Sobolev Spaces and Variational Method Applied to Elliptic Partial Differential Equations

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By
Enyi Cyril Dennis

Supervisor:
Prof. Ngalla Djitte

African University of Science and Technology
www.aust.edu.ng
P.M.B 681,Garki,Abuja F.C.T
Nigeria.

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EPIGRAPH

"Withhold not good from them to whom it is due when it is in thy power to do it "Proverb 3:27"."

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DEDICATION

This work is dedicated to God Almighty and to my family.

CONTENTS

INTRODUCTION

Variational methods have proved to be very important in the study of optimal shape, time, velocity, volume or energy. Laws existing in mechanics, physics, astronomy, economics and other fields of natural sciences and engineering obey variational principles.
The main objective of variational method is to obtain the solutions governed by these principles. Fermat postulated that light follows a part of least possible time, this is a subject in finding minimizers of a given functional. It is important to note that we are in this work concerned about solution of some Boudary Value Problem of some Partial Differential Equations.
The Boundary Value Problem is formulated in abstract form as;

$$\boldsymbol{A}(u) = 0 \text{ in } \Omega, \ \boldsymbol{B}(u) = 0 \text{ on } \partial\Omega, \ \Omega \subset \mathbb{R}^N \text{ open}, \tag{1}$$

where $\boldsymbol{A}(u) = 0$ denotes a given Partial Differential Equation for unknown u and $\boldsymbol{B}(u) = 0$ is a given boundary value condition. The problem of interest in the variational method shall be existence and the regularity of the minimizers of an associated functional.
In chapter three, we discussed Optimization in infinite dimensional spaces, a topic which is very important in the study of variational methods. We specifically studied the application of the variational methods in solving the Dirichlet Homogeneous Boundary Value Problem:

$$\begin{cases} -\Delta u = f \ \text{ in } \ \Omega, \\ \\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where Ω is bounded open subset of $\mathbb{R}^N$ of class C^1 and $f \in L^2(\Omega)$.

It is important to understand the meaning of *Linear Elliptic* Partial Differential Equations, since our work is targeted towards a method of solving such Partial Differential Equations.

Definition A partial differential equation(PDE) is an equation involving partial derivatives of an unknown function $u : \Omega \to \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^n, \ n \geq 2$.

The order of a Partial Differential Equation is the order of the highest order derivative that appears in the Partial Differential Equation.

A linear Partial Differential Equation is an equation in u with all of the terms involving u and any of its derivatives expressed as a linear combination in which the coefficients are independent of u and its derivatives. In a linear Partial Differential Equation, the coefficient depends at most on the independent variables. The following examples gives an illustration of a linear and nonlinear Partial Differential Equations.

Examples : (i) Let $u = u(x, y)$, a function of two independent variables x and y, we have that

$$\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + u^2\frac{\partial u}{\partial x} - u^3 = \cos(xy)$$

is a second order nonlinear PDE with nonconstant coefficients.

(ii) The Partial Differential Equation given by

$$\frac{\partial^2 u}{\partial x^2} + 5\frac{\partial^2 u}{\partial x \partial y} + u = 0$$

is linear with constant coefficient.

Classification of Linear Partial Differential Equations with n independent variables

It is important to note that the same way differences exist between linear and nonlinear Partial Differential Equations. Each of these classes requires different numerical methods of solution, whether linear or nonlinear. Partial Differential Equations have been classified as *elliptic, parabolic, and hyperbolic.*
A general linear Partial Differential Equation of order two in n variables has the form:

$$\sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i} + cu = d. \tag{2}$$

If

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i},$$

then the principal part of (2) can always be arranged so that

$$a_{ij} = a_{ji},$$

thus the $n \times n$ matrix $A = [a_{ij}]$ can be assumed to be symmetric. In linear algebra, it is shown that every real, symmetric $n \times n$ matrix has n real eigenvalues. These eigenvalues are the (possibly repeated) zeros of an nth-degree polynomial in λ, $\det(A - \lambda I)$, where I is the $n \times n$ identity matrix. Let p denote the number of positive eigeinvalues, and z the number of zero eigenvalues (i.e the multiplicity of the eigenvalue zero), of the matrix A. Then the Partial Differential Equation (2) is

- *hyperbolic* if $z = 0$ and $p = 1$ or $z = 0$ and $p = n - 1$

• *parabolic* if $z > 0 \Leftrightarrow \det(A) = 0$.

• *elliptic* if $z = 0$ and $p = n$ or $z = 0$ and $p = 0$.
If any of the a_{ij} is nonconstant, the type of (2) can vary with position.

Example 0.0.0.1 An illustration of the matrix of a PDE.

For the Partial Differential Equation

$$3\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + 4\frac{\partial^2 u}{\partial x_2 \partial x_3} + 4\frac{\partial^2 u}{\partial x_3^2} = 0,$$

the matrix A is

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \tag{3}$$

We have that this linear Partial Differential Equation is parabolic since $\det(A) = 0$.

Example 0.0.0.2 The Laplace equation

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x^2} = 0 \ \ (\Delta \text{ is called the Laplacian})$$

or, more generally, the Poisson equation

$$\Delta = f$$

for a given function $f : \Omega \to \mathbb{R}$. These are *elliptic* linear Partial Differential Equations.

Example 0.0.0.3 The heat equation:
we distinguish the coordinate t as the "time" coordinate, while the remaining coordinates $x_1, x_2, \cdots, x_n$ represents spatial variables. We consider

$$u : \Omega \times \mathbb{R}^+ \to \mathbb{R}, \ \ \Omega \text{ open in } \mathbb{R}^n, \mathbb{R}^+ := \{t \in \mathbb{R} : \ t > 0\},$$

and pose the equation

$$\frac{\partial u}{\partial t} = \Delta u.$$

The heat equation models heat and other diffusion processes. This is classified as *parabolic* linear Partial Differential Equation.

Example 0.0.0.4 The wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \Delta,$$

where we employed the same notion as in (3) above. The wave equation models wave and oscillation phenomena. This is classified as *hyperbolic* linear Partial Differential Equation.

In chapter one and two, we studied some function spaces which are paramount to our work. In chapter two we studied the Sobolev spaces;

$$W^{m,p}(\Omega) := \{u \in \Omega : D^{\alpha} \in L^p(\Omega),\ \forall|\alpha| \leq m, \Omega \subset \mathbb{R}^N \text{ open}\},$$

where $\alpha \in \mathbb{N}^N$ is a multi-index,i.e., $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ and $|\alpha|$ called the length of α is given by $|\alpha| = \sum_{i=1}^{N} \alpha_i$.

However in order to have a profound understanding of the Sobolev spaces, we also studied L^p**-Spaces**, and as well studied the *Theory of Distribution*.

CHAPTER 1

SPACES OF FUNCTIONS

In the following, Ω is a nonempty open subset of $\mathbb{R}^N$ with the Lebesgue measure dx.

1.1 L^p-spaces and some of its properties

1.1.1 Basic Integration Results

Theorem 1.1.1.1 (Monotone Convergence Theorem) *Let $\{f_n\}$ be a nondecreasing sequence of integrable functions such that:*

$$\sup \int_{\Omega} f_n \, dx < \infty.$$

Then $\{f_n\}$ converges pointwise to some function f. Futhermore f is integrable and

$$\lim_{n \to +\infty} \int |f_n - f| \, dx = 0.$$

Theorem 1.1.1.2 (Lebesgue Dominated Convergence Theorem) *Let $\{f_n\}$ be a sequence of integrable functions such that:*
(i) $f_n(x) \to f(x)$ a.e on Ω,
(ii) there exists a function g, integrable and $|f_n(x)| \leq g(x)$ a.e on Ω.
Then f is integrable and

$$\lim_{n \to +\infty} \int |f_n - f| \, dx = 0.$$

Theorem 1.1.1.3 (Fatou Lemma) *Let $\{f_n\}$ be a sequence of integrable functions such that:*
(i) $\forall\, n, \; f_n(x) \geq 0$ a.e on Ω,
(ii) $\sup \int f_n \, dx < \infty$.
For $x \in \Omega$, set $f(x) = \liminf_n f_n(x)$. Then f is integrable and

$$\int f \, dx \leq \liminf_n \int f_n \, dx.$$

1.1.2 Definition and basic properties

Definition Let $1 \le p < \infty$. We define:
(i) $\mathcal{L}^p(\Omega)$ as the set of measurable functions $f : \Omega \to \mathbb{R}$ such that:

$$\int_\Omega |f(x)|^p \, dx \; < +\infty$$

and
(ii) $\mathcal{L}^\infty(\Omega)$ as the set of measurable functions $f : \Omega \to \mathbb{R}$ such that:

$$\text{ess sup } |f| \; < \; +\infty$$

where

$$\text{ess sup } |f| = \inf\{K \ge \; 0, \;\; |f(x)| \le \; K, \;\; \text{a.e } x \in \; \Omega\}$$

Definition We say that two functions f and g are equivalent if $f = g$ almost everywhere. Then we define $L^p(\Omega)$ spaces as the equivalent classes for this relation.

Remark 1.1.2.1 The space $L^p(\Omega)$ can be seen as a space of functions. We do however, need to be careful sometimes. For example, saying that $f \in L^p(\Omega)$ is continuous means that f is equivalent to a continuous function. Now for $f \in L^p(\Omega)$, we define:

$$\|f\|_p \;=\; \left[\int_\Omega |f(x)|^p \, dx\right]^{\frac{1}{p}}, \;\; 1 \le p < +\infty \qquad (1.1)$$

$$\|f\|_\infty \;=\; \text{ess sup } |f|. \qquad (1.2)$$

Theorem 1.1.2.1 (Holder's Inequality) *. Let $1 \le p < +\infty$, we define p' by $1/p + 1/p' = 1$. If $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \; \le \|f\|_p \|g\|_{p'}. \qquad (1.3)$$

Proof. The cases $p = 1$ and $p' = +\infty$ are easy to prove. Now assume $1 < p < +\infty$. We use the following YOUNG's inequality: Let $1 < p < +\infty$, $a, b \ge 0$ then

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Assume that $\|f\|_p \neq 0$ and $\|g\|_{p'} \neq 0$ otherwise, nothing to do. Using YOUNG's inequality, we have

$$\frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_{p'}} \le \frac{1}{p} \frac{|f|^p}{\|g\|_p^p} + \frac{1}{p'} \frac{|g|^{p'}}{\|f\|_p^{p'}}.$$

Thus

$$\int_\Omega \frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_{p'}} \, dx \le \frac{1}{p} \int_\Omega \frac{|f|^p}{\|f\|_p^p} \, dx + \frac{1}{p'} \int_\Omega \frac{|g|^{p'}}{\|g\|_p^{p'}} \, dx = \frac{1}{p} + \frac{1}{p'} = 1.$$

Hence

$$\int_\Omega |f| \cdot |g| \, dx \le \|f\|_p \cdot \|g\|_{p'}.$$

Theorem 1.1.2.2 (Minkowski's Inequality) . *If* $1 \leq p \leq +\infty$ *and* $f, g \in L^p(\Omega)$ *then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{1.4}$$

Proof. If $f + g = 0$ $a.e$, then the statement is trivial. Assume that $f + g \neq 0$ and $p > 1$ (the case $p = 1$ is easy to check). We evaluate as follows:

$$|f + g|^p = |f + g||f + g|^{p-1} \leq (|f| + |g|)|f + g|^{p-1}.$$

Integrating over Ω, we get

$$\begin{aligned} \int_\Omega |f + g|^p \, dx &\leq \int_\Omega (|f| + |g|)|f + g|^{p-1} \, dx \\ &= \int_\Omega |f||f + g|^{p-1} \, dx + \int_\Omega |g||f + g|^{p-1} \, dx \end{aligned}$$

Using Holder's inequality in the right hand side, we obtain

$$\int_\Omega |f + g|^p \, dx \leq (\|f\|_p + \|g\|_p)\|f + g\|_p^{p/q},$$

from which it follows

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

1.1.3 The Main properties of $L^p(\Omega)$

L^p-Spaces are Banach

Theorem 1.1.3.1 *The L^p-spaces are Banach for* $1 \leq p \leq +\infty$.

Proof.

Case1. Assume that $p = \infty$. Let $\{f_n\}$ be a Cauchy sequence in L^∞. Let $k \geq 1$, there exists N_k such that

$$\|f_m - f_n\|_p \leq \frac{1}{k} \ \forall\, n, m \geq N_k.$$

There exists a set of measure zero A_k such that

$$|f_m(x) - f_n(x)|_p \leq \frac{1}{k} \ \forall\, x \in \Omega - A_k, \ \forall n, m \geq N_k. \tag{1.5}$$

Let $A = \cup A_k$ (A is of measure zero) and forall $x \in \Omega - A$ the sequence $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$. Let $f_n(x) = \lim_n f_n(x)$ forall $x \in \Omega - A$. Letting m goes to $+\infty$ in (1.5), we obtain

$$|f_n(x) - f(x)|_p \leq \frac{1}{k} \ \forall\, x \in \Omega - A_k, \ \forall n \geq N_k.$$

Thus $f \in L^\infty$ and $\|f_n - f\|_p \leq 1/k, \ \forall\, n \geq N_k$. So $\|f_n - f\|_p \to 0$.

Case2. Assume that $1 \leq p < +\infty$. Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $L^p(\Omega)$, then there exists a subsequence $(f_{n_k})_{k\geq 1}$ of (f_n) such that:

$$\|f_{n_{k+1}} - f_{n_k}\|_p \leq \frac{1}{2^k}, \ \forall\, k \geq 1. \tag{1.6}$$

To simplify the notations, let us replace f_{n_k} by f_k so that:

$$\|f_{k+1} - f_k\|_p \leq \frac{1}{2^k}, \ \forall\, k \geq 1. \tag{1.7}$$

Now set:

$$g_n(x) = \sum_{k=1}^{n} |f_{k+1}(x) - f_k(x)|.$$

It follows that:

$$\|g_n\|_p \leq 1, \ \forall\, n \geq 1.$$

Thus, from the monotone convergence theorem, $g_n(x)$ converge pointwise to some $g(x)$ almost every where and $g \in L^p$. On the other hand we have: for all $n, m \geq 2$

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_{m-1}(x)| + \cdots + |f_{n+1}(x) - f_n(x)| \leq g(x) - g_{n-1}(x).$$

It follows that $(f_n(x))$ is Cauchy in $\mathbb{R}$ and converges to some $f(x)$ a.e. Letting m goes to $+\infty$ leads to:

$$|f(x) - f_n(x)| \leq g(x), \ \forall n \geq 2.$$

Therefore $f \in L^p$ and by using dominate convergence theorem we have

$$\|f_n - f\|_p \to 0.$$

We complete the proof by applying the following lemma

Lemma 1.1.3.1 *Let E be a metric space and (x_n) be a cauchy sequence in E. If (x_n) has a convergence subsequence, then it converges to the same limit.*

The preceding proof contains a result which is interesting enough to be stated separetely:

Theorem 1.1.3.2 (Convergence criteria for L^p functions) *Let $1 \leq p < +\infty$. Let (f_n) and f in $L^p(\Omega)$ such that (f_n) converges to f in $L^p(\Omega)$. Then there exists a subsequence (f_{n_k}) of (f_n) and $h \in L^p(\Omega)$ such that $f_{n_k}(x) \to f(x)$ for a.e, $x \in \Omega$ and $f_{n_k}(x) \leq h(x)$, a.e $x \in \Omega$.*

Remark 1.1.3.1 It is in general not true that the entire sequence itself converge pointwise to the limit f, without some futher conditions holding.

Example 1.1.3.1 *Let $X = [0, 1]$, and consider the subintervals*

$$\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{3}, 1\right], \left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{2}{4}\right], \left[\frac{2}{4}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right], \left[1, \frac{1}{5}\right], \cdots$$

Let f_n denote the indicator function of the n^{th} interval of the above sequence. Then $\|f_n\|_p \to 0$, but $f_n(x)$ does not converge for any $x \in [0, 1]$.

Example 1.1.3.2 *Let $\Omega = \mathbb{R}$, and for $n \in \mathbb{N}$, set $f_n = \mathcal{X}[n, n+1]$. Then $f_n(x) \to 0$ as $n \to \infty$, but $\|f_n\|_p = 1$ for $p \in [0, \infty)$. Thus f_n converge pointwise but not in norm.*

Theorem 1.1.3.3 *Let $1 \leq p < \infty$. Let $\{f_n\}$ be a sequence in L^p such that $f_n(x) \to f(x)$ a.e. If*

$$\lim_n \|f_n\| = \|f\|$$

then $\{f_n\}$ converges to f in norm.

Theorem 1.1.3.4 *The L^p spaces are reflexive for $1 < p < \infty$.*

Proof. For $2 \leq p < \infty$. We have the follwing first Clarkson inequality:

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \leq \frac{1}{2}\Big(\|f\|_p^p + \|g\|_p^p\Big), \ \forall f, g \in L^p.$$

For $1 < p \leq 2$, we have the second Clarkson inequality:

$$\left\|\frac{f+g}{2}\right\|_p^{p'} + \left\|\frac{f-g}{2}\right\|_p^{p'} \leq \left[\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p\right]^{1/(p-1)}, \ \forall f, g \in L^p.$$

Using the Clarkson inequalities, we prove that L^p is uniformly convex for $1 < p < \infty$. So it is reflexive by Milman-Pettis Theorem

Theorem 1.1.3.5 *Let $1 \leq p < \infty$. Then L^p is separable.*

Proof. Let $(\Lambda_i)_{i \in I}$ be the family of N-cubes of $\mathbb{R}^N$ of the form $\Lambda = \prod_{k=1}^{N}]a_k, b_k[$ where $a_k, b_k \in \mathbb{Q}$ and $\Lambda \subset \Omega$. Let E be the $\mathbb{Q}$-vector space spanned by the functions $\mathcal{X}_{\Lambda_i}$.
Claim: E is a countable dense subspace of L^p.

Remark 1.1.3.2 L^∞ is not separable. To establish this, we need the following:

Lemma 1.1.3.2 *Let E be a banach space. We assume that there exists a familly $(O_i)_{i \in I}$ such that:*

(i) For all $i \in I$ O_i is a nonempty open subset of E;

(ii) $O_i \cap O_j = \emptyset$ if $i \neq j$;

(iii) I is uncountable.

Then E is not separable.

Now we apply this lemma for L^∞ as follows:

For all $a \in \Omega$, let r_a such that $0 < r_a < \mathrm{d}(a, \Omega^c)$. Set $f_a = \mathcal{X}_{B(a, r_a)}$ and

$$O_a = \{f \in L^\infty \mid \|f - f_a\|_\infty < \frac{1}{2}\}.$$

One can check that the family $(O_a)_{a \in \Omega}$ satisfies (i), (ii) and (iii).

1.1.4 Dual Space

Theorem 1.1.4.1 (Riesz representation theorem.) *Let $1 < p < +\infty$ and let $\Phi \in (L^p)'$. Then there exists a unique $g \in (L^p)'$ such that:*

$$\langle \Phi, f \rangle = \int_\Omega g \cdot f \, dx, \quad \forall f \in L^p(\Omega).$$

Futhermore

$$\|\Phi\|_{(L^1)'} = \|g\|_\infty.$$

Proof. Let $1 < p < +\infty$ and let p' such that $1/p + 1/p' = 1$. For $g \in L^{p'}(\Omega)$, we define

$$T_g : L^p(\Omega) \to \mathbb{R}, \;\; \langle T_g, f \rangle = \int_\Omega f \cdot g \, dx.$$

Using Holder's inequality, we observe that T_g is well defined, linear and

$$|\langle T_g, f \rangle| \leq \|g\|_{p'} \|f\|_p.$$

Thus

$$\|T_g\|_{(L^p)'} \leq \|g\|_{p'}.$$

In fact we have $\|T_g\|_{(L^p)'} = \|g\|_{p'}$. This follows by choosing $f = |g|^{p'-2} g$.

Now we define the map

$$T : L^{p'} \to (L^p)', \quad \text{by } \; T(g) = T_g \; \forall g \in L^{p'}.$$

We have to prove that T is onto. For this, let $E = T(L^{p'})$. We have to show that E is closed and dense in (L^p). E is closed by using the fact that $\|T_g\| = \|g\|_{p'}$ and $L^{p'}$ is Banach. For density we will show that if $L \in (L^p)''$ and $L = 0$ on E then $L = 0$ on $(L^p)'$. Since L^p is reflexive, we identify $(L^p)''$ to L^p through the canonical embeding. Thus there exists $f \in L^p$ such that $\langle L, \phi \rangle = \langle \phi, f \rangle$, for all $\phi \in (L^p)'$. So $L = 0$ on E leads $\langle T_g, f \rangle = 0$ for all $g \in L^{p'}$ and this implys that $f = 0$ so L is.

Theorem 1.1.4.2 *(Dual space of L^1). Let $\Phi \in (L^1)'$, then there exists a unique $g \in L^\infty$ such that*

$$\langle \Phi, f \rangle = \int_\Omega g \cdot f \, dx, \quad \forall f \in L^p(\Omega).$$

and

$$\|\Phi\|_{(L^1)'} = \|g\|_\infty.$$

Remark 1.1.4.1 The spaces $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive.

Indeed assume that L^1 is reflexive and let Ω open such that assume that $0 \in \Omega$. Let $f_n = \alpha_n \mathcal{X}_{B(0,1/n)}$, where $\alpha_n = \Big|B(0,1/n)\Big|^{-1}$ so that $\|f_n\|_1 = 1$. For n large enough, we have $B(0,1/n) \subset \Omega$. By reflexivity, $\{f_n\}$ has a weakly convergence subsequence f_{n_k} to some function f in $L^1(\Omega)$. Thus

$$\int_\Omega f_{n_k} \varphi \, dx \to \int_\Omega f \varphi \, dx, \; \forall \varphi \in L^\infty(\Omega). \tag{1.8}$$

So for $\varphi \in C_c(\Omega - \{0\})$, we have $\int_\Omega f_{n_k}\varphi\,dx = 0$ for k large enough. By (1.8) it follows that

$$\int_\Omega f\varphi\,dx = 0,\ \forall\varphi \in C_c(\Omega - \{0\}).$$

Thus $f = 0$ a.e on Ω. On the other hand, taking $\varphi \equiv 1$ in (1.8) leads to $\int_\Omega f\,dx = 1$. Contradiction. So $L^1(\Omega)$ is not reflexive.

Since a Banach space is reflexive if and only if its dual E' is reflexive, then $L^\infty(\Omega)$ is not reflexive.

Remark 1.1.4.2 Since $(L^1)' = L^\infty$, then from Banach-Alaogulu theorem any bounded sequence in $L^\infty(\Omega)$ has a w^*-convergence subsequence.

Proposition 1.1.4.1 *There exists a linear continuous forms on $L^\infty(\Omega)$ such that there is no $g \in L^1(\Omega)$ such that*

$$\langle T, f\rangle = \int_\Omega g\cdot f\,dx,\ \forall\, f \in L^\infty(\Omega).$$

Proof. Let Ω an open subset of $\mathbb{R}^n$ such that $0 \in \Omega$. Let

$$\Phi_0 : C_c(\Omega) \to \mathbb{R},\ \langle\Phi_0, \varphi\rangle = \varphi(0).$$

Φ_0 is a linear continuous form on $(C_c(\Omega), \|\cdot\|_\infty)$. So by the HANN-BANACH extension theorem, Φ_0 can be extended to a continuous linear form on $L^\infty(\Omega)$, say Φ. We summarize the main properties of the L^p spaces as follows:

	Completeness	Reflexivity	Separability	Dual Space
$L^p,\ 1 < p < \infty$	yes	yes	yes	$L^{p'},\ 1/p + 1/p' = 1$
L^1	yes	no	yes	L^∞
L^∞	yes	no	no	Contains strctly L^1

1.1.5 Convolutions and Mollifiers

Two usefull theorems
Let $\Omega_1 \subset \mathbb{R}^N$, $\Omega_2 \subset \mathbb{R}^N$ open subsets of $\mathbb{R}^N$ and $F : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function.

Theorem 1.1.5.1 (Tonelli) *Assume that*

$$\int_{\Omega_2} |F(x,y)|\,dy < \infty\ \ a.e\ x \in \Omega_1$$

and

$$\int_{\Omega_1}\Big(\int_{\Omega_2} |F(x,y)|\,dy\Big)\,dx < \infty.$$

Then $F \in L^1(\Omega_1 \times \Omega_2)$.

