CONTROL OF NON LINEAR OSCILLATIONS IN PLASMA GOVERNED BY A VAN DER POL EQUATION

MASTER SCIENCE PROGRAMME THESIS

Option: THEORETICAL PHYSICS

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Dedication

To God, the all Mighty for all the strength and Mercy you allow me everyday.
To my father GBEDO Hounsou Pierre and my mother EBO Omonlacho Victorine.
To my wife AGONGLOVI Adjo Josiane for her patient love and my son GBEDO Tadagbe Junias.
To my sisters and brothers HOUNSOU D. Guy, QUENUM P. B. Kevin , Astride , Elvire , GBEDO Immaculee , Virgile , Marius ,Euloge , Charlotte and Alexandre.
May you find through this work some real motive of satisfaction of your multiple sacrifies and councils.

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Abstract

This thesis deals with the control of non linear oscillations in plasma governed by a classical Van der Pol equation . The main interest devoted to such an investigation is that non linear oscillations in plasma is essential in industry.
In chapter 1 , we present some generality on the dynamical systems .
Chapter 2 is focussed on the analytical study of the Van der Pol equation in autonomous regime . The amplitude and the phase of the stable limit cycle are derived using the Averaging Method .
In chapter 3 , we investigate the oscillations in plasma .

Introduction

The dynamical systems constitute a very large field of science [1, 2, 3, 4, 5]. They are generally represented by a non linear equations . A phenomenon is non linear when his evolution don't obey to a linear Mathematic law . The non linear electric oscillators are those whom the one of the constitutive elements to the characteritics intensity - tension is of the non linear form . We can mention (diode , transistor , operational amplifier , etc) The presence of the non linear components is in the beginning of many observed phenomena .We can also mention (hystheresis , resonance [6, 7]) Another important phenomenon resulting of the presence of non linear components is the apparition of chaos , curious phenomenon , rich and complex met almost in all branches of instruction :electronic , astronomy , biology , chemistry , economy , etc Particularly in electronic , it is known the works of a Dutch electrical engineer Balthazar Van der Pol on an oscillator presenting a various mode[8] whom equation is :
$\ddot{x} + x - \mu(1 - x^2)\dot{x} = 0$
A classical Van der Pol oscillator is an example of the most studied self-maintained oscillators. Since his introduction in 1922 , he had the object of many studies ; firstly by Van der Pol himself who has established a good part of his own research on the experimental and theoretical analysis of that oscillator in the electrical circuits , secondly by many others searchers. This dissertation is composed of three chapters .
In chapter 1 , the generality on the dynamical system is presented .
In chapter 2 , we investigate the analytical study case of the Van der Pol equation in the autonomous regime . Here , we derive the amplitude and the phase of the stable limit cycle in the autonomous regime .
Chapter 3 deals essentially with the oscillations in a plasma .

GENERALITY ON THE DYNAMICAL SYSTEMS

The dynamical systems constitute a vast field for science and study . Thus , their study is very important and it generates some interesting phenomena. And then , we will study some of the dynamical systems which are involved along the work.

1.1 Definition

• A dynamical system is one of which the state is described by a vector $\vec{x}$ with n components and of which the evolution is governed by a simple differential equation of the type :

$$\dot{\vec{x}} = \overrightarrow{F}(\vec{x}) \tag{1.1}$$

For example, the Hamilton system is a dynamical system . In fact , in compact notation , we have:

$$\dot{\vec{x}} = J\left(\frac{\partial H}{\partial \vec{x}}\right) = \overrightarrow{F}(\vec{x}). \tag{1.2}$$

with

$F_i(\vec{x}) = \sum_{j=1}^{+\infty} J_{ij} \frac{\partial H}{\partial x_j}$

and

$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

Where J is the Jacobian matrix and I the unit matrix of n order.

• The solution $\vec{x}(t)$ such that $\vec{x}(t=0) = \overrightarrow{x_0}$ is called a flood and it is written down like this: $\overrightarrow{\Phi}_t(\overrightarrow{x_0})$

• A fix point (or equilibrium point or singular point) of a system described by the equation(1.1) is a $\overrightarrow{x_e}$ which is solution of the equation:

$\overrightarrow{F}(\overrightarrow{x_e}) = \overrightarrow{0}$

A fix point $\overrightarrow{x_e}$ is called well if all the values of the Jacobian matrix of the flood linearized around this fix point have their real part negative.

Similarly, if all the values of the Jacobian matrix have their real part positive , then the fix point is called source.

