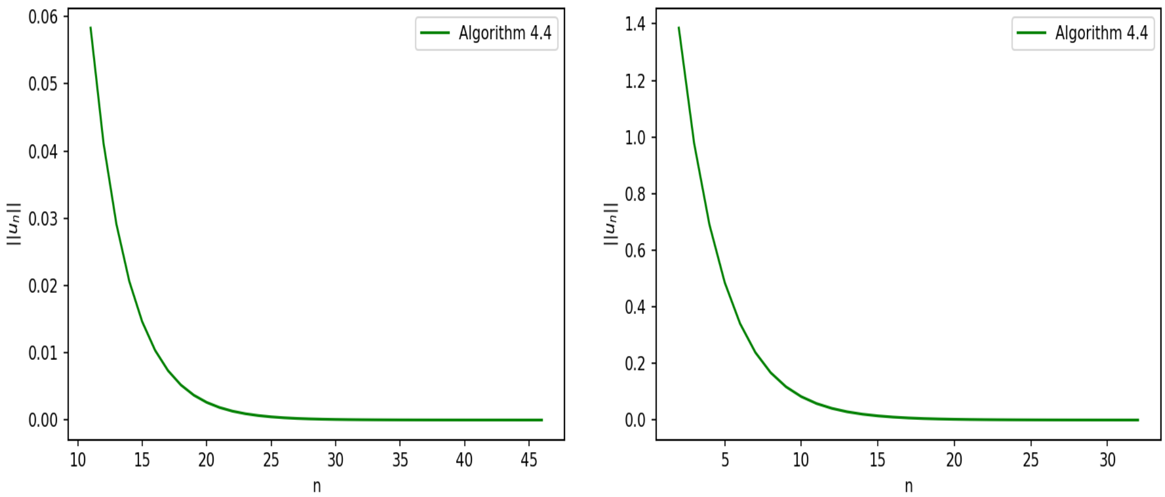
In this section, we shall numerically demonstrate the convergence of the sequence generated by the algorithm proposed in this paper. We shall also investigate the proximal point algorithm and some of its modifications.

Example 3.4.1 *Let $E = \mathbb{R}$ and $Ax = 4x$. Then, A is accretive and $0 \in A^{-1}(0)$. Taking $\lambda_n = \frac{1}{(n+1)^{0.2}}$, and $\theta_n = \frac{1}{(n+1)^{0.25}}$ we obtain the following table and graph of $|x_n|$ against number of iterations, where $\{x_n\}$ is the sequence generated by the algorithm for approximating solutions of $Au = 0$, assuming existence.*

No of iterations	Initial Points	$\lvert x_n \rvert$	Time (s)
189	2	0.12506344	0.1206655502319336
198	2	0.1247684	0.1018977165222168
600	0.5	0.02405017]	0.10132288932800293
944	-0.5	0.02157754	0.09952473640441895
1999	1.5	0.05406199	0.12050509452819824



Example 3.4.2 *Let* $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ *and* $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. *Let* $F = \begin{bmatrix} 3 & 1 \\ -1 & 8 \end{bmatrix}$ *and* $K = \begin{bmatrix} 7 & -2 \\ 2 & 5 \end{bmatrix}$. *Taking* $\lambda_n = \frac{1}{(n+1)^{0.2}}$, *and* $\theta_n = \frac{1}{(n+1)^{0.25}}$, *and the initial points* $u = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ *and* $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, *we obtain the following graph of* $|u_n|$ *against number of iterations, where* $\{u_n\}$ *is the sequence generated by Algorithm (3.3.4) for approximating solutions of* $u + KFu = 0$, *assuming existence.*



Remark 3.4.1 *We observe that from the above diagrams, Algorithm 3.1 is more desirable than Algorithms 1.4 and 1.5.*

All the results obtained in this chapter are results obtained in the following paper:

C.E. Chidume, **U.V. Nnyaba**, O.M. Romanus and A. Adamu; Approximation of zeros of m-accretive mappings, with applications to Hammerstein integral equations **(to appear)**.

CHAPTER 4

Approximation of solutions of variational inequality problems of generalized Phi-strongly monotone maps with applications

4.1 Introduction

In this chapter, we construct a new iterative algorithm and prove that the sequence generated by the algorithm converges strongly to a solution of variational inequality problem, $VI(A, \cap_{i=1}^{N} F(T_i))$ in a uniformly smooth and uniformly convex real Banach space, where $A : E \to E^*$ is a generalized Φ-strongly monotone and bounded map and let $T_i : C \to E$, $i = 1, 2, 3, ..., N$ is a finite family of quasi-ϕ-nonexpansive maps such that $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. Furthermore, results obtained are applied to a convex optimization problem. Finally, we consider a family $\{T_i\}_{i=1}^{N}$ of maps where for each i, T_i maps E into its dual space E^* and prove a strong convergence theorem for $VI(A, \cap_{i=1}^{N} F_J(T_i))$, where $F_J(T_i)$ is the set of J-fixed points of T_i.

We shall using the following lemma in this chapter.

Lemma 4.1.1 (Alber and Ryazantseva, [3]) *Let E be a uniformly convex Banach space. Then, for any $R > 0$ and any $x, y \in E$ such that $||x|| \leq R$, $||y|| \leq R$, the following inequality holds:*

$$\langle Jx - Jy, x - y \rangle \geq (2L)^{-1} \delta_E(c_2^{-1} ||x - y||),$$

where $c_2 = 2 \max\{1, R\}$, $1 < L < 1.7$, δ_E is the modulus of convexity of E.

Observe that from lemma 4.1.1, we obtain

$$||x - y|| \leq c_2 \delta_E^{-1} \big(2L ||Jx - Jy|| ||x - y|| \big). \tag{4.1.1}$$

4.2 Main result

In Theorem 4.2.1 below, the sequence $\{\theta_n\}$ in $(0,1)$ satisfies the following conditions:

$$(i) \quad \lim_{n\to\infty} \theta_n = 0; \ (ii) \sum_{n=1}^{\infty} \theta_n = \infty; \ (iii) \ \sum_{n=1}^{\infty} \theta_n \delta_E^{-1}(2KLM\theta_n) < \infty;$$
$$(iv) \quad c_2 \delta_E^{-1}(2KLM\theta_n) \leq \gamma_0,$$

where δ_E is the modulus of convexity of E and M, L, K, γ_0 are some positive constants.

Theorem 4.2.1 *Let E be a uniformly convex and uniformly smooth real Banach space and E^* be its dual. Let $A : E \to E^*$ be a generalized Φ-strongly monotone and bounded map and let $T_i : E \to E$, $i = 1, 2, 3, ..., N$ be a finite family of quasi-ϕ-nonexpansive maps such that $Q := \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence in E defined iteratively by $x_1 \in E$,*

$$x_{n+1} = J^{-1}(J(T_{[n]}x_n) - \theta_n A(T_{[n]}x_n)), \quad \forall\, n \geq 1,$$

where $T_{[n]} := T_{n \ mod \ N}$. Assume $VI(A, Q) \neq \emptyset$, then $\{x_n\}$ converges strongly to some $x^* \in VI(A, Q)$.

Proof:
Since $VI(A, Q) \neq \emptyset$, let $x^* \in VI(A, Q)$ and let $\delta > 0$ be arbitrary but fixed. Then, there exists $r > 0$ such that $max\{\phi(x^*, x_1), 4\delta^2 + ||x^*||^2\} \leq r$. Define

$$\begin{aligned} M : &= sup\{||A(T_{[n]}x)|| : ||x|| \leq \sqrt{r} + ||x^*||\} + 1, \\ \gamma_0 : &= min\Big\{1, \delta, \frac{\Phi(\delta)}{2Mc_2}\Big\}. \end{aligned}$$

We first show that $\{x_n\}$ is bounded.
Claim: $\phi(x^*, x_n) \leq r$, $\forall\ n \geq 1$. We proceed by induction. By construction, $\phi(x^*, x_1) \leq r$. Assume $\phi(x^*, x_n) \leq r$ for some $n \geq 1$. We now show that $\phi(x^*, x_{n+1}) \leq r$. Suppose for contradiction that it is not true, i.e., suppose $\phi(x^*, x_{n+1}) > r$. Then, using lemma 2.3.3 with $y^* = \theta_n A(T_{[n]}x_n)$, quasi-$\phi$-nonexpansiveness of T_i and the fact that $x^* \in VI(A, Q)$ we have that

$$\begin{aligned} r < \phi(x^*, x_{n+1}) &= V\Big(x^*, J(T_{[n]}x_n) - \theta_n A(T_{[n]}x_n)\Big) \\ &\leq \phi(x^*, T_{[n]}x_n) - 2\theta_n \langle J^{-1}(J(T_{[n]}x_n) \\ &\quad -\theta_n A(T_{[n]}x_n)) - x^*, A(T_{[n]}x_n)\rangle \\ &\leq \phi(x^*, x_n) - 2\theta_n \langle T_{[n]}x_n - x^*, A(T_{[n]}x_n)\rangle \\ &\quad -2\theta_n \langle x_{n+1} - T_{[n]}x_n, A(T_{[n]}x_n)\rangle \\ &= \phi(x^*, x_n) - 2\theta_n \langle T_{[n]}x_n - x^*, A(T_{[n]}x_n) - A(x^*)\rangle \\ &\quad -2\theta_n \langle T_{[n]}x_n - x^*, A(x^*)\rangle - 2\theta_n \langle x_{n+1} - T_{[n]}x_n, A(T_{[n]}x_n)\rangle \\ &\leq \phi(x^*, x_n) - 2\theta_n \langle T_{[n]}x_n - x^*, A(T_{[n]}x_n) - A(x^*)\rangle \\ &\quad +2\theta_n \|x_{n+1} - T_{[n]}x_n\| \|A(T_{[n]}x_n)\|. \end{aligned}$$

Using the fact that A is generalized Φ-strongly monotone and inequality (4.1.1) we obtain:

$$\begin{aligned} r &< \phi(x^*, x_{n+1}) \\ &\le \phi(x^*, x_n) - 2\theta_n \Phi(\|T_{[n]}x_n - x^*\|) + 2Mc_2\theta_n\delta_E^{-1}(2LKM\theta_n). \end{aligned} \quad (4.2.1)$$

Furthermore, $\|x_{n+1} - T_{[n]}x_n\| \le c_2\delta_E^{-1}(2LMK\theta_n)$, implies

$$\|x_{n+1} - x^*\| - \|T_{[n]}x_n - x^*\| \le c_2\delta_E^{-1}(2LMK\theta_n),$$

which yields

$$\|T_{[n]}x_n - x^*\| \ge \|x_{n+1} - x^*\| - c_2\delta_E^{-1}(2LMK\theta_n). \quad (4.2.2)$$

From lemma 2.3.12, we have

$$r < \phi(x^*, x_{n+1}) \le ||x_{n+1} - x^*||^2 + ||x^*||^2. \quad (4.2.3)$$

Using the choice of r, we obtain from inequality (4.2.3) that

$$||x_{n+1} - x^*||^2 > r - ||x^*||^2 \ge 4\delta^2 + ||x^*||^2 - ||x^*||^2.$$

Hence,

$$||x_{n+1} - x^*|| \ge 2\delta.$$

Substituting into inequality (4.2.2) and using condition (iv) and choice of γ_0, we obtain

$$||T_{[n]}x_n - x^*|| \ge 2\delta - c_2\delta_E^{-1}(2LMK\theta_n) \ge 2\delta - \gamma_0 \ge \delta.$$

Since Φ is strictly increasing, we obtain

$$\Phi(||T_{[n]}x_n - x^*||) \ge \Phi(\delta). \quad (4.2.4)$$

Substituting into inequality (4.2.1) and using condition (iv) and the choice of γ_0 we have that

$$\begin{aligned} r < \phi(x^*, x_{n+1}) &\le \phi(x^*, x_n) - 2\theta_n\Phi(\delta) + 2Mc_2\theta_n\delta_E^{-1}(2LKM\theta_n) \\ &\le r - 2\theta_n\Phi(\delta)) + 2Mc_2\theta_n\gamma_0 \le r - 2\theta_n\Phi(\delta)) + \theta_n\Phi(\delta)) < r. \end{aligned}$$

This is a contradiction. Therefore, the claim holds. Hence, the sequence $\{x_n\}$ is bounded.
We show that the sequence $\{x_n\}$ converges strongly to x^*. By the same method of computation as before, we have that

$$\begin{aligned} \phi(x^*, x_{n+1}) &\le \phi(x^*, x_n) - 2\theta_n\Phi(||T_{[n]}x_n - x^*||) \\ &\quad +2Mc_2\theta_n\delta_E^{-1}(2LKM\theta_n) \\ &\le \phi(x^*, x_n) + 2Mc_2\theta_n\delta_E^{-1}(2LKM\theta_n). \end{aligned} \quad (4.2.5)$$

By using condition (ii) and applying lemma 2.3.10 to the last inequality, we obtain that $\lim \phi(x^*, x_n)$ exists. Also, from inequality (4.2.5), we have that

$$2\theta_n \Phi(||T_{[n]}x_n - x^*||) \leq \phi(x^*, x_n) - \phi(x^*, x_{n+1}) + 2Mc_2\theta_n\delta_E^{-1}(2LKM\theta_n).$$

Claim: $\liminf \Phi(||T_{[n]}x_n - x^*||) = 0$.
Suppose not. i.e., suppose $\liminf \Phi(||T_{[n]}x_n - x^*||) := a > 0$. Then, there exists an integer $N_0 > 0$ such that for all integers $n \geq N_0$, $\Phi(||T_{[n]}x_n - x^*||) > \frac{a}{2}$. Hence, using condition (ii) and summing, we have that:

$$a\sum_{n=1}^{\infty}\theta_n \leq \sum_{n=1}^{\infty}\Big(\phi(x^*, x_n) - \phi(x^*, x_{n+1})\Big) + 2\sum_{n=1}^{\infty}Mc_2\theta_n\delta_E^{-1}(2LKM\theta_n) < \infty,$$

contradicting the hypothesis that $\sum\limits_{n=1}^{\infty}\theta_n = \infty$. Hence, $\liminf \Phi(||T_{[n]}x_n - x^*||) = 0$. So, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Phi(||T_{[n]}x_{n_k} - x^*||) \to 0, k \to \infty$.
From the property of Φ (i.e., Φ is strictly increasing and $\Phi(0)=0$), it follows that $||T_{[n]}x_{n_k} - x^*|| \to 0$ as $k \to \infty$. Recall that from Lemma 4.1.1 we have that

$$\|x_{n+1} - T_{[n]}x_n\| \leq c_2\delta_E^{-1}(2LMK\theta_n) \to 0, \quad n \to \infty.$$

Therefore,

$$\|x_{n_{k+1}} - x^*\| \leq \|x_{n_{k+1}} - T_{[n]}x_{n_k}\| + \|T_{[n]}x_{n_k} - x^*\| \to 0, \quad k \to \infty.$$

Using the definition of ϕ and the continuity of J on bounded subsets of E, we obtain

$$\phi(x^*, x_{n_{k+1}}) = \|x^*\| - 2\langle x^*, Jx_{n_{k+1}}\rangle + \|x_{n_{k+1}}\| \to 0, \quad k \to \infty,$$

which implies that $\phi(x^*, x_{n_k}) \to 0, \quad k \to \infty$. Therefore, by Lemma 2.3.10, $\phi(x^*, x_n) \to 0$, as $n \to \infty$. Hence, by Lemma 2.3.4, $\|x_n - x^*\| \to 0$, as $n \to \infty$. This completes the proof.

4.3 Application to convex minimization problem.

In this section, the following well known important results will be needed.

Lemma 4.3.1 *Let E be a real Banach space with E^* as its dual and let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper convex functional. Let $\partial f : E \to 2^{E^*}$ denote the subdifferential of f. Then, $p \in E$ is a minimizer of f if and only if $0 \in \partial f(p)$.*

Definition 4.3.1 *A function $f : E \to \mathbb{R}$ is said to be generalized h-strongly convex if there exists a strictly increasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ such that for every $x, y \in E$ with $x \neq y$ and $\gamma \in (0, 1)$, the following inequality holds:*

$$f(\gamma x + (1-\gamma)y) \leq \gamma f(x) + (1-\gamma)f(y) - \frac{1}{2}h(\|x - y\|). \tag{4.3.1}$$

Lemma 4.3.2 *Let E be a real normed space with dual space E^* and let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper generalized h-strongly convex function. Then, the subdifferential map, $\partial f : E \to 2^{E^*}$ is generalized Φ-strongly monotone.*

Proof:
Let $x, y \in E$ and let $x^* \in \partial f(x), y^* \in \partial f(y)$. Then,

$$f(x) - f(z) \leq \langle x - z, x^*\rangle, \ \ \forall\, z \in E \ \ and \ \ f(y) - f(w) \leq \langle y - w, x^*\rangle, \ \ \forall\, w \in E.$$

For $\gamma \in (0, 1)$, take in particular $z = \gamma y + (1-\gamma)x$ and $w = \gamma x + (1-\gamma)y$. Then,

$$f(x) - f(\gamma y + (1-\gamma)x) \leq \gamma\langle x - y, x^*\rangle; \tag{4.3.2}$$
$$f(y) - f(\gamma x + (1-\gamma)y) \leq \gamma\langle y - x, x^*\rangle. \tag{4.3.3}$$

Adding inequalities (4.3.2) and (4.3.3) and using the generalized h-strong convexity of f we have that for some strictly increasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0 :$ $\langle x - y, x^* - y^*\rangle \geq \Phi(\|x - y\|)$.
Therefore, ∂f is generalized Φ-strongly monotone, with $\Phi = h$.

The following lemma is well known (see e.g., Chidume and Idu [70]).

