MEASURABLE SET-VALUED FUNCTIONS AND BOCHNER INTEGRALS

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by

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Certification

This is to certify that the thesis titled **"Measurable Set-Valued Functions and Bochner Integrals"** submitted to the school of postgraduate studies, African University of Science and Technology (AUST), Abuja, Nigeria for the award of the Master's degree is a record of original research carried out by Eze Leonard Chidiebere in the Department of Pure and Applied Mathematics.

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Measurable Set-Valued Functions and Bochner Integrals

Abstract

In this thesis, several concepts from Topology, Measure Theory, Probability Theory and Functional analysis were combined in the study of the mensurability of set-valued functions and the Bochner integral. We started with a detailed study of Hausdorff metric, its properties and topology by exposing separately the case where E is a metric space and the case where E is a normed linear space. After reviewing the important theorems, we present the four convergences related to Hausdorff metric: Hausdorff convergence, Wisjman convergence, Weak convergence and Kuratowski-Mosco convergence; and then compared them. Further, set-valued random variables and their properties were studied. We study and compare five types of mensures of set-valued functions and the two forms of Bochner integral, that is, the Banach-valued and set-valued Bochner integrals.

Key Words

Hausdorff Metric, Hausdorff convergence, Wijsman and Weak convengence, Kuratowski-Mosco convergence, Set-valued random variable, selections and Bochner integrals.

Dedication

Dedicated to my parents, Mr & Mrs Remigius Eze

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Contents

CHAPTER 1

Hausdorff Metric and Topology

1 Introduction and Background Concepts

A set-valued function is a function whose output is a set rather than a point or vector. In this work, we considered a set-valued function, F defined on a measure space $(\Omega, \mathcal{A}, \mu)$ taking values as subsets of a metric space (or Banach space), E. Throughout this work, we will assume that the range of F is a sub-class $\mathcal{P}_o(E)$ of the power set $\mathcal{P}(E)$ of E. To define a measure and integration concept on a function F, we need to define at least a σ-algebra Σ on $\mathcal{P}(E)$ whose elements are classes of subsets of E.

We also need a metric on $\mathcal{P}(E)$, and a class of Borel sets $\mathcal{B}$ on $\mathcal{P}(E)$. This chapter focuses on presenting a study on the mensurability of set-valued functions. We start by requiring that E is a metric space.

Again, in this work, we will follow the development in Li *et al.* (2002) as the main source of inspiration. However, we will provide a new structure of the content by exposing separately the case where E is a metric space and the case where E is a normed linear space. We will exploit other materials where necessary.

Furthermore, in this our journey to the study of mensurability of set-valued functions and Bochner Integrals, we will need some definitions and fundamental tools. These definitions and tools are basically from Topology, Measure Theory and Integrations, Functional Analysis and Probability Theory.

We are going to stick to the following notations:
E is a metric space with a metric d and θ is the origin of E.
$\mathcal{P}_o(E)$ for power set of E.
$\mathcal{P}(E)$ for all non-empty closed elements of $\mathcal{P}_0(E)$.
$\mathcal{P}_b(E)$ for all non-empty bounded and closed elements of $\mathcal{P}_0(E)$

$\mathcal{P}_k(E)$ for all non-empty compact elements of $\mathcal{P}_0(E)$.

$\mathcal{P}_c(E)$ for all non-empty closed and convex elements of $\mathcal{P}_0(E)$.

$\mathcal{P}_{kc}(E)$ for all non-empty compact and convex elements of $\mathcal{P}_0(E)$.

$\mathcal{P}_{bc}(E)$ for all non-empty closed bounded convex elements of $\mathcal{P}_0(E)$.

Capital letters $A, B, C, \ldots$ will be reserved for subsets of E.

Calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ denote $\sigma-$ algebras.

We assume that the elementary concepts in Topology, Measure Theory and Integration, Functional Analysis and the Probability Theory are known to readers. If otherwise, then consult Chidume (2014), Chidume (2010), Lo (2017b) and Lo (2017a).

2 Hausdorff Metric and its Properties

2.1 General case of a Metric space E:

In this section, we start by presenting those properties of Hausdorff metric, which is also called Hausdorff distance that are true if E is just a metric space.

Definition 1.1. *(**Hausdorff distance**) Suppose (E, d) is a metric space and $A, B \in \mathcal{P}_o(E)$. The Hausdorff distance between A and B is defined as*

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\} \tag{2.1}$$

where

$$\rho(A, B) = \sup\{d(a, B) : a \in A\},$$

$$\rho(B, A) = \sup\{d(b, A) : b \in B\},$$

$$d(a, B) = \inf\{d(a, b) : b \in B\}$$

and

$$d(b, A) = \inf\{d(b, a) : a \in A\}.$$

Remark 1.2. *Hausdorff distance measures how far two subsets of a metric space are from each other. It is the greatest of all the distances from a point in one set to the closest point in the other set.*

The following examples illustrate how to compute the Hausdorff distance between two sets.

Example 1.3. *Let $E = \mathbb{R},\ B = [0, 3],\ if\ x = 7$. Compute $d(x, B)$*

Solution 1. $d(x, B) = \inf\{d(x, b) : b \in [0, 3]\} = \inf\{|7 - b| : b \in [0, 3]\} = |7 - 3| = 4$

Example 1.4. *Let* $E = \mathbb{R}^2$, $B =$ *Unit ball in* $\mathbb{R}^2$. *If* $x = (1, 1) \in \mathbb{R}^2$. *Compute* $d(x, B)$.

Solution 2. $d(x, B) = d((1, 1), B) = \inf\{d((1, 1), (x, y); (x, y) \in B\} = \sqrt{2} - 1$

Example 1.5. *Let* $E = \mathbb{R}$, $A = [0, 10]$, $B = [20, 100]$ *and* $C = [20, \infty)$. *Compute the following,* $\rho(A, B)$, $\rho(B, A)$, $\rho(C, A), H(A, B)$ *and* $H(A, C)$

Solution 3. *For any* $x \in A$, *we have*
$d(x, B) = \inf\{d(x, b) : b \in B\} = (10 - x) + (20 - 10) = 20 - x$
$\rho(A, B) = \sup\{d(x, B) : x \in A\} = 20 - 0 = 20.$

For any $x \in B$, *we have*
$d(x, A) = \inf\{d(x, a) : a \in A\} = (x - 20) + (20 - 10) = x - 10$
$\rho(B, A) = \sup\{d(x, A) : x \in B\} = 100 - 10 = 90.$

For any $x \in C$, *we have*
$d(x, A) = \inf\{d(x, a) : a \in A\} = (x - 20) + (20 - 10) = x - 10$
$\rho(C, A) = \sup\{d(x, A) : x \in C\} = \infty - 10 = \infty.$

$H(A, B) = \max\{\rho(A, B), \rho(B, A)\} = \{20, 90\} = 90.$
$H(C, A) = \max\{\rho(C, A), \rho(A, C)\} = \{\infty, k\} = \infty,\ k \in \mathbb{R}.$

The following facts about Hausdorff distance can be deduced from the above examples.

Fact1: Let $A, B \in \mathcal{P}_o(E)$, then we have

- ρ is not symmetric e.g In example 1.5 above

$$\rho(A, B) = 20 \neq 90 = \rho(B, A)$$

- If A and B are bounded, then ρ and $H(A, B)$ are bounded.
- If A and B are unbounded, then ρ and $H(A, B)$ may be infinite.
- If A and B are singletons, then

$$\rho(A, B) = \rho(B, A) = H(A, B)$$

- If $B = \{b\}$, then $H(A, B) = H(A, \{b\}) = \sup\{d(x, b) : x \in A\}$.

Fact2: We shall recap some fundamental facts about Metric spaces. The following facts have been proved in the first course on Topology Chidume (2010)

1. The distance between a point x in E and a subset, A of E is defined by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

When A is non-empty, $d(x, A)$ is finite. Occasionally, we will use the characterization of the infimum through a minimizing sequence $(x_n)_{n \geq 0} \subset A$ such that

$$d(x, x_n) \to d(x, A) \; as \; n \to +\infty,$$

or its alternative version

$$\forall \eta > 0, \; \exists y \in A, \; d(x, A) \leq d(x, y) < d(x, A) + \eta.$$

The minimum distance shares with the metric in the following properties :
(2) $d(x, A) = 0$ if and only if $x \in cl(A)$.

(3) For all $(x, y) \in E^2$,

$$d(x, A) \leq d(x, y) + d(y, A).$$

Definition 1.6 (ϵ-dilation of a set). *Let $A \in \mathcal{P}_b(E)$. An ϵ-dilation of a set A denoted by A_ϵ or $A + \epsilon$ is defined as*

$$A_\epsilon = \{x \in X : d(x, A) \leq \epsilon\}.$$

Definition 1.7. (*Diameter of a set*) *The diameter of a non-empty set $A \subset E$ denoted by $diam(A)$ is defined as*

$$diam\,(A) = \sup\{d(y, x) : x, y \in A\}.$$

NB: A set A is bounded if it is empty or its diameter is finite.

Definition 1.8. (*Convex hull*) *Let A be a non-empty subset of E. The convex hull of A denoted by coA is the set of all convex combinations of elements of A. The closed* *convex hull of A is denoted by $\overline{co}A$ and is defined by*

$$\overline{co}A = \{x \in E : \exists \left(x_n, y_n\right) \in A^2, \exists \left\{\lambda_n\right\}_{n\geq 1} \in (0,1)\,, \lim_{n\to\infty} \left(\lambda_n x_n + \left(1-\lambda_n\right) y_n\right) = x\}$$

Definition 1.9. (*Total Boundedness of a set A*). *A set A is said to be totally bounded if there exists $\{x_1, x_2, \ldots x_n\} \subset A$ such that $\forall\ \epsilon > 0$*

$$A \subset \bigcup_{i=1}^{n} B\left(x_i, \epsilon\right)$$

Proposition 1.10. *Let (E, d) be a metric space. The Hausdorff distance H is a metric on $\mathcal{P}_b(E)$.*

Proof: Let $A, B, C \in \mathcal{P}_b(X)$ be arbitrary. We show that

(1) $H(A, B) \geq 0$

(2) $H(A, B) = 0 \Leftrightarrow A = B$.

(3) $H(A, B) = (B, A)$

(4) $H(A, B) \leq H(A, C) + H(C, B)$.

(1) $H(A, B) = \max\{\rho(A, B), \rho(B, A)\} \geq 0$. The supremum of the infimum of positive numbers is positive.

$$\begin{aligned}
(2)\ H(A, B) = 0 &\Leftrightarrow \max\{\rho(A, B), \rho(B, A)\} = 0 \\
&\Leftrightarrow \rho(A, B) = 0 \ and\ \rho(B, A) = 0 \\
&\Leftrightarrow d(a, B) = 0\ \forall\ a \in A\ and\ d(b, A) = 0\ \forall\ b \in B \\
&\Leftrightarrow A \subset B\ and\ B \subset A.\ since\ A,\ B\ are\ closed \\
&\Leftrightarrow A = B
\end{aligned}$$

(3) $H(A, B) = \max\{\rho(A, B), \rho(B, A)\} = \max\{\rho(B, A), \rho(A, B)\} = H(B, A)$

(4) Let $a \in A,\ b \in B\ and\ c \in C$. Then,

$$d(a, b) \leq d(a, c) + d(c, b) \qquad (since\ d\ is\ a\ metric)$$

Take infimum over $b \in B$. We have

$$d(a, B) \leq d(a, c) + d(c, B)$$

Take infimum over $c \in C$. We have

$$\begin{aligned} d(a,B) &\leq d(a,C) + \inf_{c \in C} d(c,B) \\ &\leq d(a,C) + \sup_{c \in C} d(c,B) \end{aligned}$$

This is true since $\inf_{c \in C} d(c,B) \leq \sup_{c \in C} d(c,B)$. Now, take supremum over $a \in A$. We have

$$\sup_{a \in A} d(c,B) \leq \sup_{a \in A} d(a,C) + \sup_{c \in C} d(c,B)$$

This give us that $\rho(A,B) \leq \rho(A,C) + \rho(C,B)$.
Following the same procedure , we obtain that

$$\rho(B,A) \leq \rho(B,C) + \rho(C,A)$$

$$\begin{aligned} H(A,B) &= \max\{\rho(A,B), \rho(B,A)\} \\ &\leq \max\{\rho(A,C) + \rho(C,B), \rho(B,C) + \rho(C,A)\} \\ &\leq \max\{\rho(A,C), \rho(C,A)\} + \max\{\rho(B,C), \rho(C,B)\} \\ &\leq H(A,C) + H(C,B) \end{aligned}$$

Hence, H is a metric and it is called **Hausdorff metric**; $(\mathcal{P}_b(E), H)$ is a Hausdorff metric space. Observe that by the following argument, we can easily see that $(\mathcal{P}_{bc}(E), H)$, $(\mathcal{P}_k(E), H)$, and $(\mathcal{P}_{kc}(E), H)$ are all metric spaces. These metric spaces are called **Hyperspaces**.

2.2 Completeness of Hausdorff Metric Space

Here, we are going to prove that the Hausdorff metric space $(\mathcal{P}_b(E), H)$ is a complete metric space whenever E is a complete metric space. First, we look at some lemmas which we shall use in the proof.

Lemma 1.11. *$A + \epsilon$ is closed for all possible choices of $A \in \mathcal{P}(E)$.*

Proof: Let $A \in \mathcal{P}(E)$ and $\epsilon > 0$ be given. Let $\{x_n\}_{n \geq 1} \subset A + \epsilon$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. We show that $x \in A + \epsilon$. But,

$$\begin{aligned} \{x_n\}_{n \geq 1} \subset A + \epsilon &\Rightarrow x_n \in A + \epsilon \ \forall \ n \geq 1 \\ &\Rightarrow d(x_n, A) \leq \epsilon \ \forall n \geq 1 \end{aligned}$$

By triangle inequality, we have

$$d(x_n, A) \leq d(x, x_n) + d(x_n, A)$$

Take limit as $n \longrightarrow \infty$. We have $d(x, A) \leq \epsilon$ which implies that $x \in A + \epsilon$. Hence $A + \epsilon$ is closed. ■

Lemma 1.12. *$A + \epsilon$ is bounded for all $A \in \mathcal{P}_b(E)$.*

Proof: Suppose $A \in \mathcal{P}_b(E)$. Then, there exists a constant real number $\alpha > 0$ such that $diam(A) \leq \alpha$. Let $x \in A + \epsilon$. For any $a, b \in A$

$$\begin{aligned} d(x, b) &\leq d(x, a) + d(a, b) \\ &\leq d(x, a) + diam(A) \\ &\leq \inf_{a \in A} + \alpha \\ &\leq d(x, A) + \alpha \\ &\leq \epsilon + \alpha \end{aligned}$$

Thus, for any other $y \in A + \epsilon$,

$$\begin{aligned} d(x, y) &\leq d(x, b) + d(b, y) \\ &\leq 2\epsilon + 2\alpha \end{aligned}$$

Hence, $diam(A + \epsilon) \leq 2\epsilon + 2\alpha < \infty$, for all $x, y \in A + \epsilon$ ■

Lemma 1.13. *Let $A, B \in \mathcal{P}_b(E)$ and $\epsilon > 0$. Then $H(A, B) \leq \epsilon$ if and only if $A \subset B + \epsilon$ and $B \subset A + \epsilon$.*

Proof: Suppose $A, B \in \mathcal{P}_b(E)$ and $\epsilon > 0$. Then,

$$\begin{aligned} H(A, B) \leq \epsilon &\Leftrightarrow \max\{\rho(A, B), \rho(B, A)\} \leq \epsilon \\ &\Leftrightarrow \rho(A, B) \leq \epsilon \textit{ and } \rho(B, A) \leq \epsilon \\ &\Leftrightarrow \forall\ a \in A, d(a, B) \leq \epsilon \textit{ and } \forall\ b \in B, d(b, A) \leq \epsilon \\ &\Leftrightarrow A \subset B + \epsilon \textit{ and } B \subset A + \epsilon. \blacksquare \end{aligned}$$

Lemma 1.14. *If $\{x_n\}_{n \geq 1} \subset X$ such that $d(x_n, x_{n+1}) < \dfrac{\epsilon}{2^n}$, $\epsilon > 0$. Then $\{x_n\}_{n \geq 1}$ is a Cauchy sequence.*

Proof: Let $n > m$ and $\epsilon > 0$ be given.

$$\begin{aligned} d(x_n, x_m) &\leq d(x_m, x_{m+1}) + d(x_m, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq \frac{\epsilon}{2^m} + \frac{\epsilon}{2^{m+1}} + \cdots + \frac{\epsilon}{2^{n-1}} \\ &\leq \sum_{n \geq m} \frac{\epsilon}{2^{n-1}} \\ &= \epsilon \end{aligned}$$

This implies that $d(x_n, x_m) \leq \epsilon$. Hence $\{x_n\}_{n\geq 1}$ is a Cauchy sequence. ■

Theorem 1.15. Let (E, d) be a complete metric space. Then, $(\mathcal{P}_b(E), H)$ is a complete metric space.