Theorem 1.1.5.2 (Fubini) *Assume that $F \in L^1(\Omega_1 \times \Omega_2)$.*
Then for a.e $x \in \Omega_1$

$$F(x,\cdot) \in L^1(\Omega_2) \ \ \textit{and} \ \ \int_{\Omega_2} F(\cdot,y)\,dy \ \in L^1(\Omega_1).$$

Similarly, for a.e $y \in \Omega_2$

$$F(\cdot,y) \in L^1(\Omega_1) \ \ \textit{and} \ \ \int_{\Omega_1} F(x,\cdot)\,dx \ \in L^1(\Omega_2).$$

Futhermore, we have

$$\int_{\Omega_1}\int_{\Omega_2} F(x,y)\,dxdy = \int_{\Omega_2}\Big(\int_{\Omega_1} F(x,y)\,dx\Big)\,dy = \int_{\Omega_1}\Big(\int_{\Omega_2} F(x,y)\,dy\Big)\,dx.$$

Definition Let f and g be measurable functions on $\mathbb{R}^N$. We define the convolution product $f*g$ of f and g by:

$$f*g(x) = \int_{\mathbb{R}^N} f(x-y)g(y)\,dy$$

for those x, if any, for which the integral converges.

Theorem 1.1.5.3 (Minkowski's Inequality) . *Let $1 \le p < +\infty$ and let $(X,\mathcal{A},dx)$ and $(Y,\mathcal{B},dy)$ be σ-finite measure spaces. Let F be a measurable function on the product space $X \times Y$. Then*

$$\left(\int_X \left|\int_Y F(x,y)\,dy\right|^p \,dx\right)^{\frac{1}{p}} \le \int_Y \left(\int_X |F(x,y)|^p\,dx\right)^{\frac{1}{p}}\,dy,$$

in the sense that the integral on the left hand side exists if the one on the right hand side is finite, and in this case the inequality holds. Note that the inequality may also be writen as:

$$\left\|\int_Y F(\cdot,y)\,dy\right\|_p \le \int_Y \|F(\cdot,y)\|_p\,dy.$$

Theorem 1.1.5.4 *Let $1 \le p \le +\infty$. If $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$ then*

$$f*g(x) = \int_{\mathbb{R}^N} f(x-y)g(y)\,dy$$

exists for almost all x and defines a function $f*g \in L^p(\mathbb{R}^N)$. Moreover

$$\|f*g\|_p \le \|f\|_1\|g\|_p.$$

Proof.

Case1. If $p = +\infty$, we have

$$\int_{\mathbb{R}^N} |f(x-y)g(y)|\,dy \le \|g\|_\infty \int_{\mathbb{R}^N} |f(x-y)|\,dy = \|g\|_\infty\|f\|_1,$$

by invariance of LEBESGUE's measure under translation. Thus $f * g(x)$ exists a.e and

$$|f * g(x)| \leq \|g\|_\infty \|f\|_1, \text{ a.e } x \in \mathbb{R}^N.$$

So $f * g \in L^\infty(\Omega)$ and

$$\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty.$$

Case2. For $p = 1$, let

$$F(x, y) = f(x - y)g(y).$$

For almost every $y \in \mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} |F(x, y)|\, dx = |g(y)| \int_{\mathbb{R}^N} |f(x - y)|\, dx = \|f\|_1 |g(y)| < \infty$$

and

$$\int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} |F(x, y)|\, dx \Big)\, dy = \|f\|_1 \|g\|_1 < \infty.$$

Using Tonelli's Theorem, we have $F \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$. By Fubini's Theorem, we obtain

$$\int_{\mathbb{R}^N} |F(x, y)|\, dy < \infty \text{ a.e } x \in \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} |F(x, y)|\, dy \Big)\, dx \leq \|f\|_1 \|g\|_1.$$

So

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Case3. For $1 < p < +\infty$, let q be the conjugate exponent of p. From **Case2.**, we know that for a.e $x \in \mathbb{R}^N$ fixed, $y \mapsto |f(x-y)||g(y)|^p$ is integrable or equivalently $y \mapsto |f(x-y)|^{1/p}|g(y)|$ is in $L^p(\mathbb{R}^N)$. Since $y \mapsto |f(x - y)|^q$ is in $L^q(\mathbb{R}^N)$, we have from Holder's inequality that

$$|f(x - y)||g(y)| = |f(x - y)|^q \cdot |f(x - y)|^{1/p} |g(y)| \in L^1(\mathbb{R}^N)$$

and

$$|f(x - y)||g(y)| \leq \Big(\int_{\mathbb{R}^N} |f(x - y)||g(y)|^p\, dy \Big)^{1/p} \|f\|_1^{1/q}$$

i.e

$$|f * g(x)|^p \leq (|f| * |g|^p)(x) \cdot \|f\|_1^{p/q}.$$

Using again **case2.** we have

$$f * g \in L^p(\Omega) \text{ and } \|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Definition Let $\phi \in L^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} \phi(x)\, dx = 1$. Let $\phi_\epsilon(x) = \frac{1}{\epsilon^N} \phi(\frac{x}{\epsilon})$. The family of functions ϕ_ϵ, $\epsilon > 0$, is called a mollifier with kernel ϕ. Note that $\int_{\mathbb{R}^N} \phi_\epsilon\, dx = 1$.

Definition If f is a function on $\mathbb{R}^N$ and $a \in \mathbb{R}^N$, we define the translation of f by a, $\tau_a f$ as follow:

$$\tau_a f(x) = f(x-a)$$

Proposition 1.1.5.1 *Let ϕ_ϵ be a mollifier, $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^N)$. Then for each $\epsilon > 0$*

$$\|f * \phi_\epsilon - f\|_p \leq \int_{\mathbb{R}^N} \|\tau_{\epsilon y} f - f\|_p |\phi(y)|\, dy. \tag{1.9}$$

Proof. Since $\int_{\mathbb{R}^N} \phi(x)\, dx = 1$ we have

$$f * \phi_\epsilon(x) - f(x) = \int_{\mathbb{R}^N} [f(x-\epsilon y) - f(x)]\phi(y)\, dy.$$

by Minkowski's inequality (1.1.5.3)

$$\begin{aligned}
\|f * \phi_\epsilon - f\|_p &= \left(\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} [f(x-\epsilon y) - f(x)]\phi(y)\, dy \right|^p dx \right)^{\frac{1}{p}} \\
&\leq \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |f(x-\epsilon y) - f(x)|^p |\phi(y)|\, dx \right)^{\frac{1}{p}} dy \\
&= \int_{\mathbb{R}^N} \|\tau_{\epsilon y} f - f\|_p |\phi(y)|\, dy.
\end{aligned}$$

Corollary 1.1.5.1 *If ϕ is such that $\int_{\mathbb{R}^N} \phi(x)\, dx = 0$ then*

$$\|f * \phi_\epsilon\|_p \leq \int_{\mathbb{R}^N} \|\tau_{\epsilon y} f - f\|_p |\phi(y)|\, dy.$$

Theorem 1.1.5.5 *Assume that $\phi \geq 0$. Let f be a bounded continuous function on $\mathbb{R}^N$. Then $f * \phi_\epsilon$ is continous on $\mathbb{R}^N$ for each $\epsilon > 0$ and for each $x \in \mathbb{R}^N$ we have*

$$\lim_{\epsilon \to 0^+} f * \phi_\epsilon(x) = f(x).$$

Proof. Let $\epsilon > 0$, we have

$$f * \phi_\epsilon(x) = \int_{\mathbb{R}^N} f(x-y)\phi_\epsilon(y)\, dy = \int_{\mathbb{R}^N} f(x-\epsilon y)\phi(y)\, dy.$$

Let M be the bound on the absolute value of f. Then $|f(x-\epsilon y)\phi(y)| \leq M\phi(y)$ a.e. Since $\phi \in L^1(\mathbb{R}^N)$ and the function $x \to f(x-\epsilon y)\phi(y)$ is continuous a.e $y \in \mathbb{R}^N$ then by Lebesgue dominated convergence theorem $f * \phi_\epsilon$ is continuous.
Now, fix $x \in \mathbb{R}^N$. Since $\int \phi_\epsilon(y)\, dy = 1$ we have:

$$f * \phi_\epsilon(x) - f(x) = \int_{\mathbb{R}^N} [f(x-y) - f(x)]\phi_\epsilon(y)\, dy.$$

Let $\lambda > 0$. By the continuity of f at x, there is $\delta > 0$, such that

$$|f(x-y) - f(x)| \leq \frac{\lambda}{2}, \text{ for } |y| < \delta.$$

Since

$$\int_{|y|\geq\delta} \phi_\epsilon(y)\, dy = \int_{|y|\geq\frac{\delta}{\epsilon}} \phi(y)\, dy \ \to 0, \text{ as } \epsilon \to 0$$

then there exists $\epsilon_0 > 0$ such that

$$\int_{|y|\geq\delta} \phi_\epsilon(y)\, dy < \frac{\lambda}{4M}, \text{ for } \epsilon < \epsilon_0$$

It follows that for all such $\epsilon > 0$, we can write the integral as a sum over $|y| < \delta$ and $|y| \geq \delta$ and get

$$|f * \phi_\epsilon(x) - f(x)| \leq \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda.$$

1.1.6 Density of $C_c(\Omega)$ in $L^p(\Omega)$

Proposition 1.1.6.1 *Let Ω be an open subset of $\mathbb{R}^N$. Let $(U_j)_{j\in J}$ be a collection of open subsets of Ω with union U. Let $E \subset U$. If $E \cap U_j$ is a set of Lebesgue measure 0 for each $j \in J$ then E has measure 0.*

Proof. Let Q be the countable set consisting of all open balls in $\mathbb{R}^N$ with rational radius and rational center coordinates. Then for each $j \in J$

$$U_j = \bigcup \{B \mid B \in Q,\, B \subset U_j\}$$

so E is a countable union of sets of measure 0 of the form $E \cap B$.
Note that it is important that be U_j to be open.

Now let $f \in L^1(\Omega)$. Then by the proposition above there exists a largest open subset U of Ω on which f is 0 almost everywhere, just take the union of open sets on which f vanishes.

Definition The complement of U is called the support of f in Ω and is denoted by $\text{supp}(f)$.

Proposition 1.1.6.2 *If $f : \Omega \to \mathbb{R}$ is continuous then the support of f in Ω is the closure of*

$$\{x \in \Omega \mid f(x) \neq 0\}$$

Definition If Ω is an open subset of $\mathbb{R}^N$, we denote by $C_c(\Omega)$ the set of continuous functions on $\mathbb{R}^N$ with compact support in Ω. We denote by $D(\Omega)$ the set of infinitely continuously differentiable functions with compact support in Ω

Let $\phi : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases} c(1 - \|x\|) & \text{if } \|x\| \leq 1, \\ \\ 0 & \text{if } \|x\| > 1 \;. \end{cases} \tag{1.10}$$

where the constant c is chosen so that $\displaystyle\int_{\mathbb{R}^N} \phi(x)\, dx = 1$. Then ϕ_ϵ is a continuous mollifier and moreover supp (ϕ) is the ϵ-Ball $B'(0, \epsilon)$.

Lemma 1.1.6.1 (Uryshon.) *Let Ω be an open subset of $\mathbb{R}^N$ and $K \subset \Omega$ be a compact set. Then there exists $\Psi \in C_c(\Omega)$ such that $0 \leq \Psi \leq 1$ and $\Psi = 1$ on some neighborhood of K.*

Proof. Let ϕ_ϵ be a continuous mollifier as above and let L be the closed δ-neighborhood of K, that is

$$L = \{x \in \mathbb{R}^N, \;\mid \text{dist}(x, K) \leq \delta\}$$

where $\delta = \dfrac{1}{3}\text{dist}(K, \partial\Omega)$. Let

$$\Psi(x) = \mathcal{X}_L * \phi_\epsilon(x) = \int_{\mathbb{R}^N} \mathcal{X}_L(x - y)\phi_\epsilon(y)\, dy = \int_L \phi_\epsilon(x - y)\, dy$$

For $0 < \epsilon < \delta$, we have $\Psi \in C(\Omega)$, Ψ has it support in the closed 2δ-neighborhood of K and so has compact support in Ω, $0 \leq \Psi \leq 1$ and $\Psi = 1$ on the $(\delta - \epsilon)$-neighborhood of K.

Theorem 1.1.6.1 (Density of $C_c(\Omega)$ in $L^p(\Omega)$) . *Let Ω be an open subset of $\mathbb{R}^N$ and let $1 \leq p < +\infty$. Then $C_c(\Omega)$ is dense in $L^p(\Omega)$.*

Proof. We denote the Lebesgue measure of measurable set B by $m(B)$. Since the simple functions are dense in $L^p(\Omega)$ for finite p, it suffices to show that we can approximate the characteristic function $\mathcal{X}_A$ of a measurable set A of finite measure by function in $C_c(\Omega)$. Let $\epsilon > 0$. By the regularity of Lebesgue measure there exits a compact set $K \subset A$ and an open set U, $A \subset U$ such that $m(U - K) < \epsilon^p$. From Uryshon's Lemma, there is $\Psi \in C_c(U)$ such that $0 \leq \Psi \leq 1$ and $\Psi = 1$ on K. We have $|\mathcal{X}_A - \Psi| \leq \mathcal{X}_U - \mathcal{X}_K$ and so

$$\|\mathcal{X}_A - \Psi\|_p \leq m(U - K)^{\frac{1}{p}} < \epsilon.$$

Remark 1.1.6.1 If $1 \leq p < \infty$, Theorem 1.1.6.1 says that $C_c(\Omega)$ is dense in $L^p(\Omega)$, and Theorem 1.1.3.1 shows that $L^p(\Omega)$ is complete. Thus $L^p(\Omega)$ is the completion of the metric space which is obtained by endowing $C_0(\Omega)$ with the L^p-metric.

Of course, every metric space S has a completion S^* whose elements may be viewed abstractly as equivalent classes of Cauchy sequence in S. The important point in the present situation is that the various L^p-completion of $C_c(\Omega)$ again turn out to be spaces of functions on Ω.

The case $p = +\infty$ differs from the cases $p < \infty$. The L^∞-completion of $C_c(\Omega)$ is not $L^\infty(\Omega)$, but is $C_0(\Omega)$, the spaces of all continuous functions on Ω which vanish at infinity.

Definition A function $f : \Omega \to \mathbb{R}$ is said to vanish at infinity if for every $\epsilon > 0$, there exists a compact set $K \subset \Omega$ such that $|f(x)| < \epsilon$ for all x not in K.

We denote by $C_0(\Omega)$,the class of all continuous functions on Ω which vanish at infinity. It is clear that $C_c(\Omega) \subset C_0(\Omega)$.

Theorem 1.1.6.2 *$C_0(\Omega)$ is the completion of $C_c(\Omega)$, relative to the metric defined by the supremum norm:*

$$\|f\|_\infty = \sup_{x\in\Omega} |f(x)|.$$

Proof. An elementary verification shows that $C_0(\Omega)$ satisfies the axioms of a metric space if the distance between f and g is taken to be $\|f - g\|_\infty$. We have to show that (i) $C_c(\Omega)$ is dense in $C_0(\Omega)$ and (ii) C_0 is complete.
To prove (i), let $f \in C_0(\Omega)$ and $\epsilon > 0$, there exists a compact set $K \subset \Omega$ such that $|f(x)| < \epsilon$ outside K. Uryshon's lemma gives us that there exists a function $\varphi \in C_0(\Omega)$ such that $0 \le \varphi \le 1$ and $\varphi(x) = 1$ on K. Put $h = \varphi f$. Then $h \in C_c(\Omega)$ and $\|f - h\|_\infty < \epsilon$.
To prove (ii), let $\{f_n\}$ be a Cauchy sequence in $C_0(\Omega)$. Using the definition of Cauchy sequence and supremum norm, we can assume that $\{f_n\}$ converges uniformly. Then its pointwise limit function f is continuous. Given $\epsilon > 0$, there exists an N so that $\|f_N - f\|_\infty < \epsilon/2$ and there exists a compact set K so that $|f_N(x)| < \epsilon/2$ outside K. Hence $|f(x)| < \epsilon$ outside K, and we have proved that f vanishes at infinity. Thus $C_0(\Omega)$ is complete.

Proposition 1.1.6.3 (Continuity of Translation in $L^p(\Omega)$) . *Let $1 \le p < +\infty$ and $f \in L^p(\mathbb{R}^N)$. Let $\theta : \mathbb{R}^N \to L^p(\mathbb{R}^N)$ be the map defined by*

$$\theta(y) = \tau_y f, \ \forall\, y \in \mathbb{R}^N.$$

Then θ is uniformly continuous on $\mathbb{R}^N$.

Proof. Let $\epsilon > 0$. By density choose $g \in C_c$ such that $\|f - g\|_p < \dfrac{\epsilon}{3}$. Let $y, z \in \mathbb{R}^N$ and $v = y - z$, then

$$\begin{aligned}\|\theta(y) - \theta(z)\|_p = \|\tau_y f - \tau_z f\|_p &\le \|\tau_y f - \tau_y g\|_p + \|\tau_y g - \tau_z g\|_p + \|\tau_z g - \tau_z f\|_p \\ &\le \frac{2}{3}\epsilon + \|\tau_y g - \tau_z g\|_p \\ &\le \frac{2}{3}\epsilon + \|\tau_v g - g\|_p\end{aligned}$$

by translation invariance of Lebesgue measure. Since g has compact support, then the support of $\tau_v g$ stays in a fixed compact set K for $\|v\| \le 1$. Since g is bounded we have

$$|\tau_v g| \le M \cdot \mathcal{X}_K.$$

It follows that

$$|\tau_v g - g|^p \le (2M)^p \mathcal{X}_K \in L^1(\mathbb{R}^N), \ \text{ for } \|v\| \le 1$$

since g is continuous we have $\tau_v g \to g$ as $v \to 0$ pointwise. By the dominate convergence theorem $\int_{\mathbb{R}^N} |\tau_v g - g|^p \, dx \to 0$ as $v \to 0$. Thus there exists $\delta > 0$ such that $0 < \delta < 1$ and

$$\|\tau_v g - g\|_p < \frac{1}{3}\epsilon, \text{ if } \|v\| < \delta.$$

Hence, the uniform continuity of θ follows.

Theorem 1.1.6.3 *Let $\phi \in L^1(\mathbb{R}^N)$, $1 \le p < +\infty$ and let $f \in L^p(\mathbb{R}^N)$. If $\int_{\mathbb{R}^N} \phi \, dx = 1$ then $f * \phi_\epsilon \to f$ in $L^p(\mathbb{R}^N)$ as $\epsilon \to 0$. If $\int_{\mathbb{R}^N} \phi \, dx = 0$ then $f * \phi_\epsilon \to 0$ in $L^p(\mathbb{R}^N)$ as $\epsilon \to 0$*

Proof. for the first case, we know that:

$$\|f * \phi_\epsilon - f\|_p \le \int_{\mathbb{R}^N} \|\tau_{\epsilon y} f - f\|_p |\phi(y)| \, dy.$$

The integrand is bounded by $2\|f\|_p|\phi| \in L^1(\mathbb{R}^N)$ and goes to 0 as $\epsilon \to 0$ by continuity of the translation. Thus the Lebesgue dominated convergence theorem yields the desired result.

Corollary 1.1.6.1 *Let $1 < p < +\infty$ and $q \ge 1$ such that $1/p + 1/q = 1$. If $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$ then $f * g$ is uniformly continuous.*

Proof. We have

$$f * g(x) - f * g(z) = \int_{\mathbb{R}^N} (f(x-y) - f(z-y))\, g(y) dy = \int_{\mathbb{R}^N} (\tau_{-x} f(-y) - \tau_{-z} f(-y))\, g(y)\, dy$$

therefore by using HOLDER's inequality we have

$$|f * g(x) - f * g(z)| \le \|\tau_{-x} f - \tau_{-z} f\|_p \|g\|_q.$$

We conclude by using the fact that the translation is uniformly continuous.

1.1.7 Density of $D(\Omega)$ in $L^p(\Omega)$.

One important application of the convolution product is regularization of functions, that is, the approximation of functions by smooth functions. Let

$$u(t) = \begin{cases} e^{-1/t} & \text{if } t > 0, \\ 0 & \text{if } t \le 0 \,. \end{cases} \tag{1.11}$$

Since for any integer k, $\lim_{t \to 0} \frac{1}{t^k} e^{-\frac{1}{t}} = 0$, then $u \in C^\infty(\mathbb{R})$. Let $\rho(x) = cu(1 - \|x\|^2)$, $x \in \mathbb{R}^N$. Then $\rho \in C^\infty(\mathbb{R}^N)$ and $\rho(x) = 0$ if $\|x\| \ge 1$. Moreover, for a suitable choice of the constant c we have $\rho(x) \ge 0$ and $\int_{\mathbb{R}^N} \rho(x) \, dx = 1$. Let

$$\rho_\epsilon(x) = \frac{1}{\epsilon^N} \rho(\frac{x}{\epsilon}).$$

Then

1. $\rho_\epsilon \in C^\infty(\mathbb{R}^N)$,
2. $\text{supp}(\rho_\epsilon) = B'(0,\epsilon) = \{x \in \mathbb{R}^N \mid \|x\| \leq \epsilon\}$,
3. $\rho_\epsilon(x) \geq 0$,
4. $\displaystyle\int_{\mathbb{R}^N} \rho_\epsilon(x)\,dx = 1$.

Any family (ρ_ϵ) satisfying these four properties is called FRIEDRICHS's mollifier.

Theorem 1.1.7.1 *Let ρ_ϵ be a* FRIEDRICHS*'s mollifier. If $f \in L^1(\mathbb{R}^N, loc)$ the convolution*

$$f * \rho_\epsilon(x) = \int_{\mathbb{R}^N} f(x-y)\rho_\epsilon(y)\,dy = \int_{\mathbb{R}^N} f(x-\epsilon y)\rho(y)\,dy$$

exists for each $x \in R^N$. Moreover

1. *$f * \rho_\epsilon \in C^\infty(\mathbb{R}^N)$,*
2. *$\text{supp}(f * \rho_\epsilon) \subset \text{supp}(f) + B'(0,\epsilon)$,*
3. *if $1 \leq p < +\infty$ and $f \in L^p(\mathbb{R}^N)$, then $f * \rho_\epsilon \to f$ in $L^p(\mathbb{R}^N)$, as $\epsilon \to 0$. In fact we have $\|f * \rho_\epsilon - f\|_p \leq \sup\limits_{\|y\|\leq\epsilon} \|\tau_y f - f\|_p$*
4. *If K, the set of continuity points of f is compact, then $f * \rho_\epsilon \to f$ uniformly on K as $\epsilon \to 0$*

Proof. The convolution exists for each x because the mollifier has compact support. Note that $f * \rho_\epsilon(x) = \displaystyle\int_{\mathbb{R}^N} \rho_\epsilon(x-y)f(y)\,dy$ implies $f * \rho_\epsilon \in C^\infty(\mathbb{R}^N)$ by standard results on differentiating under the integral sign (since ρ has compact support). The second is obvious and the third follows from

$$\|f * \rho_\epsilon - f\|_p \leq \int_{\mathbb{R}^N} \|f_{\epsilon y} f - f\|_p \rho(y)\,dy \leq \sup_{\|y\|\leq\epsilon} \|\tau_y f - f\|_p$$

Assume that K the set of continuity points of f is compact. Then f is uniformly continuous on K and this shows a little bit more: let $\eta > 0$, then there exists $\delta > 0$ such that if $x \in K$, $z \in \mathbb{R}^N$ and $\|x - z\| < \delta$ then it follows that $|f(x) - f(z)| < \eta$. Note that we do not require z to be in K. Now

$$f * \rho_\epsilon(x) - f(x) = \int_{\|y\|\leq\epsilon} (f(x-y) - f(x))\,\rho_\epsilon(y)\,dy.$$

Hence if $0 < \epsilon < \delta$ then

$$\begin{aligned} |f * \rho_\epsilon(x) - f(x)| &\leq \int_{\|y\|\leq\epsilon} |f(x-y) - f(x)|\,\rho_\epsilon(y)\,dy \\ &\leq \eta \int_{\mathbb{R}^N} \rho_\epsilon(y)\,dy = \eta \end{aligned}$$

for each $x \in K$.

Corollary 1.1.7.1 *Let Ω be an open subset of $\mathbb{R}^N$ and K is a compact subset of $\mathbb{R}^N$ then there exits $\phi \in D(\Omega)$ with $0 \leq \phi \leq 1$ and $\phi = 1$ on K.*

Proof. Let $\delta = \frac{1}{3}\text{dist}(K, \partial\Omega)$. Let L be the closed δ-neighbborhood of K, that is:

$$L := \{x \in R^N \mid \text{dist}(x, K) \leq \delta\}$$

Let f be the characteristic function of L and let $0 < \epsilon < \delta$. then $\phi = f * \rho_\epsilon \in C^\infty(\mathbb{R}^N)$ has its support in the closed 2δ-neighborhood of K and so has compact support in Ω. We have that $0 \leq \phi \leq 1$ and $\phi = 1$ on the $(\delta - \epsilon)$-neighborhood of K.