Lemma 4.3.3 *Let E be a normed space with E^* as its dual and let $f : E \to \mathbb{R}$ be a convex function that is bounded on bounded subsets of E. Then, the subdifferential map of f, $\partial f : E \to 2^{E^*}$ is bounded on bounded subsets of E.*

Lemma 4.3.4 *Let E be real normed space with dual space E^*. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper convex function and $\partial f : E \to 2^{E^*}$, the subdifferential of f. Suppose $x^* \in VI(\partial f, E)$. Then, x^* is a minimizer of f over E.*

Proof:
Let $x^* \in VI(\partial f, E)$ and $x \in E$. then, for any $\tau_{x^*} \in \partial f$ we have that:

$$f(x) - f(x^*) \geq \langle x - x^*, \tau_{x^*}\rangle \geq 0,$$

which implies $f(x^*) \leq f(x)$. Hence, x^* is a minimizer of f over E.

We now prove the main theorem of this section.

Theorem 4.3.1 *Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^*. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper and generalized h-strongly convex function. Let $T_i : E \to E, i = 1, 2, ..., N$ be a finite family of quasi-ϕ-nonexpansive maps such that $Q := \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence in E defined iteratively by $x_1 \in E$,*

$$x_{n+1} = J^{-1}(J(T_{[n]}x_n) - \theta_n \tau_n), \quad \forall\, n \geq 1, \tau_n \in \partial f(T_{[n]}x_n),$$

where $T_{[n]} := T_{n \bmod N}$. Assume $VI(\partial f, Q) \neq \emptyset$, then $\{x_n\}$ converges strongly to some $x^* \in Q$ which minimizes f over Q.

Proof:
By lemma 4.3.2 and lemma 4.3.3, ∂f is generalized Φ-strongly monotone and bounded on bounded subsets of E. By theorem 4.2.1, $\{x_n\}$ converges strongly to some $x^* \in VI(\partial f, Q)$. By lemma 4.3.4, x^* is a minimizer of f over Q.

Corollary 4.3.1 *Let H be a real Hilbert space. Let $f : H \to \mathbb{R} \cup \{\infty\}$ be a proper and generalized h-strongly convex map and let $T_i : H \to H, i = 1, 2, ..., N$ be a finite family of nonexpansive maps such that $Q := \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence in E defined iteratively by $x_1 \in E$,*

$$x_{n+1} = T_{[n]}x_n - \theta_n \tau_n, \forall\, n \geq 1, \tau_n \in \partial f(T_{[n]}x_n).$$

Assume $VI(A, Q) \neq \emptyset$, then $\{x_n\}$ converges strongly to some $x^* \in VI(A, Q)$. Furthermore, x^* minimizes f over Q.

4.4 The case of non-self maps

In Section 4.2, we considered a finite family $\{T_i\}_{i=1}^N$ of maps, where for each i, T_i maps E to itself. In this section, we consider a finite family $\{T_i\}_{i=1}^N$ of maps where for each i, T_i maps E to its dual space, E^*. In this case, the usual notion of fixed points obviously does not make sense. However, a new notion of fixed points called J-*fixed* points has been defined for maps from a normed space E to its dual E^*, (see Chidume and Idu, [70]) for motivation and definition.

This notion turns out to be very useful in proving convergence theorems for several important classes of nonlinear maps (see e.g. Chidume and Idu, [70]). We shall employ this concept here.

Let $T : E \to E^*$ be any map. A point $p \in E$ is called a J-fixed point of T if $Tp = Jp$, where $J : E \to E^*$ is the normalized duality map. The set of J-fixed points of T will be denoted by $F_J(T)$.

Let E be a uniformly smooth and strictly convex real Banach space with dual space E^*. A map $T : C \to E^*$ will be called J-nonexpansive if the map

$J^{-1} \circ T : C \to E$ is nonexpansive, i.e., for each $x, y \in E$, the following inequality holds:

$$\|(J_* \circ T)x - (J_* \circ T)y\| \leq \|x - y\|. \tag{4.4.1}$$

Observe that since E is uniformly smooth and strictly convex, $J^{-1} : E^* \to E$ exists and $J_* = J^{-1}$.

Definition 4.4.1 *Let E be a uniformly smooth and strictly convex real Banach space with dual space E^*. A map $T : E \to E^*$ will be called generalized J-nonexpansive if $F_J(T) \neq \emptyset$ and $\phi(p, (J^{-1} \circ T)x) \leq \phi(p, x), \forall\, x \in E$ and $p \in F_J(T)$.*

We now prove the following theorem.

Theorem 4.4.1 *Let X be a uniformly smooth and uniformly convex real Banach space with dual space X^*. Let $A : X \to X^*$ be a generalized Φ-strongly monotone and bounded map. Let $S_i : E \to X^*$, $i = 1, 2, 3, ..., N$ be a finite family of generalized J-nonexpansive maps with $W := \cap_{i=1}^N F_J(S_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence in X defined iteratively by $x_1 \in X$,*

$$x_{n+1} = J^{-1}(J(J_* \circ S_{[n]})x_n - \theta_n A(J_* \circ S_{[n]})x_n), \quad \forall n \geq 0 \tag{4.4.2}$$

where $J^{-1} : X^* \to X$ is the normalized duality map on X^* and $S_{[n]} := S_{n \bmod N}$. Assume $VI(A, W) \neq \emptyset$. Then, $\{x_n\}$ converges strongly to some $x^* \in VI(A, W)$.

Proof:
Set $E = X$, then $E^* = X^*$. Define $T_{[n]} = J_* \circ S_{[n]}$. Then,

- $A : E \to E^*$ is a generalized Φ-strongly monotone and bounded map.
- Clearly, $T_{[n]} := J_* \circ S_{[n]} : E \to E$ and for each n, $T_{[n]}$ is a quasi-ϕ-nonexpansive map.
  Furthermore, $W := \cap_{i=1}^N F_J(S_i) = \cap_{i=1}^N F(T_i) = Q$, so that $VI(A, W) = VI(A, Q)$.
- The recursion formular (4.4.2) reduces to the recursion formular of theorem 4.2.1.

Hence, by theorem 4.2.1, $\{x_n\}$ converges strongly to some $x^* \in VI(A, W) = VI(A, Q)$.

Remark 4.4.1 1. *Theorem 4.4.1 complements theorem 4.2.1 in the sense that in theorem 4.2.1, the family T_i, $i = 1, 2, 3, ..., N$ for each i, maps the space E to itself while in theorem 4.4.1, T_i maps E to its dual space, E^*. In a real Hilbert space, theorem 4.4.1 and theorem 4.2.1 yield the same conclusion, basically under the same conditions.*

2. *Theorem 4.2.1 is an analogue of theorem 1.3.3 in q-uniformly smooth spaces,* $q \geq 2$. *In particular, the two theorems coincide in* L_p *spaces,* $2 \leq p < \infty$. *Furthermore, theorem 4.2.1 is applicable in* L_p *spaces,* $1 < p < 2$ *but theorem 1.3.3 is not necessarily applicable in this case, since for* $1 < p < 2$, L_p *is not q-uniformly smooth for* $q \geq 2$.

3. *Theorem 4.2.1 is a significant improvement of theorem 1.3.2 in the following sense:*
   In theorem 1.3.2, the class of η*-strongly monotone and Lipschitz maps defined on a Hilbert space is studied. In theorem 4.2.1, the much more general class of generalized* Φ*-strongly monotone and bounded maps is studied in the much more general space of uniformly smooth and uniformly convex real Banach spaces.*

Remark 4.4.2 *Unlike in Theorem 1.3.2 in which Xu and Kim [168] remarked that the canonical choice* $\alpha_n = \frac{1}{n}$, $n \geq 1$ *is not applicable, this choice is applicable in all our theorems, when* $E = L_p$, l_p *or* W_p^m, $1 < p < \infty$. *In particular, if* $\theta_n = \frac{1}{n}$, $n \geq 1$, *conditions* (i), (ii) *and* (iv) *are trivially satisfied. We verify that condition* (iii) *is satisfied.*
We have (see e.g. Lindenstrauss and Tzafriri [120], p.47) for $p > 1$, $q > 1$, $X = L^p$, $X^* = L^q$, *that*

$$\delta_{X^*}(\epsilon) = 1 - \left(1 - \left(\frac{\epsilon}{2}\right)^q\right)^{1/q},$$

and so obtain that:

$$\delta_{X^*}^{-1}(\epsilon) = 2\left[1 - (1-\epsilon)^q\right]^{1/q} \leq 2q^{1/q}\epsilon^{1/q}, \quad \text{since } (1-\epsilon)^q > 1 - q\epsilon, \ \text{for } q > 1.$$

For condition (iii), *we have that:*

$$\begin{aligned} \sum \theta_n \delta_E^{-1}(2KLM\theta_n) &= \sum 2\theta_n[1 - (1 - 2KLM\theta_n)^p]^{\frac{1}{p}} \\ &\leq 2\sum \theta_n (2pKLM)^{\frac{1}{p}} \theta_n^{\frac{1}{p}} \\ &= 2(2pKLM)^{\frac{1}{p}} \sum \left(\frac{1}{n}\right)^{1+\frac{1}{p}} < \infty. \end{aligned}$$

Hence, condition (iii) *holds.*

All the results of this chapter are the results obtained in [61], which was published in **Journal of Fixed Point Theory and Applications. DOI 10.1007/s11784-018-0502-0**.

CHAPTER 5

Parallel and cyclic hybrid subgradient extragradient algorithms for approximating solutions of variational inequality problems for Lipschitz monotone maps

5.1 Introduction

In this chapter, we introduce and study new parallel and cyclic hybrid subgradient extragradient algorithms. The sequences generated by these algorithms are proved to converge strongly to a common element of the set of solutions of variational inequality problems in a uniformly smooth and 2-uniformly convex real Banach space. The theorems proved are applied to a convex feasibility problem and to approximate a common J-fixed point for a finite family of strictly J-pseudocontractive maps. Finally, a numerical experiment is presented to illustrate the convergence of the sequence of our algorithms.

We shall use the following lemma in this chapter.

Lemma 5.1.1 (Rockafellar, [149]) *Let C be a nonempty closed convex subset of a Banach space E and let A be a monotone and hemicontinuous map from C into E^* with $C = D(A)$. Let T be a map defined by:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C, \end{cases} \tag{5.1.1}$$

where $N_C(v)$ is the normal cone for C at a point $v \in C$, defined as

$$N_C(v) := \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \text{ for all } y \in C.$$

Then, T is maximal monotone and $T^{-1}0 = VI(C, A)$.

5.2 Main Results

In the sequel, let E be a uniformly smooth and 2-uniformly convex real Banach space and C be a closed convex subset of E. Let $A_i : C \to E^*$, $i = 1, 2, ..., N$ be a finite family of monotone and L-Lipschitz maps. We denote $F := \cap_{i=1}^{\infty} VI(A_i, C) \neq \emptyset$. We define the following parallel algorithm.

$$\begin{cases} x_0 \in E,\ 0 < \lambda < \dfrac{1}{L},\ C_0 = C, \\ y_n^i = \Pi_C J^{-1}(Jx_n - \lambda A_i(x_n)),\ i = 1, ..., N, \\ T_n^i = \{v \in E : \langle (Jx_n - \lambda A_i(x_n)) - Jy_n^i, v - y_n^i \rangle \leq 0\}, \\ z_n^i = \Pi_{T_n^i} J^{-1}(Jx_n - \lambda A_i(y_n^i)),\ i = 1, ..., N, \\ i_n = argmax\{||z_n^i - x_n|| : i = 1, ..., N\}, \quad \bar{z_n} := z_n^{i_n}, \\ C_{n+1} = \{v \in C_n : \phi(v, \bar{z}_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad n \geq 0. \end{cases} \tag{5.2.1}$$

Lemma 5.2.1 *Suppose that $x^* \in F$ and the sequences $\{y_n^i\}$, $\{z_n^i\}$ are generated by the following algorithm;*

$$\begin{cases} x_0 \in E,\ 0 < \lambda < \dfrac{1}{L},\ C_0 = C, \\ y_n^i = \Pi_C J^{-1}(Jx_n - \lambda A_i(x_n)),\ i = 1, ..., N, \\ T_n^i = \{v \in E : \langle (Jx_n - \lambda A_i(x_n)) - Jy_n^i, v - y_n^i \rangle \leq 0\}, \\ z_n^i = \Pi_{T_n^i} J^{-1}(Jx_n - \lambda A_i(y_n^i)),\ i = 1, ..., N. \end{cases} \tag{5.2.2}$$

Then,

$$\phi(x^*, z_n^i) \leq \phi(x^*, x_n) - c\big(\phi(y_n^i, x_n) + \phi(x_n, y_n^i)\big), \tag{5.2.3}$$

where $c = 1 - \frac{\lambda L}{c_1} > 0$ and c_1 is the constant in Lemma 2.3.9.

Proof:
Since A_i is monotone on D_i and $y_n^i \in D_i$, we obtain

$$\langle A_i(y_n^i) - A_i(x^*), y_n^i - x^* \rangle \geq 0,\ \forall x^* \in F.$$

This together with $x^* \in VI(A_i, D_i)$ implies that

$$\langle A_i(y_n^i), y_n^i - x^* \rangle \geq 0.$$

So,

$$\langle A_i(y_n^i), z_n^i - x^* \rangle \geq \langle A_i(y_n^i), z_n^i - y_n^i \rangle. \tag{5.2.4}$$

Observe that $z_n^i \in T_n^i$ and by characterization of T_n^i, we have

$$\langle z_n^i - y_n^i, Jx_n - \lambda A_i(x_n) - Jy_n^i \rangle \leq 0.$$

Thus,

$$\langle z_n^i - y_n^i, Jx_n - \lambda A_i(y_n^i) - Jy_n^i\rangle \le \lambda\langle z_n^i - y_n^i, A_i(x_n) - A_i(y_n^i)\rangle. \quad (5.2.5)$$

Let $t_n^i = J^{-1}(Jx_n - \lambda A_i(y_n^i))$ and $z_n^i = \Pi_{T_n^i}(t_n^i)$. Using Lemma 2.3.8, definition of ϕ and inequality (5.2.4), we have

$$\begin{aligned}\phi(x^*, z_n^i) &\le \phi(x^*, t_n^i) - \phi(z_n^i, t_n^i)\\ &\le \phi(x^*, x_n) - \phi(x_n, z_n^i) + 2\lambda\langle x^* - z_n^i, A_i(y_n^i)\rangle\\ &\le \phi(x^*, x_n) - \phi(z_n^i, x_n) + 2\lambda\langle y_n^i - z_n^i, A_i(y_n^i)\rangle. \quad (5.2.6)\end{aligned}$$

Also from defintion of ϕ and using inequality (5.2.5) and Lemma 2.3.9, we have

$$\begin{aligned}\phi(z_n^i, x_n) \quad & -2\lambda\langle y_n^i - z_n^i, A_i(y_n^i)\rangle\\ = \quad & \phi(z_n^i, y_n^i) + \phi(y_n^i, x_n) + 2\langle y_n^i - z_n^i, Jx_n - Jy_n^i\rangle - 2\lambda\langle y_n^i - z_n^i, A_i(y_n^i)\rangle\\ = \quad & \phi(z_n^i, y_n^i) + \phi(y_n^i, x_n) - 2\langle z_n^i - y_n^i, Jx_n - \lambda A_i(y_n^i) - Jy_n^i\rangle\\ \ge \quad & \phi(z_n^i, y_n^i) + \phi(y_n^i, x_n) - 2\lambda\langle z_n^i - y_n^i, A_i(x_n) - A_i(y_n^i)\rangle\\ \ge \quad & \phi(z_n^i, y_n^i) + \phi(y_n^i, x_n) - 2\lambda||z_n^i - y_n^i||||A_i(x_n) - A_i(y_n^i)||\\ \ge \quad & \phi(z_n^i, y_n^i) + \phi(y_n^i, x_n) - 2L\lambda||z_n^i - y_n^i||||x_n - y_n^i||\\ \ge \quad & \phi(z_n^i, y_n^i) + \phi(y_n^i, x_n) - L\lambda\big(||z_n^i - y_n^i||^2 + ||x_n - y_n^i||^2\big)\\ \ge \quad & \phi(z_n^i, y_n^i) + \phi(y_n^i, x_n) - \frac{L\lambda}{c_1}\big(\phi(z_n^i, y_n^i) + \phi(y_n^i, x_n)\big)\\ = \quad & c\big(\phi(z_n^i, y_n^i) + \phi(y_n^i, x_n)\big). \quad (5.2.7)\end{aligned}$$

From inequalities (5.2.6) and (5.2.7), we have

$$\phi(x^*, z_n^i) \le \phi(x^*, x_n) - c\big(\phi(z_n^i, y_n^i) + \phi(y_n^i, x_n)\big). \quad (5.2.8)$$

Lemma 5.2.2 *Suppose that $\{x_n\}$, $\{y_n^i\}$, $\{z_n^i\}$ are generated by Algorithm 5.2.1. Then,*

(i) $F \subset C_n$ and x_{n+1} is well-defined for all $n \ge 0$.

(ii) $\{x_n\}$ converges strongly to a point in E.

(iii) The following hold for $i = 1, 2, ..., N$,

$$\lim ||x_{n+1} - x_n|| = \lim ||y_n^i - x_n|| = \lim ||z_n^i - x_n|| = 0.$$

Proof.

(i) Since A_i is Lipschitz continuous, A_i is continuous for each $i = 1, 2, ..., N$. It is known that $VI(A_i, D_i)$ is closed and convex for each $i = 1, ..., N$. Hence, F is closed and convex. Next we show x_n is well defined. Clearly, $C_1 = C$ is closed and convex. Suppose C_n is closed and convex for some $n \ge 1$. From definition of C_{n+1}, we have

$$\phi(v, \bar{z}_n) \le \phi(v, x_n) \Leftrightarrow 2\langle v, Jx_n - J\bar{z}\rangle \le ||x_n||^2 - ||\bar{z}_n||^2$$

This inequality is affine in v and hence, the set C_n is convex and closed. Moreover, for each $u \in F$, by Lemma 5.2.1, we obtain that $\phi(u, \bar{z_n}) \leq \phi(u, x_n)$. Thus, $F \subset C_n$, $\forall n \geq 0$. Since $F \neq \emptyset$, $\Pi_F x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$ are well-defined.