Proof: Let $\{A_n\}_{n\geq 1}$ be an arbitrary Cauchy sequence in $\mathcal{P}_b(E)$. We show that A_n converges to a point A in $\mathcal{P}_b(E)$. Let

$$A = \bigcap_{j\geq 1} \overline{\bigcup_{n\geq j}} A_n \ \ \textit{where} \ \ \overline{\bigcup_{n\geq j}} A_n = \ \textit{closure of} \ \bigcup_{n\geq j} A_n$$

Claim : We claim that A satisfy the following;

(1) $A \neq \emptyset$.

(2) A is closed.

(3) A is bounded.

(4) $H(A_n, A) \leq 2\epsilon$.

Proof of claim;

1. We show that $A \neq \emptyset$. Now, by the Cauchyness of $\{A_n\}_{n\geq 1}$, we have that there exists $N \in \mathbb{N}$ such that $H(A_n, A_m) < \epsilon \ \forall n, m \geq N$. From this, we obtain that for $k \geq 1$

$$H(A_{n_k}, A_{n_{k+1}}) < \frac{\epsilon}{2^{k+1}} \ \forall \ n_k \geq N.$$

From the definition of H, observe that $\forall \ x_{n_k} \in A_{n_k}$ there exists $x_{n_{k+1}} \in A_{n_{k+1}}$ such that $d(x_{n_k}, x_{n_{k+1}}) < \frac{\epsilon}{2^k}$. Then by lemma 1.14, $\{x_{n_k}\}_{k\geq 1}$ is a Cauchy sequence in E. Furthermore, by the completeness of E, we have that $\{x_{n_k} : k \in \mathbb{N}\}$ converges to $x \in E$. Now, let $j \geq 1$. Then, there exists $k_0 \in \mathbb{N}$ such that $n_{k_0} \geq j$.

$$x_{n_{k_o}} \in A_{n_{k_o}} \subset \bigcup_{n\geq j} A_n \Rightarrow x_{n_{k_o}} \in \bigcup_{n\geq j} A_n$$

So, $\forall\ k \geq 1$ such that $n_k > n_{k_0}$, we have $x_{n_k} \in \bigcup_{n\geq j} A_n$. This means that $x = \lim_{n\to\infty} x_{n_k} \in \overline{\bigcup_{n\geq j}} A_n,\ \forall\ j \geq 1$. Since this happens for all $j \geq 1$, we have

$$x \in \bigcap_{j\geq 1} \overline{\bigcup_{n\geq j}} A_n = A$$

Hence, $A \neq \emptyset$. ■

2. A is closed since A is an arbitrary intersection of closed sets.

3. We show that A is bounded.

$$\begin{aligned}
\{A_n\}_{n\geq 1} \ \textit{is Cauchy} &\Rightarrow \textit{Given } \epsilon > 0, \exists N \in \mathbb{N} : \forall\ i, j \geq N,\ H(A_i, A_j) \leq \epsilon \\
&\Rightarrow H(A_i, A_N) \leq \epsilon,\ \forall\ i \geq N \\
&\Rightarrow A_i \subset A_N + \epsilon \quad (\textit{By} \ \ \textit{lemma } 1.13) \\
&\Rightarrow \bigcup_{i\geq N} A_i \subset A_N + \epsilon \\
&\Rightarrow \overline{\bigcup_{i\geq N} A_i} \subset A_N + \epsilon \\
&\Rightarrow A = \bigcap_{N\geq 1} \overline{\bigcup_{i\geq N}} A_i \subset A_N + \epsilon \\
&\Rightarrow A \subset A_N + \epsilon
\end{aligned}$$

Hence, A is bounded.

4. We show that $H(A_N, A) \leq 2\epsilon$. It suffices to show that $A \subset A_n + 2\epsilon$ *and* $A_n \subset A + 2\epsilon$. Let $x \in A = \bigcap_{j\geq 1} \overline{\bigcup_{n\geq j}} A_n$. we show that $x \in A_n + 2\epsilon$.
But

$$x \in \bigcap_{j\geq 1} \overline{\bigcup_{n\geq j}} A_n \Leftrightarrow x \in \overline{\bigcup_{n\geq j}} A_n \ \forall\ j \geq 1.$$

This simply means that

$$\forall\ j \geq 1,\ \exists\ x_n \in \bigcup_{n\geq j} A_n :\ x_n \longrightarrow x \ as \ n \longrightarrow \infty.$$

This implies

$$\forall\ j \geq 1, \exists\ k \geq 1, \exists\ x_{n_k} \in A_{n_k} :\ x_{n_k} \longrightarrow x \ as \ n \longrightarrow \infty,$$

which gives us $d(x, x_{n_k}) < \epsilon$.

By triangle inequality, we have

$$d(x, A_n) \leq d(x, x_{n_k}) + d(x_{n_k}, A_n) \leq 2\epsilon.$$

Hence $A \subset A_n + 2\epsilon$.

Again, we show that $A_n \subset A + 2\epsilon$. Let $y_n \in A_n$. We show that $y_n \in A + 2\epsilon$. From the assumption, $H(A_n, A_{n_k}) \leq \frac{\epsilon}{2^{k+20}} \forall\ n_k > N$. This implies that

$$\forall\ y_n \in A_n, \exists x_{n_k} \in A_{n_k} : d(y_n, x_{n_k}) \leq \frac{\epsilon}{2^{k+1}} \leq \frac{\epsilon}{2^k} < \epsilon$$

In particular for $k = 1$, $d(y_n, x_{n_1}) < \epsilon$. But, $x = \underset{n \longrightarrow \infty}{lim} x_{n_k}$ and $x \in A$

$$\begin{aligned} d(y_n, A) &\leq d(y_n, x) \\ &\leq d(y_n, x_{n_1}) + d(x_{n_1}, x_{n_2}) + \cdots + d(x_{n_{k+1}}, x) \\ &= d(y_n, x_{n_1}) + \sum_{i=1}^{k} d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x) \\ &\leq d(y_n, x_{n_1}) + \sum_{i=1}^{\infty} d(x_{n_k}, x_{n_{k+1}}) \\ &< \epsilon + \sum_{i=1}^{k} \frac{\epsilon}{2^k} = 2\epsilon \end{aligned}$$

So, from this we have $y_n \in A + 2\epsilon$ and this actually gives us that $A_n \subset A + 2\epsilon$. Hence, $\{A_n\}$ converges to A in Hausdorff metric. ■

Proposition 1.16. *Let $A, P \in \mathcal{P}_b(E)$ and P compact. If for any $\epsilon > 0$, $A \subset P + \epsilon$. Then A is compact.*

Proof: P is compact implies that P is totally bounded. This simply means there exists $\{x_i\}_{i=1}^{n} \subset P$ such that $P \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)$.

$$A \subset P + \epsilon \Rightarrow A \subset \bigcup_{i=1}^{n} B(x_i, \epsilon) + \epsilon$$

But $a \in A$ means that there exists $b \in P$ such that $d(a, b) \leq \epsilon$. By triangle inequality, we have $d(a, x_i) \leq d(a, b) + d(b, x_i)$ which gives us that $d(a, x_i) \leq 2\epsilon$. So, $a \in B(x_i, 2\epsilon)$ for some $1 \leq i \leq n$. This implies that $B(x_i, 2\epsilon) \cap A \neq \emptyset$ for some $1 \leq i \leq n$. Let $r \leq n$ be such that $B(x_i, 2\epsilon) \cap A \neq \emptyset$ for $1 \leq j \leq p$ and $B(x_i, 2\epsilon) \cap A = \emptyset$ for $p \leq j \leq n$.

Let $y_i \in B(y_i, 2\epsilon) \cap A$

claim 1. $A \subset \bigcup_{i=1}^{p} B\left(x_i, 4\epsilon\right)$

Proof of claim: Let $a \in A$. Then, $d(a, y_i) \leq d(a, x_i) + d(x_i, y_i) \leq 4\epsilon$. This implies that $a \in B\left(y_i, 4\epsilon\right)$. Thus, $A \subset \bigcup_{i=1}^{p} B\left(y_i, 4\epsilon\right)$. Hence A is totally bounded. Also, A is a closed subset of a complete metric space X. Therefore A is complete. Hence, A is compact.

Theorem 1.17. *$\mathcal{P}_k(E)$ is a closed subset of $(\mathcal{P}_b(E), H)$*

Proof: Let $\{A_n\}_{n\geq 1} \subset \mathcal{P}_k(E)$ such that for $\epsilon > 0$, $\exists\, N \in \mathbb{N} : H(A_n, A) < \epsilon\ \forall\, n \geq N$. We show that $A \in \mathcal{P}_k(E)$. But $H(A_n, A) < \epsilon \Rightarrow A \subset A_n + \epsilon$ *and* $A_n \subset A + \epsilon$. By proposition 1.16 $A \subset A_n + \epsilon \Rightarrow A \in \mathcal{P}_k(E)$. Hence, $\mathcal{P}_k(E)$ is closed. ■

3 General case of Normed Linear Space

Let us suppose that $(E, \|.\|_E)$ is a Normed linear space over a scalar field $\mathbb{R}$, where $\|.\|_E$ denotes a norm on E. Now, we can enjoy the power of Linear Functional Analysis. Here, we require that the space $(E, \|.\|_E)$ be a Banach space.

Definition 1.18. *Let $A, B \in \mathcal{P}_o(E)$ and $\lambda \in \mathbb{R}$. We define addition and scalar multiplication by*

$$A + B = \{a + b : a \in A, b \in B\}$$

$$\lambda A = \{\lambda a : a \in A\}$$

Note. We should not confuse this with the sum of linear sub-spaces (which all contain the null vector 0), nor with the union of two disjoint subsets A and B, which is usually denoted by $A + B$ in Probability Theory notations. To avoid such confusions, we will always precise the meaning of the operation $(+)$ when it is used between two sets.
In the case of a linear space, we put the origin as θ and for any non-empty subset A of E, we define $\|A\|_K$ as

$$\|A\|_K = H(A, \{\theta\}) = \sup\{\| x \|_E : x \in A\}.$$

The notion of convexity will play an important role in this work. So, we recall the definition of the closed convex hull of a non-empty set $A \subset E$ given in definition 1.8.
All the theorems and propositions that we have proved for E being metric space also holds here.

Proposition 1.19. *$\mathcal{P}_{bc}(E)$ and $\mathcal{P}_{kc}(E)$ are closed subsets of $(\mathcal{P}_b(E), H)$.*

Proof: **Closedness of** $\mathcal{P}_{bc}(E)$:

Let $(A_n)_{n\geq 0}$ be a sequence of closed, bounded and convex subsets of E converging to A with respect to the metric H. Hence A is closed. To prove that A is convex, consider $(a,b) \in A^2$ and $\lambda \in (0,1) \subset \mathbb{R}$. Let us fix $n \geq 0$. For any $x, y \in A_n$. Since $(\lambda x + (1-\lambda)y) \in A_n$, we have

$$d(\lambda a + (1-\lambda)b, A_n) \leq \|(\lambda a + (1-\lambda)b) - (\lambda x + (1-\lambda)y)\|$$

which implies

$$d(\lambda a + (1-\lambda)b, A_n) \leq \lambda\|a - x\| + (1-\lambda)\|b - y\|.$$

By taking first the supremum over $y \in A_n$ and next over $x \in A_n$, we get

$$d(\lambda a + (1-\lambda)b, A_n) \leq \lambda \; d(a, A_n) + (1-\lambda) \; d(b, A_n) \leq \rho(A, A_n).$$

Now, as $n \to +\infty$, we get

$$d(\lambda a + (1-\lambda)b, A_n) \to 0$$

which implies $\lambda a + (1-\lambda)b \in cl(A) = A$. ■

Closedness $\mathcal{P}_{kc}(E)$ We obtain this by combining Theorem 1.17 and the convexity of $\mathcal{P}_{bc}(E)$ above. ■

Theorem 1.20. If X is a separable space, so is the Hyperspace $(\mathcal{P}_k(E), H)$.