Theorem 1.1.7.2 (Density of $D(\Omega)$ in $L^p(\Omega)$) . *Let Ω be an open subset of $\mathbb{R}^N$ and let $1 \leq p < +\infty$. Then $D(\Omega)$ is dense in $L^p(\Omega)$*

Proof. Let $f \in L^p(\Omega)$ and let $\delta > 0$. By theorem (1.1.6.1) there exits $g \in C_c(\Omega)$ such that $\|f - g\|_p < \frac{\delta}{2}$. Let $\epsilon > 0$ and define $g_\epsilon = g * \rho_\epsilon$. Then $g_\epsilon \in C^\infty(\Omega)$ and $g_\epsilon \to g$ in $L^p(\Omega)$. Moreover

$$\text{supp}(g_\epsilon) \subset \text{supp}(g) + B'(0, \epsilon)$$

Since $g_\epsilon \to g$ in $L^p(\Omega)$ as $\epsilon \to 0$, there exists $\eta > 0$ such that $\|g_\epsilon - g\|_p < \frac{\delta}{2}$ for $\epsilon < \eta$. Now let $\epsilon < \min(\eta, \text{dist}(\text{supp}(g), \partial\Omega))$. Then $g_\epsilon \in D(\Omega)$ and $\|f - g_\epsilon\|_p < \delta$

Theorem 1.1.7.3 (Partition of unity) . *Let Ω be an open subset of $\mathbb{R}^N$ and let $(U_j)_{j\in J}$ be a locally finite open cover of Ω such that each U_j has compact closure in Ω. Then there exsits ϕ_j such that*

$$\phi_j \in D(U_j),\ \ \phi_j \geq 0 \ \text{and} \ \sum_{j\in J} \phi_j(x) = 1,\ \forall\, x \in \Omega$$

Proof. There exists an open covering (w_j) of Ω such that $\bar{w}_j \subset U_j \subset \bar{U}_j$ for all $j \in U_j$. Choose $\Psi_j \in D(U_j)$ such that $0 \leq \Psi_j \leq 1$ and $\Psi_j = 1$ on $\bar{w}_j$. The sum

$$\Psi(x) = \sum_{j\in J} \Psi_j(x)$$

is locally finite and bounded below by 1. Thus $\Psi \in C^\infty(\Omega)$ and $1 \leq \Psi$. take $\phi_j = \frac{\Psi_j}{\Psi}$

The ϕ_j are called a smooth partion of unity subordinate to the locally finite open cover (U_j).

Theorem 1.1.7.4 (Finite partition of unity) . *Let K be a compact subset of $\mathbb{R}^N$ and let $(U_j)_{j=1,\cdots,N}$ be a finite open cover of K. Then there exists functions $\phi_j \in D(U_j)$ such that $\phi_j \geq 0$ and*

$$\sum_{j=1}^{N} \phi_j = 1$$

in a neighborhood of K.

Proof. For $x \in K$, let V_x be an open neighborhood of x such that $\bar{V}_x$ is a compact subset of U_j with $x \in U_j$. Since K is compact there exsit a finite set $x_1, \cdots, x_m$ in K such that

$$K \subset \bigcup_{k=1}^{m} V_{x_k}.$$

For each j let K_j be the union of those $\bar{V}_{x_k}$ which are contained in U_j. Then K_j is compact, $K_j \subset U_j$ and

$$K \subset K_1 \cup \cdots \cup K_N$$

By corollary1.1.7.1, we may choose $\Psi_j \in D(U_j)$, $0 \leq \Psi_j \leq 1$ in a neighborhood of K_j and $\Psi_j = 1$ on K_j. Finally let

$$\begin{aligned}
\phi_1 &= \Psi_1 \\
\phi_2 &= (1-\Psi_1)\Psi_2 \\
\phi_3 &= (1-\Psi_1)(1-\Psi_2)\Psi_3 \\
\cdots & \\
\phi_N &= (1-\Psi_1)(1-\psi_2)\cdots(1-\Psi_{N-1})\Psi_N
\end{aligned}$$

We have, $\phi_j \geq 0$ and

$$\phi_1 + \phi_2 + \cdots + \phi_N = 1 - (1-\Psi_1)(1-\psi_2)\cdots(1-\Psi_N).$$

For each $x \in K$ there is j so that $\Psi_j(x) = 1$. Thus $\phi_1 + \phi_2 + \cdots + \phi_N = 1$ on K To obtain the equality on a neighborhood of K, we would enlarge K a bit.

Theorem 1.1.7.5 (Dubois-Reymond) . *Let Ω be an open subset of $\mathbb{R}^N$. If $f \in L^1(\Omega, loc)$ and $\int_\Omega f(x)\phi(x) = 0$ for each $\phi \in D(\Omega)$ then $f = 0$ a.e in Ω.*

1.2 Distribution Theory

1.2.1 Test Functions

Let Ω be a nonempty open subset of $\mathbb{R}^N$.

Notations

If $m \in \mathbb{N}$, $C^m(\Omega)$ denotes the space of real-valued functions on Ω of class C^m and $C^\infty(\Omega)$ the space of those of class C^∞. By convention, $C^0(\Omega) = C(\Omega)$ the space of continuous functions on Ω.

An element $\alpha \in \mathbb{N}^n$ is called a *multiindex*. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multiindex, we define the *length* of α to be the sum $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and we put $\alpha! = \alpha_1! \cdots \alpha_n!$. We give $\mathbb{N}^n$ the product order: if $\alpha, \beta \in \mathbb{N}^n$, we write $\alpha \leq \beta$ if $\alpha_1 \leq \beta_1, \cdots, \alpha_n \leq \beta_n$.

If $1 \leq i \leq n$, we often use D_i to denote $\dfrac{\partial}{\partial x_i}$. Then if α is a multiindex, we write

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{D^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

The differential operator D^α is also denoted by $\dfrac{\partial^{|\alpha|}}{x^\alpha}$ or ∂_x^α. By convention, D^0 (the differential of order 0 with espect to any index) is the identity map. We see that each operator D^α, where $\alpha \in \mathbb{N}^n$, acts on the space $C^m(\Omega)$, for $|\alpha| \leq m$.

We recall the following classical result:

Proposition 1.2.1.1 (Leibniz' formula) . *Let $u, v \in C^m(\Omega)$. For each multiindex α such that $|\alpha| \leq m$,*

$$D^\alpha(uv) = \sum_{\beta \leq \alpha} C_\alpha^\beta D^{\alpha-\beta} u D^\beta v,$$

where

$$C_\alpha^\beta = \prod_{i=1}^{n} \frac{\alpha_i!}{\beta_i!(\alpha_i - \beta_i)!} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

☞ We denote by $D^m(\Omega)$ the space of functions of class C^m having compact support in Ω. In particular, $D^0(\Omega) = C_c(\Omega)$. Clearly, $m' \geq m$ implies $D^{m'}(\Omega) \subset D^m(\Omega)$. Now we set

$$D(\Omega) = \bigcap_{m \in \mathbb{N}} D^m(\Omega),$$

Thus $D(\Omega)$ is the space of functions of class $C^\infty(\Omega)$ having compact support in Ω; such functions are called **test functions** on Ω.

Finally, if K is a compact subset of Ω, we denote by $D_K(\Omega)$ the space of functions of class C^∞ having support contained in K.

$$D_K(\Omega) = \bigcap_{m \in \mathbb{N}} D_K^m(\Omega)$$

thus

$$D(\Omega) = \bigcup_{K\in\mathcal{K}(\Omega)} D_K(\Omega)$$

where $\mathcal{K}(\Omega)$ is the set of compact subsets of Ω.

Clearly, a function in $D^m(\Omega)$ or $D(\Omega)$, when extended with the 0 value outside Ω becomes an element of $D^m(\mathbb{R}^N)$ or $D(\mathbb{R}^N)$, respectively. Thus $D^m(\Omega)$ and $D(\Omega)$ can be considered as subspaces of $D^m(\mathbb{R}^N)$ and $D(\mathbb{R}^N)$. We will often make this identification. Conversely, an element $\varphi \in D^m(\mathbb{R}^N)$ or $D(\mathbb{R}^N)$ belong to all the spaces $D^m(\Omega)$ or $D(\Omega)$ such that $\text{supp}(\varphi) \subset \Omega$.

1.2.2 Convergence in Function Spaces

• **Convergence in $D_K^m(\Omega)$ and $D_K(\Omega)$.** Let K be a compact subset of Ω. We say that a sequence (φ_n) in $D_K^m(\Omega)$ converges to $\varphi \in D_K^m(\Omega)$, if for every multiindex α such that $|\alpha| \leq m$, the sequence $(D^\alpha\varphi_n)$ converges uniformly to $D^\alpha\varphi$. An analogous definition applies with the replacement of $D_K^m(\Omega)$ by $D_K(\Omega)$ where now there is no restriction on the multiindex $\alpha \in \mathbb{N}^n$.

The convergence thus defined on $D_K^m(\Omega)$ clearly corresponds to the convergence in the norm $\|\cdot\|^{(m)}$ defined on $D_K^m(\Omega)$ by:

$$\|\varphi\|^{(m)} = \sum_{|\alpha|\leq m} \|D^\alpha \varphi\|_\infty,$$

where $\|\cdot\|_\infty$ denote the uniform norm. In contrast, no norm on $D_K(\Omega)$ yields the notion of convergence we have defined in that space.

• **Convergence in $D^m(\Omega)$ and $D(\Omega)$.** We say that a sequence (φ_n) in $D^m(\Omega)$ converges to φ in $D^m(\Omega)$ if the followings are satisfied:

(i) there exists a compact subset K of Ω such that

$$\text{supp}(\varphi) \subset K \text{ and } \text{supp}(\varphi_n) \subset K \text{ for all } n,$$

(ii) the sequence (φ_n) converge to φ in $D_K^m(\Omega)$.

An analogous definition applies with the replacement of $D^m(\Omega)$ and $D_K^m(\Omega)$ by $D(\Omega)$ and $D_K(\Omega)$.

• **Convergence in $C^m(\Omega)$ and $C^\infty(\Omega)$.** We say that a sequence (f_n) in $C^m(\Omega)$ converge to $f \in C^m(\Omega)$, if for every multiindex α such that $|\alpha| \leq m$ and for every compact K in Ω, the sequence $(D^\alpha f_n)$ converges to $(D^\alpha f)$ uniformly on K. An analogous definition applies with the replacement of $C^m(\Omega)$ by $C^\infty(\Omega)$, where now there is no restriction on the multiindex α. For $m = 0$, the convergence in $C^0(\Omega) = C(\Omega)$ thus defined coincides with the uniform convergence on compact subsets.

Remark 1.2.2.1 The definitions of convergence of sequences just made extend immediately to families (φ_λ), where λ runs over a subset in $\mathbb{R}$ and $\lambda \to \lambda_0,\ \lambda_0 \in [-\infty, +\infty]$.

We will see that it is possible to give the spaces $D_K(\Omega)$, $C^m(\Omega)$ and $C^\infty(\Omega)$ complete metric structures for which convergence of sequences coincides with the notions just defined. In contrast, one can show that the convergence we have defined in $D^m(\Omega)$ and $D(\Omega)$ cannot come from a metric structure.

In fact, the only topological notions that we will use in connection with these function spaces are continuity and denseness, and these notions, in the case of metric spaces, can always be expressed in terms of sequences.

1.2.3 Continuity and Denseness on $D^m(\Omega)$ and $D(\Omega)$

• A subset $\mathcal{C}$ of $D^m(\Omega)$ or $D(\Omega)$ will be called dense in $D^m(\Omega)$ or $D(\Omega)$, if for every φ in $D^m(\Omega)$ or $D(\Omega)$, there exists a sequence (φ_n) in $\mathcal{C}$ converging to φ in $D^m(\Omega)$ or $D(\Omega)$.

• A function F on $D^m(\Omega)$ or $D(\Omega)$ and taking values in a metric space or in one of the spaces just introduced will be called continuous, if for every sequence (φ_n) in $D^m(\Omega)$ or $D(\Omega)$ that converges to φ in $D^m(\Omega)$ or $D(\Omega)$, the sequence $(F(\varphi_n))$ converges to $F(\varphi)$ in the space considered.

For example, the *Canonical Injection* from $D^m(\Omega)$ to $C^m(\Omega)$ is continuous. This means simply that every sequence in $D^m(\Omega)$ that converges in $D^m(\Omega)$ also converges in $C^m(\Omega)$ to the same limit.

Proposition 1.2.3.1 *For every $m \in \mathbb{N}$, the space $D(\Omega)$ is dense in $D^m(\Omega)$. In particular, $D(\Omega)$ is dense in $C_c(\Omega)$.*

Lemma 1.2.3.1 *Let Ω be an open subet of $\mathbb{R}^N$. For $n \in \mathbb{N}^*$, define*

$$K_n = \{x \in \mathbb{R}^N \mid \|x\| \leq n \text{ and } d(x, \Omega^c) \geq \frac{1}{n}\}$$

where d is the usual distance in $\mathbb{R}^N$. Then

1. *Each K_n is a compact subset of Ω and $K_n \subset \overset{\circ}{K}_{n+1}$.*

2. $\Omega = \bigcup_{n=1}^{\infty} K_n = \bigcup_{n=2}^{\infty} \overset{\circ}{K}_n$.

3. *For all compact K in Ω, there exists $N \geq 1$ such that $K \subset \overset{\circ}{K}_N$.*

Proposition 1.2.3.2 *The space $D(\Omega)$ is dense in $C^\infty(\Omega)$ and in $C^m(\Omega)$ for every $m \in \mathbb{N}$*

Proof. Let K_n be a sequence of compact subsets of Ω exhausting Ω (as above). Then there exists for each n, an element $\varphi_n \in D(\Omega)$ such that

$$0 \leq \varphi_n \leq 1, \quad \varphi_n = 1 \text{ on } K_n, \quad \text{supp}(\varphi_n) \subset \overset{\circ}{K}_{n+1}.$$

Now let $f \in C^\infty(\Omega)$, we have $(f\varphi_n) \in D(\Omega)$ for every $n \in \mathbb{N}$. If K is a compact subset of Ω, there is an integer N such that $K \subset \overset{\circ}{K}_N$. Thus for every $n \geq N$ and for every $\alpha \in \mathbb{N}^n$, we have $D^\alpha(f\varphi_n) = D^\alpha f$ on K. By the definition of convergence in $C^\infty(\Omega)$, we deduce that $(f\varphi_n)$ converge to f in $C^\infty(\Omega)$.

1.2.4 Distributions

Definition A distribution on Ω is a continuous linear mapping T from $D(\Omega)$ into $\mathbb{R}$. The set of all distributions is denoted by $D'(\Omega)$.

Remark 1.2.4.1 By Linearity, to show that T is continuous, it is enough to show that, if $\varphi_n \to 0$ in $D(\Omega)$ then $(T, \varphi_n) \to 0$ in $\mathbb{R}$.

Theorem 1.2.4.1 *Let T be a linear mapping from $D(\Omega)$ into $\mathbb{R}$. Then T is a distribution if and only if, for any compact set K in Ω, there exists an integer $n_K \in \mathbb{N}$ and a positive constant C_K such that:*

$$|(T, \varphi)| \leq C_K \sum_{|\alpha| \leq n_K} \sup_K |D^\alpha \varphi(x)|, \;\; \forall \varphi \in D_K(\Omega).$$

Definition If n_K can be chosen independent of K, then the smallest n with this property is called the *order* of the distribution T.

Example 1.2.4.1 (Distribution given by a locally integrable function) Let $f \in L^1(\Omega, loc)$, then f gives a distribution T_f defined by:

$$(T_f, \varphi) = \int_\Omega f(x) \cdot \varphi(x)\, dx, \;\; \forall \varphi \in D(\Omega).$$

The linearity of T_f follows from the linearity of integral. Now let K be a compact subset of Ω and let $\varphi \in D(\Omega)$ with supp$(\varphi) \subset K$ then we have

$$|(T_f, \varphi)| \leq \left(\int_K |f(x)|\, dx \right) \cdot \left(\sup_K |\varphi(x)| \right),$$

so T is a distribution of order 0.

We define the maping $f \to T_f$. It is linear and one to one. In fact let $f \in L^1(\Omega, loc)$ such that $T_f = 0$, Then by using *Dubois-Reymond*'s lemma, we show that $f = 0$ a.e in Ω. From now, we can identify $L^1(\Omega, loc)$ as a subset of $D'(\Omega)$.

Example 1.2.4.2 (Dirac distribution) .

Let $x_0 \in \Omega$. We denote by δ_{x_0} the linear form defined on $D(\Omega)$ by

$$(\delta_{x_0}, \varphi) = \varphi(x_0), \;\; \forall \varphi \in D(\Omega).$$

Let K be a compact subset of Ω. Since $|(\delta_{x_0}, \varphi)| \leq \sup_K |\varphi(x)|$ for all $\varphi \in D(\Omega)$ with supp$(\varphi) \subset K$, then δ_{x_0} is a distribution.

Example 1.2.4.3 (The distribution Principal Value of $1/x$) .
Consider the function $x \mapsto 1/x$ from $\mathbb{R}$ to $\mathbb{R}$. This function is clearly not locally integrable on $\mathbb{R}$ but it is on $\mathbb{R}^*$. We will see how we can extend to $\mathbb{R}$ the distribution defined by this function on $\mathbb{R}^*$.

Proposition 1.2.4.1 *For every $\varphi \in D(\mathbb{R})$, the limit*

$$(pv(1/x), \varphi) = \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\varphi(x)}{x} \, dx$$

exists. The linear form $pv(1/x)$ thus defined a distribution of order 1 on $\mathbb{R}$, and is an extension to $\mathbb{R}$ of the distribution $[1/x] \in D'(\mathbb{R}^*)$.

We call $pv(1/x)$ the **principal value** of $1/x$.

Example 1.2.4.4 (The distribution Finite part of $1/x^2$.) Let $\varphi \in D(\Omega)$, we call the distribution Finite part, the distribution denoted by $fp(1/x^2)$ and defined by:

$$(fp(1/x^2), \varphi) = \lim_{\epsilon \to 0^+} \left(\int_{|x| \geq \epsilon} \frac{\varphi(x)}{x^2} \, dx - 2\frac{\varphi(0)}{\epsilon} \right) \;\forall \varphi \in D(\mathbb{R}).$$

Proposition 1.2.4.2 *Let T be a distribution on Ω such that every point $x \in \Omega$ has an open neighborhood V_x such that $(T, \varphi) = 0$ for all $\varphi \in D(V_x)$. Then $T = 0$.*

Proof. Let $\varphi \in D(\Omega)$, we will show that $(T, \varphi) = 0$. Let $K = \text{supp}(\varphi)$ and let $x \in K$ then by hypothesis, there exists an open neighborhood V_x of x such that $(T, \varphi) = 0$ for all $\varphi \in D(V_x)$. Since K is compact, there exists $x_1, \cdots, x_N \in K$ such that

$$K \subset \bigcup_{i=1}^{N} V_{x_i}$$

Let $\alpha_1. \cdots, \alpha_N$ be a partition of unity associated to this open cover of K. Then

$$\varphi = \sum_{i=1}^{N} \alpha_i \varphi$$

Since $\text{supp}(\alpha_i \varphi) \subset V_{x_i}$ then $(T, \alpha_i \varphi) = 0$ and so is (T, φ).

1.2.5 The Support of a Distribution

Definition Let T be a distribution on Ω, an open of nullity of T is an open subset U of Ω such that $(T, \varphi) = 0$ for all $\varphi \in D(U)$.

Proposition 1.2.5.1 *Any distribution T has a largest open of nullity Ω_0. Its complement is called the support of T and denoted by supp(T).*

Proof. Let $\mathcal{U}$ be the collection of opens of nullity of T, and let $\Omega_0 = \bigcup_{U \in \mathcal{U}} U$ be there union. It suffices to show that Ω_0 is it self an open of nullity of T. Take $\varphi \in D(\Omega_0)$. By compactness of the support of φ, their exists a finite collection of opens sets $U_1, \cdots, U_N$ whose union contains the support of φ. Let $\alpha_i, i = 1 \cdots, N$ be a partion of unity associated to this open cover of $\text{supp}(\varphi)$. It follows that

$$\varphi = \sum_{i=1}^{N} \varphi \cdot \alpha_i.$$

Since each $\varphi \cdot \alpha_i$ is supported in the open of nullity U_i, this implies that

$$(T, \varphi) = \sum_{i=1}^{N} (T, \varphi \cdot \alpha) = 0.$$

This proves that Ω_0 is indeed an open of nullity of T, and by construction it is the largest of such open sets.

Somes consequences of the definition:

1. $x_0 \notin \text{supp}(T)$ if and only if there exists an open neighborhood V_{x_0} of x_0 such that

$$(T, \varphi) = 0, \ \forall \ \varphi \in D(V_{x_0}),$$

2. $x_0 \in \text{supp}(T)$ if and only for all open neighborhood V_{x_0} of x_0, there exists $\varphi \in D(V_{x_0})$ such that $(T, \varphi) \neq 0$.

Proposition 1.2.5.2 *Let T be a distribution on Ω and $\varphi \in D(\Omega)$ such that*

$$supp\,(T) \cap supp(\varphi) = \emptyset.$$

Then $(T, \varphi) = 0$.

1.2.6 Distributions with Compact Support

Theorem 1.2.6.1 *Let T be a distribution on Ω. A necessary and sufficient condition for the support of T to be compact is that T has an extension to a continuous linear form on $C^\infty(\Omega)$. The extension is then unique.*

Proof. Suppose first that the support of T is compact. Then there exists a compact K in Ω whose interior contains the support of T. It follows from corollary 1.1.7.1 that there exists $\rho \in D(\Omega)$ such that $0 \leq \rho \leq 1$ and $\rho(x) = 1$ on K. We then set, for $f \in C^\infty(\Omega)$,

$$(\bar{T}, f) = (T, \rho f). \tag{1.12}$$

It is clear that this does define a linear form $\bar{T}$ on $C^\infty(\Omega)$. On the other hand, if $\varphi \in D(\Omega)$, we have

$$\text{supp}\,(\varphi - \rho\varphi) \subset \Omega - \overset{\circ}{K} \subset \Omega - \text{supp}\,(T),$$

this implies that

$$\text{supp}\,(\varphi - \rho\varphi) \cap \text{supp}\,(T) = \emptyset.$$

So by proposition 1.2.5.2, it follows that

$$(\bar{T}, \varphi) = (T, \varphi).$$

Thus $\bar{T}$ is an extension of T to $C^\infty(\Omega)$.

Finally, if (f_n) is a sequence in $C^\infty(\Omega)$ that converges to 0 in $C^\infty(\Omega)$, then from the definitions and *Leibniz*'s formula the sequence (ρf_n) converges to 0 in $D(\Omega)$, so that

$$\lim_{n \to +\infty} (\bar{T}, f_n) = \lim_{n \to +\infty} (T, \rho f_n) = 0$$

This proves that $\bar{T}$ is continuous on $C^\infty(\Omega)$. Since $D(\Omega)$ is dense in $C^\infty(\Omega)$, the extension is unique.

For the converse, assume that T can be extended to a continuous linear form $\bar{T}$ on $C^\infty(\Omega)$. Let (K_n) be an exhausting sequence of compact subsets of Ω. If the support of T is not compact, then there exists for each $n \in \mathbb{N}$, an element $\varphi_n \in D(\Omega)$ such that

$$\text{supp}(\varphi_n) \subset \Omega - K_n \text{ and } (T, \varphi_n) \neq 0.$$

Put

$$\Psi_n = \frac{\varphi_n}{(T, \varphi_n)},$$

so we have

$$(T, \Psi_n) = 1, \ \forall n \in \mathbb{N}.$$

Now we will show that the series $\sum_{n=1}^{n} \Psi_n$ converges in $C^\infty(\Omega)$. To this end, let K be a compact subset of Ω, then there exists $N \in \mathbb{N}$ such that $K \subset K_N$. But for $n > N$, we have $\Psi_n = 0$ on K_N, and so on K, the sum $\sum_{n=0}^{\infty} \Psi_n$ reduces to a finite sum on K, and this holds for every compact subset K on Ω. So the sum converges in $C^\infty(\Omega)$. By the continuity of $\bar{T}$, it follows that the series $\sum_{n=0}^{\infty}(T, \Psi_n)$ converges, contradicting the fact that $(T, \Psi_n) = 1$.

Remark 1.2.6.1 The restriction to $D(\Omega)$ of a continuous linear form on $C^\infty(\Omega)$ is a distribution on Ω (since a sequence in $D(\Omega)$ that converges in $D(\Omega)$ also converges in $C^\infty(\Omega)$), and by the preceding theorem this distribution has compact support. Thus we can identify the space of distributions having compact support with the space of continuous linear forms on $C^\infty(\Omega)$ denoted by $C^\infty(\Omega)'$.