• A phase space can help define the state of a system by associating its coordinates such as its position and its speed.The trajectory of the phase space is a curve of the same space representing an evolution of the system . A set of trajectories constitutes a portrait of phase.
• An autonomous system is a system in which time doesn't intervene in the equation of motion explicitly , i.e the system of the form for which independent variable doesn't appears explicitly $\ddot{x} = f(x, \dot{x})$. On the contrary , we have to deal with a forced system.
• The self-maintained oscillations are oscillations in which the lost energy is recorvered during the following cycle in order to maintain these oscillations.
• Hysteresis is a phenomenon during which there is a jump from a great amplitude to a smaller one for a solution and this , vice-versa.
• A system to approach a periodic behavior which will thus appear as a closed curve in phase space is called a limit cycle .
• A limit cycle is a closed orbit in the phase space such that no other closed orbit can be found arbitrary close to it . It's a characteristic for a periodic regime.
• A close trajectory C of a dynamical system which has nearby open trajectories spiraling towards it both from inside and outside as $t \to \infty$ is called stable limit cycle .
• If they spiral towards it from one side and spiral out from the other side , it is semi - stable limit cycle .
• If nearby open trajectories spiral away from C on both side the C is unstable limit cycle .
• If nearby trajectories neither approach nor recede from C, it is Neutrally-stable limit cycle.
• An attractor is an invariable set towards which all the trajectories of the dynamical system are turned and by which they are attracted. It's included in a field of an existent volume which constitutes its attracting pool . Thus , we can have : single attractors (punctual attractors, periodic attractors , biperiodic attractors or quasi-periodic attractors) and the strange attractors (non periodic attractants , fractal attractants , chaotic attractants). The attracting point is a single point corresponding to a stationary solution of the equation of motion. The periodic behaviour is associated with a single steady attractor called limit cycle which is characterized by its amplitude and period. The third type of single attractor is the tore $T^r(r \geq 2)$ and it corresponds to a quasi-periodic regime having r frequences of independent basis. The strange attractors characterised a chaotic movement.
• The attracting pool of an attractor is the location(setting) of the phase space formed by the set of initial conditions and from which this attractor is obtained.

1.2 NOTION OF STABILITY

The notion of stability is very important and fundamental in the study case of any system.A fix point must satisfy certain criteria among which the one of stability. For a state to be observable,it must be stable,i.e this state must find an initial state after being subject to a perturbation.Now , we're going to consider a fix point$\overrightarrow{X_0}$ belonging to a system of which the evolution is generated by the following equation:

$$\dot{\overrightarrow{X}} = \overrightarrow{F}(\overrightarrow{X}) \tag{1.3}$$

This point verifies : $\dot{\overrightarrow{X_0}} = \overrightarrow{F}(\overrightarrow{X_0}) = \overrightarrow{0}$. There are two different approaches (global and local) for the study case of the stability of the fix point.As far as the global approach is concerned, let's say that there is no condition for the form and the amplitude of the perturbations ; whereas the local approach is limited to infinitesimal perturbations.

1.2.1 GLOBAL APPROACH

The concept of stability used in this study case is the one of A.M.LYAPUNOV.Thus,we speak of LYAPUNOV's notion of stability.The Mathematician A.M.LYAPUNOV in his PhD thesis in 1892,found an interesting criterion allowing the study of stability [9] . It's a generalization according to which, for a well,there is a norm for $\Re^n$ so as $\|\overrightarrow{X} - \overrightarrow{X_0}\|$ decreases for the neighbourhood solutions of $\overrightarrow{X_0}$ .LYAPUNOV showed that a number of a functions would be used as guaranty of stability in the place of the norm [9] . These are called the functions of LYAPUNOV (V).The functions of LYAPUNOV are defined as positive ones. The following theorem shows the stability conditions of $\overrightarrow{X_0}$:

• THEOREM:(STABILITY CONDITIONS OF LYAPUNOV [9])

Suppose that $\overrightarrow{X_0}$ is a fix point of the equation(1.2) and (V) a positive function of the class C^1 defined on the neighbourhood Q of $\overrightarrow{X_0}$:

(i) if $\dot{V}(\overrightarrow{X}) \leq 0$ for $\overrightarrow{X} \in Q - \{0\}$, then $\overrightarrow{X_0}$ is stable.

(ii) if $\dot{V}(\vec{X}) < 0$ for $\overrightarrow{X} \in Q - \{0\}$, then $\overrightarrow{X_0}$ is asymptotically stable.

(iii) if $\dot{V}(\overrightarrow{X}) > 0$ for $\overrightarrow{X} \in Q - \{0\}$, then $\overrightarrow{X_0}$ is unstable.

There is no general rule to determine the function of LYAPUNOV. So , we often use the local approach in order to study the stability of a fix point.

1.2.2 LOCAL APPROACH

Now ,we are going to consider the evolution of the infinitesimal perturbations . When the perturbation $\overrightarrow{X'}$ intervenes around the fix solution $\overrightarrow{X_0}$, we have:

$$\overrightarrow{X} = \overrightarrow{X_0} + \overrightarrow{X'}. \tag{1.4}$$

From the combination of the equations (1.2) and (1.3),we have the following variational equation of the linear regime:

$$\dot{\overrightarrow{X'}} = \frac{D\overrightarrow{F}}{D\overrightarrow{X}}(\overrightarrow{X_0})\overrightarrow{X'} + ... \tag{1.5}$$