Proof: Suppose that X is separable and let U be a countable dense subset of X. Let V be the set of all finite subsets of U. Consider the class V of all the non-empty and finite subsets of U. Obviously the elements of V are closed and bounded (in metric spaces singletons are closed). We now prove that V is dense in $\mathcal{P}_k(E)$. Let $P \in \mathcal{P}_k(E)$. By the Heine-Borel property, P is totally bounded. This implies that $\exists \; n \geq 1$ and a set $\{x_1, x_2, \ldots, x_n\} \subset P$ such that

$$P \subset \bigcup_{i=1}^{n} B\left(x_i, \frac{\epsilon}{2}\right)$$

Since U is dense in X, we have that $\forall \; \epsilon > 0, \; U \cap B\left(x_i, \frac{\epsilon}{2}\right) \neq \emptyset$. Let $y_i \in U \cap B\left(x_i, \frac{\epsilon}{2}\right)$. Clearly, $d(x_i, y_i) < \frac{\epsilon}{2}$. Set $P_o = \{y_1, y_2, \ldots, y_n\} \subset V$. We show that

$$H(P, P_o) < \epsilon$$

It suffice to show that $P \subset P_o + \epsilon$ and $P_o \subset P + \epsilon$. First, let $x \in P$ and y_o be any element of P_o. Then,

$$d(x, P_o) \leq d(x, y_o) \leq d(x, y_i) + d(y_i, y_o) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This implies that $x \in P_o + \epsilon$. Thus, $P \subset P_o + \epsilon$.
Secondly, let $y \in P_o$ and x_o be any element of P. Then,

$$d(y, P) \leq d(y, x_o) \leq d(y, x_i) + d(x_i, x_o) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This implies that $y \in P + \epsilon$. Thus, $P_o \subset P + \epsilon$. So, $H(P, P_o) < \epsilon$. Hence, V is dense in $(\mathcal{P}_k(E), H)$. ■

At this moment, we proceed to give some other properties of a Banach space E which we may need in the sequel. For instance, the weak topology on E, that is, the topology of dual space of E denoted by E^*, generated by the class of all continuous linear functionals on E. This weak topology is generated by open sets of the form

$$W_{\epsilon, f_1, \ldots, f_p} = \{x \in E : \forall\ i \in \{1, ..., p\},\ |\langle f_i, x - x_o \rangle| < \epsilon\}.$$

where ϵ, p are positive integers and $f_i \in E^*$. The convergence in E^*, with respect to the weak topology in E^* is operated point-wisely in the following sense

$$f_n \rightharpoonup f \Leftrightarrow\ \forall\ x \in E,\ f_n(x) \to f(x),\ as\ n \to +\infty.$$

where $\rightharpoonup$ means weakly convergent. We also have on E^* the norm defined by

$$\forall\ f \in E^*,\ \|f\|_{E^*} = \sup_{x \in E \setminus \{0\}} \frac{|f(x)|}{\|x\|_E}.$$

It may help to know that we also have, for any real number $a > 0$ and

$$\forall\ f \in E^*,\ \|f\|_{E^*} = \sup_{x \in E, \|x\|_E = 1} |f(x)| = \frac{1}{a} \sup_{x \in E, \|x\|_E = a} |f(x)|.$$

We also have for any real number $a > 0$,

$$\forall\ f \in E^*,\ \|f\|_{E^*} = \frac{1}{a} \sup_{x \in E, \|x\|_E \leq a} |f(x)|.$$

The topology on E^* with respect to this norm is called strong.

If the Banach space E is separable, then, there exists a countable family $D = \{x_j :\ j \geq 1\}$ dense everywhere in E, the weak topology is generated by the neighborhoods $W_{\epsilon, f_1, \ldots, f_p}$, where $(x_1, ..., x_p) \in D^p$, $f_i \in E^*$, $\epsilon > 0$, $1 \leq i \leq p$, $p \geq 1$. More than that, the weak topology is the metric topology of the form

$$d^*(f,g) = \sum_{j\geq 1} \frac{1}{2^j}|f(x_j) - g(x_j)|, \ \forall \ f,g \in E^*.$$

Remark. It is important to know that we might only take D as dense in $U_c = B_c(0,1)$. Let us introduce spheres and the disks for both E and E^*.

$$S^* = \{f \in E^* : \ \|f\| = 1\}; \ \ U^* = \{f \in E^* : \ \|f\| < 1\}; \ \ U_c^* = \{f \in E^* : \ \|f\| \leq 1\}$$

$$S = \{x \in E : \ \|x\| = 1\}; \ \ U = \{x \in E : \ \|x\| < 1\}; \ \ \textit{and} \ \ U_c = \{x \in E : \ \|x\| < 1\}$$

As well, we define the open and closed balls in E, respectively by

$$B(x,r) = \{y \in E : \ \|x - y\| < r\}, \ (r > 0);$$

$$B_c(x,r) = \{y \in E : \ \|x - y\| \leq r\}; \ (r \geq 0), \ x \in E$$

and their analogues in E^* by

$$B^*(u,r) = \{v \in E^* : \ \|u - v\| < r\}, \ (r > 0);$$

$$B_c^*(u,r) = \{v \in E^* : \ \|x - y\| \leq r\}, \ (r \geq 0), \ u \in E^*.$$

CHAPTER 2

Hausdorff Convergence Theory

1 Introduction

In Chapter 1, we have seen the properties of Hausdorff metric in hyperspaces. Here, we shall deal with the characterizations of Hausdorff metric, convergences emanating from the characterizations and convergences in the sense of Kuratowski Mosco.

2 Characterization of Hausdorff Metric

The aim of this section is to present two other ways of expressing Hausdorff metric on hyperspaces. First, we prove the following lemma which we shall use in the sequel.

Lemma 2.1. *If $A \neq \emptyset$ is a subset of $\mathbb{R}$. Then*

$$\sup_{x\in A}|x| = \max\{\sup_{x\in A} x,\ \sup_{x\in A}(-x)\}$$

Proof: Suppose $\emptyset \neq A \subset \mathbb{R}$. Then $\forall\ x \in A,\ x \leq |x|$.
Take supremum over $x \in A$, we obtain that

$$\sup_{x\in A} x \leq \sup_{x\in A}|x| \qquad (1)$$

Also, $\forall\ x \in A\ \ -x \leq |x|$

$$\sup_{x\in A}(-x) \leq \sup_{x\in A}|x| \qquad (2)$$

By (1) and (2) we obtain that

$$\max\{\sup_{x\in A} x,\ \sup_{x\in A}(-x)\} \leq \sup_{x\in A}|x|$$

For any $x \in A$,

$$|x| \leq \max\{\sup_{x\in A} x,\ \sup_{x\in A}(-x)\}$$

Take supremum over $x \in A$ we have

$$\sup_{x\in A}|x| \leq \max\{\sup_{x\in A} x,\ \sup_{x\in A}(-x)\}$$

Hence,

$$\sup_{x\in A}|x| = \max\{\sup_{x\in A} x,\ \sup_{x\in A}(-x)\} \ \blacksquare$$

Lemma 2.2. *Let E be a Banach space, $f \in E^*$ be nonzero and $\alpha \in \mathbb{R}$. If $H_{\alpha,f} \equiv f^{-1} := \{x \in E : \langle f, x\rangle = \alpha\}$ is the hyperplane generated by α and f, then for any $x_0 \in E$,*

$$d(x_0, H_{\alpha,f}) = \frac{|\langle f, x_0\rangle - \alpha|}{\|f\|_{E^*}} \tag{2.1}$$

Proof: Let $x_0 = 0$. If $\alpha = 0$, then $\langle f, x\rangle = 0$ implies $0 \in H_{\alpha,f}$

So that

$$d(x_0, H_{\alpha,f}) = d(0, H_{\alpha,f}) = 0 = \frac{|\langle f, 0\rangle - 0|}{\|f\|_{E^*}}.$$

If $\alpha \neq 0$, take $y = \frac{\alpha x}{\langle f,x\rangle}$. Then $x = \frac{\langle f,x\rangle y}{\alpha}$
and by linearity of f, $\langle f, y\rangle = f\left(\frac{\alpha x}{\langle f,x\rangle}\right) = \alpha$ implies $y \in H_{\alpha,f}$. Thus, by definition and for this $y \in H_{\alpha,f}$,

$$\begin{aligned}\|f\|_{E^*} &= \sup_{x\neq 0}\frac{|\langle f,x\rangle|}{\|x\|} = \sup_{\frac{\langle f,x\rangle y}{\alpha}\neq 0}\frac{\left|\langle f,\ \frac{\langle f,\ x\rangle y}{\alpha}\rangle\right|}{\left\|\frac{\langle f,\ x\rangle y}{\alpha}\right\|} = \sup_{y\neq 0}\frac{|\langle f,y\rangle|}{\|y\|}\\ &= \sup_{y\in H_{\alpha,f}}\frac{|\alpha|}{\|y\|}\\ &= \frac{|\alpha|}{\inf\limits_{y\in H_{\alpha,f}}\|y\|}\\ &= \frac{|\alpha|}{d(0,H_{\alpha,f})}\end{aligned} \tag{2.2}$$

Hence for $x_0 = 0$,

$$d(0, H_{\alpha,f}) = \frac{|\langle f, 0\rangle - \alpha|}{\|f\|_{E^*}}.$$

Let $x_0 \neq 0$. To establish 2.1 in this case, we need the following

claim 2. *With same notations as in Lemma 2.2,*

$$x_0 - H_{\alpha,f} = f^{-1}\left(\langle f, x_0\rangle - \alpha\right)$$

Proof of claim 2 $a \in x_0 - H_{\alpha,f}$ implies there is $x \in H_{\alpha,f}$ such that $a = x_0 - x$. Applying f to both sides, we have

$$\langle f, a\rangle = \langle f, x_0 - x\rangle = \langle f, x_0\rangle - \alpha$$

So that

$$a \in f^{-1}\left(\langle f, a\rangle\right) = f^{-1}\left(\langle f, x_0\rangle - \alpha\right)$$

Implies

$$(*) \qquad x_0 - H_{\alpha,f} \subseteq f^{-1}(\langle f, x_0\rangle - \alpha)$$

Similarly,

$$a \in f^{-1}(\langle f, x_0\rangle - \alpha) \ \text{ implies } \ \langle f, a\rangle = \langle f, x_0\rangle - \alpha$$

Thus,

$$\langle f, x_0 - a\rangle = \alpha \qquad \text{and} \qquad x_0 - a \in f^{-1}(\langle f, x_0 - a\rangle) = f^{-1}(\alpha)$$

Implies

$$(**) \qquad a \in x_0 - H_{\alpha,f} \qquad \text{and} \qquad f^{-1}(\langle f, x_0\rangle - \alpha) \subseteq x_0 - H_{\alpha,f}$$

Combining $(*)$ and $(**)$, it follows

$$x_0 - H_{\alpha,f} = f^{-1}(\langle f, x_0\rangle - \alpha)$$

and that ends the proof of claim 2.

Now, using the fact that $d(x_0, H_{\alpha,f}) = d(0, x_0 - H_{\alpha,f})$, the result of claim 2 and 2.2, we have

$$d(x_0, H_{\alpha,f}) = d(0, x_0 - H_{\alpha,f}) = \frac{|\langle f, x_0\rangle - \alpha|}{\|f\|_{E^*}}$$

which completes the proof of 2.1 for all values of $x_0 \in E$. ■

In what follows, for any $c \in \mathbb{R}$, $c > 0$ let

$$S^* := \{f \in E^* : \|f\|_{E^*} = 1\} \quad \text{and} \quad K^* := \{f \in E^* : \|f\|_{E^*} \leq c\}.$$

Lemma 2.3. *Let $A \in \mathcal{P}_{bc}(E)$ and $\beta = d(0, A) > 0$. Then there exists $f \in S^*$ such that*

$$\sup_{x \in \overline{B}_\beta(0)} \langle f, x\rangle \ = \ \beta \ = \ \inf_{x \in A}\langle f, x\rangle$$

Proof: We claim $A \cap B_\lambda(0) = \emptyset$. Else, we can find $x_0 \in A \cap B_\lambda(0)$ and $d(0, A) \leq \|x_0\| < \lambda$. Consequently, $d(0, A) < \lambda$ which contradicts $\lambda = d(0, A)$.

Since A is closed and $B_\lambda(0)$ is nonempty and open, by Hahn Banach theorem Chidume (2014), there is $f \in S^*$ and $\lambda \in \mathbb{R}$ such that

$$(2.3) \qquad \sup_{x \in \overline{B}_\beta(0)} \langle f, x\rangle \ \leq \ \lambda \ \leq \ \inf_{x \in A}\langle f, x\rangle.$$

Let $x \in \overline{B}_\beta(0)$. Then $y = \frac{x}{\lambda} \in \overline{B}_1(0)$.

So that

$$\begin{aligned} 1 = \|f\|_{X^*} &= \sup_{\|y\|\leq 1} |\langle f, y\rangle| \\ &= \sup_{\|\frac{x}{\lambda}\|\leq 1} \left|\langle f, \frac{x}{\lambda}\rangle\right| \\ &= \frac{1}{\lambda} \sup_{\|x\|\leq 1} |\langle f, x\rangle| \end{aligned}$$

Hence,

$$\sup_{x\in \overline{\mathrm{B}}_\beta(0)} |\langle f, x\rangle| = \lambda$$

and symmetry of balls centered at the origin, we have

$$\lambda = \sup_{x\in \overline{\mathrm{B}}_\beta(0)} |\langle f, x\rangle| = \sup_{x\in \overline{\mathrm{B}}_\beta(0)} \langle f, x\rangle \tag{2.4}$$

By Lemma 2.2,

$$d(0, f^{-1}(\lambda)) = \frac{|\langle f, 0\rangle - \lambda|}{\|f\|_{E^*}} = \lambda$$

and

$$\beta = d(0, A) \geq d(0, f^{-1}(\lambda)) = \lambda \tag{2.5}$$

By boundedness of f, for any $x \in A$, we have that

$$\langle f, x\rangle \leq \|f\|_{E^*}\|x\| = \|x\|$$

Implies

$$\inf_{x\in A}\langle f, x\rangle \leq \inf_{x\in A} \|x\| = d(0, A) = \beta \tag{2.6}$$

Combining equations 2.3, 2.4, 2.5, and 2.6, it follows

$$\sup_{x\in \overline{\mathrm{B}}_\beta(0)} \langle f, x\rangle = \beta = \inf_{x\in A}\langle f, x\rangle. \tag{$\dagger$}$$

Theorem 2.4. Let $A, B \in \mathcal{P}_b(E)$. Then the following hold

1. $\rho(A, B) = \sup_{x\in E}\{d(x, B) - d(x, A)\}$

2. $H(A, B) = \sup_{x\in E} |d(x, A) - d(x, B)|$

Proof

1. Let $x \in A$. Then $d(x,A) = 0$. By subtracting zero from $d(x,B)$ we obtain

$$\begin{aligned} d(x,B) &= d(x,B) - d(x,A) \\ &\leq \sup_{x\in A}\{d(x,B) - d(x,A)\} \\ &\leq \sup_{x\in E}\{d(x,B) - d(x,A)\} \end{aligned}$$

Take supremum over $x \in A$

$$\rho(A,B) = \sup_{x\in A} d(x,B) \leq \sup_{x\in E}\{d(x,B) - d(x,A)\}$$

Hence,

$$\rho(A,B) \leq \sup_{x\in E}\{d(x,B) - d(x,A)\}$$

Let $a \in A,\ b \in B \ and \ x \in E$. By triangle inequality we have

$$\|x - b\|_E \leq \|x - a\|_E + \|a - b\|_E$$

Take infimum over $b \in B$ and later take the infimum over $a \in A$

$$\begin{aligned} d(x,B) &\leq d(x,A) + \inf_{a\in A} d(a,B) \\ &\leq d(x,A) + \sup_{a\in A} d(a,B) \end{aligned}$$

This implis that,

$$d(x,B) - d(x,A) \leq \rho(A,B)$$

and

$$\sup_{x\in E}\{d(x,B) - d(x,A)\} \leq \rho(A,B).$$

Thus, $\rho(A,B) = \sup_{x\in E}\{d(x,B) - d(x,A)\}$ as required.