Proposition 1.2.6.1 *Every distribution T with compact support in Ω has finite order. More precisely, there exists an integer $m \in \mathbb{N}$ and a constant $C' \geq 0$ such that*

$$|(T, \varphi)| \leq C' \|\varphi\|^{(m)}, \ \forall \varphi \in D(\Omega).$$

Proof. Let K be the support of T and let K_1, K_2 be compact subsets of Ω such that

$$K \subset \mathring{K}_1 \subset K_1 \subset \mathring{K}_2 \subset K_2 \subset \Omega.$$

Then by *theorem* 1.2.4.1, there exists an integer $m \in \mathbb{N}$ and a constant $C \geq 0$ such that

$$|(T, \varphi)| \leq C \|\varphi\|^{(m)}, \ \forall \varphi \in D_{K_2}(\Omega).$$

By *corollary* 1.1.7.1, there exists $\Psi \in D(\Omega)$ such that $0 \leq \Psi \leq 1$, $\Psi = 1$ on K_1 and $\text{supp}(\Psi) \subset \mathring{K}_2$. If $\varphi \in D(\Omega)$ then $\varphi\Psi \in D_{K_2}(\Omega)$ and

$$\text{supp}\,(\varphi - \varphi\Psi) \subset \Omega - \mathring{K}_1 \subset \Omega - K.$$

Since K is the support of T, then

$$(T, \varphi - \varphi\Psi) = 0,$$

so there exists a positive constant C depending only on C', m and Ψ such that

$$|(T,\varphi)| = |(T,\varphi\Psi)| \leq C'\|\varphi\Psi\|^{(m)} \leq C\|\varphi\|^{(m)}.$$

The last inequality being a consequence of *Leibniz*' formula.

Remark 1.2.6.2 One can deduce from the preceding result that if, T is a distribution with compact support, there exists an integer $m \in \mathbb{N}$ such that T extends to a continuous linear form on $C^m(\Omega)$ and this extension is unique.

1.2.7 Convergence of Distributions

We assume that Ω is an open subset of $\mathbb{R}^N$.

Definition Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of distributions in Ω. We say that (T_n) converges to the distribution T if

$$\lim_{n\to+\infty}(T_n,\varphi) = (T,\varphi), \text{ for all } \varphi \in D(\Omega).$$

Theorem 1.2.7.1
1. Let $1 \leq p \leq +\infty$. If $f_n, f \in L^p(\Omega)$ with $f_n \to f$ in $L^p(\Omega)$ then $f_n \to f$ in $D'(\Omega)$.
2. The pointwise convergence does not imply the convergence in $D'(\Omega)$.

Proof.
1. Let q such that $1/p + 1/q = 1$. Then by *Holder*'s inequality we have:

$$\begin{aligned} |(f_n,\varphi) - (f,\varphi)| &= |(f_n - f,\varphi)| \\ &\leq \int_\Omega |f_n - f|\cdot|\varphi|\,dx \leq \|f_n - f\|_p\|\varphi\|_q \to 0. \end{aligned}$$

2. Let (f_n) be the sequence of functions defined by

$$f_n(x) = \sqrt{n}e^{-nx^2}, \;\; x \neq 0,$$

then $f_n(x) \to 0$ for all $x \neq 0$ but $f_n \to \sqrt{\pi}\delta_0$ in $D'(\Omega)$. In fact let $\varphi \in D(\Omega)$, by Lebesgue dominated convergence therorem we have

$$(f_n,\varphi) = \sqrt{n}\int_{\mathbb{R}} e^{-nx^2}\varphi(x)\,dx = \int_{\mathbb{R}} e^{-y^2}\varphi(\frac{y}{\sqrt{n}})\,dy \to \sqrt{\pi}\varphi(0) = \sqrt{\pi}(\delta_0,\varphi)$$

Examples

Example 1.2.7.1 Let $(T_n)_{n\in\mathbb{N}}$, be a sequence of distributions on $\mathbb{R}$ defined by:

$$T_n(x) = \sin(nx).$$

Let $\varphi \in D(\mathbb{R})$, we have

$$(T_n,\varphi) = \int_{\mathbb{R}} \sin(nx)\varphi(x)\,dx = \frac{1}{n}\int_{\mathbb{R}} \cos(nx)\varphi'(x)\,dx \to 0.$$

So T_n converge to 0 in $D'(\mathbb{R})$.

Example 1.2.7.2 For $\epsilon. > 0$ define

$$v_\epsilon(x) = \begin{cases} \frac{1}{\epsilon} & \text{if } x \in [0, \epsilon], \\ \\ 0 & \text{if } x \notin [0, \epsilon] \ . \end{cases} \quad \text{and} \quad w_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon} & \text{if } x \in [-\epsilon, \epsilon], \\ \\ 0 & \text{if } x \notin [-\epsilon, \epsilon] \ . \end{cases}$$

Then we have:

$$v_\epsilon \to \delta_0 \text{ in } D'(\mathbb{R}) \text{ as } \epsilon \to 0 \text{ and } w_\epsilon \to \delta_0 \text{ in } D'(\mathbb{R}) \text{ as } \epsilon \to 0.$$

1.2.8 Multiplication of Distributions

Now, we define the product of distribution by a smooth function. The definition arises from the following lemma.

Lemma 1.2.8.1 *Let $\alpha \in C^\infty(\Omega)$. The map $\varphi \to \alpha\varphi$ from $D(\Omega)$ to $D(\Omega)$ is linear and continuous. In other words if (φ_n) is a sequence in $D(\Omega)$ converging to φ in $D(\Omega)$ then the sequence $(\alpha\varphi_n)$ converges to $\alpha\varphi$ in $D(\Omega)$.*

Definition If $T \in D'(\Omega)$ and $\alpha \in C^\infty(\Omega)$, the product distribution αT on Ω is defined by setting:

$$(\alpha T, \varphi) = (T, \alpha\varphi), \ \ \forall\, \varphi \in D(\Omega).$$

The fact that αT defines a distribution follows from the preceding lemma.

The definition immediately implies that if $\alpha \in C^\infty(\Omega)$, the linear map $T \to \alpha T$ from $D'(\Omega)$ to $D'(\Omega)$ is continuous in the sense that, if (T_n) converge to T in $D'(\Omega)$ then (αT_n) converge to (αT) in $D'(\Omega)$.

Proposition 1.2.8.1 *Let $T \in D'(\Omega)$ and $\alpha \in C^\infty(\Omega)$, then we have*

$$supp(\alpha T) \subset supp(\alpha) \cap supp(T).$$

Proof. Let $\varphi \in D(\Omega)$. If $\text{supp}(\varphi) \subset \Omega\text{-supp}(\alpha)$, then $\alpha\varphi = 0$, so $(\alpha T, \varphi) = 0$. It follows that Ω -supp(α) is contained in Ω -supp(αT), so supp(αT) $\subset$supp(α).
Now if If supp$(\varphi) \subset \Omega$ -supp(T), then supp$(\alpha\varphi) \subset$ supp$(\varphi) \subset \Omega-$supp$(T)$, which implies that $(\alpha T, \varphi) = 0$. Therefore $\Omega-$supp(T) is contained in $\Omega-$supp(αT) so supp$(\alpha T) \subset$ supp(T).

The inclusion in the proposition may be strict. For example, if $T = \delta$ is the dirac distribution in $\mathbb{R}^N$, and $\alpha \in C^\infty(\mathbb{R}^N)$ is such that $\alpha(0) = 0$ and $0 \in \sup(\alpha)$ (say $\alpha(x) = x$), then $\alpha T = \alpha(0)\delta = 0$ and the support of αT is empty, whereas supp$(\alpha)\cap$supp$(T) = \{0\}$.

Proposition 1.2.8.2 *Let $S \in D'(\mathbb{R})$, then there exists a distribution $T \in D'(\mathbb{R})$ such that $xT = S$. If T_0 is such that $xT_0 = S$, the set of solutions of the equation $xT = S$ is equal to $\{T_0 + c\delta, \ c \in \mathbb{R}\}$.*

Proof. Fix $\eta \in D(\mathbb{R})$ such that $\eta(0) = 1$. To each $\varphi \in D(\mathbb{R})$, we associate $A\varphi$ defined by

$$A\varphi(x) = \int_0^1 (\varphi'(tx) - \varphi(0)\eta'(tx))\ dt.$$

One can easily check that $A\varphi \in D(\mathbb{R})$ and the map $\varphi \to A\varphi$ from $D(\mathbb{R})$ to $D(\mathbb{R})$ is continuous. Moreover if $x \in \mathbb{R}^*$,

$$A\varphi(x) = \frac{\varphi(x) - \varphi(0)\eta(x)}{x}.$$

Now put

$$(T, \varphi) = (S, A\varphi), \quad \forall\ \varphi \in\ D(\mathbb{R}).$$

Since $\varphi \to A\varphi$ is continuous then T is a distribution on $\mathbb{R}$. Since $A(x\varphi) = \varphi$, we get $(xT, \varphi) = (T, x\varphi) = (S, A(x\varphi)) = (S, \varphi)$, so $xT = S$.

Now take $T \in D'(\mathbb{R})$ with $xT = 0$. If $\varphi \in D(\mathbb{R})$, we have

$$0 = (xT, A\varphi) = (T, \varphi - \varphi(0)\eta) = (T, \varphi) - (T, \eta) \cdot (\delta, \varphi)$$

it follows that $T = (T, \eta)\delta$

Corollary 1.2.8.1 *Let $T \in D'(\mathbb{R})$. Then $xT = 1$ if and only if there exist a constant $c \in \mathbb{R}$ such that*

$$T = pv(\frac{1}{x}) + c\delta.$$

1.2.9 Differentiation of Distributions

We shall define the derivative of a distribution in such a way that it agrees with the usual notion of derivative on those distributions which arise from continuously differentiable functions. That is, we want to define

$$D^\alpha : D'(\Omega) \to D'(\Omega)$$

so that

$$D^\alpha(T_f) = T_{D^\alpha f}, \ \ |\alpha| \leq\ m, \ \ f \in C^m(\Omega).$$

But a computation with integration by parts gives

$$(T_{D^\alpha f}, \varphi) = (-1)^{|\alpha|}(T_f, D^\alpha\varphi), \ \ \varphi \in\ D(\Omega)$$

and this identity suggests the following:

Definition The α^{th} partial derivative of the distribution T is the distribution $D^\alpha T$ defined by

$$(D^\alpha T, \varphi) = (-1)^{|\alpha|}(T, D^\alpha\varphi), \ \ \varphi \in\ D(\Omega).$$

Since D^α is linear from $D(\Omega)$ into $D(\Omega)$, then it is clear that $D^\alpha T$ is a distribution. Every distribution has derivatives of all orders and so also then does every functions, e.g, in $L^1(\Omega, loc)$, when it is identified as a distribution. Furthermore, by the usual definition of derivative, it is clear that the two notions of derivative are compatible with the identification of $C^\infty(\Omega)$ in $D'(\Omega)$.

Proposition 1.2.9.1 *If (T_n) is a sequence of distribution converging to the distribution T, then for any $\alpha \in \mathbb{N}^n$, $D^\alpha T_n$ converges to $D^\alpha T$.*

Proof. Let $\alpha \in \mathbb{N}^n$. We have

$$(D^\alpha T_n, \varphi) = (-1)^{|\alpha|}(T_n, D^\alpha \varphi) \to (-1)^{|\alpha|}(T, D^\alpha \varphi) = (D^\alpha T, \varphi).$$

Proposition 1.2.9.2 (Leibniz's Formula) *Let $T \in D'(\Omega)$, $f \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}^n$. Then we have*

$$D^\alpha(fT) = \sum_{\beta \le \alpha} C_\alpha^\beta D^{\alpha-\beta} f D^\beta T.$$

Examples

We give some examples of distributions on $\mathbb{R}$. Since we do not distinguish the function $f \in L^1(\mathbb{R}, loc)$ from the distribution T_f, we have the identity

$$(f, \varphi) = \int_{\mathbb{R}} f(x) \cdot \varphi(x)\, dx, \;\; \varphi \in D(\Omega).$$

Example 1.2.9.1 If $f \in C^1(\mathbb{R})$, then

$$((T_f)', \varphi) = -(T_f, \varphi') = -\int_{\mathbb{R}} f(x) \cdot \varphi'(x)\, dx = \int_{\mathbb{R}} f'(x) \cdot \varphi(x) = (f', \varphi), \;\; \varphi \in D(\Omega).$$

where the third equality follows by the integration by parts and all others are definitions. Thus $(T_f)' = f'$, which is no surprise since the defintion of derivative of distributions was rigged to make this so.

Example 1.2.9.2 Let the ramp and *Heaviside* functions be given respectively by

$$r(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x \le 0\,. \end{cases} \qquad H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0\,. \end{cases}$$

We have

$$((T_r)', \varphi) = -(T_r, \varphi') = -\int_0^{+\infty} x\varphi(x)\, dx = \int_{\mathbb{R}} H(x) \cdot \varphi(x) = (H, \varphi), \;\; \varphi \in D(\Omega),$$

so we have $(T_r)' = H$, although $r'(0)$ does not exit.

Example 1.2.9.3 The derivative of the non-continuous function H is given by

$$((T_H)', \varphi) = -(T_H, \varphi') = -\int_0^{+\infty} \varphi'(x)\, dx = \varphi(0) = \delta(\varphi), \;\; \varphi \in D(\Omega),$$

that is $(T_H)' = \delta$, the *Dirac*'s distribution. Also, it follows directly from the definition of derivative that

$$(D^m \delta, \varphi) = (-1)^m D^m \varphi(0), \; m \ge 1.$$

4. Let $A(x) = |x|$ and $I(x) = x, \; x \in \mathbb{R}$. We observe that $A = 2r - I$ and then from above we obtain by linearity

$$(T_A)' = 2H - 1, \;\; (T_A)'' = 2\delta.$$

5. Let f defined for $x \neq 0$ by $f(x) = \ln(|x|)$. Show that $(T_f)' = pv\frac{1}{x}$.

Theorem 1.2.9.1 *Let $T \in D'(\mathbb{R})$. Then $T' = 0$ if and only if $T =$ constant, that is there exists a constant $c \in \mathbb{R}$ such that $T = c$.*

Theorem 1.2.9.2 *Let $S \in D'(\mathbb{R})$. Then there exist a distribution $T \in D'(\mathbb{R})$ such that*

$$T' = S.$$

Proof. First we remark that $T' = S$ if and only if

$$(T, \varphi') = -(S, \varphi), \quad \varphi \in D(\mathbb{R}).$$

This suggests to consider $H = \{\varphi' \ : \ \varphi \in D(\mathbb{R})\}$. By above, we have seen that if $\Psi \in D(\mathbb{R})$ then $\Psi \in H$ if and only if $\int_{\mathbb{R}} \Psi(x)\, dx = 0$. Now let $\varphi_0 \in D(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi_0(x)\, dx = 1$. We shall show that $D(\mathbb{R})$ is direct sum of H and $\mathbb{R} \cdot \varphi_0$, that is, each φ can be written in exactly one way as the sum of a $\Psi \in H$ and a constant multiple of φ_0.

To check the uniqueness of such sum, let $\Psi_1 + c_1\varphi_0 = \Psi_2 + c_2\varphi_0$ with $\Psi_1, \Psi_2 \in H$. Integrating both sides gives $c_1 = c_2$ and, hence $\Psi_1 = \Psi_2$.

To verify existence of such representation, for each $\varphi \in D(\mathbb{R})$, choose $c = \int_{\mathbb{R}} \varphi(x)\, dx$ and define $\Psi = \varphi - c\varphi_0$. Then $\Psi \in H$ and we are done.

To finish the proof, it suffices by our remark above to define T in H, so that we can extend it to all $D(\mathbb{R})$ by linearity after choosing e.g $(T, \varphi_0) = 0$

Corollary 1.2.9.1 *If $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then the derivative of f in the usual sense $f'(x)$ exits a.e, $x \in \mathbb{R}$, the function $g(x) = f'(x)$ a.e, $x \in \mathbb{R}$ is in $L^1(\mathbb{R}, loc)$ and $(T_f)' = g$ in $D'(\mathbb{R})$. Conversely, if T is a distribution on $\mathbb{R}$ with $T' \in L^1(\mathbb{R}, loc)$ then there exists an absolutely continuous function f such that $T = T_f$.*

CHAPTER 2

SOBOLEV SPACES $W^{M,P}$

2.1 Definitions and main properties

In this chapter we study some of the most important properties of a class of function spaces known as **Sobolev spaces** which will provide the proper functional for the study of the partial differential equations of the following chapters. In what follows, Ω is an open set in $\mathbb{R}^N$ and $\partial\Omega$ is its boundary.

Definition (Weak Derivative). Let $u \in L^1(\Omega, loc)$. For a given multiindex $\alpha \in \mathbb{N}^n$, a function $v \in L^1(\Omega, loc)$ is called the α^{th}-weak derivative of u if

$$\int_\Omega u \cdot D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v \cdot \varphi \, dx, \quad \forall \, \varphi \in \; D(\Omega). \tag{2.1}$$

v is also refered as the *generalized derivative* of u and we write $v = D^\alpha u$. Clearly $D^\alpha u$ is uniquely determined up to set of *Lebesgue* measure zero.

Definition Let m be a non-negative integer and $1 \le p \le \infty$. The **Sobolev space** $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega), \;\; | \;\; D^\alpha u \; \in \; L^p(\Omega), \;\; \text{for all} \;\; \alpha \; \in \; \mathbb{N}^n \; : \; |\alpha| \; \le m\}. \tag{2.2}$$

In other words, $W^{m,p}(\Omega)$ is the collection of all functions in $L^p(\Omega)$ such that all weak derivatives up to order m are also in $L^p(\Omega)$. Clearly $W^{m,p}(\Omega)$ is a vector space. (In all that follows we will consider functions with values in $\mathbb{R}$ and the corresponding function spaces as vector spaces over $\mathbb{R}$). We provide it with the norm:

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p(\Omega)}. \tag{2.3}$$

for $1 \le p < \infty$, we have

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}. \tag{2.4}$$

Finally,we also have

$$\|u\|_{W^{m,p}(\Omega)} = max\{\|D^\alpha u\|_{L^p(\Omega)} : |\alpha| \ \leq m\} \tag{2.5}$$

Remark 2.1.0.1 We will not distinguish in the future between these three norms though they are only equivalent and not equal. We will use the same notation for all and take care in any computation that we consistently use only one of the three formulas.

The case $p = 2$ will play a special role in the sequel. These spaces will be denoted by $H^m(\Omega)$. Thus

$$H^m(\Omega) = W^{m,2}(\Omega) \tag{2.6}$$

and for $u \in H^m(\Omega)$, we denote its norm by

$$\|u\|_{H^m(\Omega)} = \|u\|_{W^{m,2}(\Omega)}. \tag{2.7}$$

The spaces $H^m(\Omega)$ have a natural inner-product defined by

$$(u, v)_{H^m} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v \, dx, \quad \forall \, u \, v \ \in \ H^m(\Omega). \tag{2.8}$$

This inner-product yields the norm given by formula (2.4).

Finally, we introduce an important subspace of the space $W^{m,p}(\Omega)$. If $1 \leq p < \infty$, we know that $D(\Omega)$ is dense in $L^p(\Omega)$. Also, if $\varphi \in D(\Omega)$, so does every derivative of φ and so $D(\Omega) \subset W^{m,p}(\Omega)$, for any m and p. If $1 \leq p < \infty$, we define the space $W_0^{m,p}(\Omega)$ as the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$. Thus $W_0^{m,p}(\Omega)$ is a closed subspace of $W^{m,p}(\Omega)$ and its elements can be approximated in the $W^{m,p}(\Omega)$-norm by C^∞ functions with compact support. In general this is a strict subspace of $W^{m,p}(\Omega)$, except when $\Omega = \mathbb{R}^N$ as we will see later.

Let us return to the spaces $W^{m,p}(\Omega)$. The map

$$u \in W^{1,p}(\Omega) \to \left(u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_i} \right) \in (L^p(\Omega))^{N+1} \tag{2.9}$$

is an isometry of $W^{1,p}(\Omega)$ into $(L^p(\Omega))^{N+1}$ if we provide the later space with the norm

$$\|u\| = \sum_{i=1}^{N+1} \|u_i\|_{L^p(\Omega)} \quad \text{or} \quad \|u\| = \left(\sum_{i=1}^{N+1} \|u_i\|^p_{L^p(\Omega)} \right)^{1/p}$$

for $u = (u_i) \in (L^p(\Omega))^{N+1}$, depending on whether we use the formula (2.3) or (2.4) on $W^{1,p}(\Omega)$. This is a useful fact to remember and will be used in the proof of the folowing result.

Theorem 2.1.0.1
The spaces $W^{1,p}(\Omega)$ are *Banach* for $1 \leq p \leq +\infty$, *separable* for $1 \leq p < \infty$ and *reflexive* for $1 < p < \infty$. In particular $H^1(\Omega)$ is a *separable Hilbert* space.

Proof. Let (u_n) be a Cauchy sequence in $W^{1,p}(\Omega)$. It follows from the definition of the norm that (u_n) and $\left(\frac{\partial u_n}{\partial x_i}\right)$, $1 \leq i \leq n$, are all Cauchy sequences in $L^p(\Omega)$ which is complete. Therefore $u_n \to u$ in $L^p(\Omega)$ and $\frac{\partial u_n}{\partial x_i} \to v_i$ in $L^p(\Omega)$, $1 \leq i \leq n$.

Using theorem 1.2.7.1, we have

$$u_n \to u \quad \text{in} \quad D'(\Omega) \quad \text{and} \quad \frac{\partial u_n}{\partial x_i} \to v_i \quad \text{in} \quad D'(\Omega), 1 \leq i \leq n \tag{2.10}$$

But we know that D^α is continuous $\forall \alpha \in \mathbb{N}$ then,

$$\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{in} \quad D'(\Omega). \tag{2.11}$$

Therefore $\dfrac{\partial u}{\partial x_i} = v_i$, by the uniqueness of limit .
Thus $\dfrac{\partial u}{\partial x_i} \in L^p(\Omega)$, so $u \in W^{1,p}(\Omega)$ and $u_n \to u$ in $W^{1,p}(\Omega)$. Hence $W^{1,p}(\Omega)$ is complete .

We know that $(L^p(\Omega))^{n+1}$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. Since $W^{1,p}(\Omega)$ is complete, its image under the isometry (2.9) is a closed subspace of $(L^p(\Omega))^{n+1}$ which inherits the corresponding properties.

Remark 2.1.0.2 The results of this theorem can be proved by the same way for any integer $m \geq 2$. In Future, unless absolutely necessary, we will establish theorems only for the spaces $W^{1,p}(\Omega)$. The extension to higher order spaces is often done in anlogous manner.

Proposition 2.1.0.1
For $1 < p \leq \infty$, let (u_n) be a sequence in $W^{1,p}(\Omega)$ and $u \in L^p(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$ and for $1 \leq i \leq n$, $\left(\dfrac{\partial u_n}{\partial x_i}\right)$ is bounded in $L^p(\Omega)$. Then $u \in W^{1,p}(\Omega)$. Let us recall that the space $L^p(\Omega)$ is really only made up of equivalence classes of functions (under the equivalence relation given by equality almost everywhere). Thus by saying that u is a continuous function in $L^p(\Omega)$, we mean that the corresponding equivalence class has a representative which is a continuous function. In this spirit we prove the following result characterizing the space $W^{1,p}(I)$ where $I \subset \mathbb{R}$ is an open interval.

Theorem 2.1.0.2
Let $I \subset \mathbb{R}$ be an open interval and let $u \in W^{1,p}(I)$. Then u is absolutely continuous.

Proof. Let $x_0 \in I$ and define

$$\bar{u}(x) = \int_{x_0}^{x} u'(t)\, dt, \tag{2.12}$$

which, by definition, is absolutely continuous. Hence it classical derivative exists almost everywhere and is equal a.e to u'; this is also its distribution derivative. Hence, in the sence of distributions, $(u - \bar{u})' = 0$ and so $u - \bar{u} = c$, a constant a.e. Thus $u = \bar{u} + c$ a.e and the later function is absolutely continuous.

We can deduce an important property of $W^{1,p}(I)$ from the preceeding theorem, when I is a bounded interval. Let, for instance, $I =]0,1[$. Then if $u \in W^{1,p}(I)$, we can write

$$u(x) = u(0) + \int_0^x u'(t)\, dt. \tag{2.13}$$

Hence by Holder's inequality, if q is the Holder conjugate of p, we have

$$|u(0)| \leq |u(x)| + \|u'\|_{L^p(I)} |x|^{1/q}.$$

Thus there exists a constant $C > 0$ (not depending of u) such that

$$|u(0)| \leq C \|u\|_{W^{1,p}(I)} \tag{2.14}$$

and we also deduce that for any $x \in I$

$$|u(x)| \leq C \|u\|_{W^{1,p}(I)}, \quad C > 0 \quad \text{independent of} \ u. \tag{2.15}$$

Let B be the unit ball in $W^{1,p}(I)$, that is

$$B = \{u \in W^{1,p}(I) \ \ | \ \ \|u\|_{W^{1,p}(I)} \leq 1\}. \tag{2.16}$$

It follows that if $i : W^{1,p}(I) \to C(\bar{I})$ is the inclusion map, $B = i(B)$ is a uniformly bounded set $C(\bar{I})$. Again if $x, y \in I$, by (2.13), we have

$$|u(x) - u(y)| \leq \|u'\|_{L^p(I)} |x - y|^{1/q} \leq \|u\|_{W^{1,p}(I)} |x - y|^{1/q}, \tag{2.17}$$

from which, it follows that B is equicontinuous in $C(\bar{I})$. It follows from *Ascoli-Arzela Theorem* that B is relatively compact in $C(\bar{I})$. In other words the map $i : W^{1,p}(I) \to C(\bar{I})$ is a compact operator. This is an important property of Sobolev spaces and will be studied later.