(ii) Since $x_n = \Pi_{C_n} x_0$ and for each $u \in F \subset C_n$, we have

$$\begin{aligned} \phi(x_0, x_n) &\leq \phi(x_0, u) - \phi(x_n, u) \\ &\leq \phi(x_0, u). \end{aligned} \tag{5.2.9}$$

This implies that $\{\phi(x_0, x_n)\}$ is bounded and hence, $\{x_n\}$ is bounded.
From inequality (5.2.9), we have $\phi(x_0, x_n) \leq \phi(x_0, u)$ $\forall u \in C_n$. Since, $x_{n+1} = \Pi_{C_{n+1}} x_0$ and $C_{n+1} \subset C_n$, we take $u = x_{n+1}$, so we have $\phi(x_0, x_n) \leq \phi(x_0, x_{n+1})$ for each $n \in \mathbb{N}$. This implies that $\{\phi(x_0, x_n)\}$ is a monotone non-decreasing sequence, bounded above by $\phi(x_0, u)$, so $\lim \phi(x_0, x_n)$ exists.
Using the fact that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} \in C_n$, we have for $m > n$

$$\phi(x_n, x_m) \leq \phi(x_0, x_m) - \phi(x_0, x_n), \tag{5.2.10}$$

which implies $\lim \phi(x_n, x_m) = 0$ and by Lemma 2.3.4, we have that $\lim \|x_n - x_m\| = 0$. Hence, $\{x_n\}$ is Cauchy and so, there exists $z \in E$ such that $x_n \to z$ as $n \to \infty$.

(iii) From inequality (5.2.10), take $m = n+1$, we obtain $\lim ||x_n - x_{n+1}|| = 0$. Also, using the fact that $x_{n+1} \in C_n$, we have that

$$\phi(\bar{z_n}, x_{n+1}) \leq \phi(x_n, x_{n+1}),$$

which implies that $\lim \phi(\bar{z_n}, x_{n+1}) = 0$ and by Lemma 2.3.4, we have that $||\bar{z_n} - x_{n+1}|| \to 0$ as $n \to \infty$. Therefore, $||\bar{z_n} - x_n|| \leq ||\bar{z_n} - x_{n+1}|| + ||x_{n+1} - x_n|| \to 0,\ n \to \infty$.
Thus,

$$\lim ||\bar{z_n} - x_n|| = 0. \tag{5.2.11}$$

From definition of i_n and (5.2.11), we have

$$\lim ||z_n^i - x_n|| = 0, \quad \forall i = 1, 2, ..., N. \tag{5.2.12}$$

Using Lemma 5.2.1, inequality (5.2.9) and (5.2.12), we have

$$c\phi(y_n^i, x_n) \leq \phi(x^*, x_n) - \phi(x^*, z_n^i) \to 0, \ \text{as} n \to \infty,$$

which implies that $\phi(y_n^i, x_n) \to 0$ as $n \to \infty$ and by Lemma 2.3.4, we have that

$$\lim ||x_n - y_n^i|| = 0, \ i = 1, 2, ...N. \tag{5.2.13}$$

This completes the proof of Lemma 5.2.2.

Theorem 5.2.1 *Let E be a uniformly smooth and 2-uniformly convex real Banach space. Let C be closed and convex subset of E. Suppose $A_i : C \to E^*$, $i = 1, 2, ..., N$ be a finite family of monotone and L-Lipschitz continuous maps and the set of solution, F, is nonempty. Then, the sequences $\{x_n\}$, $\{y_n^i\}$, $\{z_n^i\}$ generated by Algorithm 5.2.1 converge strongly to $\Pi_F x_0$.*

Proof:
By Lemma 5.2.2, F and C_n are nonempty, closed and convex subsets. Besides, $F \subset C_n$ for all $n \geq 0$. Therefore, $\Pi_F x_0$ and $\Pi_{C_{n+1}} x_0$ are well-defined. From Lemma 5.2.2, we have that $\{x_n\}$ converges strongly to a point z. Since, $||y_n^i - x_n|| \to 0$, then $y_n^i \to z$, $n \to \infty$ for each $i = 1, 2, ..., N$. Now, we prove that $z \in F$. By Lemma 5.1.1, we have that the map

$$Q_i(x) = \begin{cases} A_i(x) + N_C(x), & \text{if } x \in C \\ \emptyset, & \text{if } \quad x \notin C \end{cases} \tag{5.2.14}$$

is maximal monotone, where $N_C(x)$ is the normal cone at C at $x \in C$. For all (x, v^*) in the graph of Q_i, i.e., $(x, v^*) \in G(Q_i)$, we have $v^* - A_i(x) \in N_C(x)$. By definition of $N_C(x)$, we find that

$$\langle x - w, v^* - A_i(x) \rangle \geq 0,$$

for all $w \in C$. Since $y_n^i \in C$,

$$\left\langle x - y_n^i, v^* - A_i(x) \right\rangle \geq 0.$$

Therefore,

$$\left\langle x - y_n^i, v^* \right\rangle \geq \left\langle x - y_n^i, A_i(x) \right\rangle. \tag{5.2.15}$$

Using the definition of y_n^i in Algorithm (5.2.1) and Lemma 2.3.7, we get

$$\left\langle x - y_n^i, A_i(x_n) \right\rangle \geq \left\langle x - y_n^i, \frac{Jx_n - Jy_n^i}{\lambda} \right\rangle. \tag{5.2.16}$$

Therefore, from inequalities (5.2.15), (5.2.16) and the monotonicity of A_i, we have that

$$\begin{aligned} \left\langle x - y_n^i, v^* \right\rangle &\geq \left\langle x - y_n^i, A_i(x) \right\rangle \\ &= \left\langle x - y_n^i, A_i(x) - A_i(y_n^i) \right\rangle + \left\langle x - y_n^i, A_i(y_n^i) - A_i(x_n) \right\rangle \\ &\quad + \left\langle x - y_n^i, A_i(x_n) \right\rangle \\ &\geq \left\langle x - y_n^i, A_i(y_n^i) - A_i(x_n) \right\rangle + \left\langle x - y_n^i, \frac{Jx_n - Jy_n^i}{\lambda} \right\rangle \end{aligned} \tag{5.2.17}$$

Since $||x_n - y_n^i|| \to 0$ as $n \to \infty$ and A_i is L-Lipschitz continuous, we have that

$$\lim ||A_i(y_n^i) - A_i(x_n)|| = 0. \tag{5.2.18}$$

Taking limit as $n \to \infty$ over inequality (5.2.17) and using (5.2.18), $y_n^i \to z$, we have $\langle x - z, v^* \rangle \geq 0$ for all $(x, v^*) \in G(Q_i)$. This together with the maximal monotonicity of Q_i gives that $z \in Q_i^{-1}0 = VI(A_i, C)$ for all $1 \leq i \leq N$. Hence, $z \in F = \cap_{i=1}^N VI(A_i, C)$.
Finally, we show that $z = \Pi_F x_0$. Let $p = \Pi_F x_0$. Since $z \in F$, we have

$$\phi(p, x_0) \leq \phi(z, x_0). \tag{5.2.19}$$

Also, since $x_n = \Pi_{C_n} x_0$ and $p \in F \subset C_n$, we have that $\phi(x_n, x_0) \leq \phi(p, x_0)$. Since $x_n \to z$, we have

$$\phi(z, x_0) \leq \phi(p, x_0). \tag{5.2.20}$$

From inequalities (5.2.19) and (5.2.20), we obtain $\phi(z, x_0) = \phi(p, x_0)$.
Thus, $z = p = \Pi_F x_0$. This completes the proof.

Next, we propose a cyclic hybrid subgradient extragradient algorithm for solving $CSVIP$.

$$\begin{cases} x_0 \in E, \ 0 < \lambda < \dfrac{1}{L}, \\ y_n = \Pi_{D_{[n]}} J^{-1}(Jx_n - \lambda A_{[n]}(x_n)), \ i = 1, \ldots, N, \\ T_{[n]} = \{v \in E : \langle Jx_n - \lambda A_{[n]}(x_n) - Jy_n, v - y_n \rangle \leq 0\}, \\ z_n = \Pi_{T_{[n]}} J^{-1}(Jx_n - \lambda A_{[n]}(x_n)), \ i = 1, \ldots, N, \quad [n] = n \ (mod \ N) + 1, \\ C_{n+1} = \{v \in C_n : \phi(v, z_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad n \geq 0. \end{cases} \tag{5.2.21}$$

Remark 5.2.1 *Since C_{n+1} is a half-space, then the projection $x_{n+1} = \Pi_{C_{n+1}} x_0$ in Step 3 of Algorithm 5.2.21 can be computed explicitly as in Algorithm 5.2.1.*

Theorem 5.2.2 *Let E be a uniformly smooth and 2-uniformly convex real Banach space. Let D_i, $i = 1, ..., N$ be closed and convex subsets of E such that $D = \cap_{i=1}^N D_i \neq \emptyset$. Suppose $A_i : E \to E^*$, $i = 1, 2, ..., N$ be a finite family of monotone and L-Lipschitz continuous maps and the solution set F is nonempty. Then, the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ generated by Algorithm 5.2.21 converge strongly to $\Pi_F x_0$.*

Proof:
By arguing similarly as in the proof of Theorem 5.2.1, we obtain the proof of Theorem 5.2.2.

Remark 5.2.2 *If the mapping A is α-inverse strongly monotone, then A is $1/\alpha$-Lipschitz continuous. Therefore, we can use Algorithms (5.2.1) and (5.2.21) to solve the $CSVIP$ for the α-inverse strongly monotone mappings A_i, $i = 1, ..., N$. However, in this case, instead of using the double projections as in Algorithms (5.2.1) and (5.2.21), we only need to compute the projection on D_i.*

5.3 Application to convex feasibility problems

Let E be a uniformly smooth and 2-uniformly convex real Banach space. Let $D_i,\ i = 1, ..., N$ be closed and convex subsets of E such that $D = \cap_{i=1}^N D_i \neq \emptyset$. The convex feasibility problem (CFP) is to find x^* such that $x^* \in \cap_{i=1}^N D_i$. The CFP is very important and has received a lot of attention in recent years sue to its applications in many practical problems such as signal and image processing, data recovery, communication, and geophysics, etc. Using Theorem 5.2.1, we obtain the following result.

Theorem 5.3.1 *Let E be a uniformly smooth and 2-uniformly convex real Banach space. Let $D_i,\ i = 1, ..., N$ be closed and convex subsets of E such that $D = \cap_{i=1}^N D_i \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in E,\ y_n^i = \Pi_{D_i} x_n, \quad i = 1, ..., N, \\ i_n = argmax\{||y_n^i - x_n|| : i = 1, ..., N\},\ \bar{y_n} = y_n^{i_n}, \\ C_{n+1} = \{v \in C_n : \phi(v, \bar{y_n}) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{5.3.1}$$

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_D x_0$.

In this section, we shall apply our theorem to approximate a common J-fixed point of a finite family of some class of maps. In what follows, we shall denote the set of J-fixed points of T by $F_J(T)$, i.e., $F_J(T) = \{x^* \in E : Tx^* = Jx^*\}$.

A map $T : E \to E^*$ is said to be strictly J-pseudocontractive (see [43]) if there exists $\gamma > 0$ such that for each $x,\ y \in E$, the following inequality holds;

$$\langle Tx - Ty, x - y\rangle \leq \langle Jx - Jy, x - y\rangle - \gamma||(Jx - Jy) - (Tx - Ty)||^2.$$

It is immediate that if T is strictly J-pseudocontractive, then $A := J - T$ is γ-inverse strongly monotone and zeros of A correspond to J-fixed points of T (see Chidume *et al.* [43], for more details).

Let $T_i : E \to E^*,\ i = 1, 2, ..., N$ be a finite family of strictly J-pseudocontractive maps. We consider the problem of finding $x^* \in E$ such that $x^* \in \cap_{i=1}^N F_J(T_i)$.

We have the following parallel hybrid algorithm for finding a common J-fixed point of a finite family of strictly J-pseudocontractive maps, $T_i,\ i = 1, 2, ..., N$.

$$\begin{cases} x_0 \in E, \\ y_n^i = J^{-1}(Jx_n - \lambda(Jx_n - T_i(x_n))), \quad i = 1, ..., N, \\ z_n^i = J^{-1}(Jx_n - \lambda(Jy_n - T_i(y_n))), \quad i = 1, ..., N, \\ i_n = \text{argmax}\{||z_n^i - x_n|| : i = 1, ..., N\},\ \bar{z_n} = z_n^{i_n}, \\ C_{n+1} = \{v \in C_n : \phi(v, \bar{z_n}) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{5.3.2}$$

Theorem 5.3.2 *Let E be a uniformly smooth and 2-uniformly convex real Banach space, with E^* as its dual space. Let $T_i : E \to E^*$, $i = 1, 2, ..., N$ be a finite family of strictly J-pseudocontractive maps such that $F \cap_{i=1}^N F_J(T_i) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by algorithm (5.3.2) converges strongly to $\Pi_F(x_0)$.*

Proof:
Let $D_i = E$ for all i. Since T_I is strictly J-pseudocontractive, $A_i : J - T_i$ is γ-inverse strongly monotone. So, A_i is $\dfrac{1}{\gamma}$-Lipschitz continuous. Theorem 5.2.1 ensures the proof of Theorem 5.3.2.

5.4 Numerical Experiment

In this section, we consider an example to illustrate the convergence of the proposed algorithms. The considered operators are of the form $A_i(x) = M_i(x) + q_i$, $i = 1, 2, 3, ..., N$ [91], where $M_i = B_i B_i^T + C_i + D_i$, $i = 1, 2, ..., N$. For each i, B_i is an $n \times n$ matrix, C_i is an $n \times n$ skew-symmetric matrix, D_i is an $n \times n$ diagonal matrix, whose diagonal entries arenonnegative (i.e., M_i is positive definite), q_i is a vector in $\mathbb{R}^n$ $(n = 4)$. The feasible sets are $K_i = K = \{x \in \mathbb{R}^n : ||x|| \leq n\}$. It is clear that A_i is monotone and L-Lipschitz continuous with $L = \max\{||M_i|| : i = 1, 2, ..., N\}$.

For experiments, the entries of B_i, C_i are generated randomly and uniformly in $[-m, m]$, the diagonal entries of D_i are in $[1, m]$ and q_i is equal to the zero vector. It is easy to see that the solution of the problem in this case is $x^* = 0$. We use the sequence $D_n = ||x_n - x^*||^2$, $n = 0, 1, 2, ...$ to check the convergence of $\{x_n\}$, where $x_0 = (1, 1, 1, 1) \in \mathbb{R}^4$ and $\lambda = \frac{0.5}{L}$.

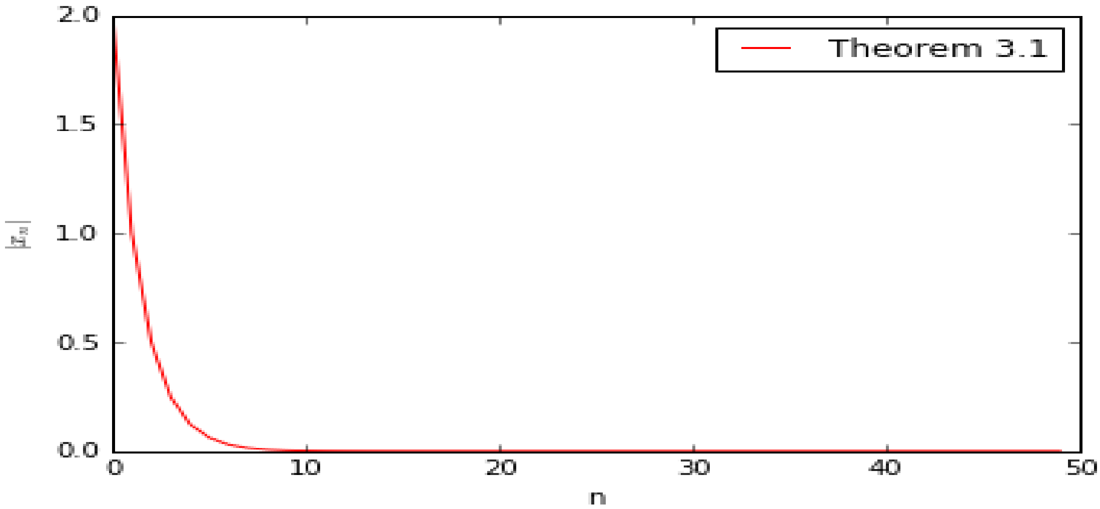


Fig 1.

Remark 5.4.1

Theorem 7.2.1 improves the results of Hieu [96] in the following ways:

- *The results of Hieu [96] are proved in Hilbert space while Theorem 7.2.1 is proved in the more general uniformly smooth and* 2-*uniformly convex real Banach spaces.*
- *In the algorithm of Hieu [96], they have projections onto the intersection of two half spaces, while in the algorithm of Theorem 7.2.1, we have projection onto one half space, which has less computation.*

All the results of this chapter are results obtained in [62], which was submitted in **Afrika Matematika**.

CHAPTER 6

Halpern-type iterative algorithms for approximating solutions of generalized split feasibility problems

6.1 Introduction

In this chapter, we construct new iterative algorithms for approximating solutions of generalized split feasibilty problems in a uniformly smooth and 2-uniformly convex real Banach space. Strong convergence of the sequences generated by these algorithms is studied. As application, we derive some algorithms and strong convergence results for some nonlinear problems, such as, split feasibility problems, equilibrium problems, etc., and a numerical experiment is given to show the implementability of the theorems. Finally, the results proved improve, complement and generalize most recent results in the literature.