Then we have the equation of evolution of $\overrightarrow{X}$:

$$\dot{\overrightarrow{X'}} = A\overrightarrow{X'}. \tag{1.6}$$

With
$A = \frac{D\overrightarrow{F}}{D\overrightarrow{X}}(\overrightarrow{X_0})$
where the elements A_{ij} of the A matrix are defined by:
$A_{ij} = \frac{\partial F_i}{\partial X_j}(\overrightarrow{X_0})$.
The solution to this equation is:

$$\overrightarrow{X'}(t) = \exp(At)\overrightarrow{X'}(0). \quad (1.7)$$

Suppose that the A matrix is diagonalizable and let's consider $a_j, j = 1, 2, 3, ..., n$, its eigenvalues which are complex numbers , and $\overrightarrow{y_j}, j = 1, 2, ..., n$,its associated eigenvectors. Its general solution in C^n is:

$$\vec{y} = \sum_{j=1}^{n} C_j(0)\exp(a_j t)\overrightarrow{y_j}. \quad (1.8)$$

As A is a real matrix , if $\vec{y}$ is solution , the real part $R_e(\vec{y})$ of $\vec{y}$ is also a solution. Then the general solution in $\Re^n$ is:

$$\overrightarrow{X'} = R_e\left(\sum_{j=1}^{n} C_j(0)\exp(a_j t)\overrightarrow{y_j}\right) \quad (1.9)$$

We can notice that the real part of a_j is equivalent to the increasing rate of the perturbation. So the stability depends on the signs of the real parts of a_j.Then , we can conclude:
• If $R_e(a_j) < 0$, for any j , the perturbation decreases , the fix point $\overrightarrow{X_0}$ is asymptotically stable.
• If $R_e(a_j) > 0$, for at least a value of j , the perturbation increases , $\overrightarrow{X_0}$ is unstable.
• If $R_e(a_j) = 0$, the perturbation neither increases nor decreases , $\overrightarrow{X_0}$ is neutral or marginal.

1.3 The concept of bifurcation

The study of the stability of the dynamical system has showed that the nature of the fix points depends on the control parameter of the system (for example , we have the control of non linearity for the Van der Pol oscillator) . According to the control parameter value , the dynamic of the system has proved to be more or less complexe . It may be an fix point attractor , a periodic attractor or a quasi - periodic attractor , a chaotic regime characterized by a strange attractor . The objectives of the bifurcation theory are to study the transitions between the different regimes when the control parameters are modified . When a solution changes qualitatively , we say that there is bifurcation . A point of the parameter space where such a phenomenon happens is defined as bifurcation point and from this point emerges many branches which are solutions ; they may be either stable or unstable . The notion of codimension is the number of parameter which is necessary to vary so as to get a stable situation , and this vice - versa . The representation of any property but characteristic of the solution(s) in function of bifurcation parameter constitute a bifurcation diagram . There are many types of bifurcation . For example , the bifurcation of types fix point $\rightarrow$ limit cycle and limit cycle $\rightarrow$ tore are called Hopf's bifurcations . In a physical system , a series of bifurcations provocated by the variation of the control parameter is called a jumping . The bifurcation diagram is very important in the study of the transition of systems from the order towards the chaos and this vice - versa .

1.4 Notion of chaos

It would be very difficult to define a chaos in general . A chaos is a persistent instability.Its main characteristics is the sensibility of the solutions with initial conditions , which is due to the non-linearity of the systems . A chaos was discovered in 1961 . Edward N. LORENTZ discovered the consequences to long period of time of slight variations of the initial conditions of numerical integration of a nonlinear differential equation . We often say that a chaotic system is a system without memory :The coming evolution of a flow is not predictable . Initially considered to be destroyable , a chaos has revealed to be useful with the discovering of its practical applications . We have for example , securing information as far as telecommunications are concerned . A regular chaotic regime can become an irregular chaotic one through many routes (by duplication of frequence , by quasi- periodicity) . The only way to determine the chaos are based on the numerical simulation . There are many litteral indicators of the chaos among which we have the Lyapunov exponent , which is the more reliable .

1.5 Notion of the Lyapunov exponent

The Lyapunov exponent is a concept of the SOVIET SCHOOL on the dynamical systems theory . Lyapunov's number gives a measure of the precise topological properties corresponding to notions such as stability or previsibility . The Lyapunov exponent of a system help determine the antagonistic effects of the extension and the contraction of an attractor in the phase space . This give an image of all the characteristics of a system conducting to stability or instability . An exponent greater than zero corresponds to an extension or to the separation of initially neighbouring points in the phase space . In this case , the system has a chaotic behaviour . On the contrary , an exponent less than zero corresponds to a contraction of initially neighbouring points , that is to say a contraction of points towards an oscillatory state : It's the periodicity or quasi - periodicity . In order to explain the Lyapunov exponent measurement , we consider a bidimensional flow defined by :

$$\frac{d\overrightarrow{\phi}}{dt} = \overrightarrow{F}(\overrightarrow{\phi}) \tag{1.10}$$

With $\overrightarrow{\phi}(t) = \left(X(t), V(t) = \frac{dX}{dt}\right)$
The linearization of the flow around its solution $\overrightarrow{\phi}(t)$ consists in this equation :

$$\overrightarrow{\phi_0}(t) = \overrightarrow{\phi}(t) + \delta\overrightarrow{\phi}(t) \tag{1.11}$$