2. By the symmetry of the role of $A \ and \ B$ we have that $\rho(B,A) = \sup_{x\in E}\{d(x,A) - d(x,B)\}$. By applying 2.1 we have

$$H(A,B) = \sup_{x\in E} |d(x,A) - d(x,B)|$$

Theorem 2.5. Let $A, B \in \mathcal{P}_b(E) \ and \ \alpha \geq 0$. Then

1. $\rho(A,B) = \inf\{\alpha > 0 : B \subset A + \alpha\}$

2. $H(A,B) = max\{\inf\{\alpha > 0 : B \subset A + \alpha\}, \inf\{\alpha > 0 : B \subset A + \alpha\}\}$

Proof

1. Let $A, B \in \mathcal{P}_b(E)$. We show that

$$\rho(A,B) \leq \inf\{\alpha > 0 : B \subset A + \alpha\}$$

Suppose $B \subset A + \alpha$. We are sure that $\rho(A,B)$ exists since A is closed and bounded. Since $\rho(A,B) = \sup_{a\in A} d(a,B)$ it follows from the consequences of supremum that for any $\epsilon > 0 \ \exists \ a_\epsilon \in A$ such that

$$\rho(A,B) \leq d(a_\epsilon, B) + \epsilon \leq \alpha + \epsilon$$

Take infimum over α such that $A \subset B + \alpha$. We have

$$\rho(A,B) \leq \inf\{\alpha : B \subset A + \alpha\}$$

($\Leftarrow$) We show that $\inf\{\alpha > 0 : A \subset B + \alpha\} \leq \rho(A,B)$ Let $\rho(A,B) = \lambda$. Clearly, $A \subset B + \lambda$. This implies that

$$\inf\{\alpha > 0 : A \subset B + \alpha\} \leq \lambda = \rho(A,B)$$

This implies that $\inf\{\alpha > 0 : A \subset B + \alpha\} \leq \rho(A,B)$.

Hence,

$$\rho(A,B) = \inf\{\alpha > 0 : B \subset A + \alpha\}$$

2. By the definition of the Hausdorff metric we have

$$\begin{aligned} H(A,B) &= \max\{\rho(A,B), \rho(B,A)\} \\ &= \max\{\inf\{\alpha > 0 : B \subset A + \alpha\}, \inf\{\alpha > 0 : B \subset A + \alpha\}\} \end{aligned}$$

Definition 2.6. *(Support Function)*
The support function $S_f(A) : E \longrightarrow \mathbb{R}$ of a closed, convex subset A of E is defined by

$$S_f(A) = \sup_{a\in A} \langle f, a\rangle, \ \ f \in E^*$$

Remark 2.7. *It is very important that we observe the following properties of support function as we shall use it in the sequel.*

$$\begin{aligned} 1. \quad S_f(\overline{A+B}) &= S_f(A+B) \\ &= \sup_{a\in A, b\in B} \langle f, a+b\rangle \\ &= \sup_{a\in A}\langle f, a\rangle + \sup_{b\in B}\langle f, b\rangle \\ &= S_f(A) + S_f(B) \end{aligned}$$

2. For $\lambda \geq 0,\ S_f(\lambda A) = \lambda S_f(A)$.

Theorem 2.8. An element $x \in \overline{co}A$ if and only if

$$\langle f, x\rangle \leq S_f(A)$$

Proof

($\Rightarrow$) Assume $x \in \overline{co}A$. Then by definition of $\overline{co}A$, we have that there exists $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \subset A$ *and* $\{\lambda_n\}_{n\geq 1} \subset [0,1]$ such that

$$\lambda_n x_n + (1-\lambda_n)y_n \longrightarrow x \text{ } as \text{ } n \longrightarrow \infty$$

But, for $f \in E^*$

$$\begin{aligned}\langle f, \lambda_n x_n + (1-\lambda_n)y_n\rangle &= \lambda_n\langle f, x_n\rangle + (1-\lambda_n)\langle f, y_n\rangle \\ &\leq \lambda_n \sup_{x\in A}\langle f, x\rangle + (1-\lambda_n)\sup_{y\in A}\langle f, y\rangle \\ &= \sup_{a\in A}\langle f, a\rangle \\ &= S_f(A)\end{aligned}$$

By the continuity of f we obtain that

$$\langle f, x\rangle = \lim_{n\to\infty}\langle f, \lambda_n x_n + (1-\lambda_n)y_n\rangle \leq S_f(A)$$

Hence,

$$\langle f, x\rangle \leq S_f(A)$$

($\Leftarrow$) Assume for $f \in E^*$, $\langle f, x\rangle \leq S_f(A)$. We show that $x \in \overline{co}A$. To achieve this we proceed by contrapositive. i.e we show that

$$x \in (\overline{co}A)^c \Rightarrow S_f(A) < \langle f, x\rangle$$

Let $x \in (\overline{co}A)^c$. Then $(\overline{co}A)^c$ is open in E since $\overline{co}A$ is closed. So, the openness of $(\overline{co}A)^c$ gives us that for any $x \in (\overline{co}A)^c$ there exists a positive number r such that the the closed ball $\overline{B}_r(x)$ centered at x with radius r is completely contained in $(\overline{co}A)^c$. Clearly,

$$\overline{B}_r(x) \cap (\overline{co}A) = \emptyset$$

By Geometric form of Hahn Banach Theorem Chidume (2014), $\exists\, f \in E^*$ *and* β such that

$$\sup_{x\in\overline{co}A}\langle f, x\rangle \leq \beta \leq \sup_{y\in\overline{B}_r(x)}\langle f, y\rangle \qquad (1)$$

Let us remember that the two members of (1) may be exchanged by replacing f by $-f$ and β by β.

$$\sup_{a\in A}\langle f,a\rangle \leq \sup_{a\in \overline{co}A}\langle f,a\rangle \leq \inf_{x\in \overline{B}_r(x)}\langle f,x\rangle \leq \inf_{x\in B_r(x)}\langle f,x\rangle \leq \langle f,x\rangle-\epsilon$$

This implies that

$$S_f(A)\leq \langle f,x\rangle-\epsilon<\langle f,x\rangle$$

Hence, $\quad S_f(A)<\langle f,x\rangle$

Theorem 2.9. Let $\{A,A_n,n\in\mathbb{N}\}\subset \mathcal{P}_c(E)$ and $\{A_n\}_{n\geq 1}$ converges to A in the Hausdorff metric. Then,

$$\lim_{n\to\infty} S_f(A_n)=S_f(A)$$

Proof: Suppose $A_n,A\in\mathcal{P}_c(E)$ such that $H(A_n,A)\longrightarrow 0\ as\ n\longrightarrow\infty$. We show that the support function of A_n converges to the support function of A. i.e $S_f(A_n)\longrightarrow S_f(A)\ as\ n\longrightarrow\infty$. It suffices to show that

$$\underline{\lim}\, S_f(A_n)=S_f(A)=\overline{\lim}\, S_f(A_n)$$

By definition, we have that $\underline{\lim}\, S_f(A_n)\leq S_f(A)\leq\overline{\lim}\, S_f(A_n)$. So, we are left to show that

$$\overline{\lim}\, S_f(A_n)\leq S_f(A)\leq\underline{\lim}\, S_f(A_n)$$

First, we show that $\overline{\lim}\, S_f(A_n)\leq S_f(A)$. By the consequences of A_n converging to A in Hausdorff metric, we obtain that $\forall\ a_n\in A_n,\ \exists\ a\in A$ such that

$$d(a_n,A)\leq\|a_n-a\|_E\leq d(a_n,A)+\frac{1}{n},\ \forall\ n\geq 1$$

Let $f\in E^*$, Then

$$\begin{aligned}\langle f,a_n\rangle &= \langle f,a_n-a+a\rangle\\ &= \langle f,a_n-a\rangle+\langle f,a\rangle\\ &\leq \|f\|_{E^*}\|a_n-a\|_E+\langle f,a\rangle\\ &\leq \|f\|_{E^*}(d(a_n,A)+\frac{1}{n})+\sup_{a\in A}\langle f,a\rangle\\ &\leq \|f\|_{E^*}(\rho(A_n,A)+\frac{1}{n})+S_f(A)\end{aligned}$$

Take supremum over $a_n \in A_n$

$$\sup_{a_n \in A_n} d(a_n, A) \leq \|f\|_{E^*}(\rho(A_n, A) + \frac{1}{n}) + \sup_{a \in A}\langle f, a\rangle \leq S_f(A)$$

This implies that

$$S_f(A_n) \leq S_f(A)$$

Hence,

$$\overline{\lim}\, S_f(A_n) \leq S_f(A)$$

In a similar way, we show that $S_f(A) \leq \overline{\lim}\, S_f(A_n)$. This is obtained by following the same argument above; interchanging A_n and A, we obtain that

$$S_f(A) \leq \|f\|_{E^*}(\rho(A, A_n) + \frac{1}{n}) + \sup_{a_n \in A_n}\langle f, a_n\rangle$$

Take the limit inferior of both sides

$$\underline{\lim}\, S_f(A) \leq \underline{\lim}\, S_f(A_n)$$

This implies that,

$$S_f(A) \leq \underline{\lim}\, S_f(A_n)$$

Hence, $\lim_{n\to\infty} S_f(A_n) = S_f(A)$. ■

Theorem 2.10. Let $A, B \in \mathcal{P}_{bc}(E)$ and $f \in S^* = \{f \in E^* : \|f\|_{E^*} = 1\}$. Then,

1. $\rho(A, B) = \sup\{S_f(A) - S_f(B) : f \in S^*\}$

2. $H(A, B) = \sup\{|S_f(A) - S_f(B)| : f \in S^*\}$

Proof: 1. Let $A, B \in \mathcal{P}_{bc}(E)$ and $f \in S^*$. We show that $\rho(A, B) = \sup\{S_f(A) - S_f(B)\}$ i.e

$$\rho(A, B) = \sup\{\sup_{a \in A}\langle f, a\rangle - \sup_{b \in B}\langle f, b\rangle : f \in S^*\}$$

First, we show that

$$\rho(A, B) \leq \sup\{\sup_{a \in A}\langle f, a\rangle - \sup_{b \in B}\langle f, b\rangle : f \in S^*\}$$

Take $a \in A$. By Lemma 2.2 and 2.3, we have $\exists\ f \in S^*$ such that

$$\begin{aligned} d(a,B) = d(0, x-B) &\leq \inf_{x \in a-B} \langle f, x\rangle \\ &= \inf_{b \in B} \langle f, a-b\rangle \\ &= \sup_{b \in B} \langle f, a-b\rangle \\ &= \langle f, a\rangle - \sup_{b \in B} \langle f, b\rangle \\ &\leq \sup_{a \in A} \langle f, a\rangle - \sup_{b \in B} \langle f, b\rangle \\ &\leq \sup\{\sup_{a \in A} \langle f, a\rangle - \sup_{b \in B} \langle f, b\rangle : f \in S^*\} \end{aligned}$$

Take supremum over $a \in A$, we have that

$$\rho(A,B) \leq \sup\{\sup_{a \in A} \langle f, a\rangle - \sup_{b \in B} \langle f, b\rangle : f \in S^*\}$$

Next, we show that

$$\sup\{\sup_{a \in A} \langle f, a\rangle - \sup_{b \in B} \langle f, b\rangle : f \in S^*\} \leq \rho(A,B)$$

Now, let $f \in S^*$, $\beta = S_f(B)$ be fixed and $\alpha = S_f(A) - \beta$. We show that $\alpha \leq \rho(A,B)$. If $\alpha \leq 0$, then the result follows from the definition of ρ. Suppose $\alpha > 0$. For each $0 < r < \alpha$, there exists $a \in A$ such that $0 < \alpha - r < \langle f, a\rangle - \beta$. By adding β all through we have $\beta < \alpha + \beta - r \leq \langle f, a\rangle$. This implies that $\beta < \langle f, a\rangle$ Thus, the hyperplane H_α separates a and B. i.e

$$\sup_{b \in B} \langle f, b\rangle \leq \beta \leq \langle f, a\rangle$$

Observe that the distance from a to B is greater than the distance from a to the hyperplane H_α. Thus,

$$d(a,B) \geq d(a, H_\alpha) = \frac{|\langle f, a\rangle - \beta|}{\|f\|_{E^*}} = |\langle f, a\rangle - \beta|$$

By the property of absolute value function it follows that

$$\beta - \langle f, a\rangle \leq |\langle f, a\rangle - \beta| \leq d(a,B) \leq \rho(A,B)$$

By Taking the supremum over $a \in A$, we have

$$\beta - \sup_{a \in A} \langle f, a\rangle \leq \rho(A,B) \Rightarrow S_f(B) - S_f(A) \leq \rho(A,B)$$

This implies that

$$\sup\{S_f(B) - S_f(A)\} \leq \rho(A,B)$$

Hence,

$$\rho(A,B) = \sup\{S_f(A) - S_f(B)\}$$

■

We also obtain that $\rho(B,A) = \sup\{S_f(B) - S_f(A)\}$ by following the same argument.

$$\begin{aligned} 2. \qquad H(A,B) &= \sup\{\rho(A,B), \rho(B,A)\} \\ &= \sup\{\sup\{S_f(A) - S_f(B)\}, \sup\{S_f(B) - S_f(A)\}\} \\ &= \sup\{|S_f(A) - S_f(B)|\} \end{aligned}$$

■

In summary, the Hausdorff metric can be characterized by the following, depending on the hyperspace:

1. If $A, B \in \mathcal{P}_c(E)$, By definition 1.1 we have

$$H(A,B) = \max\{\rho(A,B), \rho(B,A)\}$$

2. If $A, B \in \mathcal{P}_b(E)$. By Theorem 2.4 and 2.5, Hausdorff metric is given by

$$H(A,B) = \sup_{x \in E} |d(x,A) - d(x,B)|$$

and

$$H(A,B) = \max\{\inf\{\alpha > 0 : B \subset A + \alpha\}, \inf\{\alpha > 0 : B \subset A + \alpha\}\}$$

3. If $A, B \in \mathcal{P}_{bc}(E)$, the Hausdorff metric can be represented as in Theorem 2.10 by

$$H(A,B) = \sup\{|S_f(A) - S_f(B)| : f \in S^*\}$$

3 Types of Convergences in a Hyperspace

In this section, we present four types of convergences in hyperspaces: Hausdorff convergence, Weak convergence, Wijsman convergence and the Kuratowski Mosco convergence.

Let $\{A_n, A\} \subset \mathcal{P}_c(E)$ we define the four convergences in $\mathcal{P}_c(E)$ whenever they make sense as follow:

1. **Hausdorff Convergence**: The sequence $\{A_n\}_{n\geq 1}$ converges to A in Hausdorff metric if

$$H(A_n, A) \longrightarrow 0 \ as \ n \longrightarrow \infty$$

We denote it by $A_n \xrightarrow{H} A$ or $(H) - \lim_{n\to\infty} A_n = A$

2. **Weak Convergence**: The sequence $\{A_n\}_{n\geq 1}$ converges weakly to A if for all $f \in E^*$,

$$S_f(A_n) \longrightarrow S_f(A) \ as \ n \longrightarrow \infty$$

and we denote it by $A_n \xrightarrow{We} A$ or $(We) - \lim_{n\to\infty} A_n = A$.

The Weak Convergence here is an extension of weak convergence in the weak topology to the hyperspace. We recall that in weak topology, a sequence $\{x_n\}_{n\geq 1} \subset E$ converges weakly to $x \in E$ if and only if $\forall\ f \in E^*$, $\langle f, x_n\rangle$ converges to $\langle f, x\rangle$. Here, instead of $\langle f, x_n\rangle$ converging to $\langle f, x\rangle$ we demand that $\sup_{x_n\in A_n}\langle f, x_n\rangle$ converges to $\sup_{x\in A}\langle f, x\rangle$.