We shall using the following lemma in this chapter.

Definition 6.1.1 *Let C be a nonempty closed convex subset of a smooth Banach space E and $T : C \to C$ be a map. A point $p \in C$ is called an asymptotic fixed point of T if there exists a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{Fix}(T)$. The map $T : C \to C$ is said to be:*

(a) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;

(b) firmly nonexpansive if $\phi(Tx, Ty) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x)$ for all $x, y \in C$;

(c) relatively nonexpansive if the following properties are satisfied;

(i) *Fix(T)* $\neq \emptyset$;

(ii) $\phi(p, Tx) \leq \phi(p, x)$ *for* $p \in$ *Fix(T)*, $x \in C$;

(iii) $\widehat{Fix}(T) = Fix(T)$.

(d) strongly relative nonexpansive if the folloing properties are satisfied:

(i) T *is relative nonexpansive;*

(ii) $\lim_{n\to\infty} \phi(Tx_n, x_n) = 0$ *whenever* $\{x_n\}$ *is bounded sequence in* C *and* $\lim_{n\to\infty} (\phi(p, x_n) - \phi(p, Tx_n)) = 0$ *for some* $p \in Fix(T)$.

Lemma 6.1.1 (Rockafellar, [149]) *Let* E *be a smooth, strictly convex and reflexive Banch space and* $K : E \to 2^{E^*}$ *be a monotone operator. Then* K *is maximal monotone if and only if* $R(J + \lambda K) = E^*$ *for all* $\lambda > 0$, *where* $R(J + \lambda K)$ *is the range of* $J + \lambda K$.

Let E be a smooth, strictly convex and reflexive Banach space and $K : E \to 2^{E^*}$ be a maximal monotone operator. Then for $\lambda > 0$ and $x \in E$, consider

$$J_\lambda^K x := \{z \in E : Jx \in Jz + \lambda K(z)\}.$$

In other words, $J_\lambda^K = (J + \lambda K)^{-1} J$. Also, J_λ^K is known as relative resolvent of K for $\lambda > 0$. Following [116], we know the following properties:

(i) $J_\lambda^K : E \to D(K)$ is a single-valued mapping;

(ii) $K^{-1}0 = Fix(J_\lambda^K)$ for each $\lambda > 0$;

(iii) J_λ^K is strongly relatively nonexpansive,

where $D(K)$ is the domain of K.

Lemma 6.1.2 *[121] Let* $\{\alpha_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}_{i=1}^\infty$ *of* $\{n\}$ *such that* $\alpha_{n_i} < \alpha_{n_i+1}$ *forall* $i \in \mathbb{N}$. *Then there exists a nondecreasing sequence*$\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *and the following properties are satisfied for all (sufficiently large) numbersk* $\in \mathbb{N}$:

$$\alpha_{m_k} \leq \alpha_{m_k+1} \text{ and } \alpha_k \leq \alpha_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : \alpha_j < \alpha_{j+1}\}$.

6.2 Main Results

In this section, we assume E_1 and E_2 to be uniformly smooth and 2-uniformly convex real Banach spaces, and E_1^*, E_2^* be their dual spaces respectively.

We first prove the following lemma in a smooth and 2-uniformly convex real Banach space.

Lemma 6.2.1 *Let X be 2-uniformly convex and smooth real Banach space and $B : X \to X^*$ be a maximal monotone operator. Then for $\lambda > 0$ and $x \in X$, $J_\lambda^B = (J + \lambda B)^{-1}J$ is Lipschitz type.*

Proof:
From Kohsaka and Takahashi, [116], we have that J_λ^B is a firmly nonexpansive type mapping, i.e., for $x, y \in X$ and $\lambda > 0$,

$$\left\langle J_\lambda^B x - J_\lambda^B y, \frac{Jx - JJ_\lambda^B x}{\lambda} - \frac{Jy - JJ_\lambda^B y}{\lambda} \right\rangle \geq 0.$$

Hence, from Lemma 2.1.2 (3), we have that

$$\langle J_\lambda^B x - J_\lambda^B y, Jx - Jy \rangle \geq \langle J_\lambda^B x - J_\lambda^B y, JJ_\lambda^B x - JJ_\lambda^B y \rangle$$

$$\Longrightarrow ||J_\lambda^B x - J_\lambda^B y|| \leq \frac{1}{c_2}||Jx - Jy||$$

We first establish the strong convergence to a solution of problem (1.4.2).

Theorem 6.2.1 *Let K be a closed convex subset of E_1. Let E_1 and E_2 be uniformly smooth and 2-uniformly convex real Banach spaces, and E_1^*, E_2^* be their dual spaces respectively. Let $A : E_1 \to E_2$ be a bounded linear operator whose adjoint is denoted by A^* and $S : E_2 \to E_2$ be a nonexpansive map such that $F(S) \neq \emptyset$ and $T : K \to K$ be a relatively nonexpansive map such that $F(T) \neq \emptyset$. Let $B : E_1 \to 2^{E_1^*}$ be a maximal monotone mapping such that $B^{-1}0 \neq \emptyset$. Then the sequence generated by the following algorithm: for $x_1 \in K$ arbitrary and $\beta_n \in (0,1)$,*

$$\begin{cases} y_n = J_{E_1}^{-1}\big(J_{E_1}x_n - \gamma A^* J_{E_2}(I - S)Ax_n\big), \\ w_n = J_{E_1}^{-1}\big(\alpha_n J_{E_1}x_1 + (1 - \alpha_n)J_{E_1}J_\lambda^B y_n\big), \\ x_{n+1} = J_{E_1}^{-1}\big(\beta_n J_{E_1}x_n + (1 - \beta_n)J_{E_1}Tw_n\big), \quad \forall n \geq 1. \end{cases} \tag{6.2.1}$$

converges strongly to an element $z \in \Gamma$.

Proof:
Let $p \in \Gamma$ and $z_n = J_\lambda^B y_n$. Then $J_\lambda^B p = p$, $Tp = p$ and $S(Ap) = Ap$. From definition of ϕ and Lemma 2.1.3, we have

$$\begin{aligned} \phi(p, y_n) &= ||p||^2 - 2\langle p, J_{E_1}x_n - \gamma A^* J_{E_2}(I - S)Ax_n\rangle \\ &\quad + ||J_{E_1}x_n - \gamma A^* J_{E_2}(I - S)Ax_n||^2 \qquad (6.2.2) \\ &\leq ||p||^2 - 2\langle p, J_{E_1}x_n\rangle + 2\gamma\langle p, A^* J_{E_2}(I - S)Ax_n\rangle + ||x_n||^2 \\ &\quad - 2\gamma\langle x_n, A^* J_{E_2}(I - S)Ax_n\rangle + 2k^2\gamma^2||A||^2||J_{E_2}(I - S)Ax_n||^2 \\ &\leq \phi(p, x_n) + 2\gamma^2||A||^2||(I - S)Ax_n||^2 \\ &\quad + 2\gamma\langle Ap - Ax_n, J_{E_2}(I - S)Ax_n\rangle \qquad (6.2.3) \end{aligned}$$

From nonexpansiveness of S and Lemma 2.1.2 (2), we have

$$\begin{aligned}
2\langle Ap - Ax_n, J_{E_2}(I-S)Ax_n\rangle &\leq ||SAx_n - Ap||^2 - ||(I-S)Ax_n||^2 - c||Ax_n - Ap||^2 \\
&\leq (1-c)||Ax_n - Ap||^2 - ||(I-S)Ax_n||^2 \\
&\leq -||(I-S)Ax_n||^2,
\end{aligned}$$

so that,

$$2\gamma\langle Ap - Ax_n, J_{E_2}(I-S)Ax_n\rangle \leq -\gamma||(I-S)Ax_n||^2. \tag{6.2.4}$$

From inequalities (6.2.2) and (6.2.4), we have

$$\phi(p, y_n) \leq \phi(p, x_n) - \gamma(1-\gamma||A||^2)||(I-S)Ax_n||^2. \tag{6.2.5}$$

Using the fact that $\gamma \in (0, 1/||A||^2)$, relative nonexpansiveness of J_λ^B and inequality (6.2.5), we have

$$\begin{aligned}
\phi(p, z_n) &= \phi(p, J_\lambda^B y_n) \leq \phi(p, y_n) \\
&\leq \phi(p, x_n) - \gamma(1-\gamma||A||^2)||(I-S)Ax_n||^2 && (6.2.6)\\
&\leq \phi(p, x_n). && (6.2.7)
\end{aligned}$$

Using the convexity of $\phi(p, \cdot)$ and inequality (6.2.6), we have

$$\begin{aligned}
\phi(p, x_{n+1}) &\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, Tw_n) \\
&\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, w_n) \\
&\leq \beta_n\phi(p, x_n) + (1-\beta_n)\big[\alpha_n\phi(p, x_1) + (1-\alpha_n)\phi(p, z_n)\big] && (6.2.8)\\
&\leq [1-\alpha_n(1-\beta_n)]\phi(p, x_n) + \alpha_n(1-\beta_n)\phi(p, x_1) \\
&- (1-\beta_n)(1-\alpha_n)\gamma(1-\gamma||A||^2)||(I-S)Ax_n||^2 && (6.2.9)\\
&\leq [1-\alpha_n(1-\beta_n)]\phi(p, x_n) + \alpha_n(1-\beta_n)\phi(p, x_1) \\
&\leq \max\{\phi(p, x_n), \phi(p, x_1)\}.
\end{aligned}$$

This implies from induction that for each $n \geq 1$, $\phi(p, x_n) \leq \phi(p, x_1)$. Hence, $\{\phi(p, x_n)\}$ is bounded and so are $\{x_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{Tw_n\}$.

We consider the following cases:

Case 1: Suppose there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\phi(p, x_{n+1}) \leq \phi(p, x_n)$. The $\lim_{n\to\infty}\phi(p, x_n)$ exists. From inequality (6.2.9), we have that

$$(1-\beta_n)(1-\alpha_n)\gamma(1-\gamma||A||^2)||(I-S)Ax_n||^2 \leq \phi(p, x_n) - \phi(p, x_{n+1}) + \alpha_n\phi(p, x_1).$$

Using the fact that $\lim_{n\to\infty}\alpha_n = 0$ and $(1-\beta_n)(1-\alpha_n)\gamma(1-\gamma||A||^2) > 0$, we have that that

$$\lim_{n\to\infty}||(I-S)Ax_n||^2 = 0. \tag{6.2.10}$$

Using definition of ϕ, Lemmas 2.3.3 and 2.1.1, we have

$$\begin{aligned}
\phi(p,x_{n+1}) &\leq \beta_n\phi(p,x_n)+(1-\beta_n)\phi(p,Tw_n)-\beta_n(1-\beta_n)g(||J_{E_1}x_n-J_{E_1}Tw_n||)\\
&\leq \beta_n\phi(p,x_n)+(1-\beta_n)V(p,\alpha_nJ_{E_1}x_1+(1-\alpha_n)J_{E_1}z_n)\\
&\quad -\beta_n(1-\beta_n)g(||J_{E_1}x_n-J_{E_1}Tw_n||)\\
&\leq \beta_n\phi(p,x_n)+(1-\beta_n)\big[(1-\alpha_n)\phi(p,z_n)+2\alpha_n\langle w_n-p,J_{E_1}x_1-J_{E_1}p\rangle\big]\\
&\quad -\beta_n(1-\beta_n)g(||J_{E_1}x_n-J_{E_1}Tw_n||)\\
&\leq \beta_n\phi(p,x_n)+(1-\beta_n)(1-\alpha_n)\phi(p,z_n)+2\alpha_n(1-\beta_n)\langle w_n-p,J_{E_1}x_1-J_{E_1}p\rangle\\
&\quad -\beta_n(1-\beta_n)g(||J_{E_1}x_n-J_{E_1}Tw_n||)\\
&\leq \beta_n\phi(p,x_n)+(1-\beta_n)(1-\alpha_n)\phi(p,x_n)+2\alpha_n(1-\beta_n)\langle w_n-p,J_{E_1}x_1-J_{E_1}p\rangle\\
&\quad -\beta_n(1-\beta_n)g(||J_{E_1}x_n-J_{E_1}Tw_n||)\\
&= [1-\alpha_n(1-\beta_n)]\,\phi(p,x_n)+2\alpha_n(1-\beta_n)\langle w_n-p,J_{E_1}x_1-J_{E_1}p\rangle\\
&\quad -\beta_n(1-\beta_n)g(||J_{E_1}x_n-J_{E_1}Tw_n||) \qquad (6.2.11)\\
&\leq \phi(p,x_n)+2\alpha_n(1-\beta_n)\langle w_n-p,J_{E_1}x_1-J_{E_1}p\rangle\\
&\quad -\beta_n(1-\beta_n)g(||J_{E_1}x_n-J_{E_1}Tw_n||) \qquad (6.2.12)
\end{aligned}$$

From inequality (6.2.12), we have that

$$\beta_n(1-\beta_n)g(||J_{E_1}x_n-J_{E_1}Tw_n||)\leq\phi(p,x_n)-\phi(p,x_{n+1})+2\alpha_n(1-\beta_n)\langle w_n-p,J_{E_1}x_1-J_{E_1}p\rangle.$$

Since $\beta_n(1-\beta_n)>0$ and $\lim_{n\to\infty}\alpha_n=0$, taking limit as $n\to\infty$, we have that

$$g(||J_{E_1}x_n-J_{E_1}Tw_n||)\to 0 \quad \text{as} \quad n\to\infty.$$

Since g is strictly increasing and $g(0)=0$, we have that

$$||J_{E_1}x_n-J_{E_1}Tw_n||\to 0 \ \text{ as } n\to\infty \qquad (6.2.13)$$

Using the fact that $J_{E_1}^{-1}$ is uniformly continuous on bounded sets, we have that

$$\|x_n-Tw_n\|\to 0$$

Now, using definition of ϕ and Lemma 2.1.2, we have

$$\begin{aligned}
\phi(x_n,y_n) &= ||x_n||^2-2\langle x_n,J_{E_1}x_n-\gamma A^*J_{E_2}(I-S)Ax_n\rangle+||J_{E_1}x_n-\gamma A^*J_{E_2}(I-S)Ax_n||^2\\
&\leq ||x_n||^2-2\langle x_n,J_{E_1}x_n\rangle+2\gamma\langle x_n,A^*J_{E_2}(I-S)Ax_n\rangle+||x_n||^2\\
&\quad -2\gamma\langle x_n,A^*J_{E_2}(I-S)Ax_n\rangle+2k^2\gamma^2||A||^2||(I-S)Ax_n||^2 \qquad (6.2.14)\\
&\leq \phi(x_n,x_n)+2\gamma^2||A||^2||(I-S)Ax_n||^2\\
&= 2\gamma^2||A||^2||(I-S)Ax_n||^2. \qquad (6.2.15)
\end{aligned}$$

From (6.2.10) and (6.2.14), we obtain

$$\lim_{n\to\infty}\phi(x_n,y_n)=0, \qquad (6.2.16)$$

and by Lemma 2.3.4, we have that $\lim_{n\to\infty} ||x_n - y_n|| = 0$. Using the fact that J_λ^B is relatively nonexpansive and inequality (6.2.8), we have

$$\begin{aligned} 0 &\leq \phi(p, y_n) - \phi(p, J_\lambda^B y_n) \\ &\leq \phi(p, x_n) - \phi(p, z_n) \\ &\leq \phi(p, x_n) + \alpha_n \phi(p, x_0) - \phi(p, x_{n+1}). \end{aligned} \tag{6.2.17}$$

Taking limit as $n \to \infty$, we obtain that $\phi(p, y_n) - \phi(p, J_\lambda^B y_n) \to 0$, as $n \to \infty$ and by strong nonexpansiveness of J_λ^B, we have that $\phi(J_\lambda^B y_n, y_n) \to 0$, as $n \to \infty$. By Lemma 2.3.4, we have that

$$\lim_{n\to\infty} ||J_\lambda^B y_n - y_n|| = \lim_{n\to\infty} ||z_n - y_n|| = 0. \tag{6.2.18}$$

Also, we can easily see that

$$||x_n - z_n|| \leq ||x_n - y_n|| + ||y_n - z_n|| \tag{6.2.19}$$

implies $||x_n - z_n|| \to 0$ as $n \to \infty$. Also, using Lemma 6.2.1, we have

$$\begin{aligned} ||J_\lambda^B x_n - x_n|| &\leq ||J_\lambda^B x_n - z_n|| + ||z_n - x_n|| \\ &= ||J_\lambda^B x_n - J_\lambda^B y_n|| + ||z_n - x_n|| \\ &\leq ||Jx_n - Jy_n|| + ||z_n - x_n|| \\ &\leq \gamma ||A|| ||(I - S)Ax_n|| + ||z_n - x_n|| \end{aligned} \tag{6.2.20}$$

Using (6.2.10), (6.2.19) and taking limit as $n \to \infty$ over inequality (6.2.20), we have that

$$\lim_{n\to\infty} ||J_\lambda^B x_n - x_n|| = 0 \tag{6.2.21}$$

Also, we have from inequality (6.2.19) that $\|z_n - x_n\| \to 0$ as $n \to \infty$ and using the fact that J_{E_1} is uniformly continuous on bounded sets, we have $\|J_{E_1} z_n - J_{E_1} x_n\| \to 0$ as $n \to \infty$. Hence,

$$\|J_{E_1} w_n - J_{E_1} x_n\| \leq \alpha_n \|(J_{E_1} x_0 - J_{E_1} x_n)\| + (1 - \alpha_n)\|(J_{E_1} z_n - J_{E_1} x_n)\|$$

which implies $\|J_{E_1} w_n - J_{E_1} x_n\| \to 0$ as $n \to \infty$, and by uniform continuity of $J_{E_1}^{-1}$ on bounded sets, we have

$$\|w_n - x_n\| \to 0, \ n \to \infty. \tag{6.2.22}$$

From $\|w_n - Tw_n\| \leq \|w_n - x_n\| + \|x_n - Tw_n\| \to 0$ as $n \to \infty$, we have that

$$\|w_n - Tw_n\| \to 0 \text{ as } n \to \infty. \tag{6.2.23}$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup z$, $k \to \infty$. Using the fact that J_λ^B is the resolvent of B, then we have that for each $n \geq 1$, $\dfrac{Jx_n - JJ_\lambda^B x_n}{\lambda} \in BJ_\lambda^B x_n$. From monotonocity of B, we have

$$0 \leq \left\langle u - J_\lambda^B x_{n_k}, \bar{u} - \frac{Jx_{n_k} - JJ_\lambda^B x_{n_k}}{\lambda} \right\rangle, \quad \forall (u, \bar{u}) \in G(B).$$

Hence we have $\forall(u,\bar{u}) \in G(B)$, $0 \le \langle u - z, \bar{u}\rangle$ and since B is maximal monotone, we obtain that $z \in B^{-1}(0)$. Also from (6.2.23), (6.2.22) and demiclosedness of T at 0, we have that $z \in F(T)$. Hence, $z \in F(T) \cap B^{-1}(0)$. Since $x_{n_k} \rightharpoonup z$, $k \to \infty$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$, without loss of generality, we let

$$\lim_{k\to\infty} \langle x_{n_k} - z, J_{E_1}x_1 - J_{E_1}z\rangle = \limsup_{n\to\infty}\langle x_n - z, J_{E_1}x_1 - z\rangle = \limsup_{n\to\infty}\langle w_n - z, J_{E_1}x_1 - z\rangle,$$

where $z = \Pi_\Gamma x_1$. Using Lemma 2.3.7, we obtain

$$\limsup_{n\to\infty}\langle x_n - z, J_{E_1}x_1 - J_{E_1}z\rangle = \lim_{k\to\infty}\langle x_{n_k} - z, J_{E_1}x_1 - J_{E_1}z\rangle = \langle x - z, J_{E_1}x_1 - J_{E_1}z\rangle \le 0$$

By applying Lemma (2.3.13) to (6.2.11), we have that $\lim\limits_{n\to\infty} \phi(z, x_n) = 0$, and by Lemma 2.3.4, we have that $x_n \to z$, $n \to \infty$. Since A is a bounded linear map, we have that $Ax_n \to Az$ as $n \to \infty$, using the fact that $\lim_{n\to\infty} ||(I-S)Ax_n|| = 0$ and by demiclosedness of S at 0, we have that $S(Az) = Az$.