2.2 The Main Theorems

2.2.1 Approximation by smooth functions

• If Ω is an open subset in $\mathbb{R}^N$, we write $\omega \subset\subset \Omega$, if ω is open and $\bar{\omega}$ is compact and such that $\bar{\omega} \subset \Omega$.

• If u is a function defined on Ω, we denote by $\bar{u}$, the extension by 0 of u to $\mathbb{R}^N$.

Theorem 2.2.1.1 *Let ρ_ϵ be a Friedrichs's moliffier, let $u \in W^{1,p}(\Omega)$ for $1 \leq p \leq \infty$. Then we have:*

$$\rho_\epsilon * \bar{u} \to u \ \ in \ \ L^p(\Omega) \ \ as\, \epsilon \to 0. \tag{2.18}$$

$$\frac{\partial}{\partial x_i}[\rho_\epsilon * \bar{u}] \to \frac{\partial u}{\partial x_i} \ \ in \ \ L^p(\omega), \ \ as\, \epsilon \to 0, \ \ \forall\ \omega \subset\subset \Omega. \tag{2.19}$$

For $\Omega = \mathbb{R}^N$, we have

$$\rho_\epsilon * u \to u \ \ in \ \ W^{1,p}(\mathbb{R}^N), \ \ as\, \epsilon \to 0$$

Proof. Note that (2.18) follows from theorem 1.1.7.1. To prove (2.19) we apply the *Fubini theorem*, let $\omega \subset\subset \Omega$ and ϵ such that $0 < \epsilon < \text{dist}(\omega, \partial\Omega)$ so that $\omega + B(0, \epsilon) \subset \Omega$. Now let $\varphi \in D(\omega)$ we have

$$\begin{aligned}
\int_\omega [\rho_\epsilon * \bar{u}(x)] \frac{\partial \varphi}{\partial x_i}(x)\, dx &= \int_\omega \left(\int_{\mathbb{R}^N} \bar{u}(x-y)\rho_\epsilon(y)\, dy \right) \frac{\partial \varphi}{\partial x_i}(x)\, dx \\
&= \int_{\mathbb{R}^N} \frac{\partial \varphi}{\partial x_i}(x) \left(\int_{\mathbb{R}^N} \bar{u}(x-y)\rho_\epsilon(y)\, dy \right) dx \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \bar{u}(x-y)\rho_\epsilon(y) \frac{\partial \varphi}{\partial x_i}(x)\, dy dx \\
&= \int_{\mathbb{R}^N} \rho_\epsilon(y) \left(\int_{\mathbb{R}^N} \bar{u}(x-y) \frac{\partial \varphi}{\partial x_i}(x)\, dx \right) dy \\
&= \int_{B(0,\epsilon)} \rho_\epsilon(y) \left(\int_\omega u(x-y) \frac{\partial \varphi}{\partial x_i}(x)\, dx \right) dy \\
&= -\int_{B(0,\epsilon)} \rho_\epsilon(y) \left(\int_\omega \frac{\partial u}{\partial x_i}(x-y)\varphi(x)\, dx \right) dy \\
&= -\int_{B(0,\epsilon)} \rho_\epsilon(y) \left(\int_{\mathbb{R}^N} \frac{\partial u}{\partial x_i}(x-y)\varphi(x)\, dx \right) dy \\
&= -\int_{\mathbb{R}^N} \rho_\epsilon(y) \left(\int_{\mathbb{R}^N} \frac{\partial u}{\partial x_i}(x-y)\varphi(x)\, dx \right) dy \\
&= -\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \rho_\epsilon(y) \frac{\partial u}{\partial x_i}(x-y)\, dy \right) \varphi(x)\, dx \\
&= -\int_{\mathbb{R}^N} \left(\rho_\epsilon * \frac{\partial u}{\partial x_i} \right)(x)\varphi(x)\, dx.
\end{aligned}$$

So we have

$$\frac{\partial}{\partial x_i}[\rho_\epsilon * \bar{u}] = \rho_\epsilon * \frac{\partial u}{\partial x_i} \quad \text{in} \quad D'(\omega).$$

Since $\rho_\epsilon * \dfrac{\partial u}{\partial x_i} \in L^p(\omega)$ then it is the weak derivative of $\rho_\epsilon * u$ in ω. Finally by *theorem* 1.1.7 we get (2.19).

Definition There exists a non-negative function $b \in D(\mathbb{R}^N)$ such that $b \equiv 1$ on the unit ball $B(0,1)$ and $\text{supp}(b) \subset B(0,2)$. We call b a *bump function.*

Lemma 2.2.1.1 *Let b a bump function. For all $R \geq 1$, we put $b_n(x) = b(\frac{x}{n})$. Then for all $u \in W^{1,p}(\Omega)$, $b_n u \in W^{1,p}(\Omega)$, the support of $b_n u$ is bounded and moreover.*

$$b_n u \to u \quad in \quad W^{1,p}(\Omega) \quad as \quad R \to +\infty.$$

Proof. We know that for all $n \geq 1$, $\text{supp}(b_n u) \subset \text{supp}(b_n) \cap \text{supp}(u) \subset \bar{B}(0, 2n)$, therefore $\text{supp}(b_n u)$ is compact in $\mathbb{R}^N$ for each $n \geq 1$. By the continuity of b we have

$$b_n(x)u(x) = b(\frac{x}{n})u(x) \to u(x) \quad \text{for a.e} \quad x \in \Omega.$$

So $|b_n u - u|^p \to 0$ for a.e $x \in \Omega$.
There exist $C \geq 0$ such that

$$|b_n u - u|^p \leq C\left(|bu|^p + |u|^p\right) = K|u|^p \in L^1(\Omega); \quad \text{where} \quad K = C\left(1 + |b|\right).$$

Since $|b_n u - u|^p \in L^1(\Omega)\, \forall n \geq 1$, by Lebesgue dominated convergence theorem we have

$$\|b_n u - u\|_p^p = \int_\Omega |b_n u - u|^p \, dx \to 0.$$

Hence $b_n u \to u$ in $L^p(\Omega)$.
Now

$$\frac{\partial}{\partial x_i}(b_n u) = \frac{\partial}{\partial x_i} b(\frac{x}{n})u + b(\frac{x}{n})\frac{\partial u}{\partial x_i} \tag{2.20}$$

$$= \frac{1}{n}\frac{\partial b_n}{\partial x_i} u + b_n \frac{\partial u}{\partial x_i}. \tag{2.21}$$

Since $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$, by similar argument above we have

$$\frac{1}{n}\frac{\partial b_n}{\partial x_i} u + b_n \frac{\partial u}{\partial x_i} \to 0 + \frac{\partial u}{\partial x_i} \quad \text{in} \quad L^p(\Omega).$$

Hence

$$\frac{\partial}{\partial x_i}(b_n u) \to \frac{\partial u}{\partial x_i} \quad \text{in} \quad L^p(\Omega).$$

We conclude that $b_n u \to u$ in $W^{1,p}(\Omega)$.

Theorem 2.2.1.2 *Let $u \in W^{1,p}(\Omega)$ for $1 \leq p < \infty$. Then there exists a sequence (u_n) in $D(\mathbb{R}^N)$ such that:*

$$u_n \to u \quad in \;\; L^p(\Omega), \tag{2.22}$$

$$\frac{\partial u_n}{\partial x_i}|_\omega \to \frac{\partial u}{\partial x_i}|_\omega \quad in \;\; L^p(\omega), \quad for\ every\ 1 \leq i \leq n \quad and\ every\ \omega \subset\subset \Omega. \tag{2.23}$$

Proof. Let $u \in W^{1,p}(\Omega)$. We put $v_n = \rho_{\epsilon_n} * \bar{u}, \forall n \geq 1$, where $\epsilon_n \downarrow 0$. By theorem 2.2.1.1, we have that $v_n \to u$ in $L^p(\Omega)$, and $\frac{\partial v_n}{\partial x_i}$ in $L^p(\Omega)$ for $\omega \subset\subset \Omega$.
Now define $u_n = b_n v_n$, we observe that for each $n \geq 1, u_n \in C^\infty(\mathbb{R}^N)$, and $\text{supp}(u_n) \subset \bar{B}(0, 2n)$, hence compact in $\mathbb{R}^N$. Therefore (u_n) is in $D(\mathbb{R}^N)$ and has the same convergence properties as (v_n), i.e $u_n \to u$ in $L^p(\Omega)$ and $\frac{\partial u_n}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i}$ in $L^p(\omega)$ for $\omega \subset\subset \Omega, 1 \leq i \leq N$. This completes the proof.

Proposition 2.2.1.1 (Derivation of a product) . *Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, with $1 \leq p \leq \infty$. Then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and we have*

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} v + u\frac{\partial v}{\partial x_i}, \quad i = 1, \ldots, N. \tag{2.24}$$

Proof.
Case 1 :$1 \leq p < \infty$.

Let $u, v \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$. We have that $|uv| \leq |u|\|v\|_{\infty}$, and $|uv| \leq \|u\|_{\infty}\|v\|_{\infty}$.So $uv \in L^p(\Omega)$ and $uv \in L^{\infty}(\Omega)$.
We need now to show that $\frac{\partial}{\partial x_i}(uv) \in L^p(\Omega)$. We observe that $\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \in L^p(\Omega)$ and $u, v \in L^{\infty}(\Omega)$ implies $\frac{\partial u}{\partial x_i}v + u\frac{\partial v}{\partial x_i} \in L^p(\Omega), 1 \leq i \leq N$.
We only need to show that *equation* 2.2.1.1 holds.
By using *theorem* 2.2.1.2 and *theorem* 1.1.3.2 there exist two sequences (u_n) and (v_n) in $D(\mathbb{R}^N)$ such that,

$$u_n \to u, \quad v_n \to v, \quad \text{in} \quad L^p(\Omega) \text{ and } a.e \quad \text{on } \Omega.$$

and

$$\frac{\partial u_n}{\partial x_i} \to u, \quad \frac{\partial v_n}{\partial x_i} \to \frac{\partial v}{\partial x_i} \quad \text{in} \quad L^p(\omega) \quad \text{and} \quad a.e \quad \text{on } \omega, \quad \text{for all} \quad \omega \subset\subset \Omega.$$

Moreover,

$$\begin{aligned}
|u_n(x)| &= |b_n(x)\,(\rho_{\epsilon_n} * \bar{u})\,(x)| & (2.25)\\
&\leq M|\,(\rho_{\epsilon_n} * \bar{u})\,(x)| & (2.26)\\
&\leq M\int_{\mathbb{R}^N} |\rho_{\epsilon_n}(y)||\bar{u}(x-y)|\,dy & (2.27)\\
&= M\int_{\Omega} |\rho_{\epsilon_n}(y)||u(x-y)|\,dy & (2.28)\\
&\leq M\|u\|_{\infty}\int_{\Omega} |\rho_{\epsilon_n}(y)|\,dy & (2.29)\\
&\leq M\|u\|_{\infty}\int_{\mathbb{R}^N} |\rho_{\epsilon_n}(y)|\,dy & (2.30)\\
&\leq M\|u\|_{\infty}\left(\text{since} \int_{\mathbb{R}^N} |\rho_{\epsilon_n}(y)|\,dy = 1\right). & (2.31)
\end{aligned}$$

The same property is true for the sequence (v_n).
Let $\omega \subset\subset \Omega$ and $\varphi \in D(\omega)$, we have

$$\int_{\omega} u_n v_n \frac{\partial \varphi}{\partial x_i}\,dx = -\int_{\omega}\left(\frac{\partial u_n}{\partial x_i}v_n + u_n\frac{\partial v_n}{\partial x_i}\right)\varphi\,dx$$

By Lebesgue dominated convergence *theorem* and uniqueness of limits, it follows that

$$\int_{\omega} uv\frac{\partial \varphi}{\partial x_i}\,dx = -\int_{\omega}\left(\frac{\partial u}{\partial x_i}v + u\frac{\partial v}{\partial x_i}\right)\varphi\,dx, \quad \forall\,\varphi \in D(\omega).$$

Thus

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i}v + u\frac{\partial v}{\partial x_i}, \quad \text{in} \quad \omega. \qquad (\blacktriangle)$$

Since ▲ is true for all $\omega \subset\subset \Omega$, we choose $\omega_n \subset\subset \Omega$ for all $n \geq 1$ s.t $\Omega = \bigcup_{n\geq 1} \omega_n$, so ▲ holds a.e on Ω, because ▲ holds a.e on $\omega_n, \forall n \geq 1$. Thus the *proposition* is proved for $1 \leq p < \infty$.

Case 2: $p = \infty$
Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.
$|uv| \leq \|u\|_\infty \|v\|_\infty$, so $uv \in L^\infty(\Omega)$.
We note that since $\frac{\partial v}{\partial x_i}, \frac{\partial u}{\partial x_i} \in L^\infty(\Omega)$, then $\frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i} \in L^\infty(\Omega)$; $1 \leq i \leq N$.
We need only to show that

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i}, \quad \text{in } \omega, \ 1 \leq i \leq N.$$

Let $\omega \subset\subset \Omega$, we show that $W^{1,p}(\Omega) \subset W^{1,p}(\omega)$ for $1 \leq p < +\infty$.
$u \in W^{1,p}(\Omega)$, implies that $u \in L^\infty(\Omega)$ and $L^p(\omega)$ since

$$\int_\omega |u|^p \, dx \leq \|u\|_\infty^p \text{meas}(\omega) < +\infty.$$

Similarly $\int_\omega |v|^p \, dx < +\infty$.
Therefore $uv \in W^{1,p}(\omega)$, $1 \leq p < +\infty$ hence the desired result follows by **case 1** above.

Proposition 2.2.1.2 (Chain Rule) . *Let $G \in C^1(\mathbb{R})$ such that $G(0) = 0$ and $|G'(s)| \leq M \ \forall s \in \mathbb{R}$. Let $u \in W^{1,p}(\Omega)$ with $1 \leq p \leq \infty$. Then $G \circ u \in W^{1,p}(\Omega)$ and*

$$\frac{\partial}{\partial x_i}(G \circ u) = (G' \circ u) \frac{\partial u}{\partial x_i} \quad \text{for } 1 \leq i \leq n. \tag{2.32}$$

Proof. Since the derivative of G is bounded and $G(0) = 0$, by the Mean Value theorem, we have

$$|G(s)| \leq M|s|, \quad s \in \mathbb{R}.$$

Thus $|G \circ u(x)| = |G(u(x))| \leq M|u(x)|$ for every $x \in \Omega$ and so $G \circ u \in L^p(\Omega)$. Simillary $(G' \circ u) \frac{\partial u}{\partial x_i} \in L^p(\Omega)$ for $1 \leq i \leq N$.
Assume now that $1 \leq p < \infty$. Then by *theorem* 2.2.1.2, there exists a sequence (u_n) in $D(\mathbb{R}^N)$ such that $u_n \to u$ in $L^p(\Omega)$ and $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ in $L^p(\omega)$ for every $\omega \subset\subset \Omega$. Let $\varphi \in D(\Omega)$. Then choose $\omega \subset\subset \Omega$ such that $\text{supp}(\varphi) \subset \omega \subset\subset \Omega$. Since u_n is smooth, we have by the usual chain rule,

$$\int_\Omega (G \circ u_n) \frac{\partial \varphi}{\partial x_i} \, dx = \int_\omega (G \circ u_n) \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\omega (G' \circ u_n) \frac{\partial u_n}{\partial x_i} \varphi \, dx. \tag{2.33}$$

Now $G \circ u_n \to G \circ u$ in $L^p(\Omega)$ since

$$|G \circ u_n(x) - G \circ u(x)| \leq M|u_n(x) - u(x)|.$$

Also $G' \circ u_n$ is uniformly bounded by M and so $(G' \circ u_n)\dfrac{\partial u_n}{\partial x_i} \to (G' \circ u)\dfrac{\partial u}{\partial x_i}$ in $L^p(\omega)$ [up to a subsequence] and so we can pass to the limit in (2.33) and thus prove (2.32).

If $p = \infty$, fix $\varphi \in D(\Omega)$ and choose ω such that $\text{supp}(\varphi) \subset \omega \subset\subset \Omega$. Then as ω is relatively compact, $u \in W^{1,\infty}(\Omega)$ implies that $u \in W^{1,p}(\omega)$ for al $1 \leq p < \infty$ and so (2.32) is valid by the preceding method.

This result can be generalized to Lipschitz continuous functions G. We will prove this in the context of the spaces $W_0^{1,p}(\Omega)$ shortly.

Lemma 2.2.1.2 *Let $1 \leq p < \infty$, and let $u \in W^{1,p}(\Omega)$ such that u vanishes outside a compact set contained in Ω. Then $u \in W_0^{1,p}(\Omega)$.*

Proof. Let $K \subset \Omega$ be a compact set such that $u = 0$ on $\Omega - K$. Let ω be such that $K \subset \omega \subset\subset \Omega$. Let $\Phi \in D(\Omega)$ be a cut-off function such that $\Phi = 1$ on K. Then we know that $\Phi u = u$. Now by theorem 2.2.1.2, there exists a sequence (u_n) in $D(\mathbb{R}^N)$ such that $u_n \to u$ in $L^p(\Omega)$ and $\dfrac{\partial u_n}{\partial x_i} \to \dfrac{\partial u}{\partial x_i}$ in $L^p(\omega)$. Consequently $\Phi u_n \to \Phi u$ in $W^{1,p}(\Omega)$ and as $\Phi u_n \in D(\Omega)$, it follows that Φu, i.e u is in $W_0^{1,p}(\Omega)$.

Theorem 2.2.1.3 (Stampacchia) *Let G be a Lipschitz continuous function of $\mathbb{R}$ into itself such that $G(0) = 0$. Then if Ω is bounded, $1 < p < \infty$, and $u \in W_0^{1,p}(\Omega)$, we have $G \circ u \in W_0^{1,p}(\Omega)$.*

Proof. Let $u \in W_0^{1,p}(\Omega)$ and let $u_n \in D(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. Define

$$v_n = G \circ u_n.$$

Since u_n has compact support and $G(0) = 0$, v_n has compact support. Also v_n is Lipschitz continuous; for

$$|v_n(x) - v_n(y)| \leq K|u_n(x) - u_n(y)| \leq K_n|x - y|$$

as u_n is a smooth function with compact support and G is Lipschitz continuous. Hence $v_n \in L^p(\Omega)$. Also it follows that

$$\left|\frac{\partial v_n}{\partial x_i}\right| \leq K_m, \quad 1 \leq i \leq N$$

and since Ω is bounded, $\dfrac{\partial v_n}{\partial x_i} \in L^p(\Omega)$. Thus $v_n \in W^{1,p}(\Omega)$ and has compact support. Thus by lemma 2.2.1.2, $v_n \in W_0^{1,p}(\Omega)$.

Also from

$$|v_n(x) - G \circ u(x)| \leq K|u_n(x) - u(x)|,$$

it follows that $v_n \to G \circ u$ in $L^p(\Omega)$. Further if e_i is the standard basis vector of $\mathbb{R}^N$, we have

$$\frac{|v_n(x + he_i) - v_n(x)|}{|h|} \leq \frac{K|u_n(x + he_i) - u_n(x)|}{|h|}$$

and so

$$\limsup_{n\to\infty} \left\|\frac{\partial v_n}{\partial x_i}\right\|_{L^p(\Omega)} \leq K \limsup_{n\to\infty} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^p(\Omega)} \tag{2.34}$$

But $\left(\dfrac{\partial u_n}{\partial x_i}\right)$ is a convergent sequence in $L^p(\Omega)$ and so from (2.34), it follows that $\left(\dfrac{\partial v_n}{\partial x_i}\right)$ is bounded for each $1 \leq i \leq N$. Hence by proposition 2.1 it follows that (v_n) has a subsequence that converges in $W^{1,p}(\Omega)$ and that $G \circ u \in W_0^{1,p}(\Omega)$.

Corollary 2.2.1.1 *Let $u \in H_0^1(\Omega)$, Ω is a bounded open set in $\mathbb{R}^N$. Then $|u|, u^+$ and u^- belong to $H_0^1(\Omega)$ where*

$$u^+(x) = \max(u(x), 0),$$
$$u^-(x) = \max(-u(x), 0).$$

Proof. We apply the preceding theorem with $p = 2$ and $G(t) = |t|$. Thus $|u| \in H_0^1(\Omega)$ for $u \in H_0^1(\Omega)$. Now

$$u^+ = \frac{|u| + u}{2}, \quad u^- = \frac{|u| - u}{2},$$

and so $u^+, u^- \in H_0^1(\Omega)$.

Proposition 2.2.1.3 (Change of Variable) *Let Ω and Ω' be two open subsets of $\mathbb{R}^N$ and let $H : \Omega' \longrightarrow \Omega$ a bijective mapping, $x = H(y) = (H_1(y), \cdots, H_n(y))$ such that*

$$H \in C^1(\Omega'), \quad H^{-1} \in C^1(\Omega), \quad J(H) \in L^\infty(\Omega'), \quad J(H^{-1}) \in L^\infty(\Omega)$$

Let $u \in W^{1,p}(\Omega)$, then $uoH \in W^{1,p}(\Omega')$ and we have

$$\frac{\partial}{\partial y_j}(u \circ H)(y) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(H(y)) \frac{\partial H_i}{\partial y_j}(y), \quad j = 1, \ldots, n. \tag{2.35}$$

Moreover, there exists a constant $C > 0$ such that:

$$\|u \circ H\|_{W^{1,p}(\Omega')} \leq C \|u\|_{W^{1,p}(\Omega)} \tag{2.36}$$

Proposition 2.2.1.4 *Let $u \in L^p(\Omega)$, with $1 < p \leq \infty$. Then the following properties are equivalent:*

a. $u \in W^{1,p}(\Omega)$

b. *There exists a constant $C \geq 0$ such that*

$$\left| \int_\Omega u D_i \varphi \, dx \right| \leq C \, \|\varphi\|_{p'}, \quad \forall \varphi \in D(\Omega), \; \forall i = 1, \cdots, N.$$

c. *There exists a constant $C \geq 0$ such that for all $\omega \subset\subset \Omega$ and for all $h \in \mathbb{R}^N$ with $\|h\| < d(\omega, \Omega^c)$ we have*

$$\|\tau_h u - u\|_{L^p(\omega)} \leq C \|h\|.$$

Proof.

☞ **a. ⇒ b.**
Let $\varphi \in D(\Omega)$, since $u \in W^{1,p}(\Omega)$, then $D_i u \in L^p(\Omega)$ and

$$\int_\Omega u D_i \varphi \, dx = -\int_\Omega D_i u \varphi \, dx$$

so by *Holder's* inequality we have **b.** with $C = \|D_i u\|_{L^p(\Omega)}$.

☞ **b. ⇒ a.**
For $1 \leq i \leq n$, let $T_i : D(\Omega) \to \mathbb{R}$ defined by:

$$T_i(\varphi) = -\int_\Omega u D_i \varphi \, dx \quad \forall \varphi \in D(\Omega).$$

Clearly T_i is linear and by **b.**, it is continuous in $D(\Omega)$ with the $L^{p'}$-norm. So by density it can be extended as a continuous linear form to $L^{p'}(\Omega)$. By *Riesz* representation *theorem*, there exists $v_i \in L^p(\Omega)$ such that

$$T_i(\varphi) = -\int_\Omega u D_i \varphi \, dx = \int_\Omega v_i \varphi \, dx, \quad \forall \varphi \, D(\Omega).$$

Therefore $D_i u = v_i$, so $u \in W^{1,p}(\Omega)$.