With $\delta\overrightarrow{\phi}(t)$ being the perturbation .
Let's replace the expression(1.11) into the expression (1.10) , we have the generalized form of the variational equation of the flow as follows :

$$\frac{d[\delta\overrightarrow{\phi}]}{dt} = \frac{\partial\overrightarrow{F}(\overrightarrow{\phi})}{\partial\overrightarrow{\phi}}|_{\overrightarrow{\phi}(t)}\left(\delta\overrightarrow{\phi}(t)\right) \tag{1.12}$$

The matrix $\frac{\partial\overrightarrow{F}(\overrightarrow{\phi})}{\partial\overrightarrow{\phi}}|_{\overrightarrow{\phi}(t)}$ corresponds to the Jacobian matrix of the system .
If $\delta\overrightarrow{\phi_0}(t) = (\varepsilon_x(t), \varepsilon_v(t))$ is solution of the equation (1.12) by a period of time t , the Lyapunov exponent is defined by the following formular :

$$\lambda_{max} = \lim_{t\to+\infty}\frac{1}{t}\ln\|\delta\overrightarrow{\phi_0}(t)\| \tag{1.13}$$

Thus , we have :

$$\lambda_{max} = \lim_{t\to+\infty}\frac{1}{t}\ln\left(\varepsilon_x^2 + \varepsilon_v^2\right)^{\frac{1}{2}} \tag{1.14}$$

We also have the following equation which is used more often :

$$\lambda_{max} = \lim_{t\to+\infty}\frac{1}{t}\ln\left(|\varepsilon_x| + |\varepsilon_v|\right) \tag{1.15}$$

We can notice that we have asymptotically the following equation

$$|\varepsilon_x| + |\varepsilon_v| = \exp\left(\lambda_{max}t\right) \tag{1.16}$$

If $\lambda_{max} > 0$, the perturbation increases exponentially with time . This confirms the notion of extension inside the attractor in a chaotic regime . In the contrary ,
If $\lambda_{max} < 0$, $|\varepsilon_x|+|\varepsilon_v| \longrightarrow 0$, then we have a contraction in the phase space . This contraction conducts towards a fix point or towards a regular state .

1.5.1 Application to a logstic map

The Lyapunov exponent λ , of the Logistic map is used to obtain a measure of the very sensitive dependence upon initial conditions that is characteristic of chaotic behavior . Consider a general 1-dimensional map

$$X_{n+1} = f(X_n). \tag{1.17}$$

Let X_0 and Y_0 be two nearby initial points in the phase space and consider n iterations with the map to form

$$X_n = f^{(n)}(X_0) \tag{1.18}$$

$$Y_n = f^{(n)}(Y_0) \tag{1.19}$$

For a chaotic situation , nearby initial points will rapidly separate , while for a periodic solution the opposite will occur . Therefore assume , for large n , an approximatively exponential dependence on n of the separation distance ,

$$|X_n - Y_n| = |X_0 - Y_0| \exp(\lambda n) \tag{1.20}$$

with $\lambda > 0$ for the chaotic situation and $\lambda < 0$ for the periodic case . Taking n large (limit as $n \to \infty$) , λ can be extrated from (1.20)

$$\lambda = \lim_{n\to+\infty} \frac{1}{n} \ln \frac{|X_n - Y_n|}{|X_0 - Y_0|} \tag{1.21}$$

However , for trajectories confined to a bounded region such as in our logistic map , such exponential separation for the chaotic case cannot occur for very large n , unless the initial points X_0 and Y_0 are very close . Therefore , the limit $|X_0 - Y_0| \to 0$ must also be taken . Modifying (1.21), we have

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \lim_{|X_0-Y_0|\to 0} \ln \frac{|X_n - Y_n|}{|X_0 - Y_0|} \tag{1.22}$$

Substituting (1.18) and (1.19) into (1.22) we get

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \lim_{|X_0-Y_0|\to 0} \ln \frac{|f^{(n)}(X_0) - f^{(n)}(Y_0)|}{|X_0 - Y_0|} \tag{1.23}$$

Or

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln \frac{|df^{(n)}(X_0)|}{|dX_0|} \tag{1.24}$$

Now , $f(X_0) = X_1$, $f(X_1) = X_2$ or $f^{(2)}(X_0) = X_2$, so that for example

$$\frac{df^{(2)}(X_0)}{dX_0} = \frac{df(X_1)}{dX_1}\frac{dX_1}{dX_0} = \frac{df(X_1)}{dX_1}\frac{df(X_0)}{dX_0} \tag{1.25}$$

Generalizing (1.25) , we have

$$\frac{df^{(n)}(X_0)}{dX_0} = \prod_{k=0}^{n-1} \frac{df(X_k)}{dX_k} \tag{1.26}$$

and the Lyapunov exponent λ is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \frac{|df(X_k)|}{|dX_k|} \tag{1.27}$$

For periodic solutions , which starting point X_0 is chosen doesn't matter , but for chaotic trajectories , the precise value of λ will depend on X_0, i.e., in general $\lambda = \lambda(X_0)$. One can , if desired , define an average λ , averaged over all starting points . Whether this is done or not , $\lambda > 0$ should correspond to chaos , $\lambda < 0$ to periodic behavior . Figure shows λ (vertical axis) as a function of a for the logistic map for the starting point $X_0 = 0.2$. The figure was generated using the Mathematica File , MF38 .