Case 2: Suppose there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\phi(p, x_{n_{j+1}}) > \phi(p, x_{n_j}) \ \forall\ j \in \mathbb{N}.$$

By Lemma 6.1.2, there exists a nondecreasing $\{n_j\}$ in $\mathbb{N}$ such that $\phi(p, x_{n_j}) \le \phi(p, x_{nj+1})$ and $\phi(p, x_j) \le \phi(p, x_{n_{j+1}})$. By discarding the repeated terms of $\{n_j\}$, but still denoted by $\{n_j\}$, we can view $\{x_{n_j}\}$ as a subsequence of $\{x_n\}$. We show that $\limsup_{j\to\infty}\langle x_{n_j} - z, J_{E_1}x_1 - J_{E_1}z\rangle \le 0$, where $z = \Pi_\Gamma x_1$. Since $\{x_{n_j}\}$ is bounded, there exists $\{x_{nj_k}\} \subset \{x_{n_j}\}$ such that $x_{n_{j_k}} \rightharpoonup w$ for some $w \in B_1$, without loss of generality, we let

$$\lim_{k\to\infty}\langle x_{n_{j_k}} - z, J_{E_1}x_1 - J_{E_1}z\rangle = \limsup_{j\to\infty}\langle x_{n_j} - z, J_{E_1}x_1 - J_{E_1}z\rangle.$$

Following similar arguments as in Case 1, we have that

$$\lim_{n\to\infty}\|x_{n_j} - J_\lambda^B x_{n_j}\| = \lim_{n\to\infty}\|w_{n_j} - Tw_{n_j}\| = \lim_{n\to\infty}\|x_{n_j} - z_{n_j}\| = \lim_{n\to\infty}\|x_{n_j} - w_{n_j}\| = 0$$

and $w \in F(T) \cap B^{-1}(0)$.Now, using Lemma 2.3.7, we obtain

$$\lim_{k\to\infty}\langle x_{n_{j_k}} - z, J_{E_1}x_1 - J_{E_1}z\rangle = \limsup_{j\to\infty}\langle x_{n_j} - z, J_{E_1}x_1 - J_{E_1}z\rangle = \langle w - z, J_{E_1}x_1 - J_{E_1}z\rangle \le 0.$$

From inequality (6.2.11), we have

$$\begin{aligned}\phi(z, x_{n_j+1}) &\le \left[1 - \alpha_{n_j}(1-\beta_{n_j})\right]\phi(p, x_{n_j}) + 2\alpha_{n_j}(1-\beta_{n_j})\langle w_{n_j} - p, J_{E_1}x_1 - J_{E_1}p\rangle\\ &\le \left[1 - \alpha_{n_j}(1-\beta_{n_j})\right]\phi(p, x_{n_j+1}) + 2\alpha_{n_j}(1-\beta_{n_j})\langle w_{n_j} - p, J_{E_1}x_1 - J_{E_1}p\rangle.\end{aligned}$$

Since $\alpha_{n_j}(1-\beta_{n_j}) > 0$, we have

$$\phi(z, x_{n_j}) \le \phi(z, x_{n_j+1}) \le 2\langle w_{n_j} - p, J_{E_1}x_1 - J_{E_1}p\rangle.$$

So for each $j \in \mathbb{N}$, we have

$$0 \leq \limsup_{j\to\infty} \phi(z, x_{n_j}) \leq \limsup_{j\to\infty} \phi(z, x_{n_j+1}) \leq 2 \limsup_{j\to\infty} \langle w_{n_j} - p, J_{E_1}x_1 - J_{E_1}p\rangle \leq 0.$$

This implies that $\phi(z, x_j)$ as $j \to \infty$ and by Lemma 2.3.4, we have that $x_j \to z = \Pi_\Gamma x_1$ as $l \to \infty$. Hence, the conclusion follows as in proof of Case 1. This completes the proof.

Next, as a consequence of Theorem 6.2.1, we establish the strong convergence to a solution of problem (1.4.3).

Theorem 6.2.2 *Let E_1 and E_2 be uniformly smooth and 2-uniformly convex real Banach spaces, and E_1^*, E_2^* be their dual spaces respectively. Let $A : E_1 \to E_2$ be a bounded linear operator whose adjoint is denoted by A^* and $S : B_2 \to B_2$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. Let $B : E_1 \to 2^{E_1^*}$ be a maximal monotone mapping such that $B^{-1}0 \neq \emptyset$. Then the sequence generated the following algorithm: for $x_1 \in C$ arbitrary, $\beta_n \in (0, 1)$ and $\gamma \in \left(0, \frac{1}{||A||^2}\right)$,*

$$\begin{cases} y_n = J_{E_1}^{-1}\big(J_{E_1}x_n - \gamma A^* J_{E_2}(I - S)Ax_n\big), \\ x_{n+1} = J_{E_1}^{-1}\big(\alpha_n J_{E_1}x_1 + (1 - \alpha_n)J_{E_1}J_\lambda^B y_n\big), \quad \forall n \geq 1. \end{cases} \tag{6.2.24}$$

converges strongly to an element $z \in \Omega$.

Proof:
Putting $T \equiv I$, the identity map and $\beta_n = 0$ in Theorem 6.2.1. Hence, we obtain the conclusion follows from Theorem 6.2.1.

6.3 Application to Split Feasibility Problem

Let E be a smooth strictly convex and reflexive real Banach space and K be a nonempty closed convex subset of E. Then the indicator function $i_K : E \to (-\infty, \infty]$ defined by

$$i_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ \infty, & \text{if otherwise,} \end{cases}$$

is a proper lowers semicontinuous convex function. By Rockafellar [149], we have that the subdifferential of i_K, ∂i_K is a maximal monotone. It is known that for any $x \in K$,

$$\begin{aligned} \partial i_K &= \{x^* \in E^* : i_K(x) + \langle y - x, x^*\rangle \leq i_K(y) \;\; \forall \;\; y \in E\} \\ &= \{x^* \in E^* : \langle y - x, x^*\rangle \leq 0 \;\; \forall \;\; y \in K\} = N_K(x), \end{aligned}$$

where N_K is the normal operator for K. It is known that Π_K is the resolvent of N_K. Infact $\Pi_K = (J + 2^{-1}N_K)^{-1}J$ (see [116]).

Let K be a nonempty closed convex subset of E_1. Consider $K = \partial i_K$ and $S = P_Q$, where P_Q is the metric projection onto a nonempty closed convex subset Q of E_2. Then we have $J_\lambda^B = \Pi_K$ and $F(T) = Q$. Now we recover the split feasibility problem in the setting of Banach spaces as follows:

$$\text{Find } x^* \in K \text{ such that } Ax^* \in Q, \tag{6.3.1}$$

and Algorithm 6.4.5 reduces to the following; choose $x_1 \in E_1$ arbitrary,

$$\begin{cases} z_n = \Pi_K\Big(J_{E_1}^{-1}\big(J_{E_1}x_n - \gamma A^* J_{E_2}(I - P_Q)Ax_n\big)\Big), \\ x_{n+1} = J_{E_1}^{-1}\big(\alpha_n J_{E_1}x_1 + (1-\alpha_n)J_{E_1}z_n\big), \quad \forall n \geq 1. \end{cases} \tag{6.3.2}$$

Theorem 6.3.1 *Let E_1 and E_2 be uniformly smooth and 2-uniformly convex real Banach space. Let K and Q be nonempty closed convex subsets of E_1 and E_2, respectively, $A : E_1 \to E_2$ be a bounded linear operator, and $\gamma \in \left(0, \dfrac{2}{||A||^2}\right)$. If $\Omega \neq \emptyset$, then the sequence generated by Algorithm (6.3.2) converges strongly to an element $z \in \Omega$.*

Proof:
Letting $K = \partial i_K$, in Theorem 6.2.2, we have that $J_\lambda^B = \Pi_K$ for all $\lambda > 0$. Since Π_K is strongly relative nonexpansive, therefore the result follows from the arguments in the proof of Theorem 6.2.2.

6.4 Application to equilibrium problems

Let K be a nonempty closed convex subset of a smooth, strictly convex and reflexive real Banach space E. Let $f : K \times K \to \mathbb{R}$ be a bifunction. The equilibrium problem (abbreviated EP) is to find $x \in K$ such that

$$f(x, y) \geq 0, \text{ for all } y \in K. \tag{6.4.1}$$

The set of solutions of EP is denoted by $EP(f)$. For solving Problem (6.4.1), we assume that the bifunction f satisfies the following conditions:

$(A1)$ $f(x, x) = 0, \ \forall x \in K$,
$(A2)$ f is monotone, that is, $f(x, y) + f(y, x) \leq 0, \ \forall x, y \in K$,
$(A3)$ for all $x, y, z \in K$, $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$,
$(A4)$ for all $x \in E$, $y \longmapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 6.4.1 (See e.g., Takahashi and Zembayashi [?]) *Let $f : K \times K \to \mathbb{R}$ be a bi-function satisfying $(A1)-(A4)$. Let $r > 0$, define a resolvent operator of f, $T_r : C \to C$ by*

$$T_r(x) = \Big\{z \in K : f(z, y) + \frac{1}{r}\Big\langle y - z, Jz - Jx\Big\rangle \geq 0 \quad \forall \ \ y \in K\Big\},$$

for all $x \in E$. Then, the map has the following properties:

1. T_r is single-valued,

2. T_r is a firmly nonexpansive-type map, that is, for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle,$$

3. $F(T_r)$ =EP(f) is closed and convex.

Lemma 6.4.2 *Let $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A)$. Let $A_f : E \times E \to 2^{E^*}$ be a set-valued map defined by*

$$A_f(x) = \begin{cases} x^* \in E^* : f(x,y) \geq \langle y - x, x^*\rangle \text{ for all } y \in K, & \text{if } x \in K, \\ \emptyset, \quad \text{if } x \notin K. \end{cases} \tag{6.4.2}$$

Then, A_f is a maximal monotone operator with $D(A_f) \subset K$ and $EP(f) = A_f^{-1}0$. Furthermore, for $r > 0$, the resolvent T_r of f coincides with the resolvent $(J + rA_f)^{-1}J$ of A_f, that is,

$$T_r(x) = (J + rA_f)^{-1}J(x) \tag{6.4.3}$$

As a consequence of Theorem 6.2.1, we have the following results.

Theorem 6.4.1 *Let E_1 and E_2 be uniformly smooth and 2-uniformly convex real Banach spaces. Let K be a nonempty closed convex subset of E_1, $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A)$, and T_λ denote the resolvent (as defined in (6.4.3)) of A_f of index $\lambda > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator whose adjoint is denoted by A^* and $S : B_2 \to B_2$ be a nonexpansive map such that $F(S) \neq \emptyset$ and $T : K \to K$ be a relatively nonexpansive map such that $F(T) \neq \emptyset$. Then the sequence generated by the following algorithm: for $x_1 \in K$ arbitrary, $\beta_n \in (0, 1)$ and $\gamma \in \Big(0, \dfrac{1}{||A||^2}\Big)$,*

$$\begin{cases} y_n = J_{E_1}^{-1}\big(J_{E_1}x_n - \gamma A^* J_{E_2}(I - S)Ax_n\big), \\ w_n = J_{E_1}^{-1}\big(\alpha_n J_{E_1}x_1 + (1 - \alpha_n)J_{E_1}T_\lambda y_n\big), \\ x_{n+1} = J_{E_1}^{-1}\big(\beta_n J_{E_1}x_n + (1 - \beta_n)J_{E_1}Tw_n\big), \quad \forall n \geq 1. \end{cases} \tag{6.4.4}$$

converges strongly to an element $z \in \{x \in EP(f) \cap F(T) : Ax \in F(S)\}$.

Proof:
Letting $B \equiv A_f$ in Theorem 6.2.1, we have that $J_\lambda^B \equiv T_\lambda$ for all $\lambda > 0$. Since T_λ is firmly nonexpansive type, so by [116], it is strongly relative nonsexpansive. Hence, the conclusion follows from Theorem 6.2.1.

Theorem 6.4.2 *Let E_1 and E_2 be uniformly smooth and 2-uniformly convex real Banach spaces. Let K be a nonempty closed convex subset of E_1, $f : K \times K \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$, and T_λ denote the* *resolvent (as defined in (6.4.3)) of A_f of index $\lambda > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator whose adjoint is denoted by A^* and $S : B_2 \to B_2$ be a nonexpansive map such that $F(S) \neq \emptyset$. Then the sequence generated by the following algorithm: for $x_1 \in K$ arbitrary, $\beta_n \in (0,1)$ and $\gamma \in \left(0, \frac{1}{||A||^2}\right)$,*

$$\begin{cases} y_n = J_{E_1}^{-1}\big(J_{E_1}x_n - \gamma A^* J_{E_2}(I-S)Ax_n\big), \\ x_{n+1} = J_{E_1}^{-1}\big(\alpha_n J_{E_1}x_1 + (1-\alpha_n)J_{E_1}T_\lambda y_n\big), \quad \forall n \geq 1. \end{cases} \tag{6.4.5}$$

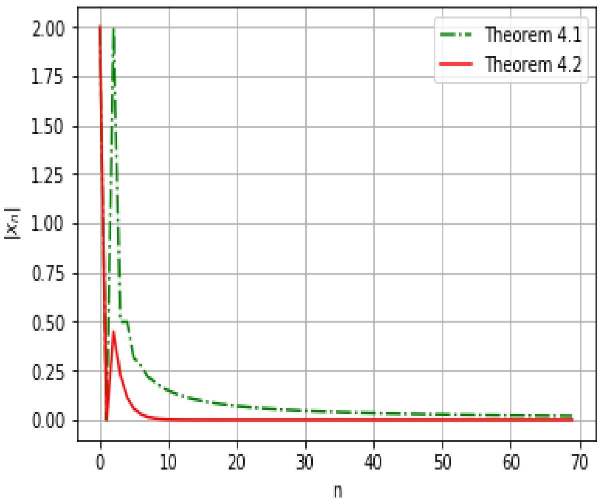
converges strongly to an element $z \in \Xi$, where $\Xi = \{x \in EP(f) : Ax \in F(S)\}$.

Proof:
Letting $T \equiv I$ and $B \equiv A_f$ in Theorem 6.2.1, we have that $J_\lambda^B \equiv T_\lambda$ for all $\lambda > 0$. Hence, the conclusion follows from Theorem 6.2.1.

6.5 Numerical Experiment

In this section, we give some numerical example to establish the implementability of the iterative algorithms proposed in this paper.

Example 6.5.1 *Let $E_1 = E_2 = \mathbb{R}$, $K = [-2, 2]$, $Tx = sinx$, $Sx = \frac{1}{3}x$, $Ax = 3x$, $Bx = 4x$. It is easy to see that A is bounded and linear, $A^* = A$, T, S and B satisfy the condition in Theorems 6.2.2 and 6.2.1. By taking $x_1 = 2$, $\gamma = 1/2$, $r = 1/2$.*



Remark 6.5.1 *1. Theorems 6.2.2 and 6.2.1 complement and improve the results of Ansari and Rehan [7] in the following sense:*

(i) The condition that the normalized duality map is weakly sequentially continuous in the theorems of Ansari and Rehan [7] was dispensed with in Theorems 6.2.2 and 6.2.1.

(ii) Weak convergence theorems were proved in the result of Ansari and Rehan [7], whereas in Therorems 6.2.2 and 6.2.1, strong convergence was established.