☞ **a. ⇒ c.**
We assume first that $u \in D(\mathbb{R}^N)$. Let $h \in \mathbb{R}^N$, by the fundamental *theorem* of calculus

$$\tau_h u(x) - u(x) = \int_0^1 h \cdot \nabla u(x + th) \, dt.$$

By *Holder's* inequality, it follows

$$|\tau_h u(x) - u(x)|^p \leq \|h\|^p \int_0^1 \|\nabla u(x + th)\|^p \, dt$$

and

$$\begin{aligned} \|\tau_h u - u\|^p_{L^p(\omega)} &\leq \|h\|^p \int_\omega dx \int_0^1 h \|\nabla u(x+th)\|^p \, dt \\ &= \|h\|^p \int_0^1 dt \int_\omega \|\nabla u(x+th)\|^p \, dx = \|h\|^p \int_0^1 dt \int_{\omega + th} \|\nabla u(y)\|^p \, dy. \end{aligned}$$

If $\|h\| <$ dist(ω, Ω^c), there exists $\omega' \subset\subset \Omega$ such that $\omega + th \subset \omega'$ forall $t \in [0, 1]$. So

$$\|\tau_h u - u\|_{L^p(\omega)} \leq \|h\| \|\nabla u\|_{L^p(\omega')}. \tag{2.37}$$

Now, for $u \in W^{1,p}(\Omega)$ and $p \neq \infty$, there exists (u_n) in $D(\mathbb{R}^N)$ such that $u_n \to u$ in $L^p(\Omega)$ and $\nabla u_n \to \nabla u$ in $L^p(\omega)$, forall ω *subset* $\subset \Omega$. We apply (2.37) to (u_n) and we pass to the limit. For $p = \infty$, we use the same process as before for $p < \infty$ and we pass to the limit for p.

☞ **c.** $\Rightarrow$ **b.**
Let $\varphi \in D(\Omega)$. Let ω be open sucth that $\mathrm{supp}(\varphi) \leq \omega \subset\subset \Omega$ and h such that $\|h\| < \mathrm{dist}(\omega, \Omega^c)$. Then by *Holder's* inequality and **c.**, we have

$$\left| \int_\Omega (\tau_h u - u)\varphi \, dx \right| = \left| \int_\omega (\tau_h u - u)\varphi \, dx \right| \leq C\|h\| \cdot \|\varphi\|_{L^{p'}(\Omega)}$$

On the other hand

$$\int_\Omega (\tau_h u(x) - u(x))\, \varphi(x) \, dx = \int_\Omega u(y) \left(\varphi(y-h) - \varphi(y) \right) \, dy,$$

it follows that

$$\left| \int_\Omega u(y) \frac{\varphi(y-h) - \varphi(y)}{\|h\|} \, dy \right| \leq C\|\varphi\|_{L^{p'}(\Omega)}$$

We choose, $h = te_i$, $t \in \mathbb{R}$ and e_i is the ith canonical basis element of $\mathbb{R}^N$. If $t \to 0$ we get **b.** by using the dominated convergence theorem.

2.2.2 Extension Theorems

Most of the important *Sobolev* inequalities and imbedding *theorem*s that we will derive in the next section are most easily derived for the space $W_0^{1,p}(\Omega)$ which can be viewed as being a subspace of $W^{1,p}(\mathbb{R}^N)$ (see *proposition* 2.2.2.2). In contrast, for the space $W^{1,p}(\Omega)$, direct derivations of these results are tedious and difficult because of the boundary behavior of the functions. In this section we investigate the existence of an extension operator that allows us to extend functions in $W^{1,p}(\Omega)$ to functions in $W^{1,p}(\mathbb{R}^N)$. This allows us to deduce the *Sobolev* imbedding *theorem*s for the spaces $W^{1,p}(\Omega)$ from the corresponding results for $W^{1,p}(\mathbb{R}^N)$.

Definition Let Ω be an open set in $\mathbb{R}^N$. An *Extension Operator* P for $W^{1,p}(\Omega)$ is a bounded linear operator

$$P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$$

such that $Pu|_\Omega = u$ for every $u \in W^{1,p}(\Omega)$. By virtue of the fact that P is a bounded linear operator, it follows that

$$\|Pu\|_{W^{1,p}(\mathbb{R}^N)} \leq \|u\|_{W^{1,p}(\Omega)} \quad \forall\, u \in W^{1,p}(\Omega) \tag{2.38}$$

where $C > 0$ is a constant, which, in general, will only depend on Ω and p. Thus if Ω is such that an extension operator exists, then we consider $W^{1,p}(\Omega)$ as a subspace of $W^{1,p}(\mathbb{R}^N)$. A sufficient condition for the existence of P is the smoothness of the boundary $\partial\Omega$ and these consideration will be taken up later.

Proposition 2.2.2.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in W^{1,p}(\Omega)$. If $K \subset \Omega$ is a closed subset and u vanishes outside K, then the function $\bar{u}$ is in $W^{1,p}(\mathbb{R}^N)$.*

Proof. Let $\varphi \in D(\mathbb{R}^N)$ and let $K_1 = K \cap \operatorname{supp}(\varphi)$. Then $K_1 \subset \Omega$ is compact. Let $\Psi \in D(\Omega)$ such that $\Psi = 1$ on K_1. Now,

$$\begin{aligned}\int_{\mathbb{R}^N} \bar{u}\frac{\partial \varphi}{\partial x_i}\,dx &= \int_\Omega u\frac{\partial \varphi}{\partial x_i}\,dx = \int_{K_1} u\frac{\partial \varphi}{\partial x_i}\,dx = \int_\Omega u\Psi\frac{\partial \varphi}{\partial x_i}\,dx \\ &= \int_\Omega u\frac{\partial}{\partial x_i}(\Psi\varphi)\,dx - \int_\Omega u\varphi\frac{\partial \Psi}{\partial x_i}\,dx.\end{aligned}$$

But $\varphi\Psi \in D(\Omega)$ and $\dfrac{\partial \Psi}{\partial x_i} = 0$ on K_1. Thus

$$\begin{aligned}\int_{\mathbb{R}^N} \bar{u}\frac{\partial \varphi}{\partial x_i}\,dx &= \int_\Omega u\frac{\partial}{\partial x_i}(\Psi\varphi)\,dx \\ &= -\int_\Omega \frac{\partial u}{\partial x_i}\varphi\Psi\,dx = -\int_\Omega \frac{\partial u}{\partial x_i}\varphi\,dx.\end{aligned}$$

Thus it follows that

$$\frac{\partial \bar{u}}{\partial x_i} = \overline{\frac{\partial u}{\partial x_i}} \in L^p(\mathbb{R}^N).$$

Remark 2.2.2.1 Given an arbitrary element of $W^{1,p}(\Omega)$, the extension by *zero* does not belong to $W^{1,p}(\mathbb{R}^N)$. For example if $\Omega =]0,1[\subset \mathbb{R}$ and $u(x) = 1$ on Ω, then $\bar{u} \in L^p(\mathbb{R})$ but $\dfrac{d\,\bar{u}}{dx} = \delta_1 - \delta_0$, which cannot be given by a locally integrable function. We have seen in proposition 2.2.2.1 that if u is supported away from the boundary, then the extension by zero provides an element of $W^{1,p}(\mathbb{R}^N)$. But such functions, if the support is compact, are in $W_0^{1,p}(\Omega)$. We will show that the extension by zero is an extension operator for $W_0^{1,p}(\Omega)$ irrespective of the nature of Ω. Thus for functions in $W_0^{1,p}(\Omega)$, we always will have a canonical extension to $W^{1,p}(\mathbb{R}^N)$ which help us to prove several important properties of these functions without supplementary hypothesis on the smoothness of Ω.

Proposition 2.2.2.2 *Let $1 < p < \infty$ and $u \in W_0^{1,p}(\Omega)$, Ω an open set in $\mathbb{R}^N$. Then $\bar{u} \in W^{1,p}(\mathbb{R}^N)$. Furthermore for any $1 \le i \le N$,*

$$\frac{\partial \bar{u}}{\partial x_i} = \overline{\frac{\partial u}{\partial x_i}}. \tag{2.39}$$

Proof. Let $u \in W_0^{1,p}(\Omega)$, then clearly we have $\bar{u} \in L^p(\mathbb{R}^N)$ and

$$\|\bar{u}\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\Omega)}$$

Now let (u_n) in $D(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. Let $\varphi \in D(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \bar{u}\frac{\partial \varphi}{\partial x_i}\,dx &= \int_\Omega u\frac{\partial \varphi}{\partial x_i}\,dx = \lim_{n\to+\infty}\int_\Omega u_n\frac{\partial \varphi}{\partial x_i}\,dx \\ &= -\lim_{n\to+\infty}\int_\Omega \frac{\partial u_n}{\partial x_i}\varphi\,dx = -\int_\Omega \frac{\partial u}{\partial x_i}\varphi\,dx \quad \text{(by L.D.C.T.)} \\ &= -\int_{\mathbb{R}^N} \overline{\frac{\partial u}{\partial x_i}}\varphi\,dx\end{aligned}$$

so we have

$$\frac{\partial \bar{u}}{\partial x_i} = \overline{\frac{\partial u}{\partial x_i}}$$

then $\bar{u} \in W^{1,p}(\mathbb{R}^N)$ and this finish the proof.

One of the fundamental methods of providing extension operator is the *method of reflection*. We will now use it to show that half-unit open ball has the extension property.

Notation: Let $x \in R^N$, we write

$$x = (x', x_N), \quad \text{with } x' \in \mathbb{R}^{N-1}, \quad x' = (x_1, \cdots, x_{N-1})$$

We denote

$$\begin{aligned}
\mathbb{R}^N_+ &= \{x = (x', x_N) \in \mathbb{R}^N \ \mid \ x_N > 0\} \\
B &= \{x = (x', x_N) \in \mathbb{R}^N \ \mid \ \|x'\| < 1 \text{ and } |x_N| < 1\} \\
B_+ &= \{x = (x', x_N) \in B \ \mid \ x_N > 0\} \\
B_- &= \{x = (x', x_N) \in B \ \mid \ x_N < 0\}
\end{aligned}$$

Proposition 2.2.2.3 (Extension by reflection) *Let $u \in W^{1,p}(B_+)$. Define on B the function u^* by*

$$u^*(x', x_N) = \begin{cases} u(x', x_N) & \text{if } x_N > 0, \\ u(x', -x_N) & \text{if } x_N < 0. \end{cases} \tag{2.40}$$

Then $u^* \in W^{1,p}(B)$ and

$$\|u^*\|_{L^p(B)} \leq 2\|u\|_{L^p(B_+)}, \quad \|u^*\|_{W^{1,p}(B)} \leq 2\|u\|_{W^{1,p}(B_+)}$$

Proof. We are going to show that

$$\frac{\partial u^*}{\partial x_i} = \left(\frac{\partial u}{\partial x_i}\right)^*, \ 1 \leq i \leq N-1 \tag{2.41}$$

$$\frac{\partial u^*}{\partial x_N} = \begin{cases} \dfrac{\partial u}{\partial x_N}(x', x_N) & \text{if } x_N > 0, \\ -\dfrac{\partial u}{\partial x_N}(x', -x_N) & \text{, if } x_N < 0. \end{cases} \tag{2.42}$$

Let $\eta \in C^\infty(\mathbb{R})$ such that

$$\eta(t) = \begin{cases} 0 & \text{if } t < 1/2, \\ 1 & \text{if } t > 1. \end{cases}$$

Define the sequence (η_k) as follow

$$\eta_k(t) = \eta(kt), \quad t \in \mathbb{R}, \quad k \in \mathbb{N}.$$

☞**Proof of** (2.41). Let $\varphi \in D(B)$. We have for $1 \leq i \leq N-1$

$$\int_B u^* \frac{\partial \varphi}{\partial x_i}\, dx = \int_{B_+} u \frac{\partial \Psi}{\partial x_i}\, dx \tag{2.43}$$

where $\Psi(x', x_N) = \varphi(x', x_N) + \varphi(x', -x_N)$. Define $\varphi_k(x) = \eta_k(x_N)\Psi(x', x_N)$. Then $\varphi_k \in D(B_+)$ we have

$$\int_{B_+} u \frac{\partial \varphi_k}{\partial x_i}\, dx = -\int_{B_+} \frac{\partial u}{\partial x_i} \varphi_k\, dx.$$

On the other hand $\dfrac{\partial \varphi_k}{\partial x_i} = \eta_k \dfrac{\partial \Psi}{\partial x_i}$, therefore

$$\int_{B_+} u\eta_k \frac{\partial \Psi}{\partial x_i}\, dx = -\int_{B_+} \frac{\partial u}{\partial x_i} \eta_k \Psi\, dx. \tag{2.44}$$

By dominated convergence *theorem*, it follows

$$\int_{B_+} u \frac{\partial \Psi}{\partial x_i}\, dx = -\int_{B_+} \frac{\partial u}{\partial x_i} \Psi\, dx. \tag{2.45}$$

If we Combine (2.43) and (2.45), we get

$$\int_B u^* \frac{\partial \varphi}{\partial x_i}\, dx = -\int_{B_+} \frac{\partial u}{\partial x_i} \Psi\, dx = -\int_B \left(\frac{\partial u}{\partial x_i} \right)^* \varphi\, dx.$$

☞**Proof of** (2.42). Let $\varphi \in D(B)$. We have

$$\int_B u^* \frac{\partial \varphi}{\partial x_N}\, dx = \int_{B_+} u \frac{\partial \phi}{\partial x_N}\, dx, \tag{2.46}$$

where $\phi(x', x_N) = \varphi(x', x_N) - \varphi(x', -x_N)$. Note that $\phi(x', 0) = 0$, so there exists a constant M such that $|\phi(x', x_N)| \leq M$ on B. Since $\eta_k \phi \in D(B_+)$, it follows

$$\int_{B_+} u \frac{\partial}{\partial x_N}(\eta_k \phi)\, dx = -\int_{B_+} \frac{\partial u}{\partial x_N} \eta_k \phi\, dx \tag{2.47}$$

with

$$\frac{\partial}{\partial x_N}(\eta_k \phi) = \eta_k \frac{\partial \phi}{\partial x_N} + k\eta'(kx_N)\phi \tag{2.48}$$

We are going to show that

$$\int_{B_+} uk\eta'(kx_N)\phi\, dx \;\to\; 0 \;\text{ as }\; k \to \infty. \tag{2.49}$$

Indeed

$$\left|\int_B u\eta'(kx_N)\phi\,dx\right| \le kMC\int_{0<x_N<1/k}|u|x_N\,dx \le MC\int_{0<x_N<1/k}|u|\,dx \to 0, \text{ as } k\to\infty.$$

with $C = \sup_{t\in[0,1]}|\eta'(t)|$. From (2.47), (2.48) and (2.49), it follows that

$$\int_{B_+} u\frac{\partial\phi}{\partial x_N}\,dx = -\int_{B_+}\frac{\partial u}{\partial x_N}\phi\,dx$$

by using (2.46), we get (2.42).

Definition We say that an open subset $\Omega \subset \mathbb{R}^N$ is of class C^m (m integer) if for every $x\in\partial\Omega$, there exists an open neigborhood U of x in $\mathbb{R}^N$ and a map $\varphi : B \to U$ such that:

(i) φ is a bijection,

(ii) $\varphi \in C^m(\bar{B}, U)$, $\varphi^{-1}\in C^m(\bar{U}, B)$,

(iii) $\varphi(B_+) = \Omega\cap U$, $\varphi(B_0) = \partial\Omega\cap U$.

Lemma 2.2.2.1 (Fundamental) *Let Ω be a bounded open subset $\mathbb{R}^N$. Assume that Ω is of class C^m and lies (locally) on one side of its boundary $\partial\Omega$. Then*

1. *there exists a collection $\{U_i,\ 0\le i\le N\}$ of open bounded sets in $\mathbb{R}^N$ and a collection of bijective maps $\varphi_i : B\to U_i$, $1\le i\le N$ such that*

$$U_0 = \Omega,\quad \bar{\Omega}\subset\bigcup_{i=0}^{N}U_i,\quad \textit{and}\quad \partial\Omega\subset\bigcup_{i=1}^{N}U_i,$$

For each i, $1\le i\le N$, $\varphi_i\in C^m(\bar{B}, U_i)$, $(i=1,\cdots,N)$ with positive jacobian $J(\varphi_i)$ and

$$\varphi_i(B_+) = \Omega\cap U_i,\ \ \varphi_i(B_0) = \partial\Omega\cap U_i;$$

2. *there exists $\alpha_i\in D(\mathbb{R}^N)$, $(i=0,\cdots,N)$ such that $\{\alpha_i,\ 0\le i\le N\}$ is a partition of unity of $\bar{\Omega}$ subordinated to the open cover $\{U_i,\ 0\le i\le N\}$ and $\{\alpha_i,\ 1\le i\le N\}$ is a partition of unity of $\partial\Omega$ subordinated to the open cover $\{U_i,\ 1\le i\le N\}$.*

Remark 2.2.2.2 Let u be function in $W^{1,p}(\Omega)$. Then $u = \sum_{i=0}^{N}\alpha_i u$ on Ω and each term $\alpha_i u$ of the sum is in $W^{1,p}(\Omega\cap U_i)$ with support in U_i.
Consider the following operator:

$$W^{1,p}(\Omega)\to W_0^{1,p}(\Omega)\times\prod_{i=1}^{N}W^{1,p}(\Omega\cap U_i):\ u\mapsto(\alpha_0 u,\alpha_1 u,\cdots,\alpha_N u).$$

Clearly, it is linear and injective.

Since $\alpha_i u \in W^{1,p}(\Omega \cap U_i)$ with support in U_i, it follows that the composite function $(\alpha_i u) \circ \varphi_i$ belongs to $W^{1,p}(B_+)$ with support in B. So we define the following linear injection operator:

$$\begin{aligned} \Lambda : W^{1,p}(\Omega) &\longrightarrow W_0^{1,p}(\Omega) \times \left[W^{1,p}(B_+)\right]^N \\ u &\longmapsto (\alpha_0 u, (\alpha_1 u) \circ \varphi_1, \cdots, (\alpha_N u) \circ \varphi_N) . \end{aligned}$$

Moreover, the product norm on Λu is equivalent to the $W^{1,p}(\Omega)$-norm. so Λ is a continuous linear injection operator from $W^{1,p}(\Omega)$ onto a closed subspace of the indicated product, and its inverse is continuous.

In a similar manner we can localize the discussion of functions on the boundary. In particular, the space $C^1(\partial\Omega)$, of continously differentiable functions on $\partial\Omega$, is the set of functions $f : \partial\Omega \to \mathbb{R}$ such that $(\alpha_i f) \circ \varphi_i \in C^1(B_0)$ for each $i,\ 1 \le i \le N$.

The manifold $\partial\Omega$ has an intrinsic measure denoted by *ds* for which integrals are given by

$$\int_{\partial\Omega} f\, ds = \sum_{i=1}^{N} \int_{\partial\Omega\cap U_i} (\alpha_i f) ds = \sum_{i=1}^{N} \int_{B_0} (\alpha_i f) \circ \varphi_i(y') J(\varphi_i) dy'$$

where $J(\varphi_i)$ is the indicated jacobian and dy' denotes the usual *Lebesgue* measure on $B_0 \subset \mathbb{R}^{N-1}$. Thus, we obtain a norm on $C(\partial\Omega)$ given by

$$\|f\|_{L^p(\partial\Omega)} = \left(\int_{\partial\Omega} |f|^p\, ds\right)^{\frac{1}{p}} .$$

The completion of $C(\partial\Omega)$ with respect to this normed is the Banach space $L^p(\partial\Omega)$. Futhermore, We have the following linear injection operator

$$\begin{aligned} \Gamma : L^p(\partial\Omega) &\longrightarrow [L^p(B_0)]^N \\ f &\longmapsto ((\alpha_1 f) \circ \varphi_1, \cdots, (\alpha_N f) \circ \varphi_N) \end{aligned}$$

onto a closed subspace of the product $[L^p(B_0)]^N$. Both Γ and its inverse are continuous.

Theorem 2.2.2.1 *Let Ω be a bounded open subset of $\mathbb{R}^N$ of class C^1 (or $\Omega = \mathbb{R}^N_+$). Then there exists a unique continuous linear extension operator*

$$P : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^N)$$

such that for all $u \in W^{1,p}(\Omega)$, we have

1. *$Pu = u$ in Ω,*
2. *$\|Pu\|_{L^p(\mathbb{R}^n)} \le C\|Pu\|_{L^p(\Omega)}$,*
3. *$\|Pu\|_{W^{1,p}(\mathbb{R}^N)} \le C\|Pu\|_{W^{1,p}(\Omega)}$, where the constant C depends only on Ω and p.*

Proof. Let $u \in W^{1,p}(\Omega)$ and $(U_i, \alpha_i, \varphi_j), 0 \le i \le N,\ 1 \le j \le N$ as in *lemma* 2.2.2.1. Then we can write u as

$$u = \sum_{i=0}^{N} \alpha_i u = \sum_{i=0}^{N} u_i, \quad \text{where} \quad u_i = \alpha_i u.$$

Now we will extend each function u_i.

• **Extension of** u_0.

Since $\alpha_0 \in D(\Omega)$ and $u \in L^p(\Omega)$, then

$$u_0 = \alpha_0 u \in L^p(\Omega)$$

and also

$$\frac{\partial \bar{u}_0}{\partial x_i} = \alpha_0 \overline{\frac{\partial u}{\partial x_i}} + \frac{\partial \alpha_0}{\partial x_i} \bar{u} \in L^p(\Omega). \tag{2.50}$$

Hence $u_0 \in W^{1,p}(\Omega)$
But we have that

$$supp(u_0) = supp(\alpha_0 u) \subseteq supp(\alpha_0) \cap supp(u) \subseteq supp(\alpha_0) \subseteq \Omega$$

which implies that the $supp(u_0)$ is compact in Ω.
Combining the fact that $u_0 \in W^{1,p}(\Omega)$, *lemma 2.2.1.2*, *proposition 2.2.2.2* and *proposition 1.2.9.2*, we have

$$u_0 \in W^{1,p}(\mathbb{R}^N) \text{ and } \frac{\partial \bar{u}_0}{\partial x_i} = \alpha_0 \overline{\frac{\partial u}{\partial x_i}} + \frac{\partial \alpha_0}{\partial x_i} \bar{u}.$$

From (2.39), there exists a constant $C \geq 0$ such that

$$\|\bar{u}_0\|_{W^{1,p}(\mathbb{R}^N)} \leq C \|u\|_{W^{1,p}(\Omega)}. \tag{2.51}$$

• **Extension of** u_i for $1 \leq i \leq N$.

Take the restriction of u on $\Omega \cap U_i$, and $\varphi_i : B \to U_i$ the coordinate map, then by *proposition* 2.2.1.3 the function $v_i(y) = u(\varphi_i(y))$, $y \in B_+$ belong to $W^{1,p}(B_+)$. We define v_i^* on B, to be the extension by reflection of v_i. By *proposition* 2.2.2.3 we have $v_i^* \in W^{1,p}(B)$. Now let $w_i(x) = v_i^*(\varphi_i^{-1}(x))$ for $x \in U_i$, then by *proposition 2.2.1.3* we have

$$w_i \in W^{1,p}(U_i), \quad w_i = u \text{ on } \Omega \cap U_i \text{ and } \|w_i\|_{W^{1,p}(U_i)} \leq c \|u\|_{W^{1,p}(\Omega \cap U_i)}. \tag{2.52}$$

Finally, we define on $\mathbb{R}^N$ the function $\hat{u}_i$ by

$$\hat{u}_i(x) = \begin{cases} \alpha_i(x) w_i(x) & \text{if } x \in U_i, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{2.53}$$

By *lemma 2.2.1.2*, $\hat{u}_i \in W^{1,p}(\mathbb{R}^N)$, $\hat{u}_i = u_i$ on Ω and

$$\|\hat{u}_i\|_{W^{1,p}(\mathbb{R}^N)} \leq C \|u\|_{W^{1,p}(\Omega \cap U_i)}. \tag{2.54}$$

Thus we have the desired operator given by

$$Pu = \bar{u}_0 + \sum_{i=1}^{N} \hat{u}_i. \tag{2.55}$$

Definition Let Ω be an open subset of $\mathbb{R}^N$. We call $D(\bar{\Omega})$ the space of all functions v such that v is the restriction on Ω of a function of $D(\mathbb{R}^N)$ i.e

$$D(\bar{\Omega}) = \{\varphi_{|\Omega} : \varphi \in D(\mathbb{R}^N)\}$$

Corollary 2.2.2.1 *Let $1 \leq p < \infty$. Then we have $W^{1,p}(\mathbb{R}^N) = W_0^{1,p}(\mathbb{R}^N)$ or equivalentely $D(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$.*

Proof. Obviously, $W_0^{1,p}(\mathbb{R}^N) \subseteq W^{1,p}(\mathbb{R}^N)$
So, let $u \in W^{1,p}(\mathbb{R}^N)$. Define $u_n = b_n(\rho_{\frac{1}{n}} * u)$. We know already that $(u_n) \subset D(\mathbb{R}^N)$ and $u_n \to u$ *in* $W^{1,p}(\mathbb{R}^N)$ *by theorem 2.2.1.1 and theorem 2.2.1.2.*
Hence $u \in W_0^{1,p}(\mathbb{R}^N)$ which complete the proof.

Corollary 2.2.2.2 *Let Ω be a bounded open subset of class C^1 in $\mathbb{R}^N$. Let $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$. Then there exists a sequence (u_n) in $D(\mathbb{R}^N)$ such that $u_n|\Omega \to u$ in $W^{1,p}(\Omega)$. More precisely $D(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$.*

Proof. By theorem 2.2.2.1, there exists a unique continuous linear extension operator, P. Let $u \in W^{1,p}(\Omega)$. Then the sequence $w_n = b_n(\rho_{1/n} * Pu) \in D(\mathbb{R}^N)$ converges to Pu in $W^{1,p}(\mathbb{R}^N)$. and so has the desired properties.