3. **Wijsman Convergence**: The sequence $\{A_n\}_{n\geq 1}$ converges to A in Wijsman if for any $x \in E$,

$$d(x, A_n) \longrightarrow d(x, A) \ as \ n \longrightarrow \infty$$

and we denote this by $A_n \xrightarrow{Wj} A$ or $(Wj) - \lim_{n\to\infty} A_n = A$.

Proposition 2.11. *Let $A_n, A \in \mathcal{P}_b(E)$. The sequence $\{A_n\}_{n\geq 1}$ converges to A in Hausdorff metric if and only if $\{A_n\}_{n\geq 1}$ converges in Wijsman to A uniformly on E.*

Proof: Let $A_n, A \in \mathcal{P}_b(E)$ such that $H(A_n, A) \longrightarrow 0 \ as \ n \longrightarrow \infty$. We show that $d(x, A_n) \longrightarrow d(x, A)$ uniformly on E. That is to say

$$\sup_{x\in E} |d(x, A_n) - d(x, A)| \longrightarrow 0 \ as \ n \longrightarrow \infty$$

and vice versa.

But by Theorem 2.4, we have that

$$H(A_n, A) = \sup_{x\in E} |d(x, A_n) - d(x, A)|$$

Thus, as $n \longrightarrow \infty$,

$$H(A_n, A) \longrightarrow 0 \Leftrightarrow \sup_{x\in E} |d(x, A) - d(x, B)| \longrightarrow 0$$
$$\Leftrightarrow d(x, A_n) \longrightarrow d(x, A) \ uniformily \ on \ E$$

■

Proposition 2.12. *Let $A_n, A \in \mathcal{P}_b(E)$ such that $\{A_n\}_{n\geq 1}$ converges to A in Hausdorff metric. Then for each $f \in E^*, \{A_n\}_{n\geq 1}$ converges weakly to A.*

Proof: The proof of this has been treated in Theorem 2.10

Theorem 2.13. Let $\{A_n\}_{n\geq 1} \in \mathcal{P}_k(E)$ be a decreasing sequence and $A = \bigcap_{n\geq 1} A_n$. Then $\lim_{n\to\infty} H(A_n, A) = 0$

Proof: Let $\epsilon > 0$ be given, we find $N \in \mathbb{N}$ such that $A \subset \hat{A}_n + \epsilon$ and $A_n \subset \hat{A} + \epsilon$ where $\hat{A} + \epsilon = \{x \in E : d(x, A) < \epsilon\}, \hat{A}_n + \epsilon = \{x \in E : d(x, A_n) < \epsilon\}$.
But $A = \bigcap_{n=1}^{\infty} A_n \Rightarrow A \subset \hat{A}_n + \epsilon$. This happens for all $n \in \mathbb{N}$. So we are only required to show that $A_n \subset \hat{A} + \epsilon$. Note that the choice for $\hat{A} + \epsilon$ to be open is very crucial here. Now,
$A = \bigcap_{n=1}^{\infty} A_n \Rightarrow A^c = \bigcup_{n=1}^{\infty} A_n^c$. Since $E = A \cup A^c$, it is easy to see that $E = (\hat{A} + \epsilon) \cup A^c = (\hat{A} + \epsilon) \cup (\bigcup_{n=1}^{\infty} A_n^c)$. So, it follows that

$$A_1 \subset (\hat{A} + \epsilon) \cup (\bigcup_{n=1}^{\infty} A_n^c)$$

This implies that $(\hat{A} + \epsilon) \cup (\bigcup_{n=1}^{\infty} A_n^c)$ is an open cover for A_1. By the compactness of A_1 and the fact that A_n is decreasing, there exists $N \in \mathbb{N}$ such that $A_1 \subset (\hat{A} + \epsilon) \cup A_n^c$. This gives us that

$$A_1 \cap ((\hat{A} + \epsilon) \cup A_n^c)^c = \emptyset \Rightarrow A_1 \cap (\hat{A} + \epsilon)^c \cap A_n = \emptyset$$

Since, A_n is a decreasing sequence, we have that $A_n \subset A_1$. Thus, $(\hat{A} + \epsilon)^c \cap A_n = \emptyset$. Hence, $A_n \subset \hat{A} + \epsilon$. ■

4 Convergence in the sense of Kuratowski - Mosco

The Kuratowski - Mosco convergence is basically characterized by means of the topological notions of limit inferior and limit superior in strong and weak topologies. Let us begin by defining the following concepts.

(A) For an Arbitrary Metric Space E

Definition 2.14. *Let* $\{A_n\}_{n\geq 1}$ *be a sequence of subsets of a metric space* E*.*

1. The **Limit inferior** of$\{A_n\}_{n\geq 1}$ is defined as

$$\underline{\lim}\, A_n = \bigcup_{k\geq 1} \bigcap_{n\geq k} A_n$$

2. The **Limit superior** of$\{A_n\}_{n\geq 1}$ is defined as

$$\overline{\lim}\, A_n = \bigcap_{k\geq 1} \bigcup_{n\geq k} A_n$$

(A) For Normed Linear Space E

Definition 2.15. *Let* $\{A_n\}_{n\geq 1}$ *be a sequence of subsets of a Normed space* E*.*

1. The **strong limit inferior** of $\{A_n\}_{n\geq 1}$ denoted by $s - \underline{\lim}\, A_n$ is defined as

$$s - \underline{\lim}\, A_n = \{x \in E : \exists\ n_1 \in \mathbb{N},\ \forall\ n \geq n_1,$$
$$\exists\ \ x_n \in A_n,\ x_n \xrightarrow{s} x \ as\ n \longrightarrow \infty\}$$

where $x_n \xrightarrow{s} x$ means that $\{x_n\}$ converges strongly to x.

2. The **strong limit superior** of $\{A_n\}_{n\geq 1}$ denoted by $s - \overline{\lim}\, A_n$ is defined as

$$s - \overline{\lim}\, A_n = \{x \in E :\ \forall\ k \in \mathbb{N},\ \exists\ n_k \in \mathbb{N},\ \exists\ x_{n_k} \in A_{n_k},$$
$$x_{n_k} \xrightarrow{s} x\ as\ k \longrightarrow \infty\}$$

3. The **weak limit inferior** of $\{A_n\}_{n\geq 1}$ denoted by $w - \underline{\lim}\, A_n$ is defined as

$$s - \underline{\lim}\, A_n = \{x \in E : \exists\ n_1 \in \mathbb{N},\ \forall\ n \geq n_1,$$
$$\exists\ \ x_n \in A_n,\ x_n \xrightarrow{w} x\ as\ n \longrightarrow \infty\}$$

4. The **weak limit superior** of $\{A_n\}_{n\geq 1}$ denoted by $w - \overline{\lim} A_n$ is defined as

$$s - \overline{\lim} A_n = \{x \in E : \; \forall \; k \in \mathbb{N}, \; \exists \; n_k \in \mathbb{N}, \; \exists \; x_{n_k} \in A_{n_k}, \quad x_{n_k} \xrightarrow{w} x \; as \; k \longrightarrow \infty\}$$

where $x_{n_k} \xrightarrow{w} x$ means that x_{n_k} converges weakly to x. For the sake of brevity, we shall use the symbols '$\longrightarrow$' and '$\rightharpoonup$' as in Chidume (2014) for strong and weak convergences respectively. The strong limit inferior and strong limit superior are called **Kuratowski** $\underline{\lim} A_n$ and **Kuratowski** $\overline{\lim} A_n$ respectively; the weak limit inferior and weak limit superior are called **Mosco** $\underline{\lim} A_n$ and **Mosco** $\overline{\lim} A_n$ respectively. The Kuratowski Mosco convergence occurs where the Mosco limit superior and the Kuratowski limit inferior coincide.

Kuratowski - Mosco Convergence: The sequence $\{A_n\}_{n\geq 1}$ converges to A in the Kuratowski - Mosco sense if and only if

$$w - \overline{\lim} A_n = A = \; s - \underline{\lim} A_n$$

and we denote this by $A_n \xrightarrow{K} A$ or $k - \lim A_n = A$.
At this moment, we can deduce the following facts from the definitions above.
Fact 1: Let $\{A_n\}_{n\geq 1}$ be a sequence of subsets of a metric space E. Then,

$$s - \underline{\lim} A_n \subset w - \overline{\lim} A_n. \tag{4.1}$$

This suggests to us that to show Kuratowski - Mosco convergence, we are only required to show that

$$w - \overline{\lim} A_n \subset A \subset s - \underline{\lim} A_n. \tag{4.2}$$

Theorem 2.16. Let $\{A_n, A : n \in \mathbb{N}\}$ be closed bounded convex subsets of a Banach space E. If $\{A_n\}_{n\geq 1}$ converges to A in Hausdorff metric, then $\{A_n\}_{n\geq 1}$ converges to A in Kuratowski - Mosco sense.

Proof: Let $\{A_n, A\} \subset \mathcal{P}_{bc}(E)$ and $H(A_n, A) \longrightarrow 0 \; as \; n \longrightarrow \infty$. We show that

$$w - \overline{\lim} A_n \subset A \subset s - \underline{\lim} A_n$$

Let $x \in A$. By the consequence of the Hausdorff convergence of $\{A_n\}_{n\geq 1}$ to A, we have that there exists $\{x_n \in A_n : \ n \in \mathbb{N}\}$ such that

$$\|x - x_n\|_E < d(x, A_n) + \frac{1}{n} \quad (BC)$$

We recall that in Theorem 2.4,

$H(A_n, A) \longrightarrow 0 \ as \ n \longrightarrow \infty \Rightarrow d(x, A_n) \longrightarrow d(x, A)$. But since $x \in A, \ d(x, A) = 0$. Thus, $d(x, A_n) \longrightarrow 0 \ as \ n \longrightarrow \infty$. So from (BC) we have that $\|x - x_n\|_E \longrightarrow 0 \ as \ n \longrightarrow \infty \Rightarrow x_n \longrightarrow x \ as \ n \longrightarrow \infty$. Hence,

$$x \in s - \underline{\lim}\, A_n$$

Next, we show that $x \in w - \overline{\lim}\, A_n \Rightarrow x \in A$. Suppose $x \in w - \overline{\lim}\, A_n$. Then there exist $x_{n_k} \in A_{n_k}, \ k \geq 1$ such that $x_{n_k} \rightharpoonup x \ as \ k \longrightarrow \infty$. i.e $\forall \ f \in E^*, \ \langle f, x_{n_k}\rangle \longrightarrow \langle f, x\rangle$. From Theorem 2.8

$$\langle f, x_{n_k}\rangle \leq S_f(A_n)$$

By Theorem 2.9, we recall that $S_f(A_n) \longrightarrow S_f(A)$. We obtain that $\langle f, x\rangle \leq S_f(A)$. Conclusion ?

Theorem 2.17. Let E be a finite dimensional Banach Space. If for $n \in \mathbb{N}, \ A_n$ and A are compacts subsets of E and $\{A_n\}$ converges to A in the Kuratowski - Mosco sense, then $\{A_n\}$ converges to A in Hausdorff Metric.

Proof: Assume $dimE < \infty$. Then, strong topology and weak topology coincide Chidume (2014). That is to say for $\{A_n, A\} \subset \mathcal{P}_k(E)$

$$s - \overline{\lim}\, A_n = w - \overline{\lim}\, A_n \ and \ s - \underline{\lim}\, A_n = w - \underline{\lim}\, A_n$$

Now, suppose $A_n \xrightarrow{K} A$. We show that given $\epsilon > 0 \ \exists \ N \in \mathbb{N}$:

$$H(A_n, A) < 2\epsilon \ \forall \ n \geq N$$

By lemma 1.13, it is sufficient to show that

$$A \subset A_n + 2\epsilon \ and \ A_n \subset A + 2\epsilon$$

Now,

$$A_n \xrightarrow{H} A \Rightarrow s - \overline{\lim}\, A_n = A = s - \underline{\lim}\, A_n$$

Since A is compact, A is totally bounded. i.e $\exists\ \{x_1, x_2, ...x_k\} \subset A$ and $\epsilon > 0$ such that

$$A \subset \bigcup_{i=1}^{k} B_\epsilon(x_i)$$

where $\bigcup_{i=1}^{k} B_\epsilon(x_i) = \{x \in E : \|x - x_i\|_E < \epsilon\}$. Since $s - \overline{\lim}\, A_n = A$ and for each $1 \leq i \leq k$, $x_i \in A$, it follows from the definition of $s - \overline{\lim}\, A_n$ that there exists a subsequence $\{x_{i,n}\}_{n\geq 1} \subset A_n$ such that

$$\|x_{i,n} - x_i\|_E \longrightarrow 0\ as\ n \longrightarrow \infty$$

$$\Rightarrow \exists\ N_i \in \mathbb{N} : \|x_{i,n} - x_i\|_E < \epsilon\ \forall\ n \geq N_i \text{ and for each } i,\ 1 \leq i \leq k$$

$$\Rightarrow x_i \in B_\epsilon(x_i, n) \subset A_n + \epsilon\ ,\forall\ n \geq N_i.$$

Now, take $N = \max\{N_i, 1 \leq i \leq k\}$. Then, $\{x_1, x_2, \ldots x_k\} \subset A_n + \epsilon$ whenever $n \geq N$. Thus, $A \subset \bigcup_{i=1}^{k} B_\epsilon(x_i) \subset A_n + \epsilon$. Hence $A \subset A_n + 2\epsilon$.

Next, we show that given $\epsilon > 0$ there exists $n_o \in \mathbb{N}$ such that $A_n \subset A + \epsilon,\ for\ each\ n > n_o$. Since A_n and A are compact for each n and $A_n \xrightarrow{H} A$, we have that both A_n and A are bounded. So, there exists $n > n_o$ and $M > 0$ such that $A \subset \overline{B_M(0)}$ and $A_n \subset \overline{B_M(0)}$ for each $n > n_o$, where $\overline{B_M(0)} = \{x \in E : \|x\|_E \leq M\}$.

Suppose for contradiction that there exists a subsequence $\{A_{n_k} : k \in \mathbb{N}\}$ such that $A_{n_k} \cap (A + \epsilon)^c \neq \emptyset$. Take $x_{n_k} \in A_{n_k} \cap (A + \epsilon)^c \subset \overline{B_M(0)}$. By Bolzano Weierstrass Theorem in $\mathbb{R}^n$, there exists a subsequence say x_{n_k} such that $x_{n_k} \xrightarrow{s} x$. Thus, $x \in \overline{\lim}\, A_n = A$. This implies that $x \in A$. But, $x_{n_k} \in A_{n_k} \cap (A + \epsilon)^c$ implies that $d(x_{n_k}, A) \geq \epsilon$. So, Taking limit as $k \longrightarrow \infty$, we have

$$0 = d(x, A) = \lim_{k\to\infty} d(x_{n_k}, A) \geq \epsilon$$

Hence, a contradiction.

Remark 2.18. *(1) By Theorem 2.16 and 2.17, we have established that in finite dimensional Banach space E, Hausdorff convergence(H) of a sequence of compact and convex subsets of E is equivalent to its Kuratowski - Mosco convergence(K).*

(2) By Proposition 2.11 we have also established that Hausdorff convergence(H) is equivalent to Wijsman convergence(Wi) for a sequence of closed and bounded subsets of a

Banach Space E. But in finite dimensional metric spaces, closed and boundedness implies compactness (Heine Borel Theorem) Chidume (2009).