All the results in this chapter are the results of [63], which was accepted in **Carpathian Journal of Mathematics**.

CHAPTER 7

Iterative algorithms for approximation of solutions of some equilibrium problems in Banach spaces

7.1 Introduction

In this chapter, we construct and study iterative algorithms of Krasnoselkii-type and of Halpern-type for approximating an element of the set of common zeros of a countable family of inverse strongly monotone maps, common fixed points of a countable family of totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued maps, and a solution of a system of generalized mixed equilibrium problems. Strong convergence of the sequences generated by these algorithms is established in uniformly smooth and 2-uniformly convex real Banach spaces. Furthermore, the theorems obtained extend, improve and generalize several recent important results.

We shall use the following results in this chapter.

Lemma 7.1.1 *Let K be a nonempty closed and convex subset of a real Banach space E and $G : K \to 2^K$ be a continuous map. Then, G is closed.*

Proof:
Let $\{x_n\}$ be a sequence in K such that $x_n \to x$ and $w_n \to y$, $w_n \in Gx_n$. By continuity of G, we have that $w_n \to p$, $p \in Gx$. By uniqueness of limit, we have that $p = y$. Therefore, $y \in Gx$. Hence, G is closed.

Lemma 7.1.2 (Chang *et al.* [75]) *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and K be a nonempty closed convex subset of E. Let $G : K \to 2^K$ be a closed and , $(\{\upsilon_n\}, \{\mu_n\}, \rho)$-total quasi-$\phi$-asymptotically nonexpansive multi-valued mapping. If $\mu_1 = 0$, then the fixed point set $F(G)$ of G is a closed and convex subset of K.*

Let K be a nonempty closed and convex subset of a Banach space E. For solving the generalized mixed equilibrium problem (1.5.2), we assume that a bifunction $h : K \times K \to \mathbb{R}$ satisfies the following conditions:

$(A1)$ $h(x,x) = 0, \ \forall x \in E,$
$(A2)$ h is monotone, that is, $h(x,y) + h(y,x) \leq 0, \ \forall x, y \in E,$
$(A3)$ for all $x, y, z \in E$, $\limsup_{t\downarrow 0} h(tz + (1-t)x, y) \leq h(x,y),$
$(A4)$ for all $x \in K$, $y \longmapsto h(x,y)$ is convex and lower semicontinuous.

Lemma 7.1.3 (Blum and Oettli [29]) *Let K be a closed convex subset of a smooth, strictly convex and reflexive real Banach space E and let $h : K \times K \to \mathbb{R}$ be a bifunction satisfying conditions $(A1)(A4)$. Let $r > 0$ and $x \in E$, then there exists $z \in K$ such that $h(z,y) + \frac{1}{r}\Big\langle y - z, Jz - Jx\Big\rangle \geq 0 \ \forall y \in K$.*

Lemma 7.1.4 (see e.g., Zhang [176]) *Let K be a nonempty closed and convex subset of a uniformly smooth, strictly convex and reflexive real Banach space E. Let $h : K \times K \to \mathbb{R}$ be a bi-function satisfying $(A1) - (A4)$, let $B : K \to E^*$ be a monotone map and $\Phi : K \to \mathbb{R}$ be a lower semi-continuous convex function. For $r > 0$ define a map $T_r : K \to K$ by:*

$$T_r(x) = \Big\{z \in K : h(z,y) + \Phi(y) - \Phi(z) + \langle Bz, y - z\rangle + \frac{1}{r}\Big\langle y - z, Jz - Jx\Big\rangle \geq 0 \quad \forall \ \ y \in K\Big\},$$

for all $x \in E$. Then, the following hold:

1. *T_r is single-valued,*
2. *T_r is a firmly nonexpansive-type map, that is, for all $x, y \in E$,*
$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle,$$
3. *$F(T_r) =$GMEP(h, Φ, B) is closed and convex.*
4. *$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q,x) \quad \forall q \in F(T_r), x \in E.$*

Lemma 7.1.5 (Deng and Bai [78]) *The unique solutions to the positive integer equation*

$$n = i_n + \frac{(m_n - 1)m_n}{2}, \ m_n \geq i_n, \ n = 1, 2, 3, \ldots \ \textit{are} \tag{7.1.1}$$

$$i_n = n - \frac{(m_n - 1)m_n}{2}, \ \ m_n = -\left(\frac{1}{2} - \left(2n + \frac{1}{4}\right)^{\frac{1}{2}}\right), \ n = 1, 2, 3, \ldots, \tag{7.1.2}$$

where $[x]$ denotes the maximal integer that is not larger than x.

7.2 Main Results

In what follows i_n and m_n are the solutions to the positive integer equation: $n = i + \dfrac{(m-1)m}{2}$ $(m \geq i,\ n = 1, 2, ...)$, that is, for each $n \geq 1$, there exist unique i_n and m_n such that

$$i_1 = 1,\ i_2 = 1,\ i_3 = 2,\ i_4 = 1,\ i_5 = 2,\ i_6 = 3,\ i_7 = 1,\ i_8 = 2, \cdots;$$
$$m_1 = 1,\ m_2 = 1,\ m_3 = 2,\ m_4 = 1,\ m_5 = 2,\ m_6 = 3,\ m_7 = 1,\ m_8 = 2, \cdots.$$

See Deng [77]. We prove the following strong convergence theorem *using a Krasnoselskii-type algorithm* (see Krasnoselskii [?] for the original algorithm of Krasnoselskii).

Theorem 7.2.1 *Let K be a closed convex nonempty subset of a 2-uniformly convex and uniformly smooth real Banach space E with dual space E^*. Let $h_i : K \times K \to \mathbb{R}$ $(i = 1, 2, 3, ...)$ be a sequence of bifunctions satisfying conditions $(A1) - (A4)$ and $G_i : K \to 2^E$, $i = 1, 2, 3, ...$ be a countable family of equally continuous and totally quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps with nonnegative real sequences $\{v_n^{(i)}\}, \{\mu_n^{(i)}\}$ and strictly increasing continuous functions $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $v_n^{(i)} \to 0, \mu_n^{(i)} \to 0$ as $n \to \infty$ and $\psi_i(0) = 0$. Let $A_i : K \to 2^{E^*}$, $i = 1, 2, 3, ...$ be a countable family of γ_i-inverse strongly monotone multi-valued maps and let $\gamma = inf\{\gamma_i, i = 1, 2, 3, ...\} > 0$. Let $\Phi_i : K \to \mathbb{R}$ $(i = 1, 2, 3, ...)$ be a sequence of lower semi-continuous convex functions and let $B_i : K \to E^*$ $(i = 1, 2, 3, ...)$ be a sequence of continuous monotone functions. Suppose $W := \Big(\cap_{i=1}^{\infty} F(G_i)\Big) \cap \Big(\cap_{i=1}^{\infty} A_i^{-1}(0)\Big) \cap \Big(\cap_{i=1}^{\infty} GMEP(h_i, \Phi_i, B_i)\Big) \neq \emptyset$ and the sequence $\{x_n\}$ in K is defined iteratively as follows:*

$$\begin{cases} x_0 \in K_0 = K, \\ y_n = \Pi_K J^{-1}(Jx_n - \lambda \xi_{i_n}), \qquad (\xi_{i_n} \in A_{i_n} x_n), \\ z_n = J^{-1}\big(\alpha J x_n + (1-\alpha) J \eta_{m_n}^{(i_n)}\big), \qquad (\eta_{m_n}^{(i_n)} \in G_{i_n}(PG_{i_n})^{m_n - 1} y_n), \\ u_n = T_{r_n} z_n, \\ K_{n+1} = \{z \in K_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{K_{n+1}} x_0, \quad n \geq 0, \end{cases}$$

where $\theta_n := (1-\alpha)\big[v_{m_n}^{(i_n)} \sup_{p \in W} \psi_{i_n}\big(\phi(p, x_n)\big) + \mu_{m_n}^{(i_n)}\big]$; $\lambda \in (0, \frac{c_2}{2}\gamma)$, $c_2 > 0$ is the constant in Lemma 2.1.4, $P : E \to K$ is a nonexpansive retraction and $\alpha \in (0, 1)$. Then, $\{x_n\}$ converges strongly to some element of W.

Proof:
The proof is divided into five steps.

Step 1: We show that the sequence $\{x_n\}$ is well defined.
Observe that for each $n \geq 0$, the set $K_n = \{z \in K_{n-1} : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}$

is equivalent to the set $D_n = \{z \in K_{n-1} : 2\langle z, Jx_n - Ju_n\rangle \leq ||x_n||^2 - ||u_n||^2 + \theta_n\}$ and clearly, D_n is closed and convex and so K_n is. Next, we show that $W \subset K_n$ for all $n \geq 0$. We do this by induction. Clearly, $W \subset K_0 = K$. Suppose $W \subset K_k$ for some $k \geq 0$. Let $p \in W$, then using the definition of ϕ, Lemma 2.1.1 and the fact that T_i is totally quasi-ϕ-asymptotically nonexpansive for each $i = 1, 2, 3, ...$ and $\eta_{m_k}^{(i_k)} \in G_{i_k}(PG_{i_k})^{m_k-1}y_k$, we have that:

$$\begin{aligned}
\phi(p, u_k) &= \phi(p, T_{r_k}z_k) \leq \phi(p, J^{-1}(\alpha Jx_k + (1-\alpha)J\eta_{m_k}^{(i_k)})) \\
&= ||p||^2 - 2\langle p, \alpha Jx_k + (1-\alpha)J\eta_{m_k}^{(i_k)}\rangle + ||\alpha Jx_k + (1-\alpha)J\eta_{m_k}^{(i_k)}||^2 \\
&\leq ||p||^2 - 2\alpha\langle p, Jx_k\rangle - 2(1-\alpha)\langle p, J\eta_{m_k}^{(i_k)}\rangle \\
&\quad +\alpha||x_k||^2 + (1-\alpha)||\eta_{m_k}^{(i_k)}||^2 \qquad (7.2.1) \\
&= \alpha\phi(p, x_k) + (1-\alpha)\phi(p, \eta_{m_k}^{(i_k)}) \\
&\leq \alpha\phi(p, x_k) + (1-\alpha)\big[\phi(p, y_k) + v_k\psi(\phi(p, y_k)) + \mu_k\big]. \qquad (7.2.2)
\end{aligned}$$

Moreover, by Lemmas 2.3.3 and 2.1.4, we have with $y^* = \lambda\xi_{i_k}$, $\xi_{i_k} \in A_{i_k}x_k$ that,

$$\begin{aligned}
\phi(p, y_k) &= \phi(p, \Pi_K J^{-1}(Jx_k - \lambda\xi_{i_k})) \leq \phi(p, J^{-1}(Jx_k - \lambda\xi_{i_k})) = V(p, Jx_k - \lambda\xi_{i_k}) \\
&\leq V(p, Jx_k) - 2\lambda\langle J^{-1}(Jx_k - \lambda\xi_{i_k}) - p, \xi_{i_k}\rangle \\
&= \phi(p, x_k) - 2\lambda\langle x_k - p, \xi_{i_k}\rangle - 2\lambda\langle J^{-1}(Jx_k - \lambda\xi_{i_k}) - x_k, \xi_{i_k}\rangle \\
&\leq \phi(p, x_k) - 2\lambda\gamma||\xi_{i_k}||^2 + 2\lambda||J^{-1}(Jx_k - \lambda\xi_{i_k}) - J^{-1}(Jx_k)||.||\xi_{i_k}|| \\
&\leq \phi(p, x_k) - 2\lambda\gamma||\xi_{i_k}||^2 + \frac{4\lambda^2}{c_2}\Big\|\xi_{i_k}\Big\|^2 \\
&= \phi(p, x_k) - 2\lambda\Big(\gamma - \frac{2}{c_2}\lambda\Big)||\xi_{i_k}||^2. \qquad (7.2.3)
\end{aligned}$$

Thus, using the fact that $\lambda \leq \frac{c_2}{2}\gamma$, we have that $\phi(p, y_k) \leq \phi(p, x_k)$. Using this and inequality (7.2.2), we have that $\phi(p, u_k) \leq \phi(p, x_k) + \theta_k$, which implies that $p \in K_{k+1}$. Then, by induction, $W \subset K_n$ for each $n \geq 0$ and hence the sequence $\{x_n\}$ is well defined.

Step 2: We show that the sequence $\{x_n\}$ converges to some $x^* \in K$.
From $x_n = \Pi_{K_n}x_0$ and by Lemma 2.3.8, we have for each $p \in W \subset K_n$, $\forall n \geq 0$, that

$$\phi(x_0, x_n) \leq \phi(x_0, p) - \phi(x_n, p) \leq \phi(x_0, p).$$

This implies that $\{\phi(x_0, x_n)\}$ is bounded, so $\{x_n\}$ is bounded.

Now, for each $i \geq 1$, set $K_i = \{k \geq 1 : k = i + \frac{(m-1)m}{2}, m \geq i, m \in \mathbb{N}\}$. Observe that if for each $i \geq 1$, $k \in K_i$, then $v_{m_k}^{(i_k)} = v_{m_k}^{(i)}$, $\mu_{m_k}^{(i_k)} = \mu_{m_k}^{(i)}$ and $\psi_{i_k} = \psi_i$. Thus, $m_k \uparrow \infty$ as $k \to \infty$ for $k \in K_i$. Therefore, $\lim_{n\to\infty} \theta_n = 0$.
Since, $x_n = \Pi_{K_n}x_0$ and $x_{n+1} = \Pi_{K_{n+1}}x_0 \in K_n$, we have $\phi(x_0, x_n) \leq \phi(x_0, x_n) + \phi(x_n, x_{n+1}) \leq \phi(x_0, x_{n+1})$, which implies that $\{\phi(x_0, x_n)\}$ is non-decreasing

and bounded, then $\lim \phi(x_0, x_n)$ exists. Next, for positve integers m, n such that $m \geq n$ and using Lemma 2.3.8, we have that

$$\phi(x_n, x_m) \leq \phi(x_0, x_m) - \phi(x_0, x_n) \tag{7.2.4}$$

which implies $\lim_{n,m\to\infty} \phi(x_n, x_m) = 0$ and by Lemma 2.3.4, $||x_n - x_m|| \to 0$ as $n, m \to \infty$. Hence, $\{x_n\}$ is Cauchy and so, there exists $x^* \in K$ such that $x_n \to x^*$ as $n \to \infty$.

Step 3: We show $x^* \in \cap_{i=1}^{\infty} F(G_i)$.
Taking $m = (n+1)$ in inequality (7.2.4) yields that $\phi(x_n, x_{n+1}) \leq \phi(x_0, x_{n+1}) - \phi(x_0, x_n) \to 0,\ n \to \infty$. Hence, $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$ by Lemma 2.3.4. Since $x_{n+1} \in K_{n+1}$ and from the definition of K_{n+1}, we have, $\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \theta_n \to 0$ as $n \to \infty$, which implies $||x_{n+1} - u_n|| \to 0$ as $n \to \infty$ by Lemma 2.3.4. It follows that $||x_n - u_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - u_n|| \to 0$ as $n \to \infty$. Also, by the uniform continuity of J on bounded sets, we have that $||Jx_n - Ju_n|| \to 0$ as $n \to \infty$.
Using basically the same computations as those leading to inequality (7.2.1), Lemma 2.1.1 and using the fact that $\eta_{m_k}^{(i_k)} = \eta_{m_k}^{(i)}$ whenever $k \in K_i$, for each $i \geq 1$, $\eta_{m_n}^{(i_n)} \in G_{i_n}(PG_{i_n})^{m_n - 1} y_n$, we have,

$$\begin{aligned}
\phi(p, u_n) &\leq ||p||^2 - 2\alpha\langle p, Jx_n\rangle - 2(1-\alpha)\langle p, \eta_{m_n}^{(i)}\rangle + \alpha||x_n||^2 + (1-\alpha)||\eta_{m_n}^{(i)}||^2 \\
&\quad -\alpha(1-\alpha)g(||Jx_n - J\eta_{m_n}^{(i)}||) \\
&\leq \alpha\phi(p, x_n) + (1-\alpha)\phi(p, \eta_{m_n}^{(i)}) - \alpha(1-\alpha)g(||Jx_n - J\eta_{m_n}^{(i)}||) \\
&\leq \alpha\phi(p, x_n) + (1-\alpha)\big[\phi(p, y_n) + \upsilon_n\psi(\phi(p, y_n)) + \mu_n\big] \\
&\quad -\alpha(1-\alpha)g(||Jx_n - J\eta_{m_n}^{(i)}||) \\
&\leq \alpha\phi(p, x_n) + (1-\alpha)\big[\phi(p, x_n) + \upsilon_n\psi(\phi(p, x_n)) + \mu_n\big] \\
&\quad -\alpha(1-\alpha)g(||Jx_n - J\eta_{m_n}^{(i)}||) \\
&\leq \phi(p, x_n) + \theta_n - \alpha(1-\alpha)g(||Jx_n - J\eta_{m_n}^{(i)}||),
\end{aligned}$$

which implies that

$$\alpha(1-\alpha)g(||Jx_n - J\eta_{m_n}^{(i)}||) \leq \phi(p, x_n) - \phi(p, u_n) + \theta_n, \quad \forall n \geq 0.$$

Using the definition of ϕ and the fact that $\lim ||x_n - u_n|| = 0$, we have that $\lim g(||Jx_n - J\eta_{m_n}^{(i)}||) = 0$. Since g is strictly increasing and $g(0) = 0$, we have that $||Jx_n - J\eta_{m_n}^{(i)}|| \to 0$ as $n \to \infty$. By uniform continuity of J^{-1} on bounded sets, we get that $||x_n - \eta_{m_n}^{(i)}|| \to 0$ as $n \to \infty$, for each $i \geq 1$, $\eta_{m_n}^{(i_n)} \in G_{i_n}(PG_{i_n})^{m_n - 1} y_n$.