2.2.3 Trace Theory.

We shall describe the sense in which functions in $W^{1,p}(\Omega)$ have *boundary values* on $\partial\Omega$. Note that this is impossible in $L^p(\Omega)$ since $\partial\Omega$ is a set of measure zero in $\mathbb{R}^N$.

First, we consider the situation where Ω is the half-space $\mathbb{R}^N_+ = \{(x', x_N) \mid x_N > 0\}$, so then $\partial\Omega = \{(x', x_N) \mid x_N = 0\}$ is the simplest possible. The general case can be deduced to this case by using the fundamental *lemma*.

2.2.3.1 Trace operator γ_0

We shall define the trace operator γ_0 when $\Omega = \mathbb{R}^N_+ = \{(x', x_N) \mid x_N > 0\}$, where we let x' denote the $(N-1)-$tuple $(x_1, \cdots, x_{N-1})$.

Lemma 2.2.3.1 (Trace Inequality) *Let $1 \leq p < +\infty$, then there exists a constant $C > 0$ such that for all $\varphi \in D(\overline{\Omega})$, we have the following estimation:*

$$\left(\int_{\mathbb{R}^{N-1}} |\varphi(x', 0)|^p \, dx'\right)^{\frac{1}{p}} \leq C\|\varphi\|_{W^{1,p}(\Omega)}. \tag{2.56}$$

Proof. Let $G(t) = |t|^{p-1}t$ and $\varphi \in D(\mathbb{R}^N)$. We choose $x_N = A \in \mathbb{R}$ such that $(x', x_N) = (x', A) \notin \operatorname{supp}(\varphi)$. By the fundamental *theorem* of calculus, we have

$$G(\varphi(x', A)) - G(\varphi(x'0)) = \int_0^A D_n G(\varphi(x', x_N)) \, dx_N,$$

so

$$G(\varphi(x',0)) = -\int_0^{+\infty} D_n G(\varphi(x',x_N))\,dx_N = -\int_0^{+\infty} G'(\varphi(x',x_N)) D_n\varphi(x',x_N)\,dx_N,$$

so

$$|\varphi(x',0)|^p \le p\int_0^{+\infty} |\varphi(x',x_N)|^{p-1}|D_n\varphi(x',x_N)|\,dx_N.$$

If $1 < p < \infty$, by using *Fubini theorem* and *Holder*'s inequality we get

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} |u(x',0)|^p\,dx' &\le p\left(\int_\Omega |\varphi(x)|^p\,dx\right)^{1-\frac{1}{p}}\left(\int_\Omega |D_n\varphi(x',x_N)|^p\,dx\right)^{\frac{1}{p}}\\ &= p\|\varphi\|_{L^p(\Omega)}^{p-1}\|D_n\varphi\|_{L^p(\Omega)}\\ &\le C\|\varphi\|^p_{W^{1,p}(\Omega)}.\end{aligned}$$

Since $D(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$, we have proved the essential part of the following result.

Theorem 2.2.3.1 *The trace operator* $\gamma_0 : D(\bar{\Omega}) \to L^p(\partial\Omega)$ *defined by*

$$\gamma_0(\varphi)(x') = \varphi(x',0) \tag{2.57}$$

where $\Omega = \mathbb{R}^N_+$ *has a unique extension to a continuous linear operator* γ_0 *from* $W^{1,p}(\Omega)$ *to* $L^p(\partial\Omega)$ *whose range is dense in* $L^p(\partial\Omega)$*, and it satisfies*

$$\gamma_0(\alpha\cdot u) = \gamma_0(\alpha)\cdot\gamma_0(u), \quad \alpha \in D(\bar{\Omega}),\ u \in W^{1,p}(\Omega).$$

Proof. The first part follows from the preceding *lemma* 2.2.3.1 and *lemma* 2.2.2.2.

Now let b be a bump function on $\mathbb{R}$ and $\Psi \in D(\mathbb{R}^{N-1})$, then

$$\varphi(x) = \Psi(x')b(x_N), \quad x = (x',x_N) \in \mathbb{R}^N_+$$

defines a function in $D(\bar{\Omega})$ and $\gamma_0(\varphi) = \Psi$. Thus the range of γ_0 contains $D(\mathbb{R}^{N-1})$. So it is dense in $L^p(\partial\Omega)$. The last identity follows by the continuity of γ_0 and the observation that it holds for $u \in D(\bar{\Omega})$.

Proposition 2.2.3.1 (Integration by parts) . *Let* $1 \le p < \infty$*. If* $u \in W^{1,p}(\mathbb{R}^N_+)$ *and* $v \in D(\overline{\mathbb{R}^N_+})$ *then*

$$\int_{\mathbb{R}^N_+} v\frac{\partial u}{\partial x_N}\,dx = -\int_{\mathbb{R}^N_+} \frac{\partial v}{\partial x_N}u\,dx - \int_{\mathbb{R}^{N-1}} \gamma_0 v\gamma_0 u\,dx', \tag{2.58}$$

$$\int_{\mathbb{R}^N_+} v\frac{\partial u}{\partial x_i}\,dx = -\int_{\mathbb{R}^N_+} \frac{\partial v}{\partial x_i}u\,dx, \quad 1 \le i \le N-1. \tag{2.59}$$

Proof. Suppose that $u \in D(\overline{\mathbb{R}^N_+})$, integrate by part, we get, for $x' \in \mathbb{R}^{N-1}$,

$$\int_0^{+\infty} v(x', x_N)\frac{\partial u}{\partial x_N}(x', x_N)\, dx_N = -\int_0^{+\infty} \frac{\partial v}{\partial x_N}(x', x_N)u(x', x_N)\, dx_N - v(x', 0)u(x', 0)$$

The *Fubini theorem* gives

$$\int_{\mathbb{R}^N_+} v\frac{\partial u}{\partial x_N}\, dx = -\int_{\mathbb{R}^N_+} \frac{\partial v}{\partial x_N} u\, dx - \int_{\mathbb{R}^{N-1}} v(x', 0)u(x', 0)\, dx'$$

For $1 \leq i \leq N-1$

$$\begin{aligned}\int_{-\infty}^{+\infty} v(x', x_N)\frac{\partial u}{\partial x_i}(x', x_N)\, dx_i &= -\int_{-\infty}^{+\infty} \frac{\partial v}{\partial x_i} u(x', x_N)\, dx_i + u(x', x_N)v(x', x_N)|_{-\infty}^{+\infty} \\ &= -\int_{-\infty}^{+\infty} \frac{\partial v}{\partial x_i} u(x', x_N)\, dx_i.\end{aligned}$$

The *Fubini theorem* gives

$$\int_{\mathbb{R}^N_+} v\frac{\partial u}{\partial x_i}\, dx = -\int_{\mathbb{R}^N_+} \frac{\partial v}{\partial x_i} u\, dx, \quad 1 \leq i \leq N-1.$$

If $u \in W^{1,p}(\mathbb{R}^N_+)$, we conclude by using the density of $D(\overline{\mathbb{R}^N_+})$ in $W^{1,p}(\mathbb{R}^N)$ and the continuity of γ_0.

Theorem 2.2.3.2 *Let $1 \leq p < \infty$ and $u \in W^{1,p}(\mathbb{R}^N_+)$. Then the following assertions are equivalent:*

1. $u \in W^{1,p}_0(\mathbb{R}^N_+)$

2. $\gamma_0 u = 0$

3. $\bar{u} \in W^{1,p}(\mathbb{R}^N)$ *and* $\dfrac{\partial \bar{u}}{\partial x_i} = \overline{\dfrac{\partial u}{\partial x_i}}, \quad 1 \leq i \leq N.$

Proof.

1. ⇒ 2.

If $u \in W^{1,p}_0(\mathbb{R}^N_+)$, there exists a sequence (u_n) in $D(\mathbb{R}^N_+)$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R}^N_+)$. Since $\gamma_0 u_n = 0$ and $\gamma_0 u_n \to \gamma_0 u$, it follows that $\gamma_0 u = 0$.

2. ⇒ 3.
If $\gamma_0 u = 0$, by *proposition* above we have, for all $v \in D(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} v\overline{\frac{\partial u}{\partial x_k}}\, dx = -\int_{\mathbb{R}^N} \frac{\partial v}{\partial x_k}\bar{u}\, dx, \quad 1 \leq k \leq N$$

and this gives **3.**

3. ⇒ 1.
Consider (b_n) the sequence associated to the bump function b. The sequence $u_n = b_n\bar{u}$

converges to $\bar{u}$ in $W^{1,p}(\mathbb{R}^N)$ and the support of u_n is contained in $B(0,2n)\cap\overline{\mathbb{R}^N_+}$. Thus we can assume that the support of u is compact in $\overline{\mathbb{R}^N_+}$. Setting $y_n=(0,\cdots,0,1/n)$ and $v_n=\tau_{y_n}\bar{u}$. Since $\dfrac{\partial v_n}{\partial x_k}=\tau_{y_n}\dfrac{\partial \bar{u}}{\partial x_k}$, by using the continuity of the translation, we have $v_n\to u$ in $W^{1,p}(\mathbb{R}^N_+)$. We can now assume that the support u is compact in $\mathbb{R}^N_+$.

There exist a compact K in $\mathbb{R}^N_+$ such that for n large enought, we have the support of $w_n=\rho_n * u$ is include in K. Since $w_n\in C^\infty(\mathbb{R}^N_+)$, then $w_n\in D(\mathbb{R}^N_+)$. By *theorem* 1.1.7, it follows that w_n converges to u in $W^{1,p}(\mathbb{R}^N_+)$, so $u\in W^{1,p}_0(\mathbb{R}^N_+)$.

We can extend the preceding results to the case where Ω is sufficiently smooth region in $\mathbb{R}^N$. Suppose Ω given as in *lemma* 2.2.2.1 and denote by $\{U_i,\ 0\le i\le N\}$, $\{\varphi_i,\ 1\le i\le N\}$ and $\{\alpha_i,\ 0\le i\le N\}$ the open cover, corresponding local maps and the partition of unity, respectively. Recalling the linear injections Λ and λ constructed above, we are let to consider $\gamma_0: W^{1,p}(\Omega)\to L^p(\partial\Omega)$ defined by

$$\gamma_0(u) = \sum_{i=1}^{N}\Big[\gamma_0\Big(((\alpha_i u)\circ\varphi_i\Big)\Big]\circ\varphi_i^{-1} \tag{2.60}$$

$$= \sum_{i=1}^{N}\gamma_0(\alpha_i)\cdot\Big(\gamma_0(u\circ\varphi_i)\circ\varphi_i^{-1}\Big) \tag{2.61}$$

where the equality follows from *theorem* 2.2.3.1.

Theorem 2.2.3.3 *Let Ω be a bounded open set in $\mathbb{R}^N$ with C^1-boundary and such that Ω lies on one side of its boundary $\partial\Omega$. Then there exists a unique continuous linear operator $\gamma_0: W^{1,p}(\Omega)\to L^p(\partial\Omega)$ such that for each $u\in D(\bar{\Omega})$, $\gamma_0(u)$ is the restriction of u to $\partial\Omega$. The kernel of γ_0 is $W^{1,p}_0(\Omega)$ and its range is dense in $L^p(\Omega)$.*

This result is a special case of the trace *theorem* which we brief discuss. For a function $u\in D(\bar{\Omega})$, we define the various traces of normal derivatives given by

$$\gamma_j(u)=\frac{\partial^j u}{\partial \boldsymbol{n}^j}\,|_{\partial\Omega}\,,\quad 0\le j\le m-1. \tag{2.62}$$

where $\boldsymbol{n}=(\boldsymbol{n}_1,\cdots,\boldsymbol{n}_N)$ denote the unit outward normal on the boundary $\partial\Omega$ of Ω. When $\Omega=\mathbb{R}^N_+$ (or Ω is localized as above), these are given by $\dfrac{\partial u}{\partial \boldsymbol{n}}=-\dfrac{\partial u}{\partial x_N}\,|_{x_N=0}$. Each γ_j can be extended by continuity to all $W^{m,p}(\Omega)$ and we obtain the following result.

Theorem 2.2.3.4 *Let Ω be an open bounded open subset of $\mathbb{R}^N$ which lies on one side of its boundary, $\partial\Omega$, which we assume to be of class C^m. Then there is a unique continuous linear function γ from $W^{m,p}(\Omega)$ into $\prod_{j=0}^{m-1}W^{m-1-j,p}(\partial\Omega)$ such that*

$$\gamma(u)=(\gamma_0(u),\gamma_1(u),\cdots,\gamma_{m-1}(u)),\quad u\in D(\bar{\Omega}). \tag{2.63}$$

The kernel of γ is $W^{m,p}_0(\Omega)$ and its range is dense in the indicated product.

Proposition 2.2.3.2 *Let γ_0 be the trace operator from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$. Then for all u, v in $H^1(\Omega)$, we have*

$$\int_\Omega v(x)\frac{\partial u}{\partial x_i}\,dx = -\int_\Omega u(x)\frac{\partial v}{\partial x_i}\,dx + \int_{\partial\Omega}\gamma_0(v)(x')\gamma_0(u)(x')\boldsymbol{n}_i\,d\sigma(x'). \tag{2.64}$$

Theorem 2.2.3.5 (Divergence Theorem) . *Let Ω be a bounded open set in $\mathbb{R}^N$ with C^1 boundary. If $u \in H^1(\Omega, \mathbb{R}^N)$ then*

$$\int_\Omega \mathbf{div}\,u\,dx = \int_{\partial\Omega}\gamma_0 u\cdot\boldsymbol{n}\,dx'. \tag{2.65}$$

Proof.
Let $u \in H^1(\Omega, \mathbb{R}^N)$. We have,

$$\mathbf{div}\,u(x) = \sum_{i=1}^N D_i(u_i)(x).$$

So,

$$\begin{aligned}
\int_\Omega \mathbf{div}\,(u)\,dx &= \int_\Omega \sum_{i=1}^N D_i(u_i)\,dx \\
&= \sum_{i=1}^N \int_\Omega D_i(u_i)\,dx \\
&= \sum_{i=1}^N \int_{\partial\Omega}\gamma_0(u_i).1.\boldsymbol{n}_i(\sigma)\,d\sigma \quad (by\ \ 2.64) \\
&= \int_{\partial\Omega}\left(\sum_{i=1}^N \gamma_0(u_i).1.\boldsymbol{n}_i(\sigma)\right)d\sigma \\
&= \int_{\partial\Omega}\gamma_0 u.\boldsymbol{n}(\sigma)\,d\sigma.
\end{aligned}$$

Example 2.2.3.1 Take $u(x) = w(x)v(x)$ where $v(x) =$ vector field and $w(x) =$ scalar. Then we have

$$\begin{aligned}
\int_\Omega \mathbf{div}\,(wv)\,dx &= \int_\Omega \sum_{i=1}^N D_i(wv_i)\,dx = \int_\Omega \sum_{i=1}^N \left[(D_i w)v_i + w(D_i v_i)\right]\,dx \\
&= \int_\Omega \left[\nabla w\cdot v + w\mathbf{div}\,v\right]\,dx
\end{aligned}$$

On the other hand

$$\int_\Omega div(wv)\,dx = \int_{\partial\Omega} wv\cdot\boldsymbol{n}\,dx'$$

Thus we have

$$\int_\Omega w\mathbf{div}\,v\,dx = -\int_\Omega \nabla w\cdot v\,dx + \int_{\partial\Omega} w(v\cdot\boldsymbol{n})\,dx'$$

Application. Take $u = \nabla v$ and $w = \phi$= scalar. We get from above

$$\int_\Omega \phi \,\mathbf{div}\,(\nabla v)\,dx = -\int_\Omega \nabla\phi\cdot\nabla v\,dx + \int_{\partial\Omega}\phi(\nabla v\cdot \boldsymbol{n})\,dx'$$

So we get *Green Formula*

$$\int_\Omega \phi\Delta\,v\,dx = -\int_\Omega \nabla\phi\cdot\nabla v\,dx + \int_{\partial\Omega}\phi\frac{\partial v}{\partial \boldsymbol{n}}\,dx' \tag{2.66}$$

Theorem 2.2.3.6 *Assume that Ω is an open set in $\mathbb{R}^N$ with C^1 bounded boundary. Let $1 \le p < \infty$ and $u \in W^{1,p}(\Omega)$. Then the following assertions are equivalent:*

1. $u \in W_0^{1,p}(\Omega)$

2. $\gamma_0 u = 0$

3. $\bar{u} \in W^{1,p}(\mathbb{R}^N)$ *and* $\dfrac{\partial \bar{u}}{\partial x_i} = \overline{\dfrac{\partial u}{\partial x_i}},\quad 1 \le i \le N.$

CHAPTER 3

VARIATIONAL METHOD

This chapter is devoted to the application of **variational method** in solving the Dirichlet Homogeneous boundary value problem.
As mentioned earlier in the introduction, when the abstract formulation of the boundary value problem:

$$\boldsymbol{A}(u) = 0 \textit{ in } \Omega, \;\; \boldsymbol{B}(u) = 0 \textit{ on } \partial\Omega, \;\; \Omega \subset \mathbb{R}^N \textit{ open and bounded}, \tag{3.1}$$

where $\boldsymbol{A}(u) = 0$ denotes a given partial differential equation for unknown u and $\boldsymbol{B}(u) = 0$ is a given boundary condition. The **variational Method** could be employed where the operator $\boldsymbol{A}$ can be formulated as the first variation("derivative") of an appropriate functional J on a Banach space X,i.e.,$\boldsymbol{A}(u) = J'(u) \; \forall u \in X$.Therefore the equation $\boldsymbol{A}(u) = 0$ can be formulated weakly as

$$\langle J'(u), v\rangle = 0 \; \forall v \in X. \tag{3.2}$$

The advantage of this new formulation is that, solving problem (3.2) is equivalent to finding the **critical points** of J on X. The minimization method for variational problems is to solve the problem by finding the minimizers of a related functional.
We start by discussing important concepts needed from the theory of Optimization.

3.1 Optimization in Infinite Dimensional Spaces

Let X be a real normed linear space $F : X \to \mathbb{R} \cup \{+\infty\}$ an extended real valued function. Consider the following optimization problems:

$$\inf_{x\in X} F(x), \tag{3.3}$$

or

$$\sup_{x\in X} F(x), \tag{3.4}$$

Remark 3.1.0.1 *Observing that:*

$$\sup_{x\in X} F(x) = -\inf_{x\in X}(-F)(x),$$

in what follows, we shall restrict our discussion to problem like ((3.3))

Definition We say that a point $\bar{x} \in K$ is a local minimizer of F in K if there exists $r > 0$ such that

$$F(\bar{x}) \leq F(x) \quad \forall\, x \in K \cap B(\bar{x}, r). \tag{3.5}$$

It is a global minimizer if (3.5) holds for all points x in K.

To solve an optimization problem like ((3.3)) is to find a global or a local minimizers of F in K. If $\bar{x}$ is a global minimum of F in K, we just write:

$$F(\bar{x}) = \min_{x \in K} F(x)$$

Definition The domain of F is the subset of X denoted by Dom (F) and defined by:

$$\mathrm{Dom}(F) = \{x \in X \ : \ F(x) < +\infty\}.$$

We say that F is proper if its domain is nonempty.

3.1.1 Notation

If $\bar{x}$ is a global minimum of F in X, we just write

$$F(\bar{x}) = \min_{x \in K} F(x)$$

3.1.2 Lower Semi Continuous Functions(lsc)

Definition Let X be a real vector space. The epigraph of $F : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is the subset of $X \times \mathbb{R}$ defined by:

$$\mathbf{epi(F)} \ = \ \{(x, a) \in X \times \mathbb{R} : F(x) \leq a\}$$

We say that F is lower semi continuous if **epi(F)** is closed in $X \times \mathbb{R}$.

Proposition 3.1.2.1 Let $F : X \longrightarrow \mathbb{R} \cup \{+\infty\}$. Then F *is lower semi continuous* if and only if for every sequence (x_n) in X that convergence to $x \in X$, *we have*

$$F(x) \leq \liminf_n F(x_n)$$

.

Proof. Assume that F is **lsc.**. Let $x_n \to x$ in X. Let us pick a subsequence (x_{n_k}) of (x_n) such that $\lim_k F(x_{n_k}) = \liminf_n F(x_n)$. Then it follows that $(x_{n_k}, F(x_{n_k}))$ is a sequence of **epi(F)** that converges to $(x, \liminf_n F(x_n))$. Since **epi(F)** is closed , we have $(x, \liminf F(x_n)) \in$ **epi(F)** and then

$$F(x) \leq \liminf_n F(x_n).$$

Conversely assume that $(x_n \to x \text{ in } X) \Rightarrow \ F(x) \leq \liminf_n F(x_n)$. Let (x_n, a_n) be a sequence in epi(F) that converges to (x, a) in $X \times \mathbb{R}$. Then $x_n \to x$ and $a_n \to a$. Therefore by hypothesis, we have :

$$F(x) \leq \liminf_n F(x_n) \leq \liminf_n a_n = \lim_n a_n = a$$

so $(x, a) \in$ epi(F) and then epi(F) is closed which implies that F is lsc.

Proposition 3.1.2.2 A function $F : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lsc if and only if *for all* $a \in \mathbb{R}$, $C_a = \{x \in X : \ F(x) \leq a\}$ is closed in X.

Proof. Assume that F is *lsc*. Let $a \in \mathbb{R}$ and let (x_n) be a sequence in C_a that converges to $x \in X$. Since $F(x_n) \leq a \ \forall n$ we have $\liminf_n F(x_n) \leq a$. By proposition 3.1.2.1,

$$F(x) \leq \liminf_n F(x_n) \leq a,$$

So $x \in C_a$. Therefore C_a is closed.

Conversely assume that C_a is closed for all $a \in \mathbb{R}$. Let (x_n) be a sequence in X that converges to $x \in X$. There exists a subsequence (x_{n_k}) of (x_n) such that:

$$\lim_k \ F(x_{n_k}) = \liminf_n \ F(x_n).$$

Assume that $F(x) > \liminf_n F(x_n)$. Then there exists $a \in \mathbb{R}$ such that

$$\liminf_n F(x_n) < a < F(x) \tag{3.6}$$

So there exists $N \in \mathbb{N}$ such that $F(x_{n_k}) < a$ for all $k \geq N$. Therefore $x_{n_k} \in C_a \ \forall k \geq N$. Since $x_{n_k} \to x$ for $k \geq N$ and C_a closed we have $x \in C_a$ and then $F(x) \leq a$. Using (3.6) we get a contradiction. So $F(x) \leq \liminf_n F(x_n)$.

3.1.3 Convex sets.

Definition Let X be a real linear space and $K \subset X$. The set K is called *convex* if for each $x_1, x_2 \in K$ and for each $t \in [0, 1]$, we have $tx_1 + (1-t)x_2 \in K$.

Notation. Let $x, y \in K$. We write $[x, y]$ for the geometric segment from x to y, i.e.,

$$[x, y] := \{z \in X \ : \ \exists\ t \in [0, 1] \ \textit{such that}\ z = tx + (1-t)y\}.$$

Hence, $z = tx + (1-t)y = y + t(x-y)$. We also have that $[x, y] = [y, x]$.

Remark 3.1.3.1 *K is convex if and only if $\forall\ x, y \in K, [x, y] \subset K$.*

Notation Let $n \in \mathbb{N}$. Define

$$\Lambda_n \ := \ \{(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^n_+ \ : \ \sum_{i=1}^{n} \alpha_i \ = \ 1\}.$$

Proposition 3.1.3.1 *$K \subset X$ is convex if and only if for each $n \in \mathbb{N}$, for each $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \Lambda_n$ and for each $(x_1, x_2, \cdots, x_n) \in K^n$, we have $\sum_{i=1}^{n} \alpha_i x_i \in K$*

Proof. ($\Rightarrow$) Suppose that K is convex.We prove by induction. For $n = 2$, let $(\alpha_1, \alpha_2) \in \Lambda$, *and* $(x_1, x_2) \in K^2$. Then $\displaystyle\sum_{i=1}^{2} \alpha_i x_i = \alpha_1 x_1 + \alpha_2 x_2$ *and* $\alpha_1 + \alpha_2 = 1$ (so that $\alpha_2 = 1 - \alpha_1$), i.e., $\sum_{i=1}^{2} \alpha_i x_i = \alpha_1 x_1 + (1-\alpha_1)x_2 \in K^2$. So the result holds for $n = 2$.