ANALYTICAL STUDY OF THE VAN DER POL EQUATION IN THE AUTONOMOUS REGIME

In this chapter , we are going to describe the Van der Pol oscillator . With the link's law , we're going to establish the differential equation of Van der Pol . This differential equation will then be studied analytically in this chapter.

2.1 Descripion Van der Pol oscillator

The Van der Pol oscillator is a self-maintained electrical circuit made up of an inductor (L) , a capacitor initially charged with a capacitance (C) and of a non linear resistance (R) ; all of them connect up in series as indicated by the figure (2.1) below. This oscillator was invented by VAN DER POL while he was trying to find out a new way to modelize the oscillations of a self-maintained electrical circuit.Here is the characteristics intensity-tension of the non linear resistance (R):

$$U_R = -R_0 i_0 \left[\frac{i}{i_0} - \frac{1}{3} \left(\frac{i}{i_0} \right)^3 \right] \tag{2.1}$$

With: i_0 and R_0 being the current and the resistance of normalization , and this respectively. This non linear resistance can be obtained by using the operational amplifier(op-amp)[10]. By applying the link's law to the figure 2.1 , we have :

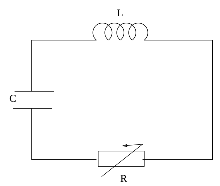

Figure 2.1: Electric circuit modelizing the Van der Pol oscillator in an autonomous regime.

$$U_L + U_R + U_C = 0 \tag{2.2}$$

with U_L being the tension to the limits of the inductor and U_C being the tension to the limits of the capacitor.

$$U_L = L\frac{di}{d\tau}$$
$$U_C = \frac{1}{C}\int id\tau \tag{2.3}$$

By replacing the equation (2.3) in the equation (2.2) ,we have :

$$L\frac{di}{d\tau} - R_0 i_0 \left[\frac{i}{i_0} - \frac{1}{3}\left(\frac{i}{i_0}\right)^3\right] + \frac{1}{C}\int id\tau = 0 \tag{2.4}$$

By deriving the equation (2.4) with respect to τ ,we have :

$$L\frac{d^2 i}{d\tau^2} - R_0\left(1 - \frac{i^2}{i_0^2}\right)\frac{di}{d\tau} + \frac{i}{C} = 0 \tag{2.5}$$

By processing the change of variables below:

$$x = \frac{i}{i_0} \tag{2.6}$$
$$t = \omega_e \tau \tag{2.7}$$

Where $\omega_e = \dfrac{1}{\sqrt{LC}}$ is an electric pulsation , we have:

$$\frac{d}{d\tau} = \frac{d}{dt}\frac{dt}{d\tau} = \omega_e \frac{d}{dt} \tag{2.8}$$
$$\frac{d^2}{d\tau^2} = \omega_e^2 \frac{d^2}{dt^2} \tag{2.9}$$

By replacing the expressions (2.8) and (2.9) in the equation (2.5) ,we have:

$$\frac{d^2 x}{dt^2} - R_0\sqrt{\frac{C}{L}}(1 - x^2)\frac{dx}{dt} + x = 0 \tag{2.10}$$

By setting $\mu = R_0\sqrt{\dfrac{C}{L}}$, the equation (2.10) take the adimensional form as follows :

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{2.11}$$

Where $\mu(\mu > 0)$ is the nonlinear control parameter for the different solutions of the equation. That equation is called the Van der Pol equation and it represents a paradigm in oscillations theory and nonlinear dynamics The classical Van der Pol oscillator as described by equation (2.11) has been subject to many studies,theoretical as well as experienced ones. From these studies ,we can notice that in the absence of any exterior excitation , the wave generated by this oscillator is periodic with sinusoidal form for the weak values of $\mu(\mu << 1)$;it is quasi-sinussoidal for the intermediate values of μ (for example: for the unit order) or relaxative for the large values of μ (μ>5)[11] with a fix amplitude equal to 2 , indicating the character of a self-maintained oscillator. Morever , we can deduce from those studies , the existence of a single stable limit cycle which is almost circular when the values of μ are very small . It's almost rectangular for the great values of μ . When the values of μ are very small , we can use this oscillator for the realization of continuous supplying , these continuous supplying being very stable . It's also valuable for the generation of an impulsion;the amplitude of which is very stable . The relaxative state of the oscillator is suited to the control of systems in which the entry impulsion generates a response of fix amplitude with ajustable frequence . This corresponds to the cardiac beats (or heart beats) where every contraction of the ventriculus is stimulated by the nervous impulsion on the contraction of the auriculus .