Moreover, using inequalities (7.2.2) and (7.2.3), we obtain that

$$\begin{aligned}
\phi(p,u_n) &\leq \alpha\phi(p,x_n)+(1-\alpha)\big[\phi(p,y_n)+v_n\psi(\phi(p,y_n))+\mu_n\big] \\
&\leq \alpha\phi(p,x_n)+(1-\alpha)\Big[\phi(p,x_n)+v_n\psi(\phi(p,x_n))+\mu_n \\
&\quad -2\lambda\Big(\gamma-\frac{2}{c_2}\lambda\Big)||\xi_{i_n}||^2\Big] \qquad (7.2.5)\\
&\leq \phi(p,x_n)+\theta_n-2(1-\alpha)\lambda\Big(\gamma-\frac{2}{c_2}\lambda\Big)||\xi_{i_n}||^2. \qquad (7.2.6)
\end{aligned}$$

Inequality (7.2.16) implies that

$$2(1-\alpha)\lambda\Big(\gamma-\frac{2}{c_2}\lambda\Big)||\xi_{i_n}||^2 \leq \phi(p,x_n)-\phi(p,u_n)+\theta_n.$$

Thus, we obtain that $\lim||\xi_{i_n}||=0$. Furthermore, since $x_n\in K$ for all $n\geq 0$, then using Lemmas 2.3.3 and 2.1.4, we have that,

$$\begin{aligned}
\phi(x_n,y_n) &= \phi(x_n,\Pi_K J^{-1}(Jx_n-\lambda\xi_{i_n})) \\
&\leq \phi(x_n,J^{-1}(Jx_n-\lambda\xi_{i_n}))=V(x_n,Jx_n-\lambda\xi_{i_n}) \qquad (7.2.7)\\
&\leq \phi(x_n,x_n)-2\lambda\langle J^{-1}(Jx_n-\lambda\xi_{i_n})-J^{-1}Jx_n,\xi_{i_n}\rangle \\
&\leq 2\lambda||J^{-1}(Jx_n-\lambda\xi_{i_n})-J^{-1}Jx_n||.||\xi_{i_n}|| \\
&\leq \frac{4}{c_2}\lambda^2\Big|\Big|\xi_{i_n}\Big|\Big|^2. \qquad (7.2.8)
\end{aligned}$$

Thus, from inequality (7.2.8) and using the fact that $\lim||\xi_{i_n}||=0$, we have that $\lim\phi(x_n,y_n)=0$ and by Lemma 2.3.4, we have that $\lim||x_n-y_n||=0$. Consequently, for each i, $k\in K_i$, $\eta^{(i)}_{m_k}\in G_i(PG_{i_n})^{m_k-1}y_k$, we have $||y_k-\eta^{(i)}_{m_k}||\to 0$, as $k\to\infty$, since $||y_k-\eta^{(i)}_{m_n}||\leq||y_k-x_k||+||x_k-\eta^{(i)}_{m_k}||\to 0$, as $k\to\infty$. Also, since $\lim_{k\to\infty}x_k=x^*$ and $\lim||x_k-y_k||=0$, we have that $\lim y_k=x^*$ and $\lim_{k\to\infty}\eta^{(i)}_{m_k}=x^*$ for each i, $\eta^{(i)}_{m_k}\in G_i(PG_{i_n})^{m_k-1}y_k$, $\eta^{(i)}_{m_{k+1}}\in G_iP\eta^{(i)}_{m_k}\subset G_i(PG_i)^{m_{k+1}-1}y_k$ and $s^{(i)}_{m_{k+1}}\in G_i(PG_i)^{m_{k+1}-1}y_{k+1}$. Now,

$$||\eta^{(i)}_{m_{k+1}}-\eta^{(i)}_{m_k}||\leq||\eta^{(i)}_{m_{k+1}}-s^{(i)}_{m_{k+1}}||+||s^{(i)}_{m_{k+1}}-y_{k+1}||+||y_{k+1}-y_k||+||y_k-\eta^{(i)}_{m_k}||$$

Since G_iP, $i=1,2,3,...$ is equally continuous, for each $i=1,2,3,...$, we have that $\lim_{k\to\infty}||\eta^{(i)}_{m_{k+1}}-\eta^{(i)}_{m_k}||=0$. Thus, $\eta^{(i)}_{m_{k+1}}\to x^*$ as $k\to\infty$, but $\eta^{(i)}_{m_{k+1}}\to s_ix^*$, $s_ix^*\in G_iPx^*=G_ix^*$, by continuity of G_iP for each i. Hence, by uniqueness of limit, $s_ix^*=x^*$ for each i, so $x^*\in\cap_{i=1}^{\infty}F(G_i)$.

Step 4: We show that $x^*\in\cap_{i=1}^{\infty}A_i^{-1}(0)$.
For each i, $k\in K_i$, $\xi_{i_k}\in A_{i_k}x_k$ and noting that $\xi_{i_k}=\xi_i$, $\xi_i\in A_ix_k$, we have from inequality (7.2.17) that $\lim_{k\to\infty}||\xi_{i_k}||=0$. Since $x_k\to x^*$, $k\to\infty$ and A_i is γ_i-inverse strongly montone for each i, it is Lipschitz continuous and thus, $\xi_{i_k}\to u_ix^*$ as $k\to\infty$ for $u_ix^*\in A_ix^*$, $i=1,2,3,...$. Thus, by the uniqueness of limit, we have that $u_ix^*=0$, $i=1,2,3,...$. Hence, $x^*\in\cap_{i=1}^{\infty}A_i^{-1}(0)$.

Step 5: Finally, we show that $x^* \in \cap_{i=1}^{\infty} GMEP(h_i, \Phi_i, B_i)$.
Define a function $\tau_i : K \times K \to \mathbb{R}$ $(i = 1, 2, 3, ...)$ by

$$\tau_i(x, y) = h_i(x, y) + \Phi_i(y) - \Phi_i(x) + \langle y - x, B_i x\rangle \; \forall x, y \in K \quad i = 1, 2, 3, ...$$

Clearly, τ_i satisfies $(A1) - (A4)$ for each i. Now, from $u_n = T_{r_n} z_n$, $p \in W$ and Lemma 7.1.4, we have

$$\begin{aligned} \phi(u_n, z_n) = \phi(T_{r_n} z_n, z_n) &\leq \phi(p, z_n) - \phi(p, T_{r_n} z_n) \\ &\leq \phi(p, x_n) - \phi(p, T_{r_n} z_n) \\ &= \phi(p, x_n) - \phi(p, u_n) \end{aligned}$$

which implies that $\lim_{n\to\infty} \phi(u_n, z_n) = 0$. Since $\{u_n\}$ is bounded, we have from Lemma 2.3.4 that $\lim_{n\to\infty} ||u_n - z_n|| = 0$. Since $x_n \to x^*$ and $u_n \to x^*$ as $n \to \infty$, we obtain that $z_n \to x^*$ as $n \to \infty$ and by uniform continuity of J on bounded sets, we get that $\lim_{n\to\infty} ||Ju_n - Jz_n|| = 0$. Again, since $r_{i_K} = r_i$ as $k \in K_i$ for each $i \in \mathbb{N}$ and $r_i \in [d, \infty)$ for some $d > 0$, we have $\lim_{k\to\infty} \dfrac{||Ju_k - Jz_k||}{r_i} = 0$.
Next, since $\tau_i(u_k, y) + \frac{1}{r_i}\Big\langle y - u_k, Ju_k - Jz_k\Big\rangle \geq 0 \, \forall y \in K$, we obtain that

$$\frac{1}{r_i}\Big\langle y - u_k, Ju_k - Jz_k\Big\rangle \geq -\tau_i(u_k, y) \geq \tau_i(y, u_k) \, \forall y \in K.$$

This implies that

$$\tau_i(y, u_k) \leq \frac{1}{r_i}\Big\langle y - u_k, Ju_k - Jz_k\Big\rangle \leq (M_0 + ||y||)\frac{||Ju_k - Jz_k||}{r_i}, \tag{7.2.9}$$

for some $M_0 \geq 0$. Since $y \longmapsto \tau_i(x, y)$ is convex and lower semi-continuous, we obtain from inequality (7.2.9) that

$$\tau_i(y, x^*) \leq \liminf \tau_i(y, u_k) \leq 0 \, \forall y \in K. \tag{7.2.10}$$

Now, for $t \in (0, 1)$ and $y \in K$, let $y_t = ty + (1-t)x^*$. Since $y \in K$ and $x^* \in K$, we have that $y_t \in K$ and so from inequality (7.2.10), $\tau_i(y_t, x^*) \leq 0$ for each i. But from conditions $(A1)$ and $(A4)$ we have that

$$0 = \tau_i(y_t, y_t) \leq t\tau_i(y_t, y) + (1-t)\tau_i(y_t, x^*) \leq t\tau_i(y_t, y).$$

So, $\tau_i(y_t, y) \geq 0 \; \forall y \in K$, $\;i = 1, 2, 3, ...$ and condition $(A3)$ implies that $\tau_i(x^*, y) \geq \limsup_{t\to 0} \tau_i(y_t, y) \geq 0 \; \forall y \in K$, $\;i = 1, 2, 3, ...$ Thus, $x^* \in EP(\tau_i) = GMEP(h_i, \Phi_i, B_i)$ for each i, so $x^* \in \cap_{i=1}^{\infty} GMEP(h_i, \Phi_i, B_i)$. Hence, $x^* \in W$. This completes the proof.

We now prove the following strong convergence theorem *using a Halpern-type algorithm* (see [?] for the original algorithm of Halpern).

Theorem 7.2.2 *Let K be a closed convex nonempty subset of a 2-uniformly convex and uniformly smooth real Banach space E with dual space E^*. Let $h_i : K \times K \to \mathbb{R}$ $(i = 1, 2, 3, ...)$ be a sequence of bifunctions satisfying conditions $(A1) - (A4)$ and $G_i : K \to 2^E$, $i = 1, 2, 3, ...$ be a countable family of equally continuous and totally quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps with nonnegative real sequences $\{v_n^{(i)}\}, \{\mu_n^{(i)}\}$ and strictly increasing continuous functions $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $v_n^{(i)} \to 0, \mu_n^{(i)} \to 0$ as $n \to \infty$ and $\psi_i(0) = 0$. Let $A_i : K \to 2^{E^*}$, $i = 1, 2, 3, ...$ be a countable family of γ_i-inverse strongly monotone multi-valued maps and let $\gamma = inf\{\gamma_i, i = 1, 2, 3, ...\} > 0$. Let $\Phi_i : K \to \mathbb{R}$ $(i = 1, 2, 3, ...)$ be a sequence of lower semi-continuous convex functions and let $B_i : K \to E^*$ $(i = 1, 2, 3, ...)$ be a sequence of continuous monotone functions. Suppose $W := \Big(\cap_{i=1}^{\infty} F(G_i)\Big) \cap \Big(\cap_{i=1}^{\infty} A_i^{-1}(0)\Big) \cap \Big(\cap_{i=1}^{\infty} GMEP(h_i, \Phi_i, B_i)\Big) \neq \emptyset$ and the sequence $\{x_n\}$ in K is defined iteratively as follows:*

$$\begin{cases} x_0 \in K_0 = K, \\ y_n = \Pi_K J^{-1}(Jx_n - \lambda \xi_{i_n}), \qquad (\xi_{i_n} \in A_{i_n} x_n), \\ z_n = J^{-1}\big(\alpha_n J x_0 + (1 - \alpha_n) J \eta_{m_n}^{(i_n)}\big), \qquad (\eta_{m_n}^{(i_n)} \in G_{i_n}(PG_{i_n})^{m_n - 1} y_n), \\ u_n = T_{r_n} z_n, \\ K_{n+1} = \{z \in K_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_0) + (1 - \alpha_n)\phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{K_{n+1}} x_0, \quad n \geq 0, \end{cases}$$

where $\theta_n := (1 - \alpha_n)\big[v_{m_n}^{(i_n)} \sup_{p \in W} \psi_{i_n}\big(\phi(p, x_n)\big) + \mu_{m_n}^{(i_n)}\big]$; $\alpha_n \in (0, 1)$ satisfying the following (i) $\lim \alpha_n = 0$ and (ii) $\liminf \alpha_n(1 - \alpha_n) > 0$; $\lambda \in (0, \frac{c_2}{2}\gamma)$ and $c_2 > 0$ is the constant in Lemma 2.1.4. Then, $\{x_n\}$ converges strongly to some element of W.

Proof:
The proof is divided into five steps.

Step 1: We show that the sequence $\{x_n\}$ is well defined.
From the proof of Theorem 7.2.1, the set $K_n = \{z \in K_{n-1} : \phi(z, u_n) \leq \alpha_n \phi(z, x_0) + (1 - \alpha_n)\phi(z, x_n) + \theta_n\}$, $\forall n \geq 0$ is closed and convex. Next, we show that $W \subset K_n$ for all $n \geq 0$. We do this by induction. Clearly, $W \subset K_0 = K$. Suppose $W \subset K_k$ for some $k \geq 0$. Let $p \in W$, then using the definition of ϕ, Lemma 2.1.1 and the fact that T_i is totally quasi-ϕ-asymptotically nonexpan-

sive for each $i = 1, 2, 3, ...$ and $\eta_{m_k}^{(i_k)} \in G_{i_k}(PG_{i_k})^{m_k-1}y_k$, we have that:

$$\begin{aligned}
\phi(p, u_k) &= \phi(p, T_{r_k}z_k) \leq \phi(p, J^{-1}(\alpha_k Jx_0 + (1-\alpha_k)J\eta_{m_k}^{(i_k)})) \\
&= ||p||^2 - 2\langle p, \alpha_k Jx_0 + (1-\alpha_k)J\eta_{m_k}^{(i_k)}\rangle + ||\alpha_k Jx_0 + (1-\alpha_k)J\eta_{m_k}^{(i_k)}||^2 \\
&\leq ||p||^2 - 2\alpha_k\langle p, Jx_0\rangle - 2(1-\alpha_k)\langle p, J\eta_{m_k}^{(i_k)}\rangle + \alpha_k||x_0||^2 \\
&\quad +(1-\alpha_k)||\eta_{m_k}^{(i_k)}||^2 - \alpha_k(1-\alpha_k)g(||Jx_0 - J\eta_{m_k}^{(i_k)}||) \qquad (7.2.11) \\
&\leq \alpha_k\phi(p, x_0) + (1-\alpha_k)\phi(p, \eta_{m_k}^{(i_k)}) \\
&\leq \alpha_k\phi(p, x_0) + (1-\alpha_k)\big[\phi(p, y_k) + v_k\psi(\phi(p, y_k)) + \mu_k\big]. \qquad (7.2.12)
\end{aligned}$$

Moreover, by Lemmas 2.3.3 and 2.1.4, we have with $y^* = \lambda\xi_{i_k}$, $\xi_{i_k} \in A_{i_k}x_k$ that,

$$\begin{aligned}
\phi(p, y_k) &= \phi(p, \Pi_K J^{-1}(Jx_k - \lambda\xi_{i_k})) \leq \phi(p, J^{-1}(Jx_k - \lambda\xi_{i_k})) = V(p, Jx_k - \lambda\xi_{i_k}) \\
&\leq V(p, Jx_k) - 2\lambda\langle J^{-1}(Jx_k - \lambda\xi_{i_k}) - p, \xi_{i_k}\rangle \\
&= \phi(p, x_k) - 2\lambda\langle x_k - p, \xi_{i_k}\rangle - 2\lambda\langle J^{-1}(Jx_k - \lambda\xi_{i_k}) - x_k, \xi_{i_k}\rangle \\
&\leq \phi(p, x_k) - 2\lambda\gamma||\xi_{i_k}||^2 + 2\lambda||J^{-1}(Jx_k - \lambda\xi_{i_k}) - J^{-1}(Jx_k)||.||\xi_{i_k}|| \\
&\leq \phi(p, x_k) - 2\lambda\gamma||\xi_{i_k}||^2 + \frac{4\lambda^2}{c_2}\Big\|\xi_{i_k}\Big\|^2 \\
&= \phi(p, x_k) - 2\lambda\Big(\gamma - \frac{2}{c_2}\lambda\Big)||\xi_{i_k}||^2. \qquad (7.2.13)
\end{aligned}$$

Thus, using the fact that $\lambda \leq \frac{c_2}{2}\gamma$, we have that $\phi(p, y_k) \leq \phi(p, x_k)$. Using this and inequality (7.2.12), we have that

$$\phi(p, u_k) \leq \alpha_k\phi(p, x_0) + (1-\alpha_k)\phi(p, x_k) + \theta_k, \qquad (7.2.14)$$

which implies that $p \in K_{k+1}$. Then, by induction, $W \subset K_n$ for each $n \geq 0$ and hence the sequence $\{x_n\}$ is well defined.

Step 2: We show that the sequence $\{x_n\}$ converges to some $x^* \in K$.
This follows as in step 2 of the proof of Theorem 7.2.1.