(3) By Proposition 2.10 we were able to deduce that Hausdorff convergence(H) implies Weak convergence(W) for a sequence of closed bounded convex subsets of a Banach space E.

Now, to complete our comparison chain we show that Weak convergence(We) implies Hausdorff convergence(H) and Wijsman convergence(Wi) implies Kuratowski - Mosco convergence(K).

Theorem 2.19. Let E be a finite dimensional Banach Space. If $\{A_n, A\} \subset \mathcal{P}_{kc}(E)$ and $\{A_n\}$ converges to A in Wijsman, then $\{A_n\}$ converges to A in the sense of Kuratowski-Mosco.

Proof: Assume $A_n \xrightarrow{Wi} A$. We show that

$$w - \overline{\lim} A_n \subset A \subset s - \underline{\lim} A_n \tag{4.3}$$

Suppose $a \in w - \overline{\lim} A_n$. Then, there exist a subsequence $\{a_{n_k}\}_{k\geq 1} \subset A_{n_k}$ such that $\{a_{n_k}\}$ converges weakly to a. Since, $dimE < \infty$, weak topology implies strong topology. Thus, $\{a_{n_k}\}$ converges strongly to a. That is $\|a_{n_k} - a\|_E \longrightarrow 0 \ as \ k \longrightarrow \infty$. Now, $\lim_{k\to\infty} d(a, A_{n_k}) = 0$. But by the assumption,

$$d(a, A) = \lim_{n\to\infty} d(a, A_n) = \lim_{k\to\infty} d(a, A_{n_k}) = 0 \tag{4.4}$$

This gives us that $d(a, A) = 0$ which implies that $a \in A$.

Next, we show that $A \subset s - \underline{\lim} A_n$. Suppose $a \in A$. By Wijsman convergence of $\{A_n\}$ we have that $d(a, A_n) \longrightarrow d(a, A) = 0$. For each $n \geq 1$, choose $a_n \in A_n$ such that

$$\|a_n - a\|_E \leq d(a, A_n) + \frac{1}{n}$$

From this, it is obvious that $\|a_n - a\|_E \longrightarrow 0 \ as \ n \longrightarrow \infty$. Thus, $\{a_n\}$ converges strongly to a. Hence, $x \in s - \underline{\lim} A_n$. ■

Theorem 2.20. Let E be a finite dimensional Banach Space. If $\{A_n, A\} \subset \mathcal{P}_{kc}(E)$ and $\{A_n\}$ converges weakly to A, then $\{A_n\}$ converges to A in Hausdorff.

Proof: Suppose for contradiction that $\{A_n\}_{n=1}^{\infty}$ converges weakly to A but fails to converge in Hausdorff. That is there exist $\epsilon_o > 0$ such that for all $N \in \mathbb{N}$, there exist $n \geq N$ such that

$$H(A_n, A) > \epsilon_o > 0$$

By Theorem 2.10,

$$H(A_n, A) = \sup_{f \in S^*} |S_f(A_n) - S_f(A)| \tag{4.5}$$

This implies, $\alpha = \sup_{f \in S^*} |S_f(A_n) - S_f(A)| > 0$. By the consequences of supremum, we have for $k \geq 1$ there exists $f_k \in S^*$ such that

$$\alpha - \frac{1}{k} < |S_{f_k}(A_{n_k}) - S_{f_k}(A)| \leq \alpha \tag{4.6}$$

$$\Rightarrow \overline{\lim} |S_{f_k}(A_{n_k}) - S_{f_k}(A)| > 0 \tag{4.7}$$

By weak convergence of $\{A_n\}_{n=1}^{\infty}$ to A, we have that the subsequence $\{A_{n_k}\}_{k=1}^{\infty}$ also converges weakly to A. We claim that $\{A_{n_k}\}_{k=1}^{\infty}$ is uniformly bounded for $k \geq 1$, since

$$\begin{aligned} A_{n_k} \xrightarrow{w} A &\Leftrightarrow \forall\ \epsilon > 0, \exists\ k_\epsilon \in \mathbb{N}\ s.t\ |S_f(A_{n_k}) - S_f(A)| < \epsilon\ \forall\ k \geq k_\epsilon \\ &\Rightarrow |S_f(A_{n_k})| - |S_f(A)| < \epsilon\ \forall\ k \geq k_\epsilon \\ &\Rightarrow |S_f(A_{n_k})| < \epsilon + |S_f(A)|\ \forall\ k \geq k_\epsilon. \end{aligned}$$

But $S_f(A)$ is bounded for $f \in S^*$ since A is compact. Let $M = \max\{S_f(A_{n_1}), S_f(A_{n_2}), \ldots, S_f(A_{n_{k_\epsilon}})\}$. Then

$$\|A_{n_k}\| = \sup_{f \in S^*} |S_f(A_{n_k})| \leq \epsilon + \sup_{f \in S^*} |S_f(A)| + M \tag{4.8}$$

So that $\sup_{k \in \mathbb{N}} \|A_{N_k}\| \leq c$. Now, $dimE < \infty$ implies that S^* is compact in E^* (Heine Borel Theorem). So, there exists a subsequence $\{f_{n_i}\}_{i=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that $\|f_{n_i} - f\| \longrightarrow 0\ as\ i \longrightarrow 0$.

Thus

$$
\begin{aligned}
|S_{f_{k_i}}(A_{n_k}) - S_f(A_{n_k})| &= |\sup_{x\in A_{n_k}} \langle f_{k_i}, x\rangle - \sup_{x\in A_{n_k}} \langle f, x\rangle| \\
&\le |\sup_{x\in A_{n_k}} \langle f_{k_i}, x\rangle - \langle f, x\rangle| \\
&\le |\sup_{x\in A_{n_k}} \{|\langle f_{k_i}, x\rangle - \langle f, x\rangle|\} \\
&\le \|f_{k_i} - f\| \sup_{x\in A_{n_k}} \|x\| \\
&= \|f_{k_i} - f\| \|A_{n_k}\| \\
&\le \|f_{k_i} - f\|\ c \longrightarrow 0\ as\ i \longrightarrow \infty
\end{aligned}
$$

From equation 4.6 above we have

$$
\begin{aligned}
0 < \overline{\lim}\, |S_{f_{k_i}}(A_{n_{k_i}}) - S_{f_{k_i}}(A)| &= \overline{\lim}\, |S_{f_{k_i}}(A_{n_{k_i}}) - S_f(A_{n_{k_i}}) + S_f(A_{n_{k_i}}) \\
&- S_f(A) + S_f(A) - S_{f_{k_i}}(A)| \\
&\le \overline{\lim}\, |S_{f_{k_i}}(A_{n_{k_i}}) - S_f(A_{n_{k_i}})| \\
&+ \overline{\lim}\, |S_f(A_{n_{k_i}}) - S_f(A)| \\
&+ \overline{\lim}\, |S_f(A) - S_{f_{k_i}}(A)| \\
&= \overline{\lim}\, |S_f(A_{n_{k_i}}) - S_f(A)|
\end{aligned}
$$

Hence, $\overline{\lim}\, |S_f(A_{n_{k_i}}) - S_f(A)| > 0$ which contradicts the hypothesis. ■

Proposition 2.21. *Let E be a finite dimensional Banach Space. If $\{A_n, A\} \subset \mathcal{P}_{kc}(E)$. Then the following are equivalent;*

(1) $A_n \xrightarrow{H} A$

(2) $A_n \xrightarrow{Wj} A$

(3) $A_n \xrightarrow{K} A$

(4) $A_n \xrightarrow{We} A$

CHAPTER 3

Set-Valued Random Variables

1 Introduction

So far, we have studied in chaphter 1 and 2 the Hausdorff metric and its properties as well as the Hausdorff convergence. This chapter focuses on set-valued random variable and its properties. We begin as usual with definitions and notations we will use in the sequel.

I. Range, Graph and Inverse Image:

Let (E, d) be a metric space, $f : \Omega \longrightarrow E$ be a single-valued function and $F : \Omega \longrightarrow \mathcal{P}(E)$ be a set-valued function.

The range of F denoted by $R(F)$ is defined by

$$R(F) = \bigcup_{\omega \in \Omega} F(\omega) \tag{1.1}$$

The graph of f denoted by $\mathcal{G}_f$ is defined as

$$\mathcal{G}_f = \{(\omega, t) \in \Omega \times E : t = f(\omega)\} \tag{1.2}$$

and the graph of F is defined by

$$\mathcal{G}_F = \{(\omega, t) \in \Omega \times E : t \in F(\omega)\} \tag{1.3}$$

The general inverse image of f for any $\emptyset \neq A \subset E$ denoted by $f^{-1}(A)$ is defined by

$$f^{-1}(A) = \{\omega \in \Omega : f(\omega) \subset A\} \tag{1.4}$$

and that of F is defined by

$$F^{-1}(A) = \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\}. \tag{1.5}$$

Let us state some important observations about these definition we have given. Observe that the generalization of graph from single-valued function to set-valued functions is direct. Indeed, the two graphs coincide if $\forall\ \omega \in \Omega, F(\omega) = \{f(\omega)\}$. But for the inverse image, direct generalization fails. To see this clearly, we suppose

$$\forall\ A \subset E, F^{-1}(A) = \{\omega \in \Omega : F(\omega) \subset A\}. \tag{1.6}$$

Then comparing $F(\omega) \cap A \neq \emptyset$ and $F(\omega) \subset A$, we discover that there is a very big difference between the two. For instance, we do not have that $F^{-1}(A^c) = \{F^{-1}(A)\}^c$ in equation 1.5, since the possibility of obtaining $F(\omega) \cap A \neq \emptyset$ and $F(\omega) \cap A^c \neq \emptyset$ is very high. So, we have to desist from applying to F^{-1} the classical properties of f^{-1}. Although, they share some properties in common. For example. If $\{A_i\}_{i\in I}$ is a family of non-empty subset of E, then

$$F^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} F^{-1}(A_i) \tag{1.7}$$

$$F^{-1}(\emptyset) = \emptyset \textit{ and } F^{-1}(E) = \Omega. \tag{1.8}$$

Definition 3.1. *Let $(\Omega, \mathcal{B})$ and $(\mathcal{P}(E), \Sigma)$ be the two Borel spaces. A set-valued function $F : \Omega \longrightarrow \mathcal{P}(E)$ is said to be a closed, open or compact set-valued function if $\mathcal{G}_F$ is closed, open or compact with respect to the product topology $\mathcal{B} \times \Sigma$.*

Definition 3.2. *A collection $\mathcal{A}$ of subsets of Ω, is a $\sigma - algebra$ if and only if $\mathcal{A}$ contains $\emptyset$, Ω and is stable by all countable set operations Lo (2017b). A pair $(\Omega, \mathcal{A})$ is called a **measurable space***

Let E be a complete metric space. Then, the smallest $\sigma - algebra$ containing all open sets in E is called the Borel $\sigma - algebra$ on E denoted by $\mathcal{B}_E$. The measurable space $(E, \mathcal{B}_E)$ is also called the Borel measurable space on E. Suppose $A \in \mathcal{B}_E$, then we say that A is Borel measurable with respect to E.
Let τ_o and τ_c represents the class of open subset of E and closed subset of E respectively.

Definition 3.3. *The projection $\pi_1 : \Omega \times E \longrightarrow \Omega$ is said to be **perfect** if and only if*

$$\forall A \in \mathcal{A} \otimes \mathcal{B}_E,\ \pi_1(A) \in \mathcal{A}.$$

2 Set-Valued Random Variables

Definition 3.4. *Let* $F : (\Omega, \mathcal{A}) \to \mathcal{P}(E)$. *Let us define five types of measurability of* F.

(1) F *is Borel-measurable if and only if*

$$\forall\ B \in \mathcal{B}_E,\ \ F^{-1}(B) \in \mathcal{A}. \ (01)$$

(2) F *is strongly measurable, or measurable, if and only if*

$$\forall\ B \in \tau_c,\ \ F^{-1}(B) \in \mathcal{A}. \ (02)$$

(3) F *is weakly measurable if and only if*

$$\forall\ B \in \tau_o,\ \ F^{-1}(B) \in \mathcal{A}. \ (03)$$

(4) F *is pathwise measurable if and only if for any* $x \in E$ *fixed, the the path*

$$\Omega \ni \omega \mapsto d(x, F(\omega))$$

is $\mathcal{A}$*-measurable.*

(5) F *is graph-measurable if and only* $\mathcal{G}_F$ *is* $(\mathcal{A} \otimes \mathcal{B}_E)$*-measurable.*

We say that, F *is a* ***set-valued random variable*** *or* F *is a* ***random set*** *if and only if (3) holds.*

The relationship between these five types of mesurability of set-valued function are as follow:

Theorem 3.5. Let $(\Omega, \mathcal{A})$ be a measurable space, (E, d) be a metric space endowed with its Borel σ-algebra $\mathcal{B}_E$ and $F : (\Omega, \mathcal{A}) \to \mathcal{P}(E)$ be a set-valued random variable. Consider the different types of mesaurability in Definition 3.4. We have :

(a) We have : $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

(b) If E is separable, we have $(4) \Rightarrow (5)$.

(c) If π_1 is perfect, all the five types are equivalent.

Proof: We proceed the proof by points.

Proof of Point (a).

(1) $\Rightarrow$ (2) This follows from the fact that closed sets are contained in Borel $\sigma-$ algebra $\mathcal{B}_E$.

(2) $\Rightarrow$ (3). Assume (2) holds. Let $\emptyset \neq A \subsetneq E$ be open. Then, we show that $F^{-1}(A) \in \mathcal{A}$. Let $B_n = \{x \in E : d(x, A^c) \geq \frac{1}{n}\}$. We observe that A^c is closed since A is open and B_n is the pre-image of $[\frac{1}{n}, \infty)$ with the function $d(., A^c)$. By the continuity of $d(., A^c)$, we have that B_n is closed. Whence, by equation 1.7 and 1.8 we obtain,

$$F^{-1}(A) = F^{-1}(\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} F^{-1} B_n \in \mathcal{A}$$

and for the case of $A = E$, $F^{-1}(A) = F^{-1}(E) = \Omega \in \mathcal{A}$. Hence, (3) proved.

(3) $\Rightarrow$ (4) Using the fact that, for $x \in E$ and $A \subset E$, $r > 0$,

$$\left(\exists\ t_0 \in A,\ d(x, t_0) < r\right) \Leftrightarrow d(x, A) < r,$$

we have for $x \in E$ fixed

$$\begin{aligned} F^{-1}(B_c(x,r)) &= \{\omega \in \Omega,\ F(\omega) \cap B_c(x,r) \neq \emptyset\} \\ &= \{\omega \in \Omega,\ \exists t_0 \in B_c(x,r),\ d(x, F(\omega)) < r\} \\ &= \{\omega \in \Omega,\ d(x, F(\omega)) < r\} \end{aligned}$$

$(d(x, t_\circ) < r) \in \mathcal{A}$ for $r > 0$. $(d(x, t_\circ) < r) = \emptyset$ for $r < 0$ and $(d(x, t_\circ) < r) = F(\omega) \in \mathcal{A}$ for $r = 0$. Then $(d(x, t_\circ) < r) \in \mathcal{A}$ for all $r \in \mathbb{R}$. Hence, (iv) holds.