2.2 Analytical study

2.2.1 Fixed Points and Stability

The stationary state (x_0, y_0) are fix solutions for the equation (2.11) . So , by operating the following change of variable $\dot{x} = y$, we have this system as follows :

$$\begin{cases} \dot{x} = y = f_1(x,y) \\ \ddot{x} = \dot{y} = -x + \mu(1-x^2)y = f_2(x,y) \end{cases}$$

A fix point is a point so as we have $(\dot{x} = 0, \dot{y} = 0)$. Then , we have a single fix point $(x_0, y_0) = (0,0)$ which is independent of μ. The stability of the point (x_0, y_0) depends on the eigenvalues of the Jacobian matrix .Now we are going to proceed by introducing the expression of the Jacobian matrix which is :

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

Then

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 - 2xy\mu & (1-x^2)\mu \end{pmatrix}$$

For the fix point$(0,0)$, we have:

$$\mathcal{J}(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

The stability of this fix point depends on the signs of the eigenvalues of the Jacobian matrix. The eigenvalues λ are solutions for the characteristic equation:

$$\lambda^2 - \mu\lambda + 1 = 0 \tag{2.12}$$

$\Delta = \mu^2 - 4$ where $\mu>0$

• If $0<\mu<2$, then the solutions of the equation (2.12) are complex numbers :

$\lambda_1 = \dfrac{\mu + i\sqrt{\mu^2 - 4}}{2}$

$\lambda_2 = \dfrac{\mu - i\sqrt{\mu^2 - 4}}{2}$ The fix point (0 , 0) is an unstable centre .

• If $\mu = 2$, the equation (2.12) has a double solution :

$\lambda = 1$

The fix point is then unstable .

• If $\mu>2$, the equation (2.12) has two real solutions :

$\lambda_1 = \dfrac{\mu + \sqrt{\mu^2 - 4}}{2}$

$\lambda_2 = \dfrac{\mu - \sqrt{\mu^2 - 4}}{2}$

The fix point (0 , 0) is an unstable node . We conclude that the single fix point (0,0) is unstable whatever the value of the control parameter μ

2.2.2 Existence of the limit cycles

We analytically study the amplitude of the limit cycle by using the average method[12]. This is applied to the equations of the following type :

$\ddot{x} + x = \mu f(x, \dot{x}, t)$

where μ is the perturbation parameter and $f(x, \dot{x}, t)$ being an integrable arbitrary function In general , we proceed with the following transformations :

$$x(t) = A(t)\cos(t + \varphi(t)) = A\cos\psi \tag{2.13}$$

$$\dot{x}(t) = -A(t)\sin(t + \varphi(t)) = -A\sin\psi \tag{2.14}$$

Where A(t) is the amplitude , $\varphi(t)$ being the phase and with :

$\psi(t) = \varphi(t) + t$

By Supposing that the amplitude and the phase feebly vary during a period $T \simeq 2\pi$,we have the fundamental equations of the average method as follows :

$$\dot{A} = -\frac{\mu}{2\pi}\int_0^{2\pi} f(A\cos\psi, -A\sin\psi)\sin\psi d\psi \tag{2.15}$$

$$\dot{\varphi} = \frac{\mu}{2\pi A}\int_0^{2\pi} f(A\cos\psi, -A\sin\psi)\cos\psi d\psi \tag{2.16}$$

The equations help to determine the amplitude A(t) and the phase $\varphi(t)$ of the oscillator . Now , we can apply this method to the equation (2.11) for which :

$f(x, \dot{x}, t) = (1 - x^2)\dot{x}$

Then , we have :

$f(A, \psi) = -A\sin\psi + A^3\sin\psi\cos^2\psi$

By replacing the last expression above in the equations (2.15) and (2.16) , we get :

$$\dot{A} = -\frac{\mu}{2\pi}\int_0^{2\pi}(-A\sin^2\psi + A^3\sin^2\psi\cos^2\psi)d\psi \tag{2.17}$$

$$\dot{\varphi} = \frac{\mu}{2\pi A}\int_0^{2\pi}(-A\sin\psi\cos\psi + A^3\sin\psi\cos^3\psi)d\psi \tag{2.18}$$

The integration of the relations (2.17) and (2.18) help to obtain the evolution equation of the amplitude A(t) and the phase $\varphi(t)$:

$$\dot{A}(t) = \frac{\mu A(t)}{2}\left(1 - \frac{A^2(t)}{4}\right) \tag{2.19}$$

$$\dot{\varphi}(t) = 0 \tag{2.20}$$

The average method state : The amplitude and the phase are feebly vary during a period . Therefore $\dot{A}(t) = 0$, and the amplitude is eventually : $A(t) = 2$

Analytically , the amplitude $A(t)$ of the Van der Pol oscillator limit cycle is equal to 2 .

Oscillations in plasma

In what follows , we are dealing with oscillations in plasma which can be defined as the fourth state of the matter . Interests according to such state of matter are due to their potential applications . Indeed , radio - wave propagation in the ionosphere was really an early stimulus for the development of the theory of plasma . Nowadays , plasma processing is viewed as a critical technology in a large number of industries , whilst semiconductor device fabrication for computers may be the best known . It is also important in order sectors such as bio - medecine , automobiles , defence , aerospace optics , solar energy , telecommunications , textiles , papers , polymers and waste managment [13].