Step 3: We show $x^* \in \cap_{i=1}^{\infty} F(G_i)$.
Following the same argument as in step 3 of the proof of Theorem 7.2.1, we obtain that $||Jx_n - Ju_n|| \to 0$ as $n \to \infty$. Also, using basically the same computataions as those leading to inequality (7.2.11) and using the fact that $\eta_{m_k}^{(i_k)} = \eta_{m_k}^{(i)}$ whenever $k \in K_i$, for each $i \geq 1$, $\eta_{m_n}^{(i_n)} \in G_{i_n}(PG_{i_n})^{m_n-1}y_n$, we

have,

$$\begin{aligned}
\phi(p,u_n) &\le ||p||^2 - 2\alpha_n\langle p, Jx_0\rangle - 2(1-\alpha_n)\langle p, \eta^{(i)}_{m_n}\rangle + \alpha_n||x_0||^2 + (1-\alpha_n)||\eta^{(i)}_{m_n}||^2 \\
&\quad -\alpha_n(1-\alpha_n)g(||Jx_0 - J\eta^{(i)}_{m_n}||) \\
&\le \alpha_n\phi(p,x_0) + (1-\alpha_n)\phi(p,\eta^{(i)}_{m_n}) - \alpha_n(1-\alpha_n)g(||Jx_0 - J\eta^{(i)}_{m_n}||) \\
&\le \alpha_n\phi(p,x_0) + (1-\alpha_n)\big[\phi(p,y_n) + \upsilon_n\psi(\phi(p,y_n)) + \mu_n\big] \\
&\quad -\alpha(1-\alpha)g(||Jx_n - J\eta^{(i)}_{m_n}||) \\
&\le \alpha_n\phi(p,x_0) + (1-\alpha_n)\big[\phi(p,x_n) + \upsilon_n\psi(\phi(p,x_n)) + \mu_n\big] \\
&\quad -\alpha_n(1-\alpha_n)g(||Jx_0 - J\eta^{(i)}_{m_n}||) \\
&\le \alpha_n\phi(p,x_0) + (1-\alpha_n)\phi(p,x_n) + \theta_n - \alpha_n(1-\alpha_n)g(||Jx_0 - J\eta^{(i)}_{m_n}||),
\end{aligned}$$

which implies that

$$\begin{aligned}
\alpha_n(1-\alpha_n)g(||Jx_n - J\eta^{(i)}_{m_n}||) &\le \alpha_n\phi(p,x_0) + (1-\alpha_n)\phi(p,x_n) \\
&\quad +\theta_n - \phi(p,u_n), \ \forall n \ge 0.
\end{aligned}$$

Using the definition of ϕ, the fact that $\lim ||x_n - u_n|| = 0$ and $\liminf \alpha_n(1-\alpha_n) > 0$, we have that $\lim g(||Jx_n - J\eta^{(i)}_{m_n}||) = 0$. Since g is strictly increasing and $g(0) = 0$, we have that $||Jx_n - J\eta^{(i)}_{m_n}|| \to 0$ as $n \to \infty$. By uniform continuity of J^{-1} on bounded sets, we get that $||x_n - \eta^{(i)}_{m_n}|| \to 0$ as $n \to \infty$, for each $i \ge 1$, $\eta^{(i_n)}_{m_n} \in G_{i_n}(PG_{i_n})^{m_n-1}y_n$.
Moreover, using inequalities (7.2.12) and (7.2.13), we obtain that

$$\begin{aligned}
\phi(p,u_n) &\le \alpha_n\phi(p,x_0) + (1-\alpha_n)\big[\phi(p,y_n) + \upsilon_n\psi(\phi(p,y_n)) + \mu_n\big] \\
&\le \alpha_n\phi(p,x_0) + (1-\alpha_n)\Big[\phi(p,x_n) + \upsilon_n\psi(\phi(p,x_n)) + \mu_n \\
&\quad -2\lambda\Big(\gamma - \frac{2}{c_2}\lambda\Big)||\xi_{i_n}||^2\Big] \qquad (7.2.15) \\
&\le \alpha_n\phi(p,x_0) + (1-\alpha_n)\phi(p,x_n) + \theta_n \\
&\quad -2(1-\alpha_n)\lambda\Big(\gamma - \frac{2}{c_2}\lambda\Big)||\xi_{i_n}||^2. \qquad (7.2.16)
\end{aligned}$$

Inequality (7.2.16) implies that

$$\begin{aligned}
2(1-\alpha_n)\lambda\Big(\gamma - \frac{2}{c_2}\lambda\Big)||\xi_{i_n}||^2 &\le \alpha_n\phi(p,x_0) + (1-\alpha_n)\phi(p,x_n) \\
&\quad -\phi(p,u_n) + \theta_n. \qquad (7.2.17)
\end{aligned}$$

Thus, we obtain that $\lim ||\xi_{i_n}|| = 0$. Furthermore, since $x_n \in K$ for all $n \ge 0$, then using Lemmas 2.3.3 and 2.1.4, we have that,

$$\begin{aligned}
\phi(x_n,y_n) &= \phi(x_n, \Pi_K J^{-1}(Jx_n - \lambda\xi_{i_n})) \\
&\le \phi(x_n, J^{-1}(Jx_n - \lambda\xi_{i_n})) = V(x_n, Jx_n - \lambda\xi_{i_n}) \\
&\le \phi(x_n,x_n) - 2\lambda\langle J^{-1}(Jx_n - \lambda\xi_{i_n}) - J^{-1}Jx_n, \xi_{i_n}\rangle \\
&\le 2\lambda||J^{-1}(Jx_n - \lambda\xi_{i_n}) - J^{-1}Jx_n||.||\xi_{i_n}|| \\
&\le \frac{4}{c_2}\lambda^2\Big\|\xi_{i_n}\Big\|^2. \qquad (7.2.18)
\end{aligned}$$

Thus, from inequality (7.2.18) and using the fact that $\lim_{n\to\infty}||\xi_{i_n}|| = 0$, we have that $\lim \phi(x_n, y_n) = 0$ and by Lemma 2.3.4, we have that $\lim ||x_n - y_n|| = 0$.

Consequently, for each i, $k \in K_i$, $\eta^{(i)}_{m_k} \in G_i(PG_{i_n})^{m_k-1}y_k$, we have $||y_k - \eta^{(i)}_{m_k}|| \to 0$, as $k \to \infty$, since $||y_k - \eta^{(i)}_{m_n}|| \leq ||y_k - x_k|| + ||x_k - \eta^{(i)}_{m_k}|| \to 0$, as $k \to \infty$. Also, since $\lim_{k\to\infty} x_k = x^*$ and $\lim ||x_k - y_k|| = 0$, we have that $\lim y_k = x^*$ and $\lim_{k\to\infty} \eta^{(i)}_{m_k} = x^*$ for each i, $\eta^{(i)}_{m_k} \in G_i(PG_{i_n})^{m_k-1}y_k$, $\eta^{(i)}_{m_{k+1}} \in G_iP\eta^{(i)}_{m_k} \subset G_i(PG_i)^{m_{k+1}-1}y_k$ and $s^{(i)}_{m_{k+1}} \in G_i(PG_i)^{m_{k+1}-1}y_{k+1}$. Now,

$$||\eta^{(i)}_{m_{k+1}} - \eta^{(i)}_{m_k}|| \leq ||\eta^{(i)}_{m_{k+1}} - s^{(i)}_{m_{k+1}}|| + ||s^{(i)}_{m_{k+1}} - y_{k+1}|| + ||y_{k+1} - y_k|| + ||y_k - \eta^{(i)}_{m_k}||$$

Since G_iP, $i = 1, 2, 3, ...$ is equally continuous, for each $i = 1, 2, 3, ...$, we have that $\lim_{k\to\infty} ||\eta^{(i)}_{m_{k+1}} - \eta^{(i)}_{m_k}|| = 0$. Thus, $\eta^{(i)}_{m_{k+1}} \to x^*$ as $k \to \infty$, but $\eta^{(i)}_{m_{k+1}} \to s_ix^*$, $s_ix^* \in G_iPx^* = G_ix^*$, by continuity of G_iP for each i. Hence, by uniqueness of limit, $s_ix^* = x^*$ for each i, so $x^* \in \cap_{i=1}^{\infty} F(G_i)$.

Step 4: We show that $x^* \in \cap_{i=1}^{\infty} A_i^{-1}(0)$.
The verification follows as in the verifcation of step 4 of the proof of Theorem 7.2.1.

Step 5: Finally, we show that $x^* \in \cap_{i=1}^{\infty} GMEP(h_i, \Phi_i, B_i)$.
This follows as in step 5 of the proof of Theorem 7.2.1. This completes the proof.

Remark 7.2.1 *We note here that one reason for developing an algorithm for finding an element in W as defined in Theorems 7.2.1 and 7.2.2 is that such an algorithm unifies various algorithms for approximating solutions of several problems of interest in nonlinear operator theory, for example, convex feasibility problems [**?**]; zeros of gamma inverse strongly monotone maps; equilibrium problems; optimization problems, etc.*

7.3 Applications

7.3.1 A system of convex optimization problems

By setting $B_i \equiv 0, \quad h_i \equiv 0 \quad (i = 1, 2, 3, ...)$ in Theorem 7.2.1, the sequence $\{x_n\}$ defined in Theorem 7.2.1 converges strongly to some element of W, where $W := \Big(\cap_{i=1}^{\infty} F(G_i)\Big) \cap \Big(\cap_{i=1}^{\infty} A_i^{-1}(0)\Big) \cap \Big(\cap_{i=1}^{\infty} MP(\Phi_i)\Big)$.

7.3.2 A system of equilibrium problems

By setting $B_i \equiv 0, \quad \Phi_i \equiv 0 \quad (i = 1, 2, 3, ...)$ in Theorem 7.2.1, the sequence $\{x_n\}$ defined in Theorem 7.2.1 converges strongly to some element of W, where $W := \Big(\cap_{i=1}^{\infty} F(G_i)\Big) \cap \Big(\cap_{i=1}^{\infty} A_i^{-1}(0)\Big) \cap \Big(\cap_{i=1}^{\infty} EP(h_i)\Big)$.

7.3.3 A system of variational inequality problems

By setting $h_i \equiv 0, \ \ \Phi_i \equiv 0$ for each $i = 1, 2, 3, ...$ in Theorem 7.2.1, the sequence $\{x_n\}$ defined in Theorem 7.2.1 converges strongly to some element of W, where $W := \Big(\cap_{i=1}^{\infty} F(G_i)\Big) \cap \Big(\cap_{i=1}^{\infty} A_i^{-1}(0)\Big) \cap \Big(\cap_{i=1}^{\infty} VIP(B_i, K)\Big)$.

7.3.4 Application in Classical Banach spaces

Let $E = L_p$, l_p, or $W_p^m(\Omega)$, $1 < p < \infty$, where $W_p^m(\Omega)$ denotes the usual Sobolev space and let E^* be the dual space of E. Cleary, E is uniformly convex and uniformly smooth. Consequently, Theorem 7.2.1 is applicable in these spaces.

Remark 7.3.1 *(see e.g.,) The analytical representations of duality maps are known in a number of Banach spaces. For instance, in the spaces l^p, $L^p(G)$ and Sobolev spaces $W_m^p(G)$, $p \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, the representations can be seen Alber and Ryazantseva, [3]; page 36 and Cioranescu [**?**].*

Application in Hilbert spaces

The following theorem follows immediately from Theorem 7.2.1.

Corollary 7.3.1 *Let H be a real Hilbert space and K be a nonempty closed and convex subset of H. Let $h_i : K \times K \to \mathbb{R}$, $i = 1, 2, 3, ...$ be a sequence of bifunctions satisfying $(A1) - (A4)$. Let $B_i : K \to H$ be a sequence of continuous monotone maps and $G_i : K \to 2^H, i = 1, 2, ...,$ be a countable family of equally continuous and totally quasi-ϕ-asymptotically nonexpansive multi-valued nonself maps with nonnegative real sequences $\{v_n^{(i)}\}, \{\mu_n^{(i)}\}$ and strictly increasing continuous functions $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $v_n^{(i)} \to 0, \mu_n^{(i)} \to 0$ as $n \to \infty$ and $\psi_i(0) = 0$. Let $A_i : K \to 2^H$, $i = 1, 2, 3, ...$ be a countable family of γ_i-inverse strongly monotone multi-valued maps and let $\gamma = inf\{\gamma_i, i = 1, 2, 3, ...\} > 0$. Let $\Phi_i : K \to \mathbb{R}$ $\ (i = 1, 2, 3, ...)$ be a sequence of lower semi-continuous convex functions. Suppose $W := \Big(\cap_{i=1}^{\infty} F(G_i)\Big) \cap \Big(\cap_{i=1}^{\infty} A_i^{-1}(0)\Big) \cap \Big(\cap_{i=1}^{\infty} GMEP(h_i, \Phi_i, B_i)\Big) \neq \emptyset$ and the sequence $\{x_n\}$ in K is defined iteratively as follows:*

$$\begin{cases} x_0 \in K_0 = K, \\ y_n = \Pi_K(x_n - \lambda\xi_i), \qquad (\xi_i \in A_i x_n), \\ z_n = \alpha x_n + (1-\alpha)\eta_n^{(i)}, \qquad (\eta_n^{(i)} \in G_i(GT_i)^{n-1} y_n), \\ u_n = T_{r_n} z_n, \\ K_{n+1} = \{z \in K_n : ||u_n - z||^2 \leq ||x_n - z||^2 + \theta_n\}, \\ x_{n+1} = P_{K_{n+1}} x_0, \quad n \geq 0, \end{cases}$$

where $\theta_n := (1-\alpha)\big[v_{m_n}^{(i_n)} \sup_{p\in W} \psi_{i_n}\big(||x_n - p||^2\big) + \mu_{m_n}^{(i_n)}\big]$; $\lambda \in (0, \frac{c_2}{2}\gamma)$, $c_2 > 0$ *is the constant in Lemma 2.1.4,* P_K *is the metric projection of* H *onto* K *and* $\alpha \in (0,1)$*. Then,* $\{x_n\}$ *converges strongly to some element of* W*.*

7.4 Numerical experiment

In this section, we give a numerical example to establish the implementability of the algorithm proposed in this paper.

Example 7.4.1 *In Theorem 7.2.1 and Theorem 7.2.2, we assume that* $E = \mathbb{R}$ *and* $[-1,1]$*. Define* $Bx = x$, $h(x,y) = (y+x)(y-x)$, $\Phi \equiv 0$, $A(x) = 3x$ *and* $Gx = sinx$*. It is easy to see that* f *satisfies* $(A1) - (A4)$, A *is* $1/4$*-inverse strongly montone and* B *is continuous montone. Thus, for any* $y \in K$ *and* $r > 0$*, we have*

$$\begin{aligned} h(u,y) \quad & +\langle B(x), y-u\rangle + \frac{1}{r}\langle u-x, y-u\rangle \geq 0 \\ \Leftrightarrow \quad & h(u,y) + \frac{1}{r}\langle u-(I-rB)x, y-u\rangle \geq 0 \\ \Leftrightarrow \quad & ry^2 + y[u-(1-r)x] - u[(1+r)u-(1-r)x] \geq 0 \end{aligned}$$

Let $Q(y) = ry^2 + y[u-(1-r)x] - u[(1+r)u-(1-r)x]$*. Since* Q *is a quadratic function relative to* y, $Q(y) \geq 0$ *for all* $y \in K$*, if and only if the coefficient of* y^2 *is positive and the discriminant* $\triangle \leq 0$*. But*

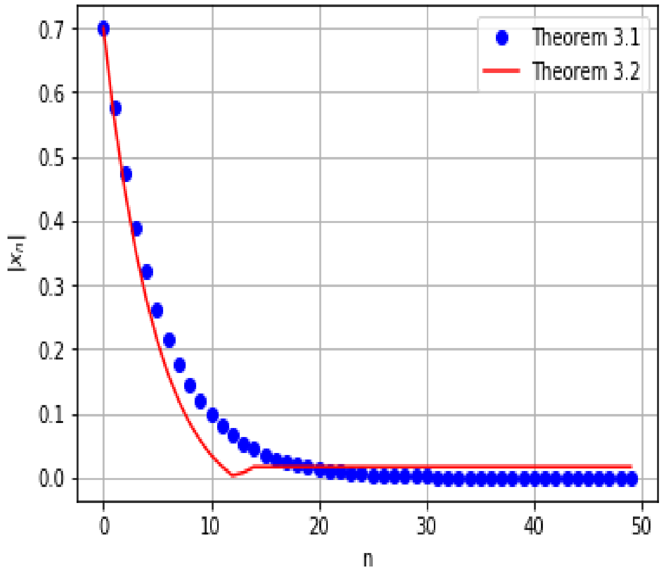
$$\begin{aligned} \triangle &= [u-(1-r)x]^2 + 4ru[(1+r)u-(1-r)x] \\ &= [(1+2r)u-(1-r)x]^2. \end{aligned}$$

Thus, we obtain that

$$u = \frac{1-r}{1+2r}x$$

and then $u_n = T_{r_n} z_n = \dfrac{1-r}{1+2r} z_n$*. Taking* $x_1 = 0.7$, $r = 2$, $\lambda = 1/15$, $\alpha_n = \dfrac{1}{n+1}$*, we obtain the following graph.*

No of iterations	$\lvert x_n\rvert$ for Theorem 7.2.1	$\lvert x_n\rvert$ for Theorem 7.2.2
3	0.474663617	0.44668362
18	0.01997619	0.0167733
25	0.00497983	0.0167733
37	0.000460236793	0.0167733
48	0.0000518720861	0.0167733



Remark 7.4.1 *1. Theorem 7.2.2 complements and improves the results of Bo and Yi [30] in the following sense:*

- *The map $T : K \to E$, considered in Bo and Yi [30] is single-valued, whereas in Theorem 7.2.2 the class of maps, $T_i : K \to 2^E$, $i = 1, 2, 3, \ldots$ considered is multi-valued.*
- *The map $T : K \to E$ such that T is uniformly L-Lipschitz continuous studied in Bo and Yi [30] is extended to the much more general class of maps $T_i : K \to 2^E$ such that for each i, T_i is equally continuous in 2-uniformly convex and uniformly smooth real Banach space.*

2. Theorem 7.2.1 is also a significant improvement of the results of Deng [76] in the following sense: The class of self-maps $T_i : K \to K$ such that for each i is uniformly L_i-Lipschitz continuous is generalized to the more general class of nonself maps $T_i : K \to 2^E$ such that for each i, T_i is equally continuous; and also the assumption that T_i is closed in the result of Deng [76] is dispensed with in Theorem 7.2.1.

All the results in this chapter are results in [64], published in **Journal of Fixed Point Theory and Applications**.

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