Assume now it holds for n=k for some $k > 2$. Let $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \Lambda_{k+1}$ (so that $\sum_{i=1}^{k+1} \alpha_i = 1$), and $(x_1, x_2, \cdots, x_k, x_{k+1}) \in K^{k+1}$. Then,

$$y_k \; := \sum_{i=1}^{k+1} \alpha_i x_i = \sum_{i=1}^{k+1} \alpha_i x_i + \alpha_{k+1} x_{k+1}.$$

Since $\sum_{i=1}^{k+1} \alpha_i = 1$, we have $\alpha_{k+1} = 1 - \sum_{i=1}^{k} \alpha_i$.

Case 1. If α_{k+1}, then $\alpha_1 = \alpha_2 = \alpha_k = 0$, and so $y_k = X_{k+1} \in K$ (by our hypothesis.)

Case 2. If $\alpha_{k+1} \leq 1$, then $\sum_{i=1}^{k} \alpha_i = 1 - \alpha_{k+1} \geq 0$. Hence,

$$y_k = (1-\alpha_{k+1}) \frac{1}{(1-\alpha_{k+1})} \sum_{i=1}^{k} \alpha_i x_i + \alpha_{k+1} x_{k+1}$$

$$= (1-\alpha_{k+1}) \sum_{i=1}^{k} \frac{\alpha_i}{(1-\alpha_{k+1})} x_i + \alpha_{k+1} x_{k+1}.$$

Observe that $(\frac{\alpha_1}{(1-\alpha_{k+1})}, \cdots, \frac{\alpha_k}{(1-\alpha_{k+1})}) \in \Lambda_k$. Thus, $x \; := \sum_{i=1}^{k} \frac{\alpha_i}{(1-\alpha_{k+1})} x_i \in K$ (by induction hypothesis).
Now, $y_k = (1-\alpha_{k+1})x + \alpha_{k+1} x_{k+1}$ and $\alpha_{k+1} \in [0,1]$. Hence, $y_k \in K$.

$(\Leftarrow)$ Let $n = 2, t \in [0,1], (x_1, x_2) \in K^2$. Put $\alpha_1 = t \geq 0, \alpha_2 = 1-t \geq 0, \alpha_1+\alpha_2 = 1, (\alpha_1, \alpha_2) \in \Lambda_2$. But $\alpha_1 x_1 + \alpha_2 x_2 \in K$. Thus, $tx_1 + (1-t)x_2 \in K$. // Since $\alpha_1 x_1 + \alpha_2 x_2 = tx_1 = (1-t)x_2$. Hence, $\sum_{i=1}^{2} \alpha_i x_i \in K$ and this implies that K is convex. Hence, the proposition holds.

Example 3.1.3.1 *a. Every affine space is convex.*

Recall that an affine space is simply a translation of a linear space by a point, e.g., if X is a linear space, then $V := X + x_0 := \{v + x_0 \; : \; v \in X\}$ is an affine space.

b. Every hyperplane is convex
Given that X is a real normed linear space. Recall $X^* = \{f \; : \; X \to \mathbb{R} \; : \; fislinearandbounded(continuo$
For $\alpha \in \mathbb{R}$, define

$$H_{f,\alpha} := \{x \in X \; : \; f(x) = \alpha, f \in X^*\} = \{x \in X \; : \; \langle x, f \rangle = \alpha\}$$

where $\langle x, f \rangle \equiv f(x)$. Then $H_{f,\alpha}$ is called a hyperplane of X.

c. Every half space is convex.

Given X a real linear space, define the following half-spaces.

$D^{+}_{f,\alpha} := \{x \in X \, : \, \langle x, f\rangle \geq \alpha\}$

$D^{+*}_{f,\alpha} := \{x \in X \, : \, \langle x, f\rangle > \alpha\}$

$D^{-}_{f,\alpha} := \{x \in X \, : \, \langle x, f\rangle \leq \alpha\}$

$D^{-*}_{f,\alpha} := \{x \in X \, : \, \langle x, f\rangle < \alpha\}.$

d. If $\bar{x} \in X -$ a real linear space and $r > 0$ then the ball centered at $\bar{x}$ with radius r

$$B'(\bar{x}, r) = \{x \in X \, : \, \|x - \bar{x}\| \leq r\}$$

is a convex set in X.

3.1.4 Convex Functions

Definition Let $F : X \to \mathbb{R} \cup \{+\infty\}$ a real-value function. Then
a.the function F is convex if

$$F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y)$$

for all $x, y \in X$ and all λ with $0 \leq \lambda \leq 1$.
b.The function F is strictly convex if

$$F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y)$$

for all $x, y \in X$ with $x \neq y$ and all λ with $0 < \lambda < 1$. If the inequalities in the above definitions are reversed, we obtain the definitions of concave functions.

Remark 3.1.4.1 *Note that F is convex (resp. strictly convex) on a convex set X if and only if $-F$ is is concave (resp. strictly concave) on X. Because of this close connection, we will formulate all results in terms of convex functions only. Corresponding results for concave functions will be clear.*

Proposition 3.1.4.1 *F is convex if and only if* **epi(F)** *is convex in $V \times \mathbb{R}$. where*

$$\mathbf{epi(F)} = \{(x, a) \in V \times \mathbb{R} : F(x) \leq a\}$$

Proposition 3.1.4.2 *Let F_1, F_2 be convex functions on X. Let $\lambda > 0$, and φ a increasing convex function on $\mathbb{R}$. Then $F_1 + F_2$, $\max(F_1, F_2)$, λF_1 and $F_1 o F_2$ are convex functions.*

Theorem 3.1.4.1 (Link to Optimization Problems)
1. If F is strictly convex on X, then F has at most one minimizer, that is if the minimizer exists, then it must be unique.
2. Any local minimizer of a convex function is also a global minimizer.
3. If F is convex, then the set of minimizers is a convex subset of X.

Proof.
1. Let x_1 and x_2 two different minimizers of F and let λ with $0 < \lambda < 1$. Because of the strict convexity of F and the fact that

$$F(x_1) = F(x_2) = \min_{x \in X} F(x)$$

we have

$$F(x_1) \leq F(\lambda x_1 + (1-\lambda)x_2) < \lambda F(x_1) + (1-\lambda)F(x_2) = F(x_1)$$

which is a contradiction, therefore, $x_1 = x_2$.
2.Suppose that $\bar{x}$ is a local minimizer of F in X. Then there is a positive number r such that

$$F(\bar{x}) \leq F(x), \ \forall x \in B(\bar{x}, r).$$

Given any $x \in X$, we want to show that $F(\bar{x}) \leq F(x)$. To this end select λ, with $0 < \lambda < 1$ and so small that

$$\bar{x} + \lambda(x - \bar{x}) = \lambda x + (1-\lambda)\bar{x} \in B(\bar{x}, r)$$

Then

$$F(\bar{x}) \leq F(\bar{x} + \lambda(x - \bar{x})) = F(\lambda x + (1-\lambda)\bar{x}) \leq \lambda F(x) + (1-\lambda)F(\bar{x})$$

because F is convex. Now subtract $F(\bar{x})$ from both sides of the preceding inequality and divide the result by λ to obtain $0 \leq F(x) - F(\bar{x})$. This establishes the desired results.

Proposition 3.1.4.3 *Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be convex and lower semi continuous. Then F is weakly lower semi continuous, that is* **epi(F)** *is weakly closed in $X \times \mathbb{R}$.*

Proof. Assume that $F : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ lower semi continuous and convex. Then **epi(F)** is closed and convex in $X \times \mathbb{R}$, therefore **epi(F)** is weakly closed and then F is weakly lower semi continuous.

3.1.5 Gateaux Differentiability

Definition Let X and Y be Banach spaces, and $f : U \subset X \to Y$, where U is an open subset of X. The *directional derivative* of f at $x \in U$ in the direction $h \in X$ is given by

$$\delta f(x, h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}, \text{ if this limit exists.}$$

If all directional derivatives at x exists for every $h \in X$, and the function $f'_G(x) : X \to Y$ defined by $f'_G(x)h = \delta f(x, h)$ is a linear and continuous map, then we say that f is *Gateaux differentiable* at x, and we call $f'_G(x)$ the *Gateaux derivative* of f at x.
The directional derivative may also be written as

$$\delta f(x, h) = \frac{d}{dt} f(x + th)|_{t=0}.$$

3.1.6 Existence Results

Theorem 3.1.6.1 *Let X be a real reflexive Banach space and let K be a closed convex bounded and nonempty subset of X. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous and convex. Then, $\exists\, \bar{x} \in K$ such that $f(\bar{x}) \leq f(x)\, \forall x \in K$, i.e.,*

$$f(\bar{x}) = \inf_{x \in k} f(x) = \min_{x \in k} f(x).$$

Proof. f is lower semi-continuous and convex $\Rightarrow$ f is weakly lower semi-continuous.
Put $m := \inf_{x \in K} f(x)$.
First suppose $m = -\infty$. Then for $n \in \mathbb{N}, \exists\, x_n \in K$ such that

$$f(x_n) < -n \tag{3.7}$$

Boundedness of K implies $\{x_n\}$ is bounded and *Eberlein − Smul′yan Theorem* implies $\exists\, \{x_{n_k}\}$ subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x \in X$. But K is convex and closed which implies that K is weakly closed. Hence $x \in K$. By weak lower semi-continuity of f, we have

$$f(x) \leq \liminf_{k \to +\infty} f(x_{n_k}) < -\infty$$

by (3.7), and this is impossible. Hence $m \in \mathbb{R}$.

We now use the definition of **inf**. Let $n \in \mathbb{N}$ and take $\epsilon_n = \frac{1}{n}$. Then $\exists\, x_n \in K$ such that $m \leq f(x_n) < m + \frac{1}{n}$. $\{x_n\}$ in K implies $\{x_n\}$ is bounded and so $\exists \{x_{n_k}\}_{k \in \mathbb{N}}$, subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$. Since f is weakly lower semi-continuous, we have

$$\begin{aligned} f(\bar{x}) &\leq \liminf_{k \to +\infty} f(x_{n_k}) \\ &\leq \liminf_{k \to +\infty} (m + \frac{1}{n_k}) \\ &= \lim_{k \to +\infty} (m + \frac{1}{n_k}) = m. \end{aligned}$$

Thus, $f(\bar{x}) \leq m = \inf_{x \in K} f(x)$. But $m = f(\bar{x})$. Hence,

$$f(\bar{x}) = m = \inf_{x \in K} f(x).$$

Next, we prove the second important existence theorem.

Theorem 3.1.6.2 *Let X be a real reflexive Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex proper lower semi-continuous function. Suppose*

$$\lim_{\|x\| \to \infty} f(x) = +\infty$$

Then, $\exists\, \bar{x} \in x$ such that $f(\bar{x}) \leq f(x)\, \forall x \in X$, i.e.,

$$f(\bar{x}) = \inf_{x \in X} f(x).$$

Proof. We shall apply Theorem 3.1.6.1. Since f is proper, $\exists\, x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. Let

$$K := \{x \in X \,:\, f(x) \leq f(x_0)\}.$$

We now show that K is closed convex nonempty and bounded, so that we can apply Theorem 3.3.0.2. But K is convex and closed since f is convex and lower semi-continuous. This implies K is a section with $f(x_0) = \alpha$

Claim K is bounded.
Suppose this is not the case. Then for each $n \in \mathbb{N}$, $\exists x_n \in K$ such that $\|x_n\| > n$. Thus,

$$f(x_n) \leq f(x_0), \|x_n\| > n. \tag{3.8}$$

This implies that $\lim\limits_{n\to+\infty} \|x_n\| = +\infty$ and so (by hypothesis),

$$\lim_{n\to+\infty} f(x_n) = +\infty \tag{3.9}$$

contradicting inequality (3.8). Hence K is boundeded. Theorem 3.3.0.2 then implies $\exists \bar{x} \in K \subset X$ such that $\forall\, x \in K$,

$$f(\bar{x}) \leq f(x), \forall\, x \in X.$$

Now, let $x \in X \backslash K$. Then $f(x) > f(x_0)$. But $x_0 \in K$. So, $f(\bar{x}) \leq f(x_0)$. Hence, $f(x) > f(\bar{x})$, $\forall x \in X$, i.e.,

$$f(\bar{x}) \leq f(x), \forall x \in X.$$

Remark 3.1.6.1 *If* $\lim\limits_{\|x\|\to+\infty} f(x) = +\infty$, *we say that* f *is* <u>*coercive*</u>.

3.1.7 Optimality Conditions

Theorem 3.1.7.1 *(First Optimality Condition.) Let* X *a real Banach space,* $U \subset X$ *open and* $f : U \subset X \to \mathbb{R}$. *Let* $x_0 \in U$ *a local minimum of* f *on* U.
(i). If f *is Gateaux differentiable at* x_0, *then*

$$\delta f(x_0, h) = \frac{d}{dt} f(x_0 + th)|_{t=0} = 0\ \forall h \in X. \tag{3.10}$$

(ii). If in addition f *is convex, then* (3.10) *becomes a sufficient condition for* x_0 *to be a minimizer of* f *on* U.

Proof. (i) Let $x_0 \in U$ be a local minimizer of f on U. This implies that $\exists\, r > 0$ such that $f(x_0) \leq f(x)\ \forall\, x \in B(x_0, r) \subset U$.

Let $h \in X$. We have that $x_0 + th \in B(x_0, r)$ if and only if $|t| < \frac{r}{\|h\|}$. Define $\delta := \frac{r}{\|h\|}$, then $x_0 + th \in B(x_0, r)\, \forall t \in (-\delta, \delta)$.

So, $f(x_0 + th) \geq f(x_0)$.

Let $t \in (0, \delta)$, then $\frac{f(x_0+th)-f(x_0)}{t} \geq 0$, letting $t \to 0^+$, we obtain that

$$\delta f(x_0, h) \geq 0. \tag{3.11}$$

Let $t \in (-\delta, 0)$, then $\frac{f(x_0+th)-f(x_0)}{t} \leq 0$, letting $t \to 0^-$, we obtain that

$$\delta f(x_0, h) \leq 0. \tag{3.12}$$

Combining (3.11) and (3.12), we have that

$$\delta f(x_0, h) = 0 \, \forall h \in X$$

(ii) Assume that $\delta f(x_0, h) = 0 \, \forall h \in X$.
We have to prove that $f(x_0) \leq f(x) \, \forall x \in U$,i.e., $x_0 \in U$ is a minimizer of f on U.
Let $x \in U$, using the convex inequality, we have

$$f(x) \geq f(x_0) + \delta f(x_0, x - x_0).$$

From our assumption, we have $\delta f(x_0, x - x_0) = 0$
So, $f(x) \geq f(x_0)$

Theorem 3.1.7.2 *Let X a real Banach space, $U \subset X$ open and $f : U \subset X \to \mathbb{R}$, be a C^2 function. Let $x_0 \in U$ such that $f'(x_0) = 0$. If x_0 is a local minimizer of f on U, then*

$$f''(x_0)(h, h) \geq 0 \;\; \forall \, h \in X.$$

Proof. Let $h \in X$. Let $t \in \mathbb{R}$. Using Taylor's Theorem, we have

$$f(x_0+th) = f(x_0)+tf'(x_0)h+\frac{t^2}{2}f(x_0)(h,h)+t^2\|h\|^2\epsilon(x_0+\theta th), \; for \, \theta \in (0,1) \; where \; \lim_{x \to x_0} \epsilon(x) = 0 \tag{3.13}$$

Since x_0 is a local minimizer,$\exists \, r > 0$ such that

$$f(x) \geq f(x_0) \, \forall x \in B(x_0, r).$$

Define $\delta := \frac{r}{\|h\|}$. $\forall t \in (0, \delta)$ we have $x_0 + th \in B(x_0, r)$, therefore $f(x_0 + th) \geq f(x_0) \, \forall t \in (0, \delta)$.
Using (3.13) and the fact that $f'(x_0) = 0$, we have

$$0 \leq f(x_0 + th) - f(x_0) = \frac{t^2}{2} f''(x_0)(h, h) + t^2\|h\|^2\epsilon(x_0 + \theta th)$$

So,

$$t^2(\frac{1}{2}f''(x_0)(h, h) + \|h\|^2\epsilon(x_0 + \theta th)) \geq 0$$

This implies that $\frac{1}{2}f''(x_0)(h, h) + \|h\|^2\epsilon(x_0 + \theta th) \geq 0$, letting $t \to 0$ and using the fact that $\epsilon(x) \to 0$ as $x \to x_0$, we obtain

$$\frac{1}{2}f''(x_0)(h, h) \geq 0.$$

So, $f''(x_0)(h, h) \geq 0$.

3.2 Application to Elliptic Partial Differential Equation

In this section we shall consider a method of solving PDE problems by taking into account an optimization problem. We shall show that to solve the PDE problem it is enough to solve an optimization problem and more precisely, the solution of the optimization problem turns out to be a solution to the PDE under consideration.

We consider the following Partial Differential Equation;

$$(P)\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is bounded open subset of $\mathbb{R}^N$ of class C^1 and $f \in L^2(\Omega)$.

We define the functional J for each $u \in H_0^1(\Omega)$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega fu\,dx.$$

Now we consider the following optimization problem;

$$(P')\begin{cases} \min J(u) \\ \\ u \in H_0^1(\Omega). \end{cases}$$

In $H_0^1(\Omega)$, we use the norm defined by

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2\,dx\right)^2 = \|\nabla u\|_{L^2(\Omega)}.$$

We shall solve problem P' and show that the solution of problem (P') is a solution of problem (P).

Theorem 3.2.0.3 (P') *has a unique solution, characterized by* $\delta J(u;v) = 0,\ \forall v \in H_0^1(\Omega)$.

Proof. We shall show that;
(1) J has a unique minimizer u in $H_0^1(\Omega)$.

(2) J is *Gâteaux* differentiable at u in $H_0^1(\Omega)$, and that $\delta J(u;v) = 0$ for all $v \in H_0^1(\Omega)$.

We have that for each $u \in H_0^1(\Omega)$,

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\nabla|^2\,dx - \int_\Omega fu\,dx \\ &= \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - \int_\Omega fu\,dx \\ &= \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \int_\Omega fu\,dx. \end{aligned}$$

1.

(a) We show that J is continuous on $H_0^1(\Omega)$.

We have that $u \mapsto -\int_\Omega fu\,dx$ is linear and $\left|-\int_\Omega fu\,dx\right| \leq \int_\Omega |f||u|\,dx \leq \|f\|_{L^2(\Omega)}\|u\|_{H^10(\Omega)}$ (by Holder's and Poincaré Inequality), therefore continuous. Also $u \mapsto \frac{1}{2}\|u\|^2_{H_0^1(\Omega)}$ is continuous using the continuity of the norm function. Therefore, it follows that J is continuous on $H_0^1(\Omega)$.

(b) We show that J is strictly convex.

Let $u, v \in H_0^1(\Omega)$ and $\lambda \in (0,1)$, $u \neq v$.

$u, v \in H_0^1(\Omega)$ implies that $u, v \in H^1(\Omega)$.

$$
\begin{aligned}
J(\lambda u + (1-\lambda)v) &= \frac{1}{2}\|\lambda u + (1-\lambda)v\|^2_{H_0^1(\Omega)} - \int_\Omega f \cdot (\lambda u + (1-\lambda)v)\,dx \\
&= \frac{\lambda}{2}\|u\|^2_{H_0^1(\Omega)} + \frac{(1-\lambda)}{2}\|v\|^2_{H_0^1(\Omega)} - \frac{\lambda(1-\lambda)}{2}\|u-v\|^2_{H_0^1(\Omega)} \\
&- \lambda\int_\Omega fu\,dx - (1-\lambda)\int_\Omega fv\,dx \\
&= \lambda J(u) + (1-\lambda)J(v) - \lambda(1-\lambda)\|u-v\|^2_{H_0^1(\Omega)} \\
&< \lambda J(u) + (1-\lambda)J(v).
\end{aligned}
$$

Therefore J is strictly convex.

(c) We show that J is coercive.

$$
\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \int_\Omega fu\,dx \\
&\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C\|f\|_{L^2(\Omega)}\|u\|_{H_0^1(\Omega)} \quad \text{(by Hölder's and Poincaré Inequality)} \\
&= \|u\|_{H_0^1(\Omega)}\left(\frac{1}{2}\|u\|_{H_0^1(\Omega)} - C\|f\|_{L^2(\Omega)}\right).
\end{aligned}
$$

So that as $\|u\| \to +\infty$, we have that $J(u) \to +\infty$. Therefore J is coercive.

Hence, by (a) J is *lower semi continuous* on $H_0^1(\Omega)$, $H_0^1(\Omega)$ is a real reflexive Banach space as a closed subspace of a real reflexive Banach space $H^1(\Omega)$, by (b) J is convex, by (c) J is coercive, $J(0) = 0 \in \mathbb{R}$, i.e., J is proper.
Therefore by Theorem 3.1.7.2, and the fact that J is strictly convex, there exists a unique $u \in H_0^1(\Omega)$ such that

$$u = \min_{w \in H_0^1(\Omega)} J(w).$$

(2) Let $v \in H_0^1(\Omega)$, let $t \neq 0 \in \mathbb{R}$

$$\begin{aligned}
\frac{J(u+tv)-J(u)}{t} &= \frac{1}{2}\int_\Omega \frac{|\nabla(u+tv)(x)|^2 - |\nabla u(x)|^2}{t}\,dx - \int_\Omega fv\,dx \\
&= \frac{1}{2}\int_\Omega |\nabla v(x)|^2\,dx + \int_\Omega \nabla u(x)\nabla v(x)\,dx - \int_\Omega fv\,dx.
\end{aligned}$$

Therefore as $t \to 0$, we have

$$\delta J(u;v) = \lim_{t\to 0}\frac{J(u+tv)-J(u)}{t} = \int_\Omega \nabla u(x)\nabla v(x)\,dx - \int_\Omega fv\,dx$$

i.e., the directional derivatives of J at u exists for all $v \in H_0^1(\Omega)$. The function $v \mapsto \delta J(u;v)$ is linear, by the linearity of integral and the gradient function ∇, hence we have

$$\begin{aligned}
|\delta J(u;v)| &= \left|\int_\Omega \nabla u \nabla v\,dx - \int_\Omega fv\,dx\right| \\
&\leq \int_\Omega |\nabla u||\nabla v|\,dx - \int_\Omega |f||v|\,dx \\
&= C_1\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} + C_2\|f\|_{L^2(\Omega)}\|v\|_{H_0^1(\Omega)} \quad \text{(by Hölder's and Poincaré Inequality)} \\
&\leq K\|v\|_{H_0^1(\Omega)} \quad \left(K = C_1\|u\|_{H_0^1(\Omega)} + C_2\|f\|_{L^2(\Omega)}\right),
\end{aligned}$$

which implies that $v \mapsto \delta J(u;v)$ is continuous on $H_0^1(\Omega)$. Therefore J is *Gateaux* differentiable on $H_0^1(\Omega)$, hence J is *Gateaux* differentiable at u.

Let u the solution of problem (P′).
By Theorem 3.1.7.1 and (2) above, $\delta J(u;v) = 0\ \forall v \in H_0^1(\Omega)$, i.e.,

$$\int_\Omega \nabla u \nabla v\,dx - \int_\Omega fv\,dx = 0 \ \ \forall v \in H_0^1(\Omega). \tag{3.14}$$

This completes the proof.

Now, from (3.14) if u is of class C^2, then u solves problem (P).
Using the fact that $D(\Omega) \subset H_0^1(\Omega)$ we obtain that

$$\int_\Omega \nabla u \nabla v\,dx - \int_\Omega fv\,dx = 0 \ \ \forall v \in D(\Omega).$$

Hence $-\Delta u = f$ in the sense of distribution, since u is of class C^2, it then follows that $-\Delta u = f$ in Ω. Since $u \in H_0^1(\Omega)$, $u = 0$ on $\partial\Omega$, i.e.,

$$\begin{cases} -\Delta u = f \ \text{ in } \ \Omega \\ \\ u = 0 \ \text{ on } \ \partial\Omega \end{cases}$$

Therefore u solves problem (P).

Theorem 3.2.0.4 *Any classical solution of problem (P) is the solution of problem (P')*

Proof. Let u a classical solution of problem (P).
Therefore u satisfies $-\Delta u = f$ in Ω. Let $v \in H_0^1(\Omega)$, multiplying both sides of $-\Delta u = f$ by v and integrating by part, we obtain

$$\int_\Omega \nabla u \nabla v \, dx = \int_\Omega f v \, dx, \tag{3.15}$$

which implies that

$$\delta J(u; v) = \int_\Omega \nabla u \nabla v \, dx - \int_\Omega f v \, dx = 0. \tag{3.16}$$

Hence, since J is convex, by (ii) of Theorem 3.1.7.1, u is a minimizer of the functional J on $H_0^1(\Omega)$ or equivalently u solves problem (P').

CONCLUSION

We have seen that, given any Boundary Value Problem for some Partial Differential Equations, we can formulate variational problems which will assist us in solving these Partial Differential Equations.
It is important to note that, although how powerful the variational method may be, not all Partial Differential Equations could be formulated as variational problems.There are lots of other(nonvariational) important methods for studying Partial Differential Equations.

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