3.1 The classical Van der Pol equation

For this survey , we consider the two - fluid model which treats the plasma as two inter penetrating conducting fluids . The Eulerian equations of motion in electric field E and magnetic field B are given as follows [14] :

$$n_\kappa M_\kappa \frac{d\upsilon_\kappa}{d\tau} = n_\kappa q_\kappa (E + \upsilon_\kappa \times B - \eta J) - \nabla p_\kappa \tag{3.1}$$

$$\frac{dn_\kappa}{d\tau} + \bigtriangledown.(n_\kappa \upsilon_\kappa) = S \tag{3.2}$$

$$\frac{d(p_\kappa n_\kappa^{-\gamma})}{d\tau} = 0 \tag{3.3}$$

Where S is the source term due to ionization or to large amplitude oscillations present in the plasma . The suffix “κ“ stands to label the species and it will be denoted by i and respectively for positive ions with charge $+e$ and for the negative ions (electron) with charge $-e$. n_κ represents the number density of the species , υ_κ their velocity , p_κ their pressure , γ the usual specific heat ratio and η the resistive collision which is defined as :

$$\eta = \frac{M\nu_\kappa}{ne^2} \tag{3.4}$$

Where ν_κ is the collision frequency of the speciesκ . The electric charge density ρ and current J are given by :

$$\rho = \sum_\kappa n_\kappa q_\kappa \tag{3.5}$$

$$J = \sum_\kappa n_\kappa q_\kappa \upsilon_\kappa \tag{3.6}$$

These quantities are the source terms for Maxwell's equations :

$$\nabla.B = 0 \tag{3.7}$$

$$\nabla.E = \frac{\rho}{\varepsilon_0} \tag{3.8}$$

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \dot{E} \tag{3.9}$$

$$\nabla \times E = -\dot{B} \tag{3.10}$$

In order to deal with small amplitude waves , we consider a " background" situation representing a uniform infinite plasma . The values of n_κ , υ_κ , p_κ , E and B for this will be denoted by $n_{0\kappa}$, $\upsilon_{0\kappa}$, $p_{0\kappa}$ ... ; however , here we shall take $\upsilon_\kappa = E = 0$ in the unperturbated state . We then have $J = 0$ and all of equations(3.1) - (3.10) are satisfied except equations (3.5) and (3.6) which requires $\rho = 0$, hence :

$$\sum_\kappa n_\kappa q_\kappa = 0 \tag{3.11}$$

For our simple two - species plasma , that condition of charge neutrality becomes :
$n_{0\kappa} = n_{0i} = n_0$ We now consider the $M_e = 0$ ion sound instability and introducing perturbations terms which are denoted by the suffix 1 , namely

$$n_i = n_0 + n_1 \tag{3.12}$$

$$p_i = p_0 + p_1 \tag{3.13}$$

$$B = B_0 + B_1 \tag{3.14}$$

Let us note that for other variables which vanish at the unperturbated state , the labels 0 and 1 are not necessary . We then insert the expressions (3.12) - (3.14) into equations (3.1) -(3.3) and after all of the second order perturbative terms have been discarded , we obtain the following equations :

$$n_0 M_i \frac{d\upsilon_i}{d\tau} = n_0 e(E + \upsilon_i \times B_0 - \eta J) - \nabla p_1 \tag{3.15}$$

$$\frac{dn_1}{d\tau} + n_0 \nabla.\upsilon_i = S \tag{3.16}$$

$$\frac{p_1}{p_0} = \gamma \frac{n_1}{n_0} \tag{3.17}$$

In dealing with equation(3.3) and taking each species to be a perfect gas with unperturbed temperature T (which could be different for each species) , we have $p_0 = n_0 k_B T$ (k_B is Boltzmann's constant) and equation (3.15) can be rewritten as follows :

$$n_0 M_i \frac{d\upsilon_i}{d\tau} = n_0 e(E + \upsilon_i \times B_0 - \eta J) - \gamma k_B T_i \nabla n_1 \tag{3.18}$$

To investigate the two - fluid model , we assume that $E = -\nabla\phi$ (ϕ is the potential perturbation) and consider the Boltzmann distribution equation of electron given as follows :

$$\frac{n_1}{n_0} = \frac{e\phi}{k_B T_e} \tag{3.19}$$

By deriving equation (3.16) with respect to τ , we get :

$$\frac{dS}{d\tau} = \frac{d^2 n_1}{d\tau^2} + n_0 \nabla.(\frac{d\upsilon_i}{d\tau}) \tag{3.20}$$

From equation (3.18) . we have :

$$\frac{d\upsilon_i}{d\tau} = \frac{e}{M_i}(-\nabla\phi + \upsilon_i \times B_0 - \eta J) - \frac{\gamma k_B T_i}{n_0 M_i}\nabla n_1 \tag{3.21}$$