Proof of point (b)

Suppose E is separable and (4) holds. Let $D = \{x_n,\ n \geq 1\}$ dense in E. For $r \in \mathbb{N}$, we have

$$E = \bigcup_{n \geq 1} B(x_n, 1/r) = \sum_{n \geq 1} C_n(r),$$

where

$$C_1(r) = B(x_1, 1/r),\ C_2(r) = (B(x_1, 1/r))^c B(x_2, 1/r),$$

$$C_n(r) = (B(x_1, 1/r))^c \cdots (B(x_{n-1}, 1/r))^c B(x_n, 1/r),\ n \geq 3.$$

We want to show that $\mathcal{G}_F$ is $(\mathcal{A} \otimes \mathcal{B})$-measurable. But since $F(\omega)$ is closed, we have

$$\begin{aligned} \mathcal{G}_F &= \{(\omega, x) \in \Omega \times E,\ (\omega, x) \in \Omega \times F(\omega)\} \\ &= \{(\omega, x) \in \Omega \times E, d(x, F(\omega)) = 0\}. \end{aligned}$$

So, it is enough to show that

$$\Omega \times E \ni (\omega, x) \mapsto d(x, F(\omega))$$

is $(\mathcal{A} \otimes \mathcal{B}_E)$-measurable. Since

$$\Omega \times E = \sum_{n \geq 1} \Omega \times C_n(r).$$

Set $d_n(x, F(\omega)) = d(x_n, F(\Omega))$ whenever $(\omega, x) \in \Omega \times C_n(r)$. We have for $\epsilon \geq 0$,

$$\begin{aligned} \{(\omega, x) \in \Omega \times E,\ d(x, F(\omega)) < \epsilon\} &= \sum_{n \geq 1} \{(\omega, x) \in \Omega \times E,\ d(x, F(\omega)) < \epsilon\} \\ &\qquad \bigcap \Omega \times C_n(r) \\ &= \sum_{n \geq 1} \{(\omega, x) \in \Omega \times E,\ d(x_n, F(\omega)) < \epsilon\} \bigcap \Omega \times C_n(r) \end{aligned}$$

So, the real-valued random variable

$$\Omega \times E \ni (\omega, x) \mapsto d(x, F(\omega))$$

is measurable. Finally, for any $(\omega, x) \in \Omega \times E$, there exists a unique $n(x) \geq 0$ such that

$$d_r(x, F(\omega)) = d(x_{n(x,r)}, F(\omega)) \text{ and } d(x, x_{n(x,r)}) < 1/r.$$

By taking limit as $r \longrightarrow \infty$ we see that for any $(\omega, x) \in \Omega \times E$, $x_{n(x,r)} \longrightarrow x$. Thus,

$$d_r(x, F(\omega)) \to d(x, F(\omega)).$$

Hence, $\mathcal{G}_F$ is $(\mathcal{A} \otimes \mathcal{B}_E)$-measurable.

Proof of Point (c). Suppose that E is separable, π_1 is perfect and (5) holds. To show that all the five types are equivalent, we only need to prove that (5) $\to$ (1) and the circular argument will be complete. Suppose that (5) hold and let $B \in \mathcal{B}_E$, we have

$$\begin{aligned} F^{-1}(B) &= \{\omega \in \Omega,\ F(\omega) \cap B \neq \emptyset\} \\ &= \{\omega \in \Omega,\ \exists x \in B,\ x \in F(\omega)\} \\ &= \{\omega \in \Omega,\ \exists x \in B,\ (\omega, x) \in \mathcal{G}_F\} \\ &= \pi_1(\mathcal{G}_F) \in \mathcal{A}, \end{aligned}$$

by the perfection of π_1. This completes the proof. ■

3 Measurable Selection and its Properties

In the next section, we introduce the notion of selections of set-valued mappings and characterize measurable selections for set-valued random variables.

Definition 3.6. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, E a metric space with its usual Borel $\sigma-$ algebra $\mathcal{B}_E$. An E-valued function $f : \Omega \longrightarrow E$ is called a **selection** for a set-valued mapping $F : \Omega \longrightarrow \mathcal{P}(E)$ if and only if*

$$\forall\ \omega \in \Omega,\ \ f(\omega) \in F(\omega)$$

.

*A function f is called an **almost everywhere selection** of F if and only if*

$$\forall\ \omega \in \Omega,\ \ f(\omega) \in F(\omega)\ \ - \mu - a.e.$$

A selection f is said to be measurable if the function f is measurable.

We now proceed to present two important properties of selection f of a set-valued random variables F. First, we state and describe without proofs the following known results which we shall use for the proof.

Lemma 3.7. *Let (E,d) be a complete metric space and $\{F_k\}_{k\geq 1}$ be a sequence of subsets of E. Suppose for each $k \in \mathbb{N}, F_k$ is a non-empty closed set,*

$$F_{k+1} \subset F_k \text{ and } \lim_{k\to\infty} diam(F_k) = 0$$

. Then, $\bigcap_{k=1}^{\infty} F_k$ contains only one element.

Lemma 3.8. *Let (E,d) be a separable metric space with $D = \{x_1, x_2, \dots\}$ being a countable dense subset of E. Suppose $A \subset E,\ A \neq \emptyset$, then for each fixed $k \in \mathbb{N}$,*

$$A \cap \bigcup_{r\in\mathbb{N}} B_C(x_r, \frac{1}{k}) \neq \emptyset$$

.

Lemma 3.9. *Suppose for each $\alpha \in \triangle,\ F_\alpha : \Omega \longrightarrow \mathcal{P}(E)$ is a measurable maps. Then, the intersection map*

$$F(\omega) = \bigcap_{\alpha\in\triangle} F_\alpha(\omega),\ for\ \omega \in \Omega.$$

is measurable.

Theorem 3.10. Assume that $(\Omega, \mathcal{A})$ is a measurable space, E is separable complete metric space and $F : \Omega \longrightarrow \mathcal{P}(E)$ is a set-valued random variable. Then F has a measurable selection.

Proof: Let $D = \{x_1, x_2, \dots\}$ be a countable dense subset of E and $B_c(x_r, \frac{1}{k+1})$ be a closed ball centred at x_r and radius $\frac{1}{k+1}$. By lemma 3.8 we have

$$F(\omega) \cap \bigcup_{r\in\mathbb{N}} B_c\left(x_r, \frac{1}{k+1}\right) \neq \emptyset \tag{3.1}$$

for each fixed $k \in \mathbb{N}$. This implies that there exists some $r \in \mathbb{N}$ such that

$$F(\omega) \cap B_c\left(x_r, \frac{1}{k+1}\right) \neq \emptyset. \tag{3.2}$$

Now, set

$$r_k(\omega) = \min\{r \in \mathbb{N} : F(\omega) \cap B_c\left(x_r, \frac{1}{k+1}\right) \neq \emptyset\}. \tag{3.3}$$

We define a decreasing sequence of set-valued maps as

$$F_0 = F \ and \ F_{k+1}(\omega) = F_k(\omega) \cap B_c\left(x_{r_k(\omega)}, \frac{1}{k+1}\right). \tag{3.4}$$

we obtained this by induction. So, for each $\omega \in \Omega$, we have

$$F_{k+1}(\omega) \subset F_k(\omega),\ diam(F_k(\omega)) \leq \frac{1}{k} \longrightarrow 0 \ as \ n \longrightarrow \infty \ and \ F_k(\omega) \in \mathcal{P}(E)$$

Then, by lemma 3.7, $\bigcap_{k=1}^{\infty} F_k(\omega)$ contains a single element. So, we define $\{f(\omega)\} = \bigcap_{k=1}^{\infty} F_k(\omega) \subset F(\omega)$. Hence, f is a selection of F.

Next, we show that f is measurable. To do this, we need to show that F_k is measurable for each $k \in \mathbb{N}$. Using induction, we prove the measurability of F_k as follows: $F_0 = F$ is measurable by assumption. Assume F_k is measurable for some $k \in \mathbb{N}$ and $A \subset \mathcal{P}(E)$. Then, we show that F_{k+1} is measurable. But

$$\begin{aligned}\{\omega \in \Omega : F_{k+1}(\omega) \cap A \neq \emptyset\} &= \{\omega \in \Omega : F_k(\omega) \cap B_c(x_{r_k(\omega)}, \frac{1}{k+1}) \cap A \neq \emptyset\} \\ &= \bigcup_{k \in \mathbb{N}} [\{\omega \in \Omega : F_k(\omega) \cap B_c(x_{r_k(\omega)}, \frac{1}{k+1}) \cap A \neq \emptyset\} \\ &\qquad \cap \{\omega \in \Omega : r_k(\omega) = r\}]\end{aligned}$$

Since, F_k is measurable for some $k \in \mathbb{N}$ by the induction assumption above, we have

$$\{\omega \in \Omega : F_k(\omega) \cap B_c(x_{r_k(\omega)}, \frac{1}{k+1}) \cap A \neq \emptyset\} \in \mathcal{A}$$

Furthermore,

$$\begin{aligned}\{\omega \in \Omega : r_k(\omega) = r\} = \bigcap_{i=1}^{r-1} \{\omega \in \Omega : F_k(\omega) \cap B_c(x_{r_i(\omega)}, \frac{1}{k+1}) = \emptyset\} \cap \\ \{\omega \in \Omega : F_k(\omega) \cap B_c(x_{r_k(\omega)}, \frac{1}{k+1}) \neq \emptyset\}\end{aligned}$$

This implies that $\{\omega \in \Omega : r_k(\omega) = r\} \in \mathcal{A}$. Therefore,

$$\{\omega \in \Omega : F_{k+1}(\omega) \cap A \neq \emptyset\} \in \mathcal{A}$$

Hence, for each $k \in \mathbb{N}$, F_k is measurable.

Now, we show that f is measurable. Define $G(\omega) = \{f(\omega)\}$. By the measurability of intersection in lemma 3.9, measurability of F_k for each $k \in \mathbb{N}$ and using the fact that $G(\omega) = \bigcap_{k \in \mathbb{N}} F_k(\omega)$ we conclude that G is measurable. Whence, for $A \in \mathcal{P}(E)$ we have

$$\begin{aligned} G^{-1}(A) &= \{\omega \in \Omega : G(\omega) \cap A \neq \emptyset\} \\ &= \{\omega \in \Omega : \{f(\omega)\} \cap A \neq \emptyset\} \\ &= \{\omega \in \Omega : \{f(\omega)\} \in A \neq \emptyset\} \\ &= f^{-1}(A) \in \mathcal{A} \end{aligned}$$

Hence, f is measurable. ■

Theorem 3.11. Suppose $(\Omega, \mathcal{A})$ is a measurable space, E is a separable complete metric space and $F : \Omega \longrightarrow \mathcal{P}(E)$ is a set-valued random variable if and only if there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of measurable selections of F such that

$$F(\omega) = cl\{f_n(\omega)\}_{n \geq 1} \ \forall \ \omega \in \Omega \tag{3.5}$$

Proof: Assume F is a set-valued random variables. Let $D = \{x_n : n \in \mathbb{N}\}$ be a countable dense subset of E and $\{B_c(x_n, 2^{-k}) : (n,k) \in \mathbb{N}^2\}$ be a countable family of closed balls of radius 2^{-k} centred at x_n. For each fixed $(n,k) \in \mathbb{N}^2$, the set

$$A_{n,k} = \{\omega \in \Omega : F(\omega) \cap B_c(x_n, 2^{-k}) \neq \emptyset\} \in \mathcal{A}.$$

Since $A_{n,k}$ is the inverse image of a closed ball $B_c(x_n, 2^{-k})$, it follows from theorem 3.5(a) that $A_{n,k}$ is measurable. Now, we define a collection of closed set-valued mapping as follows:

$$F_{n,k}(\omega) = \begin{cases} F(\omega) \cap B_c(x_n, 2^{-k}), \ \ if \ \omega \in A_{n,k} \\ F(\omega) \qquad\qquad\qquad if \ \omega \notin A_{n,k}. \end{cases} \tag{3.6}$$

At this moment, we observe that for each $(n,k) \in \mathbb{N}^2$, $F_{n,k}$ is a set-valued random variables, since for any closed subset B of E, we have

$$F_{n,k}^{-1}(\omega) = \{\omega \in \Omega : F_{n,k}(\omega) \cap B_c(x_n, 2^{-k}) \neq \emptyset\}$$
$$= \{\omega \in \Omega : F(\omega) \cap (B \cap B_c(x_n, 2^{-k})) \neq \emptyset\}$$
$$\cup \{\omega \in \Omega : F(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$$

Since $\{\omega \in \Omega : F(\omega) \cap (B \cap B_c(x_n, 2^{-k})) \neq \emptyset\} \in \mathcal{A}$ and $\{\omega \in \Omega : F(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$. Hence, $F_{n,k}$ is measurable for each $(n,k) \in \mathbb{N}^2$. Applying theorem 3.10, we obtain for each $(n,k) \in \mathbb{N}^2$, there exists a measurable selection $f_{n,k}$ of $F_{n,k}$. We now show that $\{f_{n,k}(\omega) : (n,k) \in \mathbb{N}^2\}$ is dense in $F(\omega)$ for each $\omega \in \Omega$ i.e $F(\omega) = cl\{f_{n,k}(\omega) : (n,k) \in \mathbb{N}^2\}$. Assume $x \in F(\omega)$. We show that there exist a sequence $\{f_{n,k}(\omega)\}_{n \geq 1}$ such that $f_{n,k}$ converges to x as $k \longrightarrow \infty$ *and* $n \longrightarrow \infty$. But,

$$x \in F(\omega) \Rightarrow \ for\ each\ k \geq 1, \exists x_n\ s.t\ d(x_n, x) < 2^{-k}$$

This implies that $\omega \in A_{n,k}$. Thus $f_{n,k}(\omega) \in B_c(x_n, 2^{-k})$. This follows from (2.8) above. So, this gives us that $d(x_n, f_{n,k}(\omega)) < 2^{-k}$. By the triangle inequality, we obtain

$$d(x, f_{n,k}(\omega)) \leq d(x, x_n) + d(x_n, f_{n,k}(\omega))$$
$$< 2^{-k} + 2^{-k} = 2^{-k+1}$$

This implies that $f_{n,k}(\omega) \longrightarrow x\ as\ k \longrightarrow \infty,\ n \longrightarrow \infty$. Hence, $F(\omega) = cl\{f_{n,k}(\omega) : (n,k) \in \mathbb{N}^2\}$.

Conversely, assume there exist a countable family $\{f_n : n \in \mathbb{N}\}$ of measurable selections of F such that $F(\omega) = cl\{f_n(\omega) : n \in \mathbb{N}\}$ for all $\omega \in \Omega$. Since for any given $x \in E$, $d(x, f_n(\omega))$ is measurable for each $n \in \mathbb{N}$,

$$d(x, F(\omega)) = \inf\{d(x, f_n)(\omega) : n \in \mathbb{N}\}$$

is measurable. Thus, by applying theorem 3.5, we conclude that F is a set-valued random variables. ■

Remark 3.12. *These two theorems 3.10 and 3.11 are very important in the study of set-valued integration. Theorem 3.10 gives the condition for the existence of a measurable* *selection of a set-valued random variable. In theorem 3.11 we obtain a characterization of F in terms of sequence of selections for each $\omega \in \Omega$.*

(A) Other properties of set-valued random variables.