From equation (3.6) ,we have :

$$J = n_i \upsilon_i e \tag{3.22}$$

From equation (3.19) ,we have :

$$\phi = \frac{k_B T_e n_1}{e n_0} \tag{3.23}$$

Plugging equations (3.4) , (3.22) and (3.23) into equation (3.21) , we get :

$$\nabla.(\frac{d\upsilon_i}{d\tau}) = -\frac{k_B}{n_0 M_i}(T_e + \gamma T_i)\nabla^2 n_1 + \frac{e}{M_i}\nabla.(\upsilon_i \times B_0) - \nu_i(\nabla.\upsilon_i) \tag{3.24}$$

Plugging $\nabla.\upsilon_i = \frac{1}{n_0}(S - \frac{dn_1}{d\tau})$ into equation (3.24) we have :

$$\nabla.(\frac{d\upsilon_i}{d\tau}) = -\frac{k_B}{n_0 M_i}(T_e + \gamma T_i)\nabla^2 n_1 + \frac{e}{M_i}\nabla.(\upsilon_i \times B_0) - \frac{\nu_i}{n_0}(S - \frac{dn_1}{d\tau}) \tag{3.25}$$

Plugging equation (3.25) into equation (3.20) we get the following equation :

$$\frac{dS}{d\tau} = \frac{d^2 n_1}{d\tau^2} - \frac{k_B}{M_i}(T_e + \gamma T_i)\nabla^2 n_1 + \frac{e n_0}{M_i}\nabla.(\upsilon_i \times B_0) - \nu_i(S - \frac{dn_1}{d\tau}) \tag{3.26}$$

If we now choose the source term to be of the form :

$$S = \sigma n_1 - \zeta n_1^3 + \xi n_1^5 - \varrho n_1^7 \tag{3.27}$$

and that n_1 is proportional to the quantity $e^{-j\vec{k}.\vec{r}}$ (where j is a complex number , $\vec{k}$ and$\vec{r}$ are tridimensional vectors) , equation(3.26) becomes :

$$\frac{d^2n_1}{d\tau^2} - \frac{k_B}{M_i}(T_e + \gamma T_i)\nabla^2 n_1 + \frac{en_0}{M_i}\nabla.(\upsilon_i \times B_0) - \nu_i S + \nu_i \frac{dn_1}{d\tau} - \frac{dS}{dn_1}\frac{dn_1}{d\tau} = 0 \tag{3.28}$$

From equation (3.27) we obtain :

$$\frac{dS}{dn_1} = \sigma - 3\zeta n_1^2 + 5\xi n_1^4 - 7\varrho n_1^6 \tag{3.29}$$

Plugging equation (3.29) equation (3.28) we get

$$\frac{d^2n_1}{d\tau^2} + (\nu_i - \sigma + 3\zeta n_1^2 - 5\xi n_1^4 + 7\varrho n_1^6)\frac{dn_1}{d\tau} + \frac{k_B\vec{k}^2}{M_i}(T_e + \gamma T_i)n_1 - \nu_i(\sigma n_1 - \zeta n_1^3 + \xi n_1^5 - \varrho n_1^7) - \frac{ejn_0}{M_i}(\upsilon_i \times B_0).\vec{k} = 0 \tag{3.30}$$

If one considers the case $\nu_i \to 0$ and uses the slab geometry for which densi varies in the x-direction and the z - axis coincides with the magnetic field direction , equation(3.30) takes the following expression :

$$\frac{d^2n_1}{d\tau^2} + (-\sigma + 3\zeta n_1^2 - 5\xi n_1^4 + 7\varrho n_1^6)\frac{dn_1}{d\tau} + \omega_0^2 n_1 = 0 \tag{3.31}$$

where

$$\omega_0 = k_z C_\kappa \tag{3.32}$$

$$C_\kappa = \left[\frac{k_B(T_e + \gamma T_i)}{M_i}\right]^{\frac{1}{2}} \tag{3.33}$$

Following the rescaling
$t = \omega_0 \tau$, $n_1 = \left(\frac{\sigma}{3\zeta}\right)^{\frac{1}{2}} x$, $\mu = \frac{\sigma}{\omega_0}$, $\alpha = \frac{5\sigma\xi}{9\zeta^2}$ and $\beta = \frac{7\sigma^2\varrho}{27\zeta^3}$
the equation that governs the system is :

$$\frac{d^2x}{dt^2} - \mu(1 - x^2 + \alpha x^4 - \beta x^6)\frac{dx}{dt} + x = 0 \tag{3.34}$$

When $\alpha = \beta = 0$ equation (3.34) yields the classical Van der Pol equation which has also been used to investigate nonlinear resonance effects in a plasma by Keen and Fletcher in 1969 [15]

Conclusiom

In this thesis , we have studied the nonlinear oscillations in a plasma governed by the Van der Pol equation . For that goal , it has been given in chapter 1 some generality on the dynamical system . In chapter 2 , the first part has been focussed on the description of the Van der Pol oscillator . As concerning the description of the Van der Pol oscillator , the Van der Pol equation has been established . The second part of this chapter has been devoted to the analytically study of the Van der Pol equation . As concerning the analytical study amplitude and phase of the limit cycle oscillation have been established using the averaging method in the autonomous regime . Chapter 3 has been devoted to the oscillations in a plasma .

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