Having established these two important properties of set-valued random varibles, we are going to state and fully describe other important properties without proofs.

Before we proceed let us give some definitions. Let F and G be two set-valued functions and ψ a real-valued random variable. Define :

(a) the **addition of F and G** by

$$\forall \omega \in \Omega,\ (F \oplus G = \overline{F(\omega) + G(\omega)},$$

(b) the **multiplication of F by ψ** by

$$\forall \omega \in \Omega,\ (\psi F)(\omega) = \psi(\omega) F(\omega).$$

(c) the **closed convex hull of F** by

$$\forall \omega \in \Omega,\ (\overline{co} F)(\omega) = \overline{co} F(\omega).$$

We have

Proposition 3.13. *Let $(\Omega, \mathcal{A})$ be a measurable space, (E, d) be a metric space endowed with its Borel σ-algebra $\mathcal{B}_E$, $F, G : (\Omega, \mathcal{A}) \to \mathcal{P}(E)$ be two set-valued random variables, ψ be a real-valued random variable. Let $(F_n)_{n\geq 1}$ be a sequence of set-valued random variables. The following assertions hold.*

(a) The set-valued mapping

$$\Omega \ni \omega \mapsto cl\left(\bigcup_{n\geq 1} F_n\right)$$

is measurable.

(b) If E is a complete metric space, then the graph

$$G\left(\bigcap_{n\geq 1} F_n\right)$$

is measurable, in particular

$$\{\omega \in \Omega : \bigcap_{n\geq 1} F_n = \emptyset\}.$$

If A is not empty and if the projection is perfect, $\bigcap_{n\geq 1} F_n$, $s-\liminf F_n$ and $s-\limsup F_n$ are measurable set-valued random variables.

(c) If E is a separable Banach space. Then,

(i) the real-valued mapping

$$\omega \mapsto H(F(\omega), G(\omega))$$

$$\omega \mapsto d(x, (F(\omega)), \ x \in E \ fixed$$

and

$$\omega \mapsto S_f(F(\omega)), \ f \in E^* fixed$$

are measurable.

(ii) $F \oplus G$, ψF and $\overline{co} F$ are set-valued random variables.

(d) If E is a separable Banach space, if the projection π_1 is perfect, if E is reflexive or E^* is separable, then $w - \limsup F_n$ is seperable.

We close this section here. we introduce the integration part devoted to Bochner integrals in the next chapter.

CHAPTER 4

Introduction to the Bochner Integrals

1 Introduction

This chapter focuses on two forms of Bochner Integral namely: the Bochner integral of Banach-valued functions and the Bochner integral of set-valued random varables.

2 The Bochner Integral of Banach valued functions

This integral extends the Lebesgue integral to functions that take value in a Banach space. It is a very important concept in the study of set-valued integration for two reasons.

(a) The first kind of integral for set-valued mappings, the Debreu-Bochner integral, is a direct generalization of it.

(b) The second kind, the Auman integral which is most frequently used, give set-valued integrals whose elements are Bochner integral.

Based on these two reasons, we feel it is necessary to give a detailed account of these integrals in this thesis. In the second section, we will only describe the Debreu-Bochner Integral for set-valued random variable and let future students go deep in it.

Note: Observe that we have spent time in chapter 3 defining the inverse image of both the point-valued functions and set-valued functions but we have not used it. Here, we will use the inverse images of point-valued functions in this section and reserve that of set-valued functions for the next section.

(A) - Construction of the Bochner Integral

We are going to try to extend the construction of the classical integral for real-valued function with respect to some measure μ to an arbitrary Banach-valued functions. Although, we don't have all the properties as in $\mathbb{R}$, for instance the stunning property that any measurable and non-negative function is a limit of a non-decreasing sequence of simple functions. But we will adopt the whole theory.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and E be a real Banach space endowed with its Borel σ-algebra $\mathcal{B}_E$. We are going to deal with measurable functions of the form

$$f : (\Omega, \mathcal{A}) \to (E, \mathcal{B}_E). \tag{2.1}$$

We say that f is a **simple or elementary (measurable) functions** if

$$\forall \omega \in \Omega,\ f(\omega) = \sum_{j=1}^{p} \alpha_j 1_{A_j}(\omega) \tag{2.2}$$

where p is a finite integer, $\{\alpha_1, \alpha_2, \dots, \alpha_p\} \subset E$ are the values of f, and $\{A_1, A_2, \dots, A_p\}$ form a measurable partition of Ω.

We can also write f in its **canonical form** as

$$\forall 1 \leq j \leq p, A_j = (f = \alpha_j).$$

whenever α_i are distinct. Also, if the union of the A_j is not Ω, we assume that $f = 0$ outside that union.

Indeed, the simple function f is measurable since for all $B \in \mathcal{B}_E$,

$$\begin{aligned} f^{-1}(B) &= \sum_{i=1}^{p} f^{-1}(B) \bigcap A_i \\ &= \sum_{i=1}^{p} A_i \cap (f \in B) \\ &= \sum_{j=1}^{k} A_i \in \mathcal{A}, \quad \textit{where } 1 \leq j \leq k \leq p. \end{aligned}$$

Let us denote by $\mathcal{S}(\Omega, E)$ the class of simple functions defined from Ω to E.

Let us define the *pseudo-absolute function* $|f|$ of f by

$$\forall\ \omega \in \Omega,\ \ |f|(\omega) = \|f(\omega)\|.$$

It is clear from Formula 2.2 that we have

$$\forall\ \omega \in \Omega,\ \ |f|(\omega) = \sum_{j=1}^{p} \|\alpha_j\| 1_{A_j}(\omega)$$

and that $|f|$ is a real-valued random variable. We denote

$$\|f\|_1 = \int |f|\ d\mu.$$

Definition-Theorem. If a simple function admits two expressions

$$\forall\ \omega \in \Omega,\ \ f(\omega) = \sum_{i=1}^{p} \alpha_h 1_{A_h}(\omega),\ \ (SFA)$$

and

$$\forall\ \omega \in \Omega,\ \ f(\omega) = \sum_{i=1}^{p} b_k 1_{B_k}(\omega),$$

with the usual precisions, we have

$$\sum_{h=1}^{p} \alpha_h \mu(A_h) = \sum_{k=1}^{q} b_k \mu(B_k).$$

From Formula (SFA), we define the **Bochner integral** of the simple function f,

$$\int_{\Omega} f\ d\mu = \sum_{j=1}^{p} \alpha_j \mu(A_j),\ \ (BS01)\ \Diamond$$

NOTE: The value of this integral depends on $\alpha_j' s$. It coincides with Lebesgue integral when $\alpha_j' s$ are real number.

Proposition 4.1. *For any positive integer p, for $\{a_1, a_2, \ldots, a_p\} \subset E$, and for $\{A_1, A_2, \ldots, A_p\} \subset \mathcal{B}_E$, we have*

$$\|\sum_{j=1}^{p} a_j \mu(A_j)\| \leq \int_{\Omega} \left| \sum_{j=1}^{p} a_j 1_{A_j} \right| d\mu$$

Proof: We recall that

$$\sum_{i=1}^{p} a_i 1_{A_i} = \sum_{j=1}^{k} b_j 1_{B_j}$$

for some $\{b_1, b_2, \dots, b_k\} \subset E$, and some disjoint $\{B_1, B_2, \dots, B_k\} \subset \mathcal{B}_E$.

$$\begin{aligned}
\|\sum_{i=1}^{p} a_i \mu(A_i)\| &= \|\sum_{j=1}^{k} b_j \mu(B_j)\| \\
&\leq \sum_{j=1}^{k} \|b_j \mu(B_j)\| \\
&= \sum_{j=1}^{k} \|b_j\| \mu(B_j) \\
&= \int_\Omega \sum_{j=1}^{k} \|b_j\| 1_{B_j} d\mu \\
&= \int_\Omega |\sum_{j=1}^{k} b_j 1_{B_j}| d\mu \\
&= \int_\Omega |\sum_{i=1}^{p} a_i 1_{A_i}| d\mu.
\end{aligned}$$

Now, we want to contruct the class of Bochner-integrable functions.

Definition - Theorem. A measurable mapping $f : \Omega \to E$ is Bochner integrable if and only if : (a) There exists a sequence of simple functions $(f_n)_{n \geq 1}$ such that

$$\sum_{n \geq 1} \|f_n\|_1 < +\infty \qquad (01)$$

and, by denoting

$$\Omega_0 = \{\omega \in \Omega,\ \sum_{n \geq 1} \|f_n(\omega)\| < +\infty\},$$

we have

$$\forall \omega \in \Omega_0,\ f(\omega) = \sum_{n \geq 1} f_n(\omega) \qquad (02)$$

(b) If both (01) and (02) hold, we write for short

$$f \simeq \sum_{n \geq 1} f_n,$$

the **Bochner integral** of f as defined by

$$\int f \, d\mu = \lim_{n \to +\infty} \sum_{k=1}^{n} \Big(\int f_k \, d\mu \Big).$$

This definition is not complete unless we prove that it does not depend on the sequence $(f_n)_{n\geq 1}$. This will come as a consequence of the properties below.

To help avoid the confusion between real integrals and Banach-valued integrals, we will write the space of integrable functions as $\mathcal{L}^1(\Omega, \mu, \mathbb{R})$ and $\mathcal{L}^1(\Omega, \mu, E)$ respectively.

(II) - Immediate properties

Theorem 4.2. Suppose that $f \simeq \sum_{n\geq 1} f_n$, then :

(a) $|f| \in \mathcal{L}^1(\Omega, \mu, \mathbb{R})$,
(b) we have

$$\int |f| \, d\mu = \lim_{n \to +\infty} \int \Big(\sum_{k=1}^{n} |f_k| \Big) \, d\mu.$$

and

(c) we have

$$\| \sum_{n\geq 1} \int f_n \, d\mu \| \leq \int |f| \, d\mu.$$

Proposition 4.3. *Suppose that* $f \in \sum_{n\geq 1} f_n$ *and* $f \in \sum_{n\geq 1} g_n$, *then*

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \Big(\int f_k \, d\mu \Big) = \lim_{n \to +\infty} \sum_{k=1}^{n} \Big(\int f_g \, d\mu \Big)$$

Theorem 4.4. The following assertions hold.

(1) The integral is a linear operator from $\mathcal{L}^1(\Omega, \mu, E)$ to E and

$$\forall f \in \mathcal{L}^1(\Omega, \mu, E), \ \int f \, d\mu \leq \int |f| \, d\mu.$$

If G is a bounded linear function on $\mathcal{L}^1(\Omega, \mu, E)$, we have

$$\forall f \in \mathcal{L}^1(\Omega, \mu, E),\ G \int f\ d\mu = \int Gf\ d\mu.$$

The Banach Space $L^1(\Omega, \mu, E)$.

As in the real-valued case, we define $L^1(\Omega, \mu, E)$ as the quotient of $\mathcal{L}^1(\Omega, \mu, E)$ by the class of null events. We obtain

Theorem 4.5. $L^1(\Omega, \mu, E)$ is a Banach space.

This integral has many interesting properties we will review later.

3 An Incursion to the Bochner Integral of Set-valued functions (Debreu-Bochner Integral)

By following the ideas in the first section, we may generalize the construction of integrals of set-valued random variables.
We suppose we are given a measure space $(\Omega, \mathcal{A}, \mu)$, E a real Banach space endowed with its Borel σ-algebra $\mathcal{B}(E)$, and $F : (\Omega, \mathcal{A}) \to \mathcal{P}(E)$ a set-valued random variable.

The Debreu integral is constructed in two steps.
Step 1. We define a simple measurable real-valued mapping by

$$\forall \omega \in \Omega,\ F(\omega) = \sum_{j=1}^{p} B_j 1_{A_j}(\omega) \tag{3.1}$$

where p is a finite integer, $\{B_1, B_2, \ldots, B_p\} \subset \mathcal{P}(E)$ and $\{A_1, A_2, \ldots, A_p\}$ form a measurable partition of Ω.

We may still write F in such a way that x_i's are distinct so that we have

$$\forall 1 \leq j \leq p, A_j = (F \in B_j).$$

Such a set-valued random variable is measurable since for all $B \in \tau_o$

$$\begin{aligned} F^{-1}(B) &= \sum_{j=1}^{p} F^{-1}(B) \bigcap A_j \\ &= \sum_{j=1}^{p} \{\omega \in \Omega,\ F(\omega) \cap B \neq \emptyset\} \bigcap A_j \\ &= \sum_{j=1}^{p} \{\omega \in \Omega,\ B_j \cap B \neq \emptyset\} \bigcap A_j \\ &= \sum_{j=1}^{p} C_j \in \mathcal{A}. \end{aligned}$$

where $C_j = \{\omega \in \Omega : B_j \cap B \neq \emptyset\} \cap A_j\}$.

Let us remark that in passing, the closedness of B does not play a role in the measurability of F.

Let us denote by $\mathcal{S}(\Omega, \mathcal{P}(E))$ the class of simple set-valued mappings.

Definition 4.6. *The Debreu-Bochner integral of a simple set-valued function of the form (SF01) is given and denoted by*

$$(B) - \int_{\Omega} F d\mu = \sum_{j=1}^{p} \mu(A_j) B_j,$$

which is a closed set and is independent of the representation (SF01).

Step 2. For two set-valued random variables F and G, let us

$$\Delta(F, G) = \int H(F(\omega), G(\omega)) d\mu(\omega).$$

We point out that we have already proved the measurability of the real-valued mapping

$$\Omega \ni \omega \mapsto H(F(\omega), G(\omega)),$$

when E is a separable Banach space. It may be quickly proved that Δ is a metric on the class of set-valued random variables.

Definition 4.7. *The Debreu-Bochner integral of a set-valued random variable exists if and only if there exists a sequence of simple set-valued random variables* $(F)_{n\geq 1}$ *such that* $\Delta(F_n, F) \to 0$ *as* $n \to +\infty$ *and it is defined as*

$$(B) - \int_{\Omega} F d\mu = \lim_{n\to+\infty} (B) - \int_{\Omega} F_n d\mu.$$

CHAPTER 5

Perspectives and Conclusion

1 Perspectives

So far, we have shown many mathematical results. To mathematicians these results are their interest and this mathematical interests justify our research. But, if these results were to remain abstract, far from practical mathematical concepts then our reward will only be this mathematical interests. However, our results are not only about the abstract mathematical concepts; these results are about set-valued random variable which is a natural concept from probability theory. Li *et al.* (2002) pointed out the several applications of set-valued random variables in science and engineering. In the future , it is my ultimate desire to undertake research activities on limit theorems on set-valued random variables and its applications. Examples of such application includes image processing, artificial intelligence, optimization, robust controls etc.

2 Conclusion

This thesis provides the fundamental tools required in the study of set-valued random variables. The Hausdorff metric, its properties and the four convergences related to the Hausdorff metric were covered. This gives a good background to the study of measurability of set-valued functions and its Bochner integrals. Bochner Integrals, an extension of the Lebesgue integral to both Banach-valued functions and the set-valued functions were expounded. Similarly, the role of the two forms of Bochner integrals namely: Banach valued Bochner integral and the set-valued Bochner integral otherwise known as the Debreu-Bochner integral in set-valued integrations were emphasised